Bachelorthesis

Stacked Treewidth and the Colin de Verdière Number

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Statement of Authorship

I hereby declare that this document has been composed by myself and describes my own work, unless otherwise acknowledged in the text.

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Abstract:

For $k \in \mathbb{N}$, a k-tree is constructed by starting with a $(k+1)$-clique and then repeatedly connecting a new vertex to all vertices of an existing k-clique. A k-tree is called a stacked k-tree, if during its construction no two vertices are stacked onto the same k-clique. We introduce the stacked treewidth $\mathrm{stw}(G)$ of a graph G as the smallest integer k such that G is subgraph of a stacked k-tree. The Colin de Verdière number μ is a graph parameter introduced 1993 by Yves Colin de Verdière which has the interesting property that the outerplanar, planar and linklessly embeddable graphs are characterized by $\mu(G) \leq 2$, $\mu(G) \leq 3$, and $\mu(G) \leq 4$, respectively. Recent results by Fallat and Mitchell show that for all chordal graphs G, $\mathrm{stw}(G) = \mathrm{tw}(G) + 1$ if and only if $\mu(G) = \mathrm{tw}(G) + 1$. We show that the same relation holds for all planar graphs, but for each $k \geq 4$, we find a counterexample H, such that $\mathrm{stw}(H) = \mathrm{tw}(H)+1 = k+1$, but $\mu(H) \leq k$. Along the way, we develop new tools for handling $\mathrm{stw}(\cdot)$ and we prove that $\mathrm{stw}(\cdot)$ is minor-monotone and behaves identical to $\mu(\cdot)$ on clique-sums.

Contents

1 Introduction

A widespread notion in graph theory is the notion of k-trees. For $k \in \mathbb{N}$, a k-tree is constructed by starting with a $(k+1)$-clique and then repeatedly connecting a new vertex to all vertices of an existing k-clique. For $k = 1$, they are the normal trees, and for $k \geq 2$ they can be seen as similar to normal trees, but "thicker."

A k-tree is called a *stacked* k-tree, if during its construction no two vertices are stacked onto the same k-clique, or, equivalently, if it contains no k-clique C with three or more vertices fully connected to C. Stacked k-trees have also been called *simple k-trees* or *simple-clique k-trees* in the literature [9, 15].
Knauer and Ueckerdt noticed two relations, the first one being

$$\begin{aligned} T \text{ is a stacked 1-tree} &\Leftrightarrow T \text{ is a 1-tree, which is a path.} \\ T \text{ is a stacked 2-tree} &\Leftrightarrow T \text{ is an outerplanar 2-tree.} \\ T \text{ is a stacked 3-tree} &\Leftrightarrow T \text{ is a planar 3-tree.} \\ T \text{ is a stacked 4-tree} &\Leftrightarrow T \text{ is a linklessly embeddable 4-tree.} \end{aligned}$$

and the second one being (using results of Kratochvíl and Vaner [12])

$$\begin{aligned} G \text{ subgraph of a path and } G \text{ covered by a 1-tree} &\Rightarrow G \text{ covered by a stacked 1-tree.} \\ G \text{ outerplanar and } G \text{ covered by a 2-tree} &\Rightarrow G \text{ covered by a stacked 2-tree.} \\ G \text{ planar and } G \text{ covered by a 3-tree} &\Rightarrow G \text{ covered by a stacked 3-tree.} \end{aligned}$$

Furthermore, the *Colin de Verdière number* $\mu(G)$ of a graph G is defined in terms of eigenvalues and ranks of a class of matrices related to G. Its definition is purely algebraic, but it has the following, very interesting property.

$$\begin{aligned} \mu(G) \leq 1 &\Leftrightarrow G \text{ is subgraph of a path.} \\ \mu(G) \leq 2 &\Leftrightarrow G \text{ is outerplanar.} \\ \mu(G) \leq 3 &\Leftrightarrow G \text{ is planar.} \\ \mu(G) \leq 4 &\Leftrightarrow G \text{ is linklessly embeddable.} \end{aligned}$$

So two questions were asked naturally. Is it true for all graphs G, for all $k \in \mathbb{N}$, that

1. G is a stacked k-tree $\Leftrightarrow$ G is a k-tree and $\mu(G) = k$?
2. $\mu(G) = k$ and G is contained in a k-tree $\Rightarrow$ G is contained in a stacked k-tree?

To examine this question, we study the *stacked treewidth*, which is defined as

$$\operatorname{stw}(G) := \min\{k \in \mathbb{N}_0 : \exists \text{ stacked } k\text{-tree } T \text{ with } G \subseteq T\}.$$

The goal of this thesis was to study the Colin de Verdière number and examine its relation to the stacked treewidth in an attempt to provide answers to the two questions. As it turns out, the first question was already considered and the answer is known to be "yes" [4]. However, we show in this thesis, that one can find counterexamples to the second proposed implication for all $k \geq 4$.

To do so, we introduce concepts and prove several helpful lemmas to better handle and understand the stacked treewidth, including our so-called *lifting lemma*, which helps us to "reduce the stackedness" of graphs in certain base cases. Furthermore, we prove an equivalence theorem for the stacked treewidth, analogous to the equivalence theorem of normal treewidth, an equivalence theorem for stackedness of chordal graphs and an equivalence theorem for stackedness of k-trees. These three equivalence theorems increase the number of known characterizations for stackedness of (chordal) graphs and k-trees.

Along the way, we also prove several up to now unknown facts about the stacked treewidth, namely the minor monotony, the behavior on joins and the behavior on clique-sums (which is analogous to the behavior of μ on clique-sums).

1.1 Overview

We begin in Section 2 by giving an overview of the notions of k-trees and treewidth, as well as the properties of the so-called (smooth) tree decompositions. We present the necessary background knowledge in order to give the full picture to an outside reader, as well as in order to establish notation.

In Section 3, we introduce stacked k-trees and stacked treewidth. We also introduce compact tree decompositions, which are a small abstraction of smooth decompositions and provide an elegant view to problems concerning the stacked treewidth. We prove a weak and a strong version of our so-called *lifting lemma* (see Lemmas 3.18 and 3.20), which, roughly speaking, allows us in certain base cases to change a k-tree T which covers a graph G to a "better" k-tree T' such that T' is closer to being a stacked k-tree than T.

The main result of this Section 3 is

Theorem 3.27 (Characterizations of Stacked Treewidth)**.** *Let G be a graph. The following are equivalent characterizations for* $\operatorname{stw}(G) = k$*:*

(i) k is the smallest integer such that G is subgraph of a stacked k-tree.

(ii) k is the smallest integer such that G has a stacked, compact tree decomposition of width k.

(iii) $k = \min\{\omega(H) - 1 : H \text{ is a stacked chordal completion of } G\}$.

and here, a chordal graph is called a *stacked chordal graph*, if one of the following conditions hold, which we show all to be equivalent.

Theorem 5.7 (Stackedness of Chordal Graphs, Continuation of Theorem 3.25)**.** *Let G be a chordal graph and $k := \omega(G) - 1 (= \operatorname{tw}(G))$. The following are equivalent:*

(i) $\mathrm{stw}(G) = k$, i.e. G is covered by a stacked k-tree.

(ii) G does not contain $M_{k,3}$ as minor.

(iii) G does not contain $M_{k,3}$ as topological minor.

(iv) G does not contain $M_{k,3}$ as subgraph.

(v) G has a stacked, compact tree decomposition.

(vi) $\mu(G) = k$. (With the exception of $G = \overline{K_2}$.)

Similar to this theorem, Corollary 5.11 characterizes stacked k-trees in many different ways.

Section 4 is devoted to introducing the Colin de Verdière number and its basic properties. Proofs regarding this graph parameter are usually quite long and also quite complex, involving a lot of nontrivial linear algebra. So we skip the proofs in this section and concentrate on summarizing the known, most basic results.

At last, in Section 5, we make use of the tools acquired in the previous sections to obtain our main results. We see in Theorem 5.2 that the stacked treewidth is minor-monotone, in Theorem 5.6 how the stacked treewidth behaves on clique sums and Corollary 5.11 explains why the answer to the first question asked in the introduction is positive. Concerning these results, the Colin de Verdière number and the stacked treewidth behave identically.

Concerning the second observation, we propose the following

Question 5.12. *Is it true for all graphs G, that*

$$[\mu(G) \leq \mathrm{tw}(G)] \Rightarrow [\mathrm{stw}(G) = \mathrm{tw}(G)] \ ? \qquad (\triangle)$$

This is a generalization of the second question asked in the introduction. We examine the behavior of the stacked treewidth on joins of graphs in Theorem 5.17. Using this insight, we come to find counterexamples G to $(\triangle)$ with $\mathrm{tw}(G) = k$ for all $k \geq 4$ and counterexamples G, such that $\mathrm{tw}(G) - \mu(G)$ is arbitrarily large. So we come to the conclusion that the answer to the proposed Question 5.12 is negative. However, we can still "save" the relation in the following cases.

Theorem 6.1. *Let G be a graph. The implication $(\triangle)$ does hold for G,*

- *trivially, if* $\mu(G) > \mathrm{tw}(G)$.
- *if G is planar. (Theorem 5.13)*
- *if G is chordal. (Theorem 5.7)*
- *if* $\mathrm{tw}(G) \leq 3$. *(Theorem 5.13)*
- *if G is a complete bipartite graph. (Corollary 5.18 and Theorem 4.5)*

- *if* ($\triangle$) *holds for graphs* G_1 *and* G_2 *and* G *is a clique sum of* G_1 *and* G_2. *(Theorem 5.24)*
- *if* ($\triangle$) *holds for graphs* G_1 *and* G_2 *and* G *is a disjoint graph union of* G_1 *and* G_2. *(Theorem 5.23)*

And this theorem is the concluding result of our thesis.

1.2 Preliminaries

We introduce some notations, which are used throughout the thesis. (We assume that the reader has basic knowledge of graph theory.) In this thesis, we only consider undirected, loopless graphs $G = (V, E)$ with vertex set $V(G) := V$ and edge set $E(G) := E$. If $e := \{i, j\} \in E$ is an edge, we generally use the *shorter notation* ij instead of $\{i, j\}$ to refer to e. We define the *order* of a graph G as $|G| := |V|$. For $v \in V$, we define the *neighborhood* of v in G as $N_G(v) := \{w \in V : vw \in E\}$. We also write $N(v)$ for the neighborhood of v in G, if the graph G can be deduced from the context.

If H is a *subgraph* of G, we write $H \subseteq G$. Special subgraphs are the *induced subgraphs*: For $V' \subseteq V$, we denote the subgraph induced by V' as $G[V']$. Furthermore, for $A \subseteq V$, $v \in V$, we define $G - A := G[V \setminus A]$ and $G - v := G - \{v\}$. A set $S \subseteq V$ is called a separator of G, if $G - S$ is disconnected.

Special graphs are the complete graph on n vertices, denoted by K_n, the path of length n on $n + 1$ vertices, denoted by P_n, and the cycle on n vertices, denoted by C_n. For $p, q \in \mathbb{N}_0$, $K_{p,q}$ denotes the *complete bipartite graph* with two partitions of size p and q, respectively. We define $K_{0,q} := \overline{K_q}$ and analogously $K_{p,0}$. Here, $\overline{G}$ denotes the *complement* of G, i.e. the graph that has exactly those vertex pairs as edges which are not edges in G. The *empty graphs* are exactly all graphs $\overline{K_n}$ ($n \in \mathbb{N}$).

If a graph H is derived from G by successive edge contractions, edge deletions and vertex deletions (or if $H = G$), H is called a *minor* of G. We write $\mathcal{M}H \subseteq G$ for the fact that H is a minor of G and $\mathcal{M}H \not\subseteq G$ for the fact that G does not have H as a minor. A graph H is called a *topological minor* of G, if a graph $H' \subseteq G$ exists, such that H' can be constructed from H by edge subdivisions. For a family $\mathcal{F}$ of graphs, a *forbidden minor characterization* of $\mathcal{F}$ is a family $\mathcal{C}$ of graphs such that $\mathcal{F} = \{G : G \text{ is a graph that contains none of the graphs in } \mathcal{C} \text{ as a minor.}\}$.

A *clique* in G is a complete subgraph of G. A vertex v such that $N(v)$ is a clique is called a *simplicial* vertex. The *clique number* $\omega(G)$ is the size of the largest clique in G. A graph G is called a *clique sum* of graphs G_1 and G_2, if $V(G_1) \cap V(G_2) = S$, and $G_i[S]$ is a clique for $i = 1, 2$, and further $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. A graph G is called the *join* of G_1, G_2, denoted by $G = G_1 \vee G_2$, if G is constructed by taking a copy of G_1, a copy of G_2 and then adding all possible edges between the two copies, i.e. all edges ij with $i \in V(G_1)$ and $j \in V(G_2)$.

The *chromatic number* $\chi(G)$ of a graph is the minimum number of colors needed to color the vertices of G such that no two adjacent vertices have the same color. A graph G is called *perfect*, if for all $A \subseteq V(G)$, we have $\omega(G[A]) = \chi(G[A])$.

If C is a cycle in G and an edge ij connects vertices i and j on the cycle, but ij itself is not part of C, then ij is called a *chord*. Graphs where every cycle of length at least four has a chord are called *chordal* graphs. A supergraph $H \supseteq G$ of G such that $V(G) = V(H)$ and H is chordal is called a *chordal completion* of G. Let $s := (v_1, \dots, v_n)$ be an ordering of the vertices of V. We define the graph G_i to be the graph that is left over after deleting $\{v_1, \dots, v_{i-1}\}$ from G. The sequence s is called a *perfect elimination ordering* of G, if v_i is simplicial in G_i for all $i \in \{1, \dots, n\}$. It is a known fact that a graph is chordal if and only if it has a perfect elimination ordering and that all graphs with a perfect elimination ordering are perfect graphs [2].

A graph that can be embedded in the plane without edge crossings is called *planar*. A graph that can be embedded in the plane without edge crossings and all vertices touching the outer face is called *outerplanar*. A graph that can be embedded in three-dimensional space such that no two disjoint cycles have a non-zero linking number is called *linklessly embeddable*. (Two such circles with non-zero linking number can be imagined as linked rings that can not be separated.) Also, a graph G is *maximal (outer-)planar*, if G is (outer-)planar and no edge can be added to G without destroying this property.

The *power set* $\mathcal{P}(A)$ of a set A is defined as $\mathcal{P}(A) := \{A' : A' \subseteq A\}$. A *hypergraph* H is a tuple $H = (V, E)$ where V is a finite set and $E \subseteq \mathcal{P}(V) \setminus \emptyset$. The *adjacency graph* of a hypergraph H is a graph G defined as $V(G) := V(H) \cup E(H)$ and $E(G) := \{ve : v \in V(H), e \in E(H), v \in e\}$.

Finally, some notations not regarding graph theory: For a function $f : D \to B$ and a subset $A \subseteq D$, we define the *restriction of f to A* as $f|_A : A \to B; f|_A(a) := f(a)$. For two disjoint sets A, B, we denote the *disjoint union* of A and B by $A \dot\cup B$. We sometimes write $A + x$ instead of $A \dot\cup \{x\}$ for a set A and an object x. If M is a matrix, the *corank of M* is defined as $\dim(\ker(M))$ and the *spectrum of M* is defined as $\sigma(M) := \{\lambda : \lambda \text{ is an eigenvalue of } M\}$.

2 Treewidth and k-Trees

The *treewidth* of a graph and, closely related, the concept of k-trees, are widely used concepts in graph theory. They have broad applications to theoretical problems as well as practical applications to computer science, especially concerning dynamical programming [5]. The goal of this chapter is to give an understanding of the underlying concepts and relations regarding treewidth, though we leave out the algorithmic aspects. So let us begin.

2.1 k-Trees

We start with the definition of k-trees. These are a generalization of the well-known class of simple trees. (In a first approximation, they can be understood as similar to normal trees, but "thicker".)

Definition 2.1 (k-Tree)**.** *Let $k \in \mathbb{N}_0$. The class of k-trees is recursively defined as:*

- K_{k+1} is a k-tree.
- If $T = (V, E)$ is a k-tree and $C \subseteq T$ is a k-clique in T, then $T' = (V', E')$ with $V' := V + v$ and $E' := E \cup \{vc : c \in C\}$ is a k-tree.

Generally, if G is any graph containing a clique C, we say that G' arises from G by *stacking v onto C* for $G' := (V', E')$ with $V' := V(G) + v$ and $E' := E(G) \cup \{vc : c \in C\}$.

In the simplest case, $k = 1$, Definition 2.1 states that every 1-tree is constructed by starting with an edge and then repeatedly stacking a new vertex onto an already existing vertex. Thus the set of all 1-trees is exactly the set of all trees except K_1. (For the corner case $k = 0$, the 0-trees are the empty graphs.)

As another example, consider Figure 1, which is a 3-tree that is formed by starting with the clique 1234. The next step is stacking vertex 5 onto clique 123. Then we continue by stacking the vertices 6–9 onto the cliques 124, 124, 247, and 278, respectively. From this example we see that the same clique can be chosen twice when constructing k-trees.

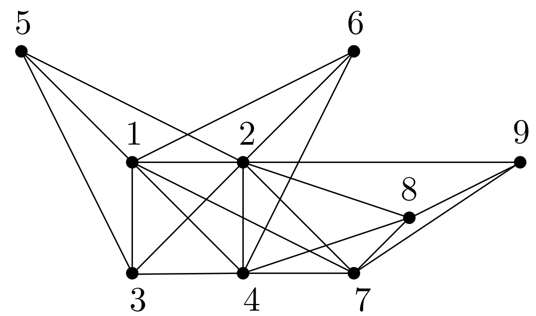


Figure 1: Example of a 3-tree

This action of building a k-tree by starting with a $(k+1)$-clique $\{v_1, \dots, v_{k+1}\}$ and repeatedly stacking a new vertex v_i onto an existing k-clique $\rho(v_i)$ will be called the *construction of a k-tree* in this thesis.

Definition 2.2 (Construction of a k-Tree)**.** *A pair $C := ((v_1, \dots, v_n), \rho)$ where $\rho : \{v_{k+2}, \dots, v_n\} \to \mathcal{P}(V)$ for $n, k \in \mathbb{N}_0$ and $n \geq k+1$ is called a construction and defines a k-tree, if one of the two following cases are met:*

- If $n = k+1$, C defines the clique on $\{v_1, \dots, v_n\}$.
- If $((v_1, \dots, v_{n-1}), \rho|_{\{v_1,\dots,v_{n-1}\}})$ defines a k-tree T' and $\rho(v_n)$ is a k-clique in T', then C defines the k-tree which arises from T' by stacking v_n onto $\rho(v_n)$.

If a construction C defines a k-tree T, we also call C *a construction of* T. By definition, every k-tree has at least one construction. For example, $((1,2,3,4,5,6,7,8,9), \rho)$ is a construction of the graph in Figure 1 and $\rho(8) = \{2,4,7\}$. Another construction would be $((1,2,3,4,7,8,6,5,9), \rho')$. (For the sake of brevity, ρ and ρ' are left out here.)
This shows that the construction is not necessarily unique. The set $\{v_1, \dots, v_{k+1}\}$ is called the *root of the construction* and later we show that it may also be chosen arbitrarily, i.e. for all k-trees T and for all $(k+1)$-cliques $C \subseteq V(T)$, there is a construction (S, ρ) of T, such that the root of (S, ρ) is C.

Having understood the notion of constructing a k-tree, we can make some easy observations:

Observation 2.3. *The last vertex v_n of a construction $((v_1, \dots, v_n), \rho)$ of a k-tree has degree k.*

Proof. Either the k-tree is simply K_{k+1} where every vertex has degree k or the last vertex v_n is connected to exactly the k-clique $\rho(v_n)$. □

Lemma 2.4. *The inclusion-maximal cliques in a k-tree T are all of size $k+1$.*

Proof. Let C be a clique and $((v_1, \dots, v_n), \rho)$ be a construction of T. Then choose $v_i \in \{v_1, \dots, v_n\} \cap C$ such that i maximal. If $i \leq k+1$, C is included in the root. Else $C - v_i \subseteq \rho(v_i)$ must hold, as v_i is the last vertex of C being stacked. Thus, $C \subseteq \rho(v_i) \cup \{v_i\}$.
In both cases C is included in a clique of size $k+1$. As C was arbitrary, no clique can have more than $k+1$ elements as well. □

Finally, note that the reverse of a construction of T is a perfect elimination ordering of T where every vertex has degree at most k when it is deleted. It follows that k-trees are chordal and perfect graphs with chromatic number $k+1$.

2.2 Treewidth

The treewidth tw(G) of a graph G is a well-studied graph parameter. It was first introduced by Bertelé and Brioschi in 1972 under the name "dimension of a graph" [1]. Later on, in 1984, it was rediscovered by Robertson and Seymour and played an important role in their Graph Minor Project [17].

Treewidth can be defined in several equivalent ways. The first characterization we give may be the easiest one to understand and is as follows:

Definition 2.5 (Treewidth)**.** *For any graph G, its **treewidth**, denoted by* tw(G)*, is the smallest k such that G is subgraph of a k-tree.*

$$\mathrm{tw}(G) := \min\{k \in \mathbb{N}_0 : \exists\ k\text{-tree } T \text{ with } G \subseteq T\}$$

An example for a graph with treewidth 3 is the graph G from Figure 2. G is clearly a subgraph of the graph from Figure 1 and thus has treewidth at most 3. On the other hand, the treewidth of G must at least 3, as G contains the 4-clique 2478. Then, due to Lemma 2.4, G cannot be contained in a 1-tree or 2-tree.

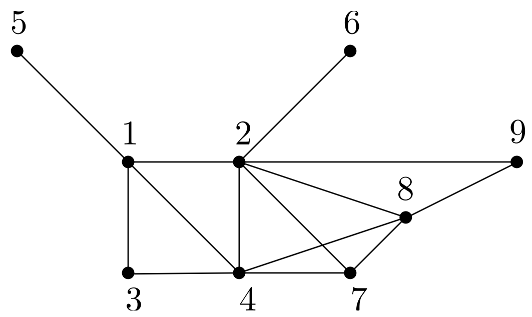


Figure 2: A graph with treewidth 3

Two easy lower bounds can quickly be established on the treewidth:

Lemma 2.6. *The treewidth is at least as large as the minimum degree, i.e. for all graphs G, we have $tw(G) \geq \delta(G)$.*

Proof. Let $T \supseteq G$ be a k-tree (for suitable k) covering G with smallest number of vertices. Assume that $k < \delta(G)$. Take any arbitrary construction $((v_1, \dots, v_n), \rho)$ and consider $v := v_n$. For the sake of contradiction, assume $v \notin V(G)$. Then, $T - v \supseteq G$ is a smaller k-tree covering G, which is a contradiction to the minimality of T. Thus $v \in V(G)$. We have seen in Observation 2.3 that $\deg_T(v) = k$. But that is a contradiction, as $\deg_T(v) = k < \delta(G) \leq \deg_G(v)$. □

Lemma 2.7. *If a graph G contains a $k+1$-clique, $tw(G) \geq k$. In other words,* $\mathrm{tw}(G) \geq \omega(G) - 1$.

Proof. By Lemma 2.4, a k-tree can have cliques of size at most $k + 1$. □

2.3 Tree Decompositions

Another way of looking at treewidth opposed to covering the graph with k-trees is provided by so-called "tree decompositions". Often times, the treewidth of a graph G is defined as the smallest k for which there exists a tree decomposition with width k of G. These two views, of course, turn out to be equivalent.

Definition 2.8 (Tree Decomposition)**.** *A **tree decomposition** of a graph $G = (V, E)$ is a tree (V_T, E_T) with vertex set V_T and a label $X_i \subseteq V$ for each vertex $i \in V_T$, such that:*

1. $\bigcup_{i \in V_T} X_i = V$.
2. $\forall uv \in E : \exists i \in V_T : u, v \in X_i$.
3. $\forall v \in V$: the subset $\Phi(v) := \{i \in V_T : v \in X_i\}$ forms a (connected) subtree of T.

The X_i are called the *bags* of the tree decomposition. We also call two bags X_i, X_j *adjacent*, if i and j are adjacent. In this thesis, the mapping $\Phi : V \to \mathcal{P}(V_T)$ shall be called the *subtree mapping* of a tree decomposition. At some times, when it is clear from the context, we will not so strictly distinguish between $\Phi(v)$ and $T[\Phi(v)]$.

For an edge $ij \in E_T$, $X_{ij} := X_i \cap X_j$ shall be called the *band* between i and j. Given a bag X_i, $|X_i|$ is called the *bagsize* of X_i. Finally, the *width* of a tree decomposition is defined as the maximum bagsize minus one:

$$\text{width}(V_T, E_T) := \max\{|X_i| - 1 : i \in V_T\}$$

Note that the second condition of Definition 2.8 can be equivalently stated as follows. For each edge $uv \in E$, the subtrees $\Phi(u)$ and $\Phi(v)$ intersect. Also, each band between i and j consists of exactly those $v \in V$, whose subtree $\Phi(v)$ uses the edge ij.

As an example of all these concepts serves Figure 3. It depicts a tree decomposition with width 3 of the graph from Figure 2. The big circles represent the elements of V_T, the edges between them represent the elements of E_T. The content of each circle represents the respective bag. Two certain vertices i and j are marked. We have $X_i = \{2, 4, 7, 8\}, X_j = \{2, 7, 8, 9\}$, and the band between i and j is $X_{ij} = \{2, 7, 8\}$. Also shown is the subtree $\Phi(1)$, which is induced by the vertices whose bags contain 1.

A quick exercise now may be to verify, that all three necessary conditions for tree decompositions from Definition 2.8 are met in this particular instance. Notice that in the original graph, although there is no edge between, e.g., 6 and 9, they may appear in the same bag of the decomposition.

As already stated, having a tree decomposition of width k is equivalent to being covered by a k-tree. Another well-known characterization is in terms of a chordal completion minimizing the maximum clique size. (Recall from the introduction that a chordal graph is a graph where every cycle of length at least 4 has a chord and a chordal completion of a graph G is obtained by adding edges to G until the resulting graph is chordal.)

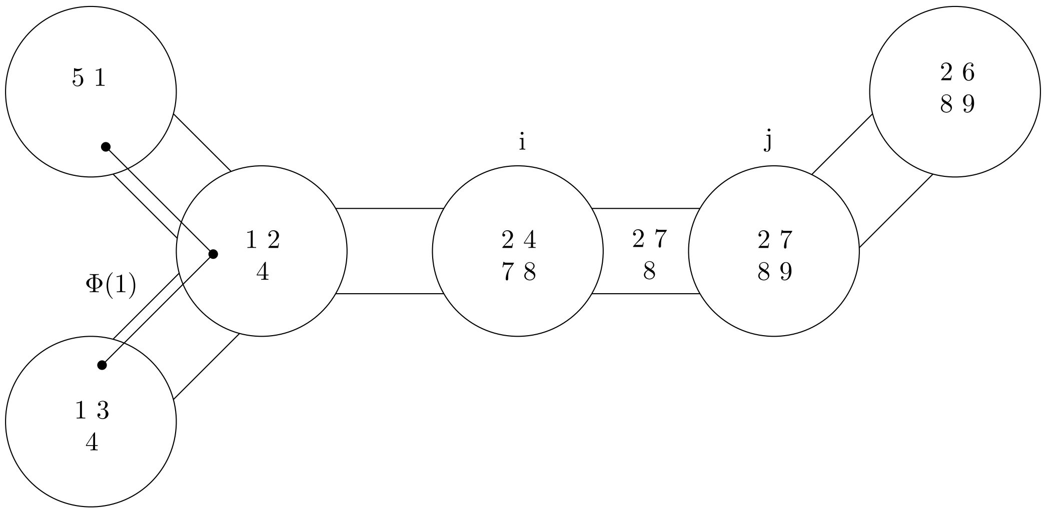


Figure 3: One possible decomposition of the graph from Figure 2. Also visualized are $\Phi(1)$ and the band between i and j.

Theorem 2.9 (Characterizations of Treewidth)**.** *Let G be a graph. The following characterizations of $tw(G) = k$ are equivalent:*

(i) k is the smallest integer such that G is subgraph of a k-tree.

(ii) k is the smallest integer such that G has a tree decomposition of width k.

(iii) $k = \min\{\omega(H) - 1 : H \text{ is a chordal completion of } G\}$

To prove Theorem 2.9, we do multiple, small steps. We loosely follow the work of Bronner and Ries [2]. To start with, we examine the behavior of the intersection of subtrees of a tree.

Observation 2.10. *Let T_1 and T_2 be subtrees of a tree T.*

a.) Their intersection $S := T_1 \cap T_2$ is a subtree again.

b.) Their union $T_1 \cup T_2$ is a subtree if and only if $S \neq \emptyset$.

Proof. For *a.)*, it suffices to show that S is connected, as every connected subgraph of T is a subtree. Indeed, if $a, b \in V(G)$ are two vertices in S, there must be an a-b-path P_i in T_i, as T_i is connected $(i = 1, 2)$. If $P_1 \neq P_2$, T would have a cycle. Thus $P_1 = P_2 \subseteq S$ and a, b are connected.

For *b.)*, notice that $T_1 \cup T_2$ is connected if and only if $S \neq \emptyset$. □

This simple observation is necessary to show that a family of subtrees of a tree satisfies the so-called *Helly property*.

Definition 2.11. *A family $F := \{T_i\}_{i\in I}$ of subsets of a set T satisfies the* ***Helly property****, if $X \subseteq F$ and $T_i \cap T_j \neq \emptyset$ for all $T_i, T_j \in X$ implies that $\bigcap_{T_x \in X} T_x \neq \emptyset$*

Theorem 2.12. *A family $F := \{T_i\}_{i\in I}$ of subtrees of a tree T satisfies the Helly property.*

Proof. We prove the theorem by induction on $n := |X|$, which is possible, as every tree has only a finite amount of different subtrees.

If $n \leq 2$, the claim trivially holds. For the induction step, assume that

$$[\forall T_i, T_j \in X : T_i \cap T_j \neq \emptyset] \Rightarrow \bigcap_{T_x \in X} T_x \neq \emptyset$$

holds for all possible subfamilies $X = \{T_1, \dots, T_n\}$ of size n.

Let then $X' := \{T_1, \dots, T_{n+1}\}$ and let all these subtrees have pairwise nonempty intersections. By the induction hypothesis, $S := \bigcap_{i=1}^{n} T_i$ is not empty. Also, by Observation 2.10, S is again a subtree of T.

Now, assume for the sake of contradiction that $S \cap T_{n+1} = \emptyset$. This means that T_{n+1} does not touch S and therefore completely lies in one component K of $T - S$. As T is connected and acyclic, K is connected to S via exactly one edge uv, where without loss of generality $u \in K$ and $v \in S$. Note that $|\{i \in \{1, \dots, n\} : v \in V(T_i)\}| = n$, but $|\{i \in \{1, \dots, n\} : u \in V(T_i)\}| < n$, as $v \in S$ but $u \notin S$. Thus, at least one of the trees $\{T_1, \dots, T_n\}$ does not intersect $K \supseteq T_{n+1}$. This is a contradiction. □

Corollary 2.13. *Let C be a clique of a graph G and T be a tree decomposition of G. There exists a bag of T that contains C.*

Proof. Let Φ be the subtree mapping belonging to the tree decomposition T and let $C = \{v_1, \dots, v_m\}$. For all $i \neq j \in \{1, \dots, m\}$, $v_i v_j \in E(G)$ and therefore $\Phi(v_i) \cap \Phi(v_j) \neq \emptyset$. By the Helly property, $\Phi(v_1) \cap \ldots \cap \Phi(v_m) \neq \emptyset$. Any element of this intersection is a vertex labeled with a bag with the desired property. □

For the step "(i) $\Rightarrow$ (ii)" of Theorem 2.9, we need the following easy lemma. It shows the explicit construction of a tree decomposition of a k-tree.

Lemma 2.14. *A k-tree T has a tree decomposition D of width k.*

Proof. Let $((v_1 \dots v_n), \rho)$ be a construction of T. Let further T_i be the k-tree that is defined by the construction $((v_1, \dots, v_i), \rho|_{\{v_1,\dots,v_i\}})$ (so the process of the construction of T can be visualized as the sequence $T_{k+1}, \dots, T_n$). We show the theorem by induction. T_{k+1} is the clique on the vertices $\{v_1, \dots, v_{k+1}\}$ and clearly has a tree decomposition with $D_{k+1} := K_1$ and $\{v_1, \dots, v_{k+1}\}$ as the label of the only vertex.

Now suppose T_i has a tree decomposition D_i of width k. We know that T_{i+1} arises from T_i by stacking v_{i+1} onto $\rho(v_{i+1}) \subseteq \{v_1, \dots, v_i\}$. By Corollary 2.13, there is a vertex j labeled with $X_j \supseteq \rho(v_{i+1})$ in the decomposition D_i.

Then the decomposition of T_{i+1} is given by $V(D_{i+1}) := V(D_i)+v$, $E(D_{i+1}) := E(D_i)+jv$. The labels for the vertices in $V(D_{i+1}) \cap V(D_i)$ stay the same and the new vertex v is labeled with $X_v := \{v\} \cup \rho(v)$.

Note that this is indeed a valid decomposition: The first condition of Definition 2.8 is clearly met. For the second condition, note that all new edges of T_{i+1} compared to T_i are covered by the new bag X_v. For the third condition, note that for all $u \in \rho(v_{i+1})$, $\Phi(u)$ contains j and v and is therefore still connected, $\Phi(v_{i+1}) = \{v\}$, and the other subtrees are not altered. □

Also, we define the *fill* of a tree decomposition T of a graph G. It denotes the graph G' on the same vertex set as G and with all possible edges that are allowed such that T is still a decomposition of G'.

Definition 2.15 (Fill of a Tree Decomposition)**.** *Let T be a tree decomposition of G with labels $\{X_i : i \in V(T)\}$. The **fill** of T is defined as* $\text{fill}(T) := (V', E')$ *with* $V' := V(G)$ *and* $E' := \{uv : \text{ there is a bag } X \text{ in } D \text{ such that } u, v \in X\}$

Lemma 2.16 (Fills are Chordal)**.** *Consider a tree decomposition T of a graph G with subtree mapping Φ. The graph $G' := \text{fill}(T)$ is chordal.*

Proof. Consider a cycle $v_1, \ldots, v_m$ in G' $(m \geq 4)$. Let $T_1 := \Phi(v_1)$, $T_2 := \Phi(v_2)$ and $T_3 := \Phi(v_3) \cup \ldots \cup \Phi(v_m)$. By Observation 2.10, T_3 is a subtree. By the Helly property, $T_1 \cap T_2 \cap T_3 \neq \emptyset$. So we find a vertex $v_i \in \{v_3, \ldots, v_m\}$ such that v_i is adjacent to both v_1 and v_2. This means we found a chord. □

After these preparations, we are now ready to prove Theorem 2.9. We do this by showing the following lemma, which clearly implies said theorem.

Lemma 2.17 (Characterizations of Treewidth)**.** *Let G be any graph. The following are equivalent:*

(i) G is subgraph of an s-tree T with $s \leq k$.

(ii) G is subgraph of an s-tree T with $s \leq k$ and $V(T) = V(G)$.

(iii) G has a tree decomposition with width at most k.

(iv) G has a chordal completion with maximal clique size at most $k + 1$.

Proof. **"(i) ⇒ (iii):"** We have seen in Lemma 2.14 that T has a tree decomposition D of width s. Now do the following: For each $v \in V(T) \setminus V(G)$, delete v from every bag. Then we arrive at a tree decomposition D' of G. (Note that some bags in this decomposition might be empty, but we did not exclude this in Definition 2.8.)

"(iii) ⇒ (iv):" Let D be the given tree decomposition. Consider $G' := \text{fill}(G)$, which has the same vertex set as G and is chordal by Lemma 2.16. Note that D is not only a decomposition of G but by construction also a decomposition of G'. Then every clique of G' must be completely inside a bag of D (due to Corollary 2.13) and therefore $\omega(G') \leq k+1$.

"(iv) ⇒ (ii):" Let G' be the chordal completion of G and let $s := \omega(G') - 1$. As G' is chordal, it has a perfect elimination ordering $(v_1, \ldots, v_n)$. Let $(v'_1, \ldots, v'_n) := (v_n, \ldots, v_1)$. We prove by induction that for each $j \in \{1, \ldots, n\}$, $G[v'_1, \ldots, v'_j]$ can be covered by an s-tree T_j.

For the induction base, note that if $j \leq \min\{s+1, n\}$, we can clearly cover $G[v'_1, \ldots, v'_j]$ with an $(s+1)$-clique. For the induction step, assume that the s-tree T_j covers $G[v'_1, \ldots, v'_j]$. As $(v'_1, \ldots, v'_n)$ is a reversed perfect elimination ordering, v'_{j+1} is a simplicial vertex in $G'' := G[v'_1, \ldots, v'_{j+1}]$. Let $S := N_{G''}(v'_{j+1})$. We have $|S| \leq s$, as $\omega(G') = s + 1$. With Lemma 2.4, we find an s-clique $K \supseteq C$ in T_j. Then T_{j+1} arises from T_j by stacking v'_{j+1} on K.

Finally, **"(ii) ⇒ (i)"** is obviously true. □

2.4 Smooth Tree Decompositions

The previous subsection showed us how to convert a tree decomposition of G into a covering k-tree making a detour via chordal completions. The relation between tree decompositions and covering k-trees can be even better understood using the notion of so-called *smooth tree decompositions*, which are tree decompositions satisfying two additional constraints. Some easy arguments show that a graph G has a tree decomposition if and only if it also has a smooth tree decomposition. Then we see the close relation between covering k-trees and smooth tree decompositions.

Definition 2.18 (Smooth Tree Decomposition)**.** *A tree decomposition* (V_T, E_T) *(of width* k*) is called* ***smooth*** *, if:*

- All bags have the same size $k + 1$.
- Adjacent bags differ by exactly one element, i.e. for all $ij \in E_T$, $|X_i \cap X_j| = k$.

As an example, consider Figure 5, which is one possible smooth tree decomposition D of the graph G in Figure 4. Again, not all pairs of vertices that appear in the same bag must be an edge in G. Figure 5 would still be a smooth tree decomposition of G, if the edges $\{1, 5\}, \{2, 6\}$ and $\{2, 10\}$ would be added to G. In other words, these three edges are the difference between G and fill(D).

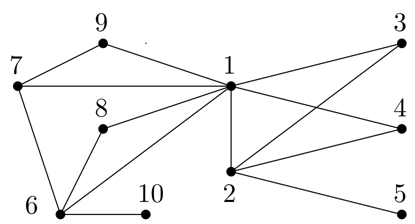


Figure 4: A graph G with treewidth 2

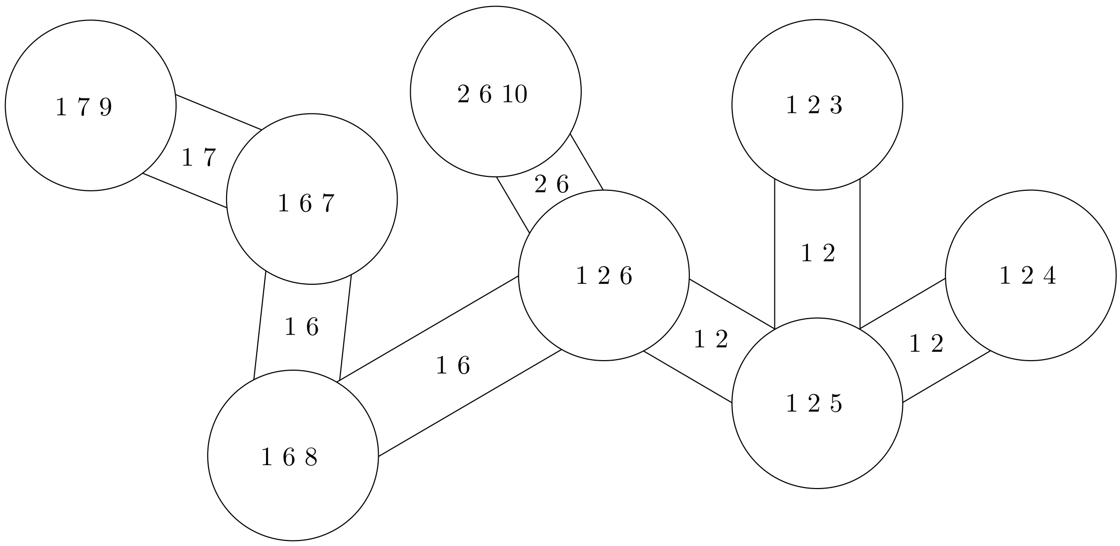


Figure 5: Smooth decomposition D of the graph G from Figure 4

The following lemma shows that the two additional conditions for a smooth tree decomposition can always be fulfilled. Therefore, we could alternatively define $\mathrm{tw}(G)$ as smallest k such that G has a smooth tree decomposition.

Lemma 2.19. *Every tree decomposition T with width k of a graph G can be transformed into a smooth tree decomposition with the same width k of G.*

Proof. Let ij be an edge of T. Observe that each of the following three operations keeps the three conditions for a tree decomposition from Definition 2.8 intact:

1. If $|X_i| < |X_j|$, add $v \in X_j \setminus X_i$ to X_i.
2. If $X_i = X_j$, contract the edge ij in T (and label the emerging vertex with X_i).
3. If $|X_i| = |X_j| = k + 1$ and $|X_i \cap X_j| < k$, choose $v \in X_i \setminus X_j$, choose $u \in X_j \setminus X_i$, subdivide ij in T and label the new, emerging vertex with $X_i - v + u$.

In fact, the first and third operations leave all subtrees $\Phi(v)$ connected and otherwise only add information, the second operation deletes only superfluous information.
To obtain the desired decomposition, apply operation 1 until all bags are of the same size size $k+1$, apply operation 2 until no two identical (adjacent) bags exist, and apply operation 3 until the decomposition is smooth. □

In fact, in a smooth tree decomposition, no two bags X_i, X_j can be identical, whether or not X_i is adjacent to X_j. For suppose $X_i = X_j$, then the subtrees $\{\Phi(v) : v \in X_j\}$ occupy the path $i \to j$ in T and thus we find two adjacent identical bags, which is forbidden.

Another nice property is the following: Let T' be a set of connected bags and suppose that we have "marked" the vertices in these bags. Then we consider a new bag

X_n, which is adjacent to a bag of T'. There is exactly one element in X_n which is not already marked.

Lemma 2.20. *Let T be a smooth tree decomposition of a graph G. Let $T' \subseteq T$ be a subtree of T and let $i \in V(T) \setminus V(T')$ be a vertex that is not in T' itself, but adjacent to a vertex j of T'. Define*

$$q(i, T') := X_i \setminus \bigcup_{t \in T'} X_t.$$

Then $|q(i, T')| = 1$.

Proof. As T is a smooth tree decomposition, $X_i \setminus X_j = \{v\}$ for some $v \in V(G)$. The subtree $\Phi(v)$ is connected and does not touch j, so $\Phi(v)$ does not touch T' either. Thus $q(i, T') = \{v\}$. □

This lemma allows us to understand the structure of smooth decompositions in the following way.

Corollary 2.21. *A smooth tree decomposition with width k of a graph G on n vertices has $n - k$ bags.*

Proof. Start with $T' = \{i\}$ for an arbitrary bag X_i and apply Lemma 2.20 repeatedly. The bag X_i has size $k + 1$ and we can therefore apply the lemma $n - k - 1$ times until there are no vertices left. □

Corollary 2.22. *The fill $G' := \text{fill}(T)$ of a smooth tree decomposition T is a k-tree and can be constructed using the following procedure:*

1. Start with $T' = \{r\}$ for arbitrary $r \in V(T)$ and initialize G' as a clique on X_r.
2. Find ij like in Lemma 2.20.
3. In G', stack $q(i, T')$ onto $X_i \cap X_j$. Set $T' \leftarrow T' \cup \{i\}$.
4. If $T' \subsetneq T$, go back to step 2.

Proof. The resulting graph equals $\text{fill}(G)$, if it has all edges between any pair of vertices in the same bag. We can go by induction on $|T'|$.

At step 1, exactly the edges between all vertices in X_r are added to G'. At every step 3, exactly the edges between $q(i, T')$ and $X_i \cap X_j$ need to added to G', as the edges between pairs of vertices in $X_i - q(i, T') = X_i \cap X_j$ are already in G'. This is exactly what happens.

Also, due to the above reasoning, at every step 3, $X_i \cap X_j$ is a k-clique. Therefore, the resulting graph is a k-tree. □

Corollary 2.23. *Let T be a smooth tree decomposition of width k of G and let $G' :=$ $\text{fill}(T)$. Let X_r be a bag of T. There is a bijection* *a.)* $f : V(T) \to \{X_r\} \cup (V(G) \setminus X_r)$

b.) $g : V(T) \to \{C : C$ *is a* $(k+1)$*-clique in* $G'\}$

Proof. The bijection f is implied by Corollary 2.22. Just map $X_r \mapsto \{X_r\}$ in step 1 and $i \mapsto q(i, T')$ in every step 3.
For the bijection g, note that every bag X_i creates the $(k+1)-$clique X_i in G' and every clique in G' is included in a bag (see Corollary 2.13). So g is given by $i \mapsto X_i$. □

The two corollaries above are in this form of course only applicable to those k-trees, which can be expressed as the fill of a tree decomposition. Fortunately, that is true for all k-trees, which a quick observation shows.

Observation 2.24. *If T is a smooth tree decomposition of a k-Tree G, we already have* $\text{fill}(T) = G$.

Proof. First, note that Lemma 2.14 of G yields a smooth decomposition T', which also has the property that $\text{fill}(T') = G$. So there exists at least one decomposition D with the property $\text{fill}(D) = G$. But T and T' have the same number $n-k$ of bags by Corollary 2.21 and all bags are different. Thus, for every bag X_i in T', we have that X_i is a $(k+1)$-clique in G and therefore X_i is a bag in T. So every bag of T corresponds to a full $(k+1)$-clique in G and T also satisfies $\text{fill}(T) = G$. □

We can use this observation, to see that for every $(k+1)$-clique $R \subseteq V(G)$, there exists a construction $((v_1, \ldots, v_n), \rho)$, such that the root of the construction is R, i.e. $R = \{v_1, \ldots, v_{k+1}\}$ (as we promised in Section 2.1). This is because R must be a bag $X_r = R$ in a smooth decomposition T and we can start with this bag when transforming a smooth decomposition into a tree in Corollary 2.22.

3 Stacked Treewidth

The smooth tree decomposition is a well-known variation on the concept of tree decompositions. In this thesis, we introduce one more variation, called a *compact tree decomposition*. These compact tree decompositions are a small abstraction from the smooth tree decompositions and provide a cleaner view on the problem of determining the stacked treewidth, which we do in the next subsections.

3.1 Compact Tree Decompositions

For a short motivation, consider the following.

Definition 3.1 (Degree of a Clique). *Let G be a graph and $C \subseteq V(G)$ a clique. The **degree** of C is defined as* $\deg(C) := |\{v \in V(G) : vc \in E(G) \text{ for all } c \in C\}|$.

Another equivalent view of $\deg(C)$ is that it equals the number of cliques with size $|C| + 1$ covering C.

Definition 3.2 (Continuation of Φ to Subsets). *Let T be a tree decomposition with labels $\{X_i : i \in V\}$ of a graph G. For a subset $S \subseteq V(G)$, we denote by $\Phi(S) := \{i \in V(T) : S \subseteq X_i\}$ the vertices with labels containing S.*

Note that for single-element sets $S = \{s\}$, this definition coincides with the function Φ from Definition 2.8, i.e. $\Phi(\{s\}) = \Phi(s)$. Also, for all sets $S \subseteq V(G)$, $\Phi(S) = \bigcap_{s \in S} \Phi(s)$, and thus, we see using the Helly property that $\Phi(S)$ is a subtree of T.

For the stacked treewidth, we later are interested in the degrees of k-cliques. In particular, we want to know whether a clique C of a covering tree T of G has degree 3 or higher. We have learned in Observation 2.24 that the tree T can be expressed as the fill of some smooth tree decomposition D. Now, every $(k+1)$-clique of T is exactly one bag in D and therefore we have that the degree of C equals the number of bags containing C, or $\deg(C) = |\Phi(C)|$.
Also, take a look at $T_C := T[\Phi(C)]$, which we already know is a subtree. As C has size k, the edges in T_C are exactly those edges ij with band $X_{ij} = C$ (The band X_{ij} is defined as $X_{ij} := X_i \cap X_j$). From this we conclude that the edge set $E(T)$ can be partitioned into $\{E(T_C)\}_{C \in \mathbf{C}}$, where $\mathbf{C}$ is the set of all k-cliques of T.

The idea now is that the exact structure of T_C is not important and we can replace T_C by a hyperedge on $\Phi(C)$. This leads to the concept of compact tree decompositions, which are hypergraphs.

Definition 3.3 (Compact Tree Decomposition). *Let $G = (V, E)$ be a graph. A **compact tree decomposition** of G is the hypergraph $H = (V_H, E_H)$ with a label (called bag) $X_i \subseteq V$ for every $i \in V_H$ such that*

1. $\bigcup_{i \in V_H} X_i = V$

2. For all $uv \in E$, there is $i \in V_T$ such that $u, v \in X_i$.

3. The adjacency graph of H is a tree.

4. For all hyperedges $e \in E_H$, there is a set $X_e \subseteq V$ such that all bags X_i $(i \in e)$ are of the form $X_e + v_i$. Also, no two X_i $(i \in e)$ are equal and no two X_e $(e \in E_H)$ are equal.

5. For all $v \in V$, $\Phi(v) := \{i \in V_H : v \in X_i\}$ is connected.

For condition 5, note that Φ is defined like before for normal and smooth tree decompositions. Also, note that if $v \in V$ and in a hyperedge $e = \{i_1, i_2, \ldots, i_m\}$, $i_1, i_2 \in \Phi(v)$, then already $e \subseteq \Phi(v)$. This is due to condition 4. Therefore, $\Phi(v)$ can always be written as a union of hyperedges and the concept of connectivity in a hypergraph leaves no room for interpretation here. Also, we can still see $\Phi(v)$ as a sub(-hyper-)tree of a (hyper-)tree. Analogously to Definition 2.8, Φ is called the *subtree mapping* and the set $X_e = \bigcap_{i \in e} X_i$ from condition 4 is called the *band of e*. The *fill of a compact decomposition H*, $\text{fill}(H)$, is defined analogous to the fill of a smooth decomposition. For a band X_e, we also define its *degree* as $\deg(X_e) := |e|$.

Example 3.4. *Figure 6 shows an example of a compact tree decomposition of the graph from Figure 4. This compact decomposition D' is similar to the smooth decomposition D from Figure 5. We have that the band $X_{\{i,j,k\}}$ belonging to the hyperedge $\{i, j, k\}$ equals $\{1, 6\}$ and has degree 3. Likewise, the degrees of the bands $\{1, 7\}$, $\{2, 6\}$, and $\{1, 2\}$ are 2, 2, and 4, respectively. Furthermore, $\Phi(6) = \{i, j, k, s\} = \{i, j, k\} \cup \{k, s\}$ and $\Phi(\{1, 6\}) = \{i, j, k\}$.*

The following theorem extends Theorem 2.9 by even more equivalent characterizations of treewidth. We can now express treewidth in terms of smooth and compact tree decompositions as well.

Theorem 3.5 (More Characterizations of Treewidth)**.** *Let G be a graph, $k \in \mathbb{N}_0$. The following are equivalent:*

(i) G has a tree decomposition of width k.

(ii) G has a smooth tree decomposition of width k.

(iii) G has a compact tree decomposition of width k.

Proof. **"(i) ⇒ (ii):"** We have seen that in Lemma 2.19.

The step **"(ii) ⇒ (i)"** is obvious.

"(ii) ⇒ (iii):" Let T with labels $X_i (i \in V(T))$ and subtree mapping Φ be a smooth tree decomposition of G. We show how to construct the compact tree decomposition H

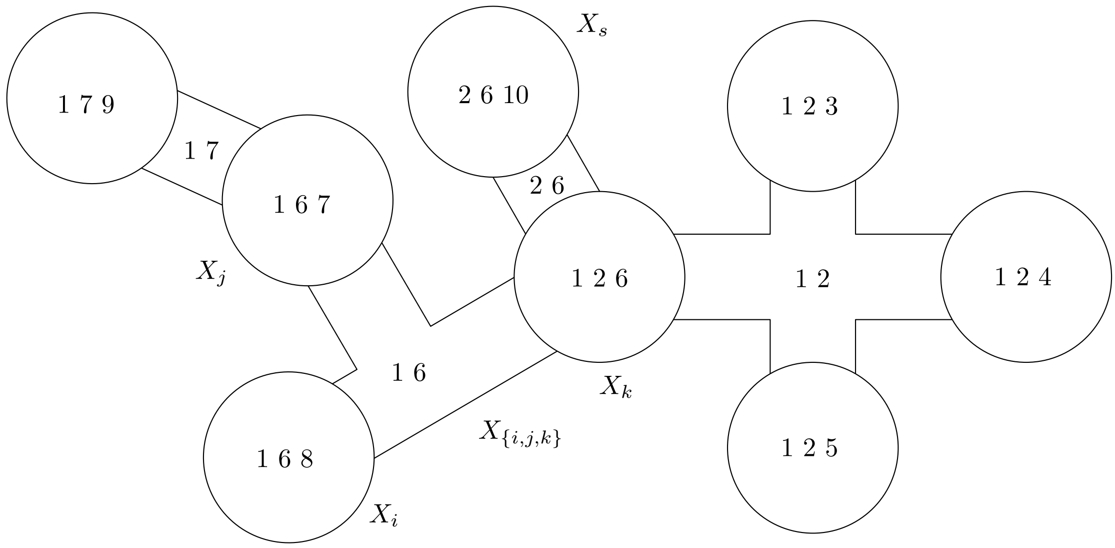


Figure 6: Compact tree decomposition D' of the graph from Figure 4

and how to transform T into another graph A such that A is a tree and the adjacency graph of H.

To achieve that, start with $V(H) = V(T)$ and $E(H) = \emptyset$. Let $\mathbf{C}$ be the family of all bands of T, namely $\mathbf{C} := \{S : S = X_{ij} \text{ for some } ij \in E(T)\}$. Repeat the following operation $(*)$ for all $S \in \mathbf{C}$:

$(*)$: Delete all edges of $T[\Phi(C)]$, add a new vertex l_C and connect $\Phi(C)$ to l_C. Add $\Phi(C)$ to $E(H)$.

This process is for example shown in the transition from Figure 5 to Figure 6. Note that $(*)$ transforms a tree into a tree. A characterizing property of a tree is to be connected and that the number of edges is one less than the number of vertices. Now, $(*)$ clearly leaves T connected. As $T[\Phi(C)]$ is a tree, operation $(*)$ removes $m - 1 := |\Phi(C)| - 1$ edges at first and then adds m edges and a new vertex. So $(*)$ transforms a tree into a tree, which gives us condition 3 of Definition 3.3. Choose the same bags for H as for T. Then one easily checks that the properties of the smooth decomposition T imply conditions 1, 2, 4 and 5 of Definition 3.3.

"(iii) $\Rightarrow$ (ii):" This step is just the reverse of (ii) $\Rightarrow$ (iii). Start with a compact decomposition H with adjacency graph A. The definition of adjacency graphs states that $V(A) = V(H) + E(H)$ and $E(A) = \{ve : v \in V(H), e \in E(H), v \in e\}$. Now, repeatedly choose $e \in V(A) \cap E(H)$, let $S := N_A(e)$ and alter A by replacing $S + e$ with an arbitrary tree on the vertex set S. This gradually transforms A into a tree T with $V(T) = V(H)$ and the properties of the compact decomposition H ensure that T (with the same bags as H) is a smooth tree decomposition of G. □

We have seen in this proof that compact and smooth compositions are very closely related: Every smooth composition can be transformed into a compact one by uniting all the bands with the same label into one band. To transform a compact composition into a smooth one, we need to transform every band into a subtree.

In a compact or a smooth tree decomposition H, there are two kinds of k-subsets of bags. The first kind are those sets S, which appear as a band, i.e. $S = X_e$ for some $e \in E(H)$. The second kind are those, which do not, i.e. $|\Phi(S)| = 1$. Sets of the second kind are called *pendant* and, by definition, appear in only one bag.

Definition 3.6 (Pendant k-Sets)**.** *Let H be a compact or a smooth tree decomposition of width k with bags X_i $(i \in V(H))$.*

- *A set $S \subseteq X_i$ for some $(i \in V_i)$ is called* ***pendant k-set****, if $|S| = k$ and $|\Phi(S)| = 1$.*
- *define* $\text{bags}(H) := \{X_i : i \in V(H)\}$.
- *Define* $\text{bands}(H) := \{S : S \textit{ is a band in } H\}$.
- *Define* $\text{pendant}(H) := \{S : S \textit{ is a pendant } k\textit{-set in } H\}$

We use these definitions to express some relations between H and $\text{fill}(H)$.

Lemma 3.7 (Bijections in compact Decompositions)**.** *Let H be a compact tree decomposition and $G' := \text{fill}(H)$. Let $R_0 \in \text{pendant}(H) \dot\cup \text{bands}(H)$ and $R_1 \in \text{bags}(H)$. We find the following identities and bijections:*

a.) $\{C : C \textit{ is a } (k+1)\textit{-clique in } G'\} = \text{bags}(H)$

b.) $\{C : C \textit{ is a } k\textit{-clique in } G'\} = \text{bands}(H) \dot\cup \text{pendant}(H)$

c.) $f : \text{bags}(H) \to \{R_1\} \cup (V(G) \setminus R_1)$

d.) $h : \text{bags}(H) \to V(G) \setminus R_0$

Proof. We have seen *a.*) and *c.*) already in Corollary 2.23.
For *b.*), we have $\text{pendant}(H) \dot\cup \text{bands}(H) = \{S \subseteq X_i : i \in V(T), |S| = k\} = \{C : C \text{ is a } k\text{-clique in } G'\}$. The first equality is due to the definition of $\text{bags}(\cdot)$ and $\text{pendant}(\cdot)$, the second equality due to the properties of a fill.
For *d.*), just find a bag R containing R_0 and apply *c.*) to R. Then h is the same function as f, with the exception that R is mapped to $R \setminus R_0$ instead of $f(R)$. □

The functions f, g from this lemma can be understood more intuitively: They correspond to constructing the k-tree G' from the compact decomposition T by starting with R (or R_0, respectively) as the root. This construction is analogous to the construction described in Corollary 2.22 from the previous section.

Sometimes, it is useful to see the pendant k-sets as labels belonging to (multi-)hyperedges of cardinality 1 in the hypergraph. Using this point of view, we obtain the following observation:

Corollary 3.8 (Connection between Bands and Cliques)**.** *Let H be a compact tree decomposition and $G' := \mathrm{fill}(H)$. Let $S \in \mathrm{bands}(H) \dot\cup \mathrm{pendant}(H)$ be the k-sized set belonging to the hyperedge e if $S = X_e \in \mathrm{bands}(H)$ or belonging to the pseudo-hyperedge $e := \Phi(S)$ if S is a pendant k-set.*

a.) The set of $(k+1)$-cliques of G which contain S is exactly the set of bags that are the labels of the vertices in e.

b.) $|\Phi(S)| = |e| = \deg(S)$.

Proof. For *a.)* : If S is a band, because of condition 4 from Definition 3.3, we must have that $\Phi(S) = e$. If S is a pendant k-set, we have $\Phi(S) = e$ by definition. But $\Phi(S)$ describes exactly the set of bags that contain S. By the previous lemma this is exactly the set of $(k+1)$-cliques in T, which contain the k-clique S.
The claim *b.)* follows from *a.)*. □

Lemma 3.9 (Uniqueness)**.** *Any k-tree T has a unique compact tree decomposition.*

Proof. Let H be a compact decomposition of T. Recall that we found out in Observation 2.24 that, if D is a smooth decomposition of G, we already have $\mathrm{fill}(D) = T$. As H can be transformed into a smooth decomposition with the same bags, $\mathrm{fill}(H) = G$. Lemma 3.7 showed that the set of bags of H must be equal to the sets of $k+1$-cliques, so we have no freedom in the choice of bags.

Due to part *a.)* of the previous corollary, we have no freedom in the choice of the hyperedges of H either. □

One can easily see that the smooth tree decomposition of a k-tree is not necessarily unique. But the relation between compact and smooth decompositions shows that a smooth decomposition D is unique up to the structure of the subtrees $D[\Phi(S)]$ where S ranges over all bands.

In general, it is also not true that a graph has a unique compact tree decomposition. In fact, to determine the stacked treewidth of a graph, we want to find out, whether there is a compact tree decomposition H with a "special" property. (This property being that no band in H has degree 3 or higher). The details of this are discussed in the next subsection. For now, consider the following, motivational example.

Example 3.10. *Consider $G := P_3 \vee \overline{K_3}$. The graph G has $\mathrm{tw}(G) = 4$. Furthermore, G has exactly those two compact tree decompositions of width 4 that are depicted in Figure 7.*

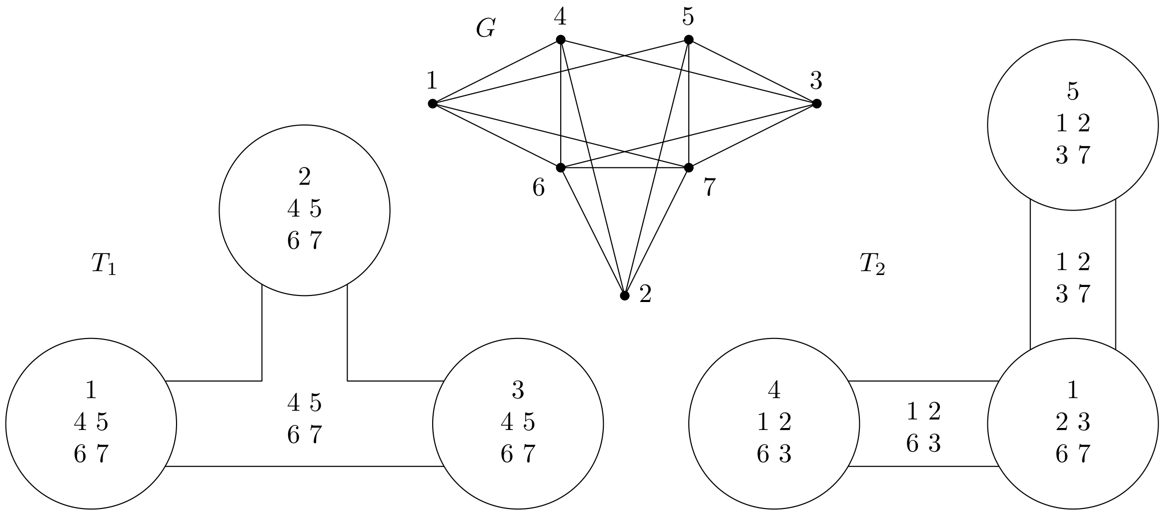


Figure 7: The graph G has exactly T_1 and T_2 as compact tree decompositions.

Proof. First note that T_1, T_2 are valid tree decompositions of G. Thus $tw(G) \leq 4$. On the other hand, $\delta(G) = 4$, so $tw(G) \geq 4$. Let T be a smooth tree decomposition with width $k = 4$ of G. The number of bags in T is $n - k = 7 - 4 = 3$. We show that either T has the same bags as T_1 and has two bands of the form $\{4, 5, 6, 7\}$ or that T has the same bags as T_2 and the two bands $\{1, 2, 3, 6\}$ and $\{1, 2, 3, 7\}$. As every compact tree decomposition can be transformed into a smooth decomposition, using the process we described in Theorem 3.5, we then can conclude that every compact decomposition must be equal to T_1 or T_2.

Note that $\{1, 4, 6\}$ and $\{3, 5, 7\}$ are cliques, so there must be a bag $X_a \supseteq \{1, 4, 6\}$ and a bag $X_b \supseteq \{3, 5, 7\}$. As these are six elements in total, $X_a \neq X_b$. Let the third bag be X_c. The only tree on three vertices is P_3 and there are only three possibilities how X_a, X_b and X_c can be arranged into a path: $X_a - X_b - X_c$, or $X_b - X_a - X_c$, or $X_a - X_c - X_b$. We make a case distinction.

Case 1: $(X_a - X_b - X_c)$**:** This case is depicted in Figure 8 As the union of $\{1, 4, 6\}$ and $\{3, 5, 7\}$ already has 6 elements, we have that $X_a \cup X_c = \{1, 3, 4, 5, 6, 7\}$. As 2 has edges to 4,5,6, and 7, but can not be in X_a or X_b, 2 must be in X_c and $X_c = \{2, 4, 5, 6, 7\}$. Thus $X_{bc} = \{4, 5, 6, 7\}$ and thus $X_b = \{3, 4, 5, 6, 7\}$. Then we have that $X_a \setminus X_b = \{1\}$ and as 1 has edges to 4,5,6,7 we have $X_a = \{1, 4, 5, 6, 7\}$. So we arrive at T_1.

Case 2: $(X_b - X_a - X_c)$**:** Observe that G is symmetrical to the y-axis. Also, if we mirror G vertically, the clique $\{1, 4, 6\}$ is mapped to $\{3, 5, 7\}$. Therefore case 2 is symmetrical to case 1.

Case 3: $(X_a - X_c - X_b)$**:** This case is depicted in Figure 9. First we observe that $6, 7$ can not be only in X_a or only in X_c, because these two have a degree higher than four

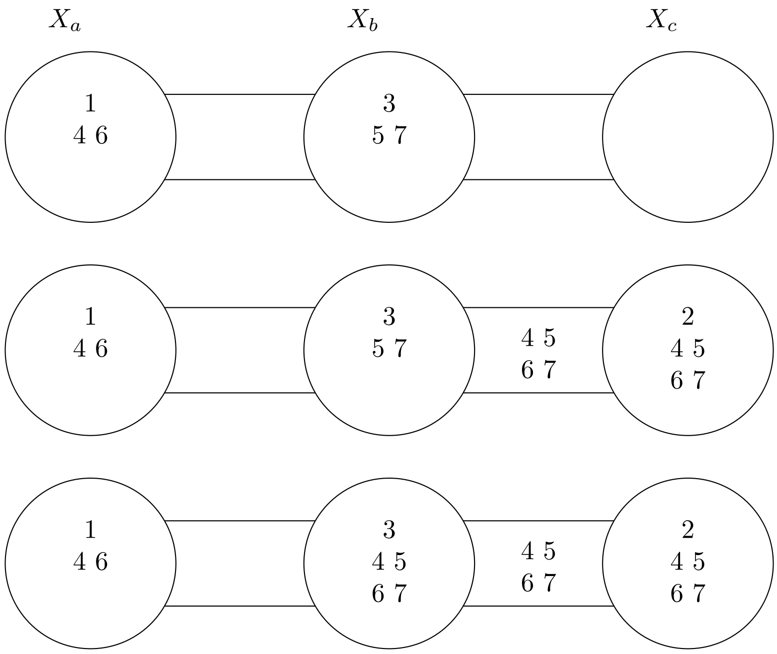


Figure 8: Deduction of the structure of T in case 1

in G. Similarly, 2 can not be only in X_a or only in X_b, because 2 has degree four, but 1 $\in X_a$ and 3 $\in X_b$ and 2 is not connected to both 1 and 3. So $\{2, 6, 7\} \subseteq X_c$.
Because T is smooth $X_a \setminus X_c = \{x\}$ for some $x \in \{1, \ldots, 7\}$. Assume $x = 1$. Then we would have $4 \in X_c$. Also we would have $X_a = \{1, 4, 5, 6, 7\}$, as 1 needs to connect to 4,5,6,7. Then $5 \in X_c$, as 5 in X_a and X_b. So then the only possibility for X_c is $\{3, 4, 5, 6, 7\}$ and we arrived at T_1.

For analogous reasons, if $X_b \setminus X_c = \{3\}$, we also arrive at T_1. So let without loss of generality $1, 3 \in X_c$. Then the only possible completion of the decomposition that is left is T_2. □

At last, we add a small observation at the end of this subsection that deals with connectivity.

Observation 3.11. *Every band X_e in a compact or smooth decomposition H is a separating set of G.*

Proof. For each $v \in X_e$, delete v in all the bags. Then X_e becomes empty. The properties of a smooth (compact) decomposition ensure that there exist some nonempty bags in every components of $H - e$. Observe that intersections of the subtrees in $\{\Phi(v) : v \in V(G) \setminus X_e\}$ characterize all possible edges of $G - X_e$. Together, these two facts tell us that $G - X_e$ is disconnected. □

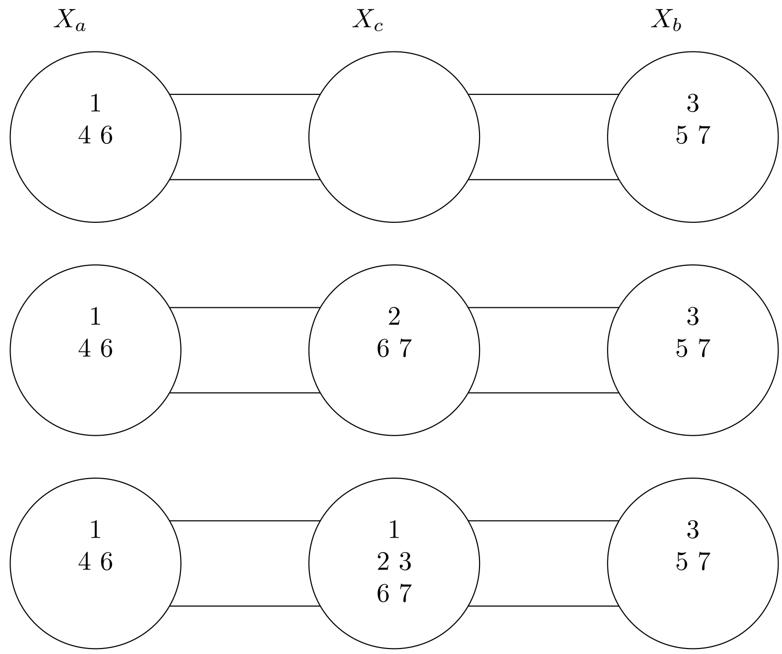


Figure 9: Deduction of the structure of T in case 2

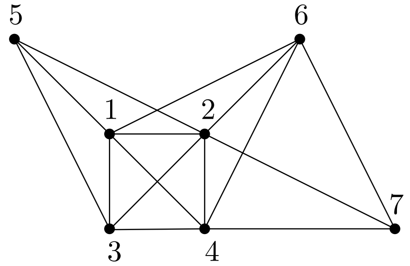
3.2 Stacked Treewidth

As already noted, the goal of this thesis was to examine the relationship between the stacked treewidth and the Colin de Verdière number. To understand why this relation might be worth considering, the first step is to define the stacked treewidth, which takes place in this section. We begin with the definition of a stacked k-tree.

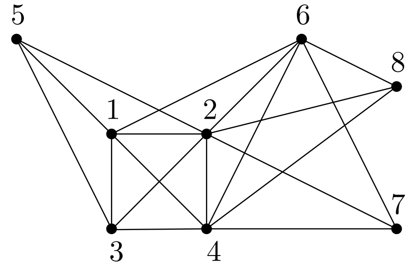
Definition 3.12 (Stacked k-Tree). *A k-tree is called* ***stacked****, if during its construction $((v_1, \dots, v_n), \rho)$ no two vertices v_i and v_j are stacked onto the same k-clique, i.e. $\rho(v_i) = \rho(v_j)$.*

As an example, consider Figure 10, where we construct two 3-trees using the sequences $(1, \dots, 7)$, and $(1, \dots, 8)$, respectively. The second 3-tree, in contrast to the first 3-tree, is not stacked, as we stack both 7 and 8 on 246 during its construction.
Stacked k-trees have also been called *simple k-trees* or *simple-clique k-trees* in the literature [9, 15]. We see in a minute that Definition 3.12 is independent of the actual construction.

Why is forbidding to stack twice a reasonable restriction to make? Consider the construction of a stacked 1-tree. We start with a single edge and are not allowed to stack on the same vertex twice. So we have that the class of stacked 1-trees is the class $\{P_n : n \geq 1\}$. For the class of stacked 2-trees, we start with a triangle and glue triangles to an existing edge at the outer face. So every obtained graph is maximal outerplanar. It is easy to see that every maximal outerplanar graph can be obtained this way, too. So the class of stacked 2-trees is the class of maximal outerplanar graphs. For $k = 3$, one



(a) A stacked 3-tree



(b) A non-stacked 3-tree

Figure 10: Comparison of a stacked and a non-stacked 3-tree

can visualize the process as starting with one tetrahedron and glue other tetrahedrons to non-occupied faces of the current polyhedron. The resulting stacked 3-tree is the 1-skeleton of the resulting polyhedron. This shows that all stacked 3-trees are planar.

The class of stacked 3-trees is better known under the name of Apollonian networks (except K_3). The Apollonian networks are obtained by starting with a single triangle and repeatedly subdividing a triangular face into three faces by connecting a new vertex to the three vertices of the face. Of course, this process is equivalent to taking 1-skeletons of stacked tetrahedrons.

In general, a k-polytope, which is obtained by starting with a k-dimensional $(k+1)$-simplex and repeatedly gluing $(k+1)$-simplices to the $(k-1)$-dimensional facets, is called a stacked polytope. Therefore the stacked k-trees are exactly the 1-skeletons of stacked polytopes. (This connection is also the reason, why we chose the names “stacked k-tree” and “stacked treewidth” for the concepts discussed in this thesis.)

What are the stacked 0-trees? We start with K_1 and then repeatedly stack on a zero-element set, such that there does not exist a zero-element set onto which we stacked twice. Thus, the stacked 0-trees are exactly $\{K_1, \overline{K_2}\}$. If the reader feels uncomfortable with this, we refer to Section 5.1, where the topic comes up again and we give some more reasoning.

Now we want to prove that the definition of a stacked tree given in Definition 3.12 is independent of the actual construction. To do this, we give four different characterizations and prove their equivalence.

Theorem 3.13 (Criteria for Stackedness of k-Trees)**.** *Let T be a k-tree. The following are equivalent:*

(i) All constructions of T suffice the criteria of Definition 3.12.

(ii) There exists a construction of T which suffices the criteria of Definition 3.12.

(iii) The bands in the unique compact tree decomposition of T have degree at most two.

(iv) All k-cliques in T have a degree of at most two.

Proof. **"(i) $\Rightarrow$ (ii):"** This is obvious.

"(ii) $\Rightarrow$ (iii):" Let $C := ((v_1, \dots, v_n), \rho)$ be such a construction. We have seen in Lemma 2.14, how to gradually construct a tree decomposition of T, if given C. We later found that this decomposition is also smooth.

During this process, the new bags being added were of the form $\rho(v_i) + v_i$ for some $i \in \{k+2, \dots, n\}$ and the new bags were adjacent to old bags also containing $\rho(v_i)$. So the bands of D are all of the form $\rho(v_i)$ for some $i \in \{k+2, \dots, n\}$. By the definition of stackedness no two $\rho(v_i)$ are equal and thus no two bands in the smooth decomposition are equal. Thus, if we transform the smooth decomposition into the unique compact decomposition, like in Theorem 3.5, no band will have degree 3 or higher.

"(iii) $\Rightarrow$ (iv):" We have seen in Corollary 3.8 how the bands of the (unique) compact tree decomposition correlate exactly to the k-cliques of T with the same degree.

"(iv) $\Rightarrow$ (i):" If a construction stacks twice on the same k-clique, it creates a k-clique of degree 3 or higher in T. (Note that every k-clique already has degree 1 when first being created during the stacking process.) □

With this insight, we are ready to define the stacked treewidth. The definition is analogous to the standard definition of treewidth that we gave at the beginning.

Definition 3.14 (Stacked Treewidth)**.** *For any graph G, its **stacked treewidth**, denoted by* $\mathrm{stw}(G)$*, is the smallest k such that G is subgraph of a stacked k-tree.*

$$\mathrm{stw}(G) := \min\{k \in \mathbb{N}_0 : \exists \text{ stacked } k\text{-tree } T \text{ with } G \subseteq T\}$$

For example, the star on $n+1$ vertices, $S_n := K_1 \vee \overline{K_n}$ has $\mathrm{stw}(K_n) = 1$ if $n = 1, 2$, because S_1 and S_2 are paths. For $n \geq 3$, however, $\mathrm{stw}(S_n) = 2$, as S_n is subgraph of a maximal outerplanar graph, but not subgraph of a path. For all $n \in \mathbb{N}$, the treewidth of S_n is 1.

We have seen that the concept of bands with degree 3 or higher in compact decompositions is closely connected to the stacked treewidth. We call a band X_e in a compact tree decomposition a *high-degree band*, if $\deg(X_e) \geq 3$. Another important role is played by the pendant k-sets: Given a k-tree T, its compact decomposition H and a k-clique $S \subseteq V(T)$, then we can stack a new vertex onto S if and only if S has degree 1, which is the case if and only if S is a pendant k-set in H.

3.3 Lifting Lemma

A natural question that may arise, is the following: Given a compact tree decomposition H of a graph G that contains a high-degree band, can we transform it into a compact

decomposition with no high-degree bands or at least one with less high-degree bands? The *lifting lemma* allows such a transformation if a basic condition is met. Roughly speaking, if one *arm* of the graph G lies on one side of a high-degree band X_e but does not "use" all of the k vertices of X_e, we can decrease the degree of X_e by one, using the so-called operation of *lifting* the arm.
What do we mean by "arm", here? As X_e is a band in a compact tree decomposition of G, it splits G into $\deg(X_e)$ parts. Each of these parts is considered an arm. More precisely, an arm is defined as follows.

Definition 3.15 (Arms in a Tree decomposition)**.** *Let H be a smooth or a compact tree decomposition of a graph G with labels X_i ($i \in V(H)$). Let X_e be a band of H, let $v \in V(G) \setminus X_e$. We define*

- $\mathrm{arm}(X_e, v) := V(K)$
  where K is the component of $\mathrm{fill}(H) - X_e$ that contains v

If additionally, G is compact, we define

- $\mathrm{bagsarm}(X_e, v) := \{X_i : i \in V(H_0)\}$
  where H_0 is the component of $H - e$ that contains $\Phi(v)$

Note that X_e is always a separating set in $\mathrm{fill}(H)$ (see Observation 3.11). So $\mathrm{arm}(X_e, v)$ is well-defined.

The lifting lemma is best understood using an example. Consider Figure 11, which shows a (familiar) graph G with a compact tree decomposition T_1. Consider the band $X_e = \{1, 2\}$ of the edge $e = \{i_4, i_6, i_7, i_8\}$. We have $\deg_{T_1}(X_e) = 4$. Furthermore, we have $A := \mathrm{arm}(X_e, 6) = \{6, 7, 8, 9, 10\}$ and $\mathrm{bagsarm}(X_e, 6) = \{X_{i_1}, \ldots, X_{i_5}\}$. We notice that the decomposition T_1 connects A to the rest of the graph via $\{1, 2\}$. But actually, this is not necessary, as $N_G(A) \cap \{1, 2\} = \{1\}$ and furthermore, the 1-clique $\{1\}$ is contained in the pendant k-set $\{1, 3\}$. So it would be better to connect the whole arm A to $\{1, 3\}$ instead of $\{1, 2\}$. This is done in T_2 and we see that the sum of the degrees of all high-degree bands has gotten smaller. The process of converting T_1 to T_2 is called *lifting the arm A* and the lifting lemma gives us a condition under which lifting an arm is allowed.

The following simple lemma was shown before (in a different form) by induction by Mitchell and Yengualp [16].

Lemma 3.16 (Lemma about pendant k-Sets)**.** *Let H be a compact (or smooth) decomposition with width k of a graph G. Let C be subset of a bag X_i of H and $|C| < k$. Then there exist at least two pendant k-sets P_1, P_2 which contain C.*

Proof. Consider $\Phi(C)$, which is a subtree of H. If $|\Phi(C)| = 1$, we find at least two k-sets $P_1, P_2 \supsetneq C$ in the bag X_i, as $|X_i| = k + 1$ and $|C| <$ k. But these sets P_1 and P_2 can not appear as bands, as $|\Phi(C)| = 1$. So P_1, P_2 are pendant k-sets.
Similarly, if $|\Phi(C)| \neq 1$, look at the leafs of $\Phi(C)$. In each leaf, we again find at least

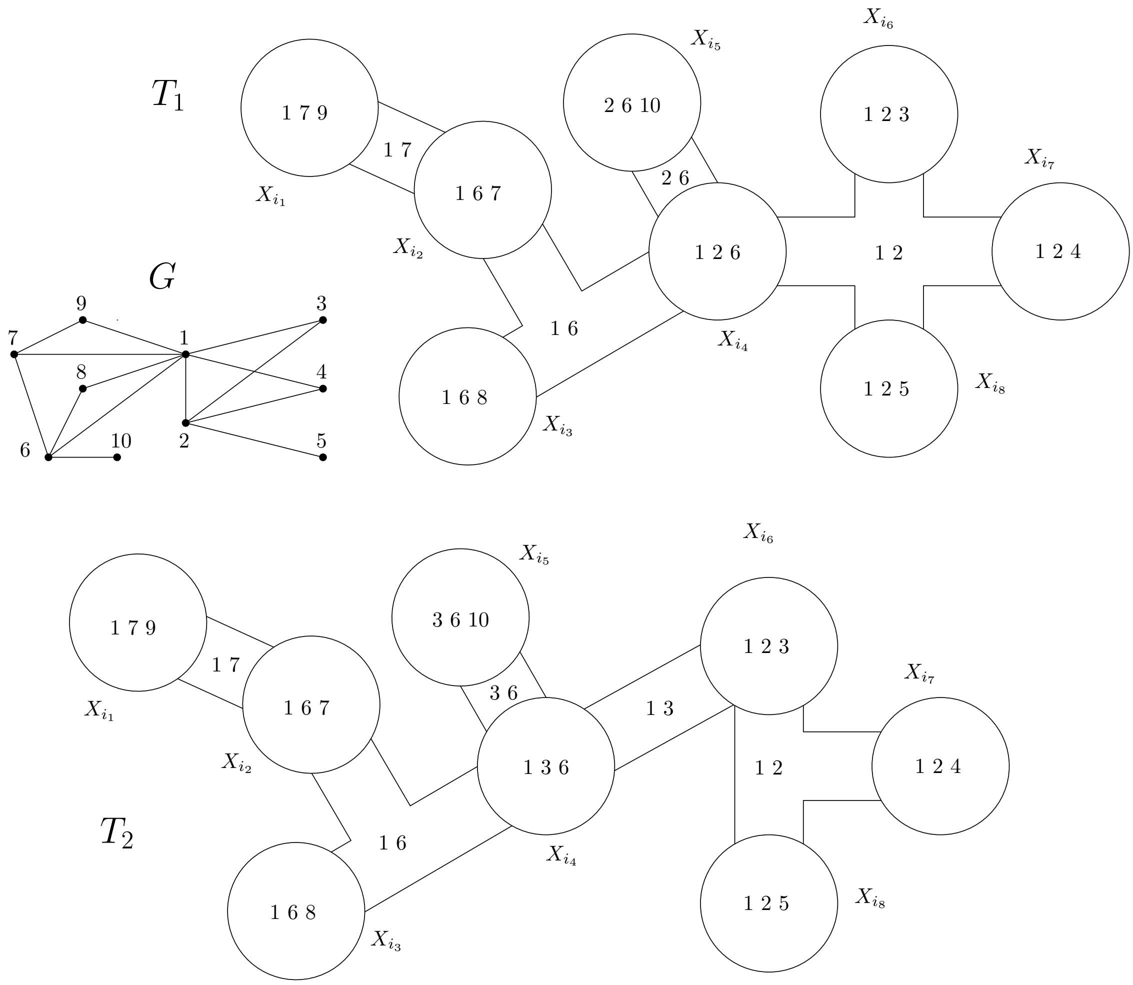


Figure 11: Application of the lifting lemma to arm$(\{1, 2\}, 6)$

.

two k-sets $P_1, P_2 \supseteq C$, but at most one of P_1, P_2 can be a band, as they are in a leaf of $\Phi(C)$. Because $\Phi(C)$ has at least two leafs, we are done. □

We want to show that lifting can be done repeatedly and the decomposition becomes "better" every time. For this, define

$$\zeta(H) := \sum_{X \in \mathbf{X}} \deg(X), \text{ where } \mathbf{X} \text{ is the set of all high-degree bands in } H.$$

This function ζ is also applicable to smooth decompositions, if we define

Definition 3.17 (Degree of a Band in a Smooth Decomposition)**.** *The **degree** of a band X_{ij} in a smooth decomposition T is*

$$\deg(X_{ij}) := |\{ab \in E(T) : X_{ab} = X_{ij}\}|$$

.

(This definition makes sense, as exactly the bands in $\{ab \in E(T) : X_{ab} = X_{ij}\}$ will be factorized into the same band of the equivalent compact decomposition.)

After these preparations, we can state the lifting lemma. We have already seen an example of the lifting process in Figure 11.

Lemma 3.18 (Lifting Lemma - Weak Version)**.** *Let H be a compact decomposition with width G. Let X_e be a band with $\deg(X_e) \geq 3$, $v \in V(G) \setminus X_e$ and $A := \mathrm{arm}(X_e, v)$. If $|N_G(A) \cap X_e| < k$, i.e. there is a vertex $u \in X_e \setminus N_G(A)$, we say A can be **lifted at** X_e **with unnecessary vertex** u. We can then construct a compact decomposition H' of G with $\zeta(H') < \zeta(H)$.*

Proof. Let $B := \mathrm{bagsarm}(X_e, v)$ and H_s be the compact tree decomposition that is implied by $\mathrm{bags}(H) \setminus B$. Apply the lemma about pendant k-sets to the decomposition H_s and the set $X_e - u$. We find a pendant k-set $P \supseteq X_e - u$ in a bag $X_i \notin B$. (X_i being the label of $i \in V(H)$.)

We can then create H' the following way: Let X_j be the bag in B, which is incident to X_e and let $j \in V(H)$ be the vertex belonging to X_j. Then, delete j from e, and add the new hyperedge $\{i, j\}$. Let $w \in P \setminus (X_e - u)$ and replace all appearances of u in all bags X in the family $\{X : X \in B\}$ with w. This last step ensures, that all subtrees $\{\Phi(v') : v' \in V(G)\}$ of H stay connected and is allowed, as every appearance of u in the bags in B was unnecessary. Note that this last step also does not alter any degrees of any bands.

So we added a new band of degree 2 and decreased the degree of a high-degree band. Therefore $\zeta(H') = \zeta(H) - 1$. □

Unfortunately, the second case of the lifting lemma has a bit of a technical proof, although being very intuitive: It states, that lifting is possible, if for $A := \mathrm{arm}(X_e, v)$, the graph $G[A]$ is disconnected into components $A_1, \ldots, A_m$, which on the one hand have $|N_G(A) \cap X_e| = k$, but on the other hand have $|N_G(A_i) \cap X_e| < k$ for all $i = 1, \ldots, m$.

This is intuitively correct, as each component $A_1, \dots, A_m$ can be lifted separately. But the difficulty is showing that we find a compact decomposition H' on the same vertex set $V(G)$ with $\zeta(H') < \zeta(H)$.

For this, consider the following helper lemma, which is also already an application of the weak lifting lemma.

Lemma 3.19 (Removing a Vertex from a Decomposition)**.** *Let H be a compact decomposition of a graph G and $v \in V(G)$. The graph $G - v$ has a compact composition H' with $\zeta(H') \leq \zeta(H)$ and $\mathrm{width}(H') \leq \mathrm{width}(H) =: k$.*

Proof. **Case 1:** The vertex v is in every bag.
Then just delete v from every bag.

Case 2: The subtree $\Phi(v) = \{i\}$ has cardinality 1 (and we do not have case 1).
In this case, all outgoing bands of X_i must be equal to $X_i - v$. So we may delete v from the bag X_i without losing information in the tree decomposition. Then we choose an outgoing band X_{ij}, contract it, and label the vertex created by the contraction with X_j.

Case 3: If $|\Phi(v)| > 1$ holds (and we do not have case 1).
In this case, we do the following: Transform H into a smooth decomposition T with the same bags (like in Theorem 3.5). Let ij be an edge in T such that i is a leaf of $\Phi(v)$ (Recall that Φ is the same for H and T). Consider $y := X_i \setminus X_{ij}$. Then we have that $\Phi(y) \cap \Phi(v) = \{i\}$. Now consider the following, unusual step: *Color* every occurrence of v in the bags of T red and color every other occurrence of a vertex in a bag green. Observe that all green-colored occurrences give enough information to be a decomposition of $G - v$. $(*)$

Now replace every occurrence of v with y. This leaves all subtrees $\Phi(v')$ $(v' \in V(G))$ connected and does not change $(*)$. The bag X_i has now only size k and $X_{ij} = X_i \subseteq X_j$. Therefore, we can contract the band X_{ij} and label the vertex created by the contraction with X_j.

The replacement $v \to y$ may have increased the degree of some bands X_r. The trick, however, is that whenever that happened, one of the bands ts labeled $X_{ts} = X_r$ must have had a vertex v that was transformed into y and one of the bags incident to ts must have a vertex colored red. Therefore we can lift the arm belonging to this bag and decrease $\zeta(T)$ by one again.

Note that lifting leaves all vertices that are colored green untouched (and changes some of the red-colored vertices). Therefore, we can repeat this process until we arrive at $\zeta(T) \leq \zeta(T')$ and still have $(*)$ fulfilled. □

With this preparation, we can prove the strong lifting lemma.

Lemma 3.20 (Lifting Lemma - Strong Version)**.** *Let H be a compact decomposition of a graph G with width k. Let X_e be a band with $\deg(X_e) \geq 3$, $v \in V(G) \setminus X_e$ and $A := \mathrm{arm}(X_e, v)$. If $|N_G(A_i) \cap X_e| < k$ for every component $A_1, \dots, A_m$ of $G[A]$, i.e.*

there is a vertex $u_i \in X_e \setminus N_G(A_i)$*, we can lift* A*. We can then construct a compact decomposition* H' *of* G *with* $\zeta(H') < \zeta(H)$*.*

Proof. At first, we create a bigger compact decomposition by replacing the bags belonging to the single arm A with m copies of $\mathrm{bagsarm}(X_e, v)$. Precisely: For every $i \in \{1, \ldots, m\}$ do the following: Create a copy B_i of $B := \mathrm{bagsarm}(X_e, v)$. For a vertex y that should be inserted into the copy, do as follows: If $y \in A_i$, use the real vertex y and color every occurrence of y in B_i green, otherwise use a new dummy-vertex $d_y^{(i)}$ instead of y in the copy and color every occurrence in B_i red. Connect B_i to the band X_e. After all copies were created, delete B.
Again, the green-colored vertices alone give enough information and every red vertex is unnecessary.

By the premise and the weak lifting lemma, each copy $B_1, \ldots, B_m$ can be lifted. After we have done this, we have decreased $\deg(X_e)$ by one compared to the beginning. But we may have increased $\zeta(H)$ in total, as a copy might have a high-degree band.

But note that if one copy has a high-degree band X_r to which the weak lifting lemma can not be applied, all elements of X_r must be colored green. So in all other copies, all vertices in the band are red and the weak lifting lemma can be applied there until the band is no high-degree band anymore. So at the end, the degree sum of all high-degree bands that can not be lifted in $B_1, \ldots, B_m$ can not exceed the degree sum of the original arm.

So we have arrived at a compact decomposition H'' with $\zeta(H'') < \zeta(H)$, but H'' covering many additional vertices. Apply Lemma 3.19 and we are done. □

After this exhausting proof, we can use the weak and strong versions of the lifting lemma to make the upcoming proofs more elegant. First, we see that the stacked treewidth may only take two values:

Theorem 3.21 (Stacked Treewidth can only take two Values)**.** *Every* k*-tree* T *is contained in a stacked* $(k+1)$*-tree.*
Thus, the stacked treewidth is bounded by: $\mathrm{tw}(G) \leq \mathrm{stw}(G) \leq \mathrm{tw}(G) + 1$*.*

Proof. It is obvious that $\mathrm{tw}(G) \leq \mathrm{stw}(G)$. For the second inequality, let H be the unique compact tree decomposition of T. Insert a new vertex $v \notin V(T)$ into every bag and color every occurrence of v red and the rest green. In every band there is an unnecessary, red-colored vertex, so we can apply the lifting lemma (the weak version suffices) until H has no high-degree bands anymore. Note that each lifting being done does not change any green vertex. Then $\mathrm{fill}(H)$ is a stacked $(k+1)$-tree containing T. □

Then, we discuss what it means for an arm to *not* fulfill the requirements of the lifting lemma.

Observation 3.22. *Let* X_e *be a band in a compact decomposition* H *of* G *with* $k := \mathrm{width}(H) = tw(G)$ *and* $d := \deg(X_e)$*.*

a.) *If for some $v \in V(G)$, the arm $A := \operatorname{arm}(X_e, v)$ can not be lifted, then we can do edge contractions in $G[A]$ to get G' such that there is a vertex $v_A \in V(G')$ adjacent to all of X_e.*

b.) *If $\deg(X_e)$ can not be decreased with the lifting lemma, then $\mathcal{M}K_{k,d} \subseteq G$.*

c.) *If G has no $K_{k,3}$-minor, then $stw(G) = k = tw(G)$.*

Proof. *a.*) Note, that if the requirements of the strong lifting lemma are not met, we find a component A_i with $N_G(A_i) \cap X_e = X_e$. Clearly, *b.*) follows from *a.*) and *c.*) follows from *b.*). □

It seems intuitive to make the following definition:

Definition 3.23 (Stacked Tree Decomposition)**.** *A smooth or a compact tree decomposition is called **stacked**, if it contains no high-degree band.*

We see in a minute that indeed having $\operatorname{stw}(G) = \operatorname{tw}(G)$ is equivalent to the existence of a stacked (smooth or compact) tree decomposition. The theorem on the equivalence of treewidth, Theorem 2.9 characterized treewidth in terms of chordal completions. Does there exist an analogous characterization for stacked treewidth? For this, we define $M_{k,d}$ as the k-clique with degree d.

Definition 3.24 ($M_{k,d}$)**.** *Let be $k, d \in \mathbb{N}_0$. We define $M_{k,d} := K_k \vee \overline{K_d}$.*

Then we have the following theorem.

Theorem 3.25 (Stackedness of Chordal Graphs)**.** *Let G be a chordal graph with $k := \omega(G) - 1 = \operatorname{tw}(G)$. The following are equivalent:*

(i) $\operatorname{stw}(G) = k$*, i.e. G is covered by a stacked k-tree.*

(ii) *G does not contain $M_{k,3}$ as subgraph.*

(iii) *G has a stacked, compact tree decomposition.*

Proof. **"(i) ⇒ (ii):"** A stacked k-tree has no cliques of degree 3 or higher, i.e. no $M_{k,3}$ as subgraph.

"(ii) ⇒ (iii):" Let H be a compact tree decomposition of G. If H is not already stacked, we show that we can apply the lifting lemma, until we arrive at a stacked tree decomposition.

Assume for the sake of contradiction that X_e is a high-degree band in H and that A_i is an arm such that we can not apply the lifting lemma to X_e and A_i ($i = 1, 2, 3$). Then, by Observation 3.22, we find a component A_i' of $G[A]$ such that $G[A_i']$ is connected and $X_e \subseteq N_G(A_i')$ for $i = 1, 2, 3$. (∗)

We claim that X_e is a clique in G. In fact, let be $x, y \in X_e$ with $y \neq x$. Because of $(*)$, we find a cycle C_0 that uses x, A'_1, y, A'_2 in this order. Choose a cycle C such that C has minimal length and uses x, A'_1, y, A'_2 in this order. Then C must have a chord, and this must be xy, as C is minimal and X_e is a separating set in G, which splits A_1 and A_2 (compare Observation 3.11).

Now, consider A'_i. We claim that there is $v_i^* \in A'_i$ such that v_i^* is connected to all of X_e. If we show this claim, we are done, as we showed that G contains $M_{k,3}$, which is a contradiction.

To show the claim, choose a set $S \subseteq A'_i$ with $|S|$ minimal such that S is connected and $X_e \subseteq N_G(S)$. This is possible due to $(*)$. If $|S| = 1$, we are done. If $|S| > 1$, let be $v \in S$ such that the deletion of v does not disconnect $G[S]$.

Define for all $u \in S$ the function $R(u) := N_G(u) \cap X_e$. (So $R(u)$ is the part of X_e that is "covered" by u.) We find, by minimality of S, an $x \in R(v)$ such that $x \notin R(v')$ for all other $v' \in S - v$. But we also find, as $|S| > 1$, a vertex $y \in X_e \setminus R(v)$.
Let then w be the vertex in $N_G(y) \cap S$ such that the distance between w and v in $G[S]$ is minimal. Let P be a shortest v-w-path in $G[S]$. Then, the cycle (x, v, P, w, y) is chordless and has length at least 4. So we finally arrived at a contradiction.

"(iii) ⇒ (i):" If G has a stacked, compact tree decomposition H, by definition H has no bands of degree 3 or higher and we have seen that fill(H) is a k-tree where the k-cliques have the same degree as the bands of H. □

So after this theorem, it seems reasonable to call a chordal graph G a *stacked chordal graph*, if it meets any of the conditions (i) – (iii). (The same name was also used in [16].) In Section 5, we will learn even more characterizations for a stacked chordal graph, one of those including the Colin de Verdière number.

Having seen what stackedness means for a chordal graph, we are ready to give the final theorem of this section, which is a characterization of stacked treewidth in terms of stacked trees, stacked compact decompositions and stacked chordal graphs, completely analogous to Theorem 2.9.

Lemma 3.26. *Let G be a graph, $k \in \mathbb{N}_0$. The following are equivalent:*

(i) There exists a stacked s-tree T that covers G, $s \le k$.

(ii) There exists a chordal completion G' of G such that G' is a stacked chordal graph and $\omega(G') - 1 \le k$.

(iii) There exists a stacked compact decomposition of G with width at most k.

(iv) There exists a tree decomposition T with width $s \le k$ of G such that T has no two identical bands of size k, i.e. X_{ij}, X_{ab} for $ij \neq ab \in E(T)$ with $|X_{ij}| = |X_{ab}| = k$ and $X_{ab} = X_{ij}$.

(v) There exists a chordal completion G' such that G' contains no $M_{k,3}$ and $\omega(G') - 1 \le k$.

Proof. **"(i) ⇒ (ii):"** The stacked s-tree T is already chordal and contains no $M_{s,3}$ as subgraph and has $\omega(T) = s + 1 \leq k + 1$. But we may have $V(T) \supseteq V(G)$, so T is not yet a chordal completion. But if we consider a stacked tree decomposition H_T of T, and repeatedly apply Lemma 3.19, we can transform H_T into a composition H'_T such that H'_T is still stacked and only uses vertices from G. Then we have (iii) and also (ii), if we consider $\text{fill}(H'_T)$.

"(ii) ⇒ (iii):" Let G' be the stacked chordal completion of G. We have seen in Theorem 3.25 that G' has a stacked compact decomposition and so has G.

"(iii) ⇒ (i):" The fill of this compact tree decomposition is a stacked k-tree covering G.

So we have shown the equivalence of (i) – (iii). This leaves us with (iv) and (v).

"(iii) ⇒ (iv):" (For condition (iv), note that T isn't necessarily smooth.)
This is clear, as we have seen in Section 3.1 how to transform a compact into a smooth tree decomposition. If the compact decomposition has no high-degree bands, the smooth decomposition has no two identical bands.

"(iv) ⇒ (v):" Let $G' := \text{fill}(T)$. We have already shown that G' is chordal. Let C be a k-clique in G'. If C had a degree of 3 or higher, C would be contained in 3 different $(k+1)$-cliques and therefore we would find at least 3 different bags containing C. But we have $|\Phi(C)| \leq 2$ by premise. Therefore, G' doesn't contain an $M_{k,3}$. Finally, $\omega(G') \leq k + 1$ is true, as $\text{width}(T) \leq k$.

"(v) ⇒ (i):" We can do the following trick: Add $k + 1$ independent vertices to G' to obtain $G'' \supseteq G'$. If we connect these $k + 1$ independent vertices into a $(k + 1)$-clique, we see that G'' has a chordal completion H with $\omega(H) = k + 1$ and H contains no k-clique of degree 3 or higher, so H is stacked. Thus, $\text{stw}(G'') \leq k$ and as G' is a subgraph of G'', we finally have $\text{stw}(G') \leq k$. □

Now, cases (i)–(iii) of the lemma clearly imply the final theorem.

Theorem 3.27 (Characterizations of Stacked Treewidth)**.** *Let G be a graph. The following are equivalent characterizations for* $\text{stw}(G) = k$*:*

(i) k is the smallest integer such that G is subgraph of a stacked k-tree.

(ii) k is the smallest integer such that G has a stacked, compact tree decomposition of width k.

(iii) $k = \min\{\omega(H) - 1 : H \text{ is a stacked chordal completion of } G\}$.

4 The Colin de Verdière Number

The Colin de Verdière number is a graph parameter defined in terms of eigenvalues and ranks of matrices and was first introduced in 1993 by Yves Colin de Verdière [3]. We denote the Colin de Verdière number of a graph G by $\mu(G)$ and this number is most famous for the fact that it characterizes the linklessly embeddable, the planar, the outerplanar graphs, and those graphs which are a disjoint union of paths by $\mu(G) \leq 4$, $\mu(G) \leq 3$, $\mu(G) \leq 2$, and $\mu(G) \leq 1$, respectively. In this section, we cite the most important results regarding the Colin de Verdière number. (The proofs of these results are all quite lengthy and are therefore omitted here for the sake of brevity.)

4.1 Definition

Recall that the corank of a matrix M is defined as $\operatorname{corank}(M) := \dim(\ker(M))$. Furthermore, let in this section $\mathbb{R}^{(n)}$ denote the space of all symmetric, real n-by-n matrices. For matrices $A, B \in \mathbb{R}^{(n)}$ with $A = (a_{ij})_{i,j=1,\dots,n}$ and $B = (b_{ij})_{i,j=1,\dots,n}$, define the *entrywise product*, also called the *Hadamard product*, $A \circ B = C$ as $C := (a_{ij}b_{ij})_{i,j=1,\dots,n}$. We define the Colin de Verdière number the following way:

Definition 4.1 (1.2.3 of [6]). *For $X, M \in \mathbb{R}^{(n)}$, we say that X **fully annihilates** M, if the following holds:*

$$MX = M \circ X = I \circ X = 0$$

Definition 4.2 (1.2.4 of [6]). *For a graph $G = (V, E)$ on vertex set $V = \{1, \dots, n\}$, define*

$$\mathcal{S}_G := \{M \in \mathbb{R}^{(n)} : M \text{ satisfies (M1)–(M3)}\}$$

where the conditions (M1)–(M3) are given by

- ***(M1):*** *For all $i \neq j$: $m_{ij} < 0$ if $ij \in E$ and $m_{ij} = 0$ if $ij \notin E$.*
- ***(M2):*** *M has exactly one negative eigenvalue, counting multiplicity.*
- ***(M3):*** *If $X \in \mathbb{R}^{(n)}$ fully annihilates M, then $X = 0$.*

*we define the **Colin de Verdière number** of G, $\mu(G)$, as*

$$\mu(G) := \max\{\operatorname{corank}(M) : M \in \mathcal{S}_G\}.$$

We add a few notes clarifying (M1)–(M3): The condition (M1) does not restrict the diagonal entries of M. The condition (M2) states that there exists exactly one negative eigenvalue of M and that additionally this eigenvalue has multiplicity 1. At last, condition (M3) has its own name: It is called the *Strong Arnold Property (SAP)*. It can be shown using the so-called *Perron-Frobenius theorem* that $\mathcal{S}_G \neq \emptyset$ [18, p. 4].

4.2 Examples

We calculate $\mu(G)$ for some easy examples, following [18, pp. 4-5].

Observation 4.3. *For all $n \in \mathbb{N}$, we have $\mu(K_n) = n - 1$.*

Proof. Set $G := K_n$. We need to determine M with maximal corank (i.e. minimal rank) in S_G. We claim that

$$M := \begin{pmatrix} -1 & \dots & -1 \\ \vdots & \ddots & \vdots \\ -1 & \dots & -1 \end{pmatrix}$$

is such a matrix. Indeed, (M1) is fulfilled, as every entry of M is negative. Observe that $M(1, \dots, 1)^\intercal = -n$ and clearly $\operatorname{rank}(M) = n - 1$. So we have for the spectrum of M, that $\sigma(M) = \{0, -n\}$ and thus (M2) holds. Furthermore, (M3) holds trivially, as any X fully annihilating M must be 0 at those elements where M is nonzero. Finally, note that finding M' with $\operatorname{corank}(M') = n$ is impossible, due to (M2). □

Observation 4.4. *For all $n \geq 2$, we have $\mu(\overline{K_n}) = 1$.*

Proof. Choose $n \geq 2$ and let $G := \overline{K_n}$. Due to (M1), when looking for a matrix M with maximal corank in $\mathcal{S}_G$, we can only choose diagonal entries of M, the rest must be 0. So we want to put as many zeros as possible into the diagonal. Assume that we put more than one zero in the diagonal, i.e. M is without loss of generality of the form

$$M = \left(\begin{array}{cc|ccc} 0 & 0 & & & \\ 0 & 0 & & & \\ \hline & & a_1 & & \\ & & & \ddots & \\ & & & & a_{n-2} \end{array}\right),$$

then let

$$X := \left(\begin{array}{cc|c} 0 & 1 & \\ 1 & 0 & \\ \hline & & \\ & & \end{array}\right)$$

and observe that X fully annihilates M. So in this case we do not have the Strong Arnold Property (M3). On the other hand, if we choose $M := \operatorname{diag}(0, \lambda, a_1, \dots, a_{n-2})$ for $\lambda < 0$ and $a_i > 0$ $(i = 1, \dots, n-2)$, we see that M satisfies (M1) and (M2). To show that M satisfies (M3), choose $X \in \mathbb{R}^{(n)}$, such that X fully annihilates M. Then, as $MX = 0$, we have that the bottom $n-1$ rows of X must be zero. As X is symmetric, the last $n - 1$ columns must be zero. And due to $X \circ I = 0$, we also have that $x_{1,1} = 0$. □

We cite another fact from Van der Holst, Lovász and Schrijver, which is not hard to obtain and whose proof is relatively easy:

Theorem 4.5 ((2) on p.16 in [18]). *For $1 \leq p \leq q$, we have*

$$\mu(K_{p,q}) = \begin{cases} p, & \text{if } q \leq 2 \\ p+1, & \text{if } q \geq 3 \end{cases}$$

4.3 Basic Properties

We summarize basic properties of the Colin de Verdière number. To start with, the Colin de Verdière number is *minor-monotone*, i.e.

Theorem 4.6 (2.4 of [18]). *If G' is a minor of G, then $\mu(G') \leq \mu(G)$.*

Furthermore, for disconnected graphs, the Colin de Verdière number can be obtained by taking the maximum over the connected components. (Given that G is not empty. In this case, said maximum would be zero, as $\mu(K_1) = 2$, but we have $\mu(K_n) = 1$ for $n \geq 2$.)

Theorem 4.7 (2.5 of [18]). *If G has at least one edge, then*

$$\mu(G) = \max_K \{\mu(K)\}$$

where K runs over the connected components.

As already noted multiple times, the Colin de Verdière number has the very interesting property that it characterizes topological properties of graphs algebraically. To be precise:

Theorem 4.8 (1.4 of [18]). *For all graphs G,*

$\mu(G) \leq 1$ *if and only if G is a disjoint union of paths.*
$\mu(G) \leq 2$ *if and only if G is outerplanar.*
$\mu(G) \leq 3$ *if and only if G is planar.*
$\mu(G) \leq 4$ *if and only if G is linklessly embeddable.*

For $k > 4$, it is unknown, which exact classes are described by $\mu(G) \leq k$. Compare this theorem to the known forbidden minor characterizations of the disjoint union of paths, and that of outerplanar, planar and linklessly embeddable graphs:

G is a disjoint union of paths if and only if $\mathcal{M}K_{1,3} \not\subseteq G$ and $\mathcal{M}K_3 \not\subseteq G$.
G is outerplanar if and only if $\mathcal{M}K_{2,3} \not\subseteq G$ and $\mathcal{M}K_4 \not\subseteq G$.
G is planar if and only if $\mathcal{M}K_{3,3} \not\subseteq G$ and $\mathcal{M}K_5 \not\subseteq G$.
G is linklessly embeddable if and only if $\mathcal{M}H \not\subseteq G$ for all graphs $H \in P$.

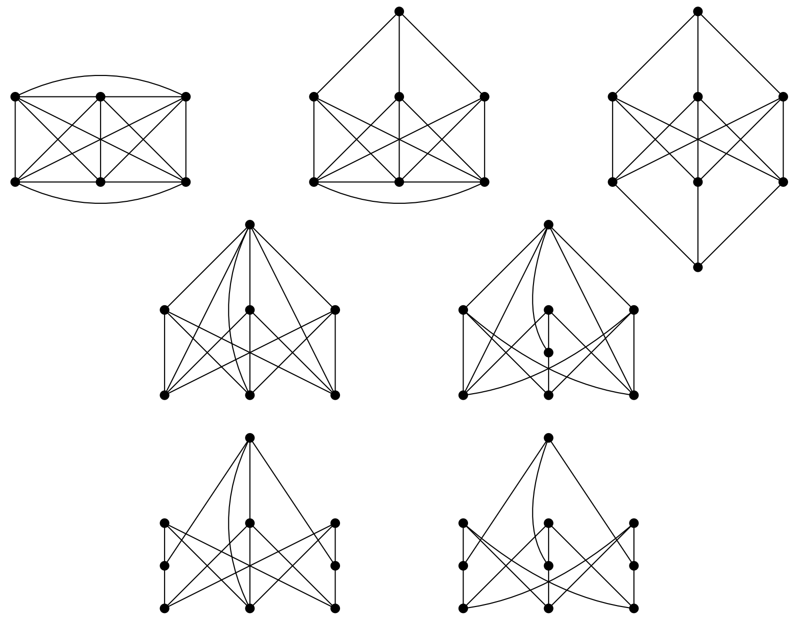


Figure 12: The Petersen family P.

Here, P denotes the Petersen family, which consists out of seven different graphs, including K_6 and the Petersen graph; see Figure 12.

When adding a vertex to a graph, its Colin de Verdière number can at most increase by one.

Theorem 4.9 (2.7 of [18]). *Let $G = (V, E)$ be a graph and let $v \in V$. Then*

$$\mu(G) \leq \mu(G - v) + 1.$$

If v is connected to all other vertices and $G - v$ has at least one edge, then equality holds.

We saw that the graphs $M_{k,d}$ and especially $M_{k,3}$ were connected to the stacked treewidth. We can use this theorem to figure out $\mu(M_{k,d})$.

Corollary 4.10. *For $k, d \in \mathbb{N}$, we have*

$$\mu(M_{k,d}) = \begin{cases} k, & \text{if } d \leq 2 \\ k+1, & \text{if } d \geq 3. \end{cases}$$

So especially $\mu(M_{k,3}) = k + 1$.

Proof. We have by Theorem 4.8 that the equation is true for $k = 1$, as $M_{1,2}$, $M_{1,1}$ are paths but not K_1 and as $M_{1,d}$ for $d \geq 3$ is outerplanar, but not a disjoint union of paths. Then note that $M_{k+1,d}$ is obtained from $M_{k,d}$ by adding a vertex to everything and the claim follows by induction with Theorem 4.9. □

Van der Holst, Lovász and Schrijver determined $\mu(G)$, if G is a clique-sum of two other graphs:

Theorem 4.11 (1 and 2 of [7])**.** *Let G be a clique sum of G_1 and G_2 and let $S := V(G_1) \cap V(G_2)$. Let $t := \max\{\mu(G_1), \mu(G_2)\}$. Then*

$$\mu(G) = \begin{cases} t, & \text{if } G \text{ does not contain } \mathcal{M}M_{t,3} \\ t+1, & \text{if } G \text{ contains } \mathcal{M}M_{t,3} \end{cases}$$

They also introduced the following parameter ν_H related to μ. Let $G = (V, E)$ be a graph and $H := \overline{G} = (V, F)$.

Definition 4.12 (Section 3 of [11])**.** *For a graph $H = (V, F)$, we denote by $\nu_H(H)$ the smallest dimension d such that there exists a vector labeling $(u_i : i \in V)$ in dimension d, satisfying*

(U1): $u_i^\intercal u_j \begin{cases} = 1, & \text{if } ij \in F, \\ < 1, & \text{if } ij \notin F. \end{cases}$

(U2): *if X is a symmetric $n \times n$ matrix such that $X_{ij} = 0$ for $ij \notin F$ and for $i = j$ and $\sum_j x_{ij} u_j = 0$ for every node i, then $X = 0$.*

Here, (U1) can be seen as a statement about hyperplanes and vectors being either on or below these hyperplanes, (U2) can be seen as a non-degeneracy condition, similar to the Strong Arnold Property. Van der Holst, Lovász and Schrijver later showed in the same article:

Theorem 4.13 (3.3 of [11])**.** *For every graph $G \neq \overline{K_2}$, we have*

$$\nu_H(\overline{G}) = n - \mu(G) - 1.$$

They use these and other geometrical ideas to show that results similar to Theorem 4.8 hold true for the complement graph of G, if $\mu(G)$ is close to $n - 1$. For details, we refer to [11]. It is important, not to confuse ν_H with another parameter ν_C related to the Colin de Verdière number, which was introduced by Colin de Verdière himself [20]. Both parameters are referred to in literature simply as "ν", and both are related to the Colin de Verdière number, but they are different.

5 Relation between stacked Treewidth and the Colin de Verdière number

After having defined the stacked treewidth and the Colin de Verdière number, having seen basic properties of these two and having introduced and proven basic methods to handle the stacked treewidth, we can make good use of our acquired abilities in this section.
This section contains our main results (apart from the lifting lemma and the equivalent characterizations of stacked treewidth, which can be seen as preparation).

5.1 General Similarities

In the first subsection, we want to give a reason why to consider the relation between μ and the stacked treewidth at all. We list several similarities here, which encourage and motivate to examine said relation more deeply. These similarities include the minor monotony, the behavior on the graph union, the behavior on clique sums and the equality $\mu(G) = \mathrm{stw}(G)$ for all chordal graphs G. We will see in the following subsections, however, that there are also many differences between the Colin de Verdière number and the stacked treewidth. But this shall not be of our concern for now, all we aim for in this subsection is to find as many similarities as possible.

We begin with the minor monotony. We saw in Theorem 4.6 that the Colin de Verdière number is indeed minor-monotone. It seems a natural question to ask, whether the same holds for the stacked treewidth, so we were surprised to not find a proof of this fact in literature.

First of all, it is a well-known fact that the treewidth itself is minor-monotone.

Theorem 5.1. *If G' is a minor of G, then* $\mathrm{tw}(G') \leq \mathrm{tw}(G)$.

Proof. If we delete an edge or a vertex from G, the theorem clearly holds. If we contract an edge uv into a new vertex w, let T be a tree decomposition of G. Replace all occurrences of u, v in the bags of T with w to obtain a new tree decomposition T'. Note that T' clearly is a tree decomposition of G' with smaller or equal width. □

Theorem 5.2. *If G' is a minor of G, then* $\mathrm{stw}(G') \leq \mathrm{stw}(G)$.

Proof. The idea is the same: If we delete an edge or a vertex, the theorem clearly holds. If $\mathrm{stw}(G) = \mathrm{tw}(G) + 1$, we are also done, as the treewidth is minor-monotone and

$$\mathrm{stw}(G') \leq \mathrm{tw}(G') + 1 \leq \mathrm{tw}(G) + 1 = \mathrm{stw}(G).$$

So the case that is left to show is the following: Let $\mathrm{stw}(G) = \mathrm{tw}(G) =: k$. If we contract an edge uv into a new vertex w, then $\mathrm{stw}(G') \leq k$.

For this, let T be a stacked, **smooth** decomposition of G, i.e., no band of the same kind appears twice in G. Now, do the following to transform T into another tree decomposition T': Replace, like in the proof of the previous theorem, each occurrence of

u or v in a bag by w. But remember where w came from. As a notation, write $w(u)$, $w(v)$ or $w(uv)$ if the bag where w was created contained only u, only v, or both u and v, respectively.

Before the replacement $\{u, v\} \to \{w\}$, the decomposition T had no duplicate bands and was smooth, but after the replacement, it is possible that two bands X_{ab}, $X_{ij}, ij \neq ab$ of T' may be equal and that T' is not smooth anymore. Duplicate bands of T' may have size k or $k-1$, but we are only interested in duplicate bands of T' which have size k.

So let $X_{ab} = X_{ij}, (ij \neq ab)$ be a problematic pair with $|X_{ab}| = |X_{ij}| = k$. As both bands have size k, before the transformation they did not contain both u and v. So they must be without loss of generality of the form $X_{ab} = S + w(u)$ and $X_{ij} = S + w(v)$ for some $(k-1)$-set S. Note that then there does not exist a third band X_3 that is equal to X_{ij}, as X_3 would then also have to be of the form $S + w$, but T was smooth. As T' is a valid tree decomposition and as X_{ij} and X_{ab} both contain S, X_{ij} and X_{ab} are adjacent bands. So, without loss of generality, $i = b$.

In other words, X_{ab} and X_{bj} are incident to a single bag X_b. But as $w(u) \in X_{ab}$ and $w(v) \in X_{bj}$, we see that X_b had contained both u, v before the transformation, so $|X_b| = k$.

Finally, consider $H := \text{fill}(T')$. We claim that H contains no k-clique of degree 3 or higher. In fact, if C is a k-clique in H, we have seen that if the band of T' belonging to C appears as a duplicate, one of the three bags in $\Phi(C)$ has only size k. So $\deg_G(C) \leq 2$. Thus, we have that G' has a chordal completion H such that H has no k-clique of degree 3 or higher and $\omega(H) \leq k+1$. This is case (v) of Lemma 3.26 and we are done. □

The next similarity that we want to discuss concerns the behavior of the Colin de Verdière number and the stacked treewidth when a graph G has multiple connected components. Then both these parameters are obtained by taking the maximum over the connected components.

Some of the components may be independent vertices, so we quickly want to discuss the meaning of $\text{stw}(G) = 0$ again, like we promised in Section 3.2. We saw three different characterizations of the stacked treewidth in Theorem 3.27. The third characterization used chordal graphs without k-cliques of degree 3 or higher. Now observe that

$$\begin{aligned} & C \text{ is a } k\text{-clique of degree } d \\ \Leftrightarrow\ & \exists \text{ exactly } d \text{ different } v_1, \ldots, v_d \text{ such that } \forall c \in C : v_i c \in E \ (i = 1, \ldots, d). \end{aligned}$$

If we set $k = 0$ here, the 0-cliques of degree 2 or lower are exactly $\{K_1, \overline{K_2}\}$.

Another argument can be made concerning stacked decompositions. If T is a stacked decomposition of width 0, all bands are empty, and we have that the empty band $\emptyset$ is a high-degree band, if $|\Phi(\emptyset)| = |V(T)| \geq 3$. This, too, categorizes the stacked 0-trees as $\{K_1, \overline{K_2}\}$.

The last argument we make, is that if T' is a tree decomposition derived from T by putting a new vertex into every bag of T, it should hold that T' is stacked if and only if

T is stacked. This would not be the case, if we considered $\overline{K_3}$ a stacked 0-tree.

After having seen some arguments, why it is very reasonable to define that

$$\mathrm{stw}(G) = 0 \text{ if and only if } G \in \{K_1, \overline{K_2}\},$$

we show that, analogue to the Colin de Verdière number the stacked treewidth of a (nonempty) graph is obtained by taking the maximum over the connected components. The argument is very simple.

Theorem 5.3. *If G has at least one edge, then*

$$\mathrm{stw}(G) = \max_K \{\mathrm{stw}(K)\}$$

where K runs over the connected components.

Proof. "$\geq$": By minor monotony (Theorem 5.2).
"$\leq$": Let $G_1, \dots, G_m$ be the components of G. One intuitive way to do this, would be to show that if G_i is covered by a stacked k_1-tree and G_j is covered by a stacked k_2-tree, we can connect the trees. But there exists a more elegant proof:
Define $T := \max\{\mathrm{stw}(G_i) : i \in \{1, \dots, m\}\}$. Due to Theorem 3.27 we find for every $i \in \{1, \dots, m\}$ a stacked chordal completion G_i' of G_i with $\omega(G_i') \leq \mathrm{stw}(G_i) + 1$. As $T > 0$, the union of $G_1, \dots, G_m$ does not increase the degree of any T-clique and thus is stacked and we have that $\mathrm{stw}(G) \leq T$. □

Next, we want to consider what happens when we add a vertex. We recall Theorem 4.9:

Theorem 4.9 (2.7 of [18])**.** *Let $G = (V, E)$ be a graph and let $v \in V$. Then*

$$\mu(G) \leq \mu(G - v) + 1.$$

If v is connected to all other vertices and $G - v$ has at least one edge, then equality holds.

The stacked treewidth behaves exactly the same, but we don't need to make an exception for the case that $G - v$ is empty.

Theorem 5.4. *Let $G = (V, E)$ be a graph and let $v \in V$. Then*

$$\mathrm{stw}(G) \leq \mathrm{stw}(G - v) + 1.$$

If v is connected to all other vertices, equality holds.

Proof. Let T be any stacked tree decomposition with width k of $G - v$ for some $k \in \mathbb{N}_0$. Insert v into every bag. Then we have a stacked tree decomposition of G with width $k + 1$. So therefore we conclude $\mathrm{stw}(G) \leq \mathrm{stw}(G - v) + 1$.

If additionally, v is connected to every vertex, let T' be a stacked, compact tree decomposition of G. Note that v must be in every bag of T', else we find $u \in V$

such that $\Phi(u) \cap \Phi(v) = \emptyset$ by the properties of a compact decomposition. So we can delete v from every bag and obtain a stacked decomposition of $G - v$, so we have $\mathrm{stw}(G - v) \leq \mathrm{stw}(G) - 1$. (By Theorem 3.27, $\mathrm{stw}(H)$ is the smallest integer such that H has a stacked decomposition.) □

Note that, in contrast to Theorem 4.9, we do not have to exclude any graphs in this theorem.

The next graph operation, where the Colin de Verdière number and the stacked treewidth behave identical, is the clique sum of two graphs. Recall Theorem 4.11:

Theorem 4.11 (1 and 2 of [7]). *Let G be a clique sum of G_1 and G_2 and let $S := V(G_1) \cap V(G_2)$. Let $t := \max\{\mu(G_1), \mu(G_2)\}$. Then*

$$\mu(G) = \begin{cases} t, & \text{if } G \text{ does not contain } \mathcal{M}M_{t,3} \\ t+1, & \text{if } G \text{ contains } \mathcal{M}M_{t,3} \end{cases}$$

We prove the exact same formula for the stacked treewidth, but first let us consider how the normal treewidth behaves on clique sums. The answer to this is a well-known fact.

Theorem 5.5. *Let G be a clique sum of G_1 and G_2 and let $S := V(G_1) \cap V(G_2)$. Then we have* $\mathrm{tw}(G) = \max\{\mathrm{tw}(G_1), \mathrm{tw}(G_2)\}$.

Proof. Let T_1, T_2 be tree decompositions of G_1, G_2 respectively. We find a bag $X_1 \supseteq S$ in T_1 and a bag $X_2 \supseteq S$ in T_2, as S is a clique in G_1 and G_2. We can connect X_1 and X_2 with a band to get a tree decomposition of G. So we have $\mathrm{tw}(G) \leq \max\{\mathrm{tw}(G_1), \mathrm{tw}(G_2)\}$. The other inequality $\mathrm{tw}(G) \geq \max\{\mathrm{tw}(G_1), \mathrm{tw}(G_2)\}$ is clear by minor monotony of $\mathrm{tw}(\cdot)$ (Theorem 5.1). □

Theorem 5.6. *Let G be a clique sum of G_1 and G_2 and let $S := V(G_1) \cap V(G_2)$. Let $t := \max\{\mathrm{stw}(G_1), \mathrm{stw}(G_2)\}$. Then*

$$\mathrm{stw}(G) = \begin{cases} t, & \text{if } G \text{ does not contain } \mathcal{M}M_{t,3} \\ t+1, & \text{if } G \text{ contains } \mathcal{M}M_{t,3} \end{cases}$$

Proof. By minor monotony of $\mathrm{stw}(\cdot)$ (Theorem 5.2), we clearly have $\mathrm{stw}(G) \geq t$.
With the minor monotony of $\mathrm{tw}(\cdot)$ (Theorem 5.1), the previous theorem and the fact that for all graphs G, the inequalities $\mathrm{tw}(G) \leq \mathrm{stw}(G) \leq tw(G) + 1$ hold, we also have

$$\begin{aligned} \mathrm{stw}(G) &\leq \mathrm{tw}(G) + 1 = \max\{\mathrm{tw}(G_1), \mathrm{tw}(G_2)\} + 1 \\ &\leq \max\{\mathrm{stw}(G_1), \mathrm{stw}(G_2)\} + 1 = t + 1. \end{aligned}$$

So we for sure know that $t \leq \mathrm{stw}(G) \leq t + 1$. Now we do a case distinction.
Case 1: G has $M_{t,3}$ as a minor:
Then, we observe that $\mathrm{stw}(M_{t,3}) = t + 1$ and by minor monotony (Theorem 5.2), we have $\mathrm{stw}(G) \geq t + 1$.

Case 2: G does not have $M_{t,3}$ as a minor:
In this case we show that $\mathrm{stw}(G) \leq t$. Let $\mathrm{stw}(G_1) =: t_1$, $\mathrm{stw}(G_2) =: t_2$ and without loss of generality, let $t_1 \geq t_2$. So then we have $t_1 = t$. Note that clearly $|S| \leq t+1$, because S is a clique in G_1. We do a further case distinction by the size of $|S|$:
Case 2a.) $|S| < t$:
Let for $(i = 1, 2)$, T_i be a **smooth**, stacked tree decomposition with width t_i of G_i (meaning T_i contains no duplicate bands). As S is a clique of G, we find a bag $X_i \supseteq S$ in T_i $(i = 1, 2)$. Connect X_1 and X_2 with a band containing only S. Then we're done by Lemma 3.26, case (iv).
Case 2b.) $|S| = t$.
Let for $(i = 1, 2)$, T_i be a stacked, compact tree decomposition with width t_i of G_i. If $t_2 < t_1 = t$, keep adding a new vertex into every bag of T_2, until $\mathrm{width}(T_2) = t$. Then we find in each of T_1, T_2, that S is either a band or a pendant k-set. If S is a pendant k-set in both graphs, we can connect these two pendant k-sets with a new band and we are done. Else, we can connect the two and by doing so, we form a band X of degree 3 or 4. But note that $X = S$. So $G[X]$ is a clique. And so, because G has no $M_{t,3}$-minor and $G[X]$ is a clique, we can apply the lifting lemma to X once or twice. (See Observation 3.22)
Case 2c.) $|S| = t + 1$.
Let for $(i = 1, 2)$, T_i be a stacked, compact tree decomposition with width t_i of G_i. We see that $t_1 = t_2 = t$, as $C \subseteq G_1, G_2$. So we find a bag $X_i = C$ in G_i for $i = 1, 2$. Then we can join T_1 and T_2 by laying X_1 over X_2. Note that every high-degree band X_e which is created this way is incident to X_i and so $G[X_e]$ is a clique. Then, like in case 2*b*.), we are done. □

The last, and maybe the nicest similarities between the Colin de Verdière number and the stacked treewidth that we want to show in this chapter, are the two relations $\mu(G) \leq \mathrm{stw}(G)$ and $[G \text{ chordal} \Rightarrow \mathrm{stw}(G) = \mu(G)]$. These relations were already known (though not in this form using the notation of $\mathrm{stw}(\cdot)$) and proven for all chordal graphs by Fallat and Mitchell in 2013 [4]. We can reprove these facts using the behavior of μ on clique-sums.

Let G be a chordal graph. We have seen in Theorem 3.25 three equivalent characterizations for whether G is a stacked chordal graph. We now add even more equivalent characterizations.

Theorem 5.7 (Stackedness of Chordal Graphs, Continuation of Theorem 3.25)**.** *Let G be a chordal graph and $k := \omega(G) - 1 (= \mathrm{tw}(G))$. The following are equivalent:*

(i) $\mathrm{stw}(G) = k$, i.e. G is covered by a stacked k-tree.

(ii) G does not contain $M_{k,3}$ as minor.

(iii) G does not contain $M_{k,3}$ as topological minor.

(iv) G does not contain $M_{k,3}$ as subgraph.

(v) G has a stacked, compact tree decomposition.

(vi) $\mu(G) = k$. (With the exception of $G = \overline{K_2}$.)

Proof. We have seen "(i) ⇔ (iv) ⇔ (v)" already.

"(i) ⇒ (ii):" By minor monotony of stw(·) (Theorem 4.6).

The chain (ii) ⇒ (iii) ⇒ (iv) is obvious and we have seen (iv) ⇒ (i) already.

The equivalence **"(i) ⇔ (vi)"** has a really elegant proof: If G has no edge, we see that $\mu(\overline{K_n}) = \mathrm{stw}(\overline{K_n})$ for $n \neq 2$. If on the other hand, G has an edge, each component of G can be obtained by starting with a clique and taking repeated clique sums with other cliques, as G has a perfect elimination ordering. But we have for all $n \in \mathbb{N}$ that $\mu(K_n) = \mathrm{stw}(K_n) = n - 1$ and furthermore, the Colin de Verdière number and the stacked treewidth behave identically on clique sums and graph unions (Theorems 5.6 and 5.3). Thus, we have $\mathrm{stw}(G) = \mu(G)$. □

As k-trees are chordal, this immediately yields

Corollary 5.8. *Let $T \neq \overline{K_2}$ be a k-tree. Then T is stacked if and only if $\mu(T) = k$.*

Corollary 5.9. *For all graphs $G \neq \overline{K_2}$, $\mu(G) \leq stw(G) \leq \mathrm{tw}(G) + 1$.*

Proof. Having $\mathrm{stw}(G) = k$ means being covered by a stacked k-tree T, which we saw has $\mu(T) = k$. So, by minor monotony of μ (Theorem 4.6), we have $\mu(G) \leq stw(G)$. We also saw that $\mathrm{stw}(G) \leq \mathrm{tw}(G) + 1$ holds in Theorem 3.21. □

Corollary 5.10. *If $G \neq \overline{K_2}$ is a chordal graph,* $\mathrm{stw}(G) = \mu(G)$.

And finally:

Corollary 5.11 (Many Criteria for Stackedness of k-Trees)**.** *Let T be a k-tree. The following are equivalent:*

(i) All constructions of T suffice the criteria of Definition 3.12.

(ii) There exists a construction of T which suffices the criteria of Definition 3.12.

(iii) The bands in the unique compact tree decomposition of T have degree at most two.

(iv) $\mu(G) = k$. (With the exception $T = \overline{K_2}$)

(v) T does not have $M_{k,3}$ as minor.

(vi) T does not have $M_{k,3}$ as topological minor.

(vii) T does not have $M_{k,3}$ as a subgraph (= all k-cliques have degree at most two).

(viii) T is $(k, 3)$-cut-free.

A note regarding the notation in (viii): Let G be connected. We call a separating set $S \subseteq V(G)$ an *(a, b)-cut*, if S is an inclusion-minimal separating set, $|S| = k$ and $G - S$ has at least b components.

Proof. We have seen (i), (ii), (iii), (vii) in Theorem 3.13. We have seen (iv), (v), (vi), (vii) in Theorem 5.7. The easy proof of (viii) is left to the reader. □

5.2 A Conjecture about stw and μ

We have seen that for all chordal graphs G, the equality $\mu(G) = \mathrm{stw}(G)$ holds, which is a very nice relation. One goal of this thesis was to extend our current understanding of this relation onto non-chordal graph. For chordal graphs, we always have that $\mu(G) \geq \mathrm{tw}(G)$, and of course $\mu(G) \leq \mathrm{stw}(G) \leq \mathrm{tw}(G) + 1$. So the Colin de Verdière number on chordal graphs can only take two numbers.

The situation is different when we consider general graphs G. Here it can happen that $\mu(G) \ll \mathrm{tw}(G)$. For example, consider the $(n \times n)$-grid $G_{n,n}$ for $n > 3$. It is a well-known fact that $\mathrm{tw}(G_{n,n}) = n$. It is also easy to show that one can find a stacked, (compact or smooth) tree decomposition of $G_{n,n}$, so we have $\mathrm{stw}(G_{n,n}) = n$. As $G_{n,n}$ is planar, however, we have $\mu(G_{n,n}) = 3 \ll n$.

We observe that if the Colin de Verdière number is quite high, i.e. $\mu(G) > \mathrm{tw}(G)$, we already have $\mu(G) = \mathrm{stw}(G) = \mathrm{tw}(G) + 1$, because we always have $\mu(G) \leq \mathrm{stw}(G)$. A reasonable thought now might be, that if $\mu(G)$ is in a way "low", then G has low geometric complexity, as the Colin de Verdière number seems closely connected to geometric topics (e.g. planarity, linkless embeddings). So then maybe the stacked treewidth is also low in the sense $\mathrm{tw}(G) = \mathrm{stw}(G)$, as the stacked treewidth, too, seems to measure some kind of geometric complexity.

We capture this thought in the following question. The question in this form was first proposed by Knauer and Ueckerdt [10, 9].

Question 5.12. *Is it true for all graphs G, that*

$$[\mu(G) \leq \mathrm{tw}(G)] \Rightarrow [\mathrm{stw}(G) = \mathrm{tw}(G)] \; ? \qquad (\triangle)$$

As already noted in the introduction, $(\triangle)$ is not true for all graphs. Therefore we choose to present Question 5.12 in this particular form here, so we can examine the conjectured relation for different pairs of (μ, k). We will say that $(\triangle)$ holds for a graph G if the implication $[\mu(G) \leq \mathrm{tw}(G)] \Rightarrow [\mathrm{stw}(G) = \mathrm{tw}(G)]$ is true and that $(\triangle)$ holds for (μ, k), if $(\triangle)$ holds for all graphs G with $\mu(G) = \mu$ and $\mathrm{tw}(G) = k$.

Note that if $\mu = k$ here, we find the second question from our motivational Section 1. As an example, fix $\mu = k = 3$. Then Question 5.12 asks, whether every planar graph with treewidth 3 is also contained in a stacked 3-tree. As we have seen in Corollary 5.11, this is equivalent to the statement, that every planar graph with treewidth 3 is also contained in a planar 3-tree. This question and related ones were for example considered in [12].

The first thing we observe now is that (△) does hold for all pairs (μ, k) with $\mu \leq 3$, i.e. for all planar G. As a consequence, (△) also holds if $k \leq 3$, as in the case $\mu > k$, the implication (△) is trivially correct.

Theorem 5.13. *Let G be a planar graph. Then (△) holds for G.*

Proof. So let G be a graph with $\mu(G) = \mu \in \mathbb{N}_0$ and $\operatorname{tw}(G) = k \in \mathbb{N}_0$. By the forbidden minor characterization of planarity, outerplanarity, disjoint union of paths and the family $\{G : \mu(G) = 0\} = \{K_1\}$, we see that G does not contain a $\mathcal{M}K_{\mu,3}$ (check for $\mu = 0, \dots, 3$). So, as $\mu \leq k$, G does not contain a $\mathcal{M}K_{k,3}$ and we can always apply the lifting lemma (see Observation 3.22). □

Corollary 5.14. *Let G be a graph with* $\operatorname{tw}(G) \leq 3$. *Then (△) holds for G.*

We introduce the following notation.

Definition 5.15. *Let P be a logical statement. We define*

$$1\{P\} := \begin{cases} 1, & \text{if } P \text{ is true} \\ 0, & \text{if } P \text{ is false.} \end{cases}$$

Observation 5.16. *Let $G \neq \overline{K_2}$. Then (△) can be rewritten the following way:*

$$\operatorname{stw}(G) = \operatorname{tw}(G) + 1\{\mu(G) > \operatorname{tw}(G)\}$$

Proof. We know that $\operatorname{tw}(G) \leq \operatorname{stw}(G) \leq \operatorname{stw}(G) + 1$. We have seen in Corollary 5.9 that $\operatorname{stw}(G) \geq \mu(G)$ for $G \neq \overline{K_2}$. So (△) is equivalent to the statement that there are exactly two classes of graphs: Those with $\mu(G) \leq \operatorname{tw}(G)$ and $\operatorname{stw}(G) = \operatorname{tw}(G)$ and those with $\mu(G) = \operatorname{stw}(G) = \operatorname{tw}(G) + 1$. □

So we have that if (△) holds for a class $\mathcal{C}$ of graphs, then $\mathcal{C}$ can be partitioned into the class $\mathcal{C}_1$, where all graphs have "low" geometric complexity, i.e. $\mu(G) \leq \operatorname{tw}(G)$ and $\operatorname{stw}(G) = \operatorname{tw}(G)$, and the class $\mathcal{C}_2$ where all graphs have "high" geometric complexity, in the sense $\mu(G) = \operatorname{stw}(G) = \operatorname{tw}(G) + 1$.

This is indeed a very nice relation. But as we already noted, (△) does not hold for all graphs and it is our next goal to show that.

5.3 Testing the Conjecture

In order to find a counterexample to (△), we consider the join $G = G_1 \vee G_2$ of graphs. Recall that $G_1 \vee G_2$ is defined as taking a copy of G_1, a copy of G_2, and adding all possible edges between the two. There is a nice relation for the treewidth and stacked treewidth of a join. For treewidth, this result was already known [21].

Theorem 5.17. *Let G_1, G_2 be graphs on n_1, n_2 vertices. For the join $G := G_1 \vee G_2$, we have*

$$\operatorname{tw}(G) = \min\{\operatorname{tw}(G_1) + n_2, \operatorname{tw}(G_2) + n_1\}$$
$$\operatorname{stw}(G) = \min\{\operatorname{stw}(G_1) + n_2, \operatorname{stw}(G_2) + n_1\}$$

Proof. "$\leq$" is easy: We can start with a (stacked) compact tree decomposition of G_1 and add V_2 into every bag or the other way around.
"$\geq$": Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. Let D be a (stacked) smooth decomposition of G and $T := \operatorname{fill}(D)$. Consider a construction of T and let v_n be the vertex of T which is stacked last. We have $v_n \in V_i$ for some $i \in \{1, 2\}$. Let $\{j\} := \{1, 2\} \setminus \{i\}$. As v_n is stacked last and $V_j \subseteq N(v_n)$, v_n was stacked on a clique containing V_j. So we find a bag X containing V_j.

If we now consider X as a root and consider the bijection between $\operatorname{bags}(D) - \{X\}$ and $V(G) \setminus X$ (see Theorem 3.5), we see that all the other bags are in bijection to vertices in V_i, as V_j is in the root already. Then all other vertices need to be stacked onto V_j. So therefore, V_j is in every bag.
This means we get a (stacked) smooth decomposition of G_i by deleting V_j from every bag, so

$$(s)tw(G_i) \leq (s)tw(G) - n_j$$

and thus

$$\min\{(s)tw(G_1) + n_2, (s)tw(G_2) + n_1\} \leq (s)tw(G).$$

□

Corollary 5.18. *We have for $p, q \in \mathbb{N}_0$, $p \leq q$:*

$$\operatorname{stw}(K_{p,q}) = \begin{cases} p, & \text{if } q \leq 2 \\ p+1, & \text{if } q \geq 3. \end{cases}$$

Proof. If $p = 0$, or $q = 0$, the claim is correct. In the case $q \leq 2$, we have by Theorem 5.17:

$$\operatorname{stw}(K_{p,q}) = \operatorname{stw}(\overline{K_p} \vee \overline{K_q}) = \min\{0 + q, 0 + p\} = p.$$

In the case of $q \geq 3$, we have

$$\operatorname{stw}(K_{p,q}) = \operatorname{stw}(\overline{K_p} \vee \overline{K_q}) = \min\{\operatorname{stw}(\overline{K_p}) + q, 1 + p\} = p + 1.$$

□

So both the treewidth and the stacked treewidth behave very nicely on joins. What about the Colin de Verdière number? We give two inequalities. The first one is rather general.

Observation 5.19 (4.5.1 of [6, p. 44])**.** *Let G_1, G_2 be nonempty graphs on n_1, n_2 vertices. We have*

$$\mu(G_1 \vee G_2) \leq \min\{\mu(G_1) + n_2, \mu(G_2) + n_1\}.$$

Proof. For $(i = 1, 2)$ and $\{j\} := \{1, 2\} \setminus \{i\}$, start with G_i and then add a vertex connected to every other existing vertex n_j times. The obtained graph H is a supergraph of $G_1 \vee G_2$ and by Theorem 4.9 and minor monotony (Theorem 4.6), we have

$$\mu(G) \leq \mu(H) = \mu(G_i) + n_j$$

and we are done. □

In contrast to the stacked treewidth, equality is not always the case here (we see an example for that in Theorem 5.21). The second inequality is given by a nice idea mentioned by Goldberg in his research thesis.

Observation 5.20 (4.5.7 of [6, p. 45])**.** *Let $m \geq 3$, and let $G \neq K_n$ be a graph on n vertices. Then*

$$\mu(C_m \vee G) \leq n + 1.$$

Proof. As $G \neq K_n$, we see that $H := C_m \vee \overline{K_2} \subseteq C_m \vee G$. As H is planar, but not outerplanar, we have $\mu(H) = 3$. To the graph H, we can add a vertex connected to everything $(n-2)$ times to obtain $C_m \vee G$. Theorem 4.9 yields $\mu(C_m \vee G) \leq 3 + n - 2 = n + 1$. □

We can use this idea to show:

Theorem 5.21. *For all $k \geq 4$, there exists a graph G with $\operatorname{tw}(G) = k$, for which $(\triangle)$ does not hold.*

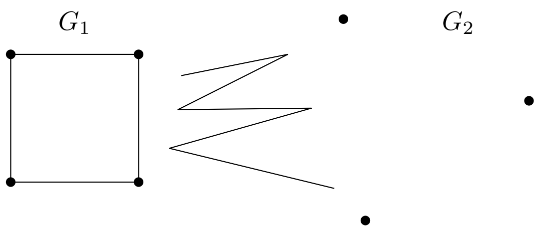
Proof. Let be $r \in \mathbb{N}_0$. We show that a counterexample G with $\mu := \mu(G) \leq 4 + r$ and $\operatorname{tw}(G) = 4 + r$ but $\operatorname{stw}(G) = 5 + r = \operatorname{tw}(G) + 1$ exists.
Indeed, let $G_1 := C_4$, $G_2 := M_{r,3}$, then said counterexample is given by $G = G_1 \vee G_2$. The situation is depicted in Figure 13.
We have $n_1 := |V(G_1)| = 4$ and one easily sees that $\operatorname{tw}(G_1) = \operatorname{stw}(G_1) = 2$.
We have $n_2 := |V(G_2)| = r + 3$ and we know that $\operatorname{tw}(G_2) = r$ and $\operatorname{stw}(G_2) = r + 1$.
By Observation 5.20, as G_2 is not complete, we have $\mu(G) \leq 3 + (n_2 - 2) = r + 4$.
Using Theorem 5.17, we have

$$\operatorname{tw}(G) = \min\{\operatorname{tw}(G_1) + n_2, \operatorname{tw}(G_2) + n_1\} = \min\{r + 5, r + 4\} = r + 4$$

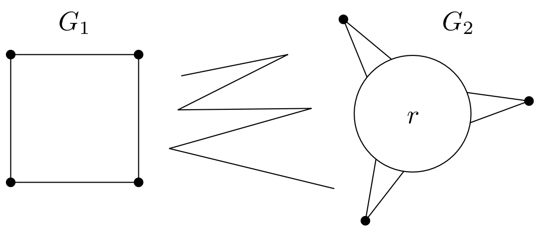
and

$$\operatorname{stw}(G) = \min\{\operatorname{stw}(G_1) + n_2, \operatorname{stw}(G_2) + n_1\} = \min\{r + 5, r + 5\} = r + 5$$

. □



	G_1	G_2	G
n	4	3	7
tw	2	0	4
stw	2	1	5
μ	2	1	≤ 4



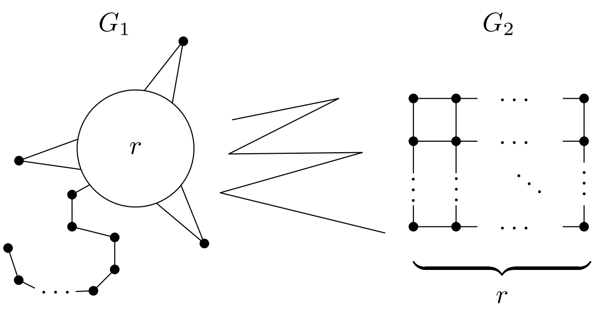
	G_1	G_2	G
n	4	$r+3$	$r+7$
tw	2	r	$r+4$
stw	2	$r+1$	$r+5$
μ	2	$r+1$	$\leq r+4$

Figure 13: A counterexample to $(\triangle)$ for $k \geq 4$.

For the case $r = 4$, we also have another proof that $\text{stw}(C_4 \vee \overline{K_3}) = 5$. Recall that we examined the graph $G' = P_3 \vee \overline{K_3}$ in Example 3.10. We saw that G' has exactly two different compact decompositions T_1, T_2 of width 4. Now, observe that after adding the edge $\{4, 5\}$ to G', the second, stacked decomposition T_2 is not a decomposition anymore, but T_1 still is. So $C_4 \vee \overline{K_3}$ has only one compact tree decomposition of width 4, and this compact tree decomposition is not stacked.

We had the idea that for all graphs G, low $\mu(G)$ compared to $\text{tw}(G)$ implied low geometric complexity, which implied low $\text{stw}(G)$. But the next counterexample lets us doubt this idea.

Theorem 5.22. *There are graphs G, such that $(\triangle)$ does not hold for G and $\text{tw}(G) - \mu(G)$ is arbitrarily large.*



	G_1	G_2	G
n	r^2+1	r^2	$2r^2+1$
tw	r	r	r^2+r
stw	$r+1$	r	r^2+r+1
μ	$r+1$	3	$\leq r^2+4$

Figure 14: A counterexample to $(\triangle)$ such that $\text{tw}(G) - \mu(G)$ arbitrarily large

Proof. The situation is depicted in Theorem 5.22.
Let be $r \geq 4$ and let G_2 be the $(r \times r)$-grid. We have $n_2 := |G_2| = r^2$ and $\text{tw}(G_2) = \text{stw}(G_2) = r$ and $\mu(G_2) = 3$.
Construct G_1 the following way: Start with $H := M_{r,3}$, which has $r + 3$ vertices and glue the first vertex of a single path to one vertex of H such that the resulting graph G_2 has $r^2 + 1 =: n_1$ vertices.
This operation can be seen as clique sum and therefore we have $\mu(G_2) = \mu(M_{r,3}) = r+1$, $\text{stw}(G_2) = \text{stw}(M_{r,3}) = r + 1$ and $\text{tw}(G_2) = \text{tw}(M_{r,3}) = r$.
By Theorem 5.17 we have

$$\text{tw}(G) = \min\{r + r^2, r + (r^2 + 1)\} = r^2 + r$$

$$\text{stw}(G) = \min\{(r + 1) + r^2, r + (r^2 + 1)\} = r^2 + r + 1$$

but by Observation 5.19, we have

$$\mu(G) \leq \min\{\mu(G_1) + n_2, \mu(G_2) + n_1\} = \min\{(r + 1) + r^2, 3 + (r^2 + 1)\} = r^2 + 4.$$

Therefore, because $r \geq 4$, we have that $\mu(G) \leq \text{tw}(G)$ although $\text{stw}(G) = \text{tw}(G) + 1$. Furthermore, $\text{tw}(G) - \mu(G) = r - 4$, can get arbitrarily large if r goes to infinity. □

So it seems safe to say that (△) is false in the general case. But it may still be too soon to dismiss Question 5.12 completely. After all, we have seen some in Section 5.1, that in some cases, the Colin de Verdière number and the stacked treewidth behave strikingly similar. So the next natural step is to examine, whether one can find large classes of graphs beside the chordal and planar graphs for which (△) holds. The last two, easy proofs of this thesis may lay the foundation for such an examination.

Theorem 5.23. *If (△) holds for G_1 and G_2, then (△) holds for the disjoint graph union $G_1 \cup G_2$.*

Proof. Let $G := G_1 \cup G_2$. If G has no edge, we are done, as (△) holds for all $\overline{K_n}$ $(n \in \mathbb{N})$. Otherwise, let for $(i = 1, 2)$, $\text{tw}_i := \text{tw}(G_i)$ and $\text{tw} := \text{tw}(G)$ and define $\text{stw}_i, \text{stw}, \mu_i$ and μ analogously. We have by Theorem 5.3 and Theorem 4.7 that

$$\text{stw} = \max\{\text{stw}_1, \text{stw}_2\} \tag{5.1}$$

$$\mu = \max\{\mu_1, \mu_2\} \tag{5.2}$$

and it is clearly

$$\text{tw} = \max\{\text{tw}_1, \text{tw}_2\}. \tag{5.3}$$

Now let without loss of generality $\text{tw}_1 \geq \text{tw}_2$. So we have $\text{tw} = \text{tw}_1$.
We claim (△) holds for G. Thus, let $\mu \leq \text{tw}$, we show that $\text{stw} = \text{tw}$. As $\mu \leq \text{tw}$, we have, by Equations 5.2 and 5.3, that

$$\max\{\mu_1, \mu_2\} \leq \max\{\text{tw}_1, \text{tw}_2\} = \text{tw}_1 \tag{5.4}$$

We now make a case distinction by

Case 1: $\mathrm{tw}_1 > \mathrm{tw}_2$.
We know that $\mu_1 \leq \mathrm{tw}_1$ by Equation (5.4). So $\mathrm{stw}_1 = \mathrm{tw}_1$, as ($\triangle$) holds for G_1.
Now consider Equation (5.1), which simplifies to

$$\mathrm{stw} = \max\{\mathrm{tw}_1, \mathrm{stw}_2\}$$

. But stw_2 can at most be 1 more than tw_2 and we have $\mathrm{tw}_2 < \mathrm{tw}_1$. We conclude, that $\mathrm{stw} = \mathrm{tw}_1 = \mathrm{tw}$.

Case 2: $\mathrm{tw}_1 = \mathrm{tw}_2$.
Then, by Equation (5.4), we already have that $\mu_1 \leq \mathrm{tw}_1$ and that $\mu_2 \leq \mathrm{tw}_2$. So we can use ($\triangle$) for G_1, G_2 to see that the equalities $\mathrm{stw}_1 = \mathrm{tw}_1$ and $\mathrm{stw}_2 = \mathrm{tw}_2$ hold. Then, by Equations 5.1 and 5.3, we see that $\mathrm{tw} = \mathrm{stw}$. □

And, finally, we conclude our thesis with

Theorem 5.24. *If ($\triangle$) holds for G_1 and G_2 and G is a clique sum of G_1 and G_2, then ($\triangle$) holds for G.*

Proof. Note that the theorem is clearly true, if $G_1 = \overline{K_2}$ or $G_2 = \overline{K_2}$. (For example by Theorem 5.23). So let without loss of generality $G_1, G_2 \neq \overline{K_2}$.
Define, like in the previous proof, for $(i = 1, 2)$, $\mathrm{tw}_i := \mathrm{tw}(G_i)$ and $\mathrm{tw} := \mathrm{tw}(G)$ and define $\mathrm{stw}_i, \mathrm{stw}, \mu_i$ and μ analogously.
Let $t := \max\{\mathrm{stw}_1, \mathrm{stw}_2\}$. We do a case distinction.

Case 1: $\mathcal{M}M_{t,3} \subseteq G$.
Then $\mu(G) \geq t + 1$ by minor monotony of μ. On the other hand,

$$\mathrm{tw}(G) = \max\{\mathrm{tw}(G_1), \mathrm{tw}(G_2)\} \leq \max\{\mathrm{stw}(G_1), \mathrm{stw}(G_2)\} = t.$$

So $\mu(G) > \mathrm{tw}(G)$ and ($\triangle$) is trivially correct.

Case 2: $\mathcal{M}M_{t,3} \nsubseteq G$. Let without loss of generality $\mathrm{tw}_1 \geq \mathrm{tw}_2$, so $\mathrm{tw} = \mathrm{tw}_1$.
By Theorem 5.5,

$$\mathrm{tw} = \max\{\mathrm{tw}_1, \mathrm{tw}_2\} = \mathrm{tw}_1\,.$$

Furthermore we see, using Theorem 5.6 about the stacked treewidth of clique sums, and using the fact that G is $M_{t,3}$-minor-free, that

$$\mathrm{stw} = t = \max\{\mathrm{stw}_1, \mathrm{stw}_2\}. \tag{5.5}$$

As $G_1, G_2 \neq \overline{K_2}$, and ($\triangle$) holds for G_1, G_2, we can use the alternative version of ($\triangle$), which we presented in Observation 5.16. Then Equation (5.5) simplifies to

$$\begin{aligned} \operatorname{stw} &= \max\{\operatorname{stw}_1,\ \operatorname{stw}_2\} \\ &= \max\{\operatorname{tw}_1 + 1\{\mu_1 > \operatorname{tw}_1\},\ \ \operatorname{tw}_2 + 1\{\mu_2 > \operatorname{tw}_2\}\} \\ &= \operatorname{tw} + 1\{\mu_1 > \operatorname{tw}_1 \vee (\operatorname{tw}_1 = \operatorname{tw}_2 \wedge \mu_2 > \operatorname{tw}_2)\}, \end{aligned} \tag{5.6}$$

where we used the fact $\operatorname{tw}_1 \geq \operatorname{tw}_2$ in the last line.

Now we proceed to show that ($\triangle$) holds for G. So assume that $\mu \leq \operatorname{tw}$. But then, by minor monotony of μ, neither of the two statements

$$\begin{aligned} \mu_1 &> \operatorname{tw}_1 \\ \operatorname{tw}_1 = \operatorname{tw}_2 \wedge \mu_2 &> \operatorname{tw}_2 \end{aligned}$$

can be true. Therefore, we get that $\operatorname{stw} = \operatorname{tw}$ with Equation (5.6). □

6 Conclusion

6.1 The Results

Our work done on the stacked treewidth and the relation between the Colin de Verdière number and the stacked treewidth has proven to be very fruitful. Not only have we built a sound and formal base for examining the stacked treewidth with our equivalence theorems (Theorem 5.7 and 3.27) and the lifting lemma (Lemma 3.20). Along our way to answer Question 5.12, we also discovered several new facts about the stacked treewidth, namely the minor monotony (Theorem 5.2), as well as behavior on clique sums (Theorem 5.6) and joins (Theorem 5.17).

Regarding our main goal, examining Question 5.12, we found counterexamples to (△) in Theorem 5.21 and Theorem 5.22. However, summarizing our results, we know that at least in the following cases (△) holds:

Theorem 6.1. *Let G be a graph. The implication (△) does hold for G,*

- *trivially, if* $\mu(G) > \mathrm{tw}(G)$.
- *if G is planar. (Theorem 5.13)*
- *if G is chordal. (Theorem 5.7)*
- *if* $\mathrm{tw}(G) \leq 3$. *(Theorem 5.13)*
- *if G is a complete bipartite graph. (Corollary 5.18 and Theorem 4.5)*
- *if (△) holds for graphs G_1 and G_2 and G is a clique sum of G_1 and G_2. (Theorem 5.24)*
- *if (△) holds for graphs G_1 and G_2 and G is a disjoint graph union of G_1 and G_2. (Theorem 5.23)*

So what about the relation between the Colin de Verdière number and the stacked treewidth? We have seen in Theorem 5.21 that for general graphs G, $\mathrm{stw}(G)$ seems to be independent of $\mu(G)$, or rather not dependent the way one would naturally expect. On the other hand, we have seen that the behavior of the Colin de Verdière number and the stacked treewidth is often symmetrical. This holds true especially for the clique-sum. For the Colin de Verdière number of a clique-sum, the graph $M_{t,3}$ plays a role, which is naturally connected to the stacked treewidth. We have seen in Theorem 5.7 that the identical behavior on clique-sums can be viewed as the reason that $\mu(G) = \mathrm{stw}(G)$ for all chordal graphs. What is the reason behind this identical behavior? We believe that the answer to this question may help us better understand how deep the connection between the Colin de Verdière number and the stacked treewidth is in reality.

6.2 Further Reading

We suggest further sources of literature covering topics, which could not be explained in full detail in this thesis.

Regarding the Colin de Verdière number: A very good, extensive survey of the Colin de Verdière number was written by Van der Holst, Lovász and Schrijver in 1999 [18]. Lovász also released a revised version with a few new facts in the year 2007, which can be found at his homepage. The research thesis of Goldberg also presents a very nicely structured overview, but omits most proofs [6]. A short proof of the planarity characterization via $\mu(G) \leq 3$ can be found [19]. Articles by Lovász, Schrijver and Izmestiev [14, 13, 8] link the Colin de Verdière number with geometric concepts and polytopes.

Regarding treewidth and stacked treewidth: A good overview on treewidth, including the algorithmic aspect is given in the lecture notes of Fiala [5]. The stacked treewidth and stacked k-trees were considered in [9, 15, 16]. Closest to our own topic, the relation between the Colin de Verdière number and stacked trees are probably [16] and [4].

6.3 Open Questions

Finally, we present a list of open questions that came up during the work on our thesis.

- Treewidth can also be defined in terms of so-called *brambles*, or the *cops and robbers game*. Do there exist variations of these concepts that describe the stacked treewidth?

- Knauer and Ueckerdt observed that if G is $(k,3)$-cut-free, then G+clique$(N(v))$ is $(k,3)$-cut-free. Can we use this idea to prove that $(\triangle)$ holds for $(k,3)$-cut-free graphs? Maybe [4] can be helpful.

- Can we find more families and graph operations where $(\triangle)$ holds?

- It is known that $\mu(G)$ is invariant under the so-called $Y\Delta$-transformations for all graphs G with $\mu(G) \geq 4$. How do $\text{tw}(G)$ and $\text{stw}(G)$ behave under $Y\Delta$-transformation? (It is easy to see that if G' is created from G with $\text{tw}(G) \geq 4$ by performing a ΔY-operation, we have $\text{stw}(G') \leq \text{stw}(G)$. But the other direction seems quite hard.)

- Can one find algorithms to compute the stacked treewidth?

- We know that there exist counterexamples to $(\triangle)$ where the absolute difference $\text{tw}(G) - \mu(G)$ is arbitrarily high. What about the relative difference?

- We mentioned the symmetry of the Colin de Verdière number and the stacked treewidth on clique-sums. Another puzzling symmetry is the following: The reason that Theorem 5.13 is not applicable for $\mu > 4$ is, in a way, that $\text{stw}(K_{k,3}) = k + 1$ for $k \leq 3$, but then does not increase anymore with $\text{stw}(K_{k,3}) = 4$ for $k > 3$. We observe that $\mu(K_{k,3})$ behaves identically. What is the reason for this and has it something to do with the unexpected switch of forbidden minors for $\mu(G) \leq 3$ to $\mu(G) \leq 4$?

- We have $\mu(\overline{K_2}) = 1$, but $\mathrm{stw}(\overline{K_2}) = 0$, so $\overline{K_2}$ is the only graph with $\mathrm{stw}(G) < \mu(G)$. An interesting detail is, however, that Theorem 4.9 and Theorem 4.13 had to exclude $\overline{K_2}$, but *would* be correct for $\overline{K_2}$, if $\mu(\overline{K_2})$ *were* 0 instead of 1. Also, the forbidden minors for $\mu(G) \leq 0$ would become $\{K_{3,0}, K_2\}$. Therefore, we propose the following (cosmetic) conjecture:

Conjecture 6.2. *There exists another characterization μ' of the Colin de Verdière number such that $\mu'(G) = \mu(G)$ for all G except that $\mu'(\overline{K_2}) = 0$.*

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