Karlsruhe Institute of Technology

On the Strong Product Theorem

Improving the Treewidth Bound

Bachelor Thesis of

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July 1, 2020 – November 2, 2020

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Abstract

A *layering* of a graph assigns each vertex an integer such that layers of adjacent vertices differ by at most one. A path in a graph with a given layering is *vertical* if its vertices are assigned pairwise distinct layers. Dujmović et al. recently proved the Strong Product Theorem, which states that for every planar graph G with a *BFS-layering* (a type of layerings that are based on distances in BFS-trees), the vertices of G can be partitioned into vertical paths $\mathcal{P}$ such that the quotient graph $G/\mathcal{P}$ has treewidth at most 8. This is equivalent to the statement that every planar graph is a subgraph of the strong product of some graph with treewidth at most 8 and some path.

By modifying the original proof of the Strong Product Theorem, we prove that the upper bound for the treewidth of the quotient graph can be improved to 7. Additionally, we show that this bound is tight when following the approach from the proof. Moreover, we prove that for a particular subclass of planar graphs, we can obtain a better upper bound.

Zusammenfassung

Die *Schichtung* eines Graphen bezeichnet eine Funktion, die jedem Knoten eine Schicht zuweist, wobei sich die Schichten von adjazenten Knoten höchstens um eins unterscheiden. Ein Pfad in einem Graphen mit gegebener Schichtung ist *vertikal*, wenn alle Knoten des Pfades in paarweise unterschiedlichen Schichten sind. Dujmović et al. bewiesen kürzlich das Strong Product Theorem. Dieses besagt, dass die Knoten jedes planaren Graphen G mit einer *BFS-Schichtung* (eine Klasse von Schichtungen, die auf Distanzen in BFS-Bäumen basieren) in vertikale Pfade $\mathcal{P}$ partitioniert werden kann, so dass der Quotientengraph $G/\mathcal{P}$ höchstens Baumweite 8 hat. Das ist äquivalent zu der Aussage, dass jeder planare Graph Subgraph eines starken Produkts von einem Graphen mit Baumweite höchstens 8 und einem Pfad ist.

Mithilfe einer Modifikation des ursprünglichen Beweises vom Strong Product Threorem zeigen wir, dass die obere Schranke für die Baumweite des Quotientengraphen auf 7 verbessert werden kann. Zusätzlich zeigen wir, dass diese Schranke für den gewählten Ansatz scharf ist. Für eine bestimmte Unterklasse von planaren Graphen können wir eine bessere obere Schranke beweisen.

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1 Introduction

The treewidth of a graph describes how "tree-like" a graph is and was introduced by Robertson and Seymour [RS86] in 1986. There are many graph-theoretical problems, for example computing the size of a maximum independent set or the chromatic number of a graph, that are $\mathcal{NP}$-hard but that can be solved efficiently on graphs with bounded treewidth using dynamic programming [AP89]. In 1990, Courcelle [Cou90] proved that every graph property that can be expressed in counting monadic second-order logic can be decided in linear time on graphs with bounded treewidth. Graph classes with bounded treewidth include outerplanar graphs and cactus graphs which both have treewidth 2 [Bod98]. Both are subclasses of the class of series parallel graphs which contains all graphs with treewidth at most 2. Graphs with treewidth at most 1 are forests.

However, the treewidth of planar graphs is unbounded since a grid graph on n vertices has treewidth $\sqrt{n}$, and many problems that have been solved for graphs with bounded treewidth remain open for planar graphs. In 2019, Dujmović et al. [Duj+20c] proved the Strong Product Theorem which states that every planar graph is a subgraph of the strong product of some graph with bounded treewidth and a path. Figure 1.1 shows that a grid graph is a subgraph of the strong product of a path, which has treewidth 1, and a path. We will define treewidth and other basic concepts in Chapter 2. In this thesis, we are going to study the bounds on the treewidth given by the Strong Product Theorem.

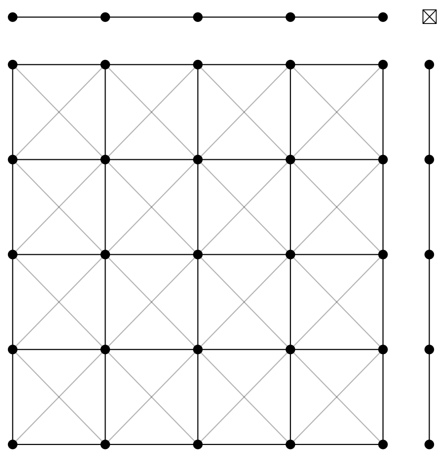


Figure 1.1: A grid graph is a subgraph of a strong product of a path and a path.

1.1 Related Work

The proof of the Strong Product Theorem given by Dujmović et al. [Duj+20c] is based on a similar result by Pilipczuk and Siebertz [PS19]. Pilipczuk and Siebertz showed that every planar graph can be partitioned into *geodesics* such that the treewidth of the graph obtained by contracting each part to a single vertex is at most eight. A path in a graph is a geodesic if it is a shortest path between its endpoints. They give a constructive proof with which the contracted graph with bounded treewidth can be found in $\mathcal{O}(n^2)$. The Strong Product Theorem is a slightly stronger statement, since it partitions a planar graph into *vertical paths* which are a special type of geodesics for certain layerings.

Although the Strong Product Theorem is rather new, it already has been used to solve many open problems. We will give a few examples of the applications of the Strong Product Theorem. It was shown in the examples mentioned below that some parameter of planar graphs is finite or the bound was improved. The proofs for the improvements use variations of a similar idea: For a graph H, let $f(H)$ be a non-negative graph parameter that is to be minimized. For the parameters we consider in the remaining section, the following has been proven: If $f(H)$ is bounded for a graph H with bounded treewidth, then $f(H \boxtimes P)$, where $H \boxtimes P$ denotes the strong product of H with some path P, is also bounded. It directly follows that the parameter is bounded for planar graphs since every planar graph is a subgraph of a strong product of some graph with bounded treewidth and a path. In most of the examples, a variation of the Strong Product Theorem was used, which states that every planar graph is a strong product of some graph with bounded treewidth, a path and a triangle.

In the following, we give an overview of graph parameters for which a better upper bound has been obtained by applying the Strong Product Theorem.

The concept of queue numbers was introduced by Heath and Rosenberg [HR92] in 1992. For a linear ordering $\preccurlyeq$ of $V(G)$, $vw, xy \in E(G)$ is a pair of *nested edges* if $v \prec x \prec y \prec w$ holds. A *queue* is a set of edges that are pairwise non-nested. A partition of $E(G)$ into k queues with respect to $\preccurlyeq$ is called a *k-queue-layout*. The *queue number* of a graph is the minimal integer k such that there is a linear ordering $\preccurlyeq$ of the vertices of G for which G admits a k-queue-layout. For a graph with bounded treewidth k, Wiechert [Wie17] proved that the queue number is bounded and at most $2^k - 1$. Heath and Rosenberg conjectured that planar graphs have bounded queue number and proved an $o(n)$ bound. This bound has been improved to $\mathcal{O}(n)$ by Dujmović [Duj15] for planar graphs, and Bekos et al. [Bek+19] showed that planar graphs with bounded maximum degree have bounded queue number. 27 years after it has been conjectured, Dujmović et al. [Duj+20c] showed that the queue number of planar graphs is bounded and at most 49.

The Strong Product Theorem has also been used for coloring problems to find an upper bound on the number of colors needed such that every planar graph has a coloring that admits certain properties.

A coloring of a graph is called *non-repetitive* if there is no path of even length such that the sequence of the colors in the first half is the same as the sequence of the colors in the second half. Graphs with bounded treewidth k are non-repetitively colorable with 4^k colors [KP08]. The problem whether planar graphs can be colored non-repetitively with a bounded number of colors remained open since 2002, when it was conjectured by Alon et al. [AGHR02].

Dujmović et al. [Duj+20b] proved in 2019 that 768 colors are sufficient for every planar graph to be colored non-repetitively. Before this result, the best known upper bound was $\mathcal{O}(\log n)$ [DFJW12].

For $p \in \mathbb{N}_+$, a coloring of a graph is called *p-centered* if every subgraph is either colored in more than p colors or if there is a color that appears exactly once in the subgraph. This type of colorings can be used to characterize classes of bounded expansion which contain sparse graphs. For any class of bounded expansion, there exists a function f such that for every $p \in \mathbb{N}_+$, $f(p)$ colors are sufficient for every graph in this class to admit a p-centered coloring [DFMS].

Pilipczuk and Siebertz [PS19] have already proven that every planar graph admits a p-centered coloring with $\mathcal{O}(p^{19})$ colors. This bound was improved by Dębski, Felsner, Micek and Schröder [DFMS] to $\mathcal{O}(p^3 \log(p))$ by using the result that for a graph with bounded treewidth k, $\mathcal{O}(p^k)$ colors are sufficient for a p-coloring [PS19]. They also give $\Omega(p^2 \log p)$ as a lower bound for the number of colors required in any p-centered coloring of a planar graph.

Another application of the Strong Product Theorem is the asymptotically optimal construction of *adjacency labelling schemes*. Informative labelling schemes, which were introduced by Peleg [Pel00], assign labels to the vertices of a graph such that the labels themselves already contain information about the graph. Apart from adjacency labelling schemes, there are also labellings that give information about ancestry [FK09] or distances [GKU16].

An adjacency labelling is a labelling of the vertices such that, given only the labels of two vertices, it is possible to decide without knowledge of the given graph, whether the vertices are adjacent or not. A family of graphs has an $f(n)$-bit adjacency labelling scheme if there exists an adjacency labelling for every graph of the family on n vertices such that each label uses at most $f(n)$ bits. This concept was introduced by Kannan, Naor and Rudich [KNR92]. For graphs with bounded treewidth, there exists a $\big((1 + o(1)) \log(n)\big)$-bit labelling scheme [GL07]. It has been shown that planar graphs have a $\big((c + o(1)) \log(n)\big)$-bit labelling scheme. The constant c has been improved twice in the last year using the Strong Product Theorem, first by Bonamy, Gavoille, and Pilipczuk [BGP], who showed that $c \leq 4/3$, then by Dujmović et al. [Duj+20a], who proved $c = 1$, which is asymptotically optimal.

For each of the problems above, there exists an algorithm that gives a solution that is at most the upper bound given in the proof. Each algorithm first partitions the graph into vertical paths as in the Strong Product Theorem. The running-time of these algorithms depend mostly on the running-time to find such a partition. Since the proof of the Strong Product Theorem is constructive, it naturally gives us an algorithm that computes a partition of a planar graph into vertical paths with the desired properties. This algorithm has the same running-time of $\mathcal{O}(n^2)$ as the constructive algorithm of obtaining a partition into geodesics [PS19]. Pat Morin [Mor20] has recently shown that the running-time to compute such a partition can be improved to $\mathcal{O}(n \log n)$. It is still open whether an algorithm with $\mathcal{O}(n)$ running-time exists.

The Strong Product Theorem only considers planar graphs, however, analogous results have been proven for other graph classes, for example graphs with bounded genus [Duj+20c], several non-minor-closed classes [DMW20] and geometrically defined graph classes [Dvo+20].

1.2 Outline

We first give some basic definitions and notations that will be used throughout the thesis.

In Chapter 3, we introduce important concepts such as layerings and partitions that are necessary to prove the Strong Product Theorem. Additionally, we prove some characteristics of vertical paths and a key lemma for a particular kind of partitions.

In the next chapter, we discuss the upper bound of the treewidth given by the Strong Product Theorem. First, we prove both the Strong Product Theorem and a variation of it. By modifying the proof, we show that this upper bound can be slightly improved. Additionally, we give a better upper bound for 2-outerplanar graphs under certain assumptions. We also discuss if the upper bound can be improved by using a different approach than the proof of the Strong Product Theorem.

By constructing an example, we show in Chapter 5 that the bound we proved in the chapter before is tight if we use the approach from the Strong Product Theorem.

2 Preliminaries

In this chapter, we give some basic definitions and notations for graphs.

A *graph* G is a pair of two sets $(V(G), E(G))$, where $V(G)$ is the vertex set and $E(G) \subseteq V \times V$ is the edge set of G. The edges are undirected, i.e. $\{u, v\}$ is an edge if and only if $\{v, u\}$ is an edge, and we denote an edge $\{u, v\} \in E(G)$ as uv. In this thesis, all graphs are finite and simple, i.e. there is at most one edge between two vertices, and every edge connects exactly two distinct vertices. Two vertices that are connected by an edge are called *adjacent*.

For a subset V' of vertices in a graph G, we define the edge set of the induced subgraph on V' as $E(V') \coloneqq \{uv \in E(G) \mid u, v \in V'\}$. We denote this induced subgraph as $G[V']$.

A *clique* K_n on n vertices in a graph G is a subset of vertices of $V(G)$ such that all vertices in $V(K_n)$ are pairwise adjacent.

We call a sequence of vertices $P = (v_1, \ldots, v_n)$ a *path* with *endpoints* v_1 and v_n if $v_i v_{i+1}$ is an edge for all $i \in \{1, \ldots, n-1\}$. The *length* of a path P is equal to the number of its edges and is denoted by $|P|$. We say that two paths P_1 and P_2 are *adjacent* if there are two vertices $u \in P_1$ and $v \in P_2$ that are adjacent. Two vertex-disjoint paths P_1 and P_2 are *consecutive* if there is an endpoint u of P_1 and an endpoint v of P_2 such that u and v are adjacent.

A graph G is *connected* if for any two vertices $u, v \in V(G)$, there is a path P such that the endpoints of P are u and v. In a connected graph G, the distance $\mathrm{dist}_G(u, v)$ of two vertices u and v in $V(G)$ is equal to the length of a shortest u-v-path in G.

A *plane graph* G is an embedding of a graph in the Euclidean plane $\mathbb{R}^2$, where the vertex set is a set of pairwise distinct points in $\mathbb{R}^2$. The edge set consists of Jordan curves in $\mathbb{R}^2$. Each curve has two endpoints in $V(G)$, and a curve crosses vertices or other curves only in its endpoints. Different edges have different sets of endpoints. By removing the edges and vertices of G from $\mathbb{R}^2$, we get connected components in $\mathbb{R}^2 \setminus G$, which are called *faces*. The unbounded face of G is the *outer face*, all other faces are *inner faces* of G. If an edge or a vertex is incident to the outer face, then it is called an *outer edge* or *outer vertex*, respectively. The remaining edges and vertices are *inner edges* and *inner vertices*, respectively. A graph is *planar* if it is isomorphic to a plane graph. In this thesis, we assume a fixed embedding for every planar graph.

We call a plane graph an *inner-triangulated* graph if the outer face is bounded by a cycle and each inner face is a triangle.

A graph is 1-*outerplanar* (or *outerplanar*) if it is isomorphic to a plane graph whose vertices lie on the outer face. A plane graph is k-*outerplanar* if the graph that is obtained by deleting all outer vertices is $(k-1)$-outerplanar.

We say that a planar graph is *bounded* by a set of vertex-disjoint paths $P = \{P_1, \ldots, P_n\}$ if the outer face is only incident to vertices in $V(P)$. A path is an *inner path* or an *outer path* if it consists only of inner vertices or outer vertices, respectively.

The treewidth of a graph is a parameter that describes how tree-like a graph is. If a graph has a clique of size k, then the treewidth is at least $k-1$.

We define a *k-tree* inductively as follows:

- K_k is a k-tree.
- Let T_k be a k-tree. Then, T'_k which is obtained by adding a new vertex v to T_k such that the neighborhood of v induces a clique in T'_k, is also a k-tree.

A graph G has *treewidth k* if G is a subgraph of a k-tree and k is minimal. Note that any tree is a 1-tree. However, an $n \times n$ grid graph has treewidth n, thus, the treewidth of planar graphs is unbounded.

3 Layerings and Partitions

In the following chapter, we introduce the concepts of layerings and partitions which are important tools we will need in the subsequent chapters, and discuss their properties. Then we define n-k-partitions for which we show a lemma that we will use extensively throughout the thesis, for instance in the proof of the upper bounds given by the Strong Product Theorem.

3.1 Layerings

Definition 3.1: *Let G be a graph. A* layering $\ell\colon V(G) \to \mathbb{Z}$ *of G is a function that assigns each vertex in G a layer. If two vertices u and v are adjacent in G, then their layers differ by at most 1, i.e. $|\ell(u) - \ell(v)| \leq 1$.*

If G is a plane graph, then we call the layering of the outer and inner vertices the outer *and* inner *layering, respectively.*

We obtain a trivial layering of a graph if we assign each vertex the same layer. An example for a non-trivial layering of a graph G is to assign each vertex its distance to a fixed, chosen vertex $v \in V(G)$. For any pair of adjacent vertices $x, y \in V(G)$, we know that the distance between x and v differs at most by 1 from the distance between y and v. Thus, this layering is valid. A type of layerings for planar graphs that generalizes this approach is the planar BFS-layering as defined below. Figure 3.1a shows an example of such a layering.

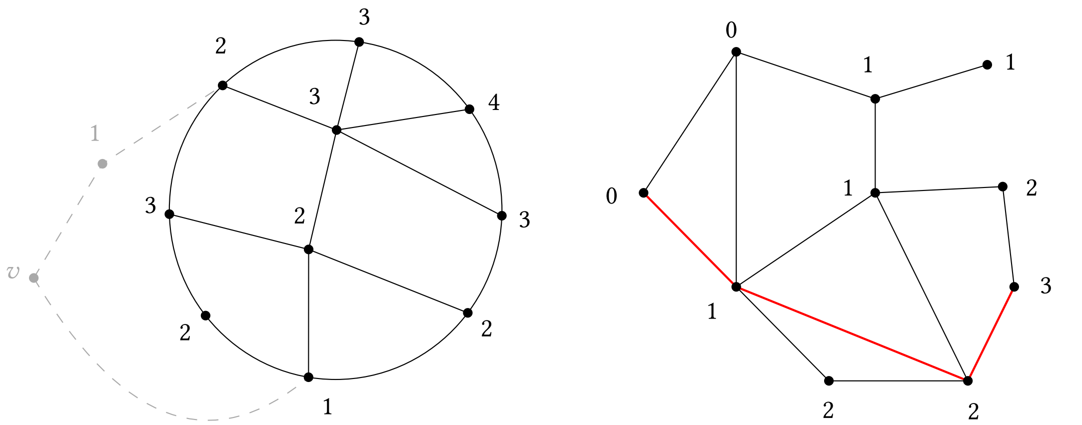
Definition 3.2: *We call a layering ℓ a* planar BFS-layering *of a planar graph G if there exists a graph G' that has the following properties:*

- *The graph G is a subgraph of G'.*
- *Vertices in $V(G') \setminus V(G)$ are not adjacent to inner vertices of G.*
- *There is a vertex $r \in V(G') \setminus V(G)$ such that $\ell(v) = \mathrm{dist}_{G'}(r, v)$ holds for all $v \in V(G)$.*

Note that G' does not have to be planar in the above definition.

For a fixed layering, we further define the concept of vertical paths. In [Duj+20c], a vertical path is a path P in a BFS-tree T such that the distance of a vertex in P to the root of T increases along the path. Note that this is only defined for BFS-layerings. We will generalize this definition for all layerings in the following.

Definition 3.3: *Let G be a graph and ℓ a layering of G. A* vertical path $P = (v_1, \dots, v_n)$ *is a path in G such that there are no two vertices in $V(P)$ that are in the same layer. Thus, the layers of the vertices on a vertical path either increase by 1 along the path or decrease by 1. Formally, it is either $\ell(v_i) = \ell(v_{i+1}) + 1$ for all $i \in \{1, \dots, n-1\}$ or $\ell(v_i) = \ell(v_{i+1}) - 1$ for all $i \in \{1, \dots, n-1\}$.*



(a) A planar BFS-layering with supergraph G' in gray.

(b) A valid layering of a graph and a red vertical path.

Figure 3.1: Examples for valid layerings.

A single vertex is a vertical path for any layering of G. For example, there is no vertical path of length at least one in a trivial layering since all vertices are in the same layer. In a BFS-layering, all shortest paths that end in the root are vertical paths. Figure 3.1b shows an example of a vertical path in a valid layering. Note that a subpath of a vertical path is also vertical.

In the following, we prove two defining properties of vertical paths. Both are helpful in identifying paths that cannot be vertical in a graph independent of the layering.

Proposition 3.4: *Let G be a graph with a layering ℓ. If P is a vertical path in G, then P is also a shortest path.*

Proof. Let u, v be vertices in $V(G)$. Consider a shortest u-v-path P in G. Since the layers of adjacent vertices differ by at most 1, we know that $|\ell(u) - \ell(v)| \leq |P| = \text{dist}(u, v)$ holds. Assume that P' is a vertical u-v-path but not a shortest path. Then the length of P' is strictly greater than the difference of the layers of u and v. Since P' is a vertical path and therefore monotonically increasing or decreasing, $\ell(w)$ is between $\ell(u)$ and $\ell(v)$ for any vertex w in path P'. Thus, there are two vertices in P' that are in the same layer. This is a contradiction to P' being a vertical path. □

Proposition 3.5: *Let G be a graph with a layering ℓ and P be a vertical path in G. Any vertex $v \in V(G)$ has at most three neighbors in P.*

Proof. Let v be a vertex in $V(G)$. Since ℓ is a valid layering of G, we know that $|\ell(v) - \ell(w)| \leq 1$ holds for any neighbor w of v. Thus, any neighbor of v is in one of three possible layers: $\ell(v) - 1$, $\ell(v)$ and $\ell(v) + 1$. Assume that $v \in V(G)$ has four neighbors in P. Then there are two vertices in the path P that are adjacent to v and in the same layer. But this is a contradiction to P being a vertical path. □

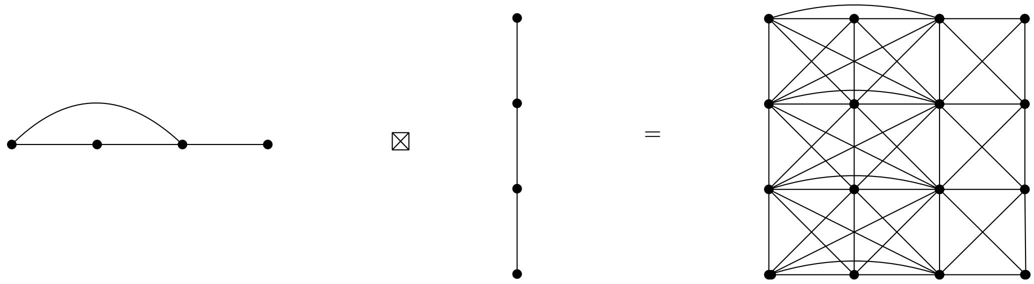


Figure 3.2: The strong product of a graph with a path.

3.2 Partitions and Strong Product

A *partition* $\mathcal{P}$ of a set X is a set of non-empty subsets of X such that each element in X is in exactly one part of $\mathcal{P}$.

We call a partition $\mathcal{P}$ of the vertices of a graph G *connected* if the induced subgraph of every part is connected. In this thesis, we will only consider connected partitions.

Definition 3.6: *For a partition $\mathcal{P}$ of a graph G, the* quotient graph *of $\mathcal{P}$ (denoted by $G/\mathcal{P}$) is a graph with vertex set $V(G/\mathcal{P}) = \mathcal{P}$. Two vertices $P, Q \in V(G/\mathcal{P})$ are adjacent if and only if there are two vertices $x \in V(P)$ and $y \in V(Q)$ that are adjacent in G.*

Dujmović et al. [Duj+20c] introduced the concept of layered partitions which was the key innovation in proving the Strong Product Theorem.

Definition 3.7: *Let G be a graph and $\mathcal{P}$ a partition of $V(G)$. The* layered width *of $\mathcal{P}$ is a minimal integer ℓ_w such that there is a layering ℓ of G with the following property: In each part of $\mathcal{P}$, there are at most ℓ_w vertices that are in the same layer.*

For example, a partition of the vertices of a graph G into vertical paths has layered width 1. A partition into triangles has at least layered width 2 since for any valid layering, the vertices of a triangle are not in pairwise distinct layers.

The following definition of the *strong product* of two graphs was first introduced by Sabidussi [Sab59] in 1959. Sabidussi also showed that the strong product is both commutative and associative.

Definition 3.8: *The* strong product *of two graphs G_1 and G_2 (denoted by $G_1 \boxtimes G_2$) is a graph $G_\boxtimes$ with the vertex set $V(G_1) \times V(G_2)$. Two vertices (u, v) and (x, y) in $G_\boxtimes$ are adjacent if*

- *$u = x$ and $vy \in E(G_2)$ or*
- *$v = y$ and $ux \in E(G_1)$ or*
- *$ux \in E(G_1)$ and $vy \in E(G_2)$.*

Note that for any subgraph $G' \subseteq G_1 \boxtimes G_2$ we can find a partition $\mathcal{P}$ of $V(G')$ such that the induced subgraph of every part is a subgraph of G_2 and G_1 is a subgraph of $G'/\mathcal{P}$. Figure 3.2 shows the strong product of a graph with a path on four vertices.

In the following, we define a particular type of partitioning the vertices of a planar graph with a given layering using vertical paths. We prove afterwards that if a planar graph admits such a partitioning, then we can give an upper bound for the treewidth of the quotient graph.

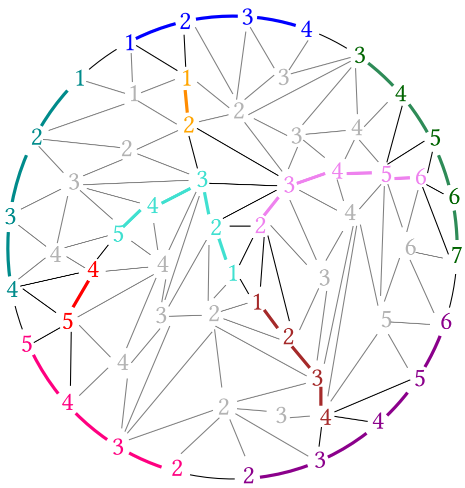


Figure 3.3: An example of a 5-4 partition with a layering ℓ. The number at a vertex is the layer of the vertex. The five outer and four inner vertical paths induce a 5-4-partition since each part is bounded by at most five vertical paths.

Definition 3.9: *Let G be a planar graph and ℓ a layering of G. Let P and Q be sets of vertex-disjoint vertical paths. Then, (P, Q) induces a* (P, Q)-partition *of $V(G') := V(G) \setminus \{V(P), V(Q)\}$ where each connected component of G' forms a part.*

We call such a partition an n-k-partition *if it has the following properties:*

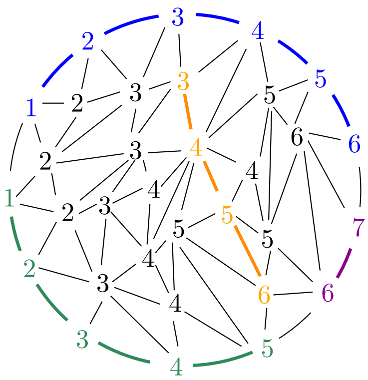
1. *P consists of $n' \leq n$ vertical paths $P_1, \ldots, P_{n'}$ that bound the outer face of G.*
2. *Q consists of $k' \leq k$ inner vertical paths $Q_1, \ldots, Q_{k'}$.*
3. *Each part in G' is incident to at most n vertical paths of $P \cup Q$.*

Let H be a planar graph and ℓ' a layering of H. Let P be a set of vertex-disjoint vertical paths that has property 1*. Assume that there is a modified layering ℓ of the inner vertices of H such that there are vertex-disjoint vertical paths Q with property* 2 *and* 3*. Then, we say that we can* extend *P to (P, Q) such that (P, Q) induces an n-k-partition using layering ℓ. Note that the layering of the outer vertices remains unchanged.*

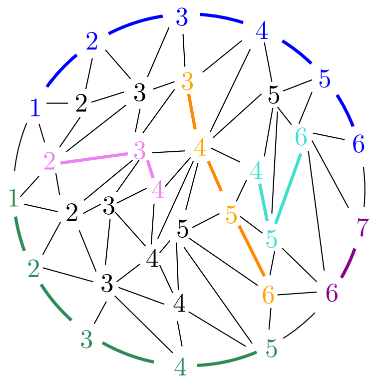
Figure 3.3 shows an example of a 5-4-partition of a graph.

If there is an n-k-partition for every planar graph, we can recursively find vertical paths that again induce an n-k-partition in a smaller part. By doing this, we obtain a partition of the vertex set of the original graph into vertex-disjoint vertical paths. An example in which such a partitioning is obtained recursively is shown in Figure 3.4. We show in the following lemma that if we contract all found vertical paths, then we obtain a graph with a treewidth that is bounded by n and k.

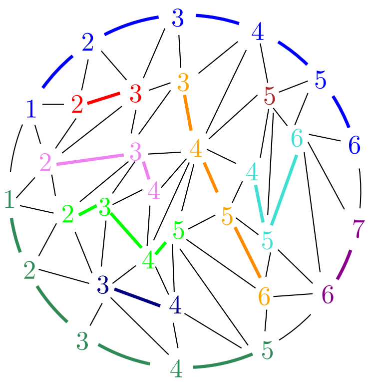
Lemma 3.10: *Let $n, k \in \mathbb{N}_+$. Let G be a planar, inner-triangulated graph with a layering ℓ such that G is bounded by $n' \leq n$ vertical paths $P = \{P_1, \ldots, P_{n'}\}$. Assume that for any subgraph G' of G that is bounded by at most n vertical paths P', P' can be extended to (P', Q), where Q is a set of vertical paths, such that (P', Q) induces an n-k-partition. Then, there exists a partition $\mathcal{P}$ of $V(G)$ into vertical paths with $\{P_1, \ldots, P_{n'}\} \subseteq \mathcal{P}$ such that there is a supergraph $G/\mathcal{P}^+$ of $G/\mathcal{P}$ that has treewidth at most $n + k - 1$. Furthermore, $P_1, \ldots, P_{n'}$ induce a clique in $G/\mathcal{P}^+$.*



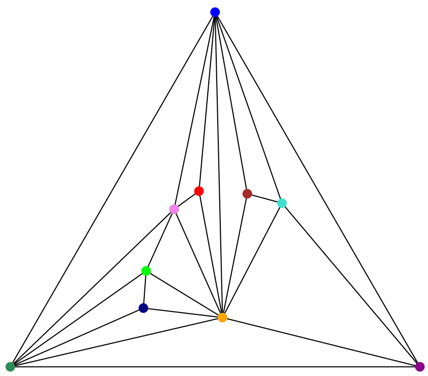
(a) Partition $\mathcal{P}$ after one step.



(b) Partition $\mathcal{P}$ after three steps.



(c) Partition $\mathcal{P}$ of all vertices in $V(G)$.



(d) The graph $G/\mathcal{P}^+$, a supergraph of $G/\mathcal{P}$ which is obtained by contracting all vertical paths in $\mathcal{P}$. Vertices have the same color as the vertical paths they correspond to.

Figure 3.4: A graph that satisfies the conditions of Lemma 3.10 with $n = 3$ and $k = 1$. The numbers correspond to the layer of a vertex. The quotient graph is a subgraph of a 3-tree.

Proof. Let G be a planar, inner-triangulated graph with a layering ℓ such that G is bounded by n vertical paths $P = \{P_1, \ldots, P_n\}$. Assume that every subgraph of G admits an n-k-partition. We will construct a partition $\mathcal{P}$ of $V(G)$ and a graph $G/\mathcal{P}^+$ as stated in the lemma.

We prove the lemma by induction on the number i of steps. The number of steps counts the number of parts we have already considered, which is equal to how often we add a set of vertical paths to $\mathcal{P}$. Let $\mathcal{P}_i \subseteq \mathcal{P}$ be the set of vertical paths in $\mathcal{P}$ after step i. Let $G/\mathcal{P}_i^+$ be the graph constructed on $\mathcal{P}_i$ after step i. Let F_i be the part we consider during step i. Assume that F_i is bounded by the vertex-disjoint vertical paths $P^i = \{P_1^i, \ldots, P_{n_i}^i\}$ with $n_i \leq n$.

For $i = 0$, we have $\mathcal{P}_0 = P$. Since P contains at most n vertical paths, $G/\mathcal{P}_0$ has at most n vertices. We obtain $G/\mathcal{P}_0^+$ by adding edges such that $G/\mathcal{P}_0^+$ is a K_{n_0} clique, which has treewidth at most $n - 1 \leq n + k - 1$.

For $i > 0$ we have two cases. If F_i is empty, then we do not add any vertical paths to $\mathcal{P}_{i-1}$, and we set $\mathcal{P}_i \coloneqq \mathcal{P}_{i-1}$ and $G/\mathcal{P}_i^+ \coloneqq G/\mathcal{P}_{i-1}^+$. If F_i is not empty, then by assumption, we can modify the layering of the inner vertices in F_i such that it has an n-k-partition. Since no inner vertex of F_i is adjacent to a vertex that is not in F_i, this modification does not affect vertices on the outside of F_i. Let $Q^i \coloneqq \{Q_1^i, \ldots, Q_{k_i}^i\}$ with $k_i \leq k$ be a set of vertex-disjoint, inner vertical paths in F_i such that (P^i, Q^i) induces an n-k-partition of F_i. Obtain $\mathcal{P}_i$ from $\mathcal{P}_{i-1}$ by adding vertices that represent the vertical paths in Q^i to $\mathcal{P}_{i-1}$. Obtain $G/\mathcal{P}_i^+$ by adding the vertical paths in Q^i one by one to $G/\mathcal{P}_{i-1}^+$ in any arbitrary order. Connect each newly added vertex in $G/\mathcal{P}_i^+$ with all vertices that correspond to the vertical paths in P^i and the paths in Q^i, that have already been added previously. The graph $G/\mathcal{P}_i$ is a subgraph of the resulting graph.

The adjacent vertices of an added vertex in $G/\mathcal{P}_{i-1}^+$ correspond to paths in P^i, which induce a clique by induction. Thus, the size of this clique increases by one after each added vertex. The vertex added last is adjacent to a clique of size $n_i + k_i - 1$ in $G/\mathcal{P}_i^+$. Thus, $G/\mathcal{P}_i^+$ has treewidth at most $n_i + k_i - 1 \leq n + k - 1$, and the vertices that correspond to $P^i \cup Q^i$ induce a clique in $G/\mathcal{P}_i^+$. Each part of the n-k-partition induced by (P^i, Q^i) in F_i is bounded by at most n vertical paths that induce a clique in $G/\mathcal{P}_i^+$. Apply induction to the new parts in F_i in any arbitrary order. $\square$

Note that this statement does not only hold for partitions into vertical paths but for any kind of partition. This can be proven by replacing vertical paths with the desired kind of partition in the proof above.

4 Upper Bounds

Although planar graphs can have unbounded treewidth, using the strong product, every planar graph can be constructed from some graph with bounded treewidth and a path, which was proven by Dujmović et al. [Duj+20c]. In the following chapter, we discuss and improve the upper bound given by the Strong Product Theorem. Then we consider 2-outerplanar graphs and give a better upper bound for this particular graph class. Additionally, we show that the upper bound might be improved by using a different approach than the proof of the Strong Product Theorem.

4.1 The Strong Product Theorem

In this section, we prove the Strong Product Theorem as stated below by showing that every planar graph has a 6-3-partition. Afterwards, we use a slightly modified proof to improve the upper bound given by the Strong Product Theorem.

Theorem 4.1 (Strong Product Theorem [Duj+20c])**:** *Every planar graph is a subgraph of $H \boxtimes P$ for some graph H with treewidth at most 8 and some path P.*

In the last chapter, we introduced a generalized definition of layerings. However, in the proofs of this section, we only consider BFS-layerings as defined in Definition 3.2. For this particular type of layerings, we will prove some properties in the following.

Lemma 4.2: *Let G be a planar graph with a planar BFS-layering ℓ. Then for any $v \in V(G)$, there exists an outer vertex u such that there is a vertical v-u-path with $\ell(v) \geq \ell(u)$.*

Proof. Since ℓ is a planar BFS-layering of G, there exists a supergraph G' of G that has the properties as stated in Definition 3.2. Let r be the vertex in $V(G') \setminus V(G)$ such that $\ell(v) = \mathrm{dist}_{G'}(v, r)$ holds for all vertices v in $V(G)$. We prove the statement by induction on the distance of a vertex in $V(G)$ to r. Consider a vertex $v \in V(G)$. If v is an outer vertex, then $P_v = (v)$ is a vertical path to an outer vertex. If v is an inner vertex, then we know that the layer of v is exactly the distance from v to r. Thus, there is a neighbor $v' \in V(G)$ of v with distance $\ell(v') = \mathrm{dist}_{G'}(v', r) = \mathrm{dist}_{G'}(v, r) - 1$. By induction, we have a vertical path $P_{v'}$ from v' to an outer vertex $u \in V(G)$. Thus, extending $P_{v'}$ with the vertex v gives us a vertical path from v to u with $\ell(v) > \ell(v') \geq \ell(u)$. $\square$

For a planar graph G with a planar BFS-layering and an arbitrary coloring of the outer vertices, we define a BFS-coloring on the inner vertices.

Definition 4.3: *Let G be a planar graph with a BFS-layering ℓ. Let c be an arbitrary coloring of the outer vertices of G. A* BFS-coloring *of $V(G)$ is obtained by the following algorithm: Let $v \in V(G)$ be an uncolored inner vertex. Consider a vertical v-u-path $P_v = (v_1 := v, \dots, v_m := u)$ for some outer vertex $u \in V(G)$. Let $v_j \in V(P_v)$ be the first colored vertex in P_v, and assume that it is colored with color i. Then we color all vertices in $\{v_1, \dots, v_{j-1}\}$ with color i. Repeat this until all vertices are colored.*

This algorithm is well-defined since by Lemma 4.2, there exists an outer vertex $u \in V(G)$ for any $v \in V(G)$ such that there is a vertical v-u-path. Every vertical path is acyclic and all outer vertices are already colored, thus, this algorithm terminates.

Lemma 4.4: *Let G be a planar graph with a planar BFS-layering ℓ and a BFS-coloring c. Then, there is a vertical path from v to an outer vertex that consists only of vertices that are colored with the same color $c(v)$ for every vertex $v \in V(G)$.*

Proof. We prove this by induction on the number of steps in the algorithm from Definition 4.3. If v is an outer vertex of G, then we are done. Assume that $v \in V(G)$ is an uncolored inner vertex and P_v is a vertical path from v to an outer vertex as in the algorithm. Let v' be the first colored vertex in P_v. By induction, there is a vertical path $P_{v'}$ colored in $c(v')$ from v' to an outer vertex u. Since the v-v'-subpath $P_{v,v'}$ of P_v is vertical, $(P_{v,v'} \setminus \{v'\}) \cup P_{v'}$ is a vertical path from v to u that is colored with $c(v')$ after coloring all uncolored vertices in P_v with $c(v')$. Since subpaths of vertical paths are also vertical, the statement of the lemma is true for all vertices on P_v. □

First, to prove the Strong Product Theorem, we show that every planar graph that is bounded by six vertical paths has a 6-3-partition, i.e. that there are three vertex-disjoint, inner vertical paths such that each part is also bounded by at most six vertical paths. In the proof of this lemma, we need the following variation of Sperner's Lemma.

Lemma 4.5 (Sperner's Lemma [AZ09])**:** *Let G be a planar, inner-triangulated graph, and let the outer edges of G be partitioned into three vertex-disjoint paths P_1, P_2 and P_3. Let c be a coloring of G that colors the vertices on P_i with color $i \in \{1, 2, 3\}$ and the inner vertices of G arbitrarily with colors in $\{1, 2, 3\}$. Then, G has an inner triangular face with vertices $\{v_1, v_2, v_3\}$ such that v_i is colored with color $i \in \{1, 2, 3\}$.*

Lemma 4.6: *Let G be a planar, inner-triangulated graph on at least three vertices with a planar BFS-layering ℓ such that the outer face of G is bounded by $n \leq 6$ vertex-disjoint vertical paths $P = \{P_1, \dots, P_n\}$. If G is not outerplanar, then we can find $k \leq 3$ vertex-disjoint vertical paths $Q = \{Q'_1, \dots, Q'_k\}$ in the interior of G such that (P, Q) induces a 6-3-partition, i.e. each part of the (P, Q)-partition is bounded by at most six vertical paths in $P \cup Q$.*

Proof. We group the n vertical paths in P into three vertex-disjoint paths $R := \{R_1, R_2, R_3\}$. Since the paths in P form a cycle, we know that we have more than one vertical path in P and that there are at least three outer vertices.

If $n = 2$, then we may assume that P_1 has more than one vertex. Thus, we can split P_1 into two vertex-disjoint vertical paths P'_1 and P''_1. We set $R_1 := P'_1$, $R_2 := P''_1$ and $R_3 := P_2$. For $n \in \{3, 4, 5, 6\}$, we group consecutive paths such that each R_i ($i \in \{1, 2, 3\}$) consists of either one or two paths in P. Color the vertices of each path R_i with color $i \in \{1, 2, 3\}$.

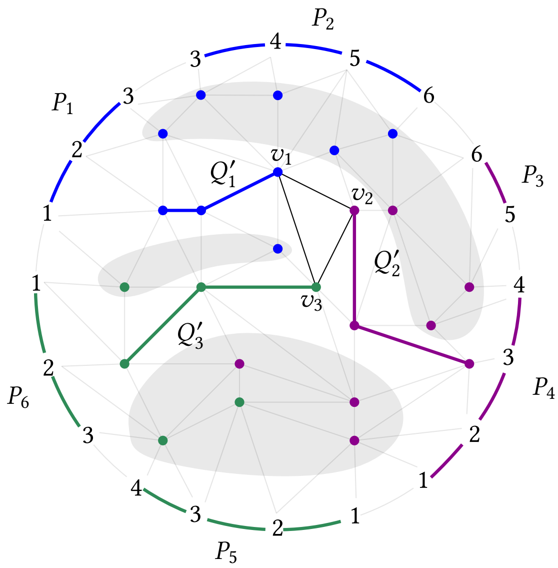


Figure 4.1: An example of a 6-3-partition that is obtained by using a BFS-coloring and the Sperner's Lemma. The numbers correspond to the layers of the outer vertices. The outer vertical paths $P = \{P_1, \ldots, P_6\}$ and the inner vertical paths $Q = \{Q'_1, Q'_2, Q'_3\}$ induce a 6-3-partition. Each part in gray is bounded by at most six vertical paths.

Since ℓ is a BFS-layering and the outer vertices of G are colored arbitrarily, we can obtain a BFS-coloring using the algorithm from Definition 4.3. We now have a 3-coloring of G that satisfies the conditions of Sperner's Lemma. Let $\{v_1, v_2, v_3\}$ be a triangular face in G such that v_i has color $i \in \{1, 2, 3\}$. Lemma 4.4 states that for each vertex, there is a vertical path to an outer vertex that only consists of vertices of the same color in a BFS-coloring.

Let Q_i be a shortest vertical v_i-u_i-path to an outer vertex $u_i \in R_i$ such that all vertices in Q_i have color $i \in \{1, 2, 3\}$. Note that Q_i contains only one outer vertex. Let Q'_i be the subpath of Q_i that does not contain the outer endpoint u_i for $i \in \{1, 2, 3\}$. We define the set of vertical paths $Q := \{Q'_1, Q'_2, Q'_3\}$. Note that Q'_i contains no vertex if v_i is an outer vertex. Since the vertices of distinct vertical paths in Q have distinct colors, all paths in Q are pairwise vertex-disjoint.

Each part in the (P, Q)-partition of G is incident to at most four outer vertical paths in P and two inner vertical paths in Q. Thus, we have three vertical paths $Q = \{Q'_1, Q'_2, Q'_3\}$ that satisfy the properties stated in the lemma. Figure 4.1 shows such a 6-3 partition for an example graph. □

The statement of the following lemma is a direct result of applying Lemma 3.10 to Lemma 4.6.

Lemma 4.7: *For any planar, inner-triangulated graph G, there exists a partition $\mathcal{P}$ of $V(G)$ into vertical paths such that there is a supergraph $G/\mathcal{P}^+$ of $G/\mathcal{P}$ that has treewidth at most* 8.

Proof. From Lemma 4.6, we know that every planar, inner-triangulated graph has a 6-3-partition induced by vertical paths. This satisfies the assumptions of Lemma 3.10. Then for every planar, inner-triangulated graph G, there exists a partition $\mathcal{P}$ into vertical paths such that there is a supergraph $G/\mathcal{P}^+$ of $G/\mathcal{P}$ that has treewidth at most $6 + 3 - 1 = 8$. □

With the lemma above, we can now prove Theorem 4.1, the Strong Product Theorem.

Proof. Let G be a planar graph and G_T a planar triangulation of G. Let $\mathcal{P}$ be the partition of $V(G_T)$ into vertical paths such that there exists a supergraph $G_T/\mathcal{P}^+$ of $G_T/\mathcal{P}$ that has treewidth at most 8. Thus, G_T is a subgraph of $(G_T/\mathcal{P}^+) \boxtimes P_l$, where l is the length of the longest path in $\mathcal{P}$. Since G is a subgraph of G_T, we get that $G \subseteq H \boxtimes P_l$ with $H = G_T/\mathcal{P}^+$ and treewidth at most 8. ▢

Such a partition into vertical paths has layered width 1, i.e. each part has at most one vertex in each layer. Dujmović et al. [Duj+20c] also proved an additional structure theorem that uses a partition with layered width three instead. In such a partition, each part has at most three vertices in the same layer.

Theorem 4.8 ([Duj+20c])**:** *Every planar graph is a subgraph of $H \boxtimes P \boxtimes K_3$ for some graph H with treewidth at most 3 and some path P.*

For a layering ℓ, we call a subgraph of G a *tripod* if it consists of at most three vertex-disjoint vertical paths whose lower endpoints are pairwise adjacent. A *bipod* is a tripod that consists of at most two vertex-disjoint vertical paths. Note that a partition of $V(G)$ into tripods has layered width three. The proof of this theorem is quite similar to the proof of Theorem 4.1.

Lemma 4.9: *Let G be a planar, inner-triangulated graph with a planar BFS-layering ℓ such that G is bounded by $n \leq 3$ bipods $P = \{P_1, \dots, P_n\}$. If G is not outerplanar, then there is a tripod T in the interior of G with the following property: Each connected component in $G[V(G) \setminus (V(P) \cup V(T))]$ is incident to at most three tripods in $P \cup \{T\}$.*

Proof. We mostly follow the proof of Lemma 4.6, using tripods instead of vertical paths. For $|V(G)| \leq 3$, the result is trivial.

In Lemma 4.6, we grouped the vertical paths first into groups of one and two. The partition of the outer vertices of G into bipods naturally gives us such a grouping. Since P contains a cycle, we know that P has at least three vertices. If $n = 1$ or $n = 2$, then we split P_1 into three or two vertical paths, respectively. Note that a single vertical path is also a bipod. Thus, we may assume $n = 3$ in the following.

We color bipod P_i with color $i \in \{1, 2, 3\}$ and obtain a BFS-coloring of the inner vertices of G using the algorithm from Definition 4.3. This gives us a 3-coloring that satisfies the conditions of Sperner's Lemma, and we get an inner triangular face $\{v_1, v_2, v_3\}$ such that v_i is colored with color $i \in \{1, 2, 3\}$. Since we have a BFS-coloring, by Lemma 4.4, we can obtain vertex-disjoint vertical paths $Q := \{Q_i \text{ without the outer endpoint} \mid i \in \{1, 2, 3\}\}$ such that Q_i is a shortest vertical path colored in only one color from v_i to an outer vertex in G.

There are at most three vertical paths in Q. Since the endpoints v_1, v_2 and v_3 of the vertical paths in Q are pairwise adjacent, Q induces a tripod T. Each connected component in the graph $G[V(G) \setminus (V(P) \cup V(T))]$ is incident to at most three tripods in $P \cup \{T\}$. ▢

By applying a generalized variant of Lemma 3.10 that uses a partition into tripods instead of a partition into vertical paths, we can prove the following lemma as in the proof of Lemma 4.7.

Lemma 4.10: *For any planar, inner-triangulated graph G, there exists a partition $\mathcal{P}$ into tripods such that there is a supergraph $G/\mathcal{P}^+$ of $G/\mathcal{P}$ that has treewidth at most* 3.

Following the proof of Theorem 4.1, we conclude that every planar graph is a subgraph of a graph $H \boxtimes T$, where H is a graph with treewidth at most three and T is a tripod. Since a tripod is a subgraph of $P \boxtimes K_3$ for some path P and the strong product is associative, the result of Theorem 4.8 follows directly.

4.2 Improving the Treewidth to 7

We modify the proof of Lemma 4.6 to show that any planar, inner-triangulated graph has a 5-3-partition. In the beginning of Lemma 4.6, we grouped the outer vertical paths into three groups. To find a 6-3-partition, any valid grouping is sufficient. Some groupings do not give a 5-3-partition, but we can show that there is one grouping that does.

Lemma 4.11: *Let G be a planar, inner-triangulated graph on at least three vertices with a planar BFS-layering ℓ such that the outer face of G is bounded by $n \leq 5$ vertex-disjoint vertical paths $P = \{P_1, \ldots, P_n\}$. If G is not outerplanar, then we can extend P to (P, Q), where Q is set of at most three vertex-disjoint, inner vertical paths, such that (P, Q) induces a* 5-3*-partition.*

Proof. For $|V(G)| \leq 3$, the result is trivial. If G is only bounded by two vertical paths P_1 and P_2, we assume without loss of generality that P_1 has more than one vertex. Thus, we can partition P_1 into two vertical paths P_1' and P_1''. In the following, we assume that G is bounded by at least three vertical paths.

We color the vertices of each outer vertical path P_i in color $i \in \{1, \ldots, n\}$. Since ℓ is a planar BFS-layering of G, we can obtain a BFS-coloring c of $V(G)$ using $n \leq 5$ colors as in Definition 4.3. Note that each induced subgraph on the set of the vertices colored with the same color is connected. By Lemma 4.4, for each vertex in $V(G)$, there is a vertical path to an outer vertex such that all vertices on that path are colored with the same color. We denote such a path that contains only one outer vertex as Q_v for a vertex $v \in V(G)$. There might be more than one path with this property. In this case, any choice is valid.

We group consecutive paths into vertex-disjoint paths $R := \{R_1, R_2, R_3\}$ such that each R_i ($i \in \{1, 2, 3\}$) consists of either one or two paths in P. To apply Sperner's lemma, we define a second coloring $c' \colon V(G) \to 2^{\{1,\ldots,n\}}$. If two paths $P_j, P_k \in P$ are merged to one path $R_i \in R$, then we also merge the colors j and k to color $\{j, k\}$. Formally, we set $c'(v) = \{j \mid P_{c(v)} \in R_i \text{ and } P_j \in R_i\}$. Each color in c appears exactly in one new color of c' and each color in c' consists of at most two colors of c. Note that colors in c' are pairwise disjoint. The coloring c' is a 3-coloring that satisfies the conditions of Sperner's Lemma.

Let $\{v_1, v_2, v_3\}$ be a triangular face in G with v_1, v_2 and v_3 colored with pairwise distinct colors in c'. Since colors in c' are pairwise disjoint, v_1, v_2 and v_3 are also colored with pairwise distinct colors in c. Let Q_{v_i}' be the subpath of Q_{v_i} that does not contain the outer endpoint of Q_{v_i} for all $i \in \{1, 2, 3\}$. We define the set of vertical paths $Q := \{Q_{v_1}', Q_{v_2}', Q_{v_3}'\}$. Note that Q_{v_i}' contains no vertex if v_i is an outer vertex. Since the vertices of distinct vertical paths in Q have distinct colors, all paths in Q are pairwise vertex-disjoint.

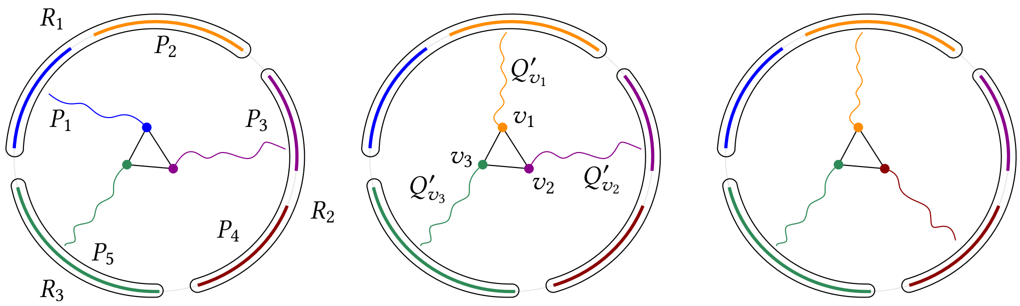
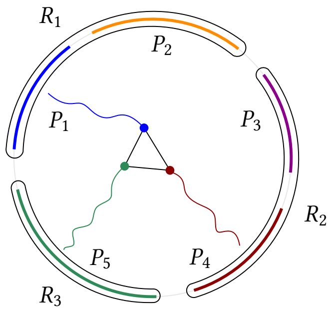
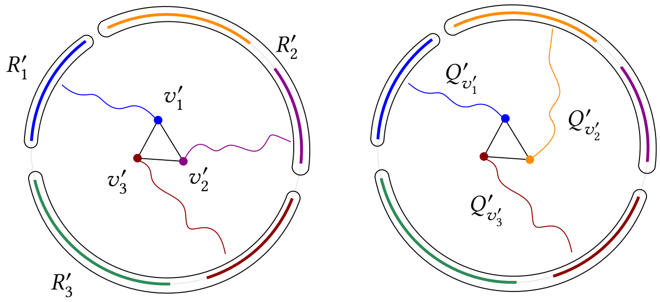


Figure 4.2: All possibilities for the colors of v_1 and v_2 such that $(P, \{Q'_{v_1}, Q'_{v_2}, Q'_{v_3}\})$ induces a 5-3-partition. Note that the color of v_3 is fixed by the grouping we chose.



(a) The only case ($c(v_1) = 1$ and $c(v_2) = 4$) that does not induce a 5-3-partition.

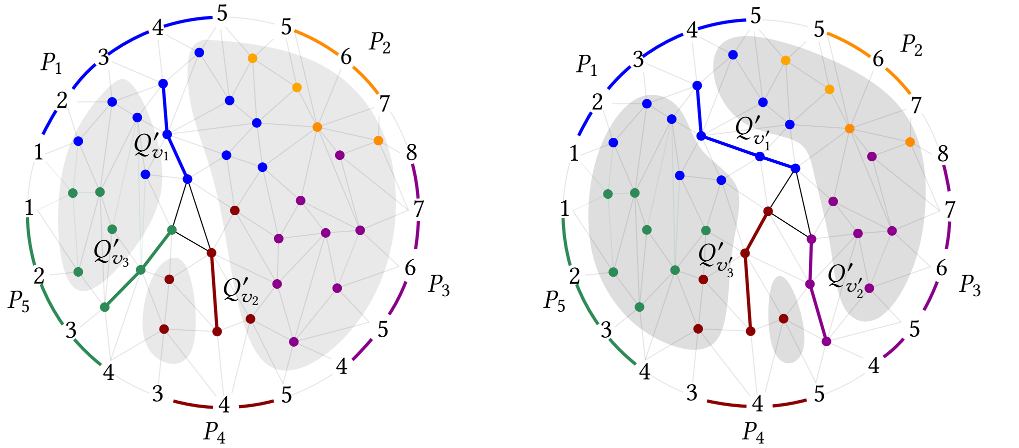


(b) Two cases in which a 5-3-partition is induced, obtained by regrouping the outer vertical path to R'. We have $c(v'_2) = 3$ and $c(v'_2) = 2$ on the left and on the right, respectively.

Figure 4.3: Regrouping the outer vertical paths from $R_1 = P_1 \cup P_2$, $R_2 = P_3 \cup P_4$ and $R_3 = P_5$ to $R'_1 := P_1$, $R'_2 := P_2 \cup P_3$ and $R'_3 := P_4 \cup P_5$ to obtain a 5-3-partition.

For $n = 3$ and $n = 4$, (P, Q) always induces a 5-3-partition, independent of the grouping we choose. For $n = 5$, we assume without loss of generality that $R_1 = P_1 \cup P_2$, $R_2 = P_3 \cup P_4$ and $R_3 = P_5$. Since R_3 consists only of one single path of P, we know that all vertices in Q_{v_3} have color 5. For v_1 and v_2, we have two possibilities each: $c(v_1) \in \{1, 2\}$ and $c(v_2) \in \{3, 4\}$. We have three cases such that (P, Q) induces a 5-3-partition, as seen in Figure 4.2: If $(c(v_1), c(v_2)) = (1, 3)$, then the first part is incident to the vertical paths P_1, P_2, P_3, Q'_{v_1} and Q'_{v_2}, the second part is incident to P_3, P_4, P_5, Q'_{v_2}, and Q'_{v_3} and the third part is incident to P_5, P_1, Q'_{v_1} and Q'_{v_2}. For $(c(v_1), c(v_2)) \in \{(2, 3), (2, 4)\}$, each part is also incident to at most five vertical paths.

In the following, we assume that $c(v_1) = 1$ and $c(v_2) = 4$ as seen in Figure 4.3a. Then the part that is incident to P_2 and P_3 is bounded by six vertical paths P_1, P_2, P_3, P_4, Q'_{v_1} and Q'_{v_2}. Since v_1v_2 is an edge, $Q_{v_1} \cup Q_{v_2}$ forms a path from a vertex in $P_{c(v_1)} = P_1$ to a vertex in $P_{c(v_2)} = P_4$. We now show that there is no vertex u with color 5 that is adjacent to a vertex v that has color 2 or 3. Assume that there exists an edge $uv \in E(G)$ with $c(u) = 5$ and $c(v) \in \{2, 3\}$. Assume without loss of generality that $c(v) = 2$. By Lemma 4.4, there is a vertical path Q_u from u to some outer vertex in P_5 that contains only vertices of color 5. There is also a vertical path Q_v, which is colored with color 2, from v to some outer vertex in P_2. Thus, $Q_u \cup Q_v$ forms a path from a vertex in P_5 to a vertex in P_2. Since P bounds the outer face of G, such a path connects P_2 and P_5 in the interior of G. But then $Q_u \cup Q_v$ and $Q_{v_1} \cup Q_{v_2}$ cross. Since vertices in $Q_u \cup Q_v$ are colored with colors in $\{2, 5\}$ and vertices in $Q_{v_1} \cup Q_{v_2}$ are colored with colors in $\{1, 4\}$, the paths do not share a vertex. But this is a contradiction to G being planar.



(a) $R_1 := P_1 \cup P_2$, $R_2 := P_3 \cup P_4$, $R_3 := P_5$. Since there is a part that is bounded by six vertical paths, this is not a 5-3-partition.

(b) $R'_1 := P_1$, $R'_2 := P_2 \cup P_3$, $R'_3 := P_4 \cup P_5$. Since each part is bounded by at most five vertical paths, this is a 5-3-partition.

Figure 4.4: An example where we have to regroup the outer vertical paths to obtain a 5-3-partition.

We group the paths in P again into three vertex-disjoint vertical paths $R' := \{R'_1, R'_2, R'_3\}$ with $R'_1 := P_1$, $R'_2 := P_2 \cup P_3$ and $R'_3 := P_4 \cup P_5$. By obtaining a new merged 3-coloring c'' (exactly as above) and applying Sperner's Lemma, we get an inner triangular face $\{v'_1, v'_2, v'_3\}$ with v'_1, v'_2 and v'_3 colored with pairwise distinct colors in c''. We know that $c(v'_1) = 1$, $c(v'_2) \in \{2, 3\}$ and $c(v'_3) \in \{4, 5\}$. Since v'_2 and v'_3 are adjacent, v'_3 is adjacent to a vertex with color 2 or a vertex with color 3. We have proven that no vertex with color 5 is adjacent to a vertex with color 2 or 3, thus it is $c(v'_3) = 4$. Then $Q_{v'_1}$ is a vertical path from v'_1 to a vertex in P_1 and $Q_{v'_3}$ is a vertical path from v'_3 to a vertex in P_4.

Let $Q'_{v'_i}$ be the subpath of $Q_{v'_i}$ that does not contain the outer endpoint of $Q_{v'_i}$ for all $i \in \{1, 2, 3\}$. We define the set of vertical paths $Q' := \{Q'_{v'_1}, Q'_{v'_2}, Q'_{v'_3}\}$. In both cases $c(v'_2) = 2$ and $c(v'_2) = 3$, (P, Q') induces a 5-3-partition, as shown in Figure 4.3b. If v'_2 has color 3, then the first part is incident to the vertical paths $P_1, P_2, P_3, Q'_{v'_1}$ and $Q'_{v'_2}$, the second part is incident to $P_3, P_4, Q'_{v'_2}$ and $Q'_{v'_3}$, and the third part is incident to $P_4, P_5, P_1, Q'_{v'_1}$ and $Q'_{v'_3}$. If v'_2 has color 2, then each part is also incident to at most five vertical paths.

In every case, we have a set Q of at most three vertex-disjoint, inner vertical paths such that (P, Q) induces a 5-3-partition in G. ▢

Figure 4.4 shows an example where a regrouping of the outer vertical paths is needed to obtain a 5-3-partition. Using the lemma above, we can now improve the bound of the treewidth given by the Strong Product Theorem.

Theorem 4.12: *Every planar graph is a subgraph of $H \boxtimes P$ for some graph H with treewidth at most 7 and some path P.*

Proof. Combining Lemma 4.11 and Lemma 3.10, we conclude that every planar graph G has a partition $\mathcal{P}$ into vertex-disjoint vertical paths such that $G/\mathcal{P}$ has treewidth at most $5+3-1 = 7$. This is equivalent to the statement of the theorem. ▢

4.3 Better Bounds for 2-Outerplanar Graphs

In this section, we only consider 2-outerplanar graphs. Recall that a 2-outerplanar graph is a planar graph that has an embedding with the following property: If we delete all outer vertices, then all remaining vertices are incident to the outer face.

We show the following: For every 2-outerplanar graph G, there exists a layering of G such that G is bounded by three vertical paths which are contained in a 3-1-partition. However, we will construct a 2-outerplanar graph afterwards with a given outer layering and three outer vertical paths, which shall not have a 3-1-partition, independent of the inner layering we choose. Note that 2-outerplanar graphs have treewidth at most 5.

First, we define a notation for any inner-triangulated, 2-outerplanar graph G on at least three vertices, which we use throughout this section.

Assume that ℓ is a layering of G such that G is bounded by $n \leq 3$ vertex-disjoint vertical paths $\{P_1, \ldots, P_n\}$. If $n = 2$, then we may assume that P_1 has more than one vertex. Thus, we can split P_1 into two vertex-disjoint vertical paths P_1' and P_1''. For the remaining section, we assume that G is bounded by exactly three vertex-disjoint vertical paths $P := \{P_1, P_2, P_3\}$.

Denote the graph that is obtained by deleting the outer vertices as G_{in}. Denote the vertices of P_i as $p_{i,1}, \ldots, p_{i,n_i}$ such that $\ell(p_{i,1}) < \ldots < \ell(p_{i,n_i})$ for all $i \in \{1, 2, 3\}$. Since P_1, P_2 and P_3 are vertical paths, the layers of adjacent vertices in P_i differ by exactly one. Assume that $p_{1,1}$ and $p_{3,1}$, p_{1,n_1} and $p_{2,1}$, and p_{2,n_2} and p_{3,n_3} are adjacent.

Denote the set of vertices in G_{in} that are adjacent to an outer vertical path P_i as V_i for all $i \in \{1, 2, 3\}$. We call the unique vertex $v_{i,i+1}$ that is both in V_i and V_{i+1} with $V_4 := V_1$ for all $i \in \{1, 2, 3\}$ a *transition vertex*. For all $i \in \{1, 2, 3\}$, we define $P_i' := (p_{i,1}', \ldots, p_{i,m_i}')$ as the shortest path from $v_{i-1,i}$ to $v_{i,i+1}$ in V_i with $v_{0,1} := v_{3,1} =: v_{3,4}$. Note that each pair P_i' and P_{i+1}' (with $P_4 := P_1$) for all $i \in \{1, 2, 3\}$ share one endpoint by construction.

In the following, we introduce some properties of particular layerings.

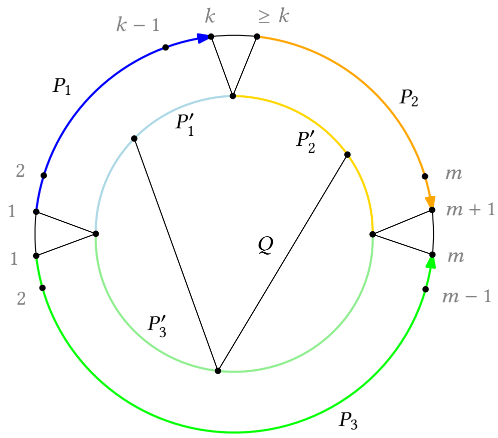
Definition 4.13: *A layering ℓ for a set $P' = \{P_1', \ldots, P_n'\}$ of paths in a graph G is* monotone *if the layers of the vertices of any path $P_i' = (p_{i,1}', \ldots, p_{i,n_i}')$ in P' are monotonically increasing along P_i', i.e. if $\ell(p_{i,1}') \leq \ldots \leq \ell(p_{i,n_i}')$ holds.*

Definition 4.14: *Let $P = (p_1, \ldots, p_n)$ be a path in a graph G. A vertex $p_j \in V(P)$ is on the* right *of a vertex $p_k \in v(P)$ if $j > k$. Otherwise, it is on the* left *of it.*

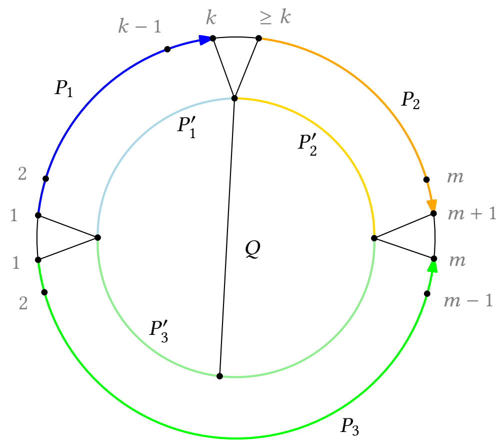
Note that if a layering ℓ is monotone for some path P, then vertices to the right are in higher layers than vertices to the left in P.

To find an inner vertical path Q such that $(P, \{Q\})$ induces a 3-1-partition, we need to modify the inner layering of a graph. There might be vertices whose layers cannot be modified to obtain a valid inner layering. We call these vertices *fixed*.

Definition 4.15: *A vertex v in a graph G is* fixed *in a layering ℓ if the layer of v is fixed by the layers of the adjacent vertices, i.e. v is fixed if and only if there are adjacent vertices u and w with $\ell(u) - \ell(w) = 2$. Then, we have $\ell(v) = \ell(w) + 1$.*



(a) The vertical path Q consists of three vertices in P'.



(b) The vertical path Q consists of a single edge. It starts at a transition vertex, which is adjacent to both P_1 and P_2.

Figure 4.5: A 2-outerplanar graph with a layering such that it is bounded by three vertical paths $P = \{P_1, P_2, P_3\}$. The layering is monotone for $\{P_1 \cup P_2, P_3\}$. The arrows indicate in which direction the layers of the vertices of a vertical path increase. In both examples, (P, Q) induces a 3-1-partition.

For the remaining proof, we assume that $\ell(p_{1,n_1}) \leq \ell(p_{2,1})$. Then, the layering ℓ is monotone for the set of paths $\{P_1 \cup P_2, P_3\}$. Our goal is to show that P can be extended to $(P, \{Q\})$ such that $(P, \{Q\})$ induces a 3-1-partition in this particular setting. We do this by finding a vertical path Q that only consists of inner vertices and that is adjacent to all outer vertical paths in P. Figure 4.5 shows two possible examples for such a path Q.

To prove that there is a layering that is monotone for $\{P'_1, P'_2, P'_3\}$, we need the following observation.

Observation 4.16: *There are no vertices $v, w \in P'_i$ for any $i \in \{1, 2, 3\}$ such that v is on the left of w and $\ell(v) > \ell(w) + 2$.*

Proof. Assume that v and w are vertices in P'_i for some $i \in \{1, 2, 3\}$ such that v is on the left of w and $\ell(v) > \ell(w) + 2$. By definition of P'_i, both v and w have a neighbor in P_i. Any neighbor $v' \in V(P_i)$ of v is in layer at least $\ell(v) - 1$ and any neighbor $w' \in V(P_i)$ of w is in layer at most $\ell(w) + 1 < \ell(v) - 1$. Since P_i is a vertical path, w' is on the left of v', which is a contradiction to G being planar. ▢

Lemma 4.17: *If the given layering ℓ is monotone for $\{P_1 \cup P_2, P_3\}$, then there is an inner monotone layering ℓ' for $\{P'_1, P'_2, P'_3\}$.*

Proof. We call a pair of vertices $(p'_{i,j}, p'_{i,j+1})$ a *defect* in a layering ℓ if $\ell(p'_{i,j+1}) < \ell(p'_{i,j})$, i.e. if the right vertex is in a strictly lower layer than the left vertex. Assume that a layering ℓ' of G is not a monotone layering and that ℓ' has a minimal number of defects. Then, there exists a defect (v, w) in some path P'_i. Since v and w are adjacent, we know that $\ell'(v) = \ell'(w) + 1$.

Assume first that both v and w are fixed. Then, there are two vertices u and x such that $uv \in E(G)$ with $\ell'(u) = \ell'(v) + 1$ and $wx \in E(G)$ with $\ell'(x) = \ell'(w) - 1$.

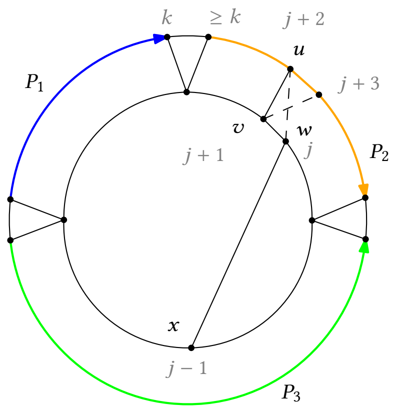


Figure 4.6: If the layering ℓ is monotone for $\{P_1 \cup P_2\}$, then there exists a layering that is also monotone for $\{P_1', P_2', P_3'\}$. Here (v, w) is a defect, and we assume that both v and w are fixed. But then, w has no neighbor in P_2.

From Observation 4.16, we know that the vertices u and x cannot be in P_i'. Since the edges uv and wx do not cross, u and x are not both vertices in P_i. If both u and x are vertices of some path $P_{i'}'$ with $i \neq i'$, then the edges uv and wx cross since by Observation 4.16, u is on the right of w.

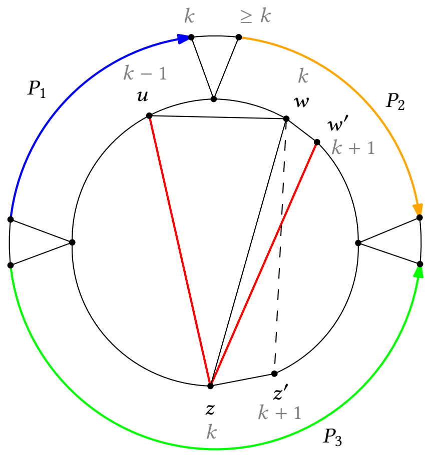
Thus, either u or x is an inner vertex, the other one is in P_i. We assume without loss of generality that u is in P_i. Since ℓ' is a layering, w has no neighbor in P_i. However, w is a vertex in P_i', which contradicts the definition of P_i'. This is shown in Figure 4.6. Thus, v and w are not both fixed. Assume without loss of generality that v is not fixed. Then, v is not adjacent to any vertex in layer $j+2$. We can modify the layering ℓ' to a layering ℓ'' with $\ell''(v) := \ell'(v) - 1 = \ell'(w)$ and $\ell''(y) := \ell'(y)$ for all $y \in V(G) \setminus \{v\}$. Then, ℓ'' has one defect less than ℓ', contradicting the assumption that ℓ' is minimal with respect to the number of defects. ▢

In the remaining section, we assume that ℓ is a monotone layering for $\{P_1', P_2', P_3'\}$ of G if not stated otherwise. Let $k \in \mathbb{N}$ be such that $\ell(v) \leq k$ for all vertices $v \in P_1'$ and $\ell(w) \geq k$ for all vertices $w \in P_2'$.

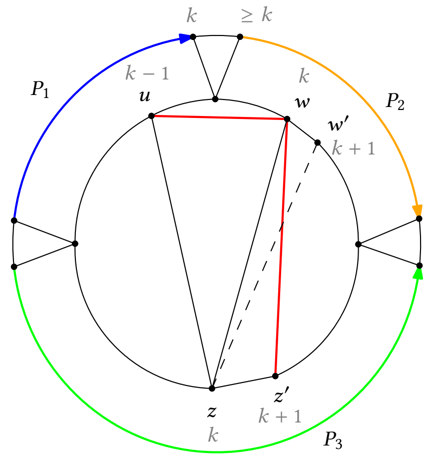
Lemma 4.18: *Assume that $u \in P_1'$, $w \in P_2'$ and $z \in P_3'$ induce a triangle with $\ell(u) = k-1$ and $\ell(w) = \ell(z) = k$. Let $z' \in P_3'$ be the right neighbor of z. If $\ell(z') = k+1$, then G contains a vertical path $P = (x_1, x_2, x_3)$ such that x_i is adjacent to a vertex in P_i for all $i \in \{1, 2, 3\}$.*

Proof. Let $w' \in P_2'$ be the right neighbor of w. Since w' is not adjacent to any vertex in layer $k-1$, we may assume that $\ell(w') = k+1$. Because G is triangulated, either zw' or wz' is an edge in G. If $zw' \in E(G)$, then $Q := (u, z, w')$ is a vertical path. If $wz' \in E(G)$, then $Q := (u, w, z')$ is a vertical path. In Figure 4.7, both cases are illustrated with the vertical paths in red. The number at a vertex corresponds to its layer in ℓ. ▢

Observation 4.19: *Let $x \in P_3'$ be the unique vertex that is adjacent to the two vertices in P_3 that are in layer k or layer $k-1$. Let $y \in P_3'$ be a vertex that is to the right of x, but not adjacent to it. If y has no neighbor in P_1' in layer $k-1$, then we may assume $\ell(y) \geq k+1$. Note that all vertices in P_2' are at least in layer k. For the right neighbor x' of x, we may assume that $\ell(x') = \ell(x)+1$. This also holds if we switch the roles of P_1, P_1' and P_3, P_3' above.*



(a) The path (u, z, w) is vertical and adjacent to all outer vertical paths.



(b) The path (u, w, z') is vertical and adjacent to all outer vertical paths.

Figure 4.7: The vertices u, v and w induce a triangle. Two vertices are connected by a dashed line if there is no edge between them. Lemma 4.18 shows that in both cases there is an inner vertical path (colored in red) as desired.

With the above lemmas and observation, we can now prove that any 2-outerplanar, inner-triangulated graph admits a 3-1-partition if we choose the outer layering.

Lemma 4.20: *Let G be a 2-outerplanar, inner-triangulated graph with a layering ℓ such that G is bounded by three vertical paths P_1, P_2 and P_3. Assume that ℓ is monotone for $\{P_1 \cup P_2, P_3\}$. Then, there is a single inner vertical path Q such that $(\{P_1, P_2, P_3\}, \{Q\})$ induces a 3-1-partition in G.*

Proof. Let v be the transition vertex that is both adjacent to a vertex in P_1 and to a vertex in P_2. If v is adjacent to a vertex $z \in P_3'$ with $|\ell(z) - \ell(v)| = 1$, then $Q := (v, z)$ is a desired vertical path. Hence, we may assume that all vertices in P_3' that are adjacent to v are in the same layer as v.

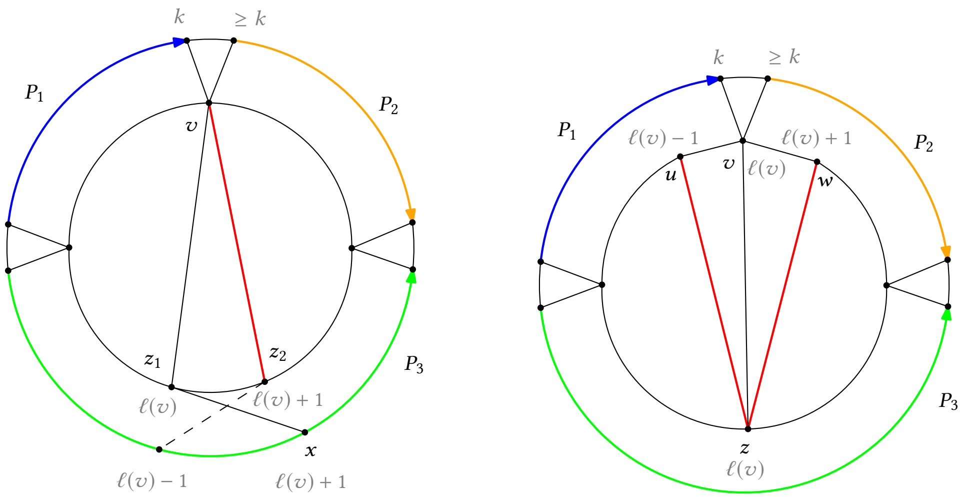
Case 1: The vertex v has a neighbor in P_3'.

Case 1a: The vertex v has more than one neighbor in P_3'.

Let z_1, z_2 be neighbors of v in P_3'. We argue that if $\ell(z_1) = \ell(z_2)$ holds as we assumed, then at least one vertex of z_1 and z_2 is not fixed. Assume that z_1 is fixed. Then, it has a neighbor x in layer $\ell(z_1) + 1$. Since G is planar, x is in P_3 (otherwise the edges z_1x and vz_2 would cross). But then z_2 is not adjacent to a vertex in layer $\ell(z_2) - 1$. If z_i for some $i \in \{1, 2\}$ is not fixed, then there exists a layering, in which $Q := (v, z_i)$ is a desired vertical path, as shown in Figure 4.8a.

Case 1b: The vertex v only has one neighbor z in P_3'.

Let $u \in P_1'$ be the left and $w \in P_2'$ be the right neighbor of v. Recall that we assumed that $\ell(v) \neq \ell(z)$. Observe that u is not adjacent to any vertex in layer $\ell(z) + 1$ and w is not adjacent to any vertex in layer $\ell(z) - 1$. Thus, we may assume $\ell(u) = \ell(z) - 1$ and $\ell(w) = \ell(z) + 1$. Since G is triangulated, both vertices u and w are adjacent to z. Thus, $Q := (u, z, w)$ is a desired vertical path. Figure 4.8b illustrates this case.



(a) Case 1a: If v has two neighbors z_1 and z_2 in P_3' and z_1 is adjacent to a vertex x in layer $\ell(v)+1$, then (v, z_2) is a vertical path in G.

(b) Case 1b: If v has only one neighbor z in P_3' and $\ell(v) = \ell(z)$, then (u, z, w) is a vertical path.

Figure 4.8: The transition vertex v that is adjacent to both P_1 and P_2 has a neighbor in P_3'.

Case 2: The vertex v has no neighbors in P_3'.

Since G is triangulated, there is an edge uw with $u \in P_1'$, $w \in P_2'$. Choose u and w such that u is leftmost in P_1' and w is rightmost in P_2'. The vertices u and w have a shared neighbor $z \in P_3'$ since the edge uw is outermost and G is triangulated.

Since $\ell(u) \leq k$ and $\ell(w) \geq k$ holds by definition of k and the layers of u and w differ by at most one, we may assume $\ell(u) = k-1$ and $\ell(w) = k$. For $\ell(u) = k$ and $\ell(w) = k+1$, we use the symmetrical version of the following proof.

If z is adjacent to a vertex $u' \in P_1'$ with $\ell(u') = k-2$, then z is in layer $k-1$ and $Q := (u', z, w)$ is a desired vertical path. This case is shown in Figure 4.9a. Hence, we may assume that all neighbors in P_1' of z are in layer $k-1$ or k.

Let $x \in P_3'$ be the unique vertex that is adjacent to the two vertices in P_3 that are in layer $k-1$ and k.

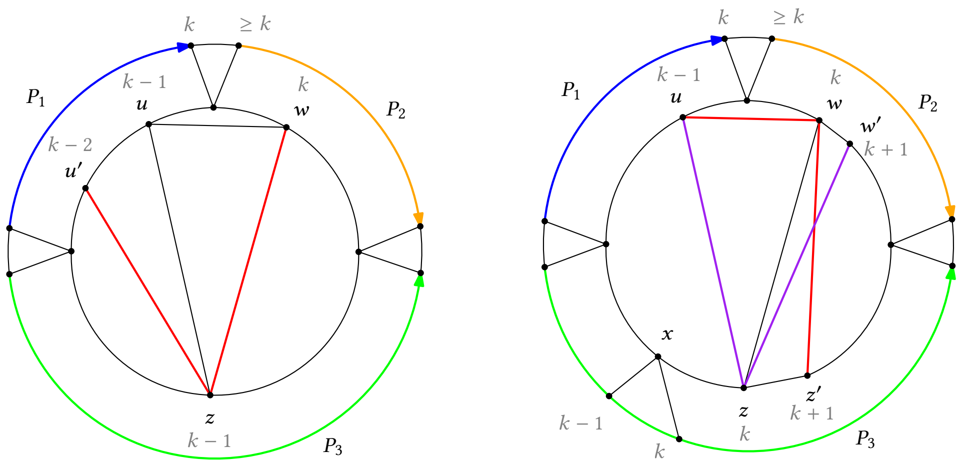
Case 2.1: z is on the right side of x.

Since we assumed that z is not adjacent to a vertex in P_1' in layer $k-2$, we may assume that $\ell(z) = k$ and $\ell(z') = k+1$ for the right neighbor z' of z in the path P_3' using Observation 4.19. Now, we can apply Lemma 4.18 and get either $Q := (u, z, w')$, where w' is the right neighbor of w in the path P_2', or $Q := (u, w, z')$ as a vertical path. In Figure 4.9b, the two possible vertical paths are marked in red and purple, respectively.

Case 2.2: $\ell(z_2) = k$ and $z_2 = x$.

Using Observation 4.19, we may assume that the right neighbor z' of z in P_3' is in layer $k+1$. With Lemma 4.18, we either get a vertical path $Q := (u, z, w')$, where w' is the right neighbor of w in the path P_2', or $Q := (u, w, z')$.

Case 2.3: $\ell(z) = k-1$ and either $z = x$ or z is on the left side of x.



(a) If z is adjacent to a vertex u' in layer $k-2$, then $\ell(z) = k-1$ and (u', z, w) is a desired vertical path.

(b) Case 2.1: If z is not adjacent to a vertex in layer $k-2$, then we may assume that $\ell(z) = k$ and $\ell(z') = k+1$. By Lemma 4.18, either the red or the purple vertical path exists.

Figure 4.9: The vertex that is adjacent to both P_1 and P_2 has no neighbor in P_3.

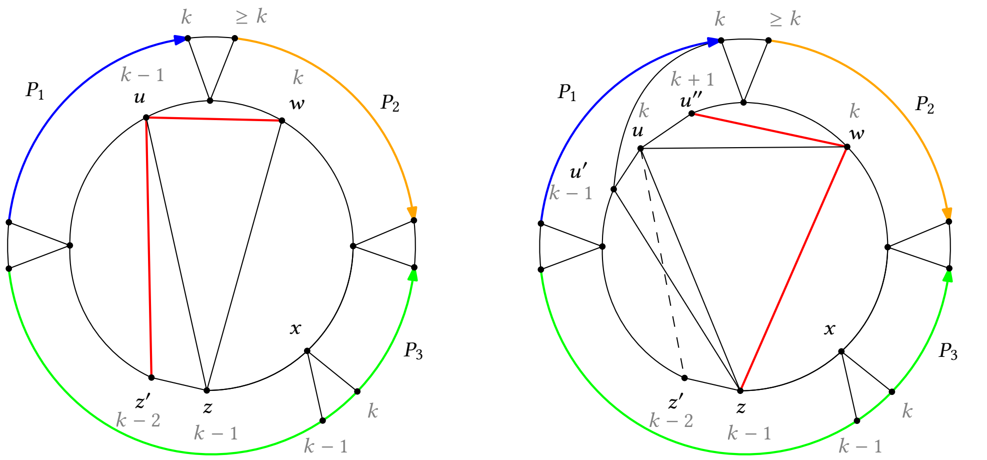
Since the left neighbor z' of z in P_2' is not adjacent to any vertex in layer k, we may assume that z' is in layer $k-2$. If $z'u \in E(G)$, then we get a vertical path $Q := (z', u, w)$, as illustrated in Figure 4.10a. We assume that $z'u \notin E(G)$. Since G is triangulated and 2-outerplanar, there is an edge between the left neighbor u' of u in P_1' and z. By assumption, z is not adjacent to any vertex in layer $k-2$, thus, we have $\ell(u') = k-1$ fixed. Then, the vertex u' is adjacent to the unique vertex in P_1 in layer k. Using Observation 4.19, we may now assume that $\ell(u) = \ell(u') + 1 = k$ and $\ell(u'') = k+1$ for the right neighbor u'' of u. Note that u'' might be v. Since G is triangulated, there is an edge between u'' and w. Then $Q := (z, w, u'')$ is a desired vertical path. This is shown in Figure 4.10b.

In every case, we can modify the inner layering such that there is an inner vertical path Q that is adjacent to all three outer vertical paths. Thus, $(P, \{Q\})$ induces a 3-1-partition. ▢

For the proof of Lemma 4.20, we have assumed that the graph has a layering ℓ that is monotone for the outer paths $\{P_1 \cup P_2, P_3\}$. However, a layering might not be monotone for the path $P_1 \cup P_2$ since $\ell(p_{1,n_1}) - 1 = \ell(p_{2,1})$ is possible. In this case, we can prove the following lemma.

Lemma 4.21: *There is a 2-outerplanar graph G with a given outer layering ℓ such that G is bounded by three vertical paths $P = \{P_1, P_2, P_3\}$, which cannot be extended to any 3-1-partition.*

Proof. We show that the graph G that is shown in Figure 4.11 is a graph with the properties as stated in the lemma. The graph G is 2-outerplanar and triangulated with a layering ℓ on the outer vertices such that G is bounded by three vertical paths P_1, P_2 and P_3. The number at a vertex corresponds to the layer of the vertex if it is fixed by the layering of the outer vertices. There are only two vertices whose layers are not fixed by the outer vertices. Since both vertices are adjacent to vertices in layer 3 and 4, they are either in layer 3 or in layer 4.



(a) If the left neighbor $z' \in P_3'$ of z is adjacent to u, then (z', u, w) is a desired vertical path.

(b) If z' is not adjacent to u, then we may assume that $\ell(u') = k-1$ and that u' is adjacent to the vertex in P_1 in layer k. By modifying the layers of u and u'', we get a vertical path $Q := (u'', w, z)$.

Figure 4.10: The vertex x that is adjacent to the vertices in P_3 in layer k and $k-1$ is on the right of z and $\ell(z) = k-1$.

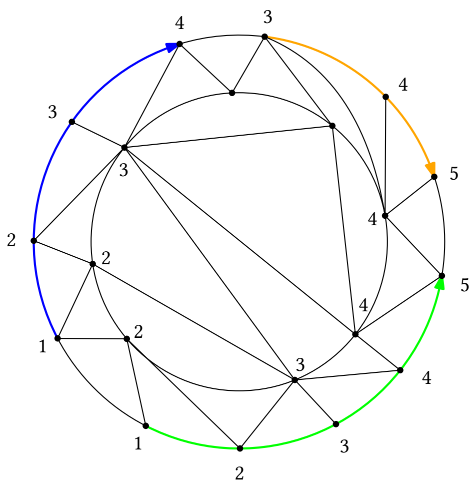


Figure 4.11: A graph with an outer layering that is not monotone for $\{P_1 \cup P_2\}$. The three outer vertical paths cannot be extended to any 3-1-partition.

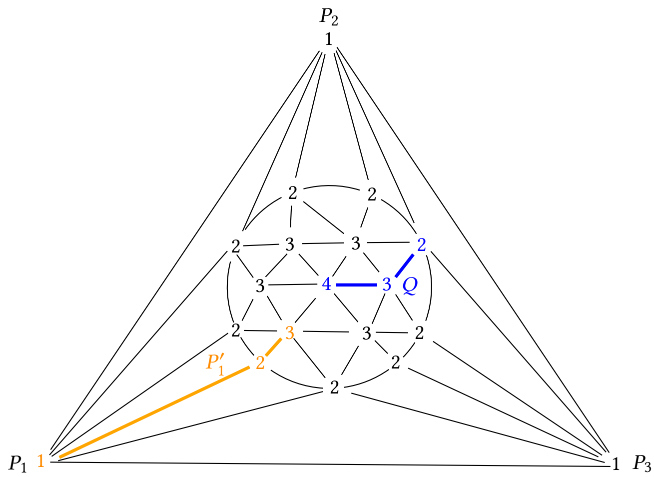


Figure 4.12: An example graph with a planar BFS-layering where we can extend the outer vertical path P_1 to the longer vertical path P_1' colored in orange. Then, $(\{P_1', P_2, P_3\}, Q)$ induces a 3-1-partition.

We argue that there is no vertical path that is adjacent to all outer vertical paths. There are no two distinct vertices that are in the same layer in a vertical path. Since every inner vertex of G is in the layer 2, 3 or 4, any vertical path consists of at most three vertices and has length at most two. A vertical path of length two starts at a vertex in layer 2. However, any such path is not adjacent to all three outer vertical paths. If there is a vertical path of length one that is adjacent to all three outer vertical paths, then it is an edge that is incident to a transition vertex. The other endpoint of the edge is adjacent to the third vertical path. It is obvious that such an edge does not exist.

We conclude that an inner vertical path which is adjacent to all three outer vertical paths in P does not exist. Thus, P cannot be extended to a 3-1-partition.

□

4.4 Extending Vertical Paths

In the proof of the Strong Product Theorem, we partitioned the vertices of a planar graph into vertical paths inductively and aimed to minimize the number of vertical paths in the partition which bound a part. Once we have added a new vertical path to the partition in an induction step, this vertical path remains unchanged in the final resulting partition. Thus, we assumed that an outer vertical path is not extended to the interior of the graph. In fact, any vertical path might be extended into two directions if the layering allows it. By giving an example in this section, we show that by extending vertical paths, we can improve the bounds for some planar graphs.

Definition 4.22: *Let G be a graph and ℓ a layering of $V(G)$. Let P_1 be a vertical path in G. We can* extend *$P_1 = (v_1, \ldots, v_{n_1})$ with another vertex-disjoint vertical path $P_2 = (u_1, \ldots, u_{n_2})$ in G if $P = (v_1, \ldots, v_{n_1}, u_1 \ldots, u_{n_2})$ is also a vertical path.*

In this section, we consider the graph as in Figure 4.12.

If the outer vertices are three singleton vertical paths P_1, P_2 and P_3, then there is no vertical path Q such that (P, Q) induces a 3-1-partition. Any inner path that is adjacent to all three outer vertices is longer than the outer path connecting the same endpoints. By Proposition 3.4, only shortest paths can be vertical paths. Thus, there is no path that satisfies the desired properties. This holds independently of the chosen layering. Therefore, using the approach of the proof for the Strong Product Theorem, the quotient graph $G/\mathcal{P}$ of any partition $\mathcal{P}$ into vertical paths has treewidth at least four.

Now, we show that we can modify the outer vertical paths by extending P_1 to the interior of G. Then, there is a layering ℓ of $V(G)$ and an inner vertical path Q such that (P', Q) induces a 3-1-partition. Let ℓ be a planar BFS-layering such that the outer vertices are in the same layer. Then, the vertical path P_1 that only consists of one vertex can be extended in the interior of G, as seen in Figure 4.12. Each part in the partition induced by (P', Q) is bounded by at most three vertical paths, thus, this is a 3-1-partition.

5 Lower Bounds

In the previous chapter, we have shown that for every planar graph G with a layering that is bounded by $n \leq 5$ vertical paths P, we can find three inner, vertex-disjoint vertical paths Q such that (P, Q) induces a 5-3-partition. We concluded that every planar graph G has a partition $\mathcal{P}$ into vertex-disjoint vertical paths such that $G/\mathcal{P}$ has treewidth at most 7. In this section, we prove by constructing a concrete graph that this bound is tight, i.e. that if every planar graph admits an n-k-partition, then $n + k$ is at least 8.

The intuition behind the construction is that in an n-k-partition induced by (P, Q), where P is fixed, $n + k$ is minimized if Q consists of vertical paths that start at a *transition vertex* or whose endpoints are both outer vertices. A transition vertex is a vertex that is adjacent to two consecutive outer vertical paths. In our constructed graph, there shall not be such vertical paths, independent of the layering.

5.1 Construction of a Tight Example

We construct two graphs G_1 and G_2 by triangulating and modifying a base graph in two steps such that G_2 is bounded by n vertical paths and for any layering and any n-k-partition of G_2, $n + k \geq 8$ holds.

Let $n \geq 2$ be the number of outer vertical paths of G_0 and $P = \{P_1, \ldots, P_n\}$ be n vertical paths of arbitrary length. We define the graph G_0 to be a planar graph that consists of a single cycle. Let ℓ_0 be a layering of $V(G_0)$ such that $V(G_0)$ can be partitioned into exactly n vertical paths $P = \{P_1, \ldots, P_n\}$, where P_i and P_{i+1} for $i \in \{1, \ldots, n-1\}$ and P_n and P_1 are consecutive. Denote the vertices of G_0 as $p_0, \ldots, p_{|V(G_0)|-1}$ counterclockwise along the cycle starting at an arbitrary vertex.

To obtain G_2, we add vertices and edges to the graph G_0 such that G_0 is the outer cycle of G_2 and G_2 is triangulated. The vertices and layering of G_0 are fixed throughout the construction of G_2. We denote the subgraph that is obtained by deleting all outer vertices of a planar graph G as G_{in}.

Now, we obtain the graph G_1 by triangulating the interior of G_0 such that no "longer" vertical path in G_1 that only uses inner vertices connects two vertices that are both in cycles of "lower" layers. Since only shortest paths can be vertical by Proposition 3.4, we achieve this by making the interior of the graph sufficiently large.

Definition 5.1: *Let C_0 be a simple cycle with vertices $\{c_0^0, \ldots, c_{s-1}^0\}$. For a constant $l \in \mathbb{N}_+$, we define a planar, inner-triangulated* net graph *G_s^l as follows:*

- *G_s^l consists of l layered cycles $C_0, \ldots, C_{l-1}$ with s vertices each and one extra center vertex x in layer l. We denote the vertices of cycle C_i as c_j^i for $j \in \{0, \ldots, s-1\}$ and $i \in \{0, \ldots, l-1\}$.*

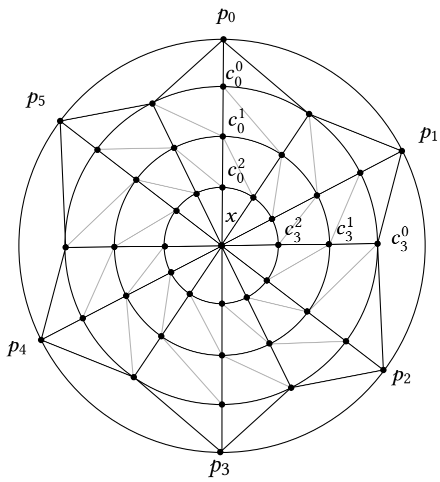


Figure 5.1: Example graph G_1 with $d := 2$ and $l := 3$. An inner triangulation of G_0 with six outer vertices $V(G_0)$, inner layered cycles C_0, C_1, C_2 with twelve vertices each and a center vertex x. The edges that are added for triangulation are colored in gray for better readability.

- *We add the edges between vertices c_j^i and c_j^{i+1} for all $j \in \{0, \ldots, s-1\}$ and $i \in \{0, \ldots, l-2\}$. Each vertex in the innermost cycle C_{l-1} is adjacent to the center vertex x.*
- *To triangulate the graph, we add edges between vertices c_j^i and c_{j+1}^{i+1} for all $j \in \{0, \ldots, s-1\}$ and $i \in \{0, \ldots, l-2\}$ with $c_k^i := c_0^i$.*

For a shortest path q between two vertices $c_{j_1}^{i_1}$ and $c_{j_2}^{i_2}$ in a net graph G_s^l, there are two possibilities:

- Starting at $c_{j_1}^{i_1}$, it goes along the cycle C_{i_1} to $c_{j_2}^{i_1}$ and then follows the path from $c_{j_2}^{i_1}$ to $c_{j_2}^{i_2}$. Note that since every layered cycle has the same number of vertices, the length of the path is the same, independent of which cycle the path follows. Since G_s^l is triangulated, such a path has length at least $\max\{\min\{|j_1 - j_2|, s - |j_1 - j_2|\}, |i_1 - i_2|\}$.
- It is the concatenated path of the shortest path from $c_{j_1}^{i_1}$ to the center x and the shortest path from x to $c_{j_2}^{i_2}$. Then, q has length $2l - |i_1 - i_2|$.

Thus, for the distance between $c_{j_1}^{i_1}$ and $c_{j_2}^{i_2}$, we have the following inequality:

$$\operatorname{dist}(c_{j_1}^{i_1}, c_{j_2}^{i_2}) \geq \min\{\max\{\min\{|j_1 - j_2|, s - |j_1 - j_2|\}, |i_1 - i_2|\}, 2l - |i_1 - i_2|\} .$$

The graph G_1 in Figure 5.1 is constructed as follows for fixed constants $d \geq 2$ and $l \geq 1$:

- It is bounded by the n vertical paths in P.
- Let C_0 be a single cycle that consists of $d \cdot |V(G_0)|$ vertices. Denote the vertices of $V(C_0)$ as $c_0^0, \ldots, c_{|V(C_0)|-1}^0$ counterclockwise along the cycle, starting at an arbitrary vertex.
- Each vertex in $V(G_0)$ has exactly $d+1$ consecutive neighbors in $V(C_0)$. Two adjacent vertices in $V(G_0)$ share exactly one neighbor in $V(C_0)$. Formally, the outer vertex p_i is adjacent to the vertices $\{c_{i \cdot d}^0, \ldots, c_{(i+1) \cdot d}^0\}$ for all $i \in \{0, \ldots, |V(G_0)| - 1\}$ with $c_{|V(C_0)|}^0 := c_0^0$.

- The subgraph induced by C_0 and its interior is a net graph $G^l_{|V(C_0)|}$. We use the same notation for the vertices as in Definition 5.1.

In the following, we prove that G_1 has the desired properties.

Lemma 5.2: *Let $m \in \mathbb{N}_0$ be a constant. There exist constants $d, l \geq 2$ such that the graph G_1 in which all outer vertices have degree $d+1$, that has l layered cycles and which is obtained by triangulating G_0 has the following properties:*

1. *The layering ℓ_0 of G_0 can be extended to a valid layering ℓ_1 of the inner vertices of G_1.*
2. *Let $c_i^{m_1} \in V(C_{m_1})$ and $c_j^{m_2} \in V(C_{m_2})$ with $m_1, m_2 \leq m$ be inner vertices such that the corresponding vertices c_i^0 and c_j^0 in $V(C_0)$ are not adjacent to the same outer vertex or to two adjacent outer vertices. Then, there is no vertical $c_i^{m_1}$-$c_j^{m_2}$-path that consists only of inner vertices of G_1 for any inner layering ℓ_1. Moreover, this inequality holds:*

$$\mathrm{dist}_{G_1}(c_i^{m_1}, c_j^{m_2}) + 1 < \mathrm{dist}_{G_{1in}}(c_i^{m_1}, c_j^{m_2})\,.$$

3. *If x and y are adjacent to two adjacent outer vertices in $V(G_0)$, then any inner vertical x-y-path has length at most three. Moreover, any such path consists only of vertices in $V(C_0)$ and two such paths are not adjacent.*

Proof.

1. We show that the layering ℓ_0 can be extended to a valid layering of the inner vertices of G_1. The distance between two vertices in $V(G_0)$ remains unchanged in G_1 since any shortest path between two vertices in $V(G_0)$ contains only outer vertices in G_1. Thus, there is a layering ℓ_1 such that $\ell_1(v) = \ell_0(v)$ for all $v \in V(G_0)$.
2. Let $c_i^{m_1} \in V(C_{m_1})$ and $c_j^{m_2} \in V(C_{m_2})$ with $m_1, m_2 \leq m$ be inner vertices. Assume that the corresponding vertices c_i^0 and c_j^0 in $V(C_0)$ are not adjacent to the same outer vertical path or two adjacent outer vertical paths. We know that c_i^0 is adjacent to the outer vertex $p_{\lfloor \frac{i}{d} \rfloor}$ and c_j^0 is adjacent to the outer vertex $p_{\lfloor \frac{j}{d} \rfloor}$. We assume without loss of generality that $|i-j| \leq |V(G_0)| - |i-j|$.

   Consider a $c_i^{m_1}$-$c_j^{m_2}$-path $q_{out} := (c_i^{m_1}, \ldots, c_i^0, p_{\lfloor \frac{i}{d} \rfloor}, \ldots, p_{\lfloor \frac{j}{d} \rfloor}, c_j^0, \ldots, c_j^{m_2})$ that uses both inner and outer vertices. This path has length $\left\lfloor \frac{|i-j|}{d} \right\rfloor + m_1 + m_2 + 1$. We compare the length of q_{out} with a shortest $c_i^{m_1}$-$c_j^{m_2}$-path q_{in} that only consists of inner vertices in $V(G_1) \setminus V(G_0)$. Since c_i^0 and c_j^0 are not adjacent to the same outer vertex or two adjacent outer vertices, we know that $|i-j| > d$ holds. Thus, if we choose $d > m$, then $|i-j| > d > m \geq |m_1 - m_2|$ holds. From above observation on the distances in a net graph, we have the following inequality:

$$|q_{in}| \geq \min\{\max\{|i-j|, |m_1 - m_2|\}, 2l - |m_1 - m_2|\} = \min\{|i-j|, 2l - |m_1 - m_2|\}\,.$$

   Since m_1 and m_2 is bounded by m, we can choose d and l sufficiently large such that $|q_{in}| > |q_{out}| + 1$ holds. Thus, we have that q_{out} is shorter than any $c_i^{m_1}$-$c_j^{m_2}$-path that uses only inner vertices. By applying Proposition 3.4, we conclude that no $c_i^{m_1}$-$c_j^{m_2}$-path using only inner vertices is a vertical path. In particular, there is no vertical path that is adjacent to two non-consecutive vertical paths.

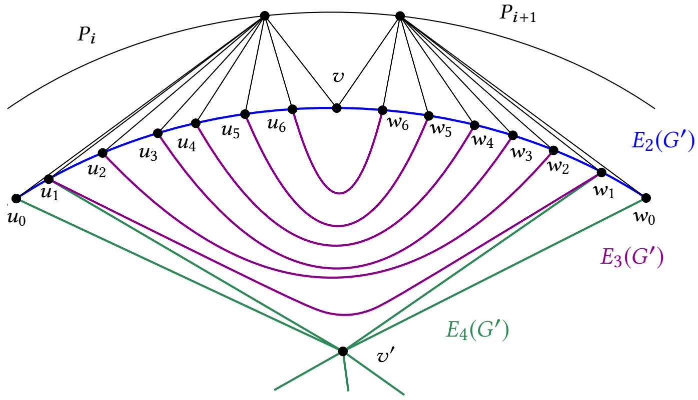


Figure 5.2: The modified graph G' after "hiding" transition vertex v with $c := 1$.

3 If c_i^0 and c_j^0 have outer neighbors that are adjacent in $V(G_0)$, then q_{out} has length three. Thus, if q_{in} is a vertical c_i^0-c_j^0-path that only consists of inner vertices, then it has length at most three. Such a shortest path only contains vertices in $V(C_0)$.

We choose d large enough such that any two transition vertices have a distance of at least 4. Then, there is no inner vertical path in G_1 that contains two transition vertices. In any case, $d \geq 5$ is sufficient to satisfy this condition.

□

We will construct the graph G_2 from G_1 by "hiding" transition vertices such that any path in G_2 from an inner vertex in G_1 to a transition vertex cannot be covered by a single vertical path. We achieve this by applying the following modification to all transition vertices in G_1.

For an inner triangulation G of G_0, a fixed constant $c \in \mathbb{N}_+$ and a transition vertex $v \in V(G)$, we define the modification to obtain G' as in Figure 5.2:

- The vertex set of G' is the union $V(G) \cup \{v'\} \cup U \cup W$ with $U := \{u_1, \ldots, u_{6\cdot c}\}$ and $W := \{w_1, \ldots, w_{6\cdot c}\}$. The new vertex v' corresponds to v in G.
- Let u and w be the two vertices in $V(C_0)$ that are adjacent to the vertex v. The edge set $E(G')$ consists of the following edge sets:
  - The set $E_1(G') := E(G) \setminus \{vx \mid x \in V(G_{in})\}$ which contains all edges in $E(G)$ except for edges between v and other inner vertices. Note that the edges uv and vw are also not in $E_1(G')$.
  - The set $E_2(G') := \{u_j u_{j+1} \mid 0 \leq j \leq 6 \cdot c\} \cup \{w_j w_{j+1} \mid 0 \leq j \leq 6 \cdot c\}$ with $u_{6\cdot c+1} := v =: w_{6\cdot c+1}$, $u_0 := u$ and $w_0 := w$. This has the same effect as subdividing the edges uv and vw.
  - The set $E_3(G') := \{u_j w_j \mid 1 \leq j \leq 6 \cdot c\}$.
  - The set $E_4(G') := \{v'x \mid vx \in E(G_{in})\}$. If uw is not an edge in $E(G)$, then we add the edges $\{v'w_1, v'u_1\}$ to $E_4(G)$.
  - We add all edges $E_5(G')$ that are needed for triangulation.

Lemma 5.3: *Let G be an inner triangulation of G_0 and $v \in V(G)$ a transition vertex with the two neighbors u and w in $V(C_0)$. Let G' be a graph that is modified using $c \in \mathbb{N}_+$ from G with the same vertex and edge set as described above. Then, G' has the following properties:*

1. *$G'[V(G)]$ is a subgraph of G and any layering ℓ of G can be extended to a valid layering of G'.*
2. *Let ℓ' be a modified inner layering of G' and $x \in V(G_{in})$ an inner vertex of G. Then, any v-x-path cannot be covered by fewer than c vertex-disjoint vertical paths.*
3. *For all vertices $x, y \in V(G) \setminus \{v\}$ that are adjacent to outer vertices in $V(G')$, the following holds:*
   a. *Both the distance and the inner distance between x and y remain the same, i.e. $\operatorname{dist}_{G'}(x, y) = \operatorname{dist}_G(x, y)$ and $\operatorname{dist}_{G'_{in}}(x, y) = \operatorname{dist}_{G_{in}}(x, y)$.*
   b. *$\operatorname{dist}_{G'}(w', x) \leq \operatorname{dist}_{G'}(w, x) + 1$ and $\operatorname{dist}_{G_{in}}(w, x) \leq \operatorname{dist}_{G'_{in}}(w', x)$ for the new neighbor w' of w.*
   c. *$\operatorname{dist}_{G'}(u', x) \leq \operatorname{dist}_{G'}(u, x) + 1$ and $\operatorname{dist}_{G_{in}}(u, x) \leq \operatorname{dist}_{G'_{in}}(u', x)$ for the new neighbor u' of u.*

Proof.

1. We prove that any layering ℓ of $V(G)$ can be extended to a valid layering ℓ' of $V(G')$. Since the neighborhood of the newly added vertices is a subset of the neighborhood of v, setting $\ell'(z) := \ell(v)$ for all new vertices $z \in V(G') \setminus V(G)$ gives us a valid layering of $V(G')$.
2. Let $x \in V(G'_{in})$ be an inner vertex of G'. Consider any v-x-path q that only uses inner vertices in G'. Any such path contains at least $6c$ vertices of $U \cup W$. Assume without loss of generality that q contains at least $3c$ vertices in W, which share one common outer neighbor in $V(G_0)$. Since any vertex has at most three neighbors in a vertical path (Proposition 3.5), the path q cannot be covered by fewer than c vertical paths.
3. We show property 3 for vertices $x, y \in V(G) \setminus \{v\}$ that are adjacent to outer vertices.
   a. We show that the distances in G' and in G'_{in} of two vertices $x, y \in V(G) \setminus \{v\}$ that are adjacent to outer vertices in $V(G')$ remains unchanged. A shortest path q between x and y does not contain any new vertex in $U \cup W$. If q contains the vertex v', then we can obtain a path q' by exchanging v' for v in q. Then, q' is a shortest path in G and $|q| = |q'|$.
   b. Since w and w' are adjacent, we have $\operatorname{dist}_{G'}(w', x) \leq \operatorname{dist}_{G'}(w, x) + 1$. We show that $\operatorname{dist}_{G_{in}}(w, x) \leq \operatorname{dist}_{G'_{in}}(w', x)$ holds for all $x \in V(G) \setminus \{v\}$. Let q' be a w'-x-path for some vertex $x \in V(G) \setminus \{v\}$. If q' contains w, then there is a subpath from x to w that is shorter than q'. If not, then q' contains v' or u. By construction, we have $\operatorname{dist}_{G_{in}}(w, x) \leq \operatorname{dist}_{G'_{in}}(w', x)$ for $x \in \{v', u\}$. Thus, there is a w-x-path q with $|q| \leq |q'|$. The same result holds for u' and u.

□

With the above lemma, we can now prove that G_2, which is obtained by applying the above modification to every transition vertex in $V(G_1)$, has the desired properties.

Lemma 5.4: *Let ℓ_2 be an extended layering ℓ_0 of the vertices in G_2. Then, the graph G_2 has the following properties:*

1. *The outer face of G_2 is bounded by P.*
2. *The statement 2 in Lemma 5.2 for G_1 is also true for G_2.*
3. *The statement 2 in Lemma 5.3 is also true for G_2.*

Proof. By Lemma 5.3, G_2 has the properties 1 and 3. From Lemma 5.2, we know that if we choose d and l in the construction of G_1 large enough, then G_1 has property 2. We need to prove that the modifications with which we obtained G_2 from G_1 do not destroy property 2.

Let $x := c_i^{m_1} \in V(C_{m_1})$ and $y := c_j^{m_2} \in V(C_{m_2})$ with $m_1, m_2 \leq m$ be inner vertices of G_2. Assume that the corresponding outer vertices c_i^0 and c_j^0 in $V(C_0)$ are not adjacent to the same outer vertex or to two adjacent outer vertices. By Lemma 5.2 property 2, the inequality $\mathrm{dist}_{G_1}(x, y) + 1 < \mathrm{dist}_{G_{1in}}(x, y)$ holds. We know from Lemma 5.3 property 3 that if x and y are not in $V(G_2) \setminus V(G_1)$, then we still have $\mathrm{dist}_{G_2}(x, y) = \mathrm{dist}_{G_1}(x, y)$ and $\mathrm{dist}_{G_{2in}}(x, y) = \mathrm{dist}_{G_{1in}}(x, y)$.

If x is not a vertex in G_1, then it has been added during the modification for some transition vertex v_x with the neighbors w_x and u_x in $V(C_0) \subset V(G_1)$. Denote the new neighbor of w_x and the new neighbor of u_x after the modification as w'_x and u'_x, respectively. Since x and y are adjacent to two non-consecutive vertical paths in P, any inner x-y-path q contains w'_x or u'_x. Assume without loss of generality that the path q contains the vertex w'_x. Assume that $y \in V(G_1)$. We show that no inner w'_x-y path is a vertical path by proving that it is not a shortest path. By applying Lemmas 5.2 and 5.3, we get the following inequality:

$$
\begin{aligned}
\mathrm{dist}_{G_2}(w'_x, y) &\leq \mathrm{dist}_{G_2}(w_x, y) + 1 && \text{(Lemma 5.3, 3b)}\\
&= \mathrm{dist}_{G_1}(w_x, y) + 1 && \text{(Lemma 5.3, 3a)}\\
&< \mathrm{dist}_{G_{1in}}(w_x, y) && \text{(Lemma 5.2, 2)}\\
&= \mathrm{dist}_{G_{2in}}(w_x, y) && \text{(Lemma 5.3, 3a)}\\
&\leq \mathrm{dist}_{G_{2in}}(w'_x, y) && \text{(Lemma 5.3, 3b)}
\end{aligned}
$$

Therefore, we have $\mathrm{dist}_{G_2}(w'_x, y) < \mathrm{dist}_{G_{2in}}(w'_x, y)$. Since vertical paths are shortest paths (Proposition 3.4), there is no inner w'_x-y path that is a vertical path. As subpaths of vertical paths are also vertical, no inner x-y-path is a vertical path. Thus, G_2 has all desired properties. □

We conclude that the constructed graph G_2 is an inner triangulation of G_0 and that it has the properties we motivated in the beginning of this chapter. In the remaining section, we show that if G_2 admits an n-k-partition, then $n + k \geq 8$.

Theorem 5.5: *There exists an inner-triangulation G of the graph G_0 with a layering ℓ_0 such that it is bounded by n vertical paths $P = \{P_1, \ldots, P_n\}$ with the following properties:*

1. *There is a layering ℓ such that $\ell(v) = \ell_0(v)$ for any $v \in V(G_0)$.*
2. *For any layering ℓ of G with property 1, the following holds: If $Q = \{Q_1, \ldots, Q_k\}$ are vertex-disjoint, inner vertical paths such that (P, Q) induces an n-k-partition, then we have $n + k \geq 8$.*

Proof. We show that G_2 is a graph that satisfies the desired properties. By Lemma 5.4, G_2 has property 1. Let ℓ_2 be such a layering that has property 1. Assume that $Q = \{Q_1, \ldots, Q_k\}$ are vertex-disjoint vertical paths on the inner vertices of G_2 such that (P, Q) induces an n-k-partition. We show that $n + k \geq 8$ holds for the graph G_2. Since $n \geq 2$, we may assume that $k \leq 6$.

From Lemma 5.4, we know that we can choose the constants for the construction of G_2 sufficiently large such that no vertical path in Q starts at a transition vertex. If Q is inclusion-minimal, i.e. (P, Q') does not induce an n-$|Q'|$-partition for any proper subset Q' of Q, then we shall see that no vertical path in Q is adjacent to two distinct outer vertical paths. We already know from Lemma 5.4 that this statement is always true for non-consecutive vertical paths. Thus, it is sufficient to consider inner vertical paths that are adjacent to two consecutive outer vertical paths. Let $Q_l \in Q$ be a vertical path that is adjacent to some vertical paths $P_i, P_{i+1} \in P$. Then, we know from Lemma 5.2 that there is no vertical path $Q_m \in Q$ that is adjacent to Q_l and another outer vertical path P_j with $j \notin \{i, i+1\}$. Thus, we may assume that Q does not contain vertical paths that are adjacent to two distinct outer vertical paths.

We construct the quotient graph $G' := G_2/(P \cup Q)$ which is obtained by contracting all paths in $P \cup Q$. Parts of the n-k-partition induced by (P, Q) and paths in G_2 correspond to faces and vertices in G', respectively. Thus, the number of paths that bound a part in the n-k-partition of G_2 is equal to the number of vertices that are incident to the corresponding face in G'. By construction, G' has exactly n outer and k inner vertices and the outer vertices form a cycle. We denote the set of inner faces in G' that are incident to an outer vertex as F'. In the remaining proof, we show that if every face in F' is incident to at most n vertices, then we have $n + k \geq 8$.

We will count the elements in the set $X := \{(v, f) \mid v \in V(G') \text{ is incident to } f \in F'\}$. Since there is no vertical path in Q that is adjacent to two outer vertical paths, every inner vertex in G' has at most one outer neighbor. Thus, every face in F' is incident to at least two inner vertices. An outer vertex in G' is incident to $i + 1$ faces in F' if it is adjacent to exactly i inner vertices. Thus, we have

$$|X| \geq \sum_{v \in V_{out}} (|\{uv \in E(G') \mid u \in V_{in}\}| + 1) + \sum_{v \in V_{in}} 2 = n + k + 2k = n + 3k\,,$$

where V_{out} and V_{in} denote the outer and inner vertices of G', respectively.

We have $|F'| \leq k$, thus, the average number of vertices that are incident to a face in F' is

$$\frac{|X|}{|F'|} \geq \frac{n + 3k}{k} = \frac{n}{k} + 3\,.$$

If $\frac{n}{k} + 3 > n$, then $\frac{|X|}{|F'|} > n$ and there is one face in F' that is incident to more than n vertices, which is a contradiction to our assumption. Thus, the following inequality holds for G' and $n \geq 4$:

$$\begin{aligned} & \frac{n}{k} + 3 \leq n \\ \iff \quad & \frac{n}{n-3} \leq k \\ \iff \quad & n + k \geq n + \frac{n}{n-3} > 7 \end{aligned}$$

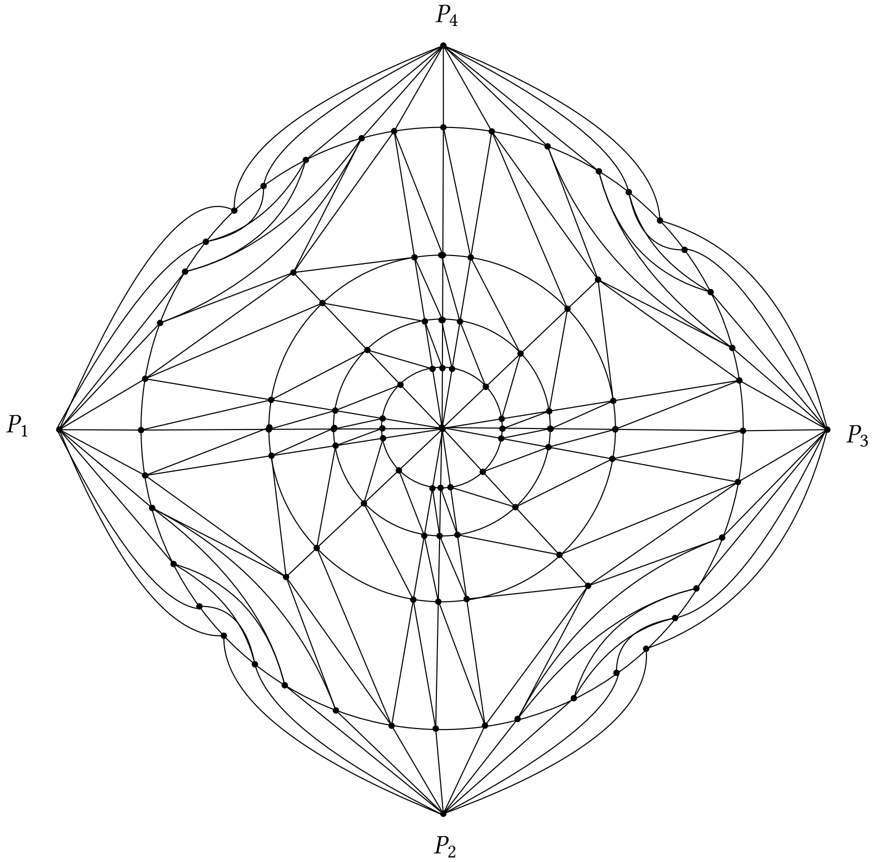


Figure 5.3: Graph G_2 for a cycle G_0 with four outer (singleton) vertical paths $P = \{P_1, P_2, P_3, P_4\}$, $d := 4$ and $l := 4$. For any inner layering ℓ of G_2, there are no vertical paths $Q = \{Q_1, Q_2, Q_3\}$ such that (P, Q) induces a 4-3-partition.

For $n = 2$ and $n = 3$, there is no such set of vertical paths Q.

We conclude that if there is an n-k-partition for G_2, then $n + k$ is at least 8. ▢

Applying Lemma 3.10, this directly implies the following theorem:

Theorem 5.6: *For every $n \geq 3$, there exists a planar graph G such that it has the following properties:*

- *It has a layering ℓ such that it is bounded by n vertical paths $P = \{P_1, \ldots, P_n\}$.*
- *For any extended inner layering and any partition $\mathcal{P}$ of $V(G)$ into vertical paths with $P \subseteq \mathcal{P}$, the treewidth of $G/\mathcal{P}$ is at least* 7.

Figure 5.3 shows an example graph for G_2 with $P = \{P_1, P_2, P_3, P_4\}$, which only consist of one vertex each. The underlying net graph G_1 is constructed with $d := 4$ and four layers. Let ℓ be an inner layering of G_2. Then there is no inner vertical path that is adjacent to two non-consecutive outer vertical paths. Any vertical path that starts at a transition vertex does not contain a vertex of $V(G_1)$. Thus, there is no set of vertical paths Q such that (P, Q) induces a 4-3-partition.

6 Conclusions

In this thesis, we discussed the treewidth bounds given by the Strong Product Theorem. By modifying the proof by Dujmović et al. [Duj+20c], we showed in Section 4.2 that every planar graph G can be partitioned into vertex-disjoint vertical paths $\mathcal{P}$ such that $G/\mathcal{P}$ has treewidth at most 7. It follows directly that every planar graph is a subgraph of the strong product of a graph with treewidth 7 and a path, which improves the upper bound of the Strong Product Theorem by one.

For the proof, we introduced the concept of n-k-partitions and proved in Section 3.2 that every planar graph that is bounded by at most five vertical paths has a 5-3-partition. It is still unknown whether there is another pair (n, k) with $n + k = 8$, for which this statement is true. In Chapter 5, we have seen that there is a graph bounded by three vertical paths that does not have a 3-5-partition. However, it could be possible that every planar graph admits a 4-4-partition or a 6-2-partition. Then, a natural question would be whether such a partition can be obtained efficiently.

In Section 4.3, we proved for 2-outerplanar graphs that they admit a 3-1-partition if we choose a certain layering and a partition into vertical paths of the outer vertices. It could be of further interest, whether similar results can be obtained for other subclasses of planar graphs.

In both the original and the modified proof of the Strong Product Theorem, we assumed that the embedding is fixed and that vertical paths that are added as parts during the partitioning will not be modified later. In Section 5.1, we proved that the bound given by the Strong Product Theorem is tight under these assumptions. However, if we drop these assumptions, we can allow vertical paths of the partition to be extended later, as we briefly discussed in Section 4.4. It is also possible to choose a certain embedding or a certain outer layering in the beginning and then construct the partitioning into vertical paths as in the proof of the Strong Product Theorem. By applying and possibly combining these strategies, we might obtain a better upper bound. It has been proven that there is a planar graph G such that for any partition $\mathcal{P}$ into vertex-disjoint vertical paths, the treewidth of $G/\mathcal{P}$ is at least three [Duj+20c]. For any other treewidth $t \in \{4, 5, 6, 7\}$ it is not known whether such a graph exists.

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