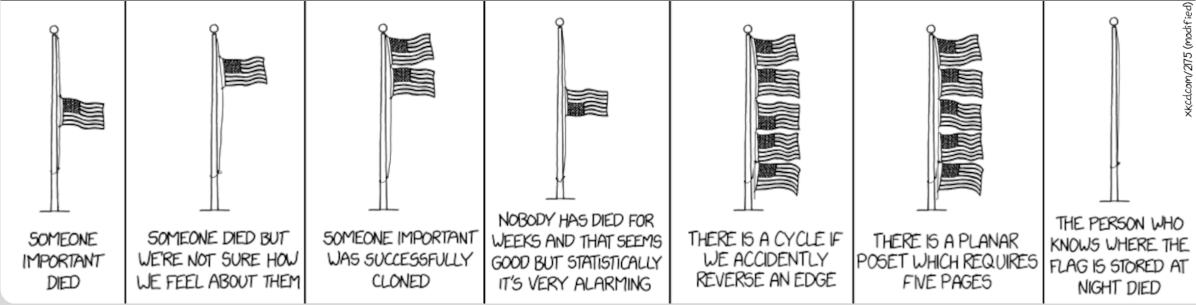
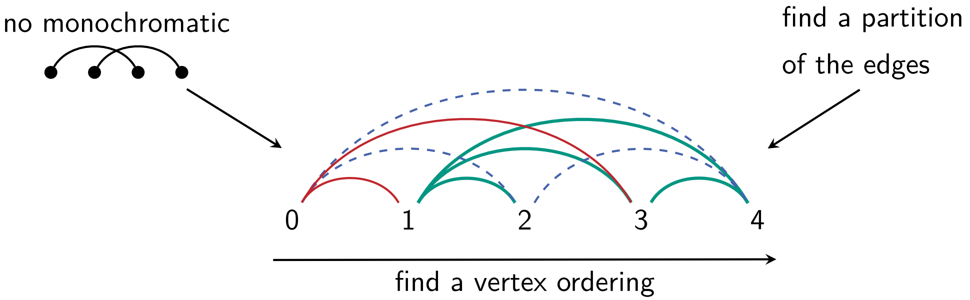
Book embeddings of upward planar graphs and planar posets

Paul Jungeblut, **Laura Merker** and Torsten Ueckerdt | 12.02.2021

Discrete Mathematics Seminar



Page Number

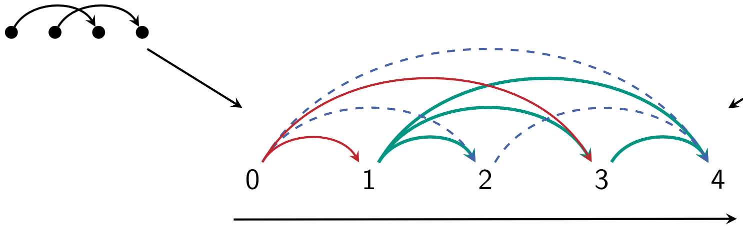


$\mathrm{pn}(G) =$ minimum number of pages

Page Number of DAGs

no monochromatic

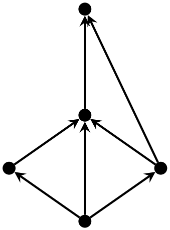
find a partition of the edges



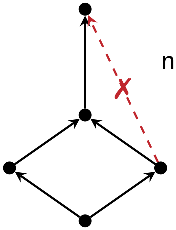
find a vertex topological ordering

pn(G) = minimum number of pages

Upward Planar



Planar Poset



no transitive edges

edges increase monotonically in y-direction

Open question

Is the page number bounded? (Nowakowski and Parker, 1989)

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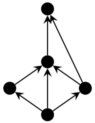
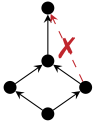
Upward planar:



Planar Poset:



Now:

Theorem

There is an upward planar graph with page number $\geqslant 5$.

Theorem

There is a planar poset with page number $\geqslant 5$.

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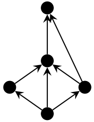
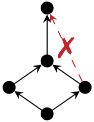
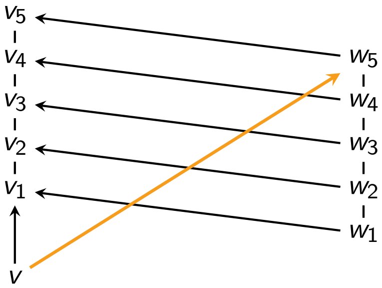
Upward planar:



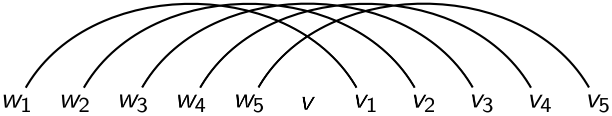
Planar Poset:



5-Flag from v to $w = w_5$



use topological ordering of augmented graph



5-twist

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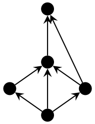
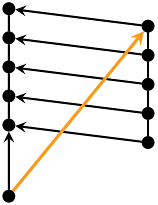
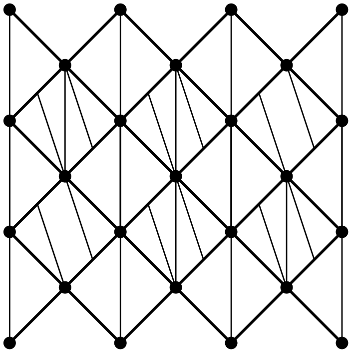
Upward planar:



5-flag:





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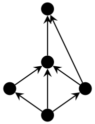
Upward planar:



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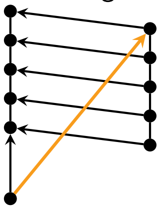

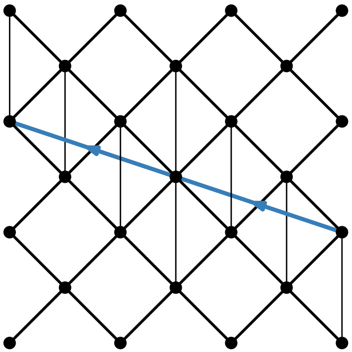

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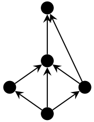
Upward planar:



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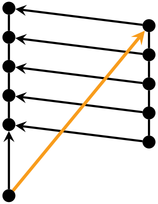

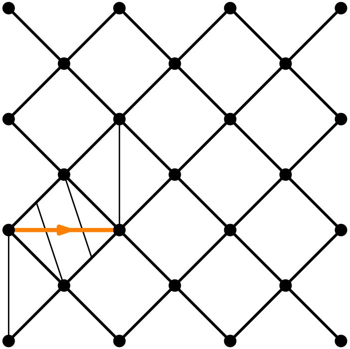

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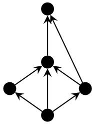
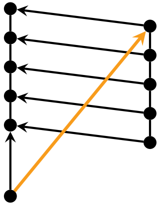
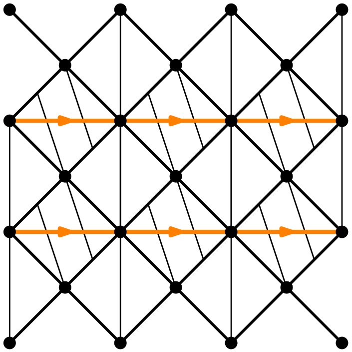
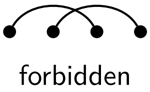
Upward planar:



5-flag:



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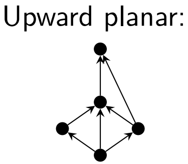
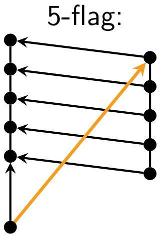

Upward planar: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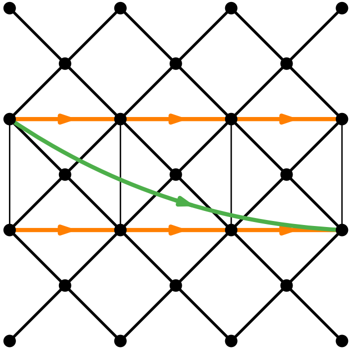

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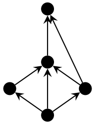
Upward planar:



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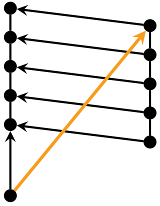

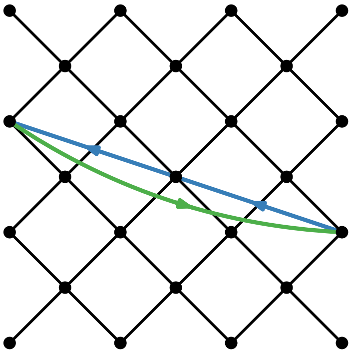

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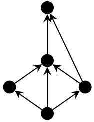
Upward planar:



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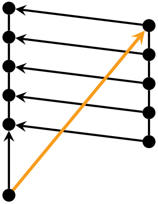

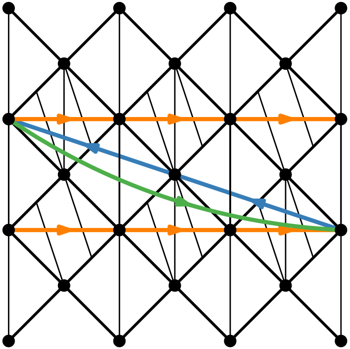

There is an upward planar graph with page number $\geqslant 4$.

(Bekos et al., 2020; Hung, 1993; Yannakakis, 2020)

There is an upward planar graph with a 4-twist in every book embedding.

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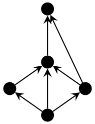
Upward planar:



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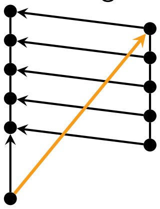

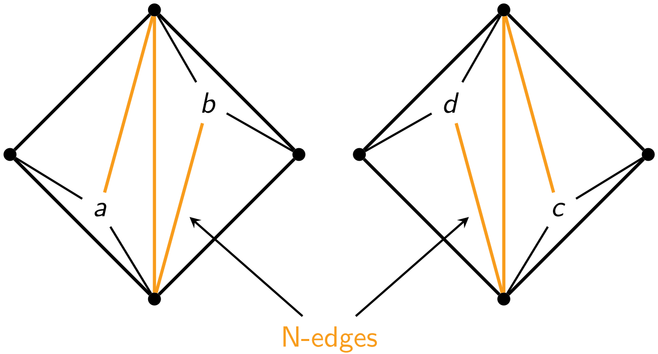

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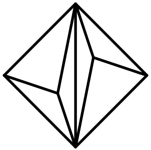
Upward planar:



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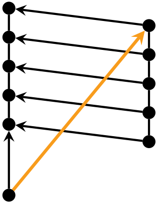

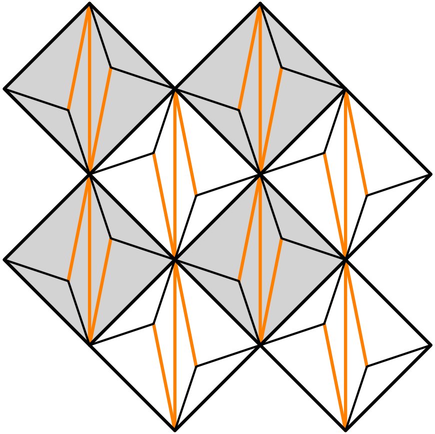

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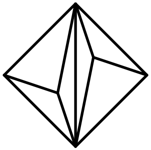
Upward planar:



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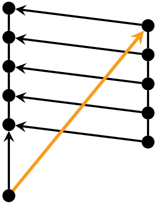
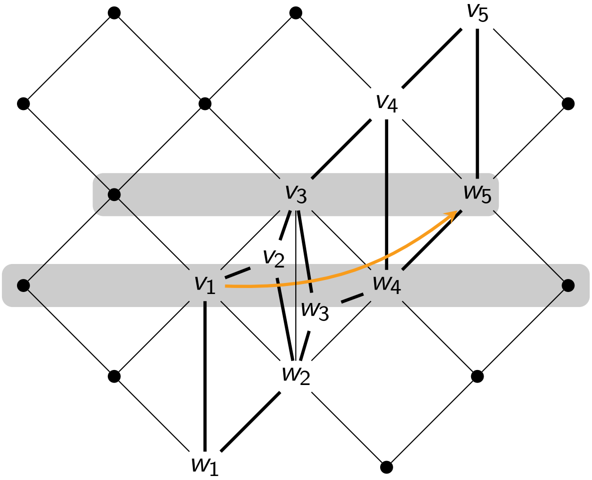


5-Flags: Separating levels



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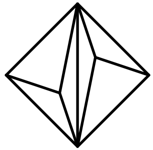
Upward planar:



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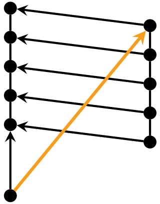

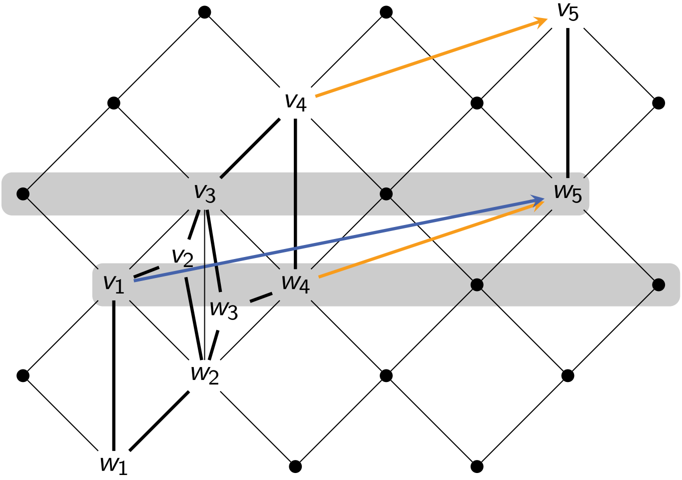

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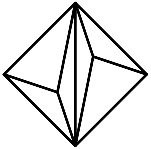
Upward planar:



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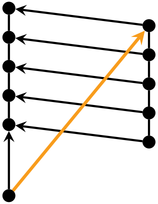
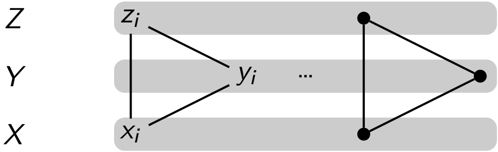
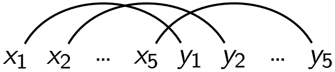


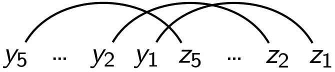


Laid out level-wise, i.e. $X \prec Y \prec Z$



Many triangles

$\rightsquigarrow y_5 \prec \cdots \prec y_2 \prec y_1$ and $z_5 \prec \cdots \prec z_2 \prec z_1$



Topological ordering of augmented graph $\implies$ large twists

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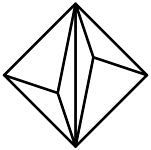
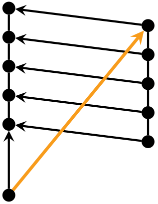
Upward planar:



5-flag:



Theorem

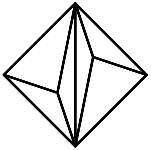
There is an upward planar graph that has a 5-twist with every vertex ordering.

Now:

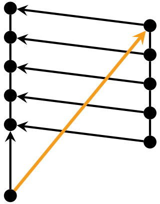
Theorem

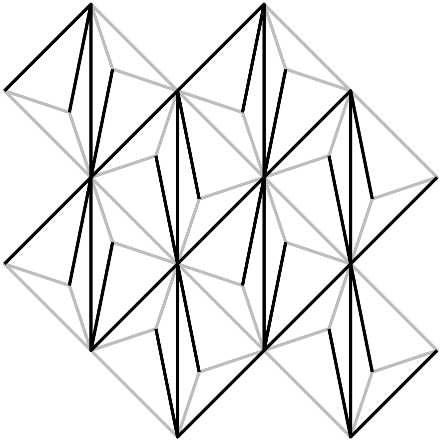
There is a planar poset that has a 5-twist with every vertex ordering.

Upward planar:



5-flag:





Idea:

- Replace gray edges by 5-flags
- Use only edges of cover graph for twists

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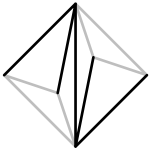
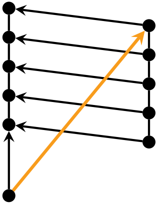
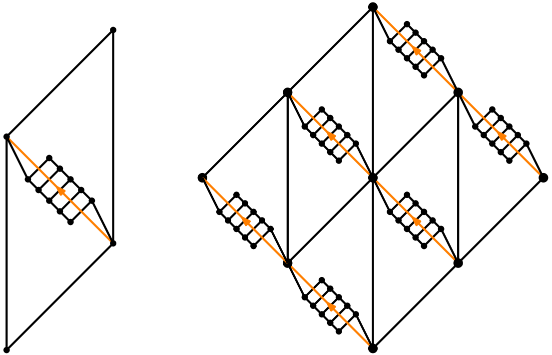
Planar poset:



5-flag:



Replace gray edges by 5-flags:



Note: Endpoints of flags are incomparable

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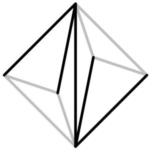
Planar poset:



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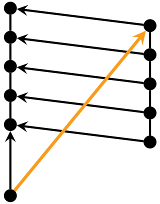

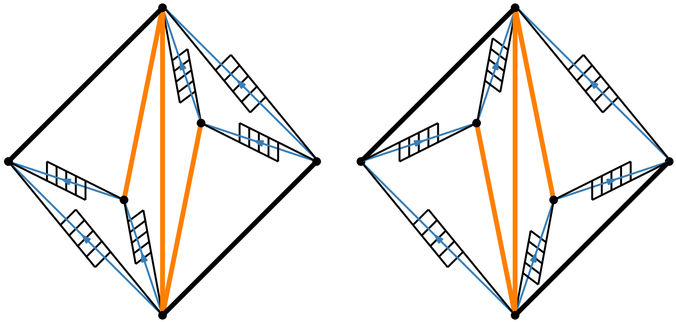

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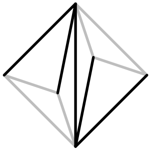
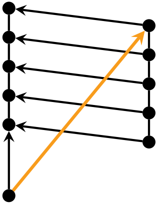
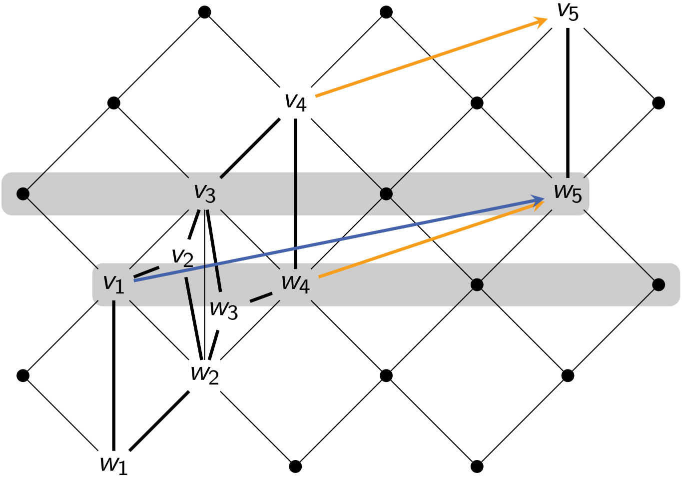
Planar poset:



5-flag:



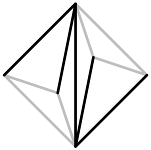


Observe: Only N-edges used for twists

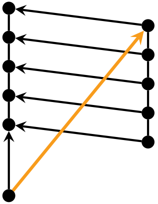
$\Longrightarrow$ w.l.o.g. vertices are laid out level-wise (5-twist otherwise)

forbidden

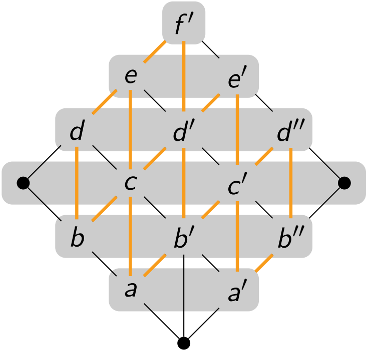
Planar poset:



5-flag:



W.l.o.g. vertices are laid out level-wise



Take many copies of this subgraph in a large grid

Find a 5-twist using only vertical and right edges

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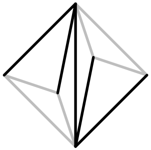
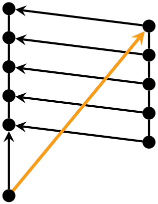
Planar poset:

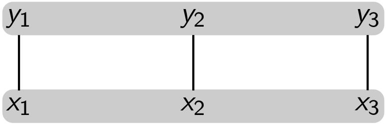


5-flag:



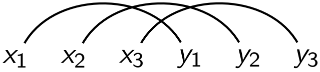
Laid out level-wise, i.e. $\{x_1, x_2, x_3\} \prec \{y_1, y_2, y_3\}$

copy G_1 copy G_2 copy G_3



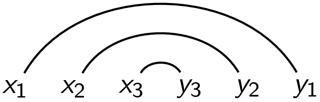
Interleaving vector $v_{ij} = \begin{cases} \text{forward} & \text{if } x_i \prec x_j \\ \text{backward} & \text{if } x_i \succ x_j \end{cases}$

Ramsey theorem: same interleaving vector for all copies



✗

or



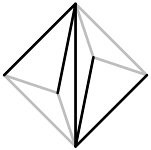
✓

forbidden

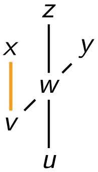
(i) Copied edges nest

(ii) Copies have pairwise the same vertex ordering

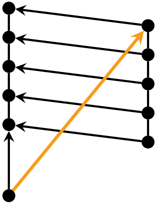
Planar poset:

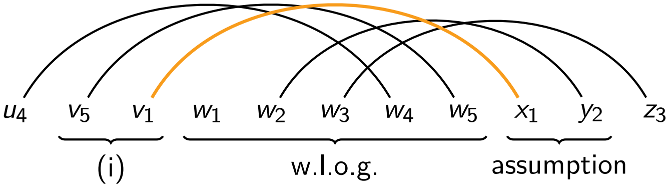


Consider five copies of



5-flag:

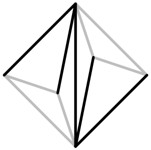




$x \prec y \implies$ 5-twist

forbidden

Planar poset:



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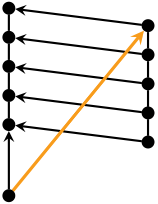
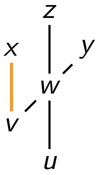
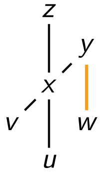




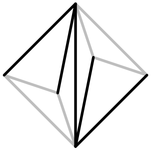
$x \prec y \implies$ 5-twist



$v \prec w \implies$ 5-twist

forbidden

Planar poset:



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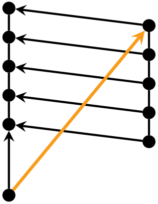

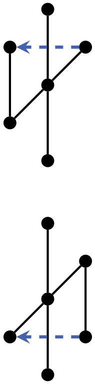

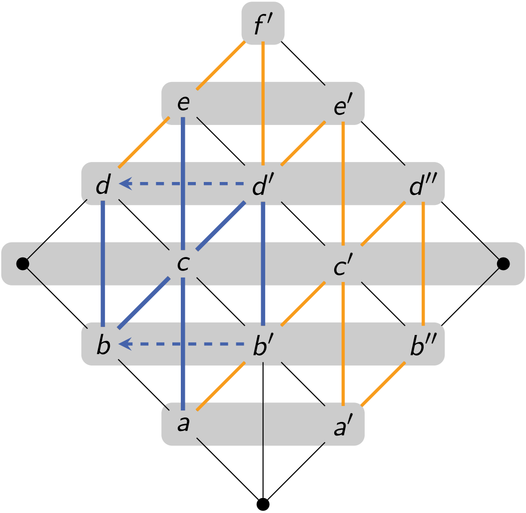

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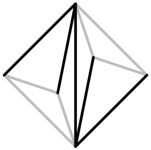
Planar poset:



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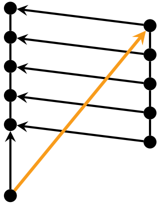
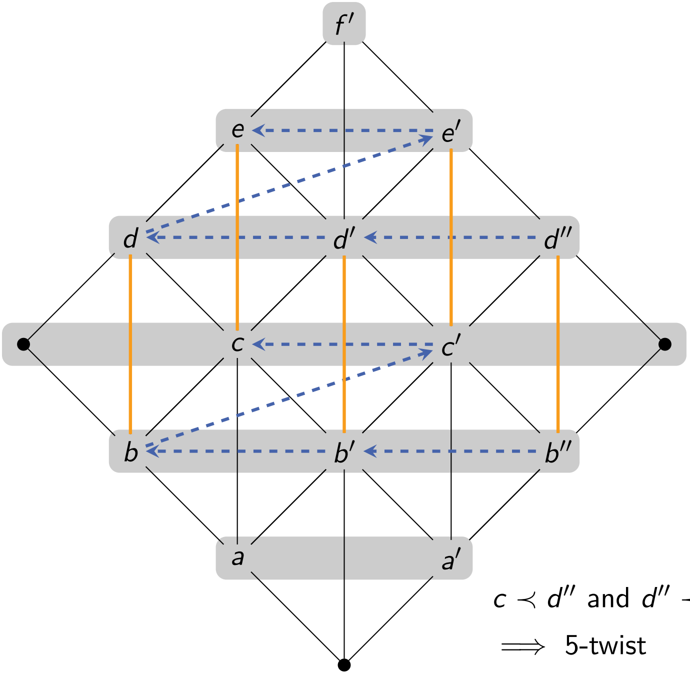


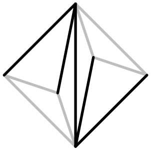


$c \prec d''$ and $d'' \prec c$

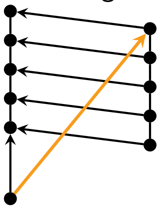
$\implies$ 5-twist

forbidden

Planar poset:

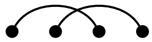


5-flag:



- Is there an upward planar graph (a planar poset) with page number 6?
- **Is there an upward planar graph (a planar poset) with page number *k* for every *k*?**

  (Nowakowski and Parker, 1989)



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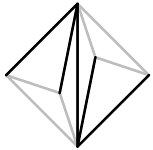
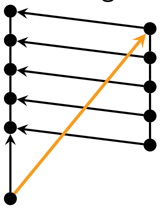
Planar poset:



5-flag:

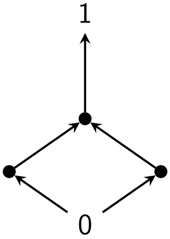


- Is there an upward planar graph (a planar poset) with page number 6?
- **Is there an upward planar graph (a planar poset) with page number *k* for every *k*?**

  (Nowakowski and Parker, 1989)
- Find more ways do augment upward planar graphs.
- Is there a k such that all augmented graphs are acyclic?

Is the page number $\omega(1)$ / $o(n)$?

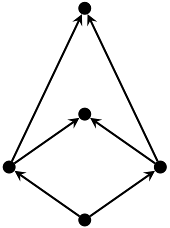
Planar lattice



single source/sink

no transitive edges

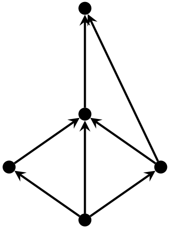
Planar poset



multiple sources/sinks

no transitive edges

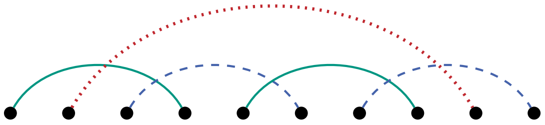
Upward planar



w.l.o.g. unique source/sink

transitive edges

Twist size vs page number



2-twist but 3 pages

For all $G, \prec$ we have

- $\mathrm{pn}(G, \prec) = \mathcal{O}((\text{max twist size})^2)$
- $\mathrm{pn}(G, \prec) = \mathcal{O}((\text{max twist size}) \cdot \log n)$

(Černý, 2007; Davies and McCarty, 2019)

There are $G, \prec$ such that

- $t =$ max twist size and
- $\mathrm{pn}(G, \prec) = \Theta(t \log t)$

(Kostochka and Kratochvíl, 1997)

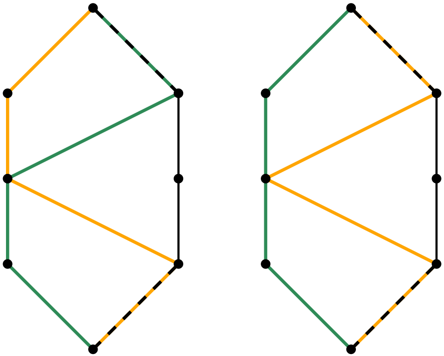
Improved bounds for upward planar graphs? $\rightsquigarrow$ restrict vertex ordering

Width of an upward planar graph

= width of its poset

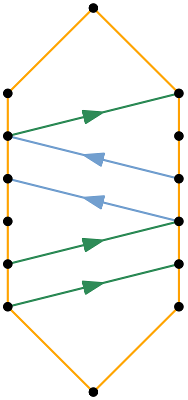
= max # of pairwise incomparable vertices

Width $w \rightsquigarrow$ partition of the vertices into

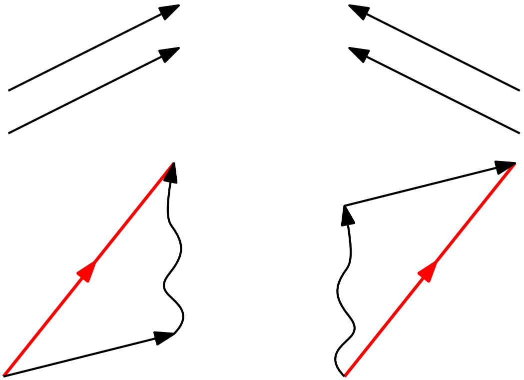


w chains $\qquad$ w non-crossing chains

Lattice with width w



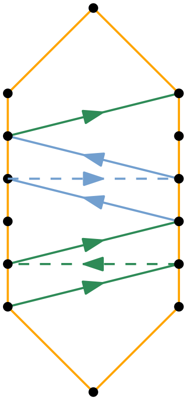
Two matchings:



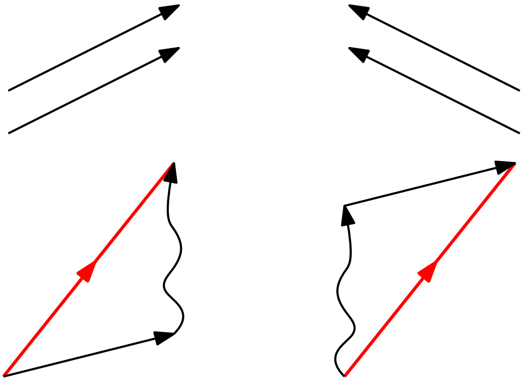
1 page per chain $+$ 1 page for each path between two neighboring chains

w pages $w-1$ pages

Lattice with width w



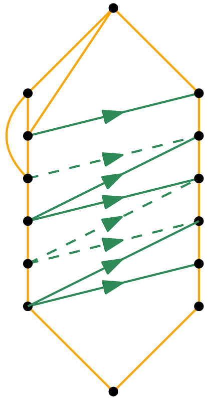
Two matchings:

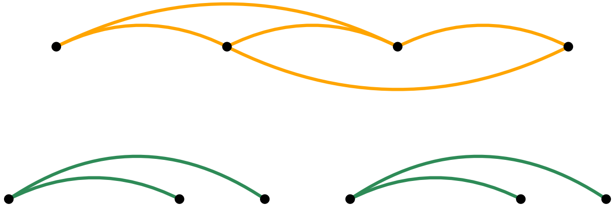


1 page per chain $+$ 1 page for each path between two neighboring chains

w pages $w-1$ pages

Upward planar graph with width w





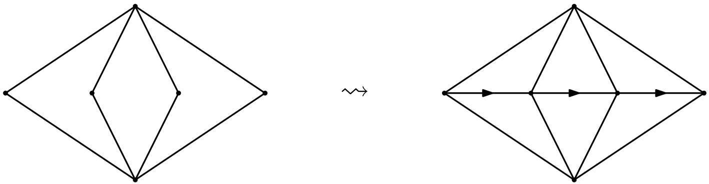
2 pages per chain + 2 pages for each path between two neighboring chains

$2w$ pages $2(w-1)$ pages

Toward a sublinear upper bound (Frati et al., 2013)

A (hypothetical) n-vertex upward planar graph with linear page number has

- width $\Theta(n)$
- height $\Omega(n/\log n)$
- a 4-connected component with page number $\Omega(n/\log n)$
- treewidth $\geqslant 4$
- w.l.o.g. max degree $\mathcal{O}(\sqrt{n})$



Toward a non-constant lower bound (Frati et al., 2013)

A (hypothetical) n-vertex upward planar graph with non-constant page number

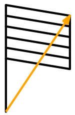
- has width $\omega(1)$
- is w.l.o.g. 4-connected
- has treewidth $\geqslant 4$

Upper bounds

- Is the page number sublinear?
  - upward planar graphs
  - planar posets
  - **planar lattices**
- Is there a k such that all *augmented* graphs are acyclic?
- Is page number bounded by a function of height?
- Triangulate lattices
  ⇝ small width/height/4-connected components?
  ⇝ flags do not create cycles?

Lower bounds

- Is there an upward planar graph (a planar poset) with page number 6?
- **Is there an upward planar graph (a planar poset) with page number k for every k?**

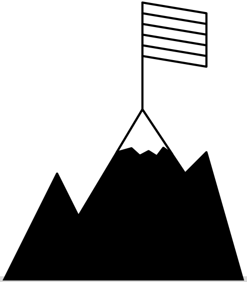


Upper bounds

- Is the page number sublinear?
  - upward planar graphs
  - planar posets
  - **planar lattices**
- Is there a k such that all *augmented* graphs are acyclic?
- Is page number bounded by a function of height?
- Triangulate lattices
  $\rightsquigarrow$ small width/height/4-connected components?
  $\rightsquigarrow$ flags do not create cycles?

Lower bounds

- Is there an upward planar graph (a planar poset) with page number 6?
- **Is there an upward planar graph (a planar poset) with page number k for every k?**



Thank you!

References I

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