Coloring the Union of Geometric Hypergraphs

Master Thesis of

Vera Chekan

At the Department of Informatics
Institute of Theoretical Informatics

Reviewers: PD Dr. Torsten Ueckerdt
Jun.-Prof. Dr. Thomas Bläsius

Advisors: PD Dr. Torsten Ueckerdt

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Statement of Authorship

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Abstract

For a range family $\mathcal{R}$ (e.g., all axis-aligned rectangles in the plane) we consider a geometric hypergraph *captured* by $\mathcal{R}$, i.e., its vertex set is a finite set of points in the plane and its hyperedges are such subsets of these points that are contained in some range in $\mathcal{R}$. In this thesis, we investigate the existence of polychromatic vertex-colorings for unions of geometric hypergraphs: in such a coloring every hyperedge contains a vertex of each color. For a range family $\mathcal{R}$ and number of colors k, the value $m_{\mathcal{R}}(k)$ is the smallest number such that every $m_{\mathcal{R}}(k)$-uniform hypergraph captured by $\mathcal{R}$ admits a polychromatic coloring with k colors. We concentrate on range families consisting of various unbounded axis-aligned rectangles and diagonal strips of slope -1: for each subfamily we either provide an upper bound on $m_{\mathcal{R}}(k)$ for all k or prove that $m_{\mathcal{R}}(2) = \infty$ holds. In particular, we show that for the range family of all bottomless rectangles and horizontal strips we have $m_{\mathcal{R}}(2) = \infty$. This strengthens the result of Chen et al. for general axis-aligned rectangles [CPST09].

Deutsche Zusammenfassung

Für eine Familie $\mathcal{R}$ von Bereichen (z.B., alle achsenparallelen Rechtecke in der Ebene) betrachten wir einen geometrischen Hypergraphen, der durch $\mathcal{R}$ *erfasst* wird: die Knotenmenge ist eine endliche Menge von Punkten in der Ebene und eine Teilmenge von Knoten bildet eine Hyperkante genau dann, wenn diese Punkte in einem Bereich aus $\mathcal{R}$ enthalten sind. In dieser Arbeit erforschen wir die polychromatischen Knotenfärbungen von Vereinigungen von geometrischen Hypergraphen: in so einer Färbung enthält jede Hyperkante einen Knoten in jeder Farbe. Für eine Familie $\mathcal{R}$ von Bereichen und eine Anzahl k von Farben ist $m_{\mathcal{R}}(k)$ die kleinste Zahl so, dass für jeden $m_{\mathcal{R}}(k)$-uniformen Hypergraphen, der durch $\mathcal{R}$ erfasst wird, eine polychromatische Färbung mit k Farben existiert. Wir konzentrieren uns auf Familien von Bereichen, die aus diversen unbeschränkten achsenparallelen Rechtecken und diagonalen Streifen mit Steigung -1 bestehen: für jede solche Teilfamilie geben wir entweder eine obere Schranke für $m_{\mathcal{R}}(k)$ für alle k an, oder wir beweisen, dass $m_{\mathcal{R}}(2) = \infty$ gilt. Unter anderem zeigen wir, dass für bodenlose Rechtecke und horizontale Streifen $m_{\mathcal{R}}(2) = \infty$ gilt. Diese Aussage verstärkt das Ergebnis von Chen et al. für allgemeine achsenparallele Rechtecke in [CPST09].

Contents

1. Introduction

A *range-capturing hypergraph* $\mathcal{H}(V, \mathcal{R})$ is defined by a finite point set $V \subset \mathbb{R}^2$ and a family of geometric *ranges* $\mathcal{R}$. A *range* is a subset of $\mathbb{R}^2$. Possible range families are for example all unit disks or all axis-aligned rectangles. The hypergraph $\mathcal{H}(V, \mathcal{R})$ has vertex set V and a subset $E \subseteq V$ is a hyperedge if it is *captured* by some range in $\mathcal{R}$, i.e., there is $R \in \mathcal{R}$ such that

$$R \cap V = E.$$

For $m \in \mathbb{N}$, the subhypergraph $\mathcal{H}(V, \mathcal{R}, m)$ of $\mathcal{H}(V, \mathcal{R})$ has the same vertex set and consists of all hyperedges of size m. We will refer to such hypergraphs as *primal*.

Similarly to graphs, a coloring of a hypergraph is called *proper* if every hyperedge (with at least two vertices) contains vertices in at least two different colors, i.e., no hyperedge is *monochromatic*.

Proper colorings of hypergraphs have been widely studied in the literature. The main question in this setting is the following:

Question 1.1. *Let $\mathcal{R}$ be a range family. What is the smallest number of colors $\chi = \chi_{\mathcal{R}}$ such that for sufficiently large m, every hypergraph $\mathcal{H}(V, \mathcal{R}, m)$ admits a proper coloring with χ colors?*

The following results are known:

1. For translates of a convex polygon: $\chi = 2$ [Pac86].
2. For translates of a non-convex polygon: $3 \leq \chi < \infty$ [Pál10].
3. For unit disks: $\chi = 3$ [DP21].
4. For arbitrary disks: $\chi = 4$ [DD20].
5. For homothets of a convex polygon: $2 \leq \chi \leq 3$ [KP19b].
6. For axis-aligned rectangles: $\chi = \infty$ [CPST09].

For a complete and up-to-date overview of existing results on colorings of geometric hypergraphs we refer to the excellent website [Zoo] which contains numerous references.

One generalization of proper colorings is polychromatic colorings: a coloring of a hypergraph with k colors is *polychromatic* if every hyperedge contains all k colors. The main question in the context of polychromatic colorings is the following:

Question 1.2. *Given a range family $\mathcal{R}$ and the number of colors $k \in \mathbb{N}$, what is the smallest uniformity $m = m(k)$ such that every hypergraph $\mathcal{H}(V, \mathcal{R}, m)$ admits a polychromatic coloring with k colors?*

Observe that if all hyperedges of a hypergraph have a size at least 2, then a coloring with 2 colors is proper if and only if it is polychromatic so:

$$\chi_{\mathcal{R}} = 2 \Leftrightarrow m(2) < \infty.$$

Both lower and upper bounds on $m(k)$ are studied. First, we want to know whether $m(k) < \infty$ holds. Note that $m(k) \leq m(k+1)$ for every $k \in \mathbb{N}$: given a polychromatic coloring with $k+1$ colors, we can recolor every point of color $k + 1$ with color, say, k so that now we use only k colors but every hyperedge still contains all k colors. For this reason, if $m(2) = \infty$ holds, then it is also true for any larger number of colors. Next, if $m(k)$ is finite, what is its asymptotic behavior: is it linear or at least polynomial in k? This problem has been widely studied, it has been shown that:

1. For vertical strips: $m(k) = k$ [ACC+11].
2. More generally, for axis-aligned strips in $\mathbb{R}^d$ [ACC+11]:
$$2\lceil \frac{(2d-1)k}{2d} \rceil - 1 \leq m(k) \leq k(4 \ln k + \ln d).$$
3. For bottomless rectangles (i.e., axis-aligned rectangles whose bottom side lies at $-\infty$): $1.67k \leq m(k) \leq 3k - 2$ [ACC+13].
4. For half-planes: $m(k) = 2k - 1$ [SY12].
5. For unit disks: $m(2) = \infty$ [PP16].
6. For axis-aligned rectangles: $m(2) = \infty$ [CPST09].
7. For translates of a convex polygon: $m(k) \in \mathcal{O}(k)$ [GV11].
8. For translates of a non-convex polygon: $m(2) = \infty$ [Pál10].
9. For homothets of a triangle: $m(k) \in \mathcal{O}(k^{4.53})$ [KP14].
10. For axis-aligned squares: $m(k) \in \mathcal{O}(k^{8.75})$ [AKV17].
11. For translates of the negative octant in $\mathbb{R}^3$: $m(k) \in \mathcal{O}(k^{5.09})$ [CKMU14].

Interestingly, for all range families studied until now, it either has been shown $m(k) < \infty$ for every $k \in \mathbb{N}$ or $m(2) = \infty$. In our results, this property will also be satisfied. However, no general explanation is known for this behavior.

One of the results of this work is the proof of $m(2) = \infty$ for the family of all bottomless rectangles and horizontal strips. This strengthens the result of Chen et al. for general axis-aligned rectangles [CPST09]. Moreover, we explicitly provide the desired point sets in contrast to their randomized proof.

All of the previously known results we mention are for the range families that consist of ranges of one type. In this work, we are interested in polychromatic colorings for a union of several range families. For example, what can we say about $m(k)$ for the range family of all bottomless and topless rectangles? We mostly concentrate on axis-aligned rectangles which are unbounded in one or two directions (i.e., quadrants, strips, and bottomless rectangles) and diagonal strips of slope -1. For two pairs of these ranges, namely south-west quadrants and diagonal strips as well as bottomless rectangles and horizontal strips we show $m(2) = \infty$.

For the remaining combinations of ranges (in which these pairs do not occur), we provide polychromatic colorings. For example, we show that for the range family of all axis-aligned quadrants and all axis-aligned strips we have $m(k) \leq 10k - 1$.

For a range family $\mathcal{R}$, there is another way to define a hypergraph. The vertex set is a finite subset $\mathcal{R}' \subseteq \mathcal{R}$ and for every point in the plane, there is a hyperedge consisting of all ranges containing (i.e., *covering*) this point. We denote this hypergraph with $\mathcal{H}^*(\mathcal{R}')$ and for $m \in \mathbb{N}$ we denote the subhypergraph containing all hyperedges of size at least m with $\mathcal{H}^*(\mathcal{R}', m)$. By contrast to primal hypergraphs, these hypergraphs are referred to as *dual*. Polychromatic colorings of dual hypergraphs are also known as *cover-decompositions* since every color class covers every point (which is covered sufficiently often). In simple words, we want to color the ranges in such a way that every point covered by at least m ranges is covered by ranges of all colors. In this setting, the question analogous to Question 1.2 is the following:

Question 1.3. *Given a range family $\mathcal{R}$ and the number of colors $k \in \mathbb{N}$, what is the smallest uniformity $m^* = m^*(k)$ such that for every finite subfamily $\mathcal{R}' \subseteq \mathcal{R}$, the hypergraph $\mathcal{H}(\mathcal{R}', m^*)$ admits a polychromatic coloring with k colors?*

The following results are known:

1. For half-planes: $m^*(k) \leq 3k - 2$ [SY12].
2. For axis-aligned strips in $\mathbb{R}^d$:
$$\lfloor k/2 \rfloor d + 1 \leq m^*(k) \leq d(k-1) + 1$$
[ACC$^+$11].
3. For translates of a convex polygon: $m^*(k) \in \mathcal{O}(k)$ [GV09].
4. For homothets of a triangle: $m^*(k) \in \mathcal{O}(k^{5.09})$ [CKMU14].
5. For translates of the negative octant in $\mathbb{R}^3$: $m^*(k) \in \mathcal{O}(k^{5.09})$ [CKMU14].
6. For unit disks: $m^*(2) = \infty$ [PP16].
7. For axis-aligned rectangles: $m^*(2) = \infty$ [PT10].
8. For homothets of a convex polygon with at least four sides: $m^*(2) = \infty$ [Kov15].

The study of this problem was initiated before the primal setting because it finds its application in sensor networks. Think of a range as a sensor that monitors a set of points and suppose every point is covered by at least $m^*(k)$ ranges. Every sensor can be turned on for one time unit. The existence of a cover decomposition into k classes means that there is a way to turn on every sensor at some time point in $\{1, 2, \ldots, k\}$ so that every point is monitored at all of k time units. This is closely related to the frequency assignment problem in cellular telephone networks, more details can be found in [ELRS03]. The notion of cover-decomposability is also closely related to ε-nets [Alo10] and conflict-free colorings [ELRS03, HPS05, Smo07]. In this work, we will take a look at cover decompositions for a union of two range families and show that if $m^*(k) < \infty$ holds for both range families, then this also holds for their union.

Another way to generalize proper colorings is to consider *strong* colorings: in such a coloring, no hyperedge contains two vertices in the same color. The analogous question for this notion is the following:

Question 1.4. *Let $\mathcal{R}$ be a range family and $m \in \mathbb{N}$ the uniformity, what is the smallest number $k = k(m)$ of colors such that every hypergraph $\mathcal{H}(V, \mathcal{R}, m)$ admits a strong coloring with k colors?*

Strong colorings were introduced by Agnarsson and Halldórsson [AH05]. It is easy to observe that a strong coloring of a hypergraph corresponds to a proper coloring of a graph on the same vertex set in which there is an edge for each pair of vertices that occur together in some hyperedge. So this problem is rather a graph coloring problem and it has not been studied much for geometric hypergraphs. Asinowski et al. [ACC$^+$13] have shown that for bottomless rectangles, $k(m) = 2m - 1$ holds. The result of Chen et al. [CPST09] implies that for axis-aligned rectangles, we have $k(2) = \infty$. We will show that if two hypergraphs admit strong colorings with k_1 and k_2 colors, respectively, then their union admits a strong coloring with $k_1 \cdot k_2$ colors. This implies that if for two range families $k(m) < \infty$ holds, then this holds for their union too. This complements the result for polychromatic colorings where, as we will see in this work, an analogous implication does not hold in general.

This thesis is structured as follows. First, in Chapter 2, we introduce the terminology and relevant results and make the first simple observations. In Chapter 3, we present an approach that is helpful to obtain a polychromatic coloring of the union of two hypergraphs. Unfortunately, we will also show that it can not be applied even in the simple case of the range family of all south-west quadrants. Next, we concentrate on positive results, i.e., exhibiting range families with $m(k) < \infty$ for all $k \in \mathbb{N}$. In Chapter 4, we explicitly construct polychromatic colorings for some subfamilies of axis-aligned quadrants. And in Chapter 5, we first show that axis-aligned quadrants admit the so-called shallow hitting sets with some nice properties and then employ these properties to prove the existence of polychromatic colorings for many range families. After that, in Chapter 6, we show that the coloring of bottomless and topless rectangles can be reduced to the coloring of axis-aligned squares. Then we move to negative results: in Chapter 7, we show that for the family of all south-west quadrants and diagonal strips as well as for the range family of all bottomless rectangles and horizontal strips, we have $m(2) = \infty$. Next, in Chapter 8, we consider strong colorings of unions of two hypergraphs as well as polychromatic colorings of dual hypergraphs for two range families. Finally, in Chapter 9, we summarize the results and pose several open questions.

2. Preliminaries

2.1 Main Definitions

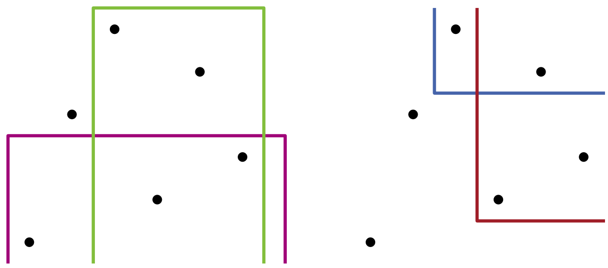
A *range* $R \subset \mathbb{R}^2$ is a set of points in the plane. A *range family* $\mathcal{R}$ is a set of ranges. These are the range families we will consider in this work:

- *axis-aligned rectangles* $\mathcal{R}_{\mathrm{R}} = \{\{(x,y) \in \mathbb{R}^2 \mid a \leq x \leq b, c \leq y \leq d\} \mid a,b,c,d \in \mathbb{R}\}$
- *axis-aligned squares* $\mathcal{R}_{\mathrm{SQ}} = \{\{(x,y) \in \mathbb{R}^2 \mid a \leq x \leq a+b, c \leq y \leq c+b\} \mid a,b,c \in \mathbb{R}\}$
- *bottomless rectangles* $\mathcal{R}_{\mathrm{BL}} = \{\{(x,y) \in \mathbb{R}^2 \mid a \leq x \leq b, y \leq c\} \mid a,b,c \in \mathbb{R}\}$
- *topless rectangles* $\mathcal{R}_{\mathrm{TL}} = \{\{(x,y) \in \mathbb{R}^2 \mid a \leq x \leq b, c \leq y\} \mid a,b,c \in \mathbb{R}\}$
- *horizontal strips* $\mathcal{R}_{\mathrm{HS}} = \{\{(x,y) \in \mathbb{R}^2 \mid a \leq y \leq b\} \mid a,b \in \mathbb{R}\}$
- *vertical strips* $\mathcal{R}_{\mathrm{VS}} = \{\{(x,y) \in \mathbb{R}^2 \mid a \leq x \leq b\} \mid a,b \in \mathbb{R}\}$
- *axis-aligned strips* $\mathcal{R}_{\mathrm{AS}} = \mathcal{R}_{\mathrm{HS}} \cup \mathcal{R}_{\mathrm{VS}}$
- *diagonal strips* (of slope -1) $\mathcal{R}_{\mathrm{DS}} = \{\{(x,y) \in \mathbb{R}^2 \mid a \leq x+y \leq b\} \mid a,b \in \mathbb{R}\}$
- *strips* $\mathcal{R}_{\mathrm{S}} = \mathcal{R}_{\mathrm{AS}} \cup \mathcal{R}_{\mathrm{DS}}$
- *north-west quadrants* $\mathcal{R}_{\mathrm{NW}} = \{\{(x,y) \in \mathbb{R}^2 \mid x \leq a, b \leq y\} \mid a,b \in \mathbb{R}\}$
- *north-east quadrants* $\mathcal{R}_{\mathrm{NE}} = \{\{(x,y) \in \mathbb{R}^2 \mid a \leq x, b \leq y\} \mid a,b \in \mathbb{R}\}$
- *south-west quadrants* $\mathcal{R}_{\mathrm{SW}} = \{\{(x,y) \in \mathbb{R}^2 \mid x \leq a, y \leq b\} \mid a,b \in \mathbb{R}\}$
- *south-east quadrants* $\mathcal{R}_{\mathrm{SE}} = \{\{(x,y) \in \mathbb{R}^2 \mid a \leq x, y \leq b\} \mid a,b \in \mathbb{R}\}$
- *axis-aligned quadrants* $\mathcal{R}_{\mathrm{Q}} = \mathcal{R}_{\mathrm{NW}} \cup \mathcal{R}_{\mathrm{NE}} \cup \mathcal{R}_{\mathrm{SW}} \cup \mathcal{R}_{\mathrm{SE}}$
- *half-planes* $\mathcal{R}_{\mathrm{HP}} = \{\{(x,y) \in \mathbb{R}^2 \mid ax+by \geq 1\} \mid a,b \in \mathbb{R}\}$

Note: we emphasize that the range family $\mathcal{R}_{\mathrm{S}}$ of strips only contains horizontal and vertical strips and strips of slope -1.

A *point set* V is a **finite** subset of $\mathbb{R}^2$. We denote the number of elements in V with $n = |V|$. Let V be a point set and let $\mathcal{R}$ be a range family. A *range-capturing hypergraph* $\mathcal{H}(V,\mathcal{R})$ has vertex set V and a subset $E \subseteq V$ forms a hyperedge if there is a range $R \in \mathcal{R}$ *capturing* E, that is,

$$V \cap R = E$$



(a) (b)

Figure 2.1: We exemplify some hyperedges captured by (a) bottomless rectangles and (b) north-east quadrants.

(see Figure 2.1 for an example). Further, for $m \in \mathbb{N}$, the subhypergraph $\mathcal{H}(V, \mathcal{R}, m)$ of $\mathcal{H}(V, \mathcal{R})$ has the same vertex set V and contains exactly the hyperedges of $\mathcal{H}(V, \mathcal{R})$ of size m. For a fixed range family $\mathcal{R}$, with *all hypergraphs* $\mathcal{H}(V, \mathcal{R}, m)$ we mean the following family of hypergraphs:

$$\{\mathcal{H}(V, \mathcal{R}, m) \mid V \text{ is a point set}, m \in \mathbb{N}\}$$

Similarly, writing about an *arbitrary* hypergraph $\mathcal{H}(V, \mathcal{R}, m)$ we mean that a point set V and $m \in \mathbb{N}$ are arbitrary.

For a point $v \in \mathbb{R}^2$, we denote its x-coordinate (respectively y-coordinate) with $x(v)$ (respectively $y(v)$). We say that the point set V is *in general position* if the points in V have pairwise distinct x- and y-coordinates:

$$\forall u \neq v \in V : x(u) \neq x(v) \wedge y(u) \neq y(v)$$

From now on, we always assume that a point set $V \subset \mathbb{R}^2$ is in general position.

Lemma 2.1. *Let V be a point set. And let $S \subseteq V$ be a subset of points captured by a north-west / south-west / north-east / south-east quadrant. Then S is also captured by a topless / bottomless / topless / bottomless rectangle.*

In particular, for every $m \in \mathbb{N}$, the hypergraph

$$\mathcal{H}(V, \mathcal{R}_{NW}, m) \;/\; \mathcal{H}(V, \mathcal{R}_{SW}, m) \;/\; \mathcal{H}(V, \mathcal{R}_{NE}, m) \;/\; \mathcal{H}(V, \mathcal{R}_{SE}, m)$$

is a subhypergraph of

$$\mathcal{H}(V, \mathcal{R}_{TL}, m) \;/\; \mathcal{H}(V, \mathcal{R}_{BL}, m) \;/\; \mathcal{H}(V, \mathcal{R}_{TL}, m) \;/\; \mathcal{H}(V, \mathcal{R}_{BL}, m).$$

Proof. We prove the statement for the case that S is captured by a north-west quadrant, the rest is then symmetrical. So let

$$R = \{(x, y) \in \mathbb{R}^2 \mid x \leq a, b \leq y\} \in \mathcal{R}_{\mathrm{NW}}$$

for $a, b \in \mathbb{R}$ be a north-west quadrant capturing S, i.e.:

$$R \cap V = S.$$

Let

$$c = \min_{v \in V} x(v).$$

We claim that the topless rectangle

$$R' = \{(x, y) \in \mathbb{R}^2 \mid c \leq x \leq a, b \leq y\} \in \mathcal{R}_{\mathrm{TL}}$$

captures S too. First, observe that $R' \subset R$ so:

$$R' \cap V \subseteq S.$$

Further, for every $p \in S$, we have:

$$x(p) \leq a, y(p) \geq b$$

due to $p \in R$ and

$$x(p) \geq c$$

by the choice of c. So $p \in R'$ and hence, $S \subset R'$. Thus, $R' \cap V = S$ holds as desired. $\square$

For simplicity, for vertices $a_1, \ldots, a_k$ of some hypergraph we will sometimes write $a_1 \ldots a_k$ instead of $\{a_1, \ldots, a_k\}$. Similarly, for $k \in \mathbb{N}$, we denote the set $\{1, \ldots, k\}$ with $[k]$. A *coloring* of a hypergraph $\mathcal{H} = (V, \mathcal{E})$ with k colors is a mapping $c : V \to [k]$. We also say that c is a *k-coloring*. The coloring c is called:

- *proper* if every hyperedge of size at least 2 contains vertices in at least two colors, i.e.:

$$\forall E \in \mathcal{E} : |E| > 1 \Rightarrow |c(E)| \geq 2,$$

- *polychromatic* if every hyperedge contains vertices in all colors (in this case we say that a hyperedge is *polychromatic* in c), i.e.:

$$\forall E \in \mathcal{E} : c(E) = [k],$$

- *strong* if there is no hyperedge containing two vertices in the same color, i.e.:

$$\forall E \in \mathcal{E}\ \forall u \neq v \in E : c(u) \neq c(v).$$

If for a hyperedge $E \in \mathcal{E}$ it holds that $|c(E)| = 1$, then we say that E is *monochromatic* in c.

Let $\mathcal{R}$ be a range family and let $k \in \mathbb{N}$ be the number of colors. The number $m_{\mathcal{R}}(k)$ is the smallest uniformity such that for every point set $V \subset \mathbb{R}^2$, and every $m \geq m_{\mathcal{R}}(k)$, the m-uniform hypergraph $\mathcal{H}(V, \mathcal{R}, m)$ admits a polychromatic coloring with k colors. If such $m_{\mathcal{R}}(k)$ does not exist (i.e., for every $n \in \mathbb{N}$, there exists a point set V_n and $n' \geq n$ such that $\mathcal{H}(V_n, \mathcal{R}, n')$ admits no polychromatic coloring with k colors), we write $m_{\mathcal{R}}(k) = \infty$. Otherwise, we write $m_{\mathcal{R}}(k) < \infty$. If the range family $\mathcal{R}$ is clear from the context, we simply write $m(k)$ instead of $m_{\mathcal{R}}(k)$.

First, note that if a hypergraph $(V, \mathcal{E})$ admits a polychromatic $(k+1)$-coloring $c : V \to [k+1]$, then it also admits a polychromatic k-coloring $c' : V \to [k]$ defined as follows:

$$c'(v) = \begin{cases} c(v) & \text{if } c(v) \neq k+1 \\ k & \text{if } c(v) = k+1 \end{cases}.$$

Since every hyperedge contains all $k+1$ colors in c, in particular, it contains all colors in $[k]$ in c and hence in c' too. So it holds that

$$m(k) \leq m(k+1) \text{ for every } k \in \mathbb{N}.$$

Let again $\mathcal{R}$ be a range family and let $m \in \mathbb{N}$ be the uniformity. The number $k_{\mathcal{R}}(m)$ is the smallest number of colors such that for every point set $V \subset \mathbb{R}^2$, the m-uniform hypergraph $\mathcal{H}(V, \mathcal{R}, m)$ admits a strong coloring with $k_{\mathcal{R}}(m)$ colors. If such $k_{\mathcal{R}}(m)$ does not exist (i.e., for every $n \in \mathbb{N}$, there exists a point set V_n such that $\mathcal{H}(V_n, \mathcal{R}, m)$ admits no strong coloring with n colors), we write $k_{\mathcal{R}}(m) = \infty$. Otherwise, we write $k_{\mathcal{R}}(m) < \infty$. If the range family $\mathcal{R}$ is clear from the context, we simply write $k(m)$ instead of $k_{\mathcal{R}}(m)$.

2.2 Vertical Strips and Wedges

Consider a strip $R = \{(x, y) \mid l \leq x \leq r\} \in \mathcal{R}_{\mathrm{VS}}$ for $l, r \in \mathbb{R}$. For a point $p \in \mathbb{R}^2$ we have:

$$p \in R \Leftrightarrow l \leq x(p) \leq r.$$

So if we order the elements of a point set V in the order of increasing x-coordinates, then R contains consecutive points with respect to this ordering.

Observation 2.2. *Let V be a point set and let $m \in \mathbb{N}$. Let $v_1, \ldots, v_n$ be the elements of V in the order of increasing x-coordinates. Then the hyperedges of the hypergraph $\mathcal{H}(V, \mathcal{R}_{VS}, m) = (V, \mathcal{E})$ are:*

$$\mathcal{E} = \{v_i v_{i+1} \ldots v_{i+m-1} \mid i \leq n - m + 1\}.$$

The analogous statement holds for horizontal strips.

Let $O \in \mathbb{R}^2$ be a fixed point called the *center*. We assume that for a point set V it holds $O \notin V$. Let the *x-ray* be the ray starting at O and pointing horizontally to the right. For a point $v \neq O$, let $\mathrm{ray}(v)$ denote the ray starting at O and passing through v and let $\alpha(v)$ denote the angle from the x-ray to $\mathrm{ray}(v)$ moving in the counterclockwise direction. For a point set V we also **assume** that:

$$u \neq v \in V \Rightarrow \alpha(u) \neq \alpha(v).$$

Let $\beta, \gamma \in [0; 360°)$ be two angles. We define the *angular interval* from β to γ as:

$$\langle \beta; \gamma \rangle = \begin{cases} [\beta; \gamma] & \text{if } \beta \leq \gamma \\ [\beta; 360°) \cup [0; \gamma] & \text{otherwise} \end{cases}$$

A *wedge* from β to γ is the set of points whose $\alpha(\cdot)$-value is in the angular interval from β to γ. Then the range family of wedges is defined as:

$$\mathcal{R}_{\mathrm{W}} = \{\{t \in \mathbb{R}^2 \mid \alpha(t) \in \langle \beta; \gamma \rangle\} \mid \beta, \gamma \in [0; 360°)\}.$$

We emphasize that all of the wedges in $\mathcal{R}_{\mathbf{W}}$ have the same center O.

Observation 2.3. *Let V be a point set, let O be the fixed center, and let $m \in \mathbb{N}$. Let $v_0, \ldots, v_{n-1}$ be the elements of V in the order of increasing $\alpha(\cdot)$-values. Then the hyperedges of the hypergraph $\mathcal{H}(V, \mathcal{R}_W, m) = (V, \mathcal{E})$ are:*

$$\mathcal{E} = \{v_i v_{i+1} \ldots v_{i+m-1} \mid i \in \{0, \ldots, n-1\}\}.$$

with indices modulo n.

2.3 Good Range Families and Shallow Hitting Sets

A range family $\mathcal{R}$ is *good* if for every point set V, every subset $U \subset V$, every range $R \in \mathcal{R}$, and every $k \leq |U| \in \mathbb{N}$ the following holds. If R captures U, i.e.,

$$R \cap V = U,$$

then there exists a range $R' \in \mathcal{R}$ in this family such that $R' \subseteq R$ and $|R' \cap V| = k$. Informally speaking, R' is a subrange of R capturing exactly k points. Notice that

$$R' \cap V \subseteq R \cap V = U.$$

The following lemma will be helpful later:

Lemma 2.4. *Range families $\mathcal{R}_{NW}$, $\mathcal{R}_{NE}$, $\mathcal{R}_{SE}$, $\mathcal{R}_{SW}$, $\mathcal{R}_{BL}$, $\mathcal{R}_{TL}$, $\mathcal{R}_{W}$, $\mathcal{R}_{VS}$, $\mathcal{R}_{HS}$, and $\mathcal{R}_{DS}$ are good.*

Proof.

- First, we show that $\mathcal{R}_{\mathrm{NW}}$ is good. Let $V \subset \mathbb{R}^2$ be a point set. Let

$$R = \{(x, y) \mid x \leq a, y \geq b\} \in \mathcal{R}_{\mathrm{NW}}$$

be a north-west quadrant for $a, b \in \mathbb{R}$, let $U = V \cap R$, and let $k \leq |U| \in \mathbb{N}$. Let $u_1, \ldots, u_{|U|}$ be the elements of U in the order of increasing x-coordinates. We claim that the north-west quadrant

$$R' = \{(x, y) \mid x \leq x(u_k), y \geq b\} \in \mathcal{R}_{\mathrm{NW}}$$

is the desired range. First, since $u_k \in R$, we have

$$x(u_k) \leq a$$

so R' is a subset of R. Second:

1. $y(u_1), \ldots, y(u_k) \geq b$ (since $u_1, \ldots, u_k \in R$)
2. For $1 \leq i \leq k$: $x(u_i) \leq x(u_k)$
3. For $i > k$: $x(u_i) > x(u_k)$

So $u_1, \ldots, u_k \in R'$, $u_{k+1}, \ldots, u_{|U|} \notin R'$, and $R' \cap V = \{u_1, \ldots, u_k\}$. Hence, R' satisfies the desired properties and $\mathcal{R}_{\mathrm{NW}}$ is good. For symmetry reasons, $\mathcal{R}_{\mathrm{NE}}$, $\mathcal{R}_{\mathrm{SE}}$, and $\mathcal{R}_{\mathrm{SW}}$ are good too.

- Now in a very similar way, we show that $\mathcal{R}_{\mathrm{BL}}$ is good as well. Let V be a point set. Let

$$R = \{(x, y) \mid a \leq x \leq b, y \leq c\} \in \mathcal{R}_{\mathrm{BL}}$$

be a bottomless rectangle for $a, b, c \in \mathbb{R}$, let $U = V \cap R$, and let $k \leq |U| \in \mathbb{N}$. Let $u_1, \ldots, u_{|U|}$ be the elements of U in the order of increasing x-coordinates. We claim that the bottomless rectangle

$$R' = \{(x, y) \mid a \leq x \leq x(u_k), y \leq c\} \in \mathcal{R}_{\mathrm{BL}}$$

is the desired range. First, since $u_k \in R$, we have $x(u_k) \leq b$ so R' is a subset of R. Second:

1. $y(u_1), \ldots, y(u_k) \leq c$, $x(u_1), \ldots, x(u_k) \geq a$ (since $u_1, \ldots, u_k \in R$)
2. For $1 \leq i \leq k$: $x(u_i) \leq x(u_k)$
3. For $i > k$: $x(u_i) > x(u_k)$

So $u_1, \ldots, u_k \in R'$, $u_{k+1}, \ldots, u_{|U|} \notin R'$ and $R' \cap V = \{u_1, \ldots, u_k\}$. Hence, R' satisfies the desired properties and $\mathcal{R}_{\mathrm{BL}}$ is good. For symmetry reasons, $\mathcal{R}_{\mathrm{TL}}$ is good too.

- Let O be a fixed center, we show that $\mathcal{R}_{\mathrm{W}}$ is good too. Let V be a point set. Let

$$R = \{t \in \mathbb{R}^2 \mid \alpha(t) \in \langle \beta; \gamma \rangle\} \in \mathcal{R}_{\mathrm{W}}$$

be a wedge for $\beta, \gamma \in [0°; 360°)$, let $U = R \cap V$, and let $k \leq |U| \in \mathbb{N}$. For an angle $\phi \in [0°; 360°)$, let r_ϕ be the ray which starts in O and points in the direction of a point v with $\alpha(v) = \phi$. Let $u_1, \ldots, u_{|U|}$ be the order of points in U in which they are hit by the following ray: we start at the ray r_β and rotate it around O in the counterclockwise direction until we reach r_γ. Note that this way the ray gets exactly the $\alpha(\cdot)$-values from $\langle\beta;\gamma\rangle$. We claim that the wedge:

$$R' = \{t \in \mathbb{R}^2 \mid \alpha(t) \in \langle\beta; \alpha(u_k)\rangle\} \in \mathcal{R}_{\mathrm{W}}$$

is the desired wedge. First, since $u_k \in R$ it holds that

$$R' \subseteq R.$$

By construction, for every $i \in [k]$, we have:

$$\alpha(u_i) \in \langle\beta; \alpha(u_k)\rangle$$

and for every $k < i \leq |U|$ we have:

$$u_i \notin \langle\beta; \alpha(u_k)\rangle.$$

So $R' \cap U = \{u_1, \ldots, u_k\}$. Thereby, $\mathcal{R}_{\mathrm{W}}$ is good.

- Finally, we show that $\mathcal{R}_{\mathrm{VS}}$ is good as well. Let V be a point set. Let

$$R = \{(x, y) \mid a \leq x \leq b\} \in \mathcal{R}_{\mathrm{VS}}$$

be a vertical strip for $a, b \in \mathbb{R}$, let $U = V \cap R$, and let $k \leq |U| \in \mathbb{N}$. Let $u_1, \ldots, u_{|U|}$ be the elements of U in the order of increasing x-coordinates. We claim that the vertical strip

$$R' = \{(x, y) \mid a \leq x \leq x(u_k)\} \in \mathcal{R}_{\mathrm{VS}}$$

is the desired range. First, since $u_k \in R$, we have $x(u_k) \leq b$ so R' is a subset of R. Second:

1. $x(u_1), \ldots, x(u_k) \geq a$ (since $u_1, \ldots, u_k \in R$)
2. For $1 \leq i \leq k$: $x(u_i) \leq x(u_k)$
3. For $i > k$: $x(u_i) > x(u_k)$

So $u_1, \ldots, u_k \in R'$, $u_{k+1}, \ldots, u_{|U|} \notin R'$ and $R' \cap V = \{u_1, \ldots, u_k\}$. Hence, R' satisfies the desired properties and $\mathcal{R}_{\mathrm{VS}}$ is good. For symmetry reasons, $\mathcal{R}_{\mathrm{HS}}$ and $\mathcal{R}_{\mathrm{DS}}$ are good too.

□

Trivially, if two range families are good, then their union is good too. This can be generalized as follows.

Observation 2.5. *Let $t \in \mathbb{N}$ and $\mathcal{R}_1, \ldots, \mathcal{R}_t$ be good range families. Then $\mathcal{R}_1 \cup \cdots \cup \mathcal{R}_t$ is a good range family too.*

Lemma 2.6. *Let $\mathcal{R}$ be a good range family and let $k, m \in \mathbb{N}$ be such that every hypergraph $\mathcal{H}(V, \mathcal{R}, m)$ admits a polychromatic coloring with k colors. Then $m(k) \leq m$.*

Proof. Let V be a point set, let $m' \geq m \in \mathbb{N}$. We show that the hypergraph $\mathcal{H}(V, \mathcal{R}, m')$ admits a polychromatic coloring with k colors. By assumption, there is a polychromatic coloring $c : V \to [k]$ of $\mathcal{H}(V, \mathcal{R}, m)$ with k colors. Let E be a hyperedge of $\mathcal{H}(V, \mathcal{R}, m')$ captured by a range $R \in \mathcal{R}$. Since $\mathcal{R}$ is good, there is a range $R' \in \mathcal{R}$ with

$$R' \subseteq R, |R' \cap V| = m.$$

So R' captures a hyperedge E' of $\mathcal{H}(V, \mathcal{R}, m)$. Since E' is polychromatic in c and

$$E' = R' \cap V \subseteq R \cap V = E,$$

the hyperedge E is polychromatic in c too. Recall that E was chosen arbitrarily. So c is a polychromatic coloring of $\mathcal{H}(V, \mathcal{R}, m')$. So every hypergraph $\mathcal{H}(V, \mathcal{R}, m')$ with $m' \geq m$ admits a polychromatic coloring with k colors and hence $m(k) \leq m$. □

Note: Since the range families considered in this work are good, by Lemma 2.6, the existence of a polychromatic k-coloring for every hypergraph $\mathcal{R}(V, \mathcal{R}, m)$ is sufficient to prove $m_{\mathcal{R}}(k) \leq m$. For this reason, we will not prove the existence of polychromatic colorings for the hypergraphs of uniformity $m' \geq m$ explicitly, we will only show this for m-uniform hypergraphs.

Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph. A set S is called a *hitting set* if every hyperedge of $\mathcal{H}$ has a non-empty intersection with S, i.e.:

$$\forall E \in \mathcal{E} : E \cap S \neq \emptyset.$$

In this case, we also say that S *hits* E.

For $t \in \mathbb{N}$, a hitting set S of $\mathcal{H}$ is called *t-shallow* if for every hyperedge of $\mathcal{H}$, the intersection with S contains at most t vertices:

$$\forall E \in \mathcal{E} : |E \cap S| \leq t.$$

We say that a range family $\mathcal{R}$ *admits t-shallow hitting sets* for $t \in \mathbb{N}$ if every hypergraph $\mathcal{H}(V, \mathcal{R}, m)$ admits a t-shallow hitting set.

Lemma 2.7. *Let $\mathcal{R}$ be a good range family and let $\mathcal{H}(V, \mathcal{R}, m) = (V, \mathcal{E})$ be an arbitrary hypergraph. Let $t \leq m \in \mathbb{N}$ and let $S \subseteq V$ be a subset of vertices such that every hyperedge has an intersection of size at most t with S (e.g., S is a t-shallow hitting set), i.e.:*

$$\forall E \in \mathcal{E} : |E \cap S| \leq t.$$

Then for every hyperedge $E \in \mathcal{E}$, there is a hyperedge $E' \in \mathcal{E}'$ of the hypergraph

$$\mathcal{H}(V \setminus S, \mathcal{R}, m - t) = (V \setminus S, \mathcal{E}')$$

such that $E' \subseteq E$, i.e.:

$$\forall E \in \mathcal{E}\ \exists E' \in \mathcal{E}' : E' \subseteq E.$$

Proof. Let $E \in \mathcal{E}$ be arbitrary and let $R \in \mathcal{R}$ be such that $V \cap R = E$. Then since

$$|S \cap E| \leq t,$$

we have

$$|R \cap (V \setminus S)| \geq m - t.$$

Since $\mathcal{R}$ is good, there is a range $R' \subseteq R \in \mathcal{R}$ such that

$$|R' \cap (V \setminus S)| = m - t.$$

Then

$$E' = R' \cap (V \setminus S)$$

is a hyperedge of $\mathcal{H}(V \setminus S, \mathcal{R}, m - t)$. Due to $R' \subseteq R$, we have

$$E' = R' \cap (V \setminus S) \subseteq R \cap (V \setminus S) \subseteq R \cap V = E.$$

□

We slightly generalize the result of Keszegh and Pálvölgyi and for the sake of completeness prove it.

Lemma 2.8 ([KP19a])**.** *Let $\mathcal{R}$ be a good range family such that there exist $k_0, m_0 \in \mathbb{N}$ with $m(k_0) \leq m_0$ and for every $m \geq m_0$, every hypergraph $\mathcal{H}(V, \mathcal{R}, m)$ admits a t-shallow hitting set. Then $m(k) \leq (k - k_0)t + m_0$ for all $k \in \mathbb{N}$ with $k \geq k_0$.*

In particular, for any good range family $\mathcal{R}$ admitting t-shallow hitting sets, we have

$$m(k) \leq (k-1)t + 1$$

for all $k \in \mathbb{N}$ (due to $m(1) = 1$).

Proof. We prove the statement by induction on k.

Base case, $k = k_0$: Due to $(k - k_0)t + m_0 = m_0$, the statement holds by assumption.

Inductive step: Assume that the statement holds for some $k \geq k_0$. We show that it also holds for $k + 1$. Let V be an arbitrary point set. Consider the hypergraph

$$\mathcal{H} = \mathcal{H}(V, \mathcal{R}, ((k+1) - k_0)t + m_0).$$

By assumption, it admits a t-shallow hitting set S. It holds that

$$(((k+1) - k_0)t + m_0) - t = (k - k_0)t + m_0.$$

So consider the hypergraph

$$\mathcal{H}' = \mathcal{H}(V \setminus S, \mathcal{R}, (k - k_0)t + m_0).$$

By induction hypothesis, it admits a polychromatic coloring $c' : V \setminus S \to [k]$ with k colors. We claim that $c : V \to [k+1]$ defined as

$$c(v) = \begin{cases} c'(v) & \text{if } v \in V - S \\ k+1 & \text{if } v \in S \end{cases}$$

is a polychromatic coloring of $\mathcal{H}$ with $k+1$ colors. First, since S is a hitting set of $\mathcal{H}$, every hyperedge of $\mathcal{H}$ contains a vertex in color $k+1$. Second, by Lemma 2.7, for every hyperedge E of $\mathcal{H}$, there is a hyperedge E' of $\mathcal{H}'$ with $E' \subseteq E$. Since c' is a polychromatic coloring of $\mathcal{H}'$, the hyperedge E' contains vertices in each of the colors in $[k]$. Since the colors of vertices in $V \setminus S$ are the same in c as in c', the hyperedge E contains vertices in each of the colors in $[k]$ too. Hence, c is the desired coloring and the statement holds for $k + 1$ too. By induction, it holds for every $k \geq k_0$. □

Lemma 2.9. *Let V be a point set, let $k, t \in \mathbb{N}$, $m \geq (k-1) \cdot t + 1 \in \mathbb{N}$, and let $\mathcal{R}$ be a good range family. Let $V_1 = V, m_1 = m$. For $i \in [k]$ let*

- *S_i be a t-shallow hitting set of $\mathcal{H}(V_i, \mathcal{R}, m_i)$,*
- *$V_{i+1} = V_i \setminus S_i$,*
- *and $m_{i+1} = m_i - t$.*

Then for every $r \in [k]$, the set S_r is a hitting set of $\mathcal{H}(V, \mathcal{R}, m)$.

Proof. First, note that due to

$$m \geq (k-1) \cdot t + 1,$$

for every $i \in [k]$, we have

$$m_i \geq m_k = m - (k-1)t \geq 1 \tag{2.1}$$

and hence, the hyperedges of $\mathcal{H}(V_i, \mathcal{R}, m_i)$ are non-empty.

Now consider an arbitrary hyperedge E of $\mathcal{H}(V, \mathcal{R}, m)$ captured by some range $R \in \mathcal{R}$. We prove the following two statements for every $r \in [k]$:

1. S_r hits E.
2. $|E \cap V_{r+1}| \geq m_{r+1}$

For $r = 1$, we have $V_r = V_1 = V$ and $m_r = m_1$ so the statements hold since S_1 is a t-shallow hitting set of $\mathcal{H}(V_1, \mathcal{R}, m_1)$ and $m_2 = m_1 - t$. Now suppose the statements hold for some $r \in [k-1]$. Let

$$E' = E \cap V_{r+1}.$$

We know

$$|E'| \geq m_{r+1} \overset{(2.1)}{\geq} 1$$

and

$$R \cap V_{r+1} = R \cap (V \cap V_{r+1}) = (R \cap V) \cap V_{r+1} = E \cap V_{r+1} = E'.$$

So

$$|R \cap V_{r+1}| \geq m_{r+1}.$$

Since $\mathcal{R}$ is good, there is a range $R'' \subseteq R \in \mathcal{R}$ such that

$$|R'' \cap V_{r+1}| = m_{r+1}.$$

Let $E'' = R'' \cap V_{r+1}$, then E'' is a hyperedge of $\mathcal{H}(V_{r+1}, \mathcal{R}, m_{r+1})$ and

$$E'' = R'' \cap V_{r+1} \subseteq R \cap V_{r+1} \subseteq R \cap V = E.$$

Since S_{r+1} is a t-shallow hitting set of $\mathcal{H}(V_{r+1}, \mathcal{R}, m_{r+1})$, it holds that:

$$1 \leq |E'' \cap S_{r+1}| \leq t.$$

So first, we have

$$|E \cap S_{r+1}| \geq |E'' \cap S_{r+1}| \geq 1$$

and second, we also have

$$|E \cap V_{r+2}| \geq |E'' \cap V_{r+2}| = |E''| - |E'' \cap S_{r+1}| \geq m_{r+1} - t = m_{r+2}.$$

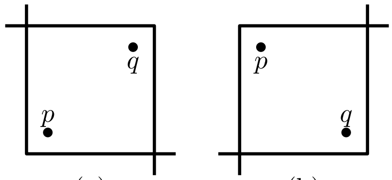
So both statements hold for $r+1$ too. By induction, we obtain that for every $r \in [k]$, the set S_r hits the hyperedge E. Since E was chosen arbitrarily, S_r is a hitting set of $\mathcal{H}(V, \mathcal{R}, m)$ for every $r \in [k]$. □

2.4 Quadrants

Observation 2.10. *1. Let $p, q \in \mathbb{R}^2$ such that $x(p) < x(q), y(p) < y(q)$ (see Figure 2.2 (a)), then:*

- *Each south-west quadrant containing q contains p too, i.e.:*

$$\forall R \in \mathcal{R}_{SW} : q \in R \Rightarrow p \in R$$



(a) (b)

Figure 2.2: (a) Every south-west / north-east quadrant containing q / p contains p / q too. (b) Every north-west / south-east quadrant containing q / p contains p / q too.

- *Each north-east quadrant containing p contains q too, i.e.:*

$$\forall R \in \mathcal{R}_{NE} : p \in R \Rightarrow q \in R$$

2. *Let $p, q \in \mathbb{R}^2$ such that $x(p) < x(q), y(p) > y(q)$ (see Figure 2.2 (b)), then:*
   - *Each north-west quadrant containing q contains p too, i.e.:*

$$\forall R \in \mathcal{R}_{NW} : q \in R \Rightarrow p \in R$$

   - *Each south-east quadrant containing p contains q too, i.e.:*

$$\forall R \in \mathcal{R}_{SE} : p \in R \Rightarrow q \in R$$

For a non-empty point set S, the *topmost* vertex $\mathrm{top}(S)$ is the vertex with the largest y-coordinate, i.e.:

$$\mathrm{top}(S) = \operatorname*{argmax}_{v \in S} y(v).$$

The *bottommost* vertex $\mathrm{bottom}(S)$, the *rightmost* vertex $\mathrm{right}(S)$, and the *leftmost* vertex $\mathrm{left}(S)$ are defined similarly. Further, let E be a hyperedge of a hypergraph $\mathcal{H}(V, \mathcal{R}_{\mathrm{SE}}, m)$. Then the *vertical* (respectively *horizontal*) *boundary* $v(E)$ (respectively $h(E)$) of E is defined as

$$v(E) = x(\mathrm{left}(E)) \text{ (respectively } h(E) = y(\mathrm{top}(E))).$$

In other words, the south-east quadrant with the left side at $x = v(E)$ and the top side at $y = h(E)$ is the inclusion-minimal south-east quadrant capturing E. Vertical and horizontal boundaries of hyperedges captured by south-west, north-east, and north-west quadrants are defined similarly.

Lemma 2.11. *Let $\mathcal{H}(V, \mathcal{R}_{SE}, m) = (V, \mathcal{E})$ be a hypergraph. Let $E_1, E_2 \in \mathcal{E}$ be two hyperedges. Then:*

1. $E_1 \neq E_2 \Leftrightarrow h(E_1) \neq h(E_2)$
2. $E_1 \neq E_2 \Leftrightarrow v(E_1) \neq v(E_2)$
3. $h(E_1) < h(E_2) \Rightarrow v(E_1) < v(E_2)$ *(in particular, the leftmost vertex of E_1 does not belong to E_2)*
4. $v(E_1) < v(E_2) \Rightarrow h(E_1) < h(E_2)$ *(in particular, the topmost vertex of E_2 does not belong to E_1)*

Proof. 1. Let $h(E_1) \neq h(E_2)$. Then the topmost vertices of E_1 and E_2 are distinct and hence the hyperedges are distinct too.

Now let $E_1 \neq E_2$. Suppose $h(E_1) = h(E_2)$. Without loss of generality, assume $v(E_2) \leq v(E_1)$ (otherwise swap the roles of E_1 and E_2). Every south-east quadrant

$$Q = \{(x, y) \mid x \geq a, y \leq b\} \in \mathcal{R}_{\text{SE}}$$

capturing E_2 necessarily contains $\text{top}(E_2)$ and $\text{left}(E_2)$. Let $v \in E_1$, then:

$$a \leq x(\text{left}(E_2)) = v(E_2) \leq v(E_1) \text{ and } b \geq y(\text{top}(E_2)) = h(E_2) = h(E_1).$$

So for every $v \in E_1$, we have:

$$a \leq v(E_1) \leq x(v) \text{ and } b \geq h(E_1) \geq y(v).$$

Therefore, every $v \in E_1$ belongs to Q so it belongs to E_2. Hence, $E_1 \subseteq E_2$ and due to $|E_1| = |E_2| = m$, we get $E_1 = E_2$ – a contradiction.

2. The proof is analogous to Item 1.

3. Due to $h(E_1) < h(E_2)$, we have $E_1 \neq E_2$. Suppose $v(E_1) \geq v(E_2)$. Every south-east quadrant

$$Q = \{(x, y) \mid x \geq a, y \leq b\} \in \mathcal{R}_{\text{SE}}$$

capturing E_2 necessarily contains $\text{top}(E_2)$ and $\text{left}(E_2)$ so

$$a \leq x(\text{left}(E_2)) = v(E_2) \leq v(E_1) \text{ and } b \geq y(\text{top}(E_2)) = h(E_2) > h(E_1).$$

Let $v \in E_1$, then

$$a \leq v(E_1) \leq x(v) \text{ and } b > h(E_1) \geq y(v).$$

So v belongs to Q and hence to E_2. Thereby, $E_1 \subseteq E_2$ and due to $|E_1| = |E_2| = m$, it holds that $E_1 = E_2$ – a contradiction.

4. The proof is analogous to Item 3.

□

Note that the symmetrical versions of the lemma hold for south-west, north-east, and north-west quadrants.

Next, we will show that the south-east quadrants capturing hyperedges of some hypergraph $\mathcal{H}(V, \mathcal{R}_{\text{SE}}, m)$ can be ordered from bottom to top (and from left to right). For this, we need the following properties.

Observation 2.12. *Let V be a point set, let $v \in V$ and let $m \in \mathbb{N}$. If there exist m pairwise distinct vertices $w_1, \ldots, w_m \in V$ such that:*

$$x(v) < x(w_i), y(v) > y(w_i) \text{ for every } i \in [m]$$

(see Figure 2.3 (a)), then v does not appear in any hyperedge of $\mathcal{H}(V, \mathcal{R}_{SE}, m)$.

Proof. Every south-east quadrant containing v contains $w_1, \ldots, w_m$ too. So it contains at least $m + 1$ vertices and does not capture a hyperedge of $\mathcal{H}(V, \mathcal{R}_{\text{SE}}, m)$. □

Lemma 2.13. *Let V be a point set, $v \in V$ a vertex, and E_1, E_2 hyperedges of*

$$\mathcal{H} = \mathcal{H}(V, \mathcal{R}_{SE}, m)$$

such that

$$v \in E_2, v \neq \text{top}(E_2), y(v) > h(E_1).$$

Then there exists a hyperedge E of $\mathcal{H}$ with $\text{top}(E) = v$.

Proof. For illustration, see Figure 2.3 (b). First note:

$$h(E_2) > y(v) > h(E_1).$$

Consider the south-east quadrant

$$Q_v = \{(x, y) \mid x \geq v(E_1), y \leq y(v)\} \in \mathcal{R}_{\mathrm{SE}}.$$

It holds that

$$E_1 \cup \{v\} \subseteq Q_v.$$

So

$$|Q_v \cap V| \geq m + 1.$$

Let $w_1, \ldots, w_q$ be the vertices in $Q_v \cap V$ in the order of decreasing x-coordinates. Consider the first m points

$$W = \{w_1, \ldots, w_m\}.$$

Observe that $v \in W$: otherwise, there would be at least m points to the bottom-right of v and by Observation 2.12, the vertex v would not belong to any hyperedge of $\mathcal{H}$ – a contradiction. For the south-east quadrant

$$Q_W = \{(x, y) \mid x \geq x(w_m), y \leq y(v)\} \subseteq Q_v \in \mathcal{R}_{\mathrm{SE}},$$

we have

$$W = V \cap Q_W \text{ and } v \in W.$$

So W is a set of m points captured by a south-east quadrant, i.e., W is a hyperedge of $\mathcal{H}$. Moreover, it holds that $\mathrm{top}(W) = v$ so W is the desired hyperedge. □

Lemma 2.14. *Let V be a point set and $m \in \mathbb{N}$. Let $E_1, \ldots, E_t$ be all hyperedges of $\mathcal{H} = \mathcal{H}(V, \mathcal{R}_{SE}, m)$ in the order of increasing horizontal boundaries, i.e.:*

$$h(E_1) < \cdots < h(E_t). \tag{2.2}$$

Then:

1. *For all $i < j$, $\mathrm{top}(E_j) \notin E_i$.*
2. *For all $i < t$, $E_{i+1} \setminus E_i = \{\mathrm{top}(E_{i+1})\}$.*
3. *For all $i < t$, $|E_i \setminus E_{i+1}| = 1$.*

Proof. For illustration, see Figure 2.3 (c). By Lemma 2.11, we have:

$$v(E_1) < \cdots < v(E_t).$$

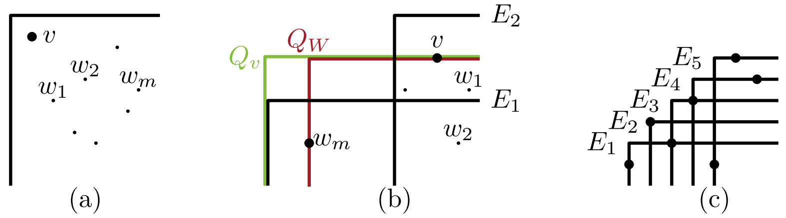


Figure 2.3: (a) Observation 2.12: If v has m further points to the bottom-right, then no hyperedge of $\mathcal{H}(V, \mathcal{R}_{\mathrm{SE}}, m)$ contains v.
(b) Lemma 2.13: Construction of a hyperedge whose topmost vertex is v.
(c) Lemma 2.14: An ordering of hyperedges $\mathcal{H}(V, \mathcal{R}_{\mathrm{SE}}, m = 3)$ such that in every step we lose one vertex and gain a new vertex.

1. If $\mathrm{top}(E_j) \in E_i$, then
$$h(E_i) \geq y(\mathrm{top}(E_j)) = h(E_j).$$
A contradiction.

2. By Item 1, we know that
$$\mathrm{top}(E_{i+1}) \in E_{i+1} \setminus E_i.$$
Suppose there exists $v \neq \mathrm{top}(E_{i+1})$ such that $v \in E_{i+1} \setminus E_i$. Due to
$$v(E_i) < v(E_{i+1}) \leq x(v),$$
it holds that
$$y(v) > h(E_i).$$
Then we have:
   - $v \in E_{i+1}$
   - $v \neq \mathrm{top}(E_{i+1})$
   - $y(v) > h(E_i)$

   So by Lemma 2.13, there is a hyperedge E of $\mathcal{H}$ with $h(E) = y(v)$. Thus,
$$h(E_i) < y(v) = h(E) < h(E_{i+1}).$$
This contradicts the fact that E_i and E_{i+1} are consecutive hyperedges in (2.2). So
$$E_{i+1} \setminus E_i = \{\mathrm{top}(E_{i+1})\}.$$

3. We know that:
$$m = |E_{i+1}| = |E_{i+1} \cap E_i| + |E_{i+1} \setminus E_i| \overset{Item\ 2}{=} |E_{i+1} \cap E_i| + 1$$
and so
$$|E_{i+1} \cap E_i| = m - 1.$$
Thus,
$$|E_i \setminus E_{i+1}| = |E_i| - |E_i \cap E_{i+1}| = m - (m-1) = 1$$
and the claim holds.

□

Definition 2.15. *Let V be a point set and let $m \in \mathbb{N}$ be such that $m \leq |V|$. Then we denote the set of m points in V with the largest (respectively smallest) y-coordinates with $E_t(V, m)$ (respectively $E_b(V, m)$).*

Lemma 2.16. *Let V be a point set and let $m \in \mathbb{N}$ with $m \leq |V|$. Let $\mathcal{H}(V, \mathcal{R}_{SE}, m) = (V, \mathcal{E})$. Then*

1. *$E_b(V, m) \in \mathcal{E}$,*
2. *$E_b(V, m)$ is the hyperedge with the lowest horizontal boundary in $\mathcal{E}$, i.e.,*
$$E_b(V, m) = \operatorname*{argmin}_{E \in \mathcal{E}} h(E),$$
3. *and $\mathrm{left}(E_b(V, m)) \notin E$ for every $E \in \mathcal{E}$ with $E \neq E_b(V, m)$.*

Proof. 1. Consider the south-east quadrant

$$Q = \{(x, y) \mid x \geq x(\text{left}(E_b(V))), y \leq y(\text{top}(E_b(V)))\} \in \mathcal{R}_{\text{SE}}.$$

First, this quadrant contains every element of $E_b(V)$ by construction so

$$E_b(V) \subset Q.$$

Further, every vertex

$$v \in V \setminus E_b(V)$$

does not belong to the m bottommost vertices of V so it lies above every vertex in $E_b(V)$. In particular, it lies above $\text{top}(E_b(V))$. So v does not belong to Q and hence:

$$Q \cap V = E_b(V).$$

By the definition of $E_b(V)$, we also have

$$|E_b(V)| = m$$

so $E_b(V)$ is indeed a hyperedge of $\mathcal{H}(V, \mathcal{R}_{\text{SE}}, m)$.

2. Let $E \in \mathcal{E}$. The hyperedge E contains exactly m vertices so its horizontal boundary lies above at least m vertices, i.e., its horizontal boundary can not lie below the horizontal line $y = y(\text{top}(E_b(V, m)))$. Further, for the hyperedge $E_b(V, m)$, we have

$$h(E_b(V, m)) = y(\text{top}(E_b(V, m))).$$

So the claim holds.

3. Let $E \neq E_b(V, m) \in \mathcal{E}$. By Lemma 2.11 Item 1, we have

$$h(E) \neq h(E_b(V, m)).$$

Since $E_b(V, m)$ is the hyperedge with the bottommost horizontal boundary in $\mathcal{E}$, we have:

$$h(E) > h(E_b(V, m))$$

and by Lemma 2.11 Item 3, we also have

$$v(E) > v(E_b(V, m)) = y(\text{left}(E_b(V, m))).$$

So we obtain

$$\text{left}(E_b(V, m)) \notin E$$

as desired.

□

2.5 Bounding Boxes

For a point set V, the *bounding box* is the inclusion-minimal axis-aligned rectangle containing all points of V, i.e., the range

$$\{(x, y) \mid x(left(V)) \leq x \leq x(\text{right}(V)), y(\text{bottom}(V)) \leq y \leq y(\text{top}(V))\}.$$

We say that a bottomless rectangle

$$\{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, y \leq c\}$$

has its *top-right corner* at (b, c) and its *left side* at a.

For $t \in \mathbb{N}$, we say that points $p_1, \ldots, p_t$ form an *increasing sequence* if

$$x(p_1) < \cdots < x(p_t) \text{ and } y(p_1) < \cdots < y(p_t).$$

We say that points $p_1, \ldots, p_t$ form a *decreasing sequence* if

$$x(p_1) < \cdots < x(p_t) \text{ and } y(p_1) > \cdots > y(p_t).$$

Lemma 2.17. *Let $a, b \in \mathbb{R}^2$ be two points forming an increasing sequence. Let A be the bounding box of $\{a, b\}$ and let B be the bottomless rectangle with the top-right corner b and the left side at $x(a)$. Then:*

1. *Every axis-aligned quadrant, every bottomless or topless rectangle, and every strip R containing both a and b contains A as a subset, i.e.:*

$$a, b \in R \Rightarrow A \subset R.$$

2. *Every bottomless rectangle, every south-west or south-east quadrant R, and every vertical strip containing both a and b contains B as a subset:*

$$a, b \in R \Rightarrow B \subseteq R.$$

Proof. First, observe that A and B have the same left, right, and top sides and B is unbounded to the bottom so

$$A \subset B.$$

1. Let

$$R = \{(x, y) \in \mathbb{R}^2 \mid x \leq c, y \leq d\} \in \mathcal{R}_{\text{SW}}$$

be a south-west quadrant for $c, d \in \mathbb{R}$ such that $a, b \in R$ (see Figure 2.4 (a)). Then for every $v \in B$, the following holds:

$$x(v) \leq x(b) \leq c, y(v) \leq y(b) \leq d.$$

So $v \in R$ and hence,

$$A \subset B \subset R.$$

2. Let

$$R = \{(x, y) \in \mathbb{R}^2 \mid x \geq c, y \leq d\} \in \mathcal{R}_{\text{SE}}$$

be a south-east quadrant for $c, d \in \mathbb{R}$ such that $a, b \in R$ (see Figure 2.4 (b)). Then for every $v \in B$, the following holds:

$$x(v) \geq x(a) \geq c, y(v) \leq y(b) \leq d.$$

So $v \in R$ and hence,

$$A \subset B \subset R.$$

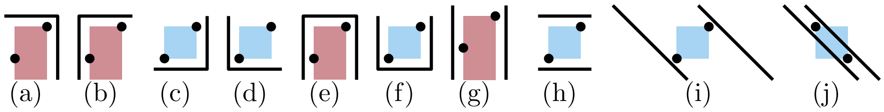


Figure 2.4: (a) – (g) If two points form an increasing sequence, then the sketched ranges containing both points contain their bounding box (in blue) or the inclusion-minimal bottomless rectangle containing these two points (in red).
(h) Important: if the points form a decreasing sequence, then a diagonal strip containing them does not necessarily contain their bounding box.

3. Let
$$R = \{(x, y) \in \mathbb{R}^2 \mid x \leq c, y \geq d\} \in \mathcal{R}_{\text{NW}}$$
be a north-west quadrant for $c, d \in \mathbb{R}$ such that $a, b \in R$ (see Figure 2.4 (c)). Then for every $v \in A$, the following holds:
$$x(v) \leq x(b) \leq c, y(v) \geq y(a) \geq d.$$
So $v \in R$ and hence,
$$A \subset R.$$

4. Let
$$R = \{(x, y) \in \mathbb{R}^2 \mid x \geq c, y \geq d\} \in \mathcal{R}_{\text{NE}}$$
be a north-east quadrant for $c, d \in \mathbb{R}$ such that $a, b \in R$ (see Figure 2.4 (d)). Then for every $v \in A$, the following holds:
$$x(v) \geq x(a) \geq c, y(v) \geq y(a) \geq d.$$
So $v \in R$ and hence,
$$A \subset R.$$

5. Let
$$R = \{(x, y) \in \mathbb{R}^2 \mid c \leq x \leq d, y \leq e\} \in \mathcal{R}_{\text{BL}}$$
be a bottomless rectangle for $c, d, e \in \mathbb{R}$ such that $a, b \in R$ (see Figure 2.4 (e)). Then for every $v \in B$, the following holds:
$$c \leq x(a) \leq x(v) \leq x(b) \leq d, y(v) \leq y(b) \leq e.$$
So $v \in R$ and hence,
$$A \subset B \subseteq R.$$

6. Let
$$R = \{(x, y) \in \mathbb{R}^2 \mid c \leq x \leq d, y \geq e\} \in \mathcal{R}_{\text{TL}}$$
be a topless rectangle for $c, d, e \in \mathbb{R}$ such that $a, b \in R$ (see Figure 2.4 (f)). Then for every $v \in A$, the following holds:
$$c \leq x(a) \leq x(v) \leq x(b) \leq d, y(v) \geq y(a) \geq e.$$
So $v \in R$ and hence,
$$A \subset R.$$

7. Let
$$R = \{(x, y) \in \mathbb{R}^2 \mid c \le x \le d\} \in \mathcal{R}_{\mathrm{VS}}$$
be a vertical strip for $c, d \in \mathbb{R}$ such that $a, b \in R$ (see Figure 2.4 (g)). Then for every $v \in B$, the following holds:
$$c \le x(a) \le x(v) \le x(b) \le d.$$
So $v \in R$ and hence,
$$A \subset B \subset R.$$

8. Let
$$R = \{(x, y) \in \mathbb{R}^2 \mid c \le y \le d\} \in \mathcal{R}_{\mathrm{HS}}$$
be a horizontal strip for $c, d \in \mathbb{R}$ such that $a, b \in R$ (see Figure 2.4 (h)). Then for every $v \in A$, the following holds:
$$c \le y(a) \le y(v) \le y(b) \le d.$$
So $v \in R$ and hence,
$$A \subset R.$$

9. Finally, let
$$R = \{(x, y) \in \mathbb{R}^2 \mid c \le x + y \le d\} \in \mathcal{R}_{\mathrm{DS}}$$
be a diagonal strip for $c, d \in \mathbb{R}$ such that $a, b \in R$ (see Figure 2.4 (i)). Then for every $v \in A$, the following holds:
$$x(a) \le x(v) \le x(b), y(a) \le y(v) \le y(b).$$
So
$$c \le x(a) + y(a) \le x(v) + y(v) \le x(b) + y(b) \le d$$
and $v \in R$. Thus, $A \subset R$ as desired.

□

The symmetrical statement holds if a, b form a decreasing sequence. **However, we emphasize that in this case, a diagonal strip containing a and b does not necessarily contain their bounding box as a subset (see Figure 2.4 (j)).**

Lemma 2.18. *Let $t \in \mathbb{N}$ and let $S = \{s_1, \ldots, s_t\}$ be a point set such that $s_1, \ldots, s_t$ form an increasing sequence. Then every axis-aligned quadrant, every bottomless or topless rectangle, and every strip R contains a subsequence of S, i.e., the elements of S occurring in R are consecutive: there exist $i, j \in \mathbb{N}$ such that for $r \in [t]$:*
$$s_r \in R \Leftrightarrow i \le r \le j.$$

Proof. If $t \le 2$ or R has an empty intersection with S, then the claim trivially holds. Otherwise, let $p \le q \in [t]$ be such that $s_p, s_q \in R$. Let A be the bounding box of $\{s_p, s_q\}$. By Lemma 2.17, it holds that
$$A \subset R.$$
Let $k \in [t]$ be such that $p \le k \le q$. Then s_p, s_k, s_q form an increasing sequence, i.e.,
$$x(s_p) \le x(s_k) \le x(s_q), y(s_p) \le y(s_k) \le y(s_q).$$
Then
$$s_k \in A \subset R.$$
So s_k belongs to R. Thereby, R contains consecutive elements of S and this concludes the proof. □

Note that given a decreasing sequence, a statement analogous to Lemma 2.18 is valid for all ranges mentioned there except for diagonal strips.

2.6 Dual Hypergraphs

As already mentioned in the introduction, the hypergraphs of form $\mathcal{H}(V, \mathcal{R}, m)$ are called *primal*. For a range family $\mathcal{R}$ and a **finite** subfamily $\mathcal{R}' \subseteq \mathcal{R}$, the *dual* hypergraph $\mathcal{H}^*(\mathcal{R}') = (\mathcal{R}', \mathcal{E})$ has vertex set $\mathcal{R}'$ and for every point in the plane, there is a hyperedge that consists of all ranges in $\mathcal{R}'$ that contain this point, i.e.:

$$\mathcal{E} = \{\{R \in \mathcal{R}' \mid p \in R\} \mid p \in \mathbb{R}^2\}.$$

If a range R contains a point p, we also say that R *covers* p. Further, for $m \in \mathbb{N}$, the subhypergraph $\mathcal{H}^*(\mathcal{R}', m)$ of $\mathcal{H}^*(\mathcal{R}')$ has the same vertex set and contains exactly the hyperedges of $\mathcal{H}^*(\mathcal{R}')$ of size at least m. **We emphasize this last condition, as it contrasts the one in primal hypergraphs $\mathcal{H}(V, R, m)$, where hyperedges are resticted to be of size exactly m.**

For $k \in \mathbb{N}$, the value $m^*_{\mathcal{R}}(k)$ is the smallest number such that for every finite subfamily $\mathcal{R}' \subseteq \mathcal{R}$, the hypergraph $\mathcal{H}^*(\mathcal{R}', m^*_{\mathcal{R}}(k))$ admits a polychromatic coloring with k colors. If such $m^*_{\mathcal{R}}(k)$ does not exist (i.e., for every $n \in \mathbb{N}$ there is a finite subfamily $\mathcal{R}'_n \subseteq \mathcal{R}$ such that $\mathcal{H}^*(\mathcal{R}'_n, n)$ does not admit a polychromatic coloring with k colors), we write $m^*_{\mathcal{R}}(k) = \infty$. Otherwise, we write $m^*_{\mathcal{R}}(k) < \infty$. If the range family $\mathcal{R}$ is clear from the context, we simply write $m^*(k)$ instead of $m^*_{\mathcal{R}}(k)$.

Observation 2.19. *Let $\mathcal{R}$ be a range family and let $k \in \mathbb{N}$ be such that $m^*(k) < \infty$. Let $\mathcal{R}' \subseteq \mathcal{R}$ be a finite subfamily of $\mathcal{R}$. Then for every $m \geq m^*(k)$, the hypergraph $\mathcal{H}^*(\mathcal{R}', m)$ is a subhypergraph of $\mathcal{H}^*(\mathcal{R}', m^*(k))$ so $\mathcal{H}^*(\mathcal{R}', m)$ also admits a polychromatic coloring with k colors.*

Observation 2.20. *Let $\mathcal{R}$ be a range family and let $k \in \mathbb{N}$ such that $m^*_{\mathcal{R}}(k) < \infty$. Then for every subfamily $\mathcal{R}' \subseteq \mathcal{R}$, it holds that $m^*_{\mathcal{R}'}(k) < \infty$.*

Proof. Let $m = m^*_{\mathcal{R}}(k) \in \mathbb{N}$ and let $\mathcal{R}'' \subseteq \mathcal{R}'$ be an arbitrary finite subfamily of $\mathcal{R}'$. So

$$\mathcal{R}'' \subseteq \mathcal{R}' \subseteq \mathcal{R},$$

i.e., $\mathcal{R}''$ is a finite subfamily of $\mathcal{R}$. Then the hypergraph $\mathcal{H}^*(\mathcal{R}'', m)$ admits a polychromatic coloring with k colors and this concludes the proof. ☐

3. Hitting Pairs

A first attempt to solve the problem for two colors is to consider the so-called *hitting pairs*. This approach works for a union of two general hypergraphs (not necessarily range-capturing) if these hypergraphs meet some special requirements. Here we define hitting pairs, present this approach, and finally show that unfortunately the requirements are not met even for a very simple range family, namely the family of all south-west quadrants.

Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph. We say that a pair of vertices $\{a, b\} \in \binom{V}{2}$ *hits* a hyperedge $E \in \mathcal{E}$ if $\{a, b\} \subseteq E$. We call a set $M \subset \binom{V}{2}$ of pairs of vertices *(a set of) hitting pairs* of $\mathcal{H}$ if the following two properties hold:

1. The elements of M are pairwise disjoint, i.e.,

$$\forall m_1, m_2 \in M : m_1 \neq m_2 \Rightarrow m_1 \cap m_2 = \emptyset.$$

2. And every hyperedge $E \in \mathcal{E}$ is hit by M, i.e.,

$$\forall E \in \mathcal{E}\ \exists m \in M : m \subseteq E.$$

Note: we emphasize that in this chapter (and only in this chapter), we use a non-standard definition of *hitting*: a hyperedge must contain a pair from M as a subset to be hit.

Let $\mathcal{R}$ be a range family and let $m \in \mathbb{N}$ be the uniformity. We say that $\mathcal{R}$ *admits hitting pairs for uniformity* m if for every point set V, the hypergraph $\mathcal{H}(V, \mathcal{R}, m)$ admits hitting pairs. We say that $\mathcal{R}$ *admits hitting pairs* if such m exists.

Note that hitting pairs generalize both vertex covers and matchings in some sense. On the one hand, similarly to edges in a matching, the elements of hitting pairs are pairwise disjoint. On the other hand, similarly to a vertex cover, every hyperedge is hit by an element of hitting pairs.

Lemma 3.1. *Let $\mathcal{H}_1 = (V_1, \mathcal{E}_1), \mathcal{H}_2 = (V_2, \mathcal{E}_2)$ be two hypergraphs such that M_i is a set of hitting pairs of $\mathcal{H}_i$ for $i \in [2]$. Then the union of $\mathcal{H}_1$ and $\mathcal{H}_2$, namely the hypergraph $\mathcal{H} = (V_1 \cup V_2, \mathcal{E}_1 \cup \mathcal{E}_2)$, admits a polychromatic coloring with 2 colors.*

Proof. For $i \in [2]$, let $G_i = (V_i, M_i)$. By the definition of hitting pairs, for $i \in [2]$, every vertex in the graph G_i has degree at most one. So the edge set of the graph

$$G = (V_1 \cup V_2, M_1 \cup M_2)$$

can be partitioned into cycles of even length and paths. Hence, it is bipartite and admits a proper 2-coloring $c : V_1 \cup V_2 \to [2]$. We claim that c is a polychromatic coloring of $\mathcal{H}$. We have to show that every hyperedge $E \in \mathcal{E}_1 \cup \mathcal{E}_2$ contains vertices in both colors. Let $E \in \mathcal{E}_i$ for some $i \in [2]$. Let $uv \in M_i$ be the pair hitting E (exists by the definition of M_i). Since c is a proper 2-coloring of G and $uv \in M_i$ is an edge of G, we have $c(u) \neq c(v)$. So

$$\{1, 2\} = \{c(u), c(v)\} \subseteq c(E).$$

Thus, E contains vertices in both colors. Since E was chosen arbitrarily, c is indeed a polychromatic coloring of $\mathcal{H}$ with 2 colors. □

This lemma can be applied to range-capturing hypergraphs as follows:

Theorem 3.2. *Let $\mathcal{R}_1, \mathcal{R}_2$ be good range families admitting hitting pairs for uniformity $m \in \mathbb{N}$. Then for the range family $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$, we have $m(2) \leq m$.*

Proof. Let V be a point set. Then the hypergraphs

$$\mathcal{H}(V, \mathcal{R}_1, m) = (V, \mathcal{E}_1)$$

and

$$\mathcal{H}(V, \mathcal{R}_2, m) = (V, \mathcal{E}_2)$$

admit hitting pairs by assumption. So by Lemma 3.1, the hypergraph

$$(V \cup V, \mathcal{E}_1 \cup \mathcal{E}_2) = (V, \mathcal{E}_1 \cup \mathcal{E}_2) = \mathcal{H}(V, \mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2, m)$$

admits a polychromatic coloring with 2 colors and this concludes the proof. □

Unfortunately, the lemma turns out not to be very helpful for our purposes: as we show next, for any $m \in \mathbb{N}$, even the simple family of all south-west quadrants does not admit hitting pairs.

3.1 No Hitting Pairs in South-West Quadrants

Lemma 3.3. *Range family $\mathcal{R}_{SW}$ does not admit hitting pairs, i.e., for every $m \geq 2$, there is a point set P_m such that the hypergraph $\mathcal{H}(P_m, \mathcal{R}_{SW}, m)$ admits no hitting pairs.*

Proof. We prove the statement by induction on m.

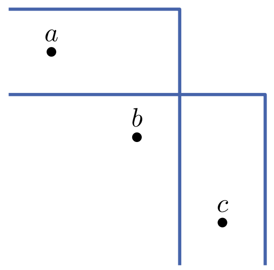
Base case, $m = 2$: Consider

$$P_2 = \{a = (1, 3), b = (2, 2), c = (3, 1)\}$$

(see Figure 3.1 (a)). Suppose there are hitting pairs M of $\mathcal{H}(P_2, \mathcal{R}_{\mathrm{SW}}, 2)$. Both hyperedges ab and bc are hit by M. The only way to hit ab is to have $ab \in M$. Similarly, the only way to hit bc is to have $bc \in M$. This contradicts the fact that the elements of M are pairwise disjoint. Hence, $\mathcal{H}(P_2, \mathcal{R}_{\mathrm{SW}}, 2)$ admits no hitting pairs.

Inductive step: Assume that for $m \geq 2$, the desired set P_m exists. We construct the set P_{m+1} using two shifted copies P_m^1 and P_m^2 of P_m as follows. First, we place P_m^1 anywhere in the plane. Then we place P_m^2 so that the bounding box B_2 of P_m^2 lies to the bottom-right of the bounding box B_1 of P_m^1 (see Figure 3.1 (b)). Let

$$\mathcal{H}_1 = \mathcal{H}(P_m^1, \mathcal{R}_{\mathrm{SW}}, m) = (P_m^1, \mathcal{E}_1),\ \mathcal{H}_2 = \mathcal{H}(P_m^2, \mathcal{R}_{\mathrm{SW}}, m) = (P_m^2, \mathcal{E}_2),$$



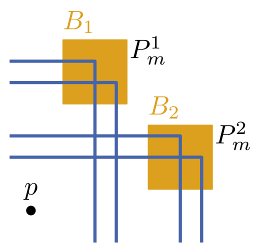


Figure 3.1: (a) Base case, $m = 2$ (b) Inductive step: the construction of P_{m+1} using two shifted copies of P_m.

and

$$\mathcal{H} = (P_m^1 \cup P_m^2, \mathcal{R}_{\mathrm{SW}}, m) = (P_m^1 \cup P_m^2, \mathcal{E}).$$

Observe that for $i \in [2]$, for every hyperedge $E \in \mathcal{E}_i$, the horizontal (respectively vertical) boundary of E lies between the bottom and top (respectively left and right) sides of B_i. In particular, there exists a south-west quadrant Q_E capturing E whose right (respectively top) side lies between the left and the right (respectively the top and the bottom) side of B_i. So for $j \neq i \in [2]$, we have:

$$Q_E \cap P_m^j = \emptyset$$

so

$$Q_E \cap (P_m^1 \cup P_m^2) = Q_E \cap P_m^i = E.$$

Hence,

$$\mathcal{E}_1, \mathcal{E}_2 \subset \mathcal{E}.$$

Finally, we place a new point p to the bottom-left of all points in $P_m^1 \cup P_m^2$ and obtain the point set P_{m+1} (see Figure 3.1 (b)). Let

$$\mathcal{H}' = \mathcal{H}(P_{m+1}, \mathcal{R}_{\mathrm{SW}}, m + 1) = (P_{m+1}, \mathcal{E}').$$

Since for every $q \in P_m^1 \cup P_m^2$ we have

$$x(p) < x(q), y(p) < y(q),$$

every south-west quadrant containing q contains p too. Hence, for every hyperedge

$$E \in \mathcal{E}_1 \cup \mathcal{E}_2,$$

it now holds $p \in Q_E$. Thereby,

$$Q_E \cap P_{m+1} = E \cup \{p\}$$

and due to

$$|E \cup \{p\}| = m + 1,$$

the set $E \cup \{p\}$ is a hyperedge of $\mathcal{H}'$. For $i \in [2]$, let

$$\mathcal{E}_i' = \{E \cup \{p\} \mid E \in \mathcal{E}_i\}.$$

So:

$$\mathcal{E}_1', \mathcal{E}_2' \subset \mathcal{E}'.$$

In simple words, we have extended every hyperedge in $\mathcal{E}_1 \cup \mathcal{E}_2$ by the new point p. Note that our construction creates further hyperedges but we are not interested in them in this proof. Suppose there is a set M of hitting pairs of $\mathcal{H}'$. We distinguish between the following two cases.

Case 1: There is no $m \in M$ with $p \in m$ (i.e., the point p is unused in the hitting pairs) or there exists $q \in P_m^2$ with $pq \in M$. Since M is a set of hitting pairs of $\mathcal{H}'$, every hyperedge in $\mathcal{E}_1'$ is hit. The hyperedges in $\mathcal{E}_1'$ can only be hit by pairs from

$$M \cap \binom{P_m^1 \cup \{p\}}{2}.$$

So by the definition of Case 1, all of these hyperedges are hit by pairs of elements from

$$M \cap \binom{P_m^1}{2}.$$

Hence, if we remove p from every hyperedge in $\mathcal{E}_1'$ (and obtain $\mathcal{E}_1$), the hyperedges stay hit. In other words,

$$M \cap \binom{P_m^1}{2}$$

is a set of hitting pairs of $\mathcal{H}_1$. This contradicts the induction hypothesis.

Case 2: There exists $q \in P_m^1$ with $pq \in M$. We apply the analogous argument to the set $\mathcal{E}_2'$ to get hitting pairs of $\mathcal{H}_2$ and obtain a contradiction.

As a result, $\mathcal{H}' = \mathcal{H}(P_{m+1}, \mathcal{R}_{\mathrm{SW}}, m+1)$ admits no hitting pairs and the statement holds for $m+1$ too. By induction, this holds for every $m \geq 2$ and this concludes the proof.

Side note: we can insert points to the bottom-right of the B_2 so that the corresponding hypergraph has arbitrarily many vertices but it still admits no hitting pairs. $\square$

By Lemma 2.1, every hyperedge of the hypergraph $\mathcal{H}_1 = \mathcal{H}(V, \mathcal{R}_{\mathrm{SW}}, m)$ is also a hyperedge of $\mathcal{H}_2 = \mathcal{H}(V, \mathcal{R}_{\mathrm{BL}}, m)$. So every set of hitting pairs of $\mathcal{H}_2$ would also be a set of hitting pairs of $\mathcal{H}_1$. Therefore, we obtain the following corollary:

Corollary 3.4. *The range family $\mathcal{R}_{BL}$ does not admit hitting pairs.*

For symmetry reasons, south-east, north-east, and north-west quadrants and topless rectangles do not admit hitting pairs too.

Theorem 3.5. *The range family $\mathcal{R}$ for $\mathcal{R} \in \{\mathcal{R}_{SW}, \mathcal{R}_{SE}, \mathcal{R}_{NW}, \mathcal{R}_{NE}, \mathcal{R}_{BL}, \mathcal{R}_{TL}\}$ does not admit hitting pairs.*

3.2 No Hitting Pairs in Half-Planes

In this subsection, we show that the family of all half-planes does not admit hitting pairs too. To show this, we prove the property for a special subfamily.

Definition 3.6. *For a non-vertical line l, the region consisting of all points below l is called the* negative half-plane *bounded by l and it is denoted with $H^-(l)$. We call l the* boundary *of $H^-(l)$. Let $\mathcal{R}_{HP}^-$ denote the family of all negative half-planes bounded by some line of slope between* 91° *and* 93°*, i.e.:*

$$\mathcal{R}_{HP}^- = \{H^-(l) \mid l \textit{ is a vertical line of slope } \in [91°, 93°]\}$$

For simplicity, we call a line of slope between 91° *and* 93° almost vertical.

Lemma 3.7. *The range family $\mathcal{R}^-_{HP}$ does not admit hitting pairs, i.e., for every $m \geq 2$, there is a point set P_m such that the hypergraph $\mathcal{H}(P_m, \mathcal{R}^-_{HP}, m)$ admits no hitting pairs.*

Proof. We prove the statement by induction on m.

Base case, $m = 2$: Consider the point set

$$P'_2 = \{a = (0,0), b = (1,1), c = (2,-1)\}.$$

There exists a line l_1 (respectively l_2) of slope in $(90°, 180°)$ such that $H^-(l_1)$ (respectively $H^-(l_2)$) captures ac (respectively ab) (see Figure 3.2 (a)). Now we first stretch the plane vertically until the slopes of l_1 and l_2 lie between $90°$ and $92°$ and then rotate it by $1°$ counterclockwise so that these slopes lie between $91°$ and $93°$. This way, we obtain the desired point set P_2. Now the hypergraph $\mathcal{H}(P_2, \mathcal{R}^-_{HP}, 2)$ contains hyperedges ac and ab. The only way to hit ac (respectively ab) is to have ac (respectively ab) in the hitting pairs. However, this is impossible since the elements of hitting pairs are pairwise disjoint. So the hypergraph does not admit hitting pairs and P_2 is the desired set.

Inductive step: Assume there exists the desired point set P_m. Similarly to the proof of Lemma 3.3, we construct the set P_{m+1} using two copies P^1_m and P^2_m of P_m. See Figure 3.2 (b) for illustration. First, we place P^1_m into the plane and let

$$\mathcal{H}_1 = (P^1_m, \mathcal{R}^-_{HP}, m) = (P^1_m, \mathcal{E}_1).$$

For every hyperedge $E \in \mathcal{E}_1$, we fix an almost vertical line l_E so that the negative half-plane

$$H_E = H^-(l_E)$$

captures E and let these lines be denoted with

$$L_1 = \{l_E \mid E \in \mathcal{E}_1\}.$$

Next, we pick the second copy P^2_m of P_m. Let

$$\mathcal{H}_2 = \mathcal{H}(P^2_m, \mathcal{R}^-_{HP}, m) = (P^2_m, \mathcal{E}_2).$$

We rotate P^2_m by $60°$ counterclockwise. Now for every hyperedge $E \in \mathcal{E}_2$, there is a line l_E with slope between $151°$ and $153°$ so that the negative half-plane

$$H_E = H^-(l_E)$$

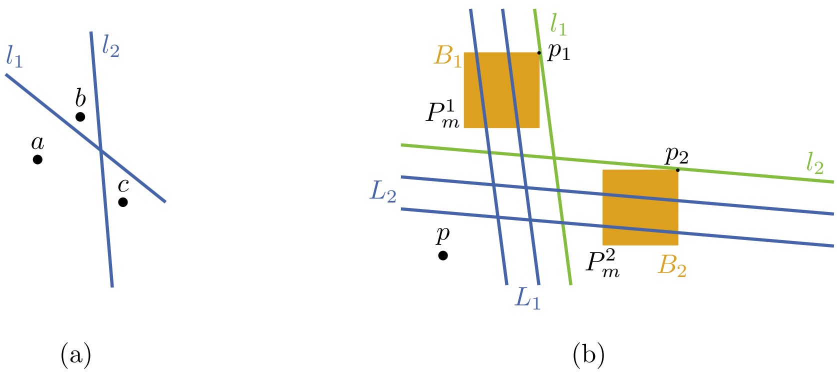


Figure 3.2: (a) Base case, $m = 2$. (b) Inductive step: the point set P_{m+1} is created from two shifted and rotated copies of P_m. The sketch before the final stretching and rotation.

captures E. So let

$$L_2 = \{l_E \mid E \in \mathcal{E}_2\}.$$

Let

$$\alpha_1 = 93°,\ \alpha_2 = 91° + 60° = 151°.$$

For $i \in [2]$, let B_i denote the bounding box of P_m^i, let p_i be its top-right corner, and let l_i be a line of slope α_i going through p_i. It is now possible to put P_m^2 (far) to the bottom-right of P_m^1 so that B_1 lies above l_2 and B_2 lies above l_1. For $i \in [2]$, let

$$H_i = \{H_E \mid E \in \mathcal{E}_i\}.$$

Informally speaking, we move P_m^2 far away so that half-planes H_1 and H_2 do not get new points, i.e., for every $i \in [2]$ and every $E \in \mathcal{E}_i$, we have:

$$H^-(l_E) \cap (P_m^1 \cup P_m^2) = E. \tag{3.1}$$

Recall that the slopes of all lines in L_1 lie between 91° and 93° and the slopes of all lines in L_2 lie between 151° and 153°. So now it is possible to stretch the point set in vertically so that the slope of every line in $L_1 \cup L_2$ lies in the inverval $(90°, 92°)$. Finally, we rotate it by 1° counterclockwise so that every line in $L_1 \cup L_2$ has slope between 91° and 93° as desired. We denote the arising point set with P'_{m+1}. Let

$$\mathcal{H} = \mathcal{H}(P'_{m+1}, \mathcal{R}^-_{HP}, m) = (P'_{m+1}, \mathcal{E}).$$

With a slight abuse of notation, we still denote the stretched point sets with P_m^1 and P_m^2, the stretched lines with L_1 and L_2, and the stretched half-planes with H_1 and H_2. Since the lines in L_1 and L_2 are almost vertical now, together with (3.1), we obtain

$$\mathcal{E}_1, \mathcal{E}_2 \subset \mathcal{E}.$$

Again, since the lines in $L_1 \cup L_2$ are almost vertical and the half-planes in $H_1 \cup H_2$ are negative, it is possible to place a new point p (far) at the bottom-left so that it lies in the intersection of all half-planes from $H_1 \cup H_2$, i.e.,

$$p \in \bigcap_{h \in H_1 \cup H_2} h$$

(see Figure 3.2 (b)). Let P_{m+1} denote the arisen point set and let

$$\mathcal{H}' = (P_{m+1}, \mathcal{R}^-_{HP}, m+1) = (P_{m+1}, \mathcal{E}').$$

Now for every $i \in [2]$ and every $E \in \mathcal{E}_i$, it holds that

$$H^-(l_E) \cap P_{m+1} = E \cup \{p\}. \tag{3.2}$$

In simple words, we added the new vertex p to every hyperedge in $\mathcal{E}_1 \cup \mathcal{E}_2$. For $i \in [2]$, let

$$\mathcal{E}'_i = \{E \cup \{p\} \mid E \in \mathcal{E}_i\},$$

then due to (3.2), it holds that

$$\mathcal{E}'_1 \cup \mathcal{E}'_2 \subset \mathcal{E}'$$

Now suppose $\mathcal{H}'$ admits a set M of hitting pairs.

Case 1: For every $q \in P_{m+1}$: $pq \notin M$; or there exists $q \in P_m^2$ such that $pq \in M$. Since M is a set of hitting pairs, every hyperedge in $\mathcal{E}'_1$ is hit. These hyperedges can only be hit by pairs from

$$M \cap \binom{P_1^m \cup \{p\}}{2}.$$

By the definition of Case 1, they are hit by pairs from

$$M \cap \binom{P_1^m}{2}.$$

Hence, if we remove p from every hyperedge in $\mathcal{E}_1'$ (and obtain $\mathcal{E}_1$), the hyperedges stay hit. So

$$M \cap \binom{P_1^m}{2}$$

is a set of hitting pairs of $\mathcal{H}_1$. This contradicts the induction hypothesis.

Case 2: There exists $q \in P_m^1$ such that $pq \in M$. We apply the analogous argument to the set $\mathcal{E}_2'$ to get hitting pairs of $\mathcal{H}_2$ and obtain a contradiction.

Hence, P_{m+1} satisfies the desired property and the statement holds for $m+1$ too. So by induction, it holds for every $m \in \mathbb{N}$. ☐

A negative half-plane with a boundary of slope between 91° and 93° is a special case of a general half-plane. So we immediately obtain the following result:

Theorem 3.8. *Range family $\mathcal{R}_{HP}$ does not admit hitting pairs.*

3.3 Hitting Pairs in Vertical Strips

Now we show that vertical strips admit hitting pairs.

Lemma 3.9. *For any $m \geq 3$ and any point set V, the hypergraph $\mathcal{H}(V, \mathcal{R}_{VS}, m)$ admits a set of hitting pairs.*

Proof. We first show that it suffices to show the claim for $m = 3$. Suppose M is a set of hitting pairs of

$$\mathcal{H} = \mathcal{H}(V, \mathcal{R}_{\mathrm{VS}}, 3) = (V, \mathcal{E}).$$

Consider a hyperedge E' of

$$\mathcal{H}' = \mathcal{H}(V, \mathcal{R}_{\mathrm{VS}}, m)$$

for $m \geq 3$. Since $\mathcal{R}_{\mathrm{VS}}$ is good (see Lemma 2.4), there is a hyperedge E of $\mathcal{H}$ with $E \subseteq E'$. Since M is a set of hitting pairs of $\mathcal{H}$, the hyperedge E is hit so E' is hit too. Thereby, M is a set of hitting pairs of $\mathcal{H}'$.

Now we show that $\mathcal{H}$ admits a set of hitting pairs. Let $v_1, \ldots, v_n$ be the elements of V in the order of increasing x-coordinates. We claim that

$$M = \{v_{2i+1}v_{2i+2} \mid 2 \leq 2i + 2 \leq n\}$$

are hitting pairs of $\mathcal{H}$ (see Figure 3.3 (a)). Clearly, the elements of M are pairwise disjoint. It remains to show that every hyperedge is hit by M. Consider an arbitrary hyperedge

$$E = v_i v_{i+1} v_{i+2}$$

for some $1 \leq i \leq n-2$ (see Observation 2.2). If i is odd, then $v_i v_{i+1} \in M$. Otherwise $i+1$ is odd and $v_{i+1}v_{i+2} \in M$. Hence, E is hit. So M is the desired set. ☐

For symmetry reasons, this also holds for horizontal and diagonal strips:

Theorem 3.10. *Let $\mathcal{R} \in \{\mathcal{R}_{VS}, \mathcal{R}_{HS}, \mathcal{R}_{DS}\}$. For any $m \geq 3$ and any point set V, the hypergraph $\mathcal{H}(V, \mathcal{R}, m)$ admits a set of hitting pairs.*

Now we can apply Theorem 3.2 and reprove the result of Aloupis et al.:

Theorem 3.11 ([ACC+11])**.** *For the range family $\mathcal{R}_{VS} \cup \mathcal{R}_{HS}$, it holds that $m(2) \leq 3$.*

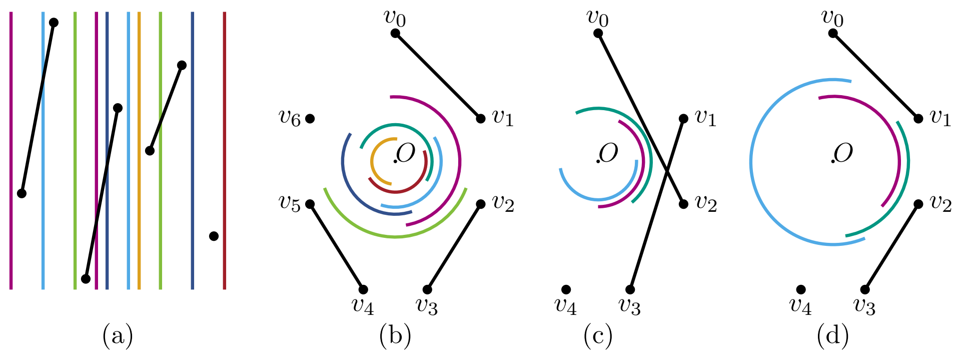


Figure 3.3: Hitting pairs are depicted with black straight-line segments. The boundaries of ranges are represented with different colors. (a) Hitting pairs in vertical strips. (b) Hitting pairs in wedges. (c) – (d) No hitting pairs in wedges for $m = 3$ and n odd (here $n = 5$).

3.4 Hitting Pairs in Wedges

Here we show that wedges admit hitting pairs too.

Lemma 3.12. *Let $m \geq 4$ and let V be a point set V. The hypergraph $\mathcal{H}(V, \mathcal{R}_W, m)$ admits a set of hitting pairs.*

The proof is very similar to the previous section but we need to be careful with wedges containing the x-ray.

Proof. Similarly to the previous section, it suffices to prove the statement for $m = 4$ since the range family of all wedges is good (see Lemma 2.4). Let $v_0, v_1, \ldots, v_{n-1}$ be the points in V in the order of increasing $\alpha(\cdot)$-values. We claim that

$$M = \{v_{2i}v_{2i+1} \mid 1 \leq 2i + 1 \leq n - 1\}$$

is a set of hitting pairs of $\mathcal{H}(V, \mathcal{R}, 4) = (V, \mathcal{E})$ (see Figure 3.3 (b)). Clearly, the elements of M are pairwise disjoint. It remains to show that every hyperedge in $\mathcal{E}$ is hit by M. Consider any hyperedge $E = v_iv_{i+1}v_{i+2}v_{i+3} \in \mathcal{E}$ with indices modulo n (see Observation 2.3).

1. If $0 \notin \{i, i+1, i+2\}$, then $v_iv_{i+1} \in M$ if i is even; and $v_{i+1}v_{i+2} \in M$ if i is odd.
2. If $0 \in \{i, i+1, i+2\}$, then $v_0v_1 \subset E$ with $v_0v_1 \in M$.

Hence, the hyperedge E is hit. So M is a set of hitting pairs of $\mathcal{H}(V, \mathcal{R}, 4)$. $\square$

This result and Theorem 3.2 reprove the result of Aloupis et al.:

Lemma 3.13 ([ACC+11]). *Let O_1 and O_2 be two centers. For $i \in [2]$, let $\mathcal{R}_i$ be the range family of wedges with center O_i. Then for the range family $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$, it holds that $m(2) \leq 4$.*

Note that the proof of Lemma 3.12 does not work for $m = 3$ if the number of points $n \geq 5$ is odd. In this case, the hyperedge $p_{n-2}p_{n-1}p_0$ is not hit. This statement can even be strengthened as follows:

Lemma 3.14. *Let V be a point set whose cardinality $|V| \geq 5$ is odd. Then the hypergraph $\mathcal{H}(V, \mathcal{R}_W, 3)$ does not admit a set of hitting pairs.*

Proof. Let $v_0, v_1, \ldots, v_{n-1}$ be the points in V in the order of increasing $\alpha(\cdot)$-values. Suppose there is a set of hitting pairs M of $\mathcal{H}(V, \mathcal{R}_{\mathrm{W}}, 3)$. Consider the hyperedge $v_0v_1v_2$ which must be hit by M.

Case 1, $v_0v_2 \in M$: For illustration, see Figure 3.3 (c). Consider the hyperedge $v_1v_2v_3$ which must also be hit. Since v_2 is already used in M, it must hold $v_1v_3 \in M$. Finally, consider the hyperedge $v_2v_3v_4$. Since v_2 and v_3 are already used in M and $v_0, v_1 \notin v_2v_3v_4$, this hyperedge is not hit. A contradiction.

Case 2: Without loss of generality, assume $v_0v_1 \in M$ (due to the cyclic ordering of points, the case $v_1v_2 \in M$ is symmetric). For illustration, see Figure 3.3 (d). We now show by induction that for every even number $i \leq n-3$, we have $v_iv_{i+1} \in M$.

Base case, $i = 0$**:** By assumption, it holds that $v_0v_1 \in M$.

Inductive step: Assume $v_iv_{i+1} \in M$ for some even $i \leq n-5$. Consider the hyperedge $v_{i+1}v_{i+2}v_{i+3}$, it is hit by M. Since v_{i+1} is already used in M, we must have $v_{i+2}v_{i+3} \in M$. So the claim holds for $i+2$ too.

By induction, it holds for every even number $i \leq n-3$. Finally, consider the hyperedge $v_{n-2}v_{n-1}v_0$. Since $n-2$ is odd, we have $v_{n-3}v_{n-2} \in M$ and we also have $v_0v_1 \in M$. Hence, the hyperedge $v_{n-2}v_{n-1}v_0$ is not hit by M. A contradiction.

Therefore, the assumption was wrong and no hitting pairs exist. □

4. Explicit Colorings

In this chapter, we explicitly construct polychromatic colorings for some subfamilies of the family $\mathcal{R}_{\mathrm{Q}}$ of all axis-aligned quadrants.

We start with an approach providing several degrees of freedom: this will allow us to apply it in different contexts.

Lemma 4.1. *Let $r \geq k \in \mathbb{N}$ and let V be a point set with $|V| \geq r$. Let $\mathcal{H} = \mathcal{H}(V, \mathcal{R}_{SE}, r)$. Let $E_1, E_2, \ldots, E_t$ be the hyperedges of $\mathcal{H}$ in the order of increasing horizontal boundaries. Further, let $V' \subseteq V$, $j \in [t]$, and $c' : V' \to [k]$ be a partial coloring such that:*

1. *For every $i \leq j$, E_i is polychromatic in c'.*
2. *For every $i > j$, $\mathrm{top}(E_i) \notin V'$, i.e., the topmost vertex is not colored.*

Then there exist a point set V'' and a partial coloring $c'' : V'' \to [k]$ such that:

1. *$V'' \subseteq V' \cup \{\mathrm{top}(E_i) \mid i > j\}$,*
2. *c'' extends c', i.e., for all $v \in V'$: $c''(v) = c'(v)$,*
3. *and c'' is a polychromatic (partial) coloring of $\mathcal{H}$.*

To create the desired coloring, we start with $c'' = c'$, $i := j + 1$ and until $i > t$, repeat:

- *color the vertex $\mathrm{top}(E_i)$ with the (unique) color from $[k] \setminus (c(E_i \cap E_{i-1}))$ if this set is non-empty,*
- *remain it uncolored otherwise.*

and set $i := i + 1$ (we call this one iteration step *i).*

Proof. We prove that this procedure indeed produces the desired coloring. We show that the following invariant is maintained: for every $i \in \{j, \ldots, t\}$, (immediately) before step $i + 1$,

- the hyperedges $E_1, \ldots, E_i$ are polychromatic in c and
- for every $r \geq i + 1$ the vertex $\mathrm{top}(E_r)$ is uncolored yet.

The property holds before step $j + 1$ due to the prerequisites of lemma. So now suppose it holds before step $i + 1$ for some $i \in \{j, \ldots, t - 1\}$. In particular, we have:

$$c\,(E_i) = [k]$$

and the vertex $\operatorname{top}(E_{i+1})$ is uncolored. First of all, this implies that after we color this vertex, the hyperedges $E_1, \ldots, E_i$ remain polychromatic and the vertices $\operatorname{top}(E_r)$ for $r \geq i+2$ remain uncolored due to

$$\operatorname{top}(E_r) \neq \operatorname{top}(E_{i+1})$$

(see Lemma 2.11 Item 1). So it now suffices to show that the hyperedge E_{i+1} is polychromatic after step $i+1$. Recall that by Lemma 2.14 Item 3, we have:

$$|E_i \setminus E_{i+1}| = 1.$$

Thus:

$$|c\left(E_i \cap E_{i+1}\right)| \geq |c(E_i)| - |c\left(E_i\right) \setminus c\left(E_{i+1}\right)| = k - 1.$$

So if

$$|c\left(E_i \cap E_{i+1}\right)| = k,$$

then the hyperedge E_{i+1} is already polychromatic and the invariant still holds before step $i+2$. Otherwise, it holds that

$$|c\left(E_i \cap E_{i+1}\right)| = k - 1.$$

So

$$|[k] \setminus \left(c\left(E_i \cap E_{i+1}\right)\right)| = 1,$$

i.e., there is exactly one color missing in E_{i+1} and the invariant holds after we color $\operatorname{top}(E_{i+1})$ with this color as described in the lemma.

Thus, the invariant holds after every step $i \in \{j+1, \ldots, t\}$ and in particular, it holds after step t. Thereby, after the process terminates, the hyperedges $E_1, \ldots, E_t$ (i.e., all hyperedges of the hypergraph $\mathcal{H}$) are polychromatic and Item 3 holds. By construction, during this process, we have only colored some vertices in

$$\{\operatorname{top}(E_i) \mid i > j\}$$

so Item 1 holds too. Finally, the invariant ensures that we only color vertices uncolored in c' so Item 2 is satisfied as well. □

We refer to the process in Lemma 4.1 as *staircase-coloring*. First, this procedure almost immediately implies the bound for south-east quadrants reproving the result of Keszegh and Pálvölgyi [KP15].

4.1 South-East Quadrants

Theorem 4.2 ([KP15])**.** *For the range family*

$$\mathcal{R} \in \{\mathcal{R}_{SE}, \mathcal{R}_{SW}, \mathcal{R}_{NE}, \mathcal{R}_{NW}\},$$

we have $m(k) = k$ for all $k \in \mathbb{N}$.

Proof. For symmetry reasons, it suffices to prove the statement for $\mathcal{R} = \mathcal{R}_{\mathrm{SE}}$. First, we recall that it trivially holds that $m(k) \geq k$ for all $k \in \mathbb{N}$: a hyperedge of size less than k can not contain vertices in all k colors.

To prove the upper bound, we will make use of Lemma 4.1 to construct a polychromatic coloring. Let V be a point set and $k \in \mathbb{N}$. Let

- $r = k$,

- $V' = E_b(V, k)$,
- $j = 1$,
- and c' colors the vertices of $E_b(V, k)$ with pairwise distinct colors.

Then the prerequisites of Lemma 4.1 are satisfied and c' can be extended to a polychromatic (partial) coloring of $\mathcal{H}(V, \mathcal{R}_{\mathrm{SE}}, r = k)$. To obtain a coloring, we color the remaining vertices arbitrarily. □

As already mentioned, in Lemma 4.1, we allow several parameters to make it possible to apply the staircase-coloring in different contexts. Note that symmetrical versions of the staircase-coloring can also be applied to hyperedges captured by south-west, north-east, or north-west quadrants. So now using the staircase-coloring, we prove the theorem about the coloring of hypergraphs determined by "adjacent" (e.g., south-east and south-west) quadrants.

4.2 South-West and South-East Quadrants

Lemma 4.3. *For the range family*

$$\mathcal{R} \in \{\mathcal{R}_{SW} \cup \mathcal{R}_{SE}, \mathcal{R}_{SE} \cup \mathcal{R}_{NE}, \mathcal{R}_{NE} \cup \mathcal{R}_{NW}, \mathcal{R}_{NW} \cup \mathcal{R}_{SW}\},$$

we have $m(k) \leq 2k$ for every $k \in \mathbb{N}$.

Proof. For symmetry reasons, it suffices to prove the statement for $\mathcal{R} = \mathcal{R}_{\mathrm{SW}} \cup \mathcal{R}_{\mathrm{SE}}$. Let V be a point set and let $k \in \mathbb{N}$. Let

$$\mathcal{H} = \mathcal{H}(V, \mathcal{R}_{\mathrm{SW}} \cup \mathcal{R}_{\mathrm{SE}}, 2k) = (V, \mathcal{E}_{\mathrm{SW}} \cup \mathcal{E}_{\mathrm{SE}})$$

where $\mathcal{E}_{\mathrm{SW}}$ (respectively $\mathcal{E}_{\mathrm{SE}}$) denotes the set of hyperedges captured by south-west (respectively south-east) quadrants. We explicitly construct a polychromatic coloring c of $\mathcal{H}$ with k colors.

We start with a partial coloring c in which no vertex is colored. If $|V| < 2k$, there is no hyperedge in $\mathcal{H}$ so every coloring of V is trivially polychromatic. Otherwise, let v be the topmost vertex in V that occurs in a hyperedge from $\mathcal{E}_{\mathrm{SW}}$ and a hyperedge from $\mathcal{E}_{\mathrm{SE}}$, i.e.:

$$v = \operatorname*{argmax}_{\substack{w \in V: \\ \exists E_1 \in \mathcal{E}_{\mathrm{SW}}, E_2 \in \mathcal{E}_{\mathrm{SE}}: \\ w \in E_1 \cap E_2}} y(w) \tag{4.1}$$

By Lemma 2.16 and its symmetrical version, it holds that

$$E_b(V, 2k) \in \mathcal{E}_{\mathrm{SW}} \cap \mathcal{E}_{\mathrm{SE}}.$$

So the vertex v is well-defined. In particular, it holds that

$$y(v) \geq h(E_b(V, 2k)). \tag{4.2}$$

Since $E_b(V, 2k)$ is the hyperedge with the bottommost horizontal boundary (see Lemma 2.16 again), for every vertex $w \in E_b(V, 2k)$ and every hyperedge $E \in \mathcal{E}_{\mathrm{SW}} \cup \mathcal{E}_{\mathrm{SE}}$, we have:

$$y(w) \leq h(E).$$

So for $E \in \mathcal{E}_{\mathrm{SE}}$, we have:

$$w \in E \Leftrightarrow x(w) \geq v(E). \tag{4.3}$$

Due to
$$|E_b(V, 2k) \setminus \{v\}| \geq 2k - 1,$$
there exist at least k points in $E_b(V, 2k) \setminus \{v\}$ lying on the same side (left or right) of v. Without loss of generality, we assume that at least k points in $E_b(V, 2k) \setminus \{v\}$ lie to the right of v (otherwise, the construction is symmetrical). So let $w_1, \ldots, w_k$ be k such points. Then for all $i \in [k]$, we have:
$$x(w_i) > x(v), y(w_i) < y(v).$$

We color the point w_i with color i for every $i \in [k]$ (see Figure 4.1 (2)). Now consider the hyperedge $E_v^{\mathrm{SE}} \in \mathcal{E}_{\mathrm{SE}}$ such that
$$\mathrm{top}(E_v^{\mathrm{SE}}) = v.$$

This hyperedge indeed exists:

- If $v \in E_b(V, 2k)$, then by (4.2) the vertex v is the topmost vertex of $E_b(V, 2k)$ and
$$E_v^{\mathrm{SE}} = E_b(V, 2k)$$
is the desired hyperedge.
- Otherwise, it holds that $y(v) > h(E_b(V, 2k))$. Recall that $v \in E$ for some $E \in \mathcal{E}_{\mathrm{SE}}$. If $\mathrm{top}(E) = v$, then $E_v^{\mathrm{SE}} = E$ is the desired hyperedge. Otherwise, we have $y(v) < h(E)$ and $y(v) > h(E_b(V, 2k))$. Due to $E_b(V, 2k) \in \mathcal{E}_{\mathrm{SE}}$ and Lemma 2.13, there exists a hyperedge $E_v^{\mathrm{SE}} \in \mathcal{E}_{\mathrm{SE}}$ such that $\mathrm{top}(E_v^{\mathrm{SE}}) = v$ as desired.

In particular, it holds that
$$v \in E_v^{\mathrm{SE}}$$
so
$$v(E_v^{\mathrm{SE}}) \leq x(v).$$

Let $E \in \mathcal{E}_{\mathrm{SE}}$ be an arbitrary hyperedge such that
$$h(E) \leq y(v) = h(E_v^{\mathrm{SE}}).$$

Then by Lemma 2.11, we have
$$v(E) \leq v(E_v^{\mathrm{SE}}).$$

So for every $i \in [k]$, it holds that
$$v(E) \leq v(E_v^{\mathrm{SE}}) \leq x(v) < x(w_i).$$

By (4.3), we obtain
$$w_i \in E.$$

So E contains points in all k colors, i.e.:
$$c(E) = [k]. \tag{4.4}$$

Now we apply the staircase-coloring from Lemma 4.1 with

- $r = 2k$,
- $V' = \{w_1, \ldots, w_k\}$,
- $c'(w_i) = i$ for all $i \in [k]$,
- and $j = 1$.

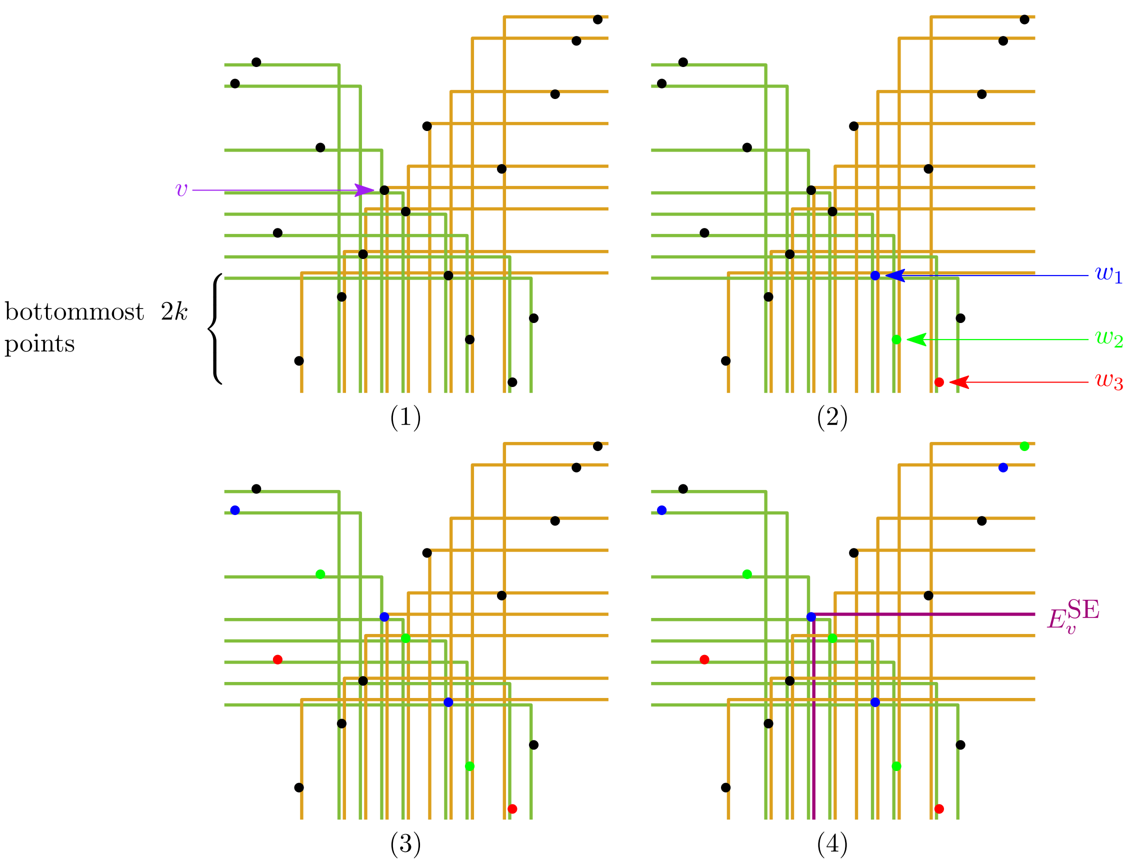


Figure 4.1: Polychromatic coloring for $\mathcal{R} = \mathcal{R}_{\mathrm{SW}} \cup \mathcal{R}_{\mathrm{SE}}$, here $k = 3$.
(1) South-west quadrants are green, south-east quadrants are orange.
(2) Among the bottommost $2k = 6$ points, we color $k = 3$ lying on the same side of v (here: right) with $k = 3$ distinct colors.
(3) Staircase-coloring of SW-quadrants.
(4) Staircase-coloring of SE-quadrants with horizontal boundary above v.

to south-west quadrants. The prerequisites of Lemma 4.1 are satisfied so the staircase-coloring is applied successfully. We obtain a partial coloring in which all hyperedges from $\mathcal{E}_{\mathrm{SW}}$ are polychromatic (see Figure 4.1 (3)). Let c'' denote the arising coloring and let V'' denote the set of colored points. Consider a vertex $\mathrm{top}(E)$ for some $E \in \mathcal{E}_{\mathrm{SE}}$ with $h(E) > y(v)$. By the choice of v (see (4.1)), the vertex $\mathrm{top}(E)$ does not occur in any hyperedge from $\mathcal{E}_{\mathrm{SW}}$ and in particular, we have:

$$\mathrm{top}(E) \notin V''. \tag{4.5}$$

Now due to (4.4), it only remains to extend the coloring so that the hyperedges $E \in \mathcal{E}_{\mathrm{SE}}$ with

$$h(E) > y(v)$$

are polychromatic too. Let $E_1, \ldots, E_t$ be the elements of $\mathcal{E}_{\mathrm{SE}}$ in the order of increasing horizontal boundaries and let $j' \in [t]$ be such that

$$E_{j'} = E_v^{\mathrm{SE}}.$$

Now we can apply the staircase-coloring from Lemma 4.1 to south-east quadrants to extend the current coloring. We use:

- $r = 2k$,

- $V' = V''$,
- $c' = c''$,
- and $j = j'$.

First, for every $i \leq j'$, we have:

$$h(E_i) \leq h(E_{j'}) = h(E_v^{\mathrm{SE}}) = y(v)$$

so by (4.4), it holds that

$$c''(E) = [k].$$

Further, for every $i > j'$, we have

$$y(\mathrm{top}(E_i)) = h(E_i) > h(E_{j'}) = h(E_v^{\mathrm{SE}}) = y(v)$$

and by (4.5), we obtain

$$\mathrm{top}(E_i) \notin V'',$$

i.e., the vertex $\mathrm{top}(E_i)$ is not colored yet. So the prerequisites of Lemma 4.1 are satisfied. As a result, we obtain an extension c of c'' in which the hyperedges from $\mathcal{E}_{\mathrm{SE}}$ are polychromatic as well. So by construction, every hyperedge of $\mathcal{H}$ is polychromatic in the obtained coloring. To make it a polychromatic (total) coloring, we can color the remaining points arbitrarily. □

4.3 2-Coloring of North-West and South-East Quadrants

In this section, we prove that for the range family of all north-west and south-east quadrants, it holds that $m(2) = 3$.

Lemma 4.4. *For the range family $\mathcal{R}_{NW} \cup \mathcal{R}_{SE}$, we have $m(2) \geq 3$.*

Proof. Consider the point set $V = \{a = (0,0), b = (1,1), c = (2,-1)\}$ (see Figure 4.2) and the hypergraph $\mathcal{H} = \mathcal{H}(V, \mathcal{R}, 2)$. Then $\mathcal{H}$ has hyperedges ac, bc captured by south-east quadrants and a hyperedge ab captured by a north-west quadrant. Thus, in every polychromatic coloring with 2 colors, the vertices a, b, c must have pairwise distinct colors (otherwise we would have a monochromatic hyperedge). A contradiction. So $\mathcal{H}$ admits no polychromatic coloring with 2 colors and $m(2) \geq 3$. □

To prove $m(2) \leq 3$, we present an algorithm creating the desired 2-coloring. At every step, a vertex v either has one of the colors $c(v) = 0$, or $c(v) = 1$, or it has not been colored yet. The latter is denoted with $c(v) = \text{none}$. As soon as a vertex is assigned a color, its color is never changed.

From now on, for a hypergraph $\mathcal{H}(V, \mathcal{R}_{\mathrm{NW}} \cup \mathcal{R}_{\mathrm{SE}}, 3)$, we denote the set of hyperedges captured by $\mathcal{R}_{\mathrm{NW}}$ (respectively $\mathcal{R}_{\mathrm{SE}}$) with $\mathcal{E}_{\mathrm{NW}}$ (respectively $\mathcal{E}_{\mathrm{SE}}$).

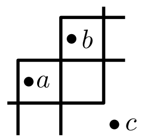


Figure 4.2: $m(2) \geq 3$ for the range family of all north-west and south-east quadrants

Definition 4.5. *Let $\mathcal{R} = \mathcal{R}_{NW} \cup \mathcal{R}_{SE}$. For a hypergraph $\mathcal{H} = \mathcal{H}(V, \mathcal{R}, 3)$, the* greedy coloring *is defined as follows. We start with all vertices being uncolored, i.e., for the initial coloring $c : V \to \{0, 1, \text{none}\}$ we have $c(v) = \text{none}$ for all $v \in V$. Now we process the hyperedges of $\mathcal{H}$ in the order of increasing horizontal boundaries. In case of a tie, we first process a hyperedge in $\mathcal{E}_{SE}$. Let E be the current hyperedge.*

If $E \in \mathcal{E}_{SE}$, for $d \in \{0, 1\}$, we repeat the following. If the hyperedge does not contain a point of color d, i.e.,

$$\{v \in E \mid c(v) = d\} = \emptyset,$$

*we color the **rightmost** uncolored vertex in E with d, i.e., we set:*

$$c(\text{right}(\{v \in E \mid c(v) = \text{none}\})) := d.$$

Otherwise, we have $E \in \mathcal{E}_{NW}$. For $d \in \{0, 1\}$, we repeat the following. If the hyperedge does not contain a point of color d, i.e.,

$$\{v \in E \mid c(v) = d\} = \emptyset,$$

*we color the **topmost** uncolored vertex in E with d, i.e., we set:*

$$c(\text{top}(\{v \in E \mid c(v) = \text{none}\})) := d.$$

We say that the vertices colored in this step are colored due to the hyperedge E*. If a vertex v has been colored due to a hyperedge E, we denote this with $E_v = E$ and write*

$$t(v) = h(E_v).$$

Then $t(v)$ stands for the timestamp of coloring v. In particular, if for two vertices a, b we have $t(a) < t(b)$, then a has been colored before b.

For $h \in \mathbb{R}$, with c_h we denote the (partial) coloring arising in this procedure if we only process hyperedges E with $h(E) < h$ (imagine that we run the algorithm and interrupt it as soon as we encounter the first hyperedge E with $h(E) \geq h$). Similarly, for a hyperedge E, with c_E we denote the coloring arising in this procedure if we only process hyperedges preceding E (imagine that we run the algorithm and interrupt it as soon as we encounter the hyperedge E). And similarly, for a vertex v such that E_v is well-defined, with c_v we denote the coloring arising in this procedure if we interrupt the process immediately before v gets colored. Finally, for the vertices colored during the greedy coloring, let $<_t$ denote the ordering in which they have been colored.

After every hyperedge has been processed, we can color the remaining vertices arbitrarily, e.g.,

$$\forall v \in V \text{ with } c(v) = \text{none} : c(v) := 0.$$

Clearly, every hyperedge is processed in some step of the greedy coloring. So if the corresponding iteration has run successfully, then afterward the hyperedge contains vertices of both colors as desired. Hence, to prove that the algorithm produces the desired polychromatic coloring with 2 colors, it suffices to show that in every iteration we are able to find the required uncolored vertex. Before we move on to the proof, we state several helpful observations.

Since the hyperedges are processed in the order of increasing horizontal boundaries, the following holds.

Observation 4.6. *For two vertices v, w such that E_v, E_w are well-defined, the following holds:*

$$t(v) < t(w) \Rightarrow v <_t w.$$

Now observe that for every hyperedge E and every color $d \in \{0, 1\}$, at most one vertex is colored with d due to E, i.e.:

Observation 4.7. *Let $a, b \in V$ and $h \in \mathbb{R}$. If $\text{none} \neq c_h(a) = c_h(b)$, then $E_a \neq E_b$.*

Next, observe that a vertex is only colored with some color d due to a hyperedge E if this hyperedge did not contain any vertex in color d before.

Observation 4.8. *Let $a \in V$ be such that E_a is well-defined and a is colored with $d \in \{0, 1\}$. Then:*

$$\forall v \in E_a : c_a(v) \neq d$$

and in particular,

$$\forall v \in E_a : c_{E_a}(v) \neq d.$$

Observation 4.9. *Let $a, b \in V$ be such that $x(a) < x(b)$, $y(a) > y(b)$ and let $h \in \mathbb{R}$. If $c_h(a), c_h(b) \neq \text{none}$ and $E_b \in \mathcal{E}_{NW}$, then $c_h(a) \neq c_h(b)$.*

Proof. For illustration, see Figure 4.3 (a). Suppose $d = c_h(a) = c_h(b)$ for some $d \in \{0, 1\}$. In particular, this implies that E_a and E_b are well-defined. Since $b \in E_b$, by Observation 2.10 it holds that $a \in E_b$.

Case 1: $a <_t b$. Then $c_b(a) = d$. This contradicts Observation 4.8.

Case 2: $b <_t a$. Then $c_b(a) = \text{none}$, i.e., at the moment b was chosen to be colored, the hyperedge E_b contained an uncolored vertex a above b – a contradiction. □

The symmetric version of this statement also holds for hyperedges in $\mathcal{E}_{\text{SE}}$.

Observation 4.10. *Let $a, b \in V$ and $h \in \mathbb{R}$ be such that $y(a) > y(b) > h$. Then one of the following properties holds:*

- $c_h(a) = \text{none}$,
- $c_h(b) = \text{none}$,
- *or* $c_h(a) \neq c_h(b)$.

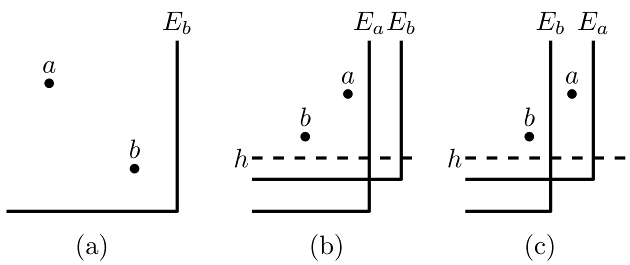


Figure 4.3: Illustrations for (a) Observation 4.9 and (b) – (c) Observation 4.10

Proof. Suppose none of these properties holds, i.e.,

$$d = c_h(a) = c_h(b) \text{ for some } d \in \{0, 1\}.$$

In particular, we have

$$t(a), t(b) < h.$$

Next, suppose that for some $v \in \{a, b\}$ it holds that $E_v \in \mathcal{E}_{\mathrm{SE}}$, then

$$h(E_v) \geq y(v) > h.$$

But then E_v has not been processed yet, i.e., $c_h(v) = \text{none}$ – a contradiction. So

$$E_a, E_b \in \mathcal{E}_{\mathrm{NW}}.$$

In particular, by Observation 4.7, we have $E_a \neq E_b$ and hence, $t(a) \neq t(b)$. If $x(a) < x(b)$, then by the previous observation, we have $c_h(a) \neq c_h(b)$ – a contradiction. So we have $x(a) > x(b)$.

Case 1: $t(a) < t(b)$ (see Figure 4.3 (b)). So $v(E_a) < v(E_b)$ (recall Lemma 2.11). Then

$$h(E_b) = t(b) < h < y(a), v(E_b) > v(E_a) \geq x(a).$$

So $a \in E_b$ and $c_{E_b}(a) = d$. This contradicts Observation 4.8.

Case 2: $t(b) < t(a)$ (see Figure 4.3 (c)). We have

$$h(E_a) < h < y(b), v(E_a) \geq x(a) > x(b).$$

So $b \in E_a$ and $c_{E_a}(b) = d$ – this contradicts Observation 4.8.

□

Observation 4.11. *Let $a, b \in V$ be such that $c(a) = d \in \{0, 1\}$, $E_a \in \mathcal{E}_{NW}$, $y(b) > y(a)$, and $b \in E_a$. Then $c_a(b) = 1 - d$ and in particular $b <_t a$.*

Proof. Suppose $c_a(b) \in \{\text{none}, d\}$. If $c_a(b) = \text{none}$, then a would not be colored due to E_a because at the moment a was chosen to be colored with d due to E_a, this hyperedge contained an uncolored vertex b above a – a contradiction. If $c_a(b) = d$, then this contradicts Observation 4.8. □

Using these properties, we can now prove the correctness of our construction:

Lemma 4.12. *For any hypergraph $\mathcal{H}(V, \mathcal{R}_{NW} \cup \mathcal{R}_{SE}, 3) = (V, \mathcal{E}_{SE} \cup \mathcal{E}_{NW})$, the greedy-coloring is well-defined, i.e., we can process every hyperedge as described in Definition 4.5.*

Proof. Suppose not. So at some step we encounter a hyperedge $E = \{e, f, g\}$ which can not be processed, i.e., at some moment when E is processed we have

$$c(e) = c(f) = c(g) = d$$

for some $d \in \{0, 1\}$. Without loss of generality, assume

$$x(e) < x(f) < x(g).$$

By Observation 4.7, at most one of these vertices could have been colored due to E. So let $k \neq l \in E$ be such that

$$c_E(k) = c_E(l) = d.$$

Suppose a vertex $r \in E \setminus \{k, l\}$ has been colored due to E, i.e., $E_p = E$. But then

$$c_r(k) = c_E(k) = d \text{ and } k \in E_p = E.$$

This contradicts Observation 4.8. So every vertex of E has been colored before E was processed.

Suppose $E \in \mathcal{E}_{\mathrm{NW}}$. Let $h \in \mathbb{R}$ such that for every $v' \in E$ with $v' \neq \mathrm{bottom}(E)$ we have

$$y(v') > h > y(\mathrm{bottom}(E)) = h(E).$$

Then for $\{m, n\} = E \setminus \{\mathrm{bottom}(E)\}$, we have

$$c_h(m) = c_h(n) = d \neq \text{none} \text{ and } y(m), y(n) > h.$$

This contradicts Observation 4.10. So

$$E \in \mathcal{E}_{\mathrm{SE}}.$$

Recall that e, f, g have been colored before E was processed and recall that in case of ties we first process hyperedges captured by south-east quadrants. Thereby, we have

$$t(e), t(f), t(g) < h(E)$$

and

$$c_{h(E)}(e) = c_{h(E)}(f) = c_{h(E)}(g) = d.$$

Next, we show that the only vertex in E that could have been colored due to a hyperedge from $\mathcal{E}_{\mathrm{SE}}$ is

$$w = \mathrm{bottom}(E).$$

Suppose for $v \neq w \in E$, we have $E_v \in \mathcal{E}_{\mathrm{SE}}$. Then it holds that

$$y(w) < y(v) \leq t(v) = h(E_v).$$

If $x(v) < x(w)$ (see Figure 4.4 (a)), then this contradicts the symmetrical version of Observation 4.9. So it holds that

$$x(w) < x(v).$$

Case 1: $w <_t v$ (see Figure 4.4 (b) and (c)). It holds that

$$t(v) = h(E_v) < h(E).$$

By Lemma 2.11, we then have $v(E_v) < v(E)$. Due to $w \in E$, it also holds that $v(E) \leq x(w)$ so

$$v(E_v) < v(E) \leq x(w).$$

Since w is the bottommost vertex of E, we also have:

$$y(w) < y(v) \leq h(E_v).$$

Thereby, $w \in E_v$ and $c_v(w) = d$. This contradicts Observation 4.8.

Case 2: $v <_t w$. In particular, by Observation 4.6, we have $t(v) \leq t(w)$. Suppose $E_w \in \mathcal{E}_{\mathrm{NW}}$ (see Figure 4.4 (c)). Then

$$t(w) = h(E_w) \leq y(w) < y(v) \leq t(v).$$

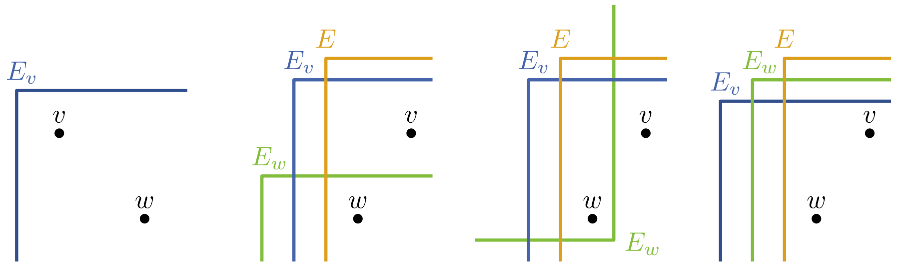


Figure 4.4: The case distinction to prove that bottom(E) is the only point that could have been colored due to a hyperedge from $\mathcal{E}_{\text{SE}}$.

A contradiction. So we have

$$E_w \in \mathcal{E}_{\text{SE}}$$

(see Figure 4.4 (d)). We know that

$$h(E) > h(E_w) = t(w) \geq t(v) \geq y(v).$$

By Lemma 2.11, we then have $v(E) > v(E_w)$ and due to $v \in E$, it holds that $x(v) > v(E)$ so

$$x(v) > v(E) > v(E_w).$$

Thereby, $v \in E_w$ and $c_w(v) = d$. This contradicts Observation 4.8.

So indeed the only vertex in E that could have been colored due to a hyperedge from $\mathcal{E}_{\text{SE}}$ is the vertex $w = \text{bottom}(E)$. Let

$$\{p, q\} = E \setminus \{\text{bottom}(E)\}$$

and without loss of generality, assume

$$x(p) < x(q).$$

Now we know that

$$E_p, E_q \in \mathcal{E}_{\text{NW}}.$$

Since $c(p) = c(q) = d$, by Observation 4.7, it holds that $E_p \neq E_q$ and in particular, by the symmetrical version of Lemma 2.11 Item 1, we have

$$t(p) = h(E_p) \neq h(E_q) = t(q).$$

The case $y(p) > y(q)$ (see Figure 4.5 (a)) contradicts Observation 4.9. So

$$y(p) < y(q).$$

Thus,

$$q = \text{top}(E) \text{ and } h(E) = y(q).$$

We again distinguish between three cases.

Case 1: $q \in E_p$. Due to $y(q) > y(p)$ and Observation 4.11, it holds that $c_p(q) = 1 - d$. This contradicts $c(q) = d$.

Case 2: $p \in E_q$

- If $t(p) < t(q)$, then $c_{E_q}(p) = d$ and $p \in E_q$ – this contradicts Observation 4.8.

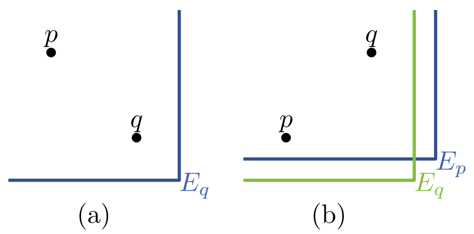


Figure 4.5: Case distinction for the non-bottommost vertices $p, q \in E$.

- If $t(q) < t(p)$ (see Figure 4.5 (b)), it holds that:

$$h(E_q) = t(q) < t(p) = h(E_p) \leq y(p) < y(q).$$

By Lemma 2.11, we have $v(E_q) < v(E_p)$. Together with $x(q) \leq v(E_q)$, this implies

$$x(q) \leq v(E_q) < v(E_p).$$

Hence, $q \in E_p$ – a contradiction due to Case 1.

Case 3: None of the above cases applies. Suppose $t(q) \leq y(p)$: together with

$$v(E_q) \geq x(q) > x(p),$$

we get $p \in E_q$ – a contradiction. So

$$t(q) > y(p) \geq t(p). \tag{4.6}$$

Now suppose $v(E_p) \geq x(q)$, then together with

$$h(E_p) \leq y(p) < y(q)$$

we obtain $q \in E_p$ – a contradiction. So

$$v(E_p) < x(q).$$

For illustration, see Figure 4.6. Let

$$E_q = \{q, a, b\}.$$

Suppose $y(a), y(b) > y(q)$, i.e., $q = \text{bottom}(E_q)$. Then by Observation 4.11, it must hold

$$c_q(a) = c_q(b) = 1 - d.$$

And for $h \in \mathbb{R}$ such that $y(a), y(b) > h > y(q)$, we have

$$c_h(a) = c_h(b) = 1 - d \neq \text{none}\,.$$

This contradicts Observation 4.10. So there must be a vertex, say a, in E_q with

$$y(a) < y(q).$$

Suppose $x(a) \geq v(E)$. Together with

$$y(a) < y(q) = h(E),$$

we get $a \in E$. Thus, $a \in \{\text{bottom}(E), p\}$. But then

$$t(q) = h(E_q) \leq y(a) \leq y(p).$$

This contradicts (4.6). So

$$x(a) < v(E) \leq x(p) \leq v(E_p)$$

and

$$y(a) \geq h(E_q) > y(p) \geq h(E_p).$$

Thus,

$$a \in E_p.$$

By Observation 4.11, it must hold

$$c_p(a) = 1 - d \text{ and } a <_t p. \tag{4.7}$$

In particular, by Observation 4.6, it holds that

$$h(E_a) = t(a) \leq t(p) = h(E_p). \tag{4.8}$$

Suppose $E_a \in \mathcal{E}_{\mathrm{SE}}$. Then

$$t(a) = h(E_a) \geq y(a) > y(p) \geq h(E_p) = t(p).$$

A contradiction. So

$$E_a \in \mathcal{E}_{\mathrm{NW}}.$$

Finally, we make a case distinction concerning the position of the remaining point

$$b \in E_q.$$

In this case distinction, the colors refer to Figure 4.6).

Case 3.1: $b \in E$. Since $b \neq q$, it holds that $b \in \{p, \text{bottom}(E)\}$, then

$$h(E_q) \leq y(b) \leq y(p), x(p) < x(q) \leq v(E_q)$$

and $p \in E_q$ – a contradiction due to Case 2. So now we may assume that $b \notin E$.

Case 3.2 (lilac): $x(b) > v(E_p)$. Then

$$x(b) > v(E_p) \geq x(p) \geq v(E).$$

Due to $b \notin E$, we have

$$y(b) > h(E) = y(q).$$

With $b \in E_q$, $c(q) = d$, and Observation 4.11, we get

$$c_q(b) = 1 - d \text{ and } b <_t q.$$

In particular, by Observation 4.6, it holds that $t(b) \leq t(q)$. Suppose $E_b \in \mathcal{E}_{\mathrm{SE}}$. Then

$$t(b) = h(E_b) \geq y(b) > y(q) \geq h(E_q) = t(q).$$

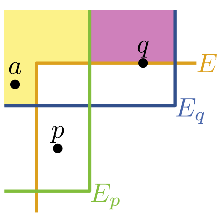


Figure 4.6: Case distinction concerning the position of the vertex $b \in E_q$.

A contradiction. So
$$E_b \in \mathcal{E}_{\mathrm{NW}}.$$
And hence,
$$h(E_b) = t(b) \leq t(q) = h(E_q) \leq y(a), x(a) < v(E_p) < x(b) \leq v(E_b)$$
so that
$$a \in E_b.$$
Due to $v(E_b) \geq x(b) > v(E_p)$ and the symmetrical version of Lemma 2.11, we obtain $h(E_b) > h(E_p)$. Thus,
$$t(b) = h(E_b) > h(E_p) = t(p) \overset{(4.8)}{\geq} t(a).$$
Thereby, $c_b(a) = 1-d$, $a \in E_b$, and recall that $c(b) = 1-d$ – this contradicts Observation 4.8.

Case 3.3 (yellow): $x(b) \leq v(E_p)$. Recall that $b \in E_q$ so
$$y(b) \geq h(E_q) > h(E_p)$$
and we obtain
$$b \in E_p.$$
Due to $y(b) \geq h(E_q) > y(p)$ and Observation 4.11, we get
$$c_p(b) = 1 - d \text{ and } b <_t p.$$
In particular, by Observation 4.6, it holds that
$$h(E_b) = t(b) \leq t(p) = h(E_p). \tag{4.9}$$
Let $h \in \mathbb{R}$ be such that $y(a), y(b) > h > h(E_p)$. By (4.8) and (4.9), we then have:
$$c_h(a) = c_h(b) = 1 - d$$
This contradicts Observation 4.10.

Altogether, every possible case leads to a contradiction. As a result, in every iteration, if a processed hyperedge misses some color, then it contains an uncolored vertex. $\square$

So for every hypergraph $\mathcal{H}(V, \mathcal{R}_{\mathrm{NW}} \cup \mathcal{R}_{\mathrm{SE}}, 3)$, the greedy coloring is well-defined so it produces a polychromatic 2-coloring of this hypergraph. With that, we obtain the following corollary:

Corollary 4.13. *For the range family $\mathcal{R}_{NW} \cup \mathcal{R}_{SE}$, we have $m(2) = 3$.*

For symmetry reasons, the following theorem holds.

Theorem 4.14. *For the range family $\mathcal{R} \in \{\mathcal{R}_{NW} \cup \mathcal{R}_{SE}, \mathcal{R}_{NE} \cup \mathcal{R}_{SW}\}$, we have $m(2) = 3$.*

5. Shallow Hitting Sets and Their Implications

Shallow hitting sets provide a powerful tool to construct polychromatic colorings of hypergraphs. In this chapter, we will show that range families

- $\mathcal{R}_{\mathrm{NW}} \cup \mathcal{R}_{\mathrm{SE}}$,
- $\mathcal{R}_{\mathrm{NW}} \cup \mathcal{R}_{\mathrm{NE}}$,
- $\mathcal{R}_{\mathrm{NW}} \cup \mathcal{R}_{\mathrm{NE}} \cup \mathcal{R}_{\mathrm{SE}}$,
- and $\mathcal{R}_{\mathrm{NW}} \cup \mathcal{R}_{\mathrm{NE}} \cup \mathcal{R}_{\mathrm{SW}} \cup \mathcal{R}_{\mathrm{SE}}$

admit t-shallow hitting sets (the value of t depends on the range family). First, this already proves $m(k) < \infty$ for every $k \in \mathbb{N}$ for these families. Second, we will show that these hitting sets have further nice properties, namely, they do not hit the hyperedges captured by certain other range families too often. This, in turn, is useful to show $m(k) < \infty$ for larger range families.

We start with a lemma about the construction of special 2-shallow hitting sets of hypergraphs captured by the range family of all north-west quadrants. Note that this way, with Lemma 2.8, we only obtain a bound $m(k) \leq 2k - 1$ (for $k \in \mathbb{N}$) for this family which is not optimal by Theorem 4.2. However, these hitting sets have other useful properties.

We say that a set A is *hit* by a set B r times if

$$|A \cap B| = r.$$

Lemma 5.1. *Let V be a point set and let $m \in \mathbb{N}$. The hypergraph*

$$\mathcal{H}(V, \mathcal{R}_{NW}, m) = (V, \mathcal{E}_{NW})$$

admits a 2-shallow hitting set S such that:

1. *The vertices in S can be ordered as $s_1, \dots, s_{|S|}$ so that the points have decreasing x-coordinates and decreasing y-coordinates along this order.*
2. *The set $E_t(V, m)$ is hit exactly once by S.*
3. *Every hyperedge $E \neq E_t(V, m)$ of $\mathcal{H} = \mathcal{H}(V, \mathcal{R}_{NE}, m)$ is not hit by S.*

4. *For any two consecutive points s_j, s_{j+1} ($j \in [t-1]$) in S, the bottomless rectangle B_j with its top-right corner s_j and its left side at $x(s_{j+1})$ satisfies*
$$|B_j \cap V| \geq m + 1.$$

5. *For any three consecutive points s_j, s_{j+1}, s_{j+2} (for $j \in [t-2]$) in S, the bounding box R_j of $\{s_j, s_{j+2}\}$ satisfies*
$$|R_j \cap V| \geq m + 2.$$

Proof. We start with $S = \emptyset$ and process hyperedges $E \in \mathcal{E}_{\mathrm{NW}}$ in the order of decreasing horizontal boundaries $h(E)$. Let E be the current hyperedge. If E is not hit by S yet, we add the leftmost vertex of E to S:
$$S := S \cup \{\mathrm{left}(E)\}.$$
We say that $\mathrm{left}(E)$ has been added *due to* E. Otherwise, we move on to the next hyperedge.

From now on, with S we refer to the state of S after all hyperedges have been processed. Let $t = |S|$. The set S is a hitting set of $\mathcal{H}(V, \mathcal{R}_{\mathrm{NW}}, m)$ by construction. Let $s_1, \ldots, s_t$ be the order in which the points have been added to S. For $j \in [t]$, let E^j denote to the hyperedge due to which s_j has been added to S. Recall that
$$h(E^1) > h(E^2) > \cdots > h(E^t).$$

For $j \in [t-1]$, consider the consecutive points s_j and s_{j+1}. We have
$$x(s_j) \geq h(E^j) > h(E^{j+1})$$
and
$$s_j \in E^j \setminus E^{j+1}$$
since at the moment E^{j+1} was processed, it was not hit by s_j contained in S. So
$$v(E^{j+1}) < x(s_j).$$
Then for every $v \in E^{j+1}$, we have:
$$x(v) \leq v(E^{j+1}) < x(s_j) \leq v(E^j). \tag{5.1}$$
In particular, it holds that $x(s_{j+1}) < x(s_j)$ and the points $s_1, \ldots, s_t$ indeed have decreasing x-coordinates. Thus, for every $r > j$, it holds that $s_r \notin E^j$ since s_j is the leftmost point in E^j. This implies
$$s_2, \ldots, s_t \notin E^1 = E_t(V, m)$$
Together with
$$s_1 \in E^1 = E_t(V, m),$$
this proves Item 2. Now suppose for $v \in E^{j+1}$ it holds that $y(v) \geq h(E^j)$. But then due to (5.1), we obtain $v \in E^j$ and s_j is not the leftmost point of E^j – a contradiction. So
$$y(v) < h(E^j) \leq y(s_j)$$
and in particular, $y(s_{j+1}) < y(s_j)$. Therefore, the points $s_1, \ldots, s_t$ indeed have decreasing y-coordinates too. This proves Item 1.

Next, consider the bottomless rectangle B_j. We claim that every vertex $v \in E^{j+1}$ belongs to B_j. Above we have shown that:
$$x(v) \leq v(E^{j+1}) < x(s_j), y(v) < y(s_j).$$

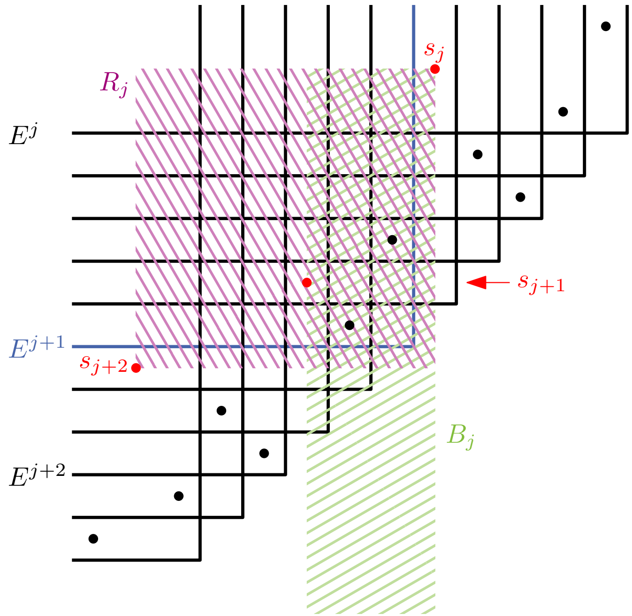


Figure 5.1: 2-shallow hitting sets for north-west quadrants, Lemma 5.1

Moreover, since s_{j+1} is the leftmost point of E^{j+1}, it also holds that $x(v) \geq x(s_{j+1})$. So $v \in B_j$. Altogether, we obtain

$$E^{j+1} \subset B_j \text{ and } s_j \in B_j \setminus E^{j+1}.$$

So $|B_j \cap V| \geq m + 1$. This proves Item 4.

Next, we show that every vertex $v \in E^{j+1}$ also belongs to R_j. Due to

$$x(s_{j+2}) < x(s_{j+1}) < x(s_j) \text{ and } E^{j+1} \subset B_j,$$

it suffices to show that $y(v) \geq y(s_{j+2})$. We know that $x(v) \geq x(s_{j+1}) > x(s_{j+2})$. Now suppose $y(v) < y(s_{j+2})$ holds. Then every north-west quadrant containing v contains s_{j+2} too. Especially, $s_{j+2} \in E^{j+1}$ – a contradiction. Altogether, we obtain

$$E^{j+1} \subset R_j \text{ and } s_j, s_{j+2} \in R_j \setminus E^{j+1}.$$

So $|R_j \cap V| \geq m + 2$. This proves Item 5.

Finally, consider a hyperedge $E \neq E_t(V, m)$ of $\mathcal{H}(V, \mathcal{R}_{\text{NE}}, m)$. First, we prove that

$$E \cap S \subseteq \{s_1\}.$$

Above we have shown that

$$y(s_2) < h(E^1) \text{ and } x(s_2) < x(s_1).$$

So for every $v \in E^1$ and every $j \in \{2, \dots, t\}$, it holds that

$$x(s_j) \leq x(s_2) < x(s_1) \leq x(v) \text{ and } y(s_j) \leq y(s_2) < y(v).$$

Thus, every north-east quadrant containing s_j contains m further vertices, namely the elements of E^1. So this quadrant does not capture a hyperedge of $\mathcal{H}(V, \mathcal{R}_{\text{NE}}, m)$, i.e., s_j does not appear in any hyperedge of this hypergraph. So

$$E \cap S \subseteq \{s_1\}.$$

Finally, recall that s_1 is the leftmost point of $E^1 = E_t(V, m)$, i.e.,

$$s_1 = \mathrm{left}(E_t(V, m)).$$

Now by Lemma 2.16 Item 3, the hyperedge $E \neq E_t(V, m)$ does not contain s_1, i.e.,

$$E \cap S = \emptyset$$

this proves Item 3. □

Since the points $s_t, \ldots, s_1$ of the above hitting set form an increasing sequence, Lemma 2.18 can be applied: every range we will consider in this chapter contains a subsequence of these points.

In particular, every south-west or south-east quadrant containing at least two points from this sequence contains at least two consecutive points s_j, s_{j+1} for some $j \in [t-1]$. Then by Lemma 2.17, it contains the bottomless rectangle B_j as a subset and by Lemma 5.1 Item 4, it contains at least $m+1$ points. In particular, such quadrant does not capture a hyperedge of $\mathcal{H}(V, \mathcal{R}_{\mathrm{SW}} \cup \mathcal{R}_{\mathrm{SE}}, m)$. This implies the following corollary.

Corollary 5.2. *Let V be a point set and let $m \in \mathbb{N}$. The hypergraph*

$$\mathcal{H}(V, \mathcal{R}_{NW}, m) = (V, \mathcal{E}_{NW})$$

admits a 2-shallow hitting set S such that:

1. *It satisfies the properties of Lemma 5.1.*
2. *Every hyperedge of $\mathcal{H} = \mathcal{H}(V, \mathcal{R}_{SW}, m)$ is hit at most once.*
3. *Every hyperedge of $\mathcal{H} = \mathcal{H}(V, \mathcal{R}_{SE}, m)$ is hit at most once.*

For symmetry reasons, 2-shallow hitting sets with analogous properties exist for hypergraphs captured by north-east or south-west, or south-east quadrants.

Corollary 5.3. *For a point set V and $m \in \mathbb{N}$, a hypergraph*

$$\mathcal{H}(V, \mathcal{R}_{NW}, m) \ / \ \mathcal{H}(V, \mathcal{R}_{NE}, m) \ / \ \mathcal{H}(V, \mathcal{R}_{SW}, m) \ / \ \mathcal{H}(V, \mathcal{R}_{SE}, m)$$

admits a hitting set

$$S_{NW} \ / \ S_{NE} \ / \ S_{SW} \ / \ S_{SE}$$

such that:

1. *It satisfies the properties of (the symmetrical version of) Lemma 5.1.*
2. *Every hyperedge $E \neq E_t(V, m)$ of $\mathcal{H}(V, \mathcal{R}_{NW}, m)$ is hit by S_{NW} / S_{NE} / S_{SW} / S_{SE} at most twice / not once / once / once.*
3. *Every hyperedge $E \neq E_t(V, m)$ of $\mathcal{H}(V, \mathcal{R}_{NE}, m)$ is hit by S_{NW} / S_{NE} / S_{SW} / S_{SE} at most not once / twice / once / once.*
4. *Every hyperedge $E \neq E_b(V, m)$ of $\mathcal{H}(V, \mathcal{R}_{SW}, m)$ is hit by S_{NW} / S_{NE} / S_{SW} / S_{SE} at most once / once / twice / not once.*
5. *Every hyperedge $E \neq E_b(V, m)$ of $\mathcal{H}(V, \mathcal{R}_{SE}, m)$ is hit by S_{NW} / S_{NE} / S_{SW} / S_{SE} at most once / once / not once / twice.*
6. *The set $E_t(V, m)$ is hit by S_{NW} / S_{NE} / S_{SW} / S_{SE} at most once / once / once / once.*
7. *The set $E_b(V, m)$ is hit by S_{NW} / S_{NE} / S_{SW} / S_{SE} at most once / once / once / once.*

Now we can state several coloring results based on shallow hitting sets.

5.1 Quadrants

First of all, we can extend the result of Corollary 4.13 about 2-coloring of hypergraphs captured by quadrants in two "non-adjacent" directions to arbitrarily many colors.

Theorem 5.4. *The family $\mathcal{R} \in \{\mathcal{R}_{NW} \cup \mathcal{R}_{SE}, \mathcal{R}_{NE} \cup \mathcal{R}_{SW}\}$ admits 3-shallow hitting sets. Further, we have $m(k) \leq 3k - 3$ for all $k \in \mathbb{N}$.*

Proof. For symmetry reasons, it suffices to prove the statement for $\mathcal{R} = \mathcal{R}_{\text{NW}} \cup \mathcal{R}_{\text{SE}}$.

Let V be a point set and $m \in \mathbb{N}$. Let S_{NW} (respectively S_{SE}) be a hitting set of $\mathcal{H}(V, \mathcal{R}_{\text{NW}}, m)$ (respectively $\mathcal{H}(V, \mathcal{R}_{\text{SE}}, m)$) satisfying the properties of Corollary 5.3. Let $S = S_{\text{NW}} \cup S_{\text{SE}}$, then S is a hitting set of $\mathcal{H}(V, \mathcal{R}, m)$. By Items 2 and 5, every hyperedge

$$E \notin \{E_b(V, m), E_t(V, m)\}$$

of $\mathcal{H}(V, \mathcal{R}, m)$ is hit at most

$$2 + 1 = 3$$

times by S while by Items 6 and 7 a hyperedge $E \in \{E_b(V, m), E_t(V, m)\}$ is hit at most

$$1 + 1 = 2$$

times. So S is a 3-shallow hitting set of $\mathcal{H}(V, \mathcal{R}, m)$ and the family $\mathcal{R}$ admits 3-shallow hitting sets. Now with $m(2) = 3$ (by Corollary 4.13), the fact that $\mathcal{R}$ is a good range family (by Lemma 2.4 and Observation 2.5), and Lemma 2.8 we get:

$$m(k) \leq (k - 2) \cdot 3 + 3 = 3k - 3$$

for every $k \in \mathbb{N}$. □

Note that without Corollary 4.13, i.e., only based on 3-shallow hitting sets and Lemma 2.8, we would obtain almost the same bound $m(k) \leq 3k - 2$ for all $k \in \mathbb{N}$.

The above theorem provides a bound for the range family of all quadrants in two "non-adjacent" directions. Similarly, we can slightly improve the bound $m(k) \leq 2k$ from Lemma 4.3 for the range family of all quadrants in two "adjacent" directions.

Theorem 5.5. *The range family*

$$\mathcal{R} \in \{\mathcal{R}_{SW} \cup \mathcal{R}_{SE}, \mathcal{R}_{SE} \cup \mathcal{R}_{NE}, \mathcal{R}_{NE} \cup \mathcal{R}_{NW}, \mathcal{R}_{NW} \cup \mathcal{R}_{SW}\}$$

admits 2-shallow hitting sets. Further, we have $m(k) \leq 2k - 1$ for all $k \in \mathbb{N}$.

Proof. For symmetry reasons, it suffices to prove the statement for $\mathcal{R} = \mathcal{R}_{\text{SW}} \cup \mathcal{R}_{\text{SE}}$.

Let V be a point set and $m \in \mathbb{N}$. Let S_{SW} (respectively S_{SE}) be a hitting set of $\mathcal{H}(V, \mathcal{R}_{\text{SW}}, m)$ (respectively $\mathcal{H}(V, \mathcal{R}_{\text{SE}}, m)$) satisfying the properties of Corollary 5.3. Let $S = S_{\text{SW}} \cup S_{\text{SE}}$, then S is a hitting set of $\mathcal{H}(V, \mathcal{R}, m)$. By Items 4 and 5, every hyperedge $E \neq E_b(V, m)$ of $\mathcal{H}(V, \mathcal{R}, m)$ is hit at most

$$2 + 0 = 2$$

times by S while by Item 7 the hyperedge $E_b(V, m)$ is hit at most

$$1 + 1 = 2$$

times. So S is a 2-shallow hitting set of $\mathcal{H}(V, \mathcal{R}, m)$ and the family $\mathcal{R}$ admits 2-shallow hitting sets. Now with the fact that $\mathcal{R}$ is a good range family (by Lemma 2.4 and Observation 2.5) and Lemma 2.8, we get:

$$m(k) \leq (k - 1) \cdot 2 + 1 = 2k - 1$$

for every $k \in \mathbb{N}$. □

After dealing with range families of quadrants in two directions, we move on to the range family of all quadrants in three directions.

Lemma 5.6. *The range family*

$$\mathcal{R} \in \{\mathcal{R}_{NW} \cup \mathcal{R}_{NE} \cup \mathcal{R}_{SE}, \mathcal{R}_{NE} \cup \mathcal{R}_{SE} \cup \mathcal{R}_{SW}, \mathcal{R}_{SE} \cup \mathcal{R}_{SW} \cup \mathcal{R}_{NW}, \mathcal{R}_{SW} \cup \mathcal{R}_{NW} \cup \mathcal{R}_{NE}\}$$

admits 4-shallow hitting sets. Further, we have $m(k) \leq 4k - 3$ for all $k \in \mathbb{N}$.

Proof. For symmetry reasons, it suffices to prove the statement for $\mathcal{R} = \mathcal{R}_{\text{NW}} \cup \mathcal{R}_{\text{NE}} \cup \mathcal{R}_{\text{SE}}$. Let V be a point set and $m \in \mathbb{N}$. Let

$$S_{\text{NW}} \;/\; S_{\text{NE}} \;/\; S_{\text{SE}}$$

be a hitting set of

$$\mathcal{H}(V, \mathcal{R}_{\text{NW}}, m) \;/\; \mathcal{H}(V, \mathcal{R}_{\text{NE}}, m) \;/\; \mathcal{H}(V, \mathcal{R}_{\text{SE}}, m)$$

satisfying the properties of Corollary 5.3. Let

$$S = S_{\text{NW}} \cup S_{\text{NE}} \cup S_{\text{SE}},$$

then S is a hitting set of $\mathcal{H}(V, \mathcal{R}, m)$. By Item 2, every hyperedge $E \neq E_t(V, m)$ of $\mathcal{H}(V, \mathcal{R}_{\text{NW}}, m)$ is hit at most

$$2 + 0 + 1 = 3$$

times by S. Similarly, by Item 3, every hyperedge $E \neq E_t(V, m)$ of $\mathcal{H}(V, \mathcal{R}_{\text{NE}}, m)$ is hit at most

$$0 + 2 + 1 = 3$$

times by S. And by Item 5, every hyperedge $E \neq E_b(V, m)$ of $\mathcal{H}(V, \mathcal{R}_{\text{SE}}, m)$ is hit at most

$$1 + 1 + 2 = 4$$

times by S. Finally, by Items 6 and 7, a hyperedge $E \in \{E_b(V, m), E_t(V, m)\}$ of $\mathcal{H}(V, \mathcal{R}, m)$ is hit at most

$$1 + 1 + 1 = 3$$

times by S.

Altogether, S is a 4-shallow hitting set of $\mathcal{H}(V, \mathcal{R}, m)$ and the family $\mathcal{R}$ admits 4-shallow hitting sets. Now with the fact that $\mathcal{R}$ is a good range family (by Lemma 2.4 and Observation 2.5) and Lemma 2.8, we get:

$$m(k) \leq (k - 1) \cdot 4 + 1 = 4k - 3$$

for every $k \in \mathbb{N}$.

□

Finally, we consider the range family of all axis-aligned quadrants. We will prove the same bound $m(k) \leq 4k - 3$ as for the range family of quadrants in three directions and thus strengthen the result from Lemma 5.6.

Lemma 5.7. *Let V be a point set and $m \in \mathbb{N}$. Let*

$$S_{NW} \;/\; S_{NE} \;/\; S_{SW} \;/\; S_{SE}$$

be a hitting set of

$$\mathcal{H}(V, \mathcal{R}_{NW}, m) \;/\; \mathcal{H}(V, \mathcal{R}_{NE}, m) \;/\; \mathcal{H}(V, \mathcal{R}_{SW}, m) \;/\; \mathcal{H}(V, \mathcal{R}_{SE}, m)$$

satisfying the properties of Corollary 5.3. Then the set

$$S = S_{NW} \cup S_{NE} \cup S_{SW} \cup S_{SE}$$

is a 4-shallow hitting set of $\mathcal{H}(V, \mathcal{R}_Q, m)$.

Proof. Let V be a point set and $m \in \mathbb{N}$. By construction, the set S is a hitting set of $\mathcal{H}(V, \mathcal{R}_\mathrm{Q}, m)$. It remains to show that it is 4-shallow. Indeed, by Items 2 to 5, every hyperedge $E \notin \{E_t(V, m), E_b(V, m)\}$ of $\mathcal{H}(V, \mathcal{R}, m)$ is hit at most

$$2 + 1 + 1 + 0 = 4$$

times by S while by Items 6 and 7, a set $E \in \{E_t(V, m), E_b(V, m)\}$ is hit at most

$$1 + 1 + 1 + 1 = 4$$

times. So S is a 4-shallow hitting set of $\mathcal{H}(V, \mathcal{R}_\mathrm{Q}, m)$. □

This immediately implies the following bound on $m(k)$ for the range family of all axis-aligned quadrants.

Theorem 5.8. *The family $\mathcal{R} = \mathcal{R}_Q$ of all axis-aligned quadrants admits 4-shallow hitting sets and in particular, we have $m(k) \leq 4k - 3$ for all $k \in \mathbb{N}$.*

Proof. Lemma 5.7 implies that the family $\mathcal{R}$ admits 4-shallow hitting sets. By Lemma 2.4 and Observation 2.5, the range family $\mathcal{R}$ is good. So by Lemma 2.8, for every $k \in \mathbb{N}$, we have:

$$m(k) \leq (k - 1)4 + 1 = 4k - 3.$$

□

The above results are an immediate implication of Corollary 5.3. Now we move on to more sophisticated implications. The proofs of the following Lemma 5.9 and Theorems 5.12 and 5.13 follow the same structure. However, the correctness requires careful analysis so for the sake of completeness we provide the full proof in every case. The idea behind it will always be the following. First, we iteratively construct hitting sets of a subhypergraph captured by quadrants with properties from Corollary 5.3, color them to obtain a polychromatic coloring of this subhypergraph, and remove these vertices from the point set. After that, we apply Lemma 5.1 to show that the hyperedges captured by the other ranges still contain sufficiently many vertices. Then we can color the remaining vertices so that these hyperedges become polychromatic too. We start by employing this idea for the range family of all north-west and north-east quadrants and bottomless rectangles.

5.2 Quadrants and Bottomless Rectangles

Lemma 5.9. *For the range family*

$$\mathcal{R} = \mathcal{R}_{NW} \cup \mathcal{R}_{NE} \cup \mathcal{R}_{BL}$$

of all north-west and north-east quadrants and bottomless rectangles, we have $m(k) \leq 5k-2$ for every $k \in \mathbb{N}$.

Proof. Let V be a point set, let $k \in \mathbb{N}$, and let $m = 5k - 2$. We construct a polychromatic coloring of $\mathcal{H} = \mathcal{H}(V, \mathcal{R}, m)$ with k colors. First, we aim at making the hyperedges captured by north-west and north-east quadrants polychromatic. Let $m_1 = 5k - 2$ and let $V_1 = V$. Then for $i \in [k]$, we repeat the following:

1. Let S^i_{NW} be a hitting set of $\mathcal{H}(V_i, \mathcal{R}_{\mathrm{NW}}, m_i)$ with properties of Corollary 5.3.
2. Let S^i_{NE} be a hitting set of $\mathcal{H}(V_i, \mathcal{R}_{\mathrm{NE}}, m_i)$ with properties of Corollary 5.3.
3. Set $S^i = S^i_{\mathrm{NW}} \cup S^i_{\mathrm{NE}}$.
4. Set $V_{i+1} = V_i \setminus S^i$.
5. Set $m_{i+1} = m_i - 2$.

Note that $m_i = m_1 - 2(i-1)$ for every $i \in [k+1]$. In particular, for every $i \in [k]$, it holds that

$$m_i \geq m_k = (5k-2) - 2(k-1) = 3k > 0.$$

By Items 2, 3 and 6 of Corollary 5.3, for every $j \in [k]$, the set S^j is a 2-shallow hitting set of $\mathcal{H}(V_j, \mathcal{R}_{\mathrm{NW}} \cup \mathcal{R}_{\mathrm{NE}}, m_j)$. It also holds that

$$m = 5k - 2 \geq 2k - 1 = (k-1) \cdot 2 + 1.$$

So by Lemma 2.9, for every $j \in [k]$, the set S^j is a hitting set of $\mathcal{H}(V, \mathcal{R}_{\mathrm{NW}} \cup \mathcal{R}_{\mathrm{NE}}, m)$. By construction, the sets $S^1, \ldots, S^k$ are pairwise disjoint. We color all vertices in S^j with color j for every $j \in [k]$. Now every hyperedge of $\mathcal{H}$ captured by a north-west or a north-east quadrant is polychromatic. Next, we need to color the remaining vertices (i.e., V_{k+1}) so that the hyperedges of $\mathcal{H}$ captured by bottomless rectangles are polychromatic too.

Let E be a hyperedge of $\mathcal{H}(V, \mathcal{R}_{\mathrm{BL}}, m)$ and let $R \in \mathcal{R}_{\mathrm{BL}}$ be a bottomless rectangle capturing it. We claim that for $i \in [k+1]$ it holds that $|E \cap V_i| = |R \cap V_i| \geq m_i$. This holds for $i = 1$ since $m_1 = m$ and $V_1 = V$ so

$$R \cap V_1 = E = E \cap V_1.$$

Suppose the claim holds for some $i \in [k]$, i.e.,

$$|R \cap V_i| \geq m_i.$$

Since the range family $\mathcal{R}_{\mathrm{BL}}$ is good (see Lemma 2.4), there exists a bottomless rectangle $R' \subseteq R \in \mathcal{R}_{\mathrm{BL}}$ such that

$$|R' \cap V_i| = m_i. \tag{5.2}$$

Suppose $|R' \cap S^i_{\mathrm{NW}}| \geq 2$. Then since the elements of S^i_{NW} form an increasing sequence (see Item 1), by Lemma 2.18, the bottomless rectangle R' contains two consecutive elements s_{j+1}, s_j of this increasing sequence. Then by Lemma 2.17 Item 2, the bottomless rectangle R' contains the bottomless rectangle B_j with left side at $x(s_{j+1})$ and top-right corner at s_j as a subset. Since S^i_{NW} is a 2-shallow hitting set of $\mathcal{H}(V_i, \mathcal{R}_{\mathrm{NW}}, m_i)$ satisfying the properties of Lemma 5.1, by Item 4, we obtain $|R' \cap V_i| \geq |B_j \cap V_i| \geq m_i + 1$ – this contradicts (5.2). So

$$|R' \cap S^i_{\mathrm{NW}}| \leq 1. \tag{5.3}$$

An analogous argument yields:

$$|R' \cap S^i_{\mathrm{NE}}| \leq 1. \tag{5.4}$$

It holds that

$$|R' \cap V_{i+1}| = |R' \cap (V_i \setminus S^i)| = |R' \cap V_i| - |R' \cap S^i| = m_i - |R' \cap (S^i_{\mathrm{NW}} \cup S^i_{\mathrm{NE}})|.$$

Thereby, we obtain

$$|R' \cap V_{i+1}| \geq m_i - |R' \cap S^i_{\mathrm{NW}}| - |R' \cap S^i_{\mathrm{NE}}| \overset{(5.3),(5.4)}{\geq} m_i - 1 - 1 = m_i - 2 = m_{i+1}. \tag{5.5}$$

Hence,

$$|E \cap V_{i+1}| = |(R \cap V) \cap V_{i+1}| = |R \cap (V \cap V_{i+1})| = |R \cap V_{i+1}| \geq |R' \cap V_{i+1}| \overset{(5.5)}{\geq} m_{i+1}.$$

So the claim holds for $i+1$ too. By induction, we obtain:

$$|R \cap V_{k+1}| = |E \cap V_{k+1}| \geq m_{k+1} = m_1 - 2k = (5k-2) - 2k = 3k - 2. \tag{5.6}$$

By [ACC$^+$13], the hypergraph

$$\mathcal{H}' = \mathcal{H}(V_{k+1}, \mathcal{R}_{\mathrm{BL}}, 3k-2)$$

admits a polychromatic coloring $c' : V_{k+1} \to [k]$ with k colors. So we color V_{k+1} according to c'. Let now E be an arbitrary hyperedge of $\mathcal{H}(V, \mathcal{R}_{\mathrm{BL}}, m)$ and let R be a bottomless rectangle capturing it. It holds that

$$|R \cap V_{k+1}| = |E \cap V_{k+1}| \overset{(5.6)}{\geq} 3k - 2.$$

Since the range family of all bottomless rectangles is good (see Lemma 2.4), there exists a bottomless rectangle $R' \subseteq R \in \mathcal{R}_{\mathrm{BL}}$ such that

$$|R' \cap V_{k+1}| = 3k - 2.$$

Let $E' = R' \cap V_{k+1}$. Then

$$E' = R' \cap V_{k+1} \subseteq R \cap V_{k+1} = R \cap (V \cap V_{k+1}) = (R \cap V) \cap V_{k+1} = E \cap V_{k+1} \subseteq E$$

and

$$|E'| = |R' \cap V_{k+1}| = 3k - 2.$$

Then E' is a hyperedge of $\mathcal{H}'$ so E' polychromatic in c'. Hence, E is polychromatic in c' too. So now every hyperedge of $\mathcal{H}$ is polychromatic in $c : V \to [k]$ with

$$c(v) = \begin{cases} j & \text{if } v \in S^j \text{ for some } j \in [k] \\ c'(v) & \text{otherwise, i.e., } v \in V_{k+1} \end{cases}$$

and this concludes the proof. □

Recall that by Lemma 2.1, every hyperedge captured by a south-west or a south-east quadrant is captured by a bottomless rectangle too. So the bound from Lemma 5.9 still holds if we extend the range family with south-west and south-east quadrants.

Corollary 5.10. *For the range family $\mathcal{R} = \mathcal{R}_Q \cup \mathcal{R}_{BL}$ of all axis-aligned quadrants and bottomless rectangles, we have $m(k) \leq 5k - 2$ for every $k \in \mathbb{N}$.*

For symmetry reasons, we obtain the following theorem

Theorem 5.11. *Let $\mathcal{R} \in \{\mathcal{R}_{BL}, \mathcal{R}_{TL}\}$. Then for the range family*

$$\mathcal{R} \cup \mathcal{R}_Q,$$

we have $m(k) \leq 5k - 2$ for every $k \in \mathbb{N}$.

5.3 Quadrants and Strips

Recall that the range family $\mathcal{R}_{\text{S}}$ consists of all vertical, horizontal, and diagonal (i.e., of slope -1) strips. With a similar idea, we will now prove an upper bound on $m(k)$ for the range family

$$\mathcal{R} = \mathcal{R}_{\text{NW}} \cup \mathcal{R}_{\text{SE}} \cup \mathcal{R}_{\text{S}}.$$

Later in Corollary 7.5, we will show that for the range family $\mathcal{R}_{\text{DS}} \cup \mathcal{R}_{\text{SW}}$, it holds that $m(2) = \infty$. For this reason, the family $\mathcal{R}$ is in some sense the maximal extension of $\mathcal{R}_{\text{DS}}$ with a subfamily of unbounded axis-aligned rectangles so that we still have $m(k) < \infty$ for all $k \in \mathbb{N}$.

Theorem 5.12. *For the range family $\mathcal{R} = \mathcal{R}_{NW} \cup \mathcal{R}_{SE} \cup \mathcal{R}_{S}$, we have*

$$m(k) \leq \lceil 4k \ln k + k \ln 3 \rceil + 4k$$

for all $k \in \mathbb{N}$.

In particular, it holds that $m(k) \in \mathcal{O}(k \ln k)$.

Proof. Let V be a point set, let $k \in \mathbb{N}$, and let $m = \lceil 4k \ln k + k \ln 3 \rceil + 4k$. We construct a polychromatic coloring of $\mathcal{H} = \mathcal{H}(V, \mathcal{R}, m)$. First, we proceed similarly to the previous proof to color the hyperedges of $\mathcal{H}(V, \mathcal{R}_{\text{NW}} \cup \mathcal{R}_{\text{SE}}, m)$ in a polychromatic way. Let $m_1 = m$ and let $V_1 = V$. Then for $i \in [k]$, we repeat the following:

1. Let S^i_{NW} be a hitting set of $\mathcal{H}(V_i, m_i, \mathcal{R}_{\text{NW}})$ with properties of Corollary 5.3.
2. Let S^i_{SE} be a hitting set of $\mathcal{H}(V_i, m_i, \mathcal{R}_{\text{SE}})$ with properties of Corollary 5.3.
3. Set $S^i = S^i_{\text{NW}} \cup S^i_{\text{SE}}$.
4. Set $V_{i+1} = V_i \setminus S^i$.
5. Set $m_{i+1} = m_i - 4$.

Note that $m_i = m_1 - 4(i-1)$ for every $i \in [k]$. In particular, for every $i \in [k]$, it holds that

$$m_i \geq m_k = m - 4(k-1) = \lceil 4k \ln k + k \ln 3 \rceil + 4k - 4(k-1) = \lceil 4k \ln k + k \ln 3 \rceil + 4 > 0.$$

By Items 2 and 5 to 7 of Corollary 5.3, for every $j \in [k]$, the set S^j is a 3-shallow hitting set of $\mathcal{H}(V_j, \mathcal{R}_{\text{NW}} \cup \mathcal{R}_{\text{SE}}, m_j)$. In particular, S^j is a 4-shallow hitting set of $\mathcal{H}(V_j, \mathcal{R}_{\text{NW}} \cup \mathcal{R}_{\text{SE}}, m_j)$. We also have

$$m = \lceil 4k \ln k + k \ln 3 \rceil + 4k \geq 4k - 3 = (k-1) \cdot 4 + 1.$$

So by Lemma 2.9, every set S^j ($j \in [k+1]$) is a hitting set of $\mathcal{H}(V, \mathcal{R}_{\text{NW}} \cup \mathcal{R}_{\text{SE}}, m)$. By construction, the sets $S^1, \ldots, S^k$ are pairwise disjoint. We color all points in S^j with color j for every $j \in [k]$. Now every hyperedge of $\mathcal{H}$ captured by a north-west or a south-east quadrant is polychromatic. Next, we need to color the remaining vertices (i.e., V_{k+1}) in such a way that every hyperedge of $\mathcal{H}$ captured by a strip is polychromatic too.

So let E be a hyperedge of $\mathcal{H}(V, \mathcal{R}_{\text{S}}, m)$ captured by a strip $R \in \mathcal{R}_{\text{S}}$. We claim that for $i \in [k+1]$ it holds that $|R \cap V_i| = |E \cap V_i| \geq m_i$. This holds for $i = 1$ since $m_1 = m$ and $V_1 = V$ so $R \cap V_1 = E = E \cap V_1$. Now suppose the claim holds for some $i \in [k]$, i.e.,

$$|E \cap V_i| = |R \cap V_i| \geq m_i.$$

Since the range family $\mathcal{R}_{\text{S}}$ is good (by Lemma 2.4 and Observation 2.5), there exists a strip $R' \subseteq R \in \mathcal{R}_{\text{S}}$ such that

$$|R' \cap V_i| = m_i. \tag{5.7}$$

Suppose $|R' \cap S^i_{\mathrm{NW}}| \geq 3$. Then since the elements of S^i_{NW} form an increasing sequence (see Item 1), by Lemma 2.18, the strip R' contains three consecutive elements s_{j+2}, s_{j+1}, s_j of this sequence. Then by Lemma 2.17 Item 1, the strip R' contains the bounding box R_j of $\{s_j, s_{j+2}\}$ as a subset. Since S^i_{NW} is a 2-shallow hitting set of $\mathcal{H}(V_i, \mathcal{R}_{\mathrm{NW}}, m_i)$ satisfying the properties of Lemma 5.1, by Item 5, we obtain $|R' \cap V_i| \geq |R_j \cap V_{i+1}| \geq m_i + 2$ – this contradicts (5.7). So

$$|R' \cap S^i_{\mathrm{NW}}| \leq 2. \tag{5.8}$$

An analogous argument yields

$$|R' \cap S^i_{\mathrm{SE}}| \leq 2. \tag{5.9}$$

So

$$|R' \cap V_{i+1}| = |R' \cap (V_i \setminus S^i)| = |R' \cap V_i| - |R' \cap S^i| = m_i - |R' \cap (S^i_{\mathrm{NW}} \cup S^i_{\mathrm{SE}})|.$$

Thereby, we obtain

$$|R' \cap V_{i+1}| \geq m_i - |R' \cap S^i_{\mathrm{NW}}| - |R' \cap S^i_{\mathrm{SE}}| \overset{(5.8),(5.9)}{\geq} m_i - 2 - 2 = m_i - 4 = m_{i+1}.$$

Hence,

$$|E \cap V_{i+1}| = |(R \cap V) \cap V_{i+1}| = |R \cap (V \cap V_{i+1})| = |R \cap V_{i+1}| \geq |R' \cap V_{i+1}| \geq m_{i+1}.$$

So the claim holds for $i + 1$ too. By induction, we obtain:

$$|R \cap V_{k+1}| = |E \cap V_{k+1}| \geq m_{k+1} = m_1 - 4k = (\lceil 4k \ln k + k \ln 3 \rceil + 4k) - 4k = \lceil 4k \ln k + k \ln 3 \rceil. \tag{5.10}$$

By the result of Aloupis et al., the hypergraph

$$\mathcal{H}' = \mathcal{H}(V_{k+1}, \mathcal{R}_{\mathrm{S}} = \mathcal{R}_{\mathrm{VS}} \cup \mathcal{R}_{\mathrm{HS}} \cup \mathcal{R}_{\mathrm{DS}}, \lceil 4k \ln k + k \ln 3 \rceil)$$

admits a polychromatic coloring $c' : V_{k+1} \to [k]$ with k colors [ACC+11]. We color the points in V_{k+1} according to c'. Let now E be an arbitrary hyperedge of $\mathcal{H}(V, \mathcal{R}_{\mathrm{S}}, m)$ and let $R \in \mathcal{R}_{\mathrm{S}}$ be a strip capturing it. It holds that

$$|R \cap V_{k+1}| = |E \cap V_{k+1}| \overset{(5.10)}{\geq} \lceil 4k \ln k + k \ln 3 \rceil.$$

Since the range family of all strips is good (by Lemma 2.4 and Observation 2.5), there exists a strip $R' \subseteq R \in \mathcal{R}_{\mathrm{S}}$ such that

$$|R' \cap V_{k+1}| = \lceil 4k \ln k + k \ln 3 \rceil.$$

Let $E' = R' \cap V_{k+1}$. Then

$$E' = R' \cap V_{k+1} \subseteq R \cap V_{k+1} = R \cap (V \cap V_{k+1}) = (R \cap V) \cap V_{k+1} = E \cap V_{k+1} \subseteq E$$

and

$$|E'| = |R' \cap V_{k+1}| = \lceil 4k \ln k + k \ln 3 \rceil.$$

Then E' is a hyperedge of $\mathcal{H}'$ so E' polychromatic in c'. Hence, E is polychromatic in c' too. So now every hyperedge of $\mathcal{H}$ is polychromatic in $c : V \to [k]$ with

$$c(v) = \begin{cases} j & \text{if } v \in S^j \text{ for some } j \in [k] \\ c'(v) & \text{otherwise, i.e., } v \in V_{k+1} \end{cases}$$

and this concludes the proof. $\square$

5.4 Quadrants and Axis-Aligned Strips

As we have already stated, for the range family $\mathcal{R}_{\mathrm{DS}} \cup \mathcal{R}_{\mathrm{SW}}$, it holds that $m(2) = \infty$. So the above result is in some sense the maximal extension of $\mathcal{R}_{\mathrm{DS}}$ such that $m(k) < \infty$. Similarly, in Corollary 7.16, we will show that for the range family $\mathcal{R}_{\mathrm{HS}} \cup \mathcal{R}_{\mathrm{BL}}$ we also have $m(2) = \infty$. For this reason, the following theorem represents the maximal extension of $\mathcal{R}_{\mathrm{HS}}$ with a subfamily of unbounded axis-aligned rectangles such that $m(k) < \infty$ still holds.

Theorem 5.13. *For the range family $\mathcal{R} = \mathcal{R}_Q \cup \mathcal{R}_{AS}$ of all axis-aligned quadrants and axis-aligned strips, we have $m(k) \leq 10k - 1$ for all $k \in \mathbb{N}$.*

Proof. Let V be a point set, let $k \in \mathbb{N}$, and let $m = 10k - 1$. We construct a polychromatic coloring of $\mathcal{H} = \mathcal{H}(V, \mathcal{R}, m)$.

First, we proceed similarly to the previous proofs to color the hyperedges of $\mathcal{H}(V, \mathcal{R}_{\mathrm{Q}}, m)$ in a polychromatic way. Let $m_1 = m$ and let $V_1 = V$. Then for $i \in [k]$, we repeat the following:

1. Let S^i_{NW} be a hitting set of $\mathcal{H}(V_i, m_i, \mathcal{R}_{\mathrm{NW}})$ with properties of Corollary 5.3.
2. Let S^i_{NE} be a hitting set of $\mathcal{H}(V_i, m_i, \mathcal{R}_{\mathrm{NE}})$ with properties of Corollary 5.3.
3. Let S^i_{SW} be a hitting set of $\mathcal{H}(V_i, m_i, \mathcal{R}_{\mathrm{SW}})$ with properties of Corollary 5.3.
4. Let S^i_{SE} be a hitting set of $\mathcal{H}(V_i, m_i, \mathcal{R}_{\mathrm{SE}})$ with properties of Corollary 5.3.
5. Set $S^i = S^i_{\mathrm{NW}} \cup S^i_{\mathrm{NE}} \cup S^i_{\mathrm{SW}} \cup S^i_{\mathrm{SE}}$.
6. Set $V_{i+1} = V_i \setminus S^i$.
7. Set $m_{i+1} = m_i - 8$.

Note that $m_i = m_1 - 8(i - 1)$ for every $i \in [k + 1]$. In particular, for every $i \in [k]$, it holds that

$$m_i \geq m_k = 10k - 1 - 8(k - 1) = 2k + 7 > 0.$$

By Lemma 5.7, for every $j \in [k]$, the set S^j is a 4-shallow hitting set of $\mathcal{H}(V_j, \mathcal{R}_{\mathrm{Q}}, m_j)$. In particular, S^j is an 8-shallow hitting set of $\mathcal{H}(V_j, \mathcal{R}_{\mathrm{Q}}, m_j)$. Moreover,

$$m = 10k - 1 \geq 8k - 7 = (k - 1) \cdot 8 + 1.$$

So by Lemma 2.9, every set S^j $(j \in [k])$ is a hitting set of $\mathcal{H}(V, \mathcal{R}_{\mathrm{Q}}, m)$. By construction, the sets $S^1, \ldots, S^k$ are pairwise disjoint. We color all points in S^j with color j for every $j \in [k]$ so that every hyperedge of $\mathcal{H}$ captured by an axis-aligned quadrant is polychromatic now. Next, we need to color the remaining vertices (i.e., V_{k+1}) in such a way that every hyperedge of $\mathcal{H}$ captured by an axis-aligned strip is polychromatic too.

So let E be a hyperedge of $\mathcal{H}(V, \mathcal{R}_{\mathrm{AS}}, m)$ captured by an axis-aligned strip $R \in \mathcal{R}_{\mathrm{AS}}$. We claim that for $i \in [k + 1]$ it holds that $|R \cap V_i| = |E \cap V_i| \geq m_i$. This holds for $i = 1$ since $m_1 = m$ and $V_1 = V$ so $R \cap V_1 = E = E \cap V_1$. Now suppose the claim holds for some $i \in [k]$, i.e.,

$$|E \cap V_i| = |R \cap V_i| \geq m_i.$$

Since the range family $\mathcal{R}_{\mathrm{AS}}$ is good (by Lemma 2.4 and Observation 2.5), there exists an axis-aligned strip $R' \subseteq R \in \mathcal{R}_{\mathrm{AS}}$ such that $R' \subseteq R$ and

$$|R' \cap V_i| = m_i. \tag{5.11}$$

Suppose $|R' \cap S^i_{\mathrm{NW}}| \geq 3$. Then since the elements of S^i_{NW} form an increasing sequence (see Item 1), by Lemma 2.18, the axis-aligned strip R' contains three consecutive elements s_{j+2}, s_{j+1}, s_j of this sequence. Then by Lemma 2.17 Item 1, the axis-aligned strip R' contains the bounding box R_j of $\{s_{j+2}, s_j\}$ as a subset. Since S^i_{NW} is a 2-shallow hitting set of $\mathcal{H}(V_i, \mathcal{R}_{\mathrm{NW}}, m_i)$ satisfying the properties of Lemma 5.1, by Item 5, we obtain $|R' \cap V_i| \geq |R_j \cap V_i| \geq m_i + 2$ – this contradicts (5.11). So

$$|R' \cap S^i_{\mathrm{NW}}| \leq 2. \tag{5.12}$$

An analogous argument yields

$$|R' \cap S^i_{\mathrm{NE}}|, |R' \cap S^i_{\mathrm{SW}}|, |R' \cap S^i_{\mathrm{SE}}| \leq 2. \tag{5.13}$$

Then

$$|R' \cap V_{i+1}| = |R' \cap (V_i \setminus S^i)| = |R' \cap V_i| - |R' \cap S^i| = m_i - |R' \cap (S^i_{\mathrm{NW}} \cup S^i_{\mathrm{NE}} \cup S^i_{\mathrm{SW}} \cup S^i_{\mathrm{SE}})|.$$

Thereby, we obtain

$$|R' \cap V_{i+1}| \geq m_i - |R' \cap S^i_{\mathrm{NW}}| - |R' \cap S^i_{\mathrm{NE}}| - |R' \cap S^i_{\mathrm{SW}}| - |R' \cap S^i_{\mathrm{SE}}| \overset{(5.12),(5.13)}{\geq} m_i - 8 = m_{i+1}.$$

Hence,

$$|E \cap V_{i+1}| = |(R \cap V) \cap V_{i+1}| = |R \cap (V \cap V_{i+1})| = |R \cap V_{i+1}| \geq |R' \cap V_{i+1}| \geq m_{i+1}.$$

So the claim holds for $i + 1$ too. By induction, we obtain:

$$|R \cap V_{k+1}| = |E \cap V_{k+1}| \geq m_{k+1} = m_1 - 8k = (10k - 1) - 8k = 2k - 1. \tag{5.14}$$

By the result of Aloupis et al., the hypergraph

$$\mathcal{H}' = \mathcal{H}\left(V_{k+1}, \mathcal{R}_{\mathrm{AS}} = \mathcal{R}_{\mathrm{VS}} \cup \mathcal{R}_{\mathrm{HS}}, 2k - 1\right)$$

admits a polychromatic coloring $c' : V_{k+1} \to [k]$ with k colors [ACC+11]. We color the points in V_{k+1} according to c'. Now consider an arbitrary hyperedge E of $\mathcal{H}$ captured by an axis-aligned strip $R \in \mathcal{R}_{\mathrm{AS}}$. It holds that

$$|R \cap V_{k+1}| = |E \cap V_{k+1}| \overset{(5.14)}{\geq} 2k - 1.$$

Since the family of all axis-aligned strips is good (by Lemma 2.4 and Observation 2.5), there exists an axis-aligned strip $R' \subseteq R \in \mathcal{R}_{\mathrm{AS}}$ such that

$$|R' \cap V_{k+1}| = 2k - 1.$$

Let $E' = R' \cap V_{k+1}$. Then

$$E' = R' \cap V_{k+1} \subseteq R \cap V_{k+1} = R \cap (V \cap V_{k+1}) = (R \cap V) \cap V_{k+1} = E \cap V_{k+1} \subseteq E$$

and

$$|E'| = |R' \cap V_{k+1}| = 2k - 1.$$

Then E' is a hyperedge of $\mathcal{H}'$ so E' polychromatic in c'. Hence, E is polychromatic in c' too. So now every hyperedge of $\mathcal{H}$ is polychromatic in $c : V \to [k]$ with

$$c(v) = \begin{cases} j & \text{if } v \in S^j \text{ for some } j \in [k] \\ c'(v) & \text{otherwise, i.e., } v \in V_{k+1} \end{cases}$$

and this concludes the proof. □

6. Bottomless and Topless Rectangles

In Lemma 5.9, we have shown that hypergraphs captured by bottomless rectangles and north-west and north-east quadrants admit polychromatic colorings with arbitrarily many colors. Moreover, the required uniformity $m(k)$ is linear in the number k of colors. Recall that by Lemma 2.1 north-west and north-east quadrants can be seen as a special case of topless rectangles. So it is now reasonable to consider the hypergraphs captured by the family of all bottomless and topless rectangles. Ackerman, Keszegh, and Vizer have proven the following bound for hypergraphs captured by axis-aligned squares:

Theorem 6.1 ([AKV17])**.** *For the range family $\mathcal{R}_{SQ}$ of all axis-aligned squares, we have*

$$m(k) \in \mathcal{O}(k^{8.75}).$$

Using this result, we will prove the same bound for the range family of all bottomless and topless rectangles. However, this bound is not linear in the number of colors k so the result in Lemma 5.9 is still interesting on its own. We first prove the following lemma.

Lemma 6.2. *Let $\mathcal{R}$ be the range family of all bottomless and topless rectangles. Let V be a point set and $m \geq 2$. There exists a point set V' and a bijective mapping $\phi : V \to V'$ such that for every hyperedge E of $\mathcal{H}(V, \mathcal{R}, m)$, the set $\phi(E)$ is a hyperedge of $\mathcal{H}(V', \mathcal{R}_{SQ}, m)$.*

Proof. For illustration, see Figure 6.1. Let

$$\mathcal{H} = \mathcal{H}(V, \mathcal{R}, m) = (V, \mathcal{E}_{\text{BL}} \cup \mathcal{E}_{\text{TL}})$$

where $\mathcal{E}_{\text{BL}}$ (respectively $\mathcal{E}_{\text{TL}}$) denotes the set of hyperedges of $\mathcal{H}$ captured by $\mathcal{R}_{\text{BL}}$ (respectively $\mathcal{E}_{\text{TL}}$).

For every hyperedge $E \in \mathcal{E}_{\text{BL}}$, fix a bottomless rectangle

$$B_E = \{(x, y) \mid a_E \leq x \leq b_E, y \leq c_E\} \in \mathcal{R}_{\text{BL}}$$

capturing E for $a_E, b_E, c_E \in \mathbb{R}$. Since no point of V lies below the horizontal line

$$y = \min_{v \in V} y(v),$$

the hyperedge E is then also captured by an axis-aligned rectangle

$$R_E = \{(x,y) \mid a_E \le x \le b_E, \min_{v\in V} y(v) \le y \le c_E\} \in \mathcal{R}_{\mathrm{R}}.$$

Similarly, for every hyperedge $E \in \mathcal{E}_{\mathrm{TL}}$, fix a topless rectangle

$$B_E = \{(x,y) \mid a_E \le x \le b_E, c_E \le y\} \in \mathcal{R}_{\mathrm{TL}}$$

capturing E for $a_E, b_E, c_E \in \mathbb{R}$. Since no point of V lies above the horizontal line

$$y = \max_{v\in V} y(v),$$

the hyperedge E is then also captured by an axis-aligned rectangle

$$R_E = \{(x,y) \mid a_E \le x \le b_E, c_E \le y \le \max_{v\in V} y(v)\} \in \mathcal{R}_{\mathrm{R}}$$

(see Figure 6.1 (b)).

For an axis-aligned rectangle

$$\{(x,y) \mid a \le x \le b, c \le y \le d\} \in \mathcal{R}_{\mathrm{R}},$$

with $a \le b, c \le d \in \mathbb{R}$ we call $b-a$ (respectively $d-c$) its *width* (respectively *height*).

Note that since $m \ge 2$, every $E \in \mathcal{E}_{\mathrm{BL}} \cup \mathcal{E}_{\mathrm{TL}}$ contains at least two vertices and the vertices are in general position, it holds that

$$a_E < b_E.$$

And if $E \in \mathcal{E}_{\mathrm{BL}}$, then

$$c_E > \min_{v\in V} y(v).$$

And if $E \in \mathcal{E}_{\mathrm{TL}}$, then

$$c_E < \max_{v\in V} y(v).$$

Hence the width and the height of every R_E are well-defined and positive.

Let

$$\mathcal{R}^*_{\mathrm{BL}} = \{R_E \mid E \in \mathcal{E}_{\mathrm{BL}}\},$$
$$\mathcal{R}^*_{\mathrm{TL}} = \{R_E \mid E \in \mathcal{E}_{\mathrm{TL}}\},$$

and

$$\mathcal{R}^* = \mathcal{R}^*_{\mathrm{BL}} \cup \mathcal{R}^*_{\mathrm{TL}}.$$

Then it holds that

$$\mathcal{H} = \mathcal{H}(V, \mathcal{R}^*, m). \tag{6.1}$$

Now we stretch V horizontally until the width of every axis-aligned rectangle in $\mathcal{R}^*$ becomes larger than its height. We obtain the point set V' (see Figure 6.1 (c)). Note that this stretching is possible since R^* contains finitely many rectangles and the width of every rectangle in R^* is positive. For $E \in \mathcal{E}_{\mathrm{BL}} \cup \mathcal{E}_{\mathrm{TL}}$, we denote the stretched version of R_E with R'_E . So let

$$\mathcal{R}'_{\mathrm{BL}} = \{R'_E \mid E \in \mathcal{E}_{\mathrm{BL}}\},$$
$$\mathcal{R}'_{\mathrm{TL}} = \{R'_E \mid E \in \mathcal{E}_{\mathrm{TL}}\},$$

and

$$\mathcal{R}' = \mathcal{R}'_{\mathrm{BL}} \cup \mathcal{R}'_{\mathrm{TL}}.$$

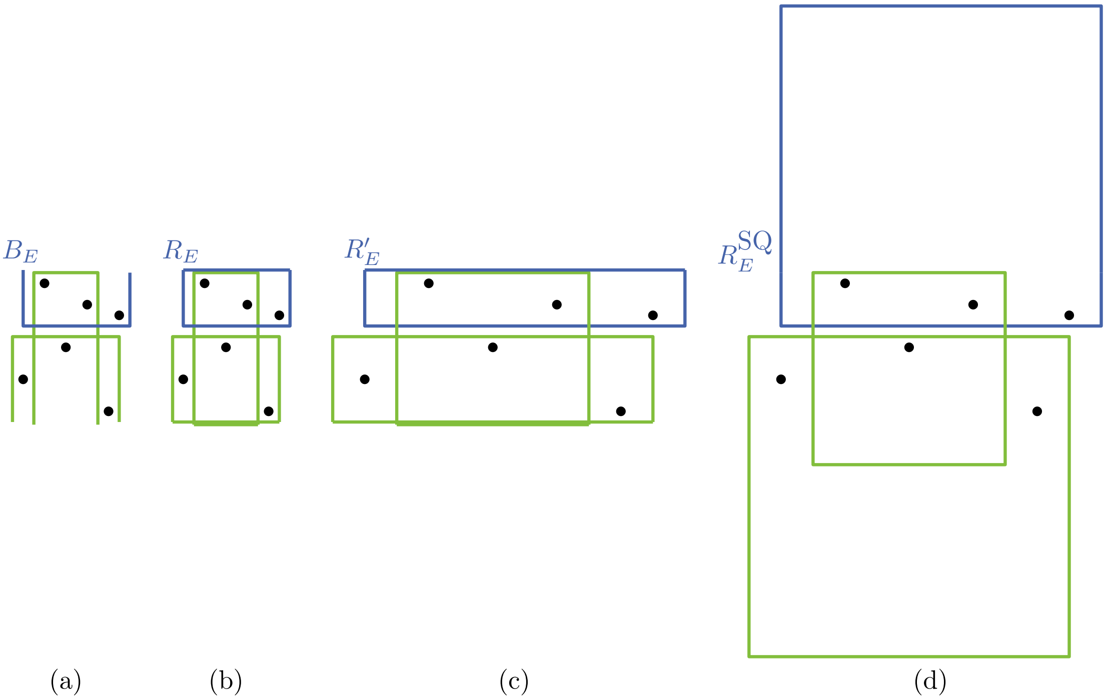


Figure 6.1: Sketch for the proof of Lemma 6.2. Here $m = 3$.
(a) The initial point set V, some of the ranges are exemplified.
(b) Making the unbounded rectangles bounded.
(c) Horizontal stretching until the width of the rectangles in $\mathcal{R}^*$ becomes larger than their height. The arising point set V'.
(d) Prolong the rectangles so that they become squares.

Let

$$\mathcal{H}' = \mathcal{H}(V', \mathcal{R}', m).$$

Since a horizontal stretching preserves the relative ordering of any two points in the plane along the x-axis and preserves their y-coordinates, we obtain:

$$\min_{v \in V} y(v) = \min_{v \in V'} y(v) \text{ and } \max_{v \in V} y(v) = \max_{v \in V'} y(v)$$

and

$$\mathcal{H} \overset{(6.1)}{=} \mathcal{H}(V, \mathcal{R}^*, m) \cong \mathcal{H}(V', \mathcal{R}', m) = \mathcal{H}'.$$

Now it remains to show that for every range in $\mathcal{R}'$, there is an axis-aligned square capturing the same subset of V'.

First, consider a range

$$R' = \{(x, y) \mid a \leq x \leq b, \min_{v \in V'} y(v) \leq y \leq c\} \in \mathcal{R}'_{\mathrm{BL}}$$

for $a, b, c \in \mathbb{R}$. Let

$$\varepsilon = (b - a) - (c - \min_{v \in V'} y(v)). \tag{6.2}$$

Since the width of R' is larger than its height, we have $\varepsilon > 0$. Consider the axis-aligned rectangle

$$R^{\mathrm{SQ}} = \{(x, y) \mid a \leq x \leq b, \min_{v \in V'} y(v) - \varepsilon \leq y \leq c\}.$$

First, we have

$$c - (\min_{v \in V'} y(v) - \varepsilon) = c - \min_{v \in V'} y(v) + \varepsilon \overset{(6.2)}{=} c - \min_{v \in V'} y(v) + (b - a) - (c - \min_{v \in V'} y(v)) = b - a.$$

So the width of R^{SQ} is equal to the height of R^{SQ} and hence, R^{SQ} is an axis-aligned square (see Figure 6.1 (d)). Let

$$R^+ = \{(x, y) \mid a \leq x \leq b, \min_{v \in V'} y(v) - \varepsilon \leq y < \min_{v \in V'} y(v)\}.$$

Since every point in R^+ lies below the bottommost point in V', it holds that

$$R^+ \cap V' = \emptyset.$$

It also holds that

$$R^{\mathrm{SQ}} = R' \cup R^+.$$

Thereby,

$$R^{\mathrm{SQ}} \cap V' = (R' \cup R^+) \cap V' = (R' \cap V') \cup (R^+ \cap V') = (R' \cap V') \cup \emptyset = R' \cap V'.$$

So R^{SQ} is an axis-aligned square capturing the same set of vertices in V' as R'.

Now similarly, consider a range

$$R' = \{(x, y) \mid a \leq x \leq b, c \leq y \leq \max_{v \in V'} y(v)\} \in \mathcal{R}'_{\mathrm{TL}}$$

for $a, b, c \in \mathbb{R}$. Let

$$\varepsilon = (b - a) - (\max_{v \in V'} y(v) - c). \tag{6.3}$$

Since the width of R' is larger than its height, we have $\varepsilon > 0$. Consider the axis-aligned rectangle

$$R^{\mathrm{SQ}} = \{(x, y) \mid a \leq x \leq b, c \leq y \leq \max_{v \in V'} y(v) + \varepsilon\}.$$

First, we have

$$\max_{v \in V'} y(v) + \varepsilon - c = \max_{v \in V'} y(v) - c + \varepsilon \overset{(6.3)}{=} \max_{v \in V'} y(v) - c + ((b - a) - (\max_{v \in V'} y(v) - c)) = b - a.$$

So the width of R^{SQ} is equal to the height of R^{SQ} and hence, R^{SQ} is an axis-aligned square (see Figure 6.1 (d)). Let

$$R^+ = \{(x, y) \mid a \leq x \leq b, \max_{v \in V'} y(v) < y \leq \max_{v \in V'} y(v) + \varepsilon\}$$

Since every point in R^+ lies above the topmost point in V', it holds that

$$R^+ \cap V' = \emptyset.$$

It also holds that

$$R^{\mathrm{SQ}} = R' \cup R^+.$$

Thereby,

$$R^{\mathrm{SQ}} \cap V' = (R' \cup R^+) \cap V' = (R' \cap V') \cup (R^+ \cap V') = (R' \cap V') \cup \emptyset = R' \cap V'.$$

So R' is an axis-aligned square capturing the same set of vertices in V' as R'.

Altogether, for every range $R' \in \mathcal{R}'$, there exists an axis-aligned square $R^{\mathrm{SQ}} \in \mathcal{R}_{\mathrm{SQ}}$ such that

$$R' \cap V' = R^{\mathrm{SQ}} \cap V'.$$

Thereby, every hyperedge of $\mathcal{H}'$ is also a hyperedge of $\mathcal{H}(V', \mathcal{R}_{\mathrm{SQ}}, m)$. Together with the fact that $\mathcal{H}' \cong \mathcal{H}$, this concludes the proof. $\square$

Theorem 6.3. *For the range family $\mathcal{R} = \mathcal{R}_{BL} \cup \mathcal{R}_{TL}$ of all bottomless and topless rectangles, we have $m(k) \in \mathcal{O}(k^{8.75})$.*

Proof. First, we prove that for every $k \geq 2$, it holds that

$$m_{\mathcal{R}}(k) \leq m_{\mathcal{R}_{\mathrm{SQ}}}(k).$$

Let V be a point set and let $m = m_{\mathcal{R}_{\mathrm{SQ}}}(k)$. Since $k \geq 2$, it also holds that $m \geq 2$. By Lemma 6.2, there exists a point set V' and a bijective mapping $\phi : V \to V'$ such that for every hyperedge E of $\mathcal{H} = \mathcal{H}(V, \mathcal{R}, m)$ the set $\phi(E)$ is a hyperedge of $\mathcal{H}' = \mathcal{H}(V', \mathcal{R}_{\mathrm{SQ}}, m)$.

By the definition of m, we know that $\mathcal{H}'$ admits a polychromatic coloring $c' : V' \to [k]$ with k colors. Consider $c : V \to [k]$ with

$$c(v) = c'(\phi(v))$$

for every $v \in V$. Let E be a hyperedge of $\mathcal{H}$. Then we know that $\phi(E)$ is a hyperedge of $\mathcal{H}'$ and hence, it is polychromatic in c'. So

$$c(E) = c'(\phi(E)) = [k].$$

Thereby, E is polychromatic in c. So c is a polychromatic coloring of $\mathcal{H}$ with k colors. Since V was an arbitrary point set, we obtain

$$m_{\mathcal{R}}(k) \leq m = m_{\mathcal{R}_{\mathrm{SQ}}}(k).$$

Since this holds for every $k \geq 2$, by Theorem 6.1, we obtain the desired bound

$$m_{\mathcal{R}}(k) \in \mathcal{O}(k^{8.75}).$$

□

7. Non-Colorable Range Families

In this chapter, we provide two range families for which $m(2) = \infty$ holds. These are the range family of all south-west quadrants and diagonal strips (see Corollary 7.5) and the range family of all bottomless rectangles and horizontal strips (see Corollary 7.16).

For $m \in \mathbb{N}$, we say that an m-uniform hypergraph $\mathcal{H} = (V, \mathcal{E})$ can be *realized* with a range family $\mathcal{R}$ if there exists an injective mapping $\phi : V \to \mathbb{R}^2$ of vertices into the plane such that $\phi(V)$ is in general position and for every hyperedge $E \in \mathcal{E}$, there is a range $R \in \mathcal{R}$ capturing $\phi(E)$ (with respect to the point set $\phi(V)$), i.e.:

$$R \cap \phi(V) = \phi(E).$$

In other words, there is a point set V' with $|V'| = |V|$ such that $\mathcal{H}(V', \mathcal{R}, m)$ contains a subhypergraph isomorphic to $\mathcal{H}$.

The following lemma is fundamental for this chapter:

Lemma 7.1. *Let $\mathcal{R}$ be a range family and let $\{\mathcal{H}_m\}_{m\in\mathbb{N}}$ be a family of hypergraphs such that for every $m \in \mathbb{N}$:*

1. *The hypergraph $\mathcal{H}_m$ is m-uniform.*
2. *The hypergraph $\mathcal{H}_m$ can be realized with $\mathcal{R}$.*
3. *And the hypergraph $\mathcal{H}_m$ admits no polychromatic coloring with 2 colors.*

Then for the range family $\mathcal{R}$, we have $m(2) = \infty$.

Proof. For every $m \in \mathbb{N}$, the hypergraph $\mathcal{H}_m = (V, \mathcal{E})$ can be realized with the range family $\mathcal{R}$ by assumption. So there is a point set V' and a bijective mapping $\phi : V \to V'$ such that for every hyperedge $E \in \mathcal{E}$ the set $\phi(E)$ is a hyperedge of $\mathcal{H}(V', \mathcal{R}, m)$. Suppose there is a polychromatic coloring $c' : V' \to [2]$ of $\mathcal{H}(V', \mathcal{R}, m)$ with 2 colors. Then consider the coloring $c : V \to [2]$ such that

$$c(v) = c'(\phi(v))$$

for every $v \in V$. Then for a hyperedge $E \in \mathcal{E}$, the set $\phi(E)$ is a hyperedge of $\mathcal{H}(V', \mathcal{R}, m)$ so $\phi(E)$ is polychromatic in c'. Then

$$c(E) = c'(\phi(E)) = [2].$$

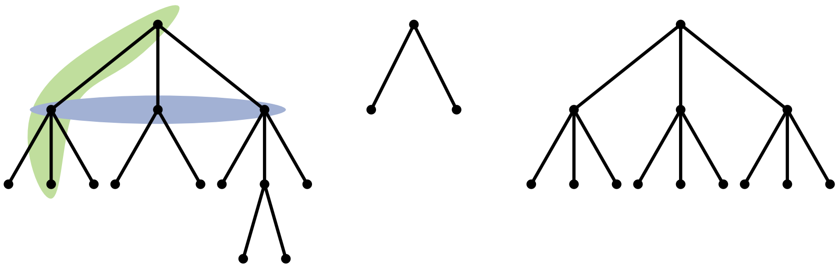


Figure 7.1: (a) An example of a tree T. A siblings-hyperedge of $\mathcal{H}(T)$ is exemplified in blue and a path-hyperedge of $\mathcal{H}(T)$ is exemplified in green. (b) The tree T_2. (c) The tree T_3.

So E is polychromatic in c. Since E was chosen arbitrarily, c is a polychromatic coloring of $\mathcal{H}_m$ with 2 colors – this contradicts Item 3. So the assumption was wrong and $\mathcal{H}(V', \mathcal{R}, m)$ does not admit a polychromatic coloring with 2 colors, hence:

$$m(2) > m.$$

Since this holds for every $m \in \mathbb{N}$, we obtain:

$$m(2) = \infty.$$

□

7.1 Diagonal Strips and South-West Quadrants

Now we show that for the range family of all diagonal strips and south-west quadrants, we have $m(2) = \infty$. To prove this, we first present a family of m-uniform hypergraphs which admit no polychromatic coloring with two colors and then show that every hypergraph in this family can be realized with diagonal strips and south-west quadrants.

Definition 7.2. *For a rooted tree $T = (V, E)$, the hypergraph $\mathcal{H}(T) = (V, \mathcal{E} = \mathcal{E}_p \cup \mathcal{E}_s)$ is defined as follows. Let $r \in V$ be the root of T. The vertices of $\mathcal{H}$ are exactly the vertices of T. There are two types of hyperedges. For every leaf l of T, there is a* path-hyperedge *in $\mathcal{E}_p$ which consists of the vertices of the (unique) path from r to l. For every vertex $v \neq r \in V$, there is a* siblings-hyperedge *in $\mathcal{E}_s$ which consists of v and its siblings (i.e., the vertices having the same predecessor as v in T). See Figure 7.1 (a) for an example.*

For every $m \in \mathbb{N}$, let T_m denote the rooted tree of height $m-1$ in which every non-leaf vertex has exactly m children (see Figure 7.1 (b) and (c)). Then $\mathcal{H}_m$ is defined as $\mathcal{H}_m = \mathcal{H}(T_m)$. Note that by construction, $\mathcal{H}_m$ is an m-uniform hypergraph.

The family has been introduced by Pach et al. in [PTT05]. They have proven that every hypergraph in this family does not admit a polychromatic 2-coloring. For the sake of completeness, we provide the proof here.

Lemma 7.3 ([PTT05])**.** *For every $m \in \mathbb{N}$, the hypergraph $\mathcal{H}_m$ admits no polychromatic coloring with 2 colors.*

Proof. Let $m \in \mathbb{N}$ and $\mathcal{H}_m = (V, \mathcal{E} = \mathcal{E}_p \cup \mathcal{E}_s)$. Suppose $\mathcal{H}_m$ admits a polychromatic 2-coloring $c : V \to \{\text{red, blue}\}$. In particular, every siblings-hyperedge in $\mathcal{E}_s$ is polychromatic in c. Without loss of generality, assume that the root r of T_m is red in c.

We prove by induction on $i \in \{0, \ldots, m-1\}$ that there is a vertex on level i in T_m such that all vertices of its root-to-leaf path in T_m are red.

Base case, $i = 0$: By assumption, the root r is the desired vertex.

Inductive step: Suppose the statement holds for some $i < m-1$ and let v be a vertex on level i whose root-to-leaf path is completely red. Let $E_v \in \mathcal{E}_s$ be the siblings-hyperedge consisting of the children of v. We know that E_v is polychromatic in c so there is a vertex $u \in E_s$ which is red in c. Its root-to-leaf path is exactly the root-to-leaf path of v together with the vertex u itself. Hence, all vertices on the root-to-leaf path of u are red and u is the desired vertex on level $i+1$.

By induction, the claim holds for $i = m-1$, i.e., there is a leaf l of T_m such that its root-to-leaf path is completely red in c. Recall that the vertices of the root-to-leaf path of l form a hyperedge in $\mathcal{E}_p$. Hence, this hyperedge is monochromatic in c and c is not a polychromatic coloring of $\mathcal{H}_m$. Thereby, the assumption was wrong and $\mathcal{H}_m$ does not admit a polychromatic 2-coloring. □

Lemma 7.4. *For every $m \in \mathbb{N}$, the hypergraph $\mathcal{H}_m$ can be realized with the family*

$$\mathcal{R} = \mathcal{R}_{DS} \cup \mathcal{R}_{SW}$$

of all diagonal strips and south-west quadrants.

Proof. To prove the lemma, we show a stronger statement: for every rooted tree $T = (V, E)$, there is an (injective) embedding of V into the plane such that for the hypergraph

$$\mathcal{H}(T) = (V, \mathcal{E}_p \cup \mathcal{E}_s),$$

the following holds:

1. For every path-hyperedge $E \in \mathcal{E}_p$, there is a south-west quadrant capturing E.
2. And for every siblings-hyperedge $E \in \mathcal{E}_s$, there is a diagonal strip capturing E.
3. And the vertices of V are embedded in general position.

We prove this stronger statement by induction on $|V|$.

Base case, $|V| = 1$: The tree with one vertex consists of the root only so any embedding satisfies the desired above properties.

Inductive step: Suppose the statement holds for every tree with at most n vertices for some $n \in \mathbb{N}$. Let T be an arbitrary tree with $n+1$ vertices, let r be its root, and let $T_1 = (V_1, E_1), \ldots, T_p = (V_p, E_p)$ be the subtrees of T rooted at the children of r (i.e., if we remove r from T, then it breaks up into the connected components $T_1, \ldots, T_p$). By induction hypothesis, there exist embeddings of $\mathcal{H}(T_1), \ldots, \mathcal{H}(T_p)$ with properties from Items 1 to 3. Let r_i denote the root of T_i for every $i \in [p]$.

Let $S_0, S_1, \ldots, S_p$ be non-overlapping diagonal (i.e., of slope -1) strips such that for every $i \in [p]$, the strip S_i is to the right of the strip S_{i-1}. For every $i \in [p]$, we repeat the following. First, we place the root r_i in S_0 so that r_i lies to the right of and below every vertex that has already been embedded if such a vertex exists (r_1 can be placed anywhere in S_0). Now consider the region A_i containing all points p such that:

1. $p \in S_i$,
2. $y(p) > y(r_i)$,

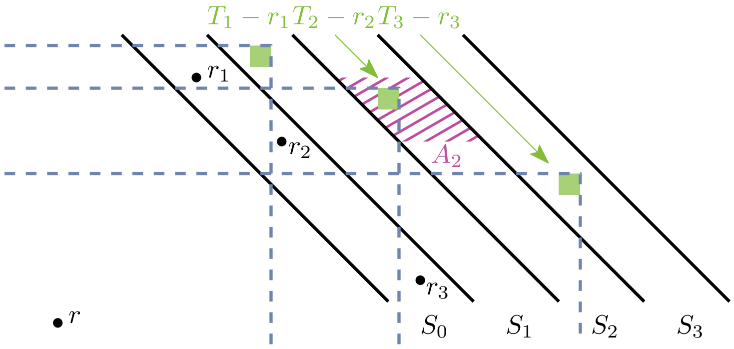


Figure 7.2: Sketch for the proof of Lemma 7.4. Here $p = 3$, for $i \in [3]$, the location of a subtree T_i (without r_i) is depicted with green.

3. $y(p) < y(r_{i-1})$ (if r_{i-1} exists),
4. and $x(p) > x(r_i)$.

By the choice of the position of r_i, the region A_i contains a non-empty open subset. Next, we pick an embedding of $\mathcal{H}(T_i)$ satisfying the properties in Items 1 to 3. We scale it down along the x- and y-axes with the same scale factor α so that $\mathcal{H}(T_i)$ can be placed inside of A_i afterward. Note that such a scaling preserves the properties in Items 1 and 2. Finally, we shift this scaled embedding so that it lies inside of A_i and remove the root r_i from it (because we already embedded it inside S_0).

For $i \in [p]$, let $v_i \in V_i - r_i$ be arbitrary. Then by construction, it holds that

$$y(v_1) > y(r_1) > y(v_2) > y(r_2) > \cdots > y(v_p) > y(r_p) \tag{7.1}$$

and

$$x(r_1) < x(v_1) < x(r_2) < x(v_2) < \cdots < x(r_p) < x(v_p). \tag{7.2}$$

Further, for every $i \in [p]$, we have $A_i \subset S_i$ so:

$$S_i \cap V = V_i - r_i. \tag{7.3}$$

Finally, we place r so that:

1. It lies to the left of S_0
2. and to the bottom-left of every other vertex in V.

Observe that this contruction satisfies Item 3. We claim that the arising embedding satisfies the desired properties from Items 1 and 2.

By construction, it holds that

$$V \cap S_0 = \{r_1, \dots, r_p\}$$

so the siblings-hyperedge $\{r_1, \dots, r_p\}$ is captured by S_0. Now consider any other siblings-hyperedge $E \neq \{r_1, \dots, r_p\}$ of $\mathcal{H}(T)$. The hyperedge E is also a siblings-hyperedge of $\mathcal{H}(T_i)$ for some $i \in [p]$ and it holds that $r_i \notin E$. Since the above-mentioned scaling preserves the property in Item 2 and we only changed the position of $r_i \notin E$, there is a diagonal strip S_E such that

$$S_E \cap (V_i - r_i) = E.$$

Since $E \subseteq V_i - r_i \subseteq S_i$, we may assume that S_E is contained in S_i and hence:

$$S_E \cap V = S_E \cap (V \cap S_i) \overset{(7.3)}{=} S_E \cap (V_i - r_i) = E.$$

So E is captured by a diagonal strip S_E and Item 2 is satisfied.

Now consider a path-hyperedge E. Observe that

$$E = E' \cup \{r\}$$

for some path-hyperedge E' of $\mathcal{H}(T_i)$ for some $i \in [p]$. In particular, it holds that $r_i \in E'$. Since the above-mentioned scaling preserves the property in Item 2 and we only changed the position of r_i, there is a south-west quadrant

$$Q_E = \{(x, y) \mid x \leq a, y \leq b\} \in \mathcal{R}_{\text{SW}}$$

for $a, b \in \mathbb{R}$ such that

$$Q_E \cap (V_i - r_i) = E' - r_i.$$

Due to (7.1), for every $v_i \in V_i - r_i$, it holds that $y(v_i) < y(r_{i-1})$ (if r_{i-1} exists). So we may assume that:

$$b < y(r_{i-1}) \text{ (if } r_{i-1} \text{ exists).}$$

Together with (7.1), we obtain:

$$(V_1 \cup V_2 \cup \cdots \cup V_{i-1}) \cap Q_E = \emptyset.$$

Similarly, due to (7.2), for every $v_i \in V_i - r_i$, it holds that $x(v_i) < x(r_{i+1})$ (if r_{i+1} exists). So we may assume that:

$$a < x(r_{i+1}) \text{ (if } r_{i+1} \text{ exists).}$$

And together with (7.2), we obtain

$$(V_{i+1} \cup V_{i+2} \cup \cdots \cup V_p) \cap Q_E = \emptyset.$$

Moreover, since r_i lies to the bottom-left of A_i, it lies to the bottom-left of every vertex in $V_i - r_i$ so by Observation 2.10 it holds that

$$r_i \in Q_E.$$

Similarly, since r lies to the bottom-left of r_i, we also have:

$$r \in Q_E.$$

Altogether, we obtain:

$$E = Q_E \cap V.$$

So E is captured by a south-west quadrant and Item 1 holds too. As a result, the constructed embedding satisfies the desired properties. By induction, every rooted tree admits such an embedding.

In particular, for every $m \in \mathbb{N}$, the hypergraph $\mathcal{H}_m$ can be embedded in such a way and hence, can be realized with the family $\mathcal{R}$ of all diagonal strips and south-west quadrants. □

With Lemma 7.1, we immediately obtain the following result:

Corollary 7.5. *For the range family $\mathcal{R} = \mathcal{R}_{DS} \cup \mathcal{R}_{SW}$ of all diagonal strips and south-west quadrants, we have $m(2) = \infty$.*

For symmetry reasons, we also obtain the following corollary:

Corollary 7.6. *For the range family $\mathcal{R} = \mathcal{R}_{DS} \cup \mathcal{R}_{NE}$ of all diagonal strips and north-east quadrants we have $m(2) = \infty$.*

Finally, recall that by Lemma 2.1 a subset of points that is captured by a south-west (respectively north-east) quadrant is also captured by a bottomless (respectively topless) rectangle. So we obtain the following theorem:

Theorem 7.7. *Let $\mathcal{R} \in \{\mathcal{R}_{SW}, \mathcal{R}_{NE}, \mathcal{R}_{BL}, \mathcal{R}_{TL}\}$ be a range family of all south-west quadrants, or all north-east quadrants, or all bottomless rectangles, or all topless rectangles. Then for range the family $\mathcal{R} \cup \mathcal{R}_{DS}$, we have $m(2) = \infty$.*

7.2 Bottomless Rectangles and Horizontal Strips

Here we proceed similarly to the previous section to show that for the range family $\mathcal{R} = \mathcal{R}_{\mathrm{BL}} \cup \mathcal{R}_{\mathrm{HS}}$ of all bottomless rectangles and all horizontal strips, we have $m(2) = \infty$. Similarly to Lemma 2.1, it can be shown that every subset of points captured by a bottomless rectangle or a horizontal strip is also captured by an axis-aligned rectangle. Kumar has proven that the family $\{\mathcal{H}_m\}_{m \in \mathbb{N}}$ from Definition 7.2 can not be realized with axis-aligned rectangles [Kum17]. This implies, in particular, that the family $\{\mathcal{H}_m\}_{m \in \mathbb{N}}$ can not be realized with bottomless rectangles and horizontal strips. So to show that for this range family $m(2) = \infty$ holds too, we require another family of hypergraphs. To provide an intuition for this family, we first show that there is no so-called semi-online algorithm for $\mathcal{R}$.

Let V be a point set and let $k \in \mathbb{N}$ be a number of colors. The elements of V are presented to a *semi-online* algorithm in the order of increasing y-coordinates. After a new point is presented, the algorithm can color an arbitrary number of already presented points – we call this iteration a *step*. However, the points are not allowed to be recolored: once a point is colored, the color remains forever. The main property is that a semi-online algorithm has no information about the points that will be presented in the future. The key observation of Asinowski et al. was that for $m \in \mathbb{N}$, a subset of m points is captured by a bottomless rectangle if and only if there is a step after which all of these points are already presented and they have consecutive x-coordinates [ACC+13]. With that, they have proven the following statement.

Theorem 7.8 ([ACC+13])**.** *There exists a semi-online algorithm that gets a point set V and the number of colors k as input and colors the points so that the following property holds: after every step, every set of $m = 3k - 2$ presented points with consecutive x-coordinates contains a point of each of k colors. In particular, this algorithm produces a polychromatic coloring of $\mathcal{H}(V, \mathcal{R}_{BL}, 3k - 2)$ with k colors.*

7.2.1 No Semi-Online Algorithm

Now we show that if we strengthen the requirement and look for a semi-online algorithm with similar property for bottomless rectangles and horizontal strips, then no such algorithm exists (for arbitrarily large uniformity m).

First, observe that it suffices to prove that such an algorithm does not exist even for two colors. Indeed, if there is a uniformity $m \in \mathbb{N}$ and a semi-online algorithm for $k > 2$ colors, then we can adapt this algorithm to color every point with color 2 instead of l for every $2 \leq l \leq k$ – this would produce the desired coloring with two colors. Next recall that a horizontal strip captures a set of vertices with consecutive y-coordinates, i.e., these vertices are presented consecutively. Thus, making horizontal strips with m points polychromatic corresponds to ensuring that after every step, among the last m presented points, there exists a point in each color.

Theorem 7.9. *There does not exist a pair $(m, \mathcal{A})$ such that $m \in \mathbb{N}$, $\mathcal{A}$ is a semi-online algorithm, and the following property holds. For every point set V, the algorithm $\mathcal{A}$ colors the points in V with two colors, say red and blue and ensures that after every step:*

1. *Every set of m presented points with consecutive x-coordinates contains a red point and a blue point.*
2. *And among the last m presented points, there is a red point and a blue point.*

Proof. Let $m \in \mathbb{N}$ and let $\mathcal{A}$ be a semi-online algorithm. To prove the statement, we design an adversary reacting to the actions of $\mathcal{A}$ and presenting points in such a way that at some moment one of Items 1 and 2 is violated. The point set arisen up to that moment then witnesses that the pair $(\mathcal{A}, m)$ does not satisfy the desired properties.

For simplicity, we call the points with consecutive x-coordinates *consecutive*. Under consecutiveness, we always refer to the points that have been presented until some step and this step will be clear from the context. After presenting some points, we "wait" until $\mathcal{A}$ makes its step and then continue. We will not define the precise coordinates of points since only the relative order of their x- and y-coordinates matters. Since the points are presented to a semi-online algorithm in the order of increasing y-coordinates, at every step the point presented by an adversary lies above every other presented point and we will not explicitly state this further.

At the beginning, the adversary presents m^2 points so that the presented point always has the largest x-coordinate (see Figure 7.3 (a)). Now if there are m consecutive points missing some color, then Item 2 is violated and we are done. Otherwise, let $p_1, \dots, p_{m^2}$ be the order in which the vertices have been presented. By construction, it holds that

$$x(p_1) < x(p_2) < \dots < x(p_{m^2}) \text{ and } y(p_1) < y(p_2) < \dots < y(p_{m^2}). \tag{7.4}$$

For $i \in [m]$, we define the set P_i as:

$$P_i = \{p_{(i-1)m+1}, \dots, p_{(i-1)m+m}\}.$$

Note that

$$|P_i| = m.$$

I.e., P_1 is the set of the first m presented points, P_2 is the set of the next m presented points, etc. Since for every $i \in [m]$, the points in P_i are consecutive (see (7.4)), the set contains a red point r_i and a blue point b_i (see Figure 7.3 (b)).

We initialize a set

$$S_i := \{r_i\}$$

for every $i \in [m]$. We will maintain the following invariant: the sets $S_1, \dots, S_m$ are pairwise disjoint and for every $i \in [m]$, the set S_i consists of consecutive red points. Clearly, the invariant holds at the beginning.

Now we repeat the following iteration. In one iteration, for every $i \in [m]$, we present a point q_i immediately to the left of points in S_i, i.e., q_i lies to the left of every point in S_i but there is no point between q_i and S_i (see Figure 7.3 (c)). For every $i \in [m]$ such that q_i is colored red after this iteration, we set

$$S_i := S_i \cup \{q_i\}$$

(see Figure 7.3 (d)). Note that for every $i \in [m]$, the set S_i still consists of consecutive red points: this is true because no point has been presented between the elements of S_i, we only

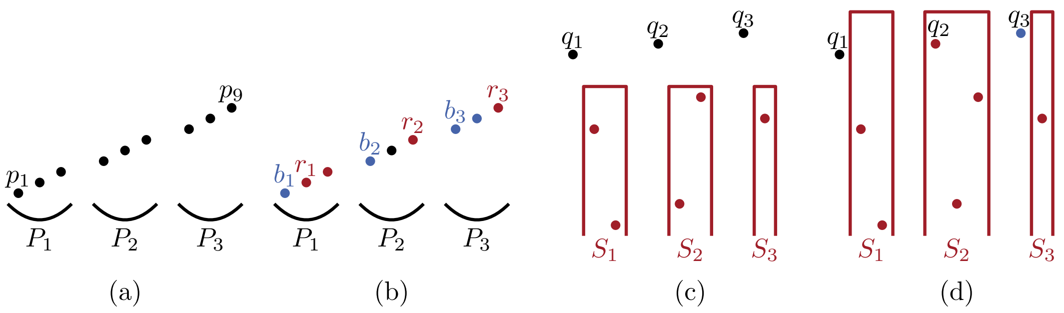


Figure 7.3: Here $m = 3$. We visualize a set S_i with a bottomless rectangle capturing it.
(a) The initial $m^2 = 9$ points.
(b) Among each $m = 3$ consecutive points, there is a red and a blue point.
(c) Present a point q_i immediately to the left of S_i for every $i \in [3]$.
(d) $\{q_1, q_2, q_3\}$ are $m = 3$ consecutively added points so one of them must be colored with red and one with blue. Suppose q_2 gets red, then we extend S_2 with q_2.

extend S_i with q_i if this point is red, and the point q_i has been presented immediately to the left of the previous state of S_i. Moreover, the sets $S_1, \ldots, S_m$ are still pairwise disjoint. If $\mathcal{A}$ colors no point q_i (for $i \in [m]$) with red, then $q_1, \ldots, q_m$ are the last m presented points and none of them is red so Item 1 is violated and we are done.

Otherwise, we repeat this process. Observe that in every iteration the size of at least one S_i (for some $i \in [m]$) increases and the size of every S_i (for all $i \in [m]$) does not decrease. Thereby, the value

$$\sum_{i \in [m]} |S_i|$$

increases by at least one in every iteration. So after at most $m^2 - m$ iterations, this sum becomes at least m^2 and by the extended pigeonhole principle, there exists $i' \in [m]$ such that

$$|S_{i'}| \geq m.$$

Recall that $S_{i'}$ is a set of consecutive red points on the line. In particular, $S_{i'}$ is a set of (at least) m consecutive points on the line containing no blue point. So Item 1 is violated and we are done. This concludes the proof.

□

7.2.2 No Polychromatic Coloring

To prove that for the range family $\mathcal{R}_{\mathrm{BL}} \cup \mathcal{R}_{\mathrm{HS}}$, we have $m(2) = \infty$, we generalize the construction from the previous subsection. There we have provided an adversary reacting to the coloring actions of a semi-online algorithm and presenting points in a smart way to ensure that after a number of steps, there would necessarily be a monochromatic hyperedge. Informally speaking, the family of hypergraphs we present here "reacts to all possible ways how a semi-online algorithm could color the points".

So we first formally define a family $\{\mathcal{D}_m\}_{m \in \mathbb{N}}$ of m-uniform hypergraphs, after that, we prove that all of them admit no polychromatic 2-coloring, and finally, we show how these hypergraphs can be realized with bottomless rectangles and horizontal strips.

Definition 7.10. *For every $m \in \mathbb{N}$, the m-uniform hypergraph $\mathcal{D}_m$ is defined as follows. First, we will define a rooted forest F_m consisting of m^m trees whose vertices are partitioned*

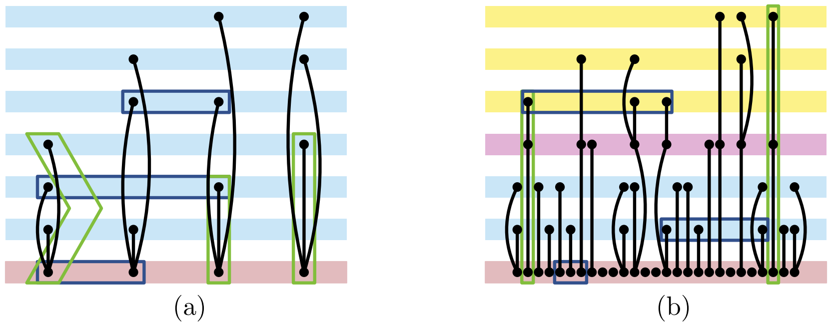


Figure 7.4: Illustration for Definition 7.10. The edges of the forests are black. Some of the hyperedges of the corresponding hypergraphs are exemplified. Path-hyperedges are green and stage-hyperedges are blue. The stages are represented by filled rectangles. The stage containing the roots is red. (a) The forest F_2. (b) Part of the forest F_3. Some of the stages containing the children of the red (respectively lilac) stage are exemplified in blue and lilac (respectively yellow).

into a set of the so-called stages. *The vertices of a stage S will be totally ordered and we denote this ordering by $<_S$. All vertices of a stage S will have the same distance to the root of the corresponding tree, we refer to this distance as the* level *of S. Every stage on level $j \in \{0, \dots, m-1\}$ will consist of m^{m-j} vertices.*

To define the rooted forest F_m and the stages, we start with m^m roots, one for each tree in F_m. They build the unique stage on level 0 and they are ordered in an arbitrary but fixed way. After that, for every stage S on level $j = 0, \dots, m-2$ and every subset

$$S' \in \binom{S}{m^{m-j-1}},$$

we add a new stage $T(S')$ consisting of m^{m-j-1} new vertices so that every vertex in S' gets exactly one child from $T(S')$ and the vertices of $T(S')$ are ordered by $<_{T(S')}$ as their parents by $<_S$. Informally speaking, every vertex in S gets a child for every (m^{m-j-1})-subset of S in which it occurs.

Now we can define the hypergraph $\mathcal{D}_m = (V, \mathcal{E} = \mathcal{E}_s \cup \mathcal{E}_p)$. The vertex set V is exactly the vertex set of F_m. There are two types of hyperedges. First, stage-hyperedges $\mathcal{E}_s$: for every stage S, each m consecutive vertices in $<_S$ constitute a stage-hyperedge. Second, path-hyperedges $\mathcal{E}_p$: for every root-to-leaf path in F_m, there is a path-hyperedge consisting of vertices on this path. Observe that by construction, every leaf in F_m is contained in some stage on level $m-1$ and its root-to-leaf path then consists of exactly m vertices. Thereby, the hypergraph $\mathcal{D}_m$ is m-uniform.

For a vertex v of F_m, we denote the root of the tree containing v with $\operatorname{root}(v)$. *Further, with* $\operatorname{path}(v)$ *we denote the set of vertices on the path from* $\operatorname{root}(v)$ *to v in F_m.*

For illustration, we refer to Figure 7.4.

Lemma 7.11. *For every $m \in \mathbb{N}$, the m-uniform hypergraph $\mathcal{D}_m = (V, \mathcal{E} = \mathcal{E}_s \cup \mathcal{E}_p)$ admits no polychromatic coloring with 2 colors.*

Proof. Let $m \in \mathbb{N}$. We show that every 2-coloring of V that makes all stage-hyperedges polychromatic necessarily produces a monochromatic path-hyperedge. Let $\phi : V \to \{\text{red}, \text{blue}\}$ be such a coloring. The key observation is that a stage S on level j (i.e., one that contains m^{m-j} vertices) can be partitioned into $m^{m-j}/m = m^{m-j-1}$ sets of m vertices that are consecutive in $<_S$, i.e., it can be partitioned into m^{m-j-1} disjoint stage-hyperedges. Hence, the stage S contains at least m^{m-j-1} red vertices.

We prove that for $j \in \{0, \ldots, m-1\}$, there is a stage S_j on level j and a subset $R_j \subset S_j$ such that $|R_j| = m^{m-j-1}$ and for every vertex $v \in R_j$, all vertices in path(v) are red.

Base case, $j = 0$: The stage consisting of the roots of F_m is on level 0 and it contains at least m^{m-1} red roots by the above observation. These vertices have the desired property.

Inductive step: Suppose the statement holds for some $j \in \{0, \ldots, m-2\}$. Consider the stage $S_{j+1} = T(R_j)$ and a set R_{j+1} of m^{m-j-2} red points in S_{j+1} (they exist by the above observation). Recall Definition 7.10: every $v \in R_{j+1} \subset S_{j+1}$ has its parent w in R_j. It holds path(v) = path$(w) \cup \{v\}$. The vertex v is red and by induction hypothesis, all vertices in path(w) are red too. So the statement holds for $j+1$ as well.

By induction, the claim holds for $j = m-1$ too. We have

$$m^{m-j-1} = m^{m-(m-1)-1} = m^0 = 1.$$

Hence, there is a vertex v on level $m-1$ (i.e., a leaf of F_m) whose root-to-leaf path is completely red. The vertices of this path form a path-hyperedge and this hyperedge is monochromatic.

As a result, every 2-coloring of V in which all stage-hyperedges are polychromatic necessarily makes some path-hyperedge monochromatic. So $\mathcal{D}_m$ admits no polychromatic coloring with 2 colors. □

Now we show that the family $\{\mathcal{D}_m\}_{m \in \mathbb{N}}$ can be realized with bottomless rectangles and horizontal strips.

Lemma 7.12. *For every $m \in \mathbb{N}$, the m-uniform hypergraph $\mathcal{D}_m = (V, \mathcal{E} = \mathcal{E}_s \cup \mathcal{E}_p)$ can be realized with the family $\mathcal{R} = \mathcal{R}_{BL} \cup \mathcal{R}_{HS}$ of all bottomless rectangles and all horizontal strips.*

Proof. To prove this statement, we construct the desired embedding of V into the plane. In this realization, the stage-hyperedges will be captured by horizontal strips while the path-hyperedges will be captured by bottomless rectangles. Under parent, predecessor, successor etc. of a vertex, we always refer to these relations in the forest F_m.

We shall embed each stage S of $\mathcal{D}_m$ into a closed horizontal strip, denoted H_S, in such a way that $H_S \cap H_{S'} = \emptyset$ whenever $S \neq S'$. Note that this way, the embedded stages are vertically ordered with some available space between any two consecutive ones.

First, we embed the roots of F_m, i.e., the unique stage S_0 on level 0, so that they form an increasing sequence along $<_{S_0}$. After that, we repeat the following. We choose some stage S that has already been embedded but the stages $T_1, \ldots, T_r$ containing the children of vertices in S not yet. In one step, we embed $T_1, \ldots, T_r$ as follows. We pick a closed horizontal strip H between H_S and the horizontal strip above H_S (if it exists) and within H identify disjoint closed horizontal strips $H_{T_1}, \ldots, H_{T_r}$. After that, every stage T_i (for $i \in [r]$) is embedded inside H_{T_i} so that every vertex gets the same x-coordinate as its parent and the vertices of T_i build an increasing sequence in H_{T_i}. Next, for every $v \in S$, we slightly shift all children of v to the right so that the children of v form a decreasing sequence but the ordering relative to the remaining points remains unchanged (i.e., there is a vertical strip that contains only v and its children). We refer to this one repetition as a *step*. This process is sketched in Figure 7.5: (a) shows the complete embedding for $m = 2$, (b) shows the embedding scheme for the general case.

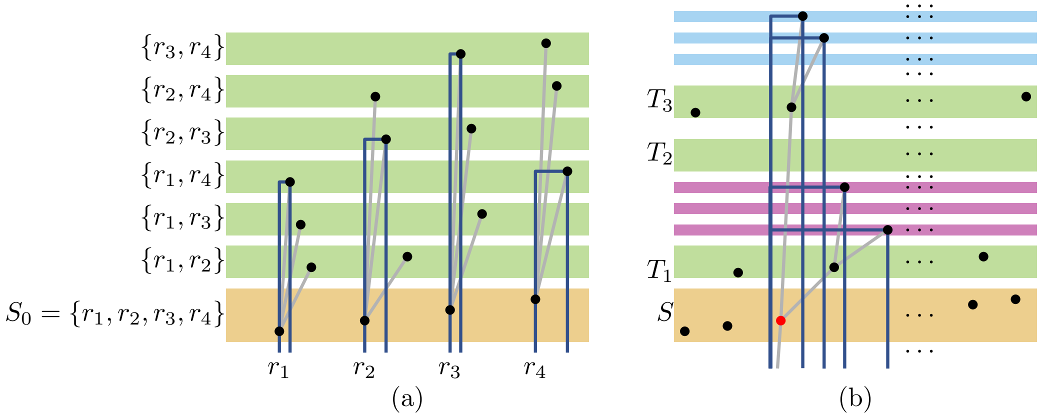


Figure 7.5: Horizontal strips of form H_S for some stage S are represented with filled rectangles. Bottomless rectangles of form $B(v)$ for some vertices v are dark blue. The edges of the forest F_m are gray. (a) The desired embedding of the hypergraph $\mathcal{D}_2$. (b) Sketch of the embedding process for general m. Stages containing the children of vertices in T_1 (respectively T_3) are embedded into the lilac (respectively light blue) horizontal strips. Note that S is not necessarily the stage containing the roots so the sketched bottomless rectangles do not need to have the red vertex on their left side.

Here are some simple observations that will be useful for the proof of correctness. We call a vertex w a *predecessor* of a vertex v in F_m if $w \neq v$ and w lies on the unique path from root(v) to v in F_m.

Observation 7.13. *Every non-root vertex is embedded after its parent. Thereby, every vertex v is embedded after its predecessors in F_m.*

Observation 7.14. *Every vertex in some stage on level* 1 *has its parent in S_0. Hence, all stages on level* 1 *are embedded in the same step. We call this step the* level-1-step. *Moreover, every vertex v in some stage on level $l \geq 2$ has a predecessor on level* 1 *so v is embedded after the level-1-step.*

Observation 7.15. *Since every non-root vertex lies to the top-right of its parent, every vertex lies to the top-right of all its predecessors.*

We show that after every step the embedding process maintains the following invariants:

1. No two embedded vertices have the same x- or y-coordinate.
2. For every embedded stage S, the vertices in S have increasing x-coordinates along $<_S$.
3. For every embedded stage S, the vertices in S build an increasing sequence along $<_S$.
4. For every embedded stage S and every stage-hyperedge $E \in \mathcal{E}_s$ with $E \subseteq S$, there is a horizontal strip capturing E.
5. For every embedded vertex v, the bottomless rectangle with the top-right corner at v and the left side at $x(\mathrm{root}(v))$ captures exactly path(v). We denote this bottomless rectangle with $B(v)$.

Clearly, all invariants are satisfied when only the roots of F_m are embedded. So now suppose the invariants hold after some step. Now we prove that every invariant (i.e., every item) also holds after the next step.

Item 1: The stages are embedded into pairwise disjoint horizontal strips and the vertices of any stage form an increasing sequence. Thus, the y-coordinates of vertices are pairwise disjoint. In one step, we first embed all children of the same vertex v at its x-coordinate and then shift all of them slightly to the right: they do not have the x-coordinate of their parent anymore, they have pairwise distinct x-coordinates (due to decreasing sequence), and do not have the same x-coordinate as any other vertex (recall that there is a vertical strip containing v and its children only). So the vertices still have pairwise distinct x-coordinates and the invariant holds.

Item 2: Note that after the step in which a stage is embedded, the positions of the vertices in this stage are not changed anymore. So it suffices to prove the statement for stages embedded in this step. Let T be an arbitrary stage embedded in this step and let S be the stage containing the parents of vertices in T. The invariant holds for S so the vertices in S have increasing x-coordinates along $<_S$. Suppose there are vertices $u \neq v \in T$ such that

$$u <_T v, x(u) > x(v).$$

Let $p_u \in S$ (respectively $p_v \in S$) be the parent of u (respectively v). Then since every child lies to the right of its parent by construction, we have:

$$x(v) > x(p_v).$$

Since the invariant holds for S and $p_u <_S p_v$ (by the construction of $<_T$), we have:

$$x(p_u) < x(p_v)$$

Altogether, we obtain

$$x(u) > x(v) > x(p_v) > x(p_u).$$

So every vertical strip containing p_u and its child u (embedded in this step) contains the point p_v. Thereby, p_v must be a child of p_u. Since p_u and p_v belong to the same stage, they are on the same level and p_v is not a child of p_u – a contradiction. So the invariant holds.

Item 3: For an embedded stage S, the vertices of S form an increasing sequence (along some ordering) by construction. Further, by Item 2, we know that along $<_S$, these vertices have increasing x-coordinates. So the vertices in S form an increasing sequence along $<_S$.

Item 4: Let S be an embedded strip and let

$$s_1 <_S \cdots <_S s_t$$

be the elements of S. Item 3 implies that $s_1, \ldots, s_t$ form an increasing sequence. Consider m consecutive vertices $s_j, s_{j+1}, \ldots, s_{j+m-1}$ in $<_S$ for an arbitrary $j \in [t-m+1]$ (i.e., arbitrary stage-hyperedge consisting of vertices in S) and consider a horizontal strip:

$$R = \{(x, y) \mid y(s_j) \leq y \leq y(s_{j+m-1})\} \in \mathcal{R}_{\text{HS}}.$$

First, note that since $s_j, s_{j+m-1} \in S$, we have:

$$R \subseteq H_S.$$

And since for every stage $S' \neq S$, we have $H_S \cap H_{S'} = \emptyset$, the horizontal strip R might only contain elements of S. Further, since $s_1, \ldots, s_t$ form an increasing sequence, the following holds:

- For $k < j$, we have $y(s_k) < y(s_j)$ so $s_k \notin R$.
- For $j \leq k \leq j + m - 1$, we have $y(s_j) \leq y(s_k) \leq y(s_{j+m-1})$ so $s_k \in R$.

- For $j+m-1<k$, we have $y(s_{j+m-1}) < y(s_k)$ so $s_k \notin R$.

Altogether, R contains exactly the vertices $s_j, s_{j+1}, \dots, s_{j+m-1}$ and captures the corresponding stage-hyperedge. So this invariant holds too.

Item 5: Suppose that the invariant held at some moment. Let the next step be the embedding of the stages $T_1, \dots, T_r$ containing the children of some stage S. Let v be a vertex embedded not later than in this step: either v has been embedded in this step (i.e., $v \in T_1 \cup \dots \cup T_r$) or before. We distinguish between these two cases to show that $B(v)$ captures $\text{path}(v)$ after this step and hence, the invariant still holds.

Case 1 $v \notin T_1 \cup \dots \cup T_r$: Since the invariant held before this step, the bottomless rectangle $B(v)$ captured exactly $\text{path}(v)$. Since in this step only the vertices in $T_1 \cup \dots \cup T_r$ have been embedded and the position of any other vertex has not been changed, we have:

$$\text{path}(v) \subset B(v).$$

Suppose $B(v)$ does not capture $\text{path}(v)$ anymore, then there is a vertex t such that:

$$t \in B(v) \text{ and } t \in T_i \text{ for some } i \in [r].$$

In particular, it holds that $t \neq v$. Suppose v is on level 0, then $B(v)$ has its top-right corner at v and the left side at $x(v)$ so $B(v)$ is a vertical ray. Since the embedded vertices have pairwise distinct x-coordinates (see Item 1) and $v \neq t$, the vertical ray $B(v)$ can not contain t – a contradiction. So v is on level at least 1 and hence,

$$x(\text{root}(v)) < x(v).$$

Since v was embedded before $T_1 \cup \dots \cup T_r$, the stages $T_1, \dots, T_r$ are on level at least 1. In particular, it holds that $\text{root}(v) \neq t$ and hence, $x(\text{root}(v)) \neq x(t)$. Due to $t \in B(v)$, we then have

$$x(\text{root}(v)) < x(t) < x(v), y(t) < y(v).$$

Let $p_t \in S$ be the parent of t. Suppose

$$p_t = \text{root}(v).$$

Since $p_t = \text{root}(v)$ is on level 0, its child $t \in T_1 \cup \dots \cup T_r$ is on level 1. Then by Observation 7.14, the current step is the level-1-step and $T_1, \dots, T_r$ are all stages on level 1. Recall that v is on level at least 1 and $v \notin T_1 \cup \dots \cup T_r$. So v is on level at least 2. By Observation 7.14, v is embedded after the level-1-step and hence it has not been embedded yet – a contradiction. Thereby,

$$p_t \neq \text{root}(v)$$

and in particular,

$$x(p_t) \neq x(\text{root}(v)).$$

By Observation 7.15, a child is always embedded to the top-right of its parent so:

$$x(p_t) < x(t), y(p_t) < y(t).$$

Now there are two possibilities for the position of p_t (see Figure 7.6 (a) and (b)). If

$$x(p_t) < x(\text{root}(v))$$

would hold (see Figure 7.6 (a)), then

$$x(p_t) < x(\text{root}(v)) < x(t)$$

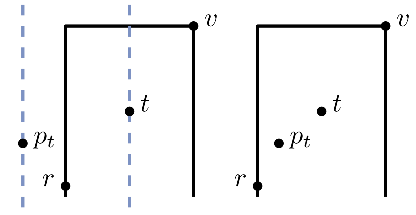


Figure 7.6: Proof of the invariant Item 5, Case 1: $v \notin T_1 \cup \cdots \cup T_r$. For space reasons we denote root(v) with r.

so every vertical strip containing t (embedded in this step) and its parent p_t would contain root(v) (not a child of p_t) – this contradicts the choice of the position of t.

So

$$x(\text{root}(v)) < x(p_t) < x(t) < x(v), y(p_t) < y(t) < y(v)$$

(see Figure 7.6 (b)) and hence,

$$p_t \in B(v).$$

Since p_t was not embedded in this step and the invariant held before, it holds that

$$p_t \in \text{path}(v).$$

So $p_t \in S$ is a predecessor of v. In this step, we have embedded all children of p_t. Recall that by Observation 7.13 a vertex is only embedded after its parents so no further successor of p_t has been embedded yet. Thus, v is either a child of p_t and belongs to $T_1 \cup \cdots \cup T_r$ or it has not been embedded yet – both cases lead to a contradiction. Altogether, for every $v \notin T_1 \cup \cdots \cup T_r$, the bottomless rectangle $B(v)$ still captures path(v) after this step. From now on, we may assume that this property holds and we will use it for Case 2.

Case 2 $v \in T_1 \cup \cdots \cup T_r$: Let $p_v \in S$ be the parent of v. From Case 1, we know that $B(p_v)$ captures path(p_v) after this step. Further, it holds that

$$\text{root}(v) = \text{root}(p_v), \text{path}(v) = \text{path}(p_v) \cup \{v\}$$

and by Observation 7.15, we have:

$$x(p_v) < x(v), y(p_v) < y(v).$$

So

$$B(p_v) \subset B(v)$$

and hence,

$$\text{path}(p_v) \subset B(v), v \in B(v).$$

It remains to prove that no further vertex is embedded inside of $B(v)$. Suppose there is a vertex

$$t \in B(v) \setminus \text{path}(v).$$

In particular, it holds that

$$t \notin \text{path}(p_v), x(t) < x(v), y(t) < y(v). \tag{7.5}$$

Since $B(p_v)$ still captures path(v), t must lie outside of $B(p_v)$, i.e.,

$$t \in B(v) \setminus B(p_v).$$

First, suppose

$$x(p_v) < x(t) < x(v)$$

(lilac area in Figure 7.7 (a)). Then t lies in every vertical strip that contains p_v and v. By the choice of the position of v (embedded in this step), t must also be a child of p_v. Since the children of p_v form a decreasing sequence by construction and

$$x(t) < x(v)$$

holds, we obtain

$$y(t) > y(v).$$

So

$$t \notin B(v).$$

This is a contradiction. Thus,

$$t \in B(v) \setminus B(p_v) \text{ and } x(\text{root}(v)) < x(t) < x(p_v) \tag{7.6}$$

(lilac area in Figure 7.7 (b) and (c)). Therefore,

$$y(p_v) < y(t) < y(v). \tag{7.7}$$

Recall that by construction, the horizontal strips $H_{T_1}, \ldots, H_{T_r}$ lie inside of a closed strip H which, in turn, lies immediately above the horizontal strip H_S, i.e., H is above H_S but it is below every horizontal strip $H_{S'}$ such that $S' \notin \{T_1, \ldots, T_r\}$ and $H_{S'}$ is above H_S. So (7.7) implies that either $t \in S$ or $t \in T_1 \cup \cdots \cup T_r$. The case $t \in S$ immediately contradicts Item 3 since $t \in S$ and $p_v \in S$ form a decreasing sequence. So

$$t \in T_1 \cup \cdots \cup T_r.$$

Consider the parent $p_t \in S$ of t. Since a child is always embedded to the right of its parent (see Observation 7.15), we obtain:

$$x(p_t) < x(t) \overset{(7.6)}{<} x(p_v).$$

Both vertices p_t and p_v belong to S so by Item 3 they build an increasing sequence and hence

$$y(p_t) < y(p_v).$$

Suppose $p_t = \text{root}(v)$. Then $t \in T_1 \cup \cdots \cup T_r$ is on level 1 and $v \in T_1 \cup \cdots \cup T_r$ is on level 1 too. Thereby, $p_v = \text{root}(v)$ so t and v have the same parent. Then by construction, t and v must form a decreasing sequence – this contradicts the fact that t and v form an increasing sequence (see (7.5)). So

$$p_t \neq \text{root}(v)$$

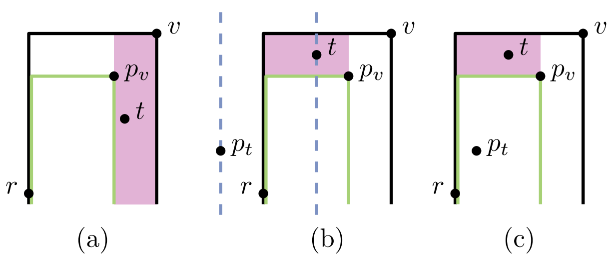


Figure 7.7: Proof of the invariant Item 5, Case 2: $v \in T_1 \cup \cdots \cup T_r$. For space reasons, we denote root(v) with r.

and in particular, by Item 1, we have

$$x(p_t) \neq x(\mathrm{root}(v)).$$

If

$$x(p_t) < x(\mathrm{root}(v)) < x(t)$$

would hold (see Figure 7.7 (b)), then every vertical strip containing t (embedded in this step) and its parent p_t would contain $\mathrm{root}(v)$ (not a child of p_t) – this contradicts the choice of the position of t.

Thus,

$$x(\mathrm{root}(v)) < x(p_t) < x(t) < x(p_v) \text{ and } y(p_t) < y(p_v)$$

(see Figure 7.7 (c)). So

$$p_t \in B(p_v).$$

Recall that $B(p_v)$ captures exactly $\mathrm{path}(p_v)$ and hence,

$$p_t \in \mathrm{path}(p_v).$$

But then $\mathrm{path}(p_v)$ contains two vertices $p_v \neq p_t$ on the same stage S, i.e., two vertices on the same level. But this is impossible for a path in a forest. Altogether, all possibilities lead to a contradiction. So $B(v)$ captures exactly $\mathrm{path}(v)$ and the invariant Item 5 still holds.

As a result, the invariants hold after each embedding step and especially, they hold for the arising embedding of $\mathcal{D}_m$. Item 1 ensures that the vertices are embedded in general position. Item 4 ensures that every stage-hyperedge is captured by a horizontal strip. Item 5 implies that in particular, for every leaf l of F_m, the bottomless rectangle $B(l)$ captures exactly $\mathrm{path}(l)$. So every path-hyperedge is captured by a bottomless rectangle. Therefore, in this embedding, every hyperedge of $\mathcal{D}_m$ is captured by a bottomless rectangle or a horizontal strip. Hence, $\mathcal{D}_m$ can be realized with the family $\mathcal{R} = \mathcal{R}_{\mathrm{BL}} \cup \mathcal{R}_{\mathrm{HS}}$.

□

With Lemma 7.1, we immediately obtain the following result:

Corollary 7.16. *For the range family $\mathcal{R} = \mathcal{R}_{BL} \cup \mathcal{R}_{HS}$ of all bottomless rectangles and horizontal strips, we have $m(2) = \infty$.*

For symmetry reasons, we get the following theorem.

Theorem 7.17. *Let $\mathcal{R} \in \{\mathcal{R}_{BL}, \mathcal{R}_{TL}\}$ be a range family of all bottomless, or all topless rectangles. Then for the range family $\mathcal{R} \cup \mathcal{R}_{HS}$, we have $m(2) = \infty$.*

8. Further Results

8.1 Strong Colorings

Recall that a coloring of a hypergraph is strong if no hyperedge contains two vertices in the same color. Here we shortly consider strong colorings and show that, unlike the polychromatic setting, if two hypergraphs admit strong colorings, so does their union.

Theorem 8.1. *Let $k_1, k_2 \in \mathbb{N}$ and let $\mathcal{H}_1 = (V_1, \mathcal{E}_1)$, $\mathcal{H}_2 = (V_2, \mathcal{E}_2)$ be two hypergraphs such that $\phi_i : V_i \to [k_i]$ is a strong coloring of $\mathcal{H}_i$ for $i \in [2]$. Then their union $\mathcal{H} = (V_1 \cup V_2, \mathcal{E}_1 \cup \mathcal{E}_2)$ admits a strong coloring with $k_1 \cdot k_2$ colors.*

Proof. Consider the coloring $\phi : V \to [k_1] \times [k_2]$ defined as

$$\phi(v) = \begin{cases} (\phi_1(v), 1) & \text{if } v \in V_1 \setminus V_2 \\ (1, \phi_2(v)) & \text{if } v \in V_2 \setminus V_1 \\ (\phi_1(v), \phi_2(v)) & \text{if } v \in V_1 \cap V_2 \end{cases}.$$

We claim that ϕ is the desired strong coloring of $\mathcal{H}$. For $i \in [2]$, consider an arbitrary hyperedge $E \in \mathcal{E}_i$ and $u \neq v \in E \subseteq V_i$. By the definition of ϕ_i, we have $\phi_i(u) \neq \phi_i(v)$ and hence, $\phi(u) \neq \phi(v)$. Thus, no hyperedge of $\mathcal{H}$ contains two distinct vertices in the same color in ϕ. So ϕ is a strong coloring of $\mathcal{H}$ with

$$|[k_1] \times [k_2]| = k_1 \cdot k_2$$

colors. □

This implies the following result.

Theorem 8.2. *Let $m \in \mathbb{N}$ and let $\mathcal{R}_1, \mathcal{R}_2$ be range families such that $k_{\mathcal{R}_i}(m) < \infty$ for $i \in [2]$. Then for the union of the range families, it holds that $k_{\mathcal{R}_1 \cup \mathcal{R}_2}(m) < \infty$.*

Proof. Let V be an arbitrary point set. By assumption, for $i \in [2]$, the hypergraph

$$\mathcal{H}(V, \mathcal{R}_i, m) = (V, \mathcal{E}_i)$$

admits a strong coloring with $k_i < \infty$ colors. Then by Theorem 8.1, the hypergraph

$$(V \cup V, \mathcal{E}_1 \cup \mathcal{E}_2) = (V, \mathcal{E}_1 \cup \mathcal{E}_2) = \mathcal{H}(V, \mathcal{R}_1 \cup \mathcal{R}_2, m)$$

admits a strong coloring with $k_1 \cdot k_2$ colors. Since V was chosen arbitrarily, we obtain

$$k_{\mathcal{R}_1 \cup \mathcal{R}_2}(m) \leq k_1 \cdot k_2 < \infty.$$

□

8.2 Dual Hypergraphs

Recall that for a range family $\mathcal{R}$, a finite subset $\mathcal{R}' \subseteq \mathcal{R}$, and $m \in \mathbb{N}$, the hypergraph $\mathcal{H}^*(\mathcal{R}', m)$ has vertex set $\mathcal{R}'$ and for every point in the plane which is covered by at least m ranges in $\mathcal{R}'$, there is a hyperedge consisting of these ranges. The number $m^*(k)$ then is the smallest number such that every hypergraph $\mathcal{H}^*(\mathcal{R}', m^*(k))$ admits a polychromatic coloring with k colors. We show that, unlike for primal hypergraphs, if for two range families, $m^*(k) < \infty$ holds, then this holds for their union too.

Lemma 8.3. *Let $\mathcal{R}_1, \mathcal{R}_2$ be disjoint range families (i.e., $\mathcal{R}_1 \cap \mathcal{R}_2 = \emptyset$) and let $k, m_1, m_2 \in \mathbb{N}$ be such that for $i \in [2]$, it holds that $m^*_{\mathcal{R}_i}(k) \leq m_i$. Then it holds that $m^*_{\mathcal{R}_1 \cup \mathcal{R}_2}(k) \leq m_1 + m_2$.*

Proof. Let $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ and let $\mathcal{R}' \subseteq \mathcal{R}$ be a finite subfamily. For $i \in [2]$, let

$$\mathcal{R}'_i = \mathcal{R}' \cap \mathcal{R}_i \subseteq \mathcal{R}_i.$$

Note that $\mathcal{R}'_i$ is also finite. By assumption and by Observation 2.19, we know that for $i \in [2]$, there exists a polychromatic coloring $c_i : \mathcal{R}'_i \to [k]$ of $\mathcal{H}^*(\mathcal{R}'_i, m_i)$. I.e., for every point $p \in \mathbb{R}^2$, if p is covered by at least m_i ranges of $\mathcal{R}'_i$, then it is covered by at least one range in every color in c_i.

Consider the coloring $c : \mathcal{R}' \to [k]$ defined as

$$c(v) = \begin{cases} c_1(v) & \text{if } v \in \mathcal{R}'_1 \\ c_2(v) & \text{if } v \in \mathcal{R}'_2 \end{cases}.$$

Note that c is well-defined since $\mathcal{R}_1$ and $\mathcal{R}_2$ are disjoint and hence, $\mathcal{R}'_1$ and $\mathcal{R}'_2$ are disjoint too.

Let E be an arbitrary hyperedge of $\mathcal{H}^*(\mathcal{R}', m_1 + m_2)$ and let p be a point in the plane such that

$$E = \{R \in \mathcal{R}' \mid p \in R\}.$$

It holds that

$$|E| = |\{R \in \mathcal{R}' \mid p \in R\}| \geq m_1 + m_2. \tag{8.1}$$

For $i \in [2]$, let

$$S_i = \{R \in \mathcal{R}'_i \mid p \in R\}.$$

Note that

$$E = S_1 \cup S_2$$

and recall that $\mathcal{R}'_1 \cap \mathcal{R}'_2 = \emptyset$. So

$$S_1 \cap S_2 = \emptyset.$$

Suppose that for every $i \in [2]$, it holds that $|S_i| < m_i$. Then

$$|E| = |S_1| + |S_2| < m_1 + m_2.$$

This contradicts (8.1). So there exists $j \in [2]$ with $|S_j| \geq m_j$, i.e., the point p is covered by at least m_j ranges from $\mathcal{R}'_j$. By the definition of c_j, the point p is covered by ranges from $\mathcal{R}'_j$ in every color in c_j. Since for every range in $\mathcal{R}'_j$ its color is the same in c_j as in c, the point p is covered by ranges from $\mathcal{R}'_j \subseteq \mathcal{R}'$ in every color in c. So E is polychromatic in c. Since E was chosen arbitrarily, c is a polychromatic coloring of $\mathcal{H}^*(\mathcal{R}', m_1 + m_2)$ with k colors. This concludes the proof. $\square$

The immediate implication of this lemma is the following theorem:

Theorem 8.4. *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be range families and let $k \in \mathbb{N}$ be such that $m^*_{\mathcal{R}_i}(k) < \infty$ for $i \in [2]$. Then $m^*_{\mathcal{R}_1 \cup \mathcal{R}_2}(k) < \infty$.*

Proof. Let

$$\mathcal{R}'_2 = \mathcal{R}_2 \setminus \mathcal{R}_1.$$

Then

$$\mathcal{R}'_2 \subseteq \mathcal{R}_2, \mathcal{R}_1 \cup \mathcal{R}_2 = \mathcal{R}_1 \cup \mathcal{R}'_2, \text{ and } \mathcal{R}_1 \cap \mathcal{R}'_2 = \emptyset.$$

By assumption, for $i \in [2]$, there exists $m_i \in \mathbb{N}$ such that

$$m^*_{\mathcal{R}_i}(k) \leq m_i.$$

Then by Observation 2.20, it also holds that

$$m^*_{\mathcal{R}'_2}(k) \leq m_2.$$

Altogether, by Lemma 8.3, it holds that

$$m^*_{\mathcal{R}_1 \cup \mathcal{R}_2}(k) = m^*_{\mathcal{R}_1 \cup \mathcal{R}'_2}(k) \leq m_1 + m_2 < \infty.$$

$\square$

9. Conclusions

In this work, we studied colorings of geometric hypergraphs. The main goal was to investigate unions of range families. We mostly concentrated on unbounded axis-aligned rectangles and diagonal strips of slope -1, these are:

- Vertical strips $\mathcal{R}_{\text{VS}}$, horizontal strips $\mathcal{R}_{\text{HS}}$, and diagonal strips of slope -1 $\mathcal{R}_{\text{DS}}$ – altogether called strips.
- Axis-aligned quadrants $\mathcal{R}_{\text{NW}}, \mathcal{R}_{\text{NE}}, \mathcal{R}_{\text{SW}}, \mathcal{R}_{\text{SE}}$ in four directions.
- Bottomless rectangles $\mathcal{R}_{\text{BL}}$ and topless rectangles $\mathcal{R}_{\text{TL}}$.

As a result of this work, for every combination of these range families, we either have proven that $m(2) = \infty$ holds or provided some upper bound on $m(k)$ for every $k \in \mathbb{N}$ (see Table 9.1).

There are two results with $m(2) = \infty$. In both cases, we have shown that there is a family of m-uniform hypergraphs (for $m \in \mathbb{N}$) such that every hypergraph in this family can be realized with the corresponding range family and does not admit a polychromatic coloring with two colors. For the range family $\mathcal{R}_{\text{SW}} \cup \mathcal{R}_{\text{DS}}$, we employed the tree-hypergraphs introduced by Pach et al. [PTT05]. After that, for the range family $\mathcal{R}_{\text{BL}} \cup \mathcal{R}_{\text{HS}}$, we have first shown that there is no semi-online algorithm producing a polychromatic coloring even for two colors. For this purpose, we provided an adversary against each such algorithm. We then generalized this approach to create a new family of m-uniform hypergraphs (for $m \in \mathbb{N}$) which admit no polychromatic 2-coloring and can be realized with $\mathcal{R}_{\text{BL}} \cup \mathcal{R}_{\text{HS}}$. The idea behind the construction is similar to the tree-hypergraphs of Pach et al. However, instead of trees we use forests and create them in a somewhat more sophisticated way: we partition the vertices into the so-called stages reflecting a huge number of relevant subsets.

It was easy to observe that, for example, south-west quadrants can be seen as special bottomless rectangles in our problem setting. So for the range family $\mathcal{R}_{\text{BL}} \cup \mathcal{R}_{\text{DS}}$, we also have $m(2) = \infty$. For symmetry reasons, there are some more of such pairs. Clearly, for every combination of range families containing one of these pairs as a subset, it still holds $m(2) = \infty$. For this reason, looking for upper bounds on $m(k)$, we only considered range families without such pairs.

For the range family $\mathcal{R}_{\text{BL}} \cup \mathcal{R}_{\text{TL}}$, we have proven that $m(k) \in \mathcal{O}(k^{8.75})$ holds. This is one of the most general results. The main idea of the proof is that if we stretch a point set along the x-axis strongly enough, each relevant bottomless or topless rectangle can be replaced by an axis-aligned square capturing the same subset of points. For the range family of all axis-aligned squares, Ackerman, Keszegh, and Vizer have proven the aforementioned bound [AKV17].

For the remaining range families, the upper bounds have been provided in Chapter 5. These results are based on t-shallow hitting sets. A subset of vertices is called a t-shallow hitting set if it hits every hyperedge in at least one but at most t vertices. This notion was introduced by Keszegh and Pálvölgyi [KP19a]. We have shown that a range family of axis-aligned quadrants in one direction admits a 2-shallow hitting set with the nice property that this set does not hit (most of the) other ranges too often. This yielded the following general approach. Let $\mathcal{R}'$ be a subfamily of axis-aligned quadrants and let $\mathcal{R}$ be another range family with some special prerequisites. We want to color a hypergraph captured by $\mathcal{R}' \cup \mathcal{R}$ with k colors in a polychromatic way. First, we build k disjoint shallow hitting sets of the subhypergraph captured by $\mathcal{R}'$ with these nice properties. Assigning each of these sets its own color makes this subhypergraph polychromatic. After that, we remove the colored points. The special prerequisites imply that the remaining hyperedges (i.e., captured by $\mathcal{R}$) have not been hit too often so they still contain many points. Therefore, we can apply another existing result (e.g., the semi-online algorithm for bottomless rectangles of Asinowski et al. [ACC+13]) to make the remaining hyperedges polychromatic too.

In Table 9.1, our results are summarized. We claim that for every combination of unbounded rectangles and diagonal strips, we either provide an upper bound on $m(k)$ or prove $m(2) = \infty$. A careful reader might notice that we have not considered the rectangles which are unbounded to the right or to the left only (i.e., bottomless rectangles rotated by 90°). However, the table is still complete if we take symmetry into account and observe that horizontal strips and axis-aligned quadrants can be seen as special cases of such rectangles.

In particular, we have shown that if for two range families we have $m(2) < \infty$, this might and might not imply $m(2) < \infty$ for their union. In Chapter 8, we complemented this fact by the result for strong colorings. We have shown that if two hypergraphs admit strong colorings with k_1 and k_2 colors, respectively, then their union admits a strong coloring with $k_1 \cdot k_2$ colors. In particular, $k(m) < \infty$ for two range families implies $k(m) < \infty$ for their union. Similarly, we considered polychromatic colorings of dual hypergraphs and we have shown that if $m^*(k) < \infty$ holds for two range families, then this also holds for their union.

9.1 Open Questions

Although we have answered the question about the existence of polychromatic colorings for every subfamily of unbounded rectangles and diagonal strips, there are still many challenging questions in this area. The most natural question would be to consider unions of further range families such as half-planes or pseudohalfplanes, homothets of a triangle or homothets of a polygon etc.

Question 9.1. *Let $\mathcal{R}_1, \mathcal{R}_2$ be your favorite range families with $m_{\mathcal{R}_1}(k), m_{\mathcal{R}_2}(k) < \infty$. What can we say about $m_{\mathcal{R}_1 \cup \mathcal{R}_2}(k)$?*

In Chapter 3, we presented one general approach to solve the problem for two colors, namely employing the so-called hitting pairs. However, we have also shown that this approach seems to be weak and can not be applied even for some simple range families, for example, the family of all quadrants in one direction. So it is worth searching for weaker sufficient conditions for the existence of polychromatic colorings. Ideally, we would like to find a criterion for this property.

Question 9.2. *Let $\mathcal{R}_1, \mathcal{R}_2$ be two range families with $m_{\mathcal{R}_1}(k), m_{\mathcal{R}_2}(k) < \infty$. What conditions are necessary / sufficient for $m_{\mathcal{R}_1 \cup \mathcal{R}_2}(k) < \infty$?*

In this work, we mostly concentrated on upper bounds on $m(k)$ or proving that $m(k) = \infty$ holds. However, in the first case, we did not care about the tightness of these bounds. Informally speaking, studying both lower and upper bounds would show us how much worse the bound on $m(k)$ becomes if we extend a range family by another range family. This can be seen as a measure of the complexity of geometric hypergraphs.

Question 9.3. *Let $\mathcal{R}_1, \mathcal{R}_2$ be two range families such that $m_{\mathcal{R}_1 \cup \mathcal{R}_2}(k)$. Find a (tight) lower bound on $m_{\mathcal{R}_1 \cup \mathcal{R}_2}(k) < \infty$.*

Positive results in this work are based on shallow hitting sets. However, in most cases we did not find shallow hitting sets for the whole range family. Instead, we constructed shallow hitting sets for some part of the family and then showed that after their removal, the remaining ranges still have enough points to become polychromatic too. But the coloring process would be much simpler if the whole range family would just admit shallow hitting sets. One of the main difficulties is that the following question is still open:

Question 9.4. *Do the hypergraphs $\mathcal{H}(V, \mathcal{R}_{BL}, m)$ admit t-shallow hitting sets for some t if m is sufficiently large?*

Keszegh and Pálvölgyi have shown that hypergraphs of form $\mathcal{H}(V, \mathcal{R}_{\mathrm{BL}})$ do not admit t-shallow hitting sets for any $t \in \mathbb{N}$ in general [KP19a]. However, their counterexamples are not uniform hypergraphs. Later, Balázs Bursics, Bence Csonka, Luca Szepessy, and Sára Tóth (personal communication) have proven that for m large enough, the hypergraphs of form $\mathcal{H}(V, \mathcal{R}_{\mathrm{BL}}, m)$ do not admit 3-shallow hitting sets in general. So the question remains open for $t \geq 4$.

If the previous question could be answered in the positive, then it would be meaningful to study how often such shallow hitting sets hit topless rectangles. If this number would be bounded, then we could proceed similarly to Chapter 5 to improve our upper bound $m(k) \in \mathcal{O}(k^{8.75})$ for the range family $\mathcal{R}_{\mathrm{BL}} \cup \mathcal{R}_{\mathrm{TL}}$.

The following questions related to shallow hitting sets are of independent interest and have been stated several times in the literature.

We know that for good range families, the existence of t-shallow hitting sets implies $m(k) < \infty$ for all k. But it is unknown whether the converse holds.

Question 9.5. *Let $\mathcal{R}$ be a range family such that $m(k) \in \mathcal{O}(k)$. Does $\mathcal{R}$ necessarily admit t-shallow hitting sets for some $t \in \mathbb{N}$?*

Another longstanding question relevant to our work is the following:

Question 9.6. *Let $\mathcal{R}$ be a range family such that $m(2) < \infty$ holds. Does this imply $m(k) < \infty$ for all k?*

For all range families studied until now, this conjecture holds, i.e., no range family is known such that $m(2) < \infty$ but $m(k) = \infty$ for some $k \in \mathbb{N}$. Our results only justify this conjecture once again. However, no general explanation for this behavior is known.

Range family	Sketch	Bound	Source
strips in one direction		$m(k) = k$	[ACC+11]
axis-aligned strips		$m(k) \leq 2k - 1$	[ACC+11]
strips		$m(k) \leq \lceil 4k \ln k + k \ln 3 \rceil$	[ACC+11]
quadrants in one direction		$m(k) = k$	Theorem 4.2
quadrants in two adjacent directions		$m(k) \leq 2k - 1$	Theorem 5.5
quadrants in two non-adjacent directions		$m(k) \leq 3k - 3$	Theorem 5.4
quadrants in three directions		$m(k) \leq 4k - 3$	Lemma 5.6
quadrants in all directions		$m(k) \leq 4k - 3$	Theorem 5.8
bottomless rectangles		$m(k) \leq 3k - 2$	[ACC+13]
quadrants in all directions and bottomless rectangles		$m(k) \leq 5k - 2$	Theorem 5.11
axis-aligned strips and quadrants		$m(k) \leq 10k - 1$	Theorem 5.13
bottomless rectangles and horizontal strips		$m(2) = \infty$	Theorem 7.17
diagonal strips and south-west quadrants		$m(2) = \infty$	Theorem 7.7
strips and north-west and south-east quadrants		$m(k) \leq \lceil 4k \ln k + k \ln 3 \rceil + 4k$	Theorem 5.12
bottomless and topless rectangles		$m(k) \in \mathcal{O}(k^{8.75})$	Theorem 6.3

Table 9.1: Bounds on $m(k)$ for unbounded axis-aligned rectangles and diagonal strips.

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