Masterthesis

Top Trumps – The Graph-Theoretical Structure behind a Children's Game

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Statement of Authorship

I hereby declare that this document has been composed by myself and describes my own work, unless otherwise acknowledged in the text.

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Abstract:

The game *Top Trumps* is a card game played between two players. A deck is a finite subset of $\mathbb{R}^c$. Each player gets dealt half of the deck. One player chooses one of the c categories, the players compare the values of the top two cards of their decks in the chosen category. The winner shuffles the two cards to the bottom of his or her deck and is choosing player for the next round. To a deck X, we associate a directed *dominance graph* on vertex set X and edge set $\{(x, y) \mid x_{c(x)} < y_{c(x)}\}$, where $c(x) := \arg\max_i x_i$. In the first part of the thesis, we relate properties of the deck, like strength of individual cards and first-player-advantage, to graph-theoretical properties of the dominance graph. In the second part, we analyze, which directed graphs can be realized as dominance graphs and if so, the minimum number of categories required for this. This notion is strongly connected to the concept of *monotone edges* and directed, acyclic cliques consisting of monotone edges. Here, a directed edge (u, v) is called monotone, if $N^{\text{out}}(u) \supseteq N^{\text{out}}(v)$. We prove an equivalence theorem, which is a generalization of Dilworth's theorem to directed graphs. As a particular consequence, the width of a poset is exactly the minimum number of categories required to realize it as a Top Trumps deck.

Zusammenfassung:

Das Spiel *Supertrumpf* ist ein Kartenspiel für zwei Spieler. Ein Deck für das Spiel ist eine endliche Teilmenge des $\mathbb{R}^c$. Jeder Spieler bekommt die Hälfte des Decks ausgeteilt. Nun wählt ein Spieler eine der c Kategorien, woraufhin die Spieler die Werte der obersten Karten ihrer jeweiligen Stapel in der gewählten Kategorie vergleichen. Der Gewinner legt die beiden Karten unter seinen eigenen Stapel und darf die Kategorie in der nächsten Runde wählen. Mit einem Deck X verknüpfen wir seinen *Dominanzgraph* auf der Knotenmenge X und der Kantenmenge $\{(x, y) \mid x_{c(x)} < y_{c(x)}\}$, wobei wir $c(x) := \arg\max_i x_i$ definieren. Im ersten Teil der Arbeit setzen wir Eigenschaften des Decks – wie beispielsweise die Spielstärke individueller Karten oder den Vorteil des beginnenden Spielers – in Beziehung mit graphentheo-

retischen Eigenschaften des Dominanzgraphen. Im zweiten Teil der Arbeit untersuchen wir, welche gerichteten Graphen als Dominanzgraphen realisiert werden können. Falls dies möglich ist, fragen wir nach der minimalen Anzahl hierfür notwendiger Kategorien. Diese Fragestellung ist eng verwandt mit dem Konzept *monotoner Kanten* und gerichteter Cliquen, die aus monotonen Kanten bestehen. Hierbei bezeichnen wir eine gerichtete Kante (u, v) als monoton, falls $N^{\mathrm{out}}(u) \supseteq N^{\mathrm{out}}(v)$ gilt. Wir beweisen eine Äquivalenzaussage, die eine Verallgemeinerung des Satzes von Dilworth auf beliebige gerichtete Graphen darstellt. Eine bemerkenswerte Konsequenz dieser Aussage ist die Feststellung, dass die Weite einer Halbordnung genau der minimalen Anzahl an benötigten Kategorien, um sie als Supertrumpf-Deck zu realisieren, entspricht.

Contents

1 Introduction

The game *Top Trumps* is a simple children's card game known to a lot of people. It consists out of a deck of cards with a common theme. For example, this theme may be "sports cars", "racing trucks", "football stars", "comic characters", or one of many others. By our personal experience, most of the themes revolve around vehicles.

On each card there is the name and a picture of one subject from the common theme, e.g. the name and picture of a certain car. Furthermore, there exists a set of fixed categories, e.g. "engine displacement", "engine power", "number of cylinders", "weight" and "top speed". Each card displays how well its subject fares in these fixed categories. An example is given in Figure 1. These statistics may be real-world statistics, or be fictional, like, for example, in the case of comic book characters.

Figure 1: Two cards from the deck "Auto Monster" by the German company Ravensburger. The categories are: "Number of seats", "engine displacement (cm^3)", "engine power (kW)", "engine speed (rpm)" and "weight (kg)".

Top Trumps is played with two or more players and abides the following rules. Each player gets dealt the same portion of the deck and places his or her own cards face down on a pile in front of them. One of the players begins and inspects his or her topmost card. He or she chooses one of the categories, say, engine power. Then every player reveals his or her topmost card. The player with the best value in the chosen category wins this round and all the involved cards. These cards are placed at the bottom of the winning player's pile. Now, the next round is played, with the player who just won deciding on the next category. The game continues in this fashion until all players but one are out of cards. (Depending on the deck, sometimes a stalemate may occur. This case will be discussed later.)

The motivation for this thesis did arise, when its advisor, Torsten Ueckerdt, played a game of Top Trumps against his son. The son noted that some cards were clearly superior to some other cards and that it would be good to have them. Ueckerdt, after playing some time and being mercilessly beaten by his son, agreed. He asked himself the question, of how one would formalize the strength of a card, regarding both its attack abilities when oneself chooses the category, as well as its defense abilities, when the opposing player chooses the category. For example: Suppose you have a card which is very good in one category but very bad in all the other categories. Is it better or worse than a card, which is average in all categories? Also: Do there exist decks in which every card has the same strength?

We could find no serious, scientific investigations regarding the mathematical properties of Top Trumps. However, considering the topic yields a theory, which is surprisingly rich and elegant, especially from a graph-theoretical point of view. In this thesis, we answer the following natural questions:

1. What is the best strategy for the game?
2. How can the strength of a card be formalized?
3. Do there exist decks in which every card has the same strength?
4. If the winner of a round wins one dollar, how much should one pay for the right to be the player choosing the category? Does this amount depend on the deck?

These questions can be answered relatively quickly. But when considering them, new natural questions arise: We will notice, that if in a Top Trumps deck X of size n, the set of values given to the cards inside of each category is exactly $\{1, \dots, n\}$, the optimal play can be expressed very easily: Simply select $c(x) := \arg\max_i x_i$ for a card $x \in \mathbb{R}^c$. We call X a *ranked deck* in this special case. Every Top Trumps deck can be transformed into an equivalent ranked deck.

We will also notice, that if we have two cards x, y in a Top Trumps deck, and we assume both players play optimally, there are exactly three cases: Either

- the player choosing the category wins,
- or the player with card x always wins, irregardless of who chooses the category,

- or the player with card y always wins, irregardless of who chooses the category.

In the latter two cases, we say that x dominates y, or y dominates x, respectively. (We will see that y dominates x, if and only if $x_{c(x)} < y_{c(x)}$). Therefore, every ranked Top Trumps deck implies a directed *dominance graph*, where the vertices are the cards and the edge set is the set of all pairs (x, y) such that the card y dominates the card x. Thus, the dominance graph encodes, which cards beat which other cards in direct combat; in some way, the structure of the complete deck is captured in the dominance graph. Therefore, we considered the following questions:

1. What necessary properties must a dominance graph have?

2. Which directed graphs can appear as the dominance graph of ranked Top Trumps decks? For some given directed graph, what is the lowest number of categories necessary to get such a representation?

3. Which graphs can appear as induced subgraphs of dominance graphs of ranked decks? For some given directed graph, what is the lowest number of categories necessary, such that it can appear as an induced subgraph in this way?

We could answer Questions 1 and 2 only partially. However, Question 3 seems to correspond to a more regular behavior: Exactly all acyclic directed graphs can be represented as induced subgraphs of dominance graphs of ranked decks. Furthermore, the lowest number of categories to represent a directed graph G in this way is equal to the width of a certain poset P related to G, which is formed by the set of all *monotone edges* in G; furthermore this number is also equal to the size of a minimal partition of G into *monotone-neighborhood-cliques*. Here, a directed edge (u, v) is called monotone, if $N^{\text{out}}(u) \supseteq N^{\text{out}}(v)$ and a directed clique in G is called monotone-neighborhood, if the clique is acyclic and all edges inside are monotone (compare Figure 2).

This result is interesting due to two reasons: First, we will see how it implies a polynomial-time algorithm to compute a minimal Top Trumps subdeck representation for a given acyclic graph. Secondly, we will see how the result can be seen as a generalization of *Dilworth's theorem*, which is a well-known theorem relating to partially ordered sets. As a particular consequence, we will see that the width of a poset is exactly the minimum number of categories required for a Top Trumps subdeck representation of the poset. This insight we deemed surprising, as it connects the initial question about a children's game with a seemingly unrelated statement from the subject of graph theory.

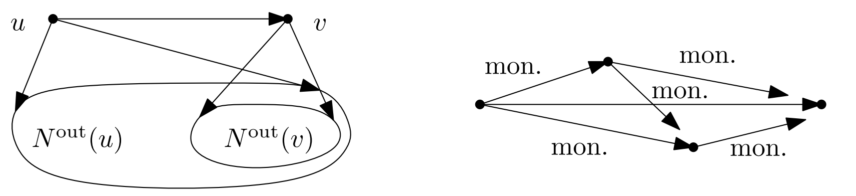

Figure 2: Depiction of a monotone edge and a monotone-neighborhood-clique.

1.1 Overview

We begin in Section 2, by introducing preliminary knowledge and notation, including the notion of partially ordered sets and Dilworth's theorem. After that, the thesis is split into three main parts:

In the first part, Section 3, we are concerned with introducing a model of the game and studying basic relationships. We begin by defining a formal mathematical model of Top Trumps (Section 3.1). We find an optimal strategy for the game and define the strength of a card under the assumption that the involved players play optimally. We introduce the concept of *dominance graphs* in Section 3.2. We then continue in Section 3.3, where we analyze the relation between dominance graphs and the strengths of individual cards. Also, we consider those decks, where every card has the same strength. We analyze properties common to all dominance graphs in Section 3.4 and conclude the first part of the thesis in Section 3.5, by considering the first-player-advantage in a game of Top Trumps.

In the second part, Section 4, we are concerned with the question, which directed graphs can be realized as dominance graphs of ranked decks. We introduce the notion of *ranked realizability* and the *ranked-realizability number* in Section 4.1. We then consider general properties of ranked-realizable, directed graphs in Section 4.2. We are not able to answer the question completely, but we make some progress concerning certain graph operations, which we present in Section 4.3.

In the third and final part, Section 5, we are concerned with the question, which directed graphs can appear as dominance graphs of not necessarily ranked decks. It turns out, that this is equivalent to asking, which directed graphs appear as induced subgraphs of dominance graphs of ranked decks. In Section 5.1, we define the concept of *realizability* and the *realizability number*. Subsequently, in Section 5.2, we show that realizability can be characterized in terms of *monotone-neighborhood-cliques*. In Section 5.3, we describe the behavior of realizabilty under some graph operations. In the final subsection, Section 5.4, we introduce the concept of *monotone edges* and show how they can be used to obtain a third characterization of realizability, creating a connection between the concept of realizability and partially ordered sets. We show a generalization of Dilworth's theorem to arbitrary directed graphs in terms of realizability. As a consequence, there exists a polynomial-time algorithm computing the realizability number of an acyclic directed graph.

1.2 Related Work

We could find almost no scientific investigations related to the topic of Top Trumps. In 2011, the mathematician James Grime, best known for his regular appearance on the Youtube channel "numberphile", published a video on his personal channel, where he analyzed some statistical properties of one specific deck and wondered about whether the game is transitive [13]. There is also a blog post regarding the topic, but to our knowledge, he did not publish any precise results. In 2014, A. B. Cardona, A. W. Hansen, J. Togelius and M. G. Friberger described a simple evolutionary algorithm to extract Top Trump decks from arbitrary data sets as a means of interactive data exploration [2]. They conducted a study and report that "the results show that players enjoy playing the game, are enthusiastic about its potential and answer questions related to decks they have played significantly better than questions related to decks they have not played." However, they also mention some problems, which may make this method unsuitable for data exploration. They do not consider any mathematical properties of the game. The parameter for which their evolutionary algorithm optimizes the extracted deck, is the average attack strength $\overline{s}_1$, whose properties we will analyze in Section 3.5. The *game of war* is an incredibly simple variant of Top Trumps: Here, the cards have only one single category (and there exist multiple suits, so stalemates are possible). The only possible choice players can make in the game of war, is the choice, in which of two possible orders the winner may shuffle the two cards back to his deck. The game can thus be seen as a Markov chain. E. Lakshtanov and V. Roshchina showed that the expected number of turns in this game is finite [10].

These are the mathematical investigations regarding Top Trumps that we know about. But there is one more connection: Further down in the thesis, we will see that the concept of pairs of vertices connected by an edge, such the neighborhood of one of the vertices includes the neighborhood of the other, will play a critical rule. Although not quite the same as our concept, a very similar concept has already been considered (for details, see Section 5.4). This concept is the *Dilworth number of an undirected graph*, which was introduced in 1978 by S. Foldes and P. Hammer [6]. There seem to be many similarities and the notions of the reduced graph (denoted by $\mathrm{mon}^*(G)$ in our thesis), as well as the polynomial-time algorithm used by C. T. Hoàng and N. V. R. Mahadev [8], as well as by S. Felsner, V. Raghavan and J. Spinrad [5] are basically identical for the Dilworth number and for our number. Understanding the connection completely is currently an open problem.

2 Preliminaries and Notation

In the following section, we introduce all necessary concepts of graph theory and other branches of Mathematics and Computer Science which are required for the main part of the thesis. This serves the purpose of building preliminary knowledge, as well as establishing the notation. Although every concept will be briefly explained, some basic knowledge of graph theory is recommended. An excellent book on graph theory was written by Diestel [3]. (Also, during all parts regarding algorithms, we assume the reader to have basic knowledge equivalent to a beginner's algorithmics course.) In this thesis, we will mainly talk about directed graphs without loops, which most of the time will be acyclic as well. However, in Section 5.4, we will also consider undirected graphs and loops. We begin by introducing undirected graphs and concepts related to them.

2.1 Undirected Graphs

An *undirected graph* G is a tuple $G = (V, E)$ of a vertex set $V(G) := V$ and an edge set $E(G) := E$, such that V is some finite set and $E \subseteq \binom{V}{2}$, where $\binom{V}{2} := \{A \subseteq V : |A| = 2\}$. An element $v \in V$ is called a *vertex* and an element $e \in E$ is called an *edge* of G. If $e := \{i, j\} \in E$ is an edge, we generally use the *shorter notation* ij instead of $\{i, j\}$ to refer to e. We define the *order* $|G|$ of a graph G as the number of its vertices, i.e. $|G| := |V(G)|$. For $v \in V$, we define the *neighborhood* of v in G as $N_G(v) := \{w \in V : vw \in E\}$. We also write $N(v)$ for the neighborhood of v in G, if the graph G can be deduced from the context. The *degree* of a vertex v in G, denoted by $d_G(v)$, is the size of its neighborhood in G. Like before, we write $d(v)$ if the graph can be deduced from the context. Two vertices are called *adjacent* if there is an edge between them. A vertex v and an edge e are called *incident*, if $v \in e$.

A graph H is called a *subgraph* of $G = (V, E)$, denoted by $H \subseteq G$, if $V(H) \subseteq V$ and $E(H) \subseteq E \cap \binom{V(H)}{2}$. Special subgraphs are the *induced subgraphs*: For $X \subseteq V$, the subgraph induced by X is denoted by $G[X]$ and defined by having the vertex set X and the edge set $E(G) \cap \binom{X}{2}$. If H is an induced subgraph of G, we denote this by $H \subseteq_{\text{ind}} G$. Furthermore, for $A \subseteq V$, $v \in V$, we define $G - A := G[V \setminus A]$ and $G - v := G - \{v\}$. The process of getting from G to $G - v$ is called *removing the vertex* v. The *complement* of $G = (V, E)$, denoted by $\overline{G}$, is the graph on the same vertex set V having exactly those vertex pairs as edges, which are not edges in G, i.e. the set $\binom{V}{2} \setminus E(G)$.

We give a name and a symbol to some special graphs and graph classes: The *complete graph* K_n on n vertices, has all possible edges between its n vertices. The *path* P_{n+1} of length $n \geq 1$ on $n+1$ vertices consists out of the distinct vertices $v_1, \dots, v_{n+1}$ and the edges $v_i v_{i+1}$ for $i = 1, \dots, n$. The *cycle* C_n on $n \geq 3$ vertices has the same vertices and edges as P_n and additionally the edge $v_n v_1$. When we say that G *contains a path (a cycle)*, we mean that G contains a subgraph, which itself is a path (a cycle). In particular, paths and cycles use no vertex more than once. For $p, q \in \mathbb{N}$, $K_{p,q}$ denotes the *complete bipartite graph* with $V(K_{p,q}) := A \cup B$, such that $A \cap B = \emptyset, |A| = p, |B| = q$ and $E(K_{p,q}) := \{ab \mid a \in A, b \in B\}$. A (sub–)graph with only vertices but no edges between them is called an *independent* set. A subgraph, which is complete, is called a *clique*.

Let $r, n \in \mathbb{N}$. The *extremal number* $\text{ext}(n, K_{r+1})$ is the largest integer m such that there exists a graph G on n vertices and m edges which does not have K_{c+1} as a subgraph, i.e. it does not contain a clique of size $c+1$. *Turáns theorem* [3] states that for all $r, n \in \mathbb{N}$, we have

$$\text{ext}(n, K_{r+1}) \leq \left(1 - \frac{1}{r}\right) \frac{n^2}{2}.$$

The *Turán graph* $T(n, r)$ is the graph on n vertices, whose vertex set is the disjoint union of r disjoint vertex sets $A_1, \dots, A_r$, such that the following three conditions hold: (i) For all i, we have $|A_i| \in \{\lfloor n/r \rfloor, \lceil n/r \rceil\}$. (ii) For all i, between the vertices of A_i there are no edges. (iii) For $i \neq j$, there are all possible edges between A_i and A_j. If a graph G on n vertices has the maximal amount of edges between all graphs not containing a K_{r+1}, Turáns theorem states, that $G = T(n, r)$.

Given a graph G, we consider several *graph parameters* of G: The *clique number* $\omega(G)$ is the size of the largest clique contained in G. The *chromatic number* $\chi(G)$ of a graph is the minimum number of colors needed to color the vertices of G such that no two adjacent vertices have the same color. Stated slightly different, the chromatic number is the size of a cardinality-minimal partition of $V(G)$ into independent sets. The *independence number* $\alpha(G)$ is the size of the biggest independent set in G. The *clique-cover number* $k(G)$ is the minimum number of cliques needed to cover all vertices of G. These last two parameters are complementary to the clique number and the chromatic number in the sense that $\omega(G) = \alpha(\overline{G})$ and $\chi(G) = k(\overline{G})$. One easily sees that $\chi(G) \geq \omega(G)$ and $k(G) \geq \alpha(G)$ holds for every graph G.

A *matching* in a graph G is a set of edges $M \subseteq E(G)$ such that for all $e, f \in M : e \cap f = \emptyset$. A *vertex cover* W in G is a set of vertices $W \subseteq V(G)$ such that for all $e \in E(G) : e \cap W \neq \emptyset$. A graph G with $\chi(G) \leq 2$ is called *bipartite*. *König's theorem* states that in a bipartite graph, the size of a cardinality-maximal matching is equal to the size of a cardinality-minimal vertex cover [3].

Two vertices u, v of a graph G are said to be *connected*, if there is a path from u to v, *disconnected* otherwise. Being connected is an equivalence relation on $V \times V$ and divides V into equivalence classes. Each such class induces a *connected component* in G.

2.2 Directed Graphs

In a directed graph, each edge has additionally one of two directions. Formally, a directed graph G is a tuple (V, E), where $E \subseteq V \times V \setminus \{(v, v) \mid v \in V\}$ and V is finite. We will *not* use the shorter notation uv for an edge (u, v) in the case of directed graphs, because (u, v) is different from (v, u) in contrast to the case of undirected graphs. The *reverse edge* of (u, v) is (v, u). If both (u, v) and (v, u) are edges for $u \neq v$, we call the set $\{(u, v), (v, u)\}$ a *double-edge between u and v*. In a directed graph, we distinguish between the incoming and the outgoing neighborhood of a vertex v. The *incoming neighborhood* of v is defined as $N_G^{\text{in}}(v) := \{u \in V \mid (u, v) \in E\}$ and the *outgoing neighborhood* of v is defined as $N_G^{\text{out}}(v) := \{u \in V \mid (v, u) \in E\}$. As before, we may also write $N^{\text{in}}(v)$ and $N^{\text{out}}(v)$, if the graph G can be deduced from the context. Analogously to the case of undirected graphs, the *outdegree* $d^{\text{out}}(v)$ of a vertex v is the size of its outgoing neighborhood, and the *indegree* $d^{\text{in}}(v)$ of a vertex v is the size of its incoming neighborhood.

We define *subgraphs, induced subgraphs,* $G - A$, and *removing vertices* analogously to the undirected case. (We skip the definition of a complement of a directed graph for now.) A directed graph G can be transformed into an undirected graph G' by taking the edge $\{u, v\}$ in G' if and only if at least one of (u, v) or (v, u) is present in G. This way, we can define the *graph parameters ω, χ, α, k of directed graphs* G as $\omega(G) := \omega(G')$, $\chi(G) := \chi(G')$, $\alpha(G) := \alpha(G')$ and $k(G) := k(G')$, where G' is the corresponding undirected graph. We say that a set $A \subseteq V(G)$ is independent in G, if it is independent in G'.

The *directed path* $\vec{P}_{n+1}$ of length $n \geq 1$ on $n + 1$ vertices from v_1 to v_{n+1} consists out of the distinct vertices $v_1, \ldots, v_{n+1}$ and the edges (v_i, v_{i+1}) for $i = 1, \ldots, n$. The *directed cycle* $\vec{C}_n$ on $n \geq 2$ vertices has the same vertices and edges as $\vec{P}_n$ and additionally the edge (v_n, v_1). When we say that G *contains a directed path (a directed cycle)*, we mean that G contains a subgraph, which itself is a directed path (a directed cycle). In particular, directed paths and directed cycles use no vertex more than once. A vertex v in a directed graph with $N^{\text{in}}(G) = \emptyset$ is called a *source*, a vertex v with $N^{\text{out}}(G) = \emptyset$ is called a *sink*.

A directed graph without directed cycles is called *acyclic* or *directed acyclic graph (DAG)*. In particular, a DAG contains no double-edges, as these are directed cycles of length 2. A *topological ordering* of a directed graph G is an ordering $(v_1, \ldots, v_n)$ of $V(G)$ such that if (v_i, v_j) is an edge in G, then $i < j$. It can easily be shown that a directed graph G is acyclic if and only if it has a topological ordering. A topological ordering or a directed cycle in G can be found in time $\mathcal{O}(|V| + |E|)$ using depth-first-search [11].

If G is an undirected graph, and H is a directed graph created from G by assigning one of the two directions to each edge, H is called an *orientation* of G. An orientation of a complete graph is called a *tournament*. We call the directed graph on the vertex set $\{v_1, \ldots, v_n\}$ and the edge set $\{(v_i, v_j) \mid i < j\}$ a *directed clique*. A directed clique can also be seen as an acyclic tournament. Sometimes, this graph is also called a *transitive tournament*.

Two vertices u, v of a directed graph G are said to be strongly connected, if there is a directed path from u to v and there is a directed path from v to u. Being strongly connected is an equivalence relation on $V \times V$ and divides V into equivalence classes. Each such class induces a *strongly connected component* in G.

Two directed graphs G_1 and G_2 are called *isomorphous*, denoted by $G_1 \cong G_2$, if there exists a bijection $f : V(G_1) \to V(G_2)$ such that for all $(u, v) \in V(G_1) \times V(G_1)$, we have $(u, v) \in E(G_1)$ if and only if $(f(u), f(v)) \in E(G_2)$.

2.3 Loops

At one point in the thesis, we will consider loops in graphs. A graph $G = (V, E)$, which allows loops is a tuple (V, E) of vertices and edges, where V is finite and $E \subseteq \binom{V}{2} \cup V$. A directed graph G, which allows loops, is a tuple (V, E) where V is finite and $E \subseteq V \times V$. An undirected or a directed graph, which contains a loop, is not acyclic. If v is a loop in an undirected graph, it holds true that $v \in N(v)$ (otherwise this is false). Analogously, if (v, v) is a loop in a directed graph, $v \in N^{\text{out}}(v)$ and $v \in N^{\text{in}}(v)$. At one point in Section 5.4, we will also need the definition of a *strong module* in a directed graph allowing loops. A subgraph $H \subseteq G$ of a directed graph G allowing loops is called a strong module, if for all $u, v \in V(H)$, we have $N_G^{\text{out}}(u) = N_G^{\text{out}}(v)$ and $N_G^{\text{in}}(u) = N_G^{\text{in}}(v)$.

Most of the time, we will not consider loops. When talking of a graph or a directed graph, we do not allow loops, unless explicitly stated otherwise.

2.4 Posets and Dilworth's Theorem

Let M be some set. A subset $R \subseteq M \times M$ is called a *binary relation* on M. If $x, y \in M$ and $(x, y) \in R$, we also write xRy. For example, the relation "smaller-than", $<$, is a relation on $\mathbb{R}$ and $(-1, 3) \in <$. A binary relation R may have multiple properties. The relation is called

- *reflexive*, if for all $x \in M : xRx$.
- *transitive*, if for all $x, y, z \in M : (xRy \wedge yRz) \Rightarrow xRz$.
- *antisymmetric*, if for all $x, y \in M : (xRy \wedge yRx) \Rightarrow x = y$.
- *irreflexive*, if for all $x \in M : (x, x) \notin R$.

Binary relations on a finite set can also be understood as directed graphs (allowing loops). If $R \subseteq M \times M$ is a binary relation on a finite set M, we say that the directed graph $G_R = (M, R)$, which allows loops, is the *graph corresponding to* R. In this context, R is reflexive, if and only if G_R has all possible loops, R is irreflexive, if and only if G_R has no loops at all, and R is antisymmetric, if and only if G_R has no double-edges. Continuing this pattern, we call a directed graph $G = (V, E)$ *transitive*, if for all $x, y, z \in V : (x, y) \in E \wedge (y, z) \in E \Rightarrow (x, z) \in E$. (Note that loops are not important for deciding whether some given directed graph is transitive.) Following this definition, we have that a binary relation R is transitive if and only if its corresponding graph G_R is transitive.

A tuple $(M, \preceq)$, where M is a set and $\preceq$ is a binary relation on M such that $\preceq$ is reflexive, transitive and antisymmetric is called a *partially ordered set*, or *poset*, for short. The relation $\preceq$ is called a *partial order* in this case. The poset is called *finite*, if M is finite. The corresponding directed graph (with loops) G_R is called the *comparability graph of* R. Two elements $x \neq y \in M$ in a poset are said to be *comparable*, if $x \preceq y$ or $y \preceq x$ holds. Otherwise, x and y are said to be *incomparable*, denoted by $x \parallel y$. As an example for these concepts, let $X := \{1, \dots, 5\}$ and let $\mathcal{P}(X)$ be the power set of X, i.e. the set of all subsets of X. Then $(\mathcal{P}(X), \subseteq)$ is a poset and the two elements $\{1, 2\}$ and $\{2, 3, 5\}$ are

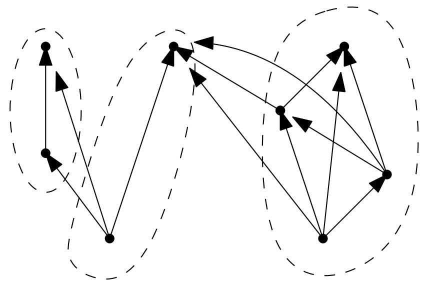

Figure 3: A poset of width 3 and height 4 partitioned into 3 chains (loops omitted).

incomparable.

Sometimes it will be convenient to look at *strict posets*: If R is a transitive and irreflexive binary relation on a set M, we call the tuple (M, R) a strict poset and R a *strict partial order*. It is easy to see that R is a strict partial order if and only if the *reflexive closure* $R \cup \{(x,x) : x \in M\}$ is a partial order. For example, on the natural numbers $\mathbb{N}$, $\leq$ is a partial order and $<$ is a strict partial order. If R is a strict partial order on a finite set, the comparability graph G_R is acyclic, and in particular it is loop-free.

Let $P := (M, \preceq)$ be a finite poset. A sequence $(x_1, \ldots, x_n) \subseteq M$ of pairwise different elements such that for all $i \in \{1, \ldots, n-1\} : x_i \preceq x_{i+1}$ is called a *chain* in P. A chain can also be seen as a set of elements such that every two of them are comparable, or alternatively as a directed clique in the comparability graph of P. An *antichain* is a set of elements such that every two of them are incomparable. An antichain can also be seen as an independent set in the comparability graph. The *height* of the poset P, denoted by $\text{height}(P)$, is defined as the size of a largest chain. The *width* of P, denoted by $\text{width}(P)$, is defined as the size of a largest antichain. There are two theorems regarding chains and antichains of a poset. We begin with the easier one, due to Mirsky [12]:

Theorem 2.1 (Mirsky's theorem)**.** *Let $P = (M, \preceq)$ be a poset. Then M can be partitioned into* $\text{height}(P)$ *antichains (and not less).*

Proof. First, note that if C is a maximal chain, any antichain can contain at most one element of C, as the elements of C are pairwise comparable, but the elements of an antichain are pairwise incomparable. Therefore, M can not be partitioned into less than $\text{height}(P)$ antichains.

Define the set $\min(P) := \{x \in M \mid \forall y \preceq x : x = y\}$ of the minimal elements of P. From its definition, it follows that $\min(P)$ is an antichain. Note that each maximal chain C must contain an element from $\min(P)$. Otherwise, the lowest element of C is not a minimal element and therefore we can extend C. Therefore, the height of $P - \min(P)$ is at least one less than height(P). By induction, we can partition $P - \min(P)$ into at most height$(P) - 1$ antichains and we are done. □

The second theorem is known as *Dilworth's theorem* and was discovered by Dilworth in 1950 [4]. We give a proof due to Fulkerson, based on König's theorem [7].

Theorem 2.2. *[Dilworth's theorem] Let $P = (M, \preceq)$ be a poset. Then M can be partitioned into* width(P) *chains (and not less).*

Proof. Similar to before, note that if A is an antichain, no chain can contain two elements of A. Therefore, at least width(P) chains are needed to cover M.

For the other direction, let $M := \{x_1, \ldots, x_n\}$. Consider the undirected, bipartite graph G on the union of the two vertex sets $U := \{u_1, \ldots, u_n\}$ and $V := \{v_1, \ldots, v_n\}$ and on the edge set $E := \{u_i v_j \mid i \neq j, x_i \preceq x_j\}$. By König's theorem, there exist a matching $M' \subseteq E$ and a vertex cover $C \subseteq U \cup V$ in G such that $|M'| = |C|$.

Let $A := \{x_i \in M \mid u_i \notin C \text{ and } v_i \notin C\}$. Then $|A| \geq n - |C|$. The set A is an antichain in P, because if $x_i, x_j \in A$, then by the properties of a vertex cover, neither $u_i v_j$ or $v_i u_j$ is an edge of G and thus x_i and x_j are incomparable in P. We conclude $|A| \leq \text{width}(P)$.

Define a chain decomposition $\mathcal{D}$ of P the following way: Start with n chains of size 1, then for each edge $u_i v_j \in M'$, unify the chains containing x_i and x_j. By the properties of a matching, whenever the chains containing x_i and x_j are unified, x_i is the maximal element of its chain, and x_j is the minimal element of its chain. By the transitivity of the poset P, the resulting family $\mathcal{D}$ is a valid chain decomposition. We start with n chains and get one chain fewer for each edge of M'. Thus, we finally have

$$|\mathcal{D}| = n - |M'| = n - |C| \leq |A| \leq \text{width}(P).$$

Therefore, P can be partitioned into at most width(P) chains. □

A closer look on the last inequality in the previous proof yields that equality must hold as well, i.e. width$(P) = n - |M'|$. This yields a polynomial-time algorithm to compute the width of a given poset, as the size of a maximum matching in a bipartite graph can be computed in polynomial time. (A classical approach is to reduce the problem to a maximum-flow-problem and use, for example, the Ford-Fulkerson-Algorithm, which takes time $\mathcal{O}(|V||E|)$ in this context, because at most $|V|$ augmenting paths can be found [11]). This can be improved asymptotically to $\mathcal{O}(|V|^{2.5})$ by using the Hopcroft and Karp algorithm for finding maximum bipartite matchings [9] and even a bit further to $O(|V|^{5/2}/\sqrt{\log|V|})$, as shown by Alt, Blum, Mehldorn and Paul [1].

2.5 Other

For $n \in \mathbb{N}$, we define $[n] := \{1, \dots, n\}$. For two disjoint sets A, B, we denote the *disjoint union* of A and B by $A \dot\cup B$. A *partition* of a set M is a family $\{M_1, \dots, M_k\}$ of subsets of M, which are pairwise disjoint and whose union is M. For a function $f : A \to B$, we denote its *image* by $f(A) := \{f(a) \mid a \in A\}$. Also, if $f : A \to \mathbb{R}^d$ is a function mapping to the d-dimensional space, and if $i \in [d]$, we denote the *i-th component function* of f by f_i, i.e. for all $x \in A$, the value $f_i(x)$ is the i-th component of the vector $f(x)$. Inside of formulas, we sometimes use the symbol $\wedge$ for the *and-operator*, and likewise $\vee$ for the *or-operator*. We denote the probability for event A to happen under condition B by $\Pr(A \mid B)$.

3 Top Trumps

This thesis is devoted to analyzing the game *Top Trumps* and the graph-theoretical structures surrounding it. The main section is divided into three parts. In this first part, we concentrate on analyzing the game directly and introducing important concepts. In the later two parts, we concentrate on analyzing the structures surrounding the game, using the concepts defined earlier.

Due to space reasons, throughout the whole thesis, we will only consider the game when played by two players, and only when played for one single round. Although this approach seems restrictive at first glance, it will yield several nontrivial results. We believe these results can be a basis for a study of the game with multiple players and multiple rounds. However, this is out of the scope of this thesis.

3.1 Mathematical Model of Top Trumps

We begin by recalling the rules of a Top Trumps game between two players. If the main target group of the game were mathematicians instead of children, the game manual would maybe state something like this:

A Top Trumps game on $c \in \mathbb{N}$ categories is played with a set of $n \in \mathbb{N}$ cards, where each card is an element of $\mathbb{R}^c$. If Alice and Bob play against each other, Alice is dealt $\lfloor n/2 \rfloor$ cards and Bob is dealt $\lceil n/2 \rceil$ cards. Each player places his or her cards top-down on a pile in front of them. In the first round, Alice takes the role of the *choosing player*. In each round, the choosing player P looks at his or her topmost card x and then chooses a category $i \in \{1, \ldots, c\}$. The other player P' then reveals his or her topmost card y. If $x_i > y_i$, then P wins. If $x_i < y_i$, then P' wins[1]. The winning player places x and y at the bottom of his or her deck and is the choosing player for the next round. If $x_i = y_i$, a *stalemate* occurs. In the case of a stalemate, x and y are placed in the center and a new round is played with the same choosing player[2]. The winner of this new round wins the cards of the new round, as well as all cards in the center. Finally, the game ends when one player has acquired all n cards. This player is the winner of the game. An example of one round can be seen in Figure 4: If Alice chooses the category "agility" of her card "Proxima Midnight", she wins,

[1] In some categories, lower values may be considered "better". For example, in the category "weight", generally the lightest car wins. This can easily be modeled by inverting all values of the category.

[2] Assuming both players have at least one card left. If only one player has a card left, this player wins. If both players are out of cards, i.e. all cards are in the center, the game ends in a draw.

because 74 is bigger than 65. If she would have chosen “strength” instead, a stalemate would have occured.

Figure 4: Two cards from the deck “Marvel Avengers Infinity War” by the UK company Winning Moves.

We want to exclude stalemates from the Top Trumps decks we consider, as they require case distinction and seem quite cumbersome to handle. For example, we will later define the strength of a card and show that a stalemate-free deck on c categories, in which every card has the same strength, has at most c cards. But if we allow stalemates, we can just assemble a deck consisting out of copies of the same card. In such a degenerate deck, every card has the same strength. We found that complications similar to the one described appear quite frequently.

Therefore, we consider:

Definition 3.1 (no-stalemate-property)**.** *Let $c \in \mathbb{N}$. A set of points $X \subseteq \mathbb{R}^c$ has the* no-stalemate-property*, if for all distinct $x, y \in X$, and for all $i \in \{1, \ldots, c\} : x_i \neq y_i$.*

Interpreted geometrically, this means no two points lie on a line parallel to a coordinate axis. In this geometrical context, the no-stalemate-property is usually called *being in general position*. However, note that the notion “general position” has a lot of different meanings, depending on the context.

Due to the above reason, we only want to consider Top Trump games without stalemates. Therefore, from now on and for the rest of the thesis, the term *Top Trumps deck* shall exclusively denote a deck without stalemates. Formally:

Definition 3.2 (Top Trumps deck)**.** *Let $c \in \mathbb{N}$. A* Top Trumps deck *on c categories is a finite set $X \subseteq \mathbb{R}^c$ with the no-stalemate-property. A Top Trumps deck is also simply called a* deck*. The elements of X are called* cards*.*

Definition 3.3 (value of a card in a category)**.** *Let X be a deck on c categories. In order to avoid using double indices later on, we define the* value *of a card $x \in X$ in the category $i \in [c]$:*

$$r_i(x) := x_i$$

We also make a technical definition which will be used in later proofs: If we have a finite set M, we can assign the elements of M labels, where the indiviudal label of an element $x \in M$ may depend on x. We can use these labels to implicitely define a deck.

Definition 3.4 (Defining a deck via a function)**.** *Let M be a finite set, let $c \in \mathbb{N}$, and let $r' : M \to \mathbb{R}^c$ be a function. We say that r'* defines the deck *X', if $X' = r'(M)$ and X' is a deck with $|M|$ cards. (Note that this is equivalent to: For all $i \in [c]$, the component function r'_i is injective.)*

During the process of modeling the game, one observes, that the individual value of a card in a category does not matter, only whether this value is better or worse than the values of the other cards in the same category. For example, say the "McLaren F1" is the fastest car in the deck with a stunning 391.21 km/h. Then it does not matter whether its speed is 391.21 km/h or 427.34 km/h. For the study of the properties of this certain deck it is only important that the McLaren is faster than all other cars from the deck. Therefore we introduce the notion of a normalized deck, called a *ranked deck*.

Definition 3.5 (ranked deck)**.** *Let $c, n \in \mathbb{N}$. A* ranked deck *on c categories with n cards is a set $X \subseteq \mathbb{R}^c$, such that $|X| = n$ and for all $i \in [c] : \{x_i \mid x \in X\} = \{1, \dots, n\}$.*

Note that, in particular, a ranked deck possesses the no-stalemate-property and is thus a deck as well. Every deck can be transformed into a ranked deck by sorting the values inside of each category. An example is depicted in Figure 5.

Theorem 3.6 (converting a deck to a ranked deck)**.** *Let X be a deck on c categories. There exists a ranked deck X', which is equivalent to X in the following sense: X' is defined by $r' : X \to \mathbb{R}^c$ and*

$$\forall x, y \in X : \forall i \in [c] : r_i(x) < r_i(y) \Leftrightarrow r'_i(x) < r'_i(y).$$

Proof. Let $n := |X|$. Fix $i \in [c]$ and sort the cards of X ascending by their value in the i-th category. (As X is a deck, this is possible without ambiguities.) Let O_i be this order. For $x \in X$, let $r'_i(x)$ be the index of x in O_i. By its definition, $r'_i(X) = \{1, \ldots, n\}$ and therefore the deck X' defined by r' is a ranked deck. As X' maintains the relative order of cards inside one category, the rest of the claim follows. $\square$

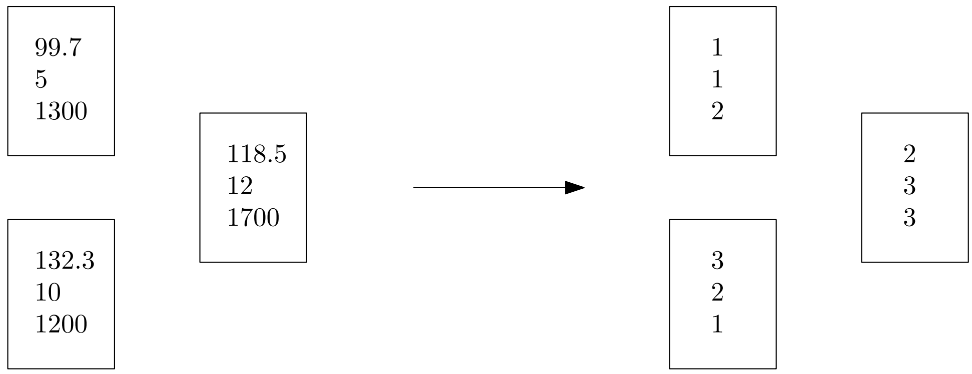

Figure 5: Example of a transformation of a deck with three cards into a ranked deck.

Finally, we introduce the notion of *subdecks*:

Definition 3.7 (subdeck)**.** *If X is a deck and $Y \subseteq X$, then Y is called a* subdeck *of X.*

Note that subdecks of decks are always decks, but subdecks of ranked decks are not necessarily ranked anymore.

After having talked extensively about modeling Top Trump decks, we move on to our model of the game itself. As noted before, due to space and time limitations, we only considered a simple model of the game, in particular, we model only a single round played between the two players *Alice* and *Bob*. We call this model of the game α-Mini-Trump, where α is a parameter in the range $[0, 1]$.

Definition 3.8 (α-Mini-Trump)**.** *Let $\alpha \in [0, 1]$. Let X be a deck on c categories. The game α-*Mini-Trump *is a game between the two players Alice and Bob with the following rules: Alice is dealt a card $a \in X$ chosen uniformly at random. Then Bob is dealt a card $b \in X \setminus \{a\}$ chosen uniformly at random. The* choosing player *is randomly determined: With probability α, Alice is the choosing player, with probability $1 - \alpha$, Bob is the choosing player. The*

$$\boxed{\begin{array}{c}3\\7\\7\\4\end{array}}\ x \qquad\qquad \begin{array}{c}r_1(x) = 3\\ h(x) = 7\\ c(x) = 2\end{array}$$

Figure 6: Example for the functions $r_i(x), c(x)$ and $h(x)$.

choosing player may look at his or her own card (but not at any other card) and choose $i \in [c]$. Subsequently, the player with the higher value in category i wins.

The game $\frac{1}{2}$-MINI-TRUMP *is simply called* MINI-TRUMP.

Due to the fact that Alice's and Bob's cards $\{a, b\}$ with $a \neq b$ are randomly chosen from all pairs in X, an optimal strategy for α-MINI-TRUMP can be easily given:

Theorem 3.9. *Let X be a ranked deck. Let P be the choosing player and let $x \in X$ be the card which was dealt to P. Then an optimal strategy for P is to choose i such that $r_i(x)$ is maximal.*

Proof. Let $A(i)$ be the set of cards, which are worse than x in category i, i.e. $A(i) := \{v \in X \mid r_i(v) < r_i(x)\}$. Because X is a ranked deck, we have that $|A(i)| = r_i(x) - 1$. Now let P' be the other player and let y be their card. As y is randomly chosen from $X \setminus \{x\}$, the player P maximizes his or her chance of winning by maximizing $|A(i)|$. □

When there exist multiple best categories, clearly, any one of them may be chosen, still resulting in optimal play. Especially, always picking the smallest of these best categories results in an unambiguous, deterministic, optimal strategy. We give names to these concepts. (An example is given in Figure 6.)

Definition 3.10 (chosen category $c(x)$, highest value $h(x)$)**.** *Let $x \in \mathbb{R}^c, c \in \mathbb{N}$. We define the* highest value of x *as*

$$h(x) := \max\{x_i \mid i \in [c]\}.$$

The chosen category of x, *denoted by $c(x)$ is the number $i \in [c]$ such that $x_i = h(x)$. If there are multiple such i, choose the smallest one. Formally:*

$$c(x) := \min\{i \in [c] \mid x_i = h(x)\}.$$

Definition 3.11 (Highest-Value strategy)**.** *The* Highest-Value *strategy for the game α-*Mini-Trump *is the strategy to always pick $c(x)$ when choosing the category for a card x.*

If X is a ranked deck, the strategy Highest-Value is an optimal strategy. If, on the other hand, X is not a ranked deck, Highest-Value is not necessarily an optimal strategy. In this case, it can be seen as the strategy of the "easily impressible" player, who always chooses the highest value of the current card. In the case of a non-ranked deck, we can tell by Theorem 3.6 and Theorem 3.9 that the optimal strategy is to select the category, which beats the largest number of other cards when chosen.

As we now have seen that for ranked decks, Highest-Value is an optimal strategy (and that indeed every optimal strategy is equal to Highest-Value except the choice of category when multiple best categories exist), it is natural to define the *strength* of a card the following way:

Definition 3.12 (strength of a card)**.** *Let X be a deck, let $x \in X$ be a card, and $\alpha \in [0,1]$. Let $W(\alpha; X)$ be the event that Alice wins in α-*Mini-Trump *when the deck X is used, under the assumption that both Alice and Bob use the strategy* Highest-Value*. We define the α-*strength of x in X *as*

$$s_\alpha(x; X) := \Pr(W(\alpha; X) \mid \text{ Alice gets dealt } x).$$

$s_1(x; X)$ is called the attack strength of x in X*, $s_0(x; X)$ is called the* defense strength of x in X*, and $s(x; X) := s_{1/2}(x; X)$ is simply called the* strength of x in X*. If the deck X is clear from the context, we may also write $s_\alpha(x), s_1(x), s_0(x)$ and $s(x)$.*

This means that in a ranked deck X, the attack strength $s_1(x)$ is the probability that Alice wins with card x, when both players play optimally and Alice chooses the category. The defense strength $s_0(x)$ is the probability that Alice wins with card x, when both players play optimally and Bob chooses the category. The strength $s(x)$ is the probability of Alice winning with card x, if the choosing player is determined by a fair coin toss.

3.2 Dominance Graphs

We have seen in the previous section, that the strategy Highest-Value is an optimal strategy for Mini-Trump on a ranked deck. Therefore, this section is devoted to understanding the structure of Top Trump decks when the strategy Highest-Value is used. We begin with an elementary observation (requiring a small definition):

Definition 3.13 (two-card-duel). *Let X be a deck. Let $x \in X$ be the card dealt to Alice and $y \in X \setminus \{x\}$ be the card dealt to Bob. If Alice and Bob play a round of Top Trumps both adhering to the strategy* Highest-Value*, we say that there is a* two-card-duel between x and y. *If Alice (Bob) chooses the category, we say that x (y) chooses the category and if Alice (Bob) wins, we say that x (y) wins. Similarly, we say that x (y) loses.*

Observation 3.14 (nature of two-card-duels). *Let X be a deck, $x, y \in X, x \neq y$. In a two-card duel between x and y there are exactly three possible results:*

1. *x wins if x chooses the category, and x loses, if y chooses the category*
2. *x wins, irregardless of who chooses the category*
3. *y wins, irregardless of who chooses the category*

Proof. As Highest-Value is a deterministic strategy, the result is uniquely determined from the cards x, y and the choosing player. Thus, we only have to show that it is impossible for x to win, when y chooses and at the same time lose, when x itself chooses. To do this, assume x wins, when y chooses. Because y chooses $c(y)$ and loses, we have $h(y) < r_{c(y)}(x)$. But we also have $r_{c(y)}(x) \leq h(x)$. Thus $h(x) > h(y) \geq r_{c(x)}(y)$, which implies that x also wins, if x itself chooses. □

This motivates the following definition: If we are in case 2 of Observation 3.14, i.e. x always wins, we say that x *dominates* y.

Definition 3.15 (dominance). *Let X be a deck and $x, y \in X$ with $x \neq y$. We say that x* dominates *y or that x* always wins against *y, if*

$$h(y) < r_{c(y)}(x)$$

and we denote this by

$$y <_D x.$$

Definition 3.16 (incomparability). *Let X be a deck and $x, y \in X, x \neq y$. If neither $x <_D y$, nor $y <_D x$, we call x and y* incomparable.

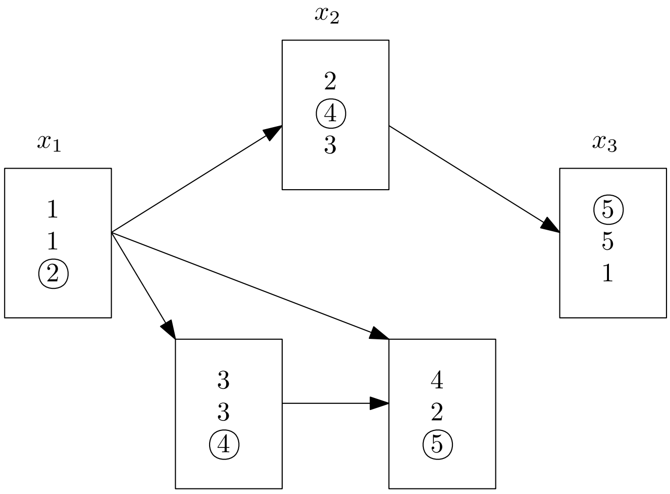

Figure 7: Example of a dominance graph: $x_1 <_D x_2$ and $x_2 <_D x_3$, but x_1 and x_3 are incomparable.

In other words, x dominates y, if x wins against y, even if y chooses the category. An example for these concepts is given in Figure 7 using a ranked deck. From this figure, one can already see the direction into which we are headed: Given a deck X, it is helpful to consider the directed graph where the vertices are the cards and the edges are the dominances, i.e. the edge set implied by the relation $<_D$.

Definition 3.17 (dominance graph). *Let X be a deck. The* dominance graph of X, *denoted by* $\mathrm{dom}(X)$ *is the directed graph $G = (V, E)$ on the vertex set $V := X$ and the edge set $E := \{(x, y) \mid x <_D y\}$.*

An example is again Figure 7.

Observation 3.18. *The dominance relation is not necessarily transitive.*

Proof. A counterexample is given in Figure 7. □

Theorem 3.19 (dominance graphs are acyclic). *Let X be a deck, and $G :=$ $\mathrm{dom}(G)$ its dominance graph. Then G is acyclic.*

Proof. If (x, y) is an edge in G, we observe that $h(x) < r_{c(x)}(y) \le h(y)$. Thus, along every directed path $x_1, \ldots, x_k$ in G, the value $h(x_i)$ strictly increases (where $i \in [k]$). Therefore G is acyclic. □

3.3 Strength of Cards

This subsection is devoted to analyzing the concept of the strength of cards. In Definition 3.12, we defined the strength $s_\alpha(x)$ of a card x in a ranked deck X as the probability, that Alice wins if she gets dealt card x, Bob gets dealt a random card $y \in X \setminus \{x\}$, both players play optimally, and Alice is the choosing player with probability α. We also defined the attack strength s_1 and defense strength s_0. Note that if X is not a ranked deck, the strength of a card means its winning probability under the assumption that both players use the Highest-Value strategy, which is not necessarily optimal. Although this is somewhat unintuitive, it can be understood as a generalization of the initial concept, because every deck can be transformed into an equivalent ranked deck (Theorem 3.6). Now let us begin. A property of α-strength is that it is a linear mixture of attack strength and defense strength:

Theorem 3.20. *Let X be a deck, $x \in X$, and $\alpha \in [0, 1]$. Then*

$$s_\alpha(x) = \alpha s_1(x) + (1 - \alpha)s_0(x).$$

Proof. The value $s_\alpha(x)$ is the probability of Alice to win α-Mini-Trump, if Alice gets dealt the card x. Here the probability space is the random card that gets dealt to Bob, as well as the weighted coin, which lets Alice be the choosing player with probability α.

As the toss of the coin and the dealing of the cards are independent, we can imagine, the coin gets tossed first. If the coin shows that Alice can choose the category, she will win with probability $s_1(x)$, by the definition of s_1. Similarly, if the coin shows "Bob", Alice will win with probability $s_0(x)$. By the basic laws of probability, we have $s_\alpha(x) = \alpha s_1(x) + (1 - \alpha)s_0(x)$. □

An important insight about the strength of a card x is that it correlates directly to the difference of indegree and outdegree of the vertex x in the dominance graph of X: High indegree means that x is "strong", high outdegree means that x is "weak". An example supplementing the next theorem is given in Figure 8.

Theorem 3.21. *Let X be a deck on n cards, $x \in X$ and $G := \mathrm{dom}(X)$ the dominance graph of X. Let $d^{\mathrm{in}}(x)$ and $d^{\mathrm{out}}(x)$ be the in- and the outdegree of x in G, repectively. Then the following equations hold:*

$$s_1(x) = 1 - \frac{d^{\mathrm{out}}(x)}{n-1} \tag{3.1}$$

$$s_0(x) = \frac{d^{\mathrm{in}}(x)}{n-1} \tag{3.2}$$

$$s(x) = \frac{1}{2} + \frac{d^{\mathrm{in}}(x) - d^{\mathrm{out}}(x)}{2n-2} \tag{3.3}$$

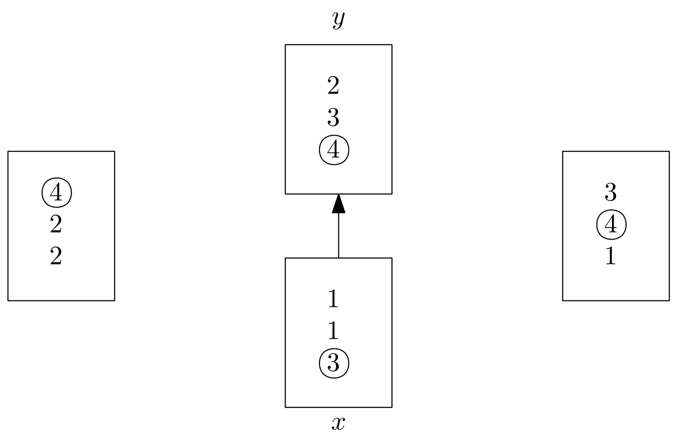

Figure 8: Example of the concept of strength. If x chooses, it wins against two cards and loses against one. Thus $s_1(x) = \frac{2}{3}$. If the opponent of x chooses, x never wins. Thus $s_0(x) = 0$. Combined, $s(x) = \frac{1}{3}$.

Proof. Let A_1 be the incoming neighborhood of x, i.e. $A_1 := \{y \in X \mid (y, x) \in E(G)\}$. Let A_2 be the outgoing neighborhood of x, i.e. $A_2 := \{y \in X \mid (x, y) \in E(G)\}$. Let B be the other vertices in $X \setminus \{x\}$, i.e. $B := X \setminus (A_1 \cup A_2 \cup \{x\})$. Now observe, that if x chooses the category, x wins against exactly the cards in $A_1 \cup B$ by the definition of dominance graphs. Similarly, if the opponent of x chooses the category, x wins against exactly the cards from A_1. As the opponent of x gets dealt a random card from $X \setminus \{x\}$, we have

$$\begin{aligned} s_1(x) &= \frac{|A_1| + |B|}{n-1} = \frac{n-1-|A_2|}{n-1} = 1 - \frac{d^{\text{out}}(x)}{n-1} \\ \text{and } s_0(x) &= \frac{|A_1|}{n-1} = \frac{d^{\text{in}}(x)}{n-1}. \end{aligned}$$

By Theorem 3.20, we have $s(x) = \frac{1}{2}s_0(x) + \frac{1}{2}s_1(x)$, which implies the third equation. $\square$

One of our motivational questions was, whether there exist decks where every card has the same strength. We therefore consider

Definition 3.22 (identical-strength-deck)**.** *Let X be a deck. It is called* identical-strength-deck*, if $s(x; X)$ is identical for all $x \in X$.*

In order to determine the nature of identical-strength-decks, we make a helpful observation:

Theorem 3.23 (average strength)**.** *Let X be a deck on n cards. The average strength of a card in X is $\frac{1}{2}$, i.e.*

$$\frac{1}{n}\sum_{x\in X} s(x) = \frac{1}{2}.$$

Proof. We give two different proofs.

First proof. Consider Alice and Bob playing MINI-TRUMP and getting dealt uniformly random cards $x, y \in X, x \neq y$. As $\alpha = 1/2$, Alice and Bob have the same chance of being the choosing player. Therefore, the situation is completely symmetrical. This means that Alice's chance of winning is equal to Bob's chance of winning. As these two chances add up to 1, Alice's chance of winning is equal to $\frac{1}{2}$. But by the definition of strength, Alice's chance of winning is equal to the average strength of the card dealt to her.

Second proof. This follows from Theorem 3.21 and the fact that the sum of the indegrees in a graph is equal to the sum of the outdegrees. In fact:

$$\begin{aligned}\frac{1}{n}\sum_{x\in X} s(x) &= \frac{1}{n}\sum_{x\in X}\left(\frac{1}{2} + \frac{d^{\text{in}}(x) - d^{\text{out}}(x)}{2n-2}\right)\\ &= \frac{1}{2} + \frac{\sum_{x\in X} d^{\text{in}}(x) - \sum_{x\in X} d^{\text{out}}(x)}{n(2n-2)} = \frac{1}{2}\end{aligned}$$

□

Corollary 3.24. *In every identical-strength-deck, every card has exactly strength $\frac{1}{2}$.*

Using this insight, we can deduce the nature of identical-strength-decks. Consider the following concept: Call the best card of a category the *champion of this category.* For example, if X is a ranked deck on n cards, the champion in category 1 is the card x with $r_1(x) = n$. Formally:

Definition 3.25 (champion)**.** *Let X be a deck on c categories, let $i \in [c]$. The* champion in category i *is the unique card $x \in X$ with $r_i(x) = \max\{r_i(y) \mid y \in X\}$. A card $x \in X$ is called a* champion*, if it is the champion in at least one category.*

Theorem 3.26 (nature of ranked identical-strength-decks)**.** *Let X be a ranked deck on n cards. The following are equivalent:*

(i) X is an identical-strength-deck

(ii) For all $x \in X : x$ is a champion

(iii) The graph $\mathrm{dom}(X)$ *has no edges.*

Proof. " $(i) \Rightarrow (ii)$ " : If X is an indentical-strength-deck, by Corollary 3.24, the strength of all cards is $\frac{1}{2}$. Now assume for the sake of contradiction, there exists $x \in X$ such that x is not a champion. Let y be the champion in category $c(x)$. Then y beats x, if x chooses, because y is the champion in category $c(x)$. On the other hand, if y chooses, we not necessarily have $c(y) = c(x)$, but we know that $h(y) = n$, because y is a champion and X is a ranked deck. Therefore y wins against every other card, if y chooses. In total, $s(y) > \frac{1}{2}$. This is a contradiction.

" $(ii) \Rightarrow (iii)$ " If x is a champion in some category i, then in particular we have $h(x) = n$, because X is a ranked deck. Then there can not be an edge (x, y) in the dominance graph, because this would mean that $h(x) < r_{c(x)}(y)$, but all values in a ranked deck are at most n.

" $(iii) \Rightarrow (i)$ " If the dominance graph has no edges, by Equation (3.3), the strength of every card is $\frac{1}{2}$. □

Corollary 3.27. *Let* X *be a ranked deck, which is identical-strength on* c *categories. Then* X *has at most* c *cards.*

Proof. There can be at most c different cards, which are a champion in some category. □

3.4 Properties of Dominance Graphs

Dominance graphs, introduced in Section 3.2, capture the underlying structure of a Top Trumps deck, or, more abstractly, the structure of a set of points in $\mathbb{R}^c$ with the dominance relation $x <_D y :\Leftrightarrow h(x) < r_{c(x)}(y)$. We hope that the previous sections were able to persuade the reader that the study of these objects seems interesting. In the following subsection, we are now interested in understanding necessary properties that are common to all dominance graphs. We begin with an easy observation:

Lemma 3.28 (monotony alongside an edge)**.** *Let X be a deck and $G := \mathrm{dom}(X)$ its dominance graph. Let $(x, y) \in E(G)$ be an edge in G. Then $h(x) < h(y)$.*

Proof. As $(x, y) \in E(G)$ we have by the definition of the dominance graph and the dominance relation, that $h(x) < r_{c(x)}(y)$. We also have $r_{c(x)}(y) \leq h(y)$ by the definition of $h(y)$. $\square$

We have already seen in Theorem 3.19 that dominance graphs are acyclic and in Observation 3.18 that they are not necessarily transitive. However, they admit a behavior similar to transitivity:

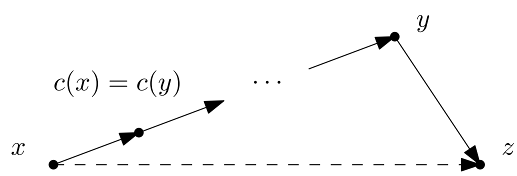

Figure 9: Depiction of Lemma 3.29.

Lemma 3.29 (weak transitivity of dominance graphs)**.** *Let X be a deck on c categories and $G := \mathrm{dom}(X)$ its dominance graph. Let $x, y, z \in X$ such that:*

- $c(x) = c(y)$
- *There is a directed path from x to y in G*
- (y, z) *is an edge in G*

Then we also have that (x, z) is an edge in G.

Proof. (A sketch of the situation is depicted in Figure 9.) By Lemma 3.28, the h-value alongside every edge strictly increases. Applying this lemma repeatedly on the directed path from x to y yields $h(x) < h(y)$. Because $(y, z) \in E(G)$, we have $h(y) < r_{c(y)}(z)$. In total, we have $h(x) < h(y) < r_{c(y)}(z) = r_{c(x)}(z)$, which proves the claim. $\square$

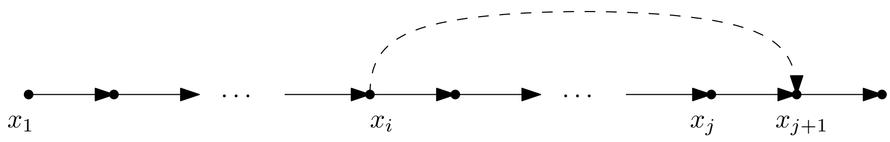

Figure 10: Depiction of Corollary 3.30.

Corollary 3.30 (length of induced paths)**.** *Let X be a deck, $G := \mathrm{dom}(X)$ its dominance graph. Each induced directed path in G has length at most c, i.e. at most $c+1$ vertices.*

Proof. Assume for the sake of contradiction, G contains an induced, directed path P on at least $c+2$ vertices. Let $x_1, \dots, x_{c+2}$ be the first $c+2$ vertices of this path. Consider the set $A := \{x_1, \dots, x_{c+1}\}$. As A has size $c+1$, there exist $x_i, x_j \in A$ with $i < j$, such that $c(x_i) = c(x_j)$. Because $j \leq c+1$, the edge (x_j, x_{j+1}) is contained in G. Applying Lemma 3.29 to x_i, x_j, x_{j+1} yields that $(x_i, x_{j+1}) \in E(G)$. This is a contradiction, as P is an induced path. □

An example that this theorem is sharp, i.e. that we can create an induced path of length c using c categories, can be found as a sketch in Figure 17. Another property that is easy to see is the following:

Lemma 3.31 (same chosen category implies edge)**.** *Let X be a deck, $G :=$ $\mathrm{dom}(X)$ its dominance graph and $x, y \in X, x \neq y$. If $c(x) = c(y)$, then either $(x, y) \in E(G)$ or $(y, x) \in E(G)$.*

Proof. This is obvious due to the definition of dominance and the no-stalemate-property of X. □

Corollary 3.32 (size of independent sets)**.** *Let X be a deck on c categories and $G := \mathrm{dom}(X)$ its dominance graph. The size of an independent set in G is at most c. In other words, $\alpha(G) \leq c$.*

Proof. If a vertex set has size $c+1$, it contains distinct vertices x, y with $c(x) = c(y)$ and thus there is an edge between x and y. □

It is easy to see that we can get an independent set of size c using c categories, so this theorem is also sharp.

For the next property, we recall from the preliminaries the concept of a *directed clique.*

Definition 3.33 (directed clique)**.** *Let $k \in \mathbb{N}$. The directed graph H_k on the vertex set $\{v_1, \dots, v_k\}$ and the edge set $\{(v_i, v_j) \mid 1 \leq i < j \leq k\}$ is called the*

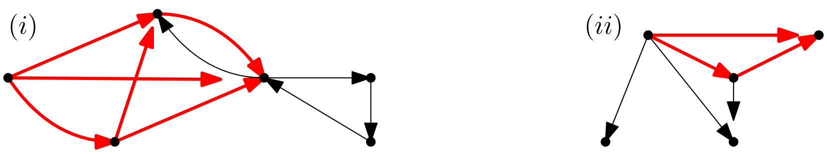

Figure 11: (i) A graph containing a directed clique of size 4. (ii) A monotone-neighborhood-clique of size 3.

directed clique on k vertices. *Note that in this case, the sequence* $(v_1, \dots, v_k)$ *is the unique topological order of* H_k*.*

Let G *be a directed graph. We say* G *contains a directed clique, if* $H_k \subseteq G$ *for some* $k \in \mathbb{N}$*.*

An example can be found in Figure 11. Now consider the following special case of directed cliques:

Definition 3.34 (monotone-neighborhood-clique)**.** *Let* G *be a directed graph,* $H \subseteq G$ *a directed clique and* $(v_1, \dots, v_k)$ *the topological vertex order of* H*. We call* H monotone-neighborhood *in* G *or a* monotone-neighborhood-clique (MNH-clique)*, if*

$$\forall i \in \{1, \dots, k-1\} : N^{\text{out}}(v_i) \supseteq N^{\text{out}}(v_{i+1}).$$

In other words: If we go along a path in the directed clique, the outgoing neighborhoods of the vertices we encounter can not grow, in the sense that whenever we encounter a new vertex, its outgoing neighborhood is included in the outgoing neighborhood of the previous vertex.

An interesting insight is now that we can cover a dominance graph with c monotone-neighborhood-cliques.

Lemma 3.35 (clique-cover of dominance graphs)**.** *Let* X *be a deck on* c *categories,* $G := \mathrm{dom}(X)$ *its dominance graph. Define for* $i \in [c]$: $X_i := \{x \in X \mid c(x) = i\}$*. Then each* X_i *is a monotone-neighborhood-clique in* G *and* $\{X_1, \dots, X_c\}$ *is a partition of* X*.*

Proof. It is obvious that $\{X_1, \dots, X_c\}$ is a partition of X. It is also easy to see that each X_i is a directed clique in G: Let O_i be the sequence that is obtained when ordering the elements of X_i ascending by their value in category i. Because we have $c(y) = i$ for all $y \in X_i$, whenever two cards from X_i compete with each other, the card with the higher value in category i wins. Therefore, all the edges $\{(x, y) \mid x, y \in X_i, r_i(x) < r_i(y)\}$ are present in G and O_i is the topological order of the directed clique X_i in G.

For the monotone-neighborhood-property, observe that if $x, y \in X_i$ for some $i \in [c]$ such that $(x, y) \in E(G)$ and $z \in X$ is another vertex such that $(y, z) \in E(G)$, then we have that $c(x) = c(y)$ and can therefore apply Lemma 3.29 (weak transitivity) to x, y, z and get that (x, z) is also an edge of G. Therefore, $N^{\mathrm{out}}(x) \supseteq N^{\mathrm{out}}(y)$. □

Corollary 3.36. *Let X be a deck on c categories, $G := \mathrm{dom}(X)$ its dominance graph. For the clique-cover number k, we have $k(G) \leq c$.*

Corollary 3.37. *Let X be a deck on n cards and c categories, $G := \mathrm{dom}(X)$ its dominance graph. Then G contains a directed clique of size at least $\lceil n/c \rceil$.*

3.5 Attack Advantage

This subsection is dedicated to the study of the attack advantage. The attack advantage is a measure on how much it is worth to be the player choosing the category in a game of Top Trumps. We motivate the introduction of attack advantage by considering the average strength of a deck. In Theorem 3.23, we showed that the average strength of a card in a deck is always $\frac{1}{2}$. What about the average α-strength?

Definition 3.38 (average α-strength)**.** *Let X be a deck on n cards, let $\alpha \in [0,1]$. We introduce the following notation for the average α-strength of a card in X:*

$$\overline{s}_\alpha(X) := \frac{1}{n}\sum_{x\in X} s_\alpha(x)$$

We also write $\overline{s}_\alpha$, if the deck X can be deduced from the context.

Lemma 3.39. *Let X be a deck on n cards, $\alpha \in [0,1]$. Then the average α-strength is a linear combination of average attack strength and average defense strength, i.e.*

$$\overline{s}_\alpha = \alpha\overline{s}_1 + (1-\alpha)\overline{s}_0$$

Proof. By Theorem 3.20, for all $x \in X$, we have $s_\alpha(x) = \alpha s_1(x) + (1-\alpha)s_0(x)$. The claim follows from summing over all $x \in X$ and dividing by n. □

Corollary 3.40. *In every deck X,*

$$\frac{1}{2} = \overline{s}_{1/2} = \frac{1}{2}\left(\overline{s}_1 + \overline{s}_0\right).$$

Proof. Apply the previous lemma with $\alpha = 1/2$ and Theorem 3.23, which tells us that the average strength is $1/2$. □

In other words, both $\overline{s}_0$ and $\overline{s}_1$ have the same absolute difference to $1/2$. As we always have $\overline{s}_0 \le \overline{s}_1$ in any deck (this follows from Observation 3.14), Corollary 3.40 tells us that we can reconstruct both $\overline{s}_0$ and $\overline{s}_1$ from only their difference $\overline{s}_1 - \overline{s}_0$. Therefore, this difference of $\overline{s}_1$ and $\overline{s}_0$ can be considered an intrinsic parameter of the deck X. We therefore consider:

Definition 3.41 (attack advantage β)**.** *Let X be a deck. We define the* attack advantage $\beta(X)$ *of X as*

$$\beta(X) := \overline{s}_1(X) - \overline{s}_0(X).$$

Why is this parameter called the attack advantage? Consider the following modification of MINI-TRUMP: Alice and Bob get dealt random cards and play one single round as before. However, the choosing player is no longer determined randomly, but by an auction before the start of the game. After the game is over, the winner of the game receives a reward of one dollar.

If Alice wins the auction and is choosing player, her probability to subsequently win the game is $\overline{s}_1$ (this follows from the definition of strength). The expected payout in this case is therefore $\overline{s}_1$ dollar. Similarly, if Alice loses the auction, her expected payout is only $\overline{s}_0$ dollar. Therefore, the expected gain of Alice if she is choosing player, in contrast to the situation where she is not choosing player, is $\overline{s}_1 - \overline{s}_0$ dollar. Therefore, Alice should bid $\overline{s}_1 - \overline{s}_0 = \beta(X)$ dollar in the auction to be the choosing player.

In Theorem 3.21, we expressed the attack and defense strength of individual cards in terms of their in- and outdegree in the dominance graph. Applying these equations to the attack advantage yields the key insight, that the attack advantage is directly correlated to the density of the dominance graph.

Theorem 3.42 ($\beta = 1-$ density)**.** *Let X be a deck on n cards, $G := dom(X)$ its dominance graph and $|E|$ the number of edges in G. Then*

$$\beta(X) = 1 - \frac{|E|}{\binom{n}{2}}.$$

Proof. By Theorem 3.21, we have $s_1(x) = 1 - d^{\text{out}}(x)/(n-1)$ and $s_0(x) = d^{\text{in}}(x)/(n-1)$ for all $x \in X$. Therefore

$$\overline{s}_0 = \frac{1}{n}\sum_{x \in X} s_0(x) = \frac{1}{n}\sum_{x \in X} \frac{d^{\text{in}}(x)}{n-1} = \frac{|E|}{n(n-1)} = \frac{1}{2}\frac{|E|}{\binom{n}{2}}$$

and

$$\overline{s}_1 = \frac{1}{n}\sum_{x \in X} s_1(x) = \frac{1}{n}\sum_{x \in X}\left(1 - \frac{d^{\text{out}}(x)}{n-1}\right) = \frac{n}{n} - \frac{|E|}{n(n-1)} = 1 - \frac{1}{2}\frac{|E|}{\binom{n}{2}}$$

and thus

$$\beta(X) = \overline{s}_1 - \overline{s}_0 = 1 - \frac{|E|}{\binom{n}{2}}.$$

$\square$

As an example to this theorem, consider Figure 12, where G_1 is a complete directed clique and G_2 is an independent set. In a deck with G_1 as dominance

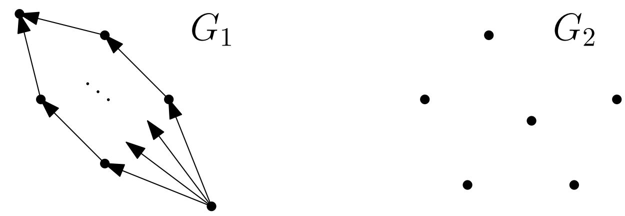

Figure 12: Example of the attack advantage $\beta(X)$.

graph, the right to be the choosing player is completely irrelevant (assuming optimal play), as the cards are linearly ordered and the player with the higher card always wins. Therefore $\beta(X) = 0$. If, on the other hand, we have a deck with G_2 as dominance graph, the choosing player always wins. Therefore, Alice should bid 1 dollar in the setting mentioned above, and $\beta(X) = 1$.

This example shows that the attack advantage can take the value 0 as well as the value 1. It is obvious by its definition that $0 \leq \beta(X) \leq 1$. However, as we have seen in Corollary 3.32, to get an independent set on n vertices, we need n categories. In a typical top trumps deck sold in store, there are few categories. So a natural question is, which values $\beta(X)$ can take on decks with a fixed number of categories. As we have just seen, this question is basically a question about the possible values which the density in dominance graphs can take. It has a nice answer using Turán's theorem: The dominance graph of a deck on c colors has density at least $1/c - o(1)$.

Lemma 3.43 (Minimum number of edges)**.** *Let X be a deck on n cards and c categories. Let $G := \mathrm{dom}(X)$ be its dominance graph and $|E|$ be the number of edges in G. Then*

$$|E| \geq \binom{n}{2} - \mathrm{ext}(n, K_{c+1}).$$

Furthermore, this bound is tight in the sense that for all $c, n \in N$, there exists a ranked deck Y on c colors and n cards such that $\mathrm{dom}(Y)$ has exactly $\binom{n}{2} - \mathrm{ext}(n, K_{c+1})$ edges.

Proof. Let X, G, n, c be like in the claim. By Corollary 3.32, $\alpha(G) \leq c$, so $\overline{G}$ does not contain a K_{c+1}. Therefore, $\overline{G}$ has at most $\mathrm{ext}(n, K_{c+1})$ edges. This proves the first part of the claim.

To see the second part, let $n, c \in \mathbb{N}$. We can assume $c \leq n$. (If $c > n$, we have $\mathrm{ext}(n, K_{c+1}) = \binom{n}{2}$. Then it is easy to see that we can create an independent set as dominance graph of a ranked deck using n categories. The rest of the categories can just be copies of existing categories.)

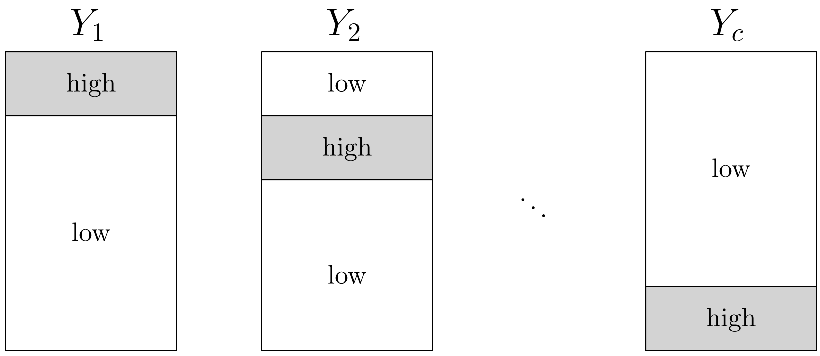

Figure 13: Sketch of the idea behind the proof of Lemma 3.43.

We construct a ranked deck Y such that the dominance graph $H := \mathrm{dom}(Y)$ of Y has the Turán graph $T(n, c)$ as complement. The Turán graph $T(n, c)$ has $\mathrm{ext}(n, K_{c+1})$ edges and is a complete multipartite graph on c parts where the parts are of almost equal size (meaning their sizes differ by at most one). So its complement $\overline{T}$ is a disjoint union of c cliques of almost equal size. We will construct a ranked deck Y such that $\mathrm{dom}(Y)$ is an orientation of $\overline{T}$.

How can we achieve this? The idea is best explained under the simplified assumption that c divides n: Consider a partition $Y = Y_1 \dot\cup \ldots \dot\cup Y_c$ into c sets of size $s := n/c$ each. In a first step, for each $i \in [c]$, assign the s highest values in category i to the cards in Y_i. In other words, $\{r_i(x) : x \in Y_i\} = \{n, n-1, \ldots, n-s+1\}$. In a second step, for each $i \in [c]$, assign the other $n - s$ values in category i to the vertices of $\bigcup_{j \neq i} Y_j$. (Note that such an assignment is always possible and Y is a ranked deck afterwards.) After these two steps, we have for $x \in Y_i$, that $r_i(x) \geq n - s + 1$ and $r_j(x) \leq n - s$ for $j \neq i$. Therefore for all $x \in Y_i$, $c(x) = i$ and for all $i \in [c]$, the $|Y_i|$ highest values of category i are inside Y_i. These two postconditions imply that in the graph H each Y_i is a directed clique and that there is no edge between Y_i and Y_j, if $i \neq j$.

If in the more general case, c does not divide n, consider a partition $Y = Y_1 \dot\cup \ldots \dot\cup Y_c$, such that $|Y_i| \in \{s, s+1\}$ for all $i \in [c]$, where $s := \lfloor n/c \rfloor$. Call those Y_i with $|Y_i| = s$ "small". Similar to before, in a first step, for each $i \in [c]$ assign the $|Y_i|$ highest values $\{n, \ldots, n - |Y_i| + 1\}$ in category i to the cards of Y_i. In a second step, for each i, assign the remaining $n - |Y_i|$ lower values in category i to the cards of $\bigcup_{j \neq i} Y_j$. But if Y_i is small, additionally make sure that the value $n - s$ in category i is assigned to a card $y \in Y_j$ such that $r_j(y) = n$. This is always possible.

By doing this, we achieved that for each $i \in [c]$ and $x \in Y_i$, we have

$r_i(x) \geq n - s$, $r_j(x) \leq n - s$ for $j \neq i$ and we also have the implication

$$r_j(x) = n - s \text{ for some } j \neq i \Rightarrow r_i(x) = n$$

As $s \geq 1$, we therefore have $c(x) = i$. As before, the $|Y_i|$ highest values of category i are assigned to Y_i for all i and therefore the claim follows. □

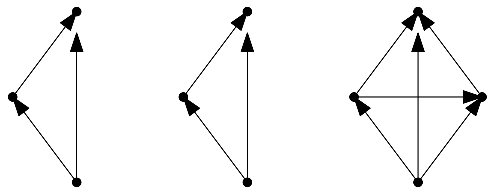

Figure 14: Example of Lemma 3.43 and Corollary 3.44: The highest attack advantage with a deck on 3 categories is achieved when the dominance graph has minimal density over all DAGs with independence number at most 3.

Corollary 3.44 (bounds on the attack advantage)**.** *Let X be a deck on c categories and n cards. Then*

$$0 \leq \beta(X) \leq \left(1 - \frac{1}{c}\right) \frac{n}{n-1}$$

Proof. By Theorem 3.42, we have $\beta(X) = 1 - d$, where d is the density of the graph $\mathrm{dom}(X)$. The density is clearly at most 1. (This bound is also tight, consider a ranked deck, where the values across all categories are equal for each card.) For the upper bound, by the previous theorem, we have

$$d \geq 1 - \frac{ext(n, K_{c+1})}{\binom{n}{2}}$$

and this bound is also tight by the same theorem. By Turán's theorem, $\mathrm{ext}(n, K_{c+1}) \leq (1 - 1/c)n^2/2$, and so

$$1 \geq d \geq 1 - \left(1 - \frac{1}{c}\right) \frac{n^2}{2} \frac{1}{\binom{n}{2}} = 1 - \left(1 - \frac{1}{c}\right) \frac{n}{n-1},$$

which together with $\beta(X) = 1 - d$ implies the result. □

4 Ranked Realizability

As seen in the previous section, the underlying structure of a Top Trumps deck X is represented by its dominance graph. Here, by the term "underlying structure", we mean the following: If both players use the HIGHEST-VALUE-strategy in the game MINI-TRUMP, the dominance graph tells us, which card beats which other card, how strong each card is, and how much one should pay to be the choosing player (compare Sections 3.2, 3.3 and 3.5).

If the deck X is additionally a ranked deck, the HIGHEST-VALUE-strategy is an optimal strategy for MINI-TRUMP. Therefore it is very natural to ask the following questions: Which graphs can be represented as dominance graphs of ranked Top Trumps decks? And given such a graph, what is the minimum possible numbers of categories needed to represent it? For example, consider the graph G from Figure 15. There exists a ranked deck on 3 categories representing G. We call the graph *3-ranked-realizable* in this case. What is the minimum $c \in \mathbb{N}$ such that G is c-ranked-realizable?

In this section, we will give our findings regarding these questions. We could find several insights, but no definite answer to said questions. However, in the next section, we will consider realizability where non-ranked decks are allowed; in this case we find a definite answer. This is partly due to the fact that plain realizability is in some sense more "natural" than ranked realizability. We will explain why that is the case later on.

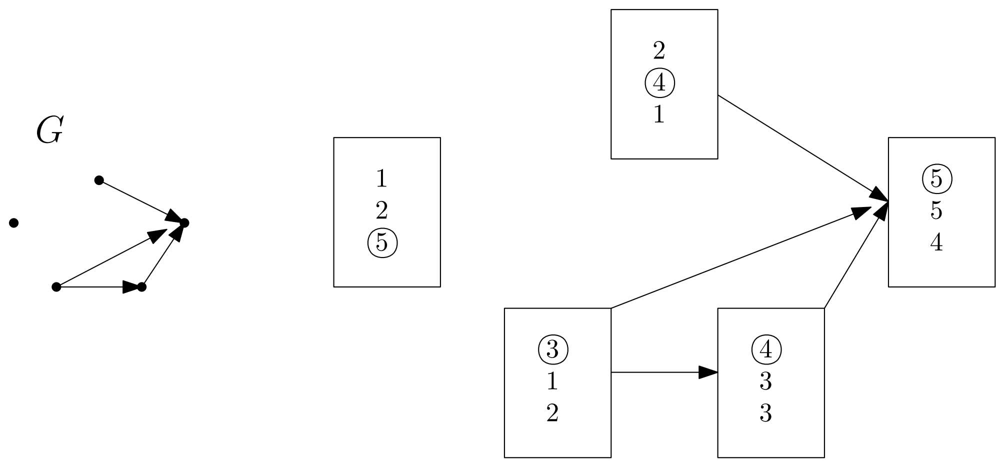

Figure 15: The directed graph G is 3-ranked-realizable, so $\Psi(G) \leq 3$.

4.1 Definition

Consider the following definitions (and Figure 15 as an example):

Definition 4.1 (realization)**.** *Let G be a directed graph and X be a deck. We say that X is a* realization of G*, or that X* realizes G*, if* $\mathrm{dom}(X) = G$*.*

Definition 4.2 (c-ranked-realizable)**.** *Let G be a directed graph, let $c \in \mathbb{N}$. We say that G is* c-ranked-realizable*, if there exists a ranked deck X on c categories such that X realizes G. We say that G is* ranked-realizable*, if there exists some $c' \in \mathbb{N}$ such that G is c'-ranked-realizable.*

Definition 4.3 (ranked-realizability number Ψ)**.** *Let G be a directed graph. The* ranked-realizibility number $\Psi(G)$ *is the minimum $c \in \mathbb{N}$ such that G is c-ranked-realizable. If such a c does not exist, then $\Psi(G) = \infty$.*

We are now interested in the question, which directed graphs G are ranked-realizable and if so, we want to determine $\Psi(G)$. The most basic insight is the fact, that all these graphs are acyclic (Theorem 3.19). Before we dive further into the topic, note that in particular, it can not happen that some directed graph G is c-ranked-realizable, but not c'-ranked-realizable for some $c' > c$:

Observation 4.4. *If a directed graph G is c-ranked-realizable for some $c \in \mathbb{N}$, it is also $(c+1)$-ranked-realizable.*

Proof. Just copy an existing category. Formally: Let X be a ranked deck on c categories realizing G, define a new deck X' via $r' : X \to \mathbb{N}^{c+1}$ such that for all $x \in X : r'_i(x) := r_i(x)$ if $i \in [c]$ and $r'_{c+1}(x) := r_c(x)$. Then X' is a ranked deck on $c+1$ categories and clearly $\mathrm{dom}(X') = \mathrm{dom}(X)$. $\square$

4.2 Properties

So what can we say about the properties of c-ranked-realizable graphs? Some properties we already know, due to results from the previous sections, especially from Section 3.4, where we analyzed the properties of dominance graphs. If G is a ranked-realizable graph on n vertices and $c := \Psi(G)$, we have in particular, that

- G is acyclic (by Theorem 3.19)
- an induced directed path in G has at most c edges (by Corollary 3.30)
- $\alpha(G) \leq k(G) \leq c$, and G can be covered with c monotone-neighborhood-cliques (by Lemma 3.35)
- G has at least

  $$\binom{n}{2} - \mathrm{ext}(n, K_{c+1}) \geq \frac{n(n-1)}{2} - \left(1 - \frac{1}{c}\right)\frac{n^2}{2} = \frac{1}{c}\frac{n^2}{2} - \frac{n}{2}$$

  edges (by Lemma 3.43). So for fixed c and $n \to \infty$, G is dense.

Also, in the previous section we have already seen a class of graphs, which are indeed ranked-realizable.

Lemma 4.5. *For all $c \in \mathbb{N}$ and for all $n \in \mathbb{N}$, let $H(n,c)$, be the directed graph on n vertices, which we get by taking the disjoint graph union of c directed cliques of almost equal size (equivalently, an acyclic orientation of the complement of the Turán graph $T(n,c)$). Then $\Psi(H(n,c)) = c$.*

Proof. We saw in Lemma 3.43 that $H(n,c)$ is ranked-realizable with c categories. So $\Psi(H(n,c)) \leq c$. On the other hand, $\Psi(H(n,c)) \geq c$, because $H(n,c)$ contains an independent set of size c. □

As special cases of this observation, we get:

- If C is a directed clique, $\Psi(C) = 1$.
- If E is an independent set on n vertices, $\Psi(E) = n$
- If M is a matching on m directed edges and $2m$ vertices, $\Psi(M) = m$.

So directed cliques, independent sets, and matchings are ranked-realizable. On the other hand, there exists a very easy family of graphs, which do not have this property: For each $n \geq 3$, the directed path on n vertices is not ranked-realizable. To see this, consider the following lemma:

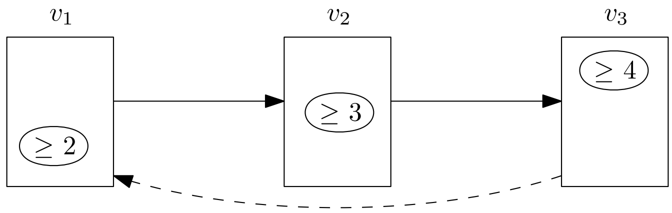

Figure 16: Sketch of the proof that $\vec{P}_3$ is not ranked-realizable.

Theorem 4.6 (length of induced directed paths)**.** *Let $r \geq 3$, let $\vec{P}_r$ be the directed path on r vertices and let G be a ranked-realizable directed graph on n vertices. If G contains an induced directed path $\vec{P}_r \subseteq_{\text{ind}} G$ then $r \leq n/2 + 1$.*

Proof. (A sketch of the proof in the case $r = 3$ is depicted in Figure 16.) Let $v_1, \ldots, v_r$ be the vertices of the induced copy of $\vec{P}_r$ in G along the path, i.e. $(v_i, v_{i+1}) \in E(G)$ for all $i \in [r-1]$. As G is ranked-realizable there exists a ranked deck X such that $\text{dom}(X) = G$. As the copy of $\vec{P}_r$ is induced, for all $i \in \{3, \ldots, r\}$, there is no edge between v_1 and v_i. This means that v_1 wins against v_i, if v_1 chooses the category. Then, in particular, $h(v_1) \geq r - 1$, because X is a ranked deck and v_1 wins against at least $r - 2$ cards, if v_1 chooses. By Lemma 3.28, the h-value of the cards strictly increases alongside any path. As v_r is connected to v_1 by a path on $r - 1$ edges, we have $h(v_r) \geq h(v_1) + r - 1 \geq 2r - 2$. As X is a ranked deck, $n \geq h(v_r)$. In total we have $n \geq 2r - 2$, which implies the claim. $\square$

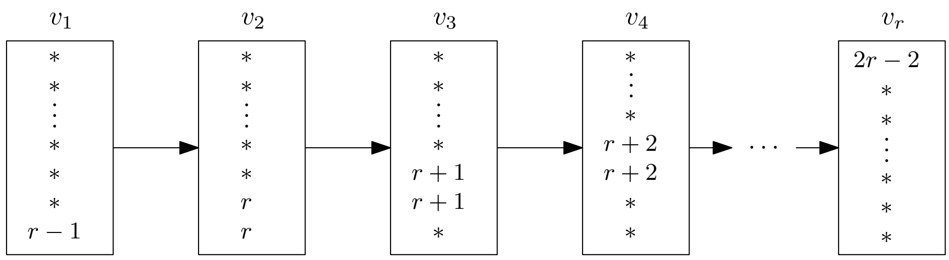

Figure 17: Sketch of the proof that Theorem 4.6 is sharp. Elements marked $*$ are $\leq r - 2$.

We note that this bound on r with respect to n sharp, a sketch of the idea for the proof is depicted in Figure 17. However, as we will not need this fact for the rest of the thesis, we chose not to present a formal proof. Instead, consider a direct corollary:

Corollary 4.7 (directed paths are not ranked-realizable)**.** *Let $n \geq 3$. The directed path $\vec{P}_n$ on n vertices is not ranked-realizable.*

Proof. By the previous theorem, all induced directed paths in a ranked-realizable graph G on n vertices have at most $n/2 + 1$ vertices. In particular, for $n \geq 3$, the directed path $\vec{P}_n$ itself is not ranked-realizable. □

As a corollary to this, we get a fact which seems counter-intuitive at first glance:

Corollary 4.8 (Ranked realizability is not preserved under vertex removal)**.** *There exists a directed graph G and $v \in V(G)$ such that G is ranked-realizable, but $G - v$ is not.*

Proof. Consider Figure 7 from the previous section. The graph H depicted in this figure is ranked-realizable and has an induced $\vec{P}_3$ as subgraph. We just saw that $\vec{P}_3$ is not ranked-realizable. So if we repeatedly remove a vertex from H to get $\vec{P}_3$, we will at some step get a graph G like in the claim. □

Differently stated, if G is ranked-realizable and $H \subseteq_{\text{ind}} G$ is an induced subgraph of G, then H is not necessarily ranked-realizable. This might be contrary to one's first intuition in the following way: Suppose we have a graph G, a ranked deck X realizing G and a vertex $v \in V(G)$, then one might expect that if we take the card x_v corresponding to v out of the deck X, we get a ranked deck realizing $G - v$. But that is not necessarily the case, because subdecks of ranked decks are not necessarily ranked anymore. Or, stated at a higher level, the optimal strategy (and thus the dominance graph) may change when we take out a card. We also can not easily "fix" the deck $X \setminus \{x_v\}$ to be ranked again by adjusting the values of the cards: Say, for example, $r_1(x_v) = 4$ and $r_2(x_v) = 7$. Then there is a set of 3 cards, which are worse than x_v in category 1 and a set of 6 cards, which are worse than x_v in category 2. But these two sets may be completely different. So if we adjust the values of the categories in some simple way such that $X \setminus \{x_v\}$ becomes ranked again, for any card $y \in X \setminus \{x_v\}$, the values in some categories may change but in other categories may stay the same. So, ultimately, $c(y)$ may change and so we have no control over the dominance graph of this new altered deck.

Up so far, we have not been able to determine $\Psi(G)$ for a large class of graphs. However, it turns out that if we know some graph G has a representation with a natural property which we call *unique-champion-property*, determining $\Psi(G)$ becomes trivial:

Definition 4.9 (unique-champion ranked deck (UCRD))**.** *If X is a ranked deck on c categories, such that for all distinct $i, j \in [c]$, the champion in* *category i and the champion in category j are not the same card, X is called a* unique-champion ranked deck (UCRD).

(Compare Definition 3.25 for the definition of a champion.) Imagine for example that Alice creates a game of Top Trumps using her favorite superheroes as protagonists, and decides on the four categories "strength", "speed", "cleverness" and "charisma". Then Alice's deck (after it has been transformed to a ranked deck according to Theorem 3.6) is a UCRD, if there does not exist a superhero, which meets two or more of the descriptions "strongest", "fastest", "smartest" or "most charismatic". Alice thinks, this is good, because otherwise the game could be frustrating to the players.

As stated before, if we have a graph G such that G can be realized with a UCRD, then determining $\Psi(G)$ becomes very easy:

Theorem 4.10. *Let G be a directed graph. If there exists a UCRD X on c categories with* $\mathrm{dom}(X) = G$, *then* $\Psi(G) = \alpha(G) = k(G) = c$.

Proof. The ranked deck X is a realization of G, so $\Psi(G) \leq c$. On the other hand, for $i \in [c]$ let $x_i \in X$ be the champion in category i. As X is a UCRD, the set $A := \{x_i \mid i \in [c]\}$ of all champions has size c. But note that A is an independent set in G, because if a card is champion, it always wins, if it chooses the category. Therefore $\alpha(G) \geq c$. Now, we have seen at the beginning of this subsection, that $\alpha(G) \leq k(G) \leq \Psi(G)$ always holds. In total

$$c \leq \alpha(G) \leq k(G) \leq \Psi(G) \leq c,$$

which proves the claim. □

As a consequence, if a UCRD deck representing some directed graph G exists, it can only have exactly $\Psi(G)$ categories. This leaves us in an interesting spot. If we know that some UCRD exists representing G, we immediately know $\Psi(G)$. But we can not exclude the possibility, that for all decks representing G, there is a card, which is champion in two or more categories.

Also note that if we fix $c \in \mathbb{N}$, let $n \in \mathbb{N}$ tend to infinity, and consider a random ranked deck X on c categories and n cards (choose c random permutations of $[n]$), X is a UCRD with probability tending to 1. From this point of view, very many ranked decks are indeed UCRDs and very few are not. This would suggest that, for fixed c, almost all c-ranked-realizable graphs G fulfill $\alpha(G) = \kappa(G) = \Psi(G)$, although this is of course, no proof at all (we have no information about how many different UCRDs realize the same graph). Could it even be true, that $\alpha(G) = k(G)$ for all c-ranked-realizable graphs G? Or even $\alpha(G) = k(G) = \Psi(G)$?

Finally, note that the previous theorem gives us $\Psi(G)$, if the directed graph G has a UCRD, but it does not help us determining, which graphs at all are ranked-realizable. This question is still open. We could only find explicit constructions for some limited classes of graphs, obtained by repeatedly applying one of three graph operations, which will be described in the next subsection.

4.3 Operations Preserving Ranked Realizability

For three graph operations, we were able to show that they preserved ranked realizability. Namely, these graph operations are the *unidirectional join*, the *disjoint graph union* and the *lexicographical graph product*. We will describe each of these operations and prove, in what way they preserve ranked realizability.

Before we begin, we make some technical definitions. These technical definitions are helpful, because all the proofs in this subsection (and several of the proofs in the next section) work using the same strategy: Namely, we consider a set M, which is often a deck or a union of decks, and we assign new values $r_i'(x)$ to the elements $x \in M$, which may depend of the old values. Then we want to argue that the collection of the new values indeed forms a deck or even a ranked deck. Finally, we want to argue that the dominance graph of this new deck X' is similar to some other graph. To avoid imprecise notation for these proofs, we make the following technical definitions:

Definition 4.11 (defining a deck via a function – part (ii))**.** *Let M be a finite set, $c \in \mathbb{N}$ and $r' : M \to \mathbb{R}^c$ be a function such that $X' := r'(M)$ is a deck on $|M|$ cards. Then we define for $i \in [c]$ and $x \in M$:*

$$c'(x) := c(r'(x)) \text{ and } h'(x) := h(r'(x)).$$

Here, the functions $c(\cdot)$ and $h(\cdot)$ are the usual ones, compare Definition 3.10. Furthermore, define $\mathrm{dom}(M; r')$ *to be the directed graph with vertex set M and edge set $\{(x, y) \mid (r'(x), r'(y)) \in E(\mathrm{dom}(X'))\}$.*

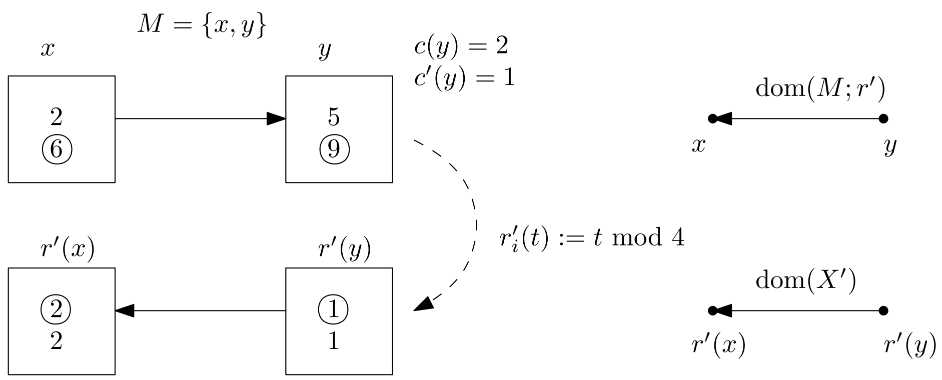

Figure 18: Example of Definition 4.11.

By this definition, $\mathrm{dom}(M; r')$ is the graph isomorphic to $\mathrm{dom}(X')$, but on the vertex set M instead of the vertex set X' (this is possible, as r' is a bijection). An example for all these concepts can be found in Figure 18. These definitions are a bit technical, but if they confuse you, do not worry: In the next proof, it will all come together.

Definition 4.12 (unidirectional join)**.** *Let G_1, G_2 be directed graphs. The* unidirectional join *of G_1 and G_2, denoted by $G_1 \vec{\nabla} G_2$, is obtained by taking a copy of G_1, a copy of G_2 and then adding all edges $\{(x, y) \mid x \in V(G_1), y \in V(G_2)\}$ between these copies.*

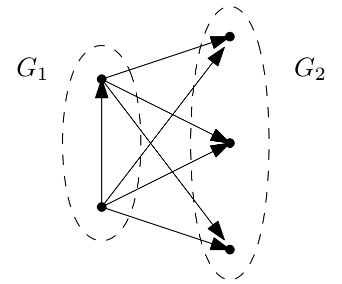

Figure 19: Unidirectional join of a directed path and an independent set.

In the world of undirected graphs, the join of undirected graphs G_1 and G_2 is obtained by taking a copy of G_1, a copy of G_2 and then adding all possible edges, i.e. a complete bipartite graph between them. The unidirectional join is similar, only that the edges all have the same direction (from G_1 to G_2). An example can be seen in Figure 19. Now we will prove, as claimed in the introduction of this subsection:

Theorem 4.13 (Unidirectional joins preserve ranked-realizability)**.** *Let $c \in \mathbb{N}$ and let both G_1, G_2 be c-ranked-realizable, directed graphs. Then $G_1 \vec{\nabla} G_2$ is c-ranked-realizable.*

Proof. Let $n_1 := |V(G_1)|$ and $n_2 := |V(G_2)|$ be the number of vertices in G_1 and G_2, respectively. There exists a ranked deck X_1 on c categories and n_1 cards realizing G_1. Likewise, there exists a ranked deck X_2 on c categories and n_2 cards realizing G_2. Let $V_1 := V(G_1)$ and $V_2 := V(G_2)$ and $M := V_1 \dot{\cup} V_2$. We will create a new ranked deck X' from X_1 and X_2 the following way: Consider the deck-defining function $r' : M \to \mathbb{R}^c$ (compare Definition 4.11) which is given by

$$r'_i(x) = \begin{cases} r_i(x) & \text{for } x \in V_1 \\ n_1 + r_i(x) & \text{for } x \in V_2 \end{cases}$$

for all $i \in [c]$. Also, define for $x \in M$ the value $c'(x)$ like in Definition 4.11. Let $X' := r'(M)$ be the image of r'. We now claim that

(i) X' is ranked deck on $n_1 + n_2$ cards and c categories,

(ii) For all $x \in M$, we have $c(x) = c'(x)$, and

(iii) $\mathrm{dom}(X') \cong \mathrm{dom}(M; r') = G_1 \vec{\nabla} G_2$.

If we can show all these claims, we are done. For claim (i), note that for each fixed $i \in [c]$, we have $r'_i(V_1) = \{1, \ldots, n_1\}$ and $r'_i(V_2) = \{n_1+1, \ldots, n_1+n_2\}$, so in total $X' = r'(V_1 \dot\cup V_2)$ is a ranked deck on n_1+n_2 cards. For claim (ii), note that for any $x \in V_1 \dot\cup V_2$, all of its values in X' compared to X either stayed the same or were all shifted by the same amount n_1. So indeed $c(x) = c'(x)$. Finally, for claim (iii), let $H := \mathrm{dom}(M; r')$. Inside of V_1, all values stayed the same, so together with claim (ii), we get $H[V_1] = G_1$. Inside of V_2, all values were shifted by n_1, so similarly $H[V_2] = G_2$. Finally, if $x \in V_1$ and $y \in V_2$, note that $r'_i(x) \leq n_1$ and $r'_i(y) \geq n_1 + 1$ for all $i \in [c]$. So we always have the edge (x, y) in H. In total, $H = G_1 \vec{\nabla} G_2$. By their definition, $\mathrm{dom}(X')$ and H are isomorphic. □

We proceed with showing that the disjoint graph union preserves ranked realizability. Consider

Definition 4.14 (disjoint graph union)**.** *Let G_1, G_2 be directed graphs. The* disjoint graph union *of G_1 and G_2, also sometimes called the* union *of G_1 and G_2 is denoted by $G_1 \dot\cup G_2$ and is the directed graph obtained by taking a copy of G_1 and adding a disjoint copy of G_2.*

For example, $K_2 \dot\cup K_2$ is a matching on two edges. We now show a construction, which proves that the disjoint graph union $G_1 \dot\cup G_2$ is ranked-realizable, if both G_1 and G_2 are ranked-realizable. To supplement the proof, an exemplary application of the theorem is depicted in Figure 20.

Theorem 4.15 (disjoint graph unions preserve ranked-realizability)**.** *Let G_1 be a c_1-ranked-realizable, directed graph and G_2 be a c_2-ranked-realizable, directed graph. Then $G_1 \dot\cup G_2$ is a $(c_1 + c_2)$-ranked-realizable graph.*

Proof. There exists a ranked deck X_1 on c_1 categories and $n_1 := |V(G_1)|$ cards realizing G_1 and a ranked deck X_2 on c_2 categories and $n_2 := |V(G_2)|$ cards realizing G_2. Let $V_1 := V(G_1)$, $V_2 := V(G_2)$ and $M := V_1 \dot\cup V_2$. We can assume $n_1, n_2, c_1, c_2 \geq 1$. Let $C_1 := \{1, \ldots, c_1\}$ and $C_2 := \{c_1 + 1, \ldots, c_1 + c_2\}$. It will be convenient to assume, that the set of categories, which X_2 uses, is C_2 instead of $\{1, \ldots, c_2\}$. By this, we mean that for $x \in X_2$, the value $r_i(x)$ is defined exactly if $i \in C_2$ and the value $c(x)$ is contained in C_2. This assumption is possible without loss of generality.

Now consider the deck-defining function $r' : M \to \mathbb{R}^{c_1+c_2}$ defined by

$$x \in V_1 \Rightarrow r'_i(x) := \begin{cases} n_2 + r_i(x) & \text{for } i \in C_1 \\ r_1(x) & \text{for } i \in C_2 \end{cases}$$

$$x \in V_2 \Rightarrow r'_i(x) := \begin{cases} r_{c_1+1}(x) & \text{for } i \in C_1 \\ n_1 + r_i(x) & \text{for } i \in C_2 \end{cases}$$

for all $i \in [c_1+c_2]$. Let $X' := r'(M)$. Let $c'(x) := c(r'(x))$ and $h'(x) := h(r'(x))$ for $x \in M$. Similar to the proof of the last theorem, Theorem 4.13, we claim that

(i) X' is ranked deck on $n_1 + n_2$ cards and $c_1 + c_2$ categories,

(ii) For all $x \in M$, we have $c(x) = c'(x)$, and

(iii) $\mathrm{dom}(X') \cong \mathrm{dom}(M; r') = G_1 \dot\cup G_2$.

If we prove all these claims, we are done. For the proof of (i), note that if $i \in C_1$, then $r'_i(V_2) = \{1, \dots, n_2\}$ and $r'_i(V_1) = \{n_2 + 1, \dots, n_2 + n_1\}$. If, on the other hand, $i \in C_2$, then $r'_i(V_1) = \{1, \dots, n_1\}$ and $r'_i(V_2) = \{n_1 + 1, \dots, n_1 + n_2\}$. In total, in every category, every of the values $\{1, \dots, n_1 + n_2\}$ is assigned exactly once, which shows the claim.

For the proof of (ii), note that if $x \in V_1$, then for all $i \in C_2$ we have $r'_i(x) < r'_1(x)$. Therefore $c'(x) = c(x)$ in this case. Similarly, if $x \in V_2$, then for all $i \in C_1$, we have $r'_i(x) < r_{c_1+1}(x)$. Therefore $c'(x) = c(x)$ in this case and this proves the claim.

For the last claim, let $H := \mathrm{dom}(M; r')$. Due to (ii) and the fact that for all $x \in V_1$ and $i \in C_1$, the value $r'_i(x) = n_2 + r_i(x)$ is just a constant shift of the previous value $r_i(x)$, we have that $H[V_1] = G_1$. Similarly, we have $H[V_2] = G_2$. Finally, let $x \in V_1$ and $y \in V_2$. Then, by claim (ii), $c'(x) \in C_1$, which implies $h'(x) \geq n_2 + 1$ and $r'_{c'(x)}(y) \leq n_2$. This shows that x wins, if x chooses. Analogously, we can show that y wins, if y chooses. So there is no edge between x and y in H. In total, $H = G_1 \dot\cup G_2$. □

Attentive readers may have noticed that the presented theorem is a generalization of Lemma 4.5. Now we see that indeed every graph, which is a disjoint union of k directed cliques, is k-ranked-realizable ($k \in \mathbb{N}$).

Definition 4.16 (lexicographical graph product). *Let G, H be directed graphs. The* lexicographical graph product *of G and H, denoted by $G \cdot H$, is obtained*

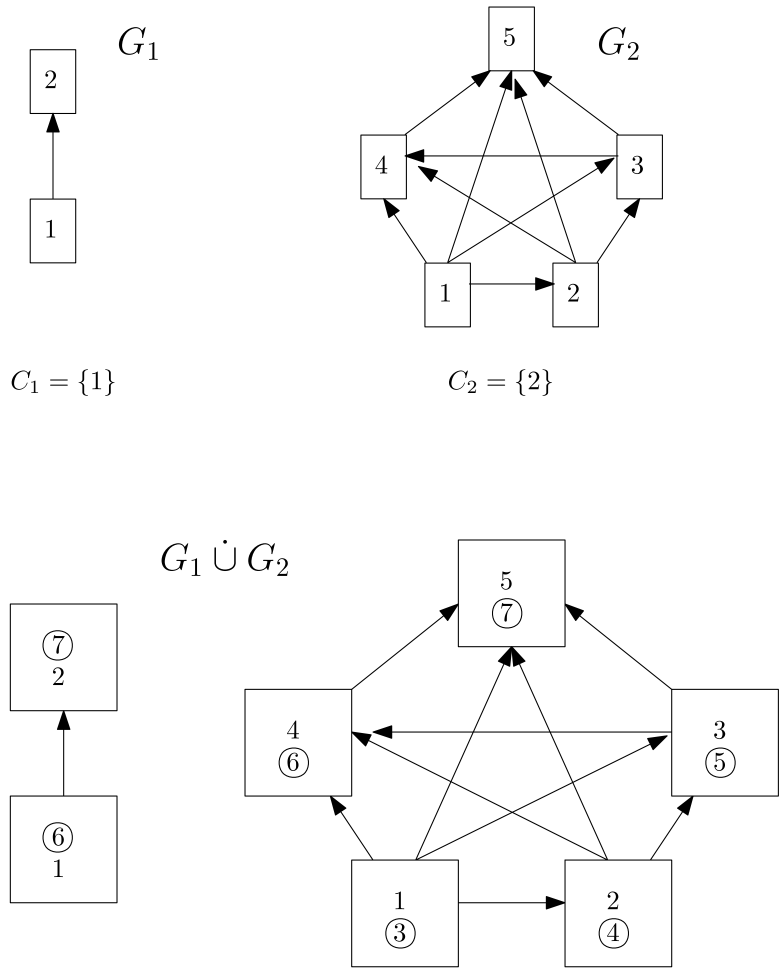

Figure 20: An application of Theorem 4.15 to two directed cliques of size 2 and 5, respectively.

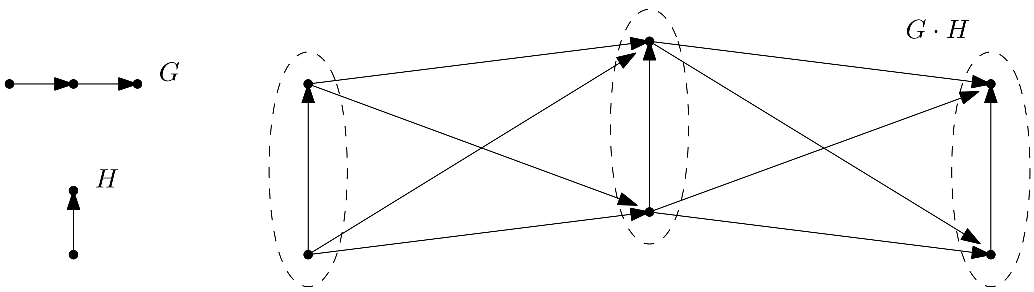

Figure 21: The lexicographical graph product $\vec{P}_3 \cdot \vec{P}_2$.

by taking a copy H_v of H for every $v \in V(G)$ and then, for all $(u,v) \in E(G)$, adding all possible directed edges going from H_u to H_v. Formally:

$$V(G \cdot H) := V(G) \times V(H), \text{ and}$$
$$((u_1,v_1),(u_2,v_2)) \in E(G \cdot H) \Leftrightarrow (u_1,u_2) \in E(G) \vee (u_1 = u_2 \wedge (v_1,v_2) \in E(H)).$$

An example can be found in Figure 21. The lexicographical graph product is related to the lexicographic order of tuples. A lexicographic order resembles the way we would sort words in a dictionary. For readers unfamiliar with the lexicographic order, we will define it here. We will only be concerned with 2-tuples:

Definition 4.17 (lexicographic order of 2-tuples)**.** *Let $x,y \in \mathbb{R}^2$, where $x = (x_1,x_2)$ and $y = (y_1,y_2)$. We say that x* comes before y in the lexicographic order*, denoted by $x <_L y$, if*

$$x_1 < y_1 \vee (x_1 = y_1 \wedge x_2 < y_2).$$

For example, $(1,7) <_L (2,0)$ and $(2,2) <_L (2,3)$. We will now use this order in the proof of the final theorem of this subsection.

Theorem 4.18 (lex. graph products preserve ranked-realizability)**.** *Let G_1 be a c_1-ranked-realizable, directed graph and G_2 be a c_2-ranked-realizable, directed graph. Then $G_1 \cdot G_2$ is a (c_1c_2)-ranked-realizable graph.*

Proof. There exists a ranked deck X_1 on c_1 categories and $n_1 := |V(G_1)|$ cards realizing G_1 and a ranked deck X_2 on c_2 categories and $n_2 := |V(G_2)|$ cards realizing G_2. Let $V_1 := V(G_1)$, $V_2 := V(G_2)$, $C_1 := [c_1]$, $C_2 := [c_2]$ and $M := V_1 \times V_2$. We wish to construct a ranked deck on c_1c_2 categories realizing $G_1 \cdot G_2$. Roughly speaking, the idea behind the proof is, that this new deck has the "category space" $C' := C_1 \times C_2$ and the values of a card in a category are elements from $\mathbb{R}^2$. Whenever two values of this new deck are compared, they are done so according to lexicographic order. We can formalize this idea the following way:

Consider for $(i,j) \in C_1 \times C_2$ the function $t_{(i,j)} : M \to \mathbb{R}^2$ given by

$$t_{(i,j)}((u,v)) := (r_i(u), r_j(v)).$$

For each fixed $(i,j) \in C'$, the image $T_{(i,j)} := t_{(i,j)}(M)$ is a set of 2-tuples. Then consider the deck-defining function $r' : M \to \mathbb{R}^{c_1 \times c_2}$ given by

$$r'_{(i,j)}((u,v)) := \text{ index of } t_{(i,j)}((u,v)) \text{ in the lex. order of } T_{(i,j)}$$

for all $(i,j) \in C'$. Let, similar to previous proofs, $X' := r'(M)$, $c'(x) := c(r'(x))$ and $h'(x) := h(r'(x))$ for $x \in M$. We now claim, that

(i) X' is ranked deck on n_1n_2 cards and c_1c_2 categories,

(ii) For all $(u,v) \in M$, we have $c'((u,v)) = (c(u), c(v))$ and

(iii) $\mathrm{dom}(X') \cong \mathrm{dom}(M; r') = G_1 \cdot G_2$.

For claim (i), note that for each fixed $(i,j) \in C'$, we have $T_{(i,j)} = t_{(i,j)}(M) = [n_1] \times [n_2]$, because $M = V_1 \times V_2$ and X_1, X_2 are ranked decks. Therefore, $r'_{(i,j)}(M) = \{1, \ldots, n_1n_2\}$. Also, $|C'| = c_1c_2$, therefore X' is indeed a ranked deck on c_1c_2 categories and n_1n_2 cards.

For claim (ii), fix $(u,v) \in M$, let $z := (u,v)$. Now, the category $c'(z) =: (i^*, j^*) \in C'$ is the uniquely determined category, such that $t_{(i*,j*)}(x)$ is the lexicographically largest of the tuples $R := \{t_{(i,j)}(z) \mid (i,j) \in C'\}$, or the first one, if multiple largest tuples exist. What does “first” mean? That depends on the way, we arrange the categories $C_1 \times C_2$ on the cards of the new deck. If we let this arrangement be the lexicographical order of $C_1 \times C_2$, then the category (i^*, j^*) is the lexicographically smallest of those categories (i,j) with the property that $t_{(i,j)}(z)$ is (lexicographically) maximal. Now, because for comparing the tuples of R, the first component is more important, and the first component is uniquely determined by i, we have that $i^* = c(u)$. If we already know that i^* is fixed, we similarly see, that $j^* = c(v)$. Thus follows the claim.

Finally, for claim (iii), let $H := \mathrm{dom}(M, r')$, so $V(H) = M = V_1 \times V_2$. (For simplification, we will write $f(x_1, y_1)$ instead of $f((x_1, y_1))$ for $f \in \{r', c', h'\}$.) For $(a,b), (c,d) \in V_1 \times V_2$, we have

$$\begin{aligned}
&((a,b),(c,d)) \in E(H) \\
\Leftrightarrow\ & h'(a,b) < r'_{c'(a,b)}(c,d) \\
\Leftrightarrow\ & t_{c'(a,b)}(a,b) <_L t_{c'(a,b)}(c,d).
\end{aligned}$$

Due to claim (ii), this is equivalent to

$$\begin{aligned}
\Leftrightarrow\ & (h(a), h(b)) <_L (r_{c(a)}(c), r_{c(b)}(d)) \\
\Leftrightarrow\ & h(a) < r_{c(a)}(c) \vee (h(a) = r_{c(a)}(c) \wedge h(b) < r_{c(b)}(d)) \\
\Leftrightarrow\ & (a,b) \in E(G_1) \vee (h(a) = r_{c(a)}(c) \wedge (b,d) \in E(G_2)) \\
\Leftrightarrow\ & (a,b) \in E(G_1) \vee (a = c \wedge (b,d) \in E(G_2)) \\
\Leftrightarrow\ & ((a,b),(c,d)) \in E(G_1 \cdot G_2).
\end{aligned}$$

In total, this proves $H = G_1 \cdot G_2$, so we are done (by its definition, $H \cong \mathrm{dom}(X')$). $\square$

This was the last of our findings about operations preserving ranked realizibility. We summarize Section 4.3:

Corollary 4.19. *Let G_1, G_2 be directed graphs. Then we have*

- $\Psi(G_1 \vec{\nabla} G_2) \leq \max\{\Psi(G_1), \Psi(G_2)\}$
- $\Psi(G_1 \dot{\cup} G_2) \leq \Psi(G_1) + \Psi(G_2)$
- $\Psi(G_1 \cdot G_2) \leq \Psi(G_1)\Psi(G_2)$,

where $\Psi(G_i) = \infty$ is allowed for $i \in \{1, 2\}$.

Proof. By Theorems 4.13, 4.15 and 4.18 . $\square$

5 Realizability

We saw in Corollary 4.8, that ranked realizability is not preserved under vertex removal. In other words, when we have a ranked-realizable, directed graph G and a vertex v of G, then $G - v$ is not necessarily ranked-realizable anymore. This fact seems to be against intuition: At first glance, one would expect that if we have a deck realizing G, we could simply take the card corresponding to v out of the deck X and get a deck realizing $G - v$. But that is not the case, because subdecks of ranked decks are not necessarily ranked anymore.

This raises the following question: What if, instead of asking whether some graph can be realized using a ranked deck, we just ask, whether the graph can be realized using a deck. This means we get rid of the requirement that the values inside of a category are exactly $\{1, \ldots, n\}$ for a deck of size n. Now, they can be any arbitrary n distinct real numbers (the no-stalemate-property requires the numbers to be distinct). For example, Figure 22 shows that we can realize the directed path $\vec{P}_4$ with a deck on 3 categories, but we know that we can not realize this graph with a ranked deck, by Corollary 4.7. Let us call a directed graph G *realizable*, if there exists a deck realizing G.

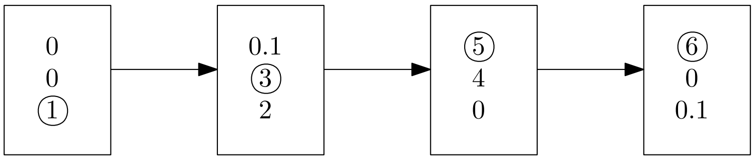

Figure 22: The directed path $\vec{P}_4$ is realizable.

We can interpret this new situation in three different ways:

- We ask for the structure of a game of Top Trumps, if both players use the strategy Highest-Value to pick their category, even if this strategy is not optimal. As stated before in Section 3.1, the strategy Highest-Value resembles the behavior of an "easily impressible player", who always picks the category with the highest value off of his or her card.

- We ask the question, which graphs can appear as induced subgraphs of ranked-realizable graphs. (For details, see below.)

- On a more abstract level, we ask for the structure of a set of points $X \subseteq \mathbb{R}^c$ equipped with the dominance relation $x <_D y \Leftrightarrow h(x) < r_{c(x)}(y)$.

5.1 Definition

Analogous to the ranked-realizability (Definitions 4.1 to 4.3), consider

Definition 5.1 (c-realizable). *Let G be a directed graph, let $c \in \mathbb{N}$. We say that G is* c-realizable, *if there exists a deck X on c categories such that X realizes G. We say that G is* realizable, *if there exists some $c' \in \mathbb{N}$ such that G is c'-realizable.*

Definition 5.2 (realizability number φ). *Let G be a directed graph. The* realizibility number $\varphi(G)$ *is the minimum $c \in \mathbb{N}$ such that G is c-realizable. If such a c does not exist, then $\varphi(G) = \infty$.*

As an example, consider again Figure 22, which shows that $\vec{P}_4$ is 3-realizable, therefore $\varphi(\vec{P}_4)$ is at most 3.

We want to make some regards concerning regularity of decks. Namely, when considering the class of all dominance graphs realized by the family of all possible decks, it does not matter, whether the values on the cards of a deck are real-valued or natural numbers: The class of their dominance graphs stays the same. To see this, consider

Lemma 5.3. *Let $X \subseteq \mathbb{R}^c$ be a deck. Then there exists a deck $X' \subseteq \mathbb{N}^c$, which is equivalent to X in the following sense: The deck X' is defined by a function $r' : X \to \mathbb{N}^c$ such that for all $x \in X : c'(x) = c(x)$ (where $c'(x) := c(r'(x))$) and furthermore for all $x, y \in X$, for all $i \in [c]$:*

$$r_i(x) < r_i(y) \Leftrightarrow r'_i(x) < r'_i(y).$$

Proof. Let $T := \{r_i(x) \mid x \in X, i \in [c]\}$ be the set of all values used in X. Sort the values in T ascending to obtain an order O_T of T. For $i \in [c]$ and $x \in X$ let the new value $r'_i(x)$ of x in category i be the index of $r_i(x)$ in the order O_T. Let $X' := r'(X)$. Then we have that for any two values $t_1 = r'_i(x)$ and $t_2 := r'_j(y)$ for arbitrary $x, y \in X$ and $i, j \in [c]$, that $t_1 < t_2$ if and only $r_i(x) < r_j(y)$. Likewise, $t_1 = t_2$, if and only $r_i(x) = r_j(y)$. This implies both of the claims. □

Note that this in particular implies $\mathrm{dom}(X') = \mathrm{dom}(X)$. This means that, roughly speaking, we can always assume that a deck uses only natural numbers as values. Furthermore, we can substitute the no-stalemate-property for a weaker condition:

Definition 5.4 (weak deck). *Let $X \subseteq \mathbb{R}^c$. The set X is called a* weak deck, *if for all $x, y \in X$, we have $r_{c(x)}(x) \neq r_{c(x)}(y)$. (The functions c, r defined as usual.)*

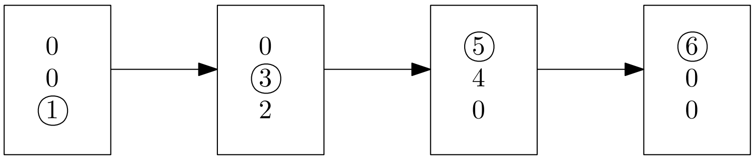

Figure 23: A weak deck realizing $\vec{P}_4$.

(An example can be found in Figure 23.) This definition is motivated by the following observation: If both players of a Top Trumps game stick to using the strategy HIGHEST-VALUE, then for a card x, the only category ever chosen will be $c(x)$. Therefore it suffices for X to be a weak deck, in order to uniquely determine, which player wins each round and which card wins against which other card. This also determines the dominance graph uniquely. Therefore, we can define

Definition 5.5 (dominance graph of weak decks)**.** *If X is a weak deck, we define the dominance graph of X, denoted by* $\mathrm{dom}(X)$*, analogously to the dominance graph of a deck (ompare Definition 3.17). Likewise, we define* $\mathrm{dom}(M; r')$ *analogously to Definition 4.11, if $r'(M)$ is only a weak deck.*

If we have a weak deck X, we can always find a (proper) deck X' with the same dominance graph as X:

Lemma 5.6. *Let X be a weak deck. Then there exists a deck X', which is equivalent to X in the following sense: The deck X' is defined by a function $r' : X \to \mathbb{R}^c$ such that for all $x \in X : c'(x) = c(x)$ and furthermore for all $x, y \in X$ the property $(\star)$, defined as*

$$(\star) : h'(x) < r_{c'(x)}(y) \Leftrightarrow h(x) < r_{c(x)}(y),$$

holds true.

Proof. Let $\delta := \min\{|h(x) - r_{c(x)}(y)| : x, y \in X\}$. Because X is a weak deck, we have $\delta > 0$. We obtain the new values r' the following way: For every $x \in X$, decrease the values in the categories $[c] \setminus \{c(x)\}$ by some amount in the range $(0, \delta/3)$. For every card x we decreased all values besides $r_{c(x)}(x)$, so clearly $c'(x) = c(x)$. If we do this decrease-step in the right way, we can also make X' have the no-stalemate-property, because we have the freedom to alter any value besides the values on cards x in category $c(x)$. But if there are two distinct cards x, y with $c(x) = c(y) =: i$, then already $r_i(x) \neq r_j(y)$. Finally, note that due to the definition of δ, we also get property $(\star)$. $\square$

Together, these two lemmas show the following:

Corollary 5.7. *Let $c \in \mathbb{N}$. The following classes of graphs are identical:*

$$\begin{aligned}
A_1 &:= \{\mathrm{dom}(X) \mid X \subseteq \mathbb{R}^c \textit{ is a deck } \} \\
A_2 &:= \{\mathrm{dom}(X) \mid X \subseteq \mathbb{N}^c \textit{ is a deck } \} \\
A_3 &:= \{\mathrm{dom}(X) \mid X \subseteq \mathbb{R}^c \textit{ is a weak deck } \} \\
A_4 &:= \{H \mid H \subseteq_{\mathrm{ind}} G, G \textit{ is c-ranked-realizable}\}
\end{aligned}$$

Proof. Obviously $A_2 \subseteq A_1$. By Lemma 5.3, $A_1 \subseteq A_2$. Obviously $A_1 \subseteq A_3$. By Lemma 5.6, $A_3 \subseteq A_1$. Obviously $A_4 \subseteq A_2$. If on the other hand, we have a graph $G \in A_2$ and a deck $X \subseteq \mathbb{N}^c$ realizing G, let M be the largest value appearing in X. We can fill in $M - |X|$ cards to get a ranked deck $Y \supseteq X$. Then $G \subseteq_{\mathrm{ind}} \mathrm{dom}(Y)$ and so $G \in A_4$. $\square$

5.2 Monotone-Neighborhood-Cliques

After having defined the realizability number φ of directed graphs, we are interested in the question, which directed graphs are realizable, and in the minimum number $\varphi(G)$ of categories to realize some given directed graph G. The aim of this subsection is to show, that exactly all acyclic directed graphs are realizable, and that the realizability number $\varphi(G)$ is exactly the minimal size of a cover of G with *monotone-neighborhood-cliques*. Recall from Section 3.4 the definition of a monotone-neighborhood-clique:

Definition 3.34 (monotone-neighborhood-clique)**.** *Let G be a directed graph, $H \subseteq G$ a directed clique and $(v_1, \ldots, v_k)$ the topological vertex order of H. We call H* monotone-neighborhood *in G or a* monotone-neighborhood-clique (MNH-clique)*, if*

$$\forall i \in \{1, \ldots, k-1\} : N^{\text{out}}(v_i) \supseteq N^{\text{out}}(v_{i+1}).$$

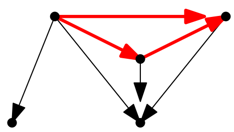

Figure 24: A monotone-neighborhood-clique of size 3.

So if we go along any directed path in such a clique, the sequence of outgoing neighborhoods of the vertices we encounter, is monotone and non-increasing with respect to inclusion. Thus the name. An example can be found in Figure 24.

Before we acquire this result however, we will prove some weaker theorems, just to make the reader familiar with the concept of realizability. Strictly speaking, these weaker theorems are not necessary. However, we felt that these easier theorems provide a nice "warm-up" for the final proof.

So let us begin. In the last subsection we defined the class of realizable, directed graphs as the class of directed graphs, which are c-realizable for some $c \in \mathbb{N}$. Clearly, the following holds:

Observation 5.8. *All realizable, directed graphs are acyclic.*

Proof. By Theorem 3.19, all dominance graphs are acyclic (indepedent from the fact, whether the corresponding deck is ranked or not). □

What operations can we perform on DAGs to preserve acyclity? For example, we can add a source or a sink. In the simplest case, this source or sink is connected to every other vertex.

Theorem 5.9 (adding a complete source/sink)**.** *Let $c \in \mathbb{N}$, let G be a c-realizable, directed graph.*

(i) If we add a source v to G, such that v is connected to every other vertex, the resulting graph H is c-realizable.

(ii) If we add a sink v to G, such that v is connected to every other vertex, the resulting directed graph H is c-realizable.

Proof. Let X be a deck on c categories realizing G. Let L be the lowest value appearing in X and M be the highest value. Now observe that

(i) we can add the card $(L-1,\ldots,L-1)$ to X.

(ii) we can the card $(M+1,\ldots,M+1)$ to X.

It is easy to see that the resulting deck realizes H. □

So we can always add a complete source/sink without increasing the realizability number. What happens when adding a source, which is not necessarily complete? We can do it by using one additional category in this case.

Theorem 5.10 (adding a source)**.** *Let $c \in \mathbb{N}$, let G be a c-realizable, directed graph. If we add a source v to G, the resulting directed graph H is $(c+1)$-realizable.*

Proof. (A sketch of the proof idea can be seen in Figure 25.) Let X be a deck on c categories realizing G. Without loss of generalization, $X \subseteq \mathbb{N}^c$ by Lemma 5.3. Let $M := V(G) \cup \{v\}$. We obtain a new (weak) deck X' on $c+1$ categories by assigning new values $r' : M \to \mathbb{R}^{c+1}$ the following way: If $i \in [c]$ is an old category, define for $x \in V(G) : r'_i(x) := r_i(x)$ and let $r'_i(v) := 0$. On the other hand, let in the new category $c+1$ be $r_{c+1}(v) := 1/3$ and for $x \in V(G)$

$$r_{c+1}(x) := \begin{cases} 2/3 & \text{if } x \in N_H^{\text{out}}(v) \\ 0 & \text{otherwise.} \end{cases}$$

Then for $x \in V(G)$, we have that $c'(x) = c(x) \in [c]$, because the old values are all at least 1, but the new value in category $c+1$ is less than one. Also, we have $c'(v) = c+1$, because $r'_i(v) = 0$ for $i \in [c]$ and $r'_{c+1}(v) = 1/3$.

Now note, that X' is a weak deck. Finally, let $T := \mathrm{dom}(M; r')$. Note that $T[V(G)] = G$, because all values and chosen categories for cards corresponding to $V(G)$ stayed the same. Also, for $x \in V(G)$, there is no directed edge (x,v) in T, because v has the value 0 in categories $1,\ldots,c$. Finally, note that due to the choice of $r_{c+1}(x)$ for $x \in V(G)$, we have that in T there are exactly the edges from v to $N_H^{\text{out}}(v)$. In total, $T = H$. By Corollary 5.7, we can also find a deck with the same dominance graph as the weak deck X', and so we are done. □

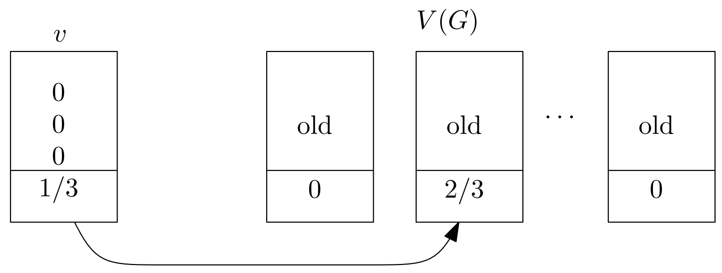

Figure 25: Idea of Theorem 5.10.

As a direct consequence of this theorem, we get that all directed acyclic graphs are realizable:

Corollary 5.11. *Let G be a DAG on n vertices. Then G is n-realizable.*

Proof. By induction on n. For the induction base, we have that K_1 is 1-realizable. For the induction step, if we have some DAG G on n vertices, we can always find a source v in G (for example the first vertex in a topological order of G). Then by induction, $G-v$ is $(n-1)$-realizable and by Theorem 5.10, G is n-realizable. □

Interestingly, when adding a sink instead of a source, we can not always manage to do so with only one additional category. At the end of this subsection, we will show such an example, where we indeed need to double the number of categories.

We are now ready to prove the promised result. Consider

Definition 5.12 (monotone-neighborhood-clique cover)**.** *Let G be a directed graph. Let $C_1, \dots, C_k \subseteq G$ be monotone-neighborhood-cliques in G. The set $\{C_1, \dots, C_k\}$ is called a* monotone-neighborhood-clique cover of size k*, if $C_i \cap C_j = \emptyset$ for all $i \neq j$ and $\bigcup_{i=1}^{k} V(C_i) = V(G)$.*

We call a monotone-neighborhood-clique cover of G *minimal*, if k is minimal over all monotone-neighborhood-clique covers of G. Note that we proved in Lemma 3.35, that if we have a deck on c categories realizing G, then we always get a monotone-neighborhood-clique cover of size c of G by considering all the cards with the same chosen category. Therefore, we have

Observation 5.13. *Let G be a DAG. Then $\varphi(G)$ is at least the size of a minimal monotone-neighborhood-clique cover.*

In this sense, the property of having a MNH-clique cover of size at most c is necessary for $\varphi(G) \leq c$. We will now show that this is also sufficient.

Theorem 5.14. *Let G be a DAG and let $\{A_1, \dots, A_k\}$ be a MNH-clique cover of G. Then G is k-realizable.*

Proof. Let $V := V(G)$ and $n := |V|$. The proof will use the same strategy like the proof of Theorem 5.10. First, we give out values $r : V \to \mathbb{R}^k$, such that $X' := r(V)$ is a weak deck. Then we will prove that $\mathrm{dom}(V; r) = G$. By Corollary 5.7, this suffices.

Let $C_i := V(A_i)$ for $i \in [k]$. Let $(v_1, \dots, v_n)$ be a topological order of G, i.e. all edges are of the form (v_i, v_j) for some $i < j$. In a first step, for all $v \in V$ do the following: Let t be the position of v in the topological order, i.e. $v = v_t$ and let $i \in [k]$ be the index of the clique, which contains v, i.e. $v_t \in C_i$. Then, define

$$r_i(v_t) := t.$$

In a second step, for $i \in [k]$ and for $v \in C_i$, let for $j \neq i$

$$r_j(v) := \begin{cases} 1/2 + \max\{r_j(w) \mid w \in N^{\mathrm{in}}(v) \cap C_j\}; & \text{if } N^{\mathrm{in}}(v) \cap C_j \neq \emptyset \\ 0 & \text{else.} \end{cases} \tag{5.1}$$

(A sketch of the assignment of these values is depicted in Figure 26.) Note that the second step is well-defined, because all recursive calls to r are for values, which were already defined in the first step. For $v \in V$, define $c(v) := c(r(v))$ and $h(v) := h(r(v))$. Now we make the following claims:

(i) For all $i \in [k]$, for all $v \in C_i$, we have $c(v) = i$.

(ii) The set X' is a weak deck.

(iii) We have $\mathrm{dom}(X') \cong \mathrm{dom}(V; r) = G$.

For the proof of (i) note, that if $v \in C_i$, then $r_i(v)$ equals the index of v in the topological order. But the maximum in Equation (5.1) runs only over vertices, who come sooner in the topological order than v. Therefore, $c(v) = i$.

For the proof of (ii), we have to show that for all distinct $x, y \in V$, we have $h(x) \neq r_{c(x)}(y)$. Let $i \in [k]$ be such that $x \in C_i$. Note that due to claim (i), the value $h(x)$ is a natural number. Due to Equation (5.1), if y were not contained in C_i, then $r_i(y)$ would not be a natural number. But if $y \in C_i$, then clearly $r_i(x) \neq r_i(y)$.

For the proof of (iii), let $H := \mathrm{dom}(V; r)$. For $i \in [k]$, we quickly see that $H[C_i] = G[C_i]$. This is due to the fact that for $v \in C_i$, we have $c(v) = i$ and $h(v)$ is the index of v in the topological order of G. So in $H[C_i]$ we get exactly all the edges of the directed clique A_i with the same direction as in A_i as well.

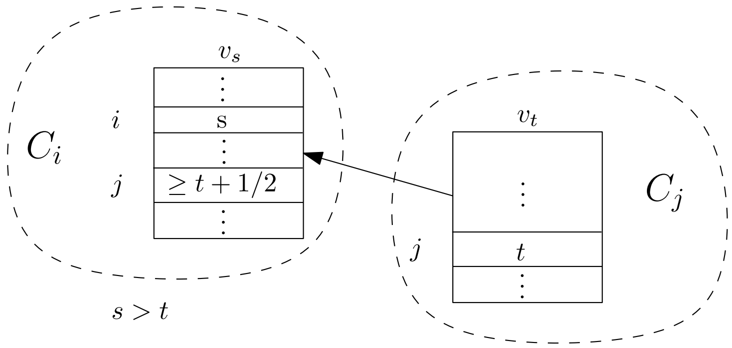

Figure 26: Central idea behind the proof of Theorem 5.14.

So let $v_l \in C_i$ and $v_r \in C_j$ for some $i \neq j$, such that l and r are the positions of v_l and v_r in the topological order. Without loss of generality, $l < r$. We have $c(v_l) = i$, $c(v_r) = j$, $h(v_l) = l$ and $h(v_r) = r$. Due to the properties of a topological order, we either have $(v_l, v_r) \in E(G)$ or no edge between v_l and v_r in G. In the first case, we also have the edge (v_l, v_r) in H, due to Equation (5.1) (applied for $v = v_r$).

In the second case, if there is no edge between v_l and v_r in G, we have to show that v_r wins, if v_r chooses and that v_l wins, if v_l chooses. If v_r chooses, we have $h(v_r) = r > l = h(v_l) \geq r_{c(v_r)}(v_l)$. Therefore, v_r wins.

So the only thing left to show is that v_l wins, if v_l chooses, when there is no edge between v_l and v_r in G. For the sake of contradiction, assume v_l loses. This would mean

$$\begin{aligned} & h(v_l) < r_{c(v_l)}(v_r) = r_i(v_r) \\ \Leftrightarrow \quad & l = h(v_l) < 1/2 + \max\{r_i(w) \mid w \in N_G^{\text{in}}(v_r) \cap C_i\} \end{aligned}$$

But this is only possible if there is a vertex $y \in C_i \cap N_G^{\text{in}}(v_r)$ such that the index of y in the topological order is at least l. But then, due to the monotone-neighborhood-property of C_i, we get that the edge (v_l, v_r) is in G as well. This is a contradiction. So we indeed have that v_l wins if v_l chooses. This completes the proof of claim (iii), which completes the proof. $\square$

Corollary 5.15. *Let G be a DAG. Then*

$$\varphi(G) = \min\{k \mid \textit{there exists a MNH-clique cover of } G \textit{ of size } k.\}$$

Proof. By Observation 5.13 and Theorem 5.14. $\square$

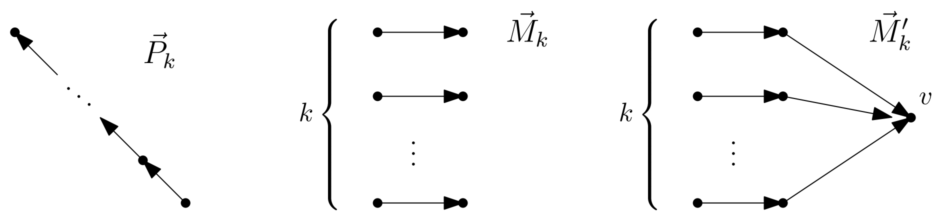

Figure 27: Determining $\varphi(G)$ using the MNH-clique characterization.

If we have a directed graph G with a cycle, we clearly have $\varphi(G) = \infty$. (Although directed graphs with cycles can not be realized as a deck, they can be covered with MNH-cliques. Take for example the trivial cover, where every clique has size 1.) Therefore, we have completely characterized realizability of directed graphs.

It is satisfying to see, how easy it becomes to determine the realizability number of some directed graphs using this new characterization:

Example 5.16. *For $k \in \mathbb{N}$, let $\vec{P}_k$ be the directed path on k vertices, $\vec{M}_k$ be a matching on k directed edges and $\vec{M}'_k$ be the directed graph obtained from $\vec{M}_k$ by adding a sink v connected to all endpoints of the edges in $\vec{M}_k$ (compare Figure 27). Then we have*

(i) $\varphi(\vec{P}_k) = k - 1$

(ii) $\varphi(\vec{M}_k) = k$

(iii) $\varphi(\vec{M}'_k) = 2k$

Proof. We use Corollary 5.15 together with the following facts: (i) The last two vertices induce a MNH-clique, the rest of the edges do not. (ii) Clearly, $\vec{M}_k$ can be covered with k MNH-cliques, but not less. (iii) All MNH-cliques of size two (or more) intersect at v. □

This is the promised example that adding a sink is not as well-behaved as adding a source when considering realizability.

5.3 Properties of Realizability

Now that we have obtained an alternative characterization of realizability, we will use this short subsection to describe some properties and graph operations concerning realizability. An important property of realizabilty is that it is preserved under vertex removal.

Observation 5.17 (realizability is monotone under vertex removal)**.** *Let $c \in \mathbb{N}$, let G be a c-realizable, directed graph and let $v \in V(G)$. Then $G - v$ is c-realizable.*

Proof. There exists a deck X on c colors realizing G. By taking the card corresponding to v out of X, we get a deck realizing $G - v$. □

In other words, if some directed graph H is an induced subgraph $H \subseteq_{\text{ind}} G$ of some directed graph G, then $\varphi(H) \leq \varphi(G)$. As, a consequence, we can reprove some of the results from Section 3.4: A c-realizable graph does not contain an induced directed path on $c+2$ or more vertices, because $\varphi(\vec{P}_{c+2}) = c+1$. By the same argument, a c-realizable graph does not contain an independent set of size $c + 1$. In general, we have

$$\alpha(G) \leq k(G) \leq \varphi(G) \leq \Psi(G)$$

for all directed graphs G. Here, $k(G) \leq \varphi(G)$ follows from Corollary 5.15 and $\varphi(G) \leq \Psi(G)$ follows from the definition of these two parameters. How big can the difference between two consecutive elements of this inequality get? For the parameters of $\alpha(G)$ and $k(G)$, this is an old question, answered for example by Mycielski's construction or Tutte's construction [3]. There exist undirected graphs G such that we can have $\alpha(G) = 2$, but $k(G)$ arbitrarily large. The same is true for the difference between $k(G)$ and $\varphi(G)$:

Theorem 5.18. *For all even $n \in \mathbb{N}$, there exists an undirected graph G on n vertices and $k(G) = 2$ such that for each acyclic orientation $\vec{G}$ of G, we have $\varphi(\vec{G}) \geq n/2$.*

Proof. (Note that a clique of G becomes a directed clique in $\vec{G}$, so we can always cover $\vec{G}$ with two directed cliques.) Let G be the undirected graph consisting out of a clique on vertex set $A := \{a_1, \ldots, a_{n/2}\}$ of size $n/2$, another clique $B := \{b_1, \ldots, b_{n/2}\}$ of size $n/2$ and a matching connecting A and B, i.e. the edges $\{a_i b_i \mid i \in [n/2]\}$. Now, fix an acyclic orientation $\vec{G}$ of G. For each $i \in [n/2]$, the edge $a_i b_i$ receives an orientation (s_i, t_i), where the starting point $s_i \in \{a_i, b_i\}$ and the end point t_i has $\{t_i\} = \{a_i, b_i\} \setminus \{s_i\}$. Consider the set $X := \{s_1, \ldots, s_{n/2}\}$. No two elements of X can be contained in the same

MNH-clique. This is because if $s_i \in A$ and $s_j \in B$ for some $i \neq j$, then there is no edge between s_i and s_j in G, so they can not be in the same MNH-clique. If both $s_i, s_j \in A$, say, then let without loss of generality $(s_i, s_j) \in E(G)$ (otherwise, swap i and j in the argument). But then s_i and s_j can not be in the same MNH-clique, because $(s_i, t_j) \notin E(\vec{G})$. In total, we need at least $|X| = n/2$ MNH-cliques to cover $\vec{G}$. This completes the proof.

(For an upper bound on φ, it is not hard to show, that $\varphi(\vec{G}) \leq n/2 + 2$: Take the cliques on the two vertex sets $C_1 := \{t_1, \ldots, t_{n/2}\} \cap A$ and $C_1 := \{t_1, \ldots, t_{n/2}\} \cap B$ and a clique of size 1 for every element in X.) □

This settles our question about the inequality $k(G) \leq \varphi(G)$. For the inequality $\varphi(G) \leq \Psi(G)$, we have already seen that all directed paths on at least three vertices are not ranked-realizable, so $\Psi(G) = \infty$, but they are of course realizable.

Another property of realizabilty that we want to mention, is how it behaves on the graph operations introduced in Section 4.3.

Theorem 5.19. *Let G_1, G_2 be directed, acyclic graphs. Then*

(i) $\varphi(G_1 \vec{\nabla} G_2) = \max\{\varphi(G_1), \varphi(G_2)\}$

(ii) $\varphi(G_1 \dot{\cup} G_2) = \varphi(G_1) + \varphi(G_2)$

(iii) $\varphi(G_1 \cdot G_2) \leq \varphi(G_1)\varphi(G_2)$

Proof. (i) Let $c_1 := \varphi(G_1)$ and $c_2 := \varphi(G_2)$. By Corollary 5.7, there exists c_1-ranked-realizable G_1' with $G_1 \subseteq_{\text{ind}} G_1'$ and c_2-ranked-realizable G_2' with $G_2 \subseteq G_2'$. By Theorem 4.13, the graph $G_1' \vec{\nabla} G_2'$ is $(\max\{c_1, c_2\})$-ranked-realizable. Because $G_1 \vec{\nabla} G_2 \subseteq_{\text{ind}} G_1' \vec{\nabla} G_2'$, we get that $\varphi(G_1 \vec{\nabla} G_2) \leq \max\{c_1, c_2\}$. The other direction is true because $G_1 \subseteq_{\text{ind}} G_1 \vec{\nabla} G_2$ and $G_2 \subseteq_{\text{ind}} G_1 \vec{\nabla} G_2$.

(ii) The minimal number of MNH-cliques to cover $G_1 \dot{\cup} G_2$ is clearly the sum of the minimal number of MNH-cliques to cover G_1 and the minimal number of MNH-cliques to cover G_2.

(iii) The proof is analogous to the first direction of the proof of (i): We use, that if $G_1 \subseteq_{\text{ind}} G_1'$ and $G_2 \subseteq_{\text{ind}} G_2'$, then $G_1 \cdot G_2 \subseteq_{\text{ind}} G_1' \cdot G_2'$. The rest of the proof is a reduction to Theorem 4.18. □

The last proof in this subsection regards the question of adding sinks. We saw in Theorem 5.10, that adding a source increases the realizability number by at most one, but adding a sink is not so well-behaved: We saw in Example 5.16, that the realizability number may double. This is the worst that can happen:

Theorem 5.20. *Let G be a DAG. Let H be a graph which is obtained from G by adding a sink v, which is not an isolated vertex. Then*

$$\varphi(H) \leq \varphi(G) + \min\{\varphi(G), d^{\text{in}}(v)\}.$$

Proof. Let $c := \varphi(G)$. By Corollary 5.15, there exists a MNH-clique-cover $\{C_1, \ldots, C_c\}$ of size c of G. For $i \in [c]$, let $A_i := V(C_i) \cap N^{\text{in}}(v)$ and $B_i := V(C_i) \setminus N^{\text{in}}(v)$. Let k be the number of indices $i \in [c]$ such that $A_i \neq \emptyset$. Then $k \leq \min\{c, d^{\text{in}}(v)\}$. Now observe that each of the nonempty A_i or B_i induces a MNH-clique. For B_i this is the case, as C_i was a MNH-clique beforehand, then we deleted vertices, but added no new outgoing edges to the vertices of B_i. Deleting vertices preserves the monotone-neighborhood-property. For A_i this is the case, as we started with the MNH-clique C_i, then deleted vertices, and then added a new outgoing edge for each of the vertices in A_i. Because $k \neq 0$, we can integrate the vertex v into one of the MNH-cliques induced by the A_i. In total, we have a MNH-clique-cover of size $c + k$, which proves the claim. □

Example 5.16 tells us, that this theorem is sharp. Note that if v is an isolated vertex, $\varphi(G)$ increases by exactly 1, due to Theorem 5.19, case (ii).

5.4 Monotone Edges and Dilworth's Theorem

The aim of this last subsection is to prove yet another equivalent characterization of realizability, which will imply a polynomial-time algorithm computing both $\varphi(G)$, as well as a Top Trumps deck realization on $\varphi(G)$ categories, given some directed acyclic graph G. This idea can be seen as a generalization of the well-known Dilworth's theorem (which is a statement about posets) to arbitrary directed or even undirected graphs. We presented Dilworth's theorem as part of the preliminaries in Section 2.4.

The new characterization is a characterization in terms of *monotone edges*.

Definition 5.21 (monotone edge)**.** *Let G be a directed graph (which may allow loops). An edge $(u, v) \in E(G)$ is called,* monotone*, if $N^{\text{out}}(u) \supseteq N^{\text{out}}(v)$.*

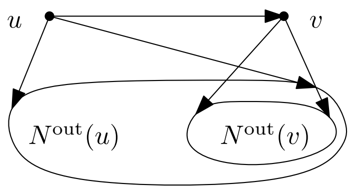

Figure 28: A monotone edge.

An example is depicted in Figure 28. The case where G can have loops will be required later, but can be ignored for the first few results. Every loop is a monotone edge by this definition.

We also define the monotone subgraph of some directed graph G as the subgraph with all the monotone edges:

Definition 5.22 (monotone subgraph)**.** *Let G be a directed graph (which may allow loops). The* monotone subgraph *of G, denoted by* $\text{mon}(G)$ *is the subgraph $H \subseteq G$ on the same vertex set as G and on the edge set*

$$\{(u, v) \in E(G) \mid (u, v) \textit{ is monotone in } G\ \}.$$

An example is given in Figure 29. Again, by this definition, all loops which are present in G, are also present in $\text{mon}(G)$. Finally, consider the following definition of an *anti-monotone chain*, which in our result will be the equivalent to an antichain in a poset:

Definition 5.23 (anti-monotone chain)**.** *Let G be a directed graph (which may allow loops). A vertex set $A \subseteq V(G)$ is called an* anti-monotone chain *in G, if there do not exist distinct $u, v \in A$, such that (u, v) is a monotone edge in G.*

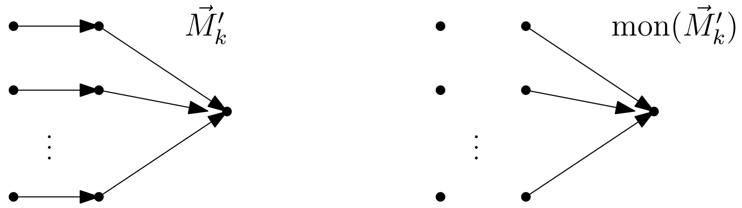

Figure 29: Monotone subgraph of the graph $\vec{M}_k'$ from Example 5.16.

Recall, that if we have a directed graph G with two distinct vertices u, v, such that both (u, v) and (v, u) are present, we called this a *double-edge* between u and v. Our final result can be very elegantly expressed in directed graphs without double-edges and without loops. In the other case, the result is a bit more technical. We begin with the case that there are no double edges and no loops:

Lemma 5.24. *Let G be a directed graph (not allowing loops) without a double edge, let $X \subseteq V(G)$. Then we have, that*

(i) X is an anti-monotone chain in G if and only X is independent in $\mathrm{mon}(G)$*.*

(ii) $G[X]$ is a MNH-clique if and only if $\mathrm{mon}(G)[X]$ *is a directed clique.*

Proof. (i) "⇒" By the definition of an anti-monotone chain. "⇐" By the definition of $\mathrm{mon}(G)$.

(ii) "⇒" If $G[X]$ is a MNH-clique, every edge inside $G[X]$ is monotone and $G[X]$ is a directed clique. "⇐" If $\mathrm{mon}(G)[X]$ is a directed clique, then $G[X]$ has exactly the edges as in $\mathrm{mon}(G)[X]$ and no more (because there exist no double-edges) and each of these is monotone, so we get a MNH-clique. □

We also have the following lemma:

Lemma 5.25. *Let G be a directed graph (which may allow loops). Then* $\mathrm{mon}(G)$ *is transitive.*

Proof. If $(u, v) \in E(\mathrm{mon}(G))$ and $(v, w) \in E(\mathrm{mon}(G))$, we have that the edge (u, w) exists in G, because (u, v) is monotone and $w \in N^{\mathrm{out}}(v)$. We also have $N^{\mathrm{out}}(u) \subseteq N^{\mathrm{out}}(v) \subseteq N^{\mathrm{out}}(w)$, because (u, v) and (v, w) are monotone, so (u, w) is monotone as well and thus $(u, w) \in E(\mathrm{mon}(G))$. □

Corollary 5.26. *Let G be a directed graph (not allowing loops) without double edges. Then* $\mathrm{mon}(G)$ *is the comparability graph of a strict poset.*

Proof. By the last lemma, $\mathrm{mon}(G)$ is transitive. Because G does not allow double-edges, $\mathrm{mon}(G)$ is antisymmetric. Because there are no loops, $\mathrm{mon}(G)$ is irreflexive. □

We will from now on identify $\mathrm{mon}(G)$ with the poset, of whom it is the comparability graph. Now we are ready for the final result (case without double edges):

Theorem 5.27 (Generalized Dilworth's theorem – no double edges)**.** *Let G be a directed graph (not allowing loops) without double-edges. Then the following numbers are equal:*

(i) Maximal size of an anti-monotone chain in G

(ii) The width of the strict poset $\mathrm{mon}(G)$.

(iii) Minimal size of a MNH-clique-cover of G

(iv) If additionally G is acyclic, $\varphi(G)$.

Proof. Case (iii) and (iv) are equivalent by Corollary 5.15. By Lemma 5.24, the minimal size of a MNH-clique-cover equals the minimum number of directed cliques to cover $\mathrm{mon}(G)$. As $\mathrm{mon}(G)$ is a strict poset by Corollary 5.26, this is equal to $\mathrm{width}(\mathrm{mon}(G))$, which, by Dilworth's theorem, equals the maximum size of an antichain in $\mathrm{mon}(G)$. Again, by Lemma 5.24, this equals the maximum size of an anti-monotone chain in G. □

As a corollary, we get

Corollary 5.28. *There exists an algorithm with time-complexity $\mathcal{O}(nm)$, which for all DAGs G on n vertices and m edges computes a deck X on $\varphi(G)$ categories, such that X realizes G.*

Proof. The poset $\mathrm{mon}(G)$ can be created in time $\mathcal{O}(nm)$ by checking for each edge, whether it is monotone. There exist known algorithms computing the width of this poset together with a minimal chain decomposition in time $\mathcal{O}(nm)$ (compare Theorem 2.2). By the previous theorem, this chain decomposition is a MNH-clique-cover of G of size $\varphi(G)$. Then, we can create a weak deck realizing G by giving out the ranks like in the proof of Theorem 5.14, and transform this weak deck into a deck. These two steps take time $\mathcal{O}(n+m)$, including the required topological sorting. □

Consider also the special case, when G itself was already a comparability graph of a strict poset. In this case, we have

Observation 5.29. *In a poset, all edges are monotone.*

Proof. This follows from transitivity: If (u, v) is an edge, and $w \in N^{\text{out}}(v)$, then (u, w) is an edge. □

This implies that for comparability graphs G of (strict or non-strict) posets, we have $\text{mon}(G) = G$. Then, Theorem 5.27 simply reduces to Dilworth's theorem, but with the additional characterization that the width of this poset is exactly $\varphi(G)$.

Corollary 5.30. *Let G be the comparability graph of a strict poset. The width of the poset is equal to $\varphi(G)$, i.e. the minimal c such that there exists a Top Trumps deck X on c categories realizing G.*

Proof. By Observation 5.29 and Theorem 5.27. □

The *dimension* of a poset is a well-studied parameter of posets. Our new characterization of the width of a poset has striking similarities to the characterization of the *dimension* of a poset:

Definition 5.31 (Pareto-dominance and dimension)**.** *For two points $x, y \in \mathbb{R}^c$, we say that y* Pareto-dominates *x, denoted by $x <_P y$, if $x_i < y_i$ for all $i \in [c]$.*

Let G be the comparability graph of a strict poset. The dimension *of the poset is equal to the minimal c such that there exists a set $X \subseteq \mathbb{R}^c$, such that X realizes G via Pareto-dominance. (By this, we mean $V(G) = X$ and $E(G) = \{(x, y) \mid x <_P y\}$.)*

The rest of this subsection is now devoted to generalizing this very elegant result to graphs with double-edges and undirected graphs. So suppose that we have a double edge between two vertices u and w, like in Figure 30. If we want to allow that the edge (u, v) is monotone, this means by our definition, that $N^{\text{out}}(u) \supseteq N^{\text{out}}(v)$. But because u is an outgoing neighbor of v, the loop at u must be present for the edge (u, v) to be considered monotone. This means that if we want to consider the MNH-clique-cover problem in directed graphs with double edges, we either have to change our definition of a monotone edge, or we have to include loops. We chose the latter. Especially, a MNH-clique in a directed graph G allowing loops is a subgraph of G, which is a directed clique, such that every edge in the directed clique is monotone.

With this insight in mind, we define the transformation of an undirected graph to a directed graph the following way:

Definition 5.32 (transformation of undirected graph to directed graph with loops)**.** *Let G be an undirected graph. We denote by* $\text{dir}(G)$ *the directed graph allowing loops, which has the same vertex set as G, the double edge $\{(u, v), (v, u)\}$ for every edge $uv \in E(G)$ and additionally a loop at every vertex.*

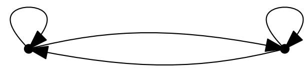

Figure 30: If there is a double edge between u and v and (u, v) is monotone, then there is a loop at u.

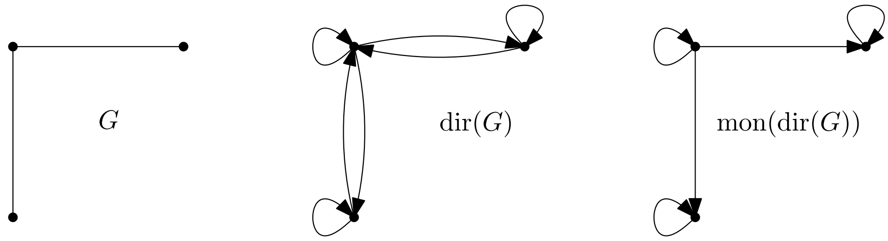

Figure 31: Turning an undirected into a directed graph.

(An example can be found in Figure 31.) We also need to say what we mean by a MNH-clique cover in a directed graph allowing loops:

Definition 5.33 (MNH-clique-covers in directed graph with loops)**.** *Let G be a directed graph allowing loops. A MNH-clique cover of G of size k is a set $\{C_1, \dots, C_k\}$ of subgraphs of G, which are MNH-cliques in G, such that they are pairwise vertex-disjoint and $\bigcup_{i=1}^{k} V(C_i) = V(G)$. (In particular, the C_i are not necessarily induced subgraphs.)*

This means that in the directed graph (with loops) G from Figure 30, there are actually two MNH-cliques: Both have vertex set $\{u, v\}$, but one has the edge (u, v) and the other the edge (v, u). Each one of these two MNH-cliques forms a MNH-clique cover of G of size 1.

Regarding anti-monotone chains in directed graph G allowing loops, we have already defined what we mean by this term in Definition 5.23. Namely, this is a set $A \subseteq V(G)$, such that no two distinct vertices of A are connected by a monotone edge in G.

So if G is an undirected graph, which is modeled as $\mathrm{dir}(G)$ like in Definition 5.32, we have that a vertex set $A \subseteq V(G)$ is an anti-monotone chain, if and only if there do not exist distinct $x, y \in A$, such that $N(x) \cup \{x\} \subseteq N(y)$. As it turns out, a very close variant of this parameter has already been considered in the literature. The *Dilworth number of a graph* is a graph parameter of undirected graphs, introduced in 1978 by Foldes and Hammer as

Definition 5.34 (Dilworth number $\nabla(G)$ [6])**.** *Let G be an undirected graph. The Dilworth number of G is the maximum size of a set $A \subseteq V(G)$ such that there do not exist distinct $x, y \in A$ with $N(x) \subseteq N(y) \cup \{y\}$.*

But $\nabla(G)$ is not quite the size of a maximal anti-monotone chain, because we consider the direction (u, v) in an undirected graph G to be monotone, when $uv \in E(G)$ and the neighborhoods are monotone, but for the Dilworth number, it is only important that the neighborhoods are monotone, independent of the edge uv being in G. So the two numbers are not quite the same. However, as noted in Section 1.2, the authors of [6, 5, 8] used very similar techniques to the ones we use in this subsection.

Now we are ready to prove the promised generalization of MNH-clique covers to arbitrary directed graph allowing loops, which by our definition of $\mathrm{dir}(G)$ also includes undirected graphs. Recall from the preliminaries that a strong module in a directed graph G (allowing loops) is a subgraph S of G, which is induced by a subset $X \subseteq V(G)$, such that for all $x, y \in X$, $N^{\mathrm{in}}(x) = N^{\mathrm{in}}(y)$ and $N^{\mathrm{out}}(x) = N^{\mathrm{out}}(y)$. Now consider

Lemma 5.35. *Let G be a directed graph allowing loops. Then each strongly connected component S in $\mathrm{mon}(G)$ is a a bidirectionally connected clique and a a strong module.*

Proof. Let $H := \mathrm{mon}(G)$ and S be a strongly connected component in H. By the definition of a strongly connected component, for all $x, y \in V(S)$, there exists a directed path from x to y in H and vice versa a path from y to x. Then, by the definition of monotony, we have that $N_G^{\mathrm{out}}(x) = N_G^{\mathrm{out}}(y)$. So if $z \in N_G^{\mathrm{out}}(x)$ and the edge (x, z) is monotone in G, we have that the edge (y, z) is present in G and also monotone in G. This implies $N_H^{\mathrm{out}}(x) = N_H^{\mathrm{out}}(y)$. So all vertices in S have the same outgoing neighborhood, both in G as well as in H. Together with the fact that S is a strongly connected component, we get that S is a bidirectionally connected clique. (If $|V(S)| > 1$, we also have that there exists the loop at x and the loop at y in both G and H.)

If on the other hand there is a vertex $y \in V(H) \setminus V(S)$ and a vertex $x \in V(S)$, such that $(y, x) \in E(H)$, then the edge (y, x) is monotone in G, so $N_G^{\mathrm{out}}(y) \supseteq N_G^{\mathrm{out}}(x) \supseteq V(S)$, so y is connected to all of $V(S)$ in G. Because all vertices of S have the same outgoing neighborhood in G and the edge (y, x) is monotone in G, we get that all the other edges $\{(y, s) \mid s \in V(S)\}$ are monotone in G as well, and so they "survive" into H. This means that $y \in N_H^{\mathrm{in}}(s)$ for all $s \in V(S)$. This implies $N_H^{\mathrm{in}}(v) = N_H^{\mathrm{in}}(w)$ for all $v, w \in V(S)$.

In total, we have $N_H^{\mathrm{out}}(v) = N_H^{\mathrm{out}}(w)$ and $N_H^{\mathrm{in}}(v) = N_H^{\mathrm{in}}(w)$ for all $v, w \in V(S)$. So S is indeed a strong module in H. $\square$

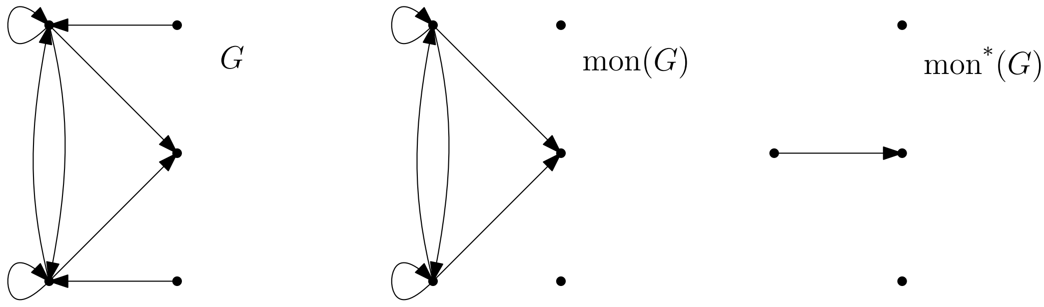

Figure 32: Getting from $\mathrm{mon}(G)$ to $\mathrm{mon}^*(G)$.

In other words, if we have a strongly connected component S in $\mathrm{mon}(G)$, the component behaves "just like a single vertex". This has the following implications

Lemma 5.36. *Let G be a directed graph allowing loops and $H := \mathrm{mon}(G)$. Then*

(i) No anti-monotone chain in G can use two vertices from the same strongly connected component of H.

(ii) If there exists a MNH-clique cover M of G of size k for some $k \in \mathbb{N}$, then there also exists a MNH-clique cover M' of G of size k such that there is no strongly connected component in H which has a nonempty intersection with two different MNH-cliques of M'.

Proof. (i) Because each strongly connected component is a bidirectional clique in H, so any two vertices are connected by a monotone edge in G. (ii) If there exists a strongly connected component S of H such that in M there are distinct MNH-cliques C_1 and C_2 with $V(C_1) \cap V(S) \neq \emptyset$ and $V(C_2) \cap V(S) \neq \emptyset$, then we can integrate the vertices of S into C_1 and remove them from C_2. After this operation, C_1 is still an MNH-cliques, because all of the vertices in S have the same outgoing neighborhood in G. Repeating this operation yields the claim. □

Definition 5.37. *Let G be a directed graph allowing loops. Let $\mathrm{mon}^*(G)$ be the directed graph obtained from $\mathrm{mon}(G)$ by first contracting all strongly connected components (this is possible, because each of them is a strong module, compare Lemma 5.35) and then deleting all loops.*

An example can be seen in Figure 32.

Observation 5.38. *Let G be a directed graph allowing loops. Then* $\mathrm{mon}^*(G)$ *is the comparability graph of a strict poset.*

Proof. Let $H := \mathrm{mon}^*(G)$. The graph $\mathrm{mon}(G)$ is transitive by Lemma 5.25. This property is preserved when contracting all strongly connected components, which are strong modules, so H is transitive. It is antisymmetric, because contracting all strongly connected components of a directed graph yields an acyclic graph. So H is transitive and antisymmetric, hence the reflexive closure is a poset. □

This leads to our final theorem:

Theorem 5.39 (generalization of Dilworths theorem to arbitrary graphs)**.** *Let G be a directed graph, allowing loops, or an undirected graph, which is modeled as a directed graph with loops, like in Definition 5.32. Then the following numbers are equal:*

(i) Maximal size of an anti-monotone chain in G

(ii) Minimal size of a MNH-clique cover of G

(iii) Width of the strict poset $\mathrm{mon}^*(G)$

(iv) If additionally G directed and acyclic without loops, the minimum number $\varphi(G)$ to realize G as a Top Trumps deck

Proof. We already know (i) $\Leftrightarrow$ (iv) in Theorem 5.27. As a consequence of Lemma 5.36, we know that the maximal size of an anti-monotone chain is equal to the maximum size of an antichain in $\mathrm{mon}^*(G)$. By the same lemma, we also know that the minimum size of a MNH-clique cover of G equals the minimum size of a chain decomposition of $\mathrm{mon}^*(G)$. Therefore the claim follows analogously to Theorem 5.27. □

6 Conclusion

The research involved in creating this thesis has been a fascinating journey for the author and advisor. Who could have guessed, that starting with some questions about a children's game, we would end up in the domain of realizability, of monotone edges and Dilworth's theorem? We revisit the central topics of the thesis and state open problems regarding each of them.

First, in Section 3, we defined a simple model of the game Top Trumps and established the strategy HIGHEST-VALUE as an optimal strategy in this model. Then we introduced the dominance graph of a Top Trumps deck and showed that properties of the deck, like the strength of individual cards and the attack advantage, can be elegantly expressed as graph-theoretical properties of the dominance graph. We also determined the nature of identical-strength-decks.

It can be argued, that we made our life intentionally simple. That is to say, in our model we only considered a single round played between only two players on a deck without stalemates. Also, when there exist multiple equally best choices of a category for a card x, i.e. the set $\{i \in [c] \mid r_i(x) = h(x)\}$ has cardinality strictly greater than 1, we simply define $c(x)$ to be the smallest category, when in reality any choice results in optimal play. These simplifications most likely contribute to the fact, that many of the later theorems seem quite elegant. So it is an open question, whether our observations still hold true, when the conditions are relaxed. For example, is the strategy HIGHEST-VALUE still optimal, when the game is played for more than one round? Could the situation occur, that if on top of Alice's pile there is a particularly bad card, she should intentionally lose the next round, in order to improve the quality of her own pile? Another question is, what happens if we extend the model to include mixed strategies? In particular, what happens when the category is picked uniformly at random by both players?

In Section 4, we asked the question, which directed graphs are realizable using ranked decks, and what is the minimum number $\Psi(G)$ of categories to do so. This was motivated by the fact, that in a ranked deck, the strategy HIGHEST-VALUE resembles optimal play. For the graph operations of the unidirectional join, the disjoint graph union and the lexicographical graph product, we determined the behavior of Ψ. When there exists a unique-champion ranked deck realizing G, determining $\Psi(G)$ is also easy. In general, it remains an open problem, to determine which graphs are ranked-realizable and if so, to determine $\Psi(G)$. This question might be hard, and solving the following, maybe easier problems would also be significant progress:

- Motivated by the observation, that $\alpha(G) \leq k(G) \leq \varphi(G) \leq \Psi(G)$ for all directed graphs G, we ask whether each of these inequalities can be strict on ranked-realizable graphs. I.e., do there exist directed graphs G with $\Psi(G) < \infty$, but $\varphi(G) < \Psi(G)$? Do there exist directed graphs G with $\Psi(G) < \infty$, but $k(G) < \varphi(G)$? And directed graphs G with $\Psi(G) < \infty$, but $\alpha(G) < k(G)$? All examples of ranked-realizable graphs, we could find, have the property $\alpha(G) = \Psi(G)$ and this is maintained by the three graph operations.

- We saw in Theorem 4.15, that $\Psi(G_1 \dot\cup G_2) \leq \Psi(G_1) + \Psi(G_2)$. Is actually equality true here, or do there exist counterexamples with $\Psi(G_1 \dot\cup G_2) < \Psi(G_1) + \Psi(G_2)$ (where $\Psi(G_1) = \infty$ or $\Psi(G_2) = \infty$ is allowed)? Note that we cannot easily prove equality: Although in a realization of $G_1 \dot\cup G_2$ the sets of used categories must be disjoint, we cannot easily extract ranked decks realizing G_1 and G_2.

In the last part, Section 5, we relaxed the condition of ranked realizability to realizability, by allowing decks instead of ranked decks, i.e. allowing arbitrary values on the cards instead of the set $\{1, \dots, n\}$. This is a conceptual step away from the initial game Top Trumps to a more abstract idea. Very interestingly, every DAG G is realizable and the realizability number $\varphi(G)$ is equal to the minimal size of a monotone-neighborhood-clique cover, which is also the width of the underlying poset $\mathrm{mon}(G)$, which consists of the monotone edges in G. Here, a directed edge (u, v) is called monotone, if $N^{\mathrm{out}}(u) \supseteq N^{\mathrm{out}}(v)$ and a monotone-neighborhood-clique is an acyclic directed clique consisting out of monotone edges. This connects the idea of realizability to Dilworth's theorem and the width of a poset. In particular, the width of a poset is exactly the minimal number of categories such that there exists a deck realizing the poset. Our theorem can be generalized to arbitrary directed and undirected graphs. However, as soon as we have a cycle in the graph, we lose the representation as a Top Trumps deck, because all dominance graphs are acyclic. Regarding this final section, we have the following open questions:

- The *Dilworth number*, introduced by Foldes and Hammer, is almost, but not quite equal to the generalization of φ, which we considered in Section 5.4. What are the similartities, what are the differences? Can one number be expressed in terms of the other? Unfortunately, we discovered the Dilworth number at a very late stage of the thesis, so we had no time left to answer these questions.

- Can the idea of Top Trumps deck realizations be generalized, such that all directed graphs can be realized, consistent with Theorem 5.39?

- The generalization of φ to arbitrary directed graphs in Section 5.4 is a bit technical, especially because loops are involved. Can these concepts be expressed in an easier, more elegant way, and without using loops?
- Is there some connection between perfect graphs and Top Trump deck realizations?
- How does $\varphi(G)$ behave on random graphs?

As a final note, we want to mention, that due to its nature, a lot of equivalences and connections were discovered at a late stage of the thesis. Therefore, we can not fully exclude the possibility, that several of our results were obtained before by different authors. We nonetheless hope that we were able to convince the reader, that the graph-theoretical theory behind the children's game Top Trumps is by far deeper and richer than the name suggests. We therefore hope that this thesis is not the last, but rather the first scientific publication regarding the topic.

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