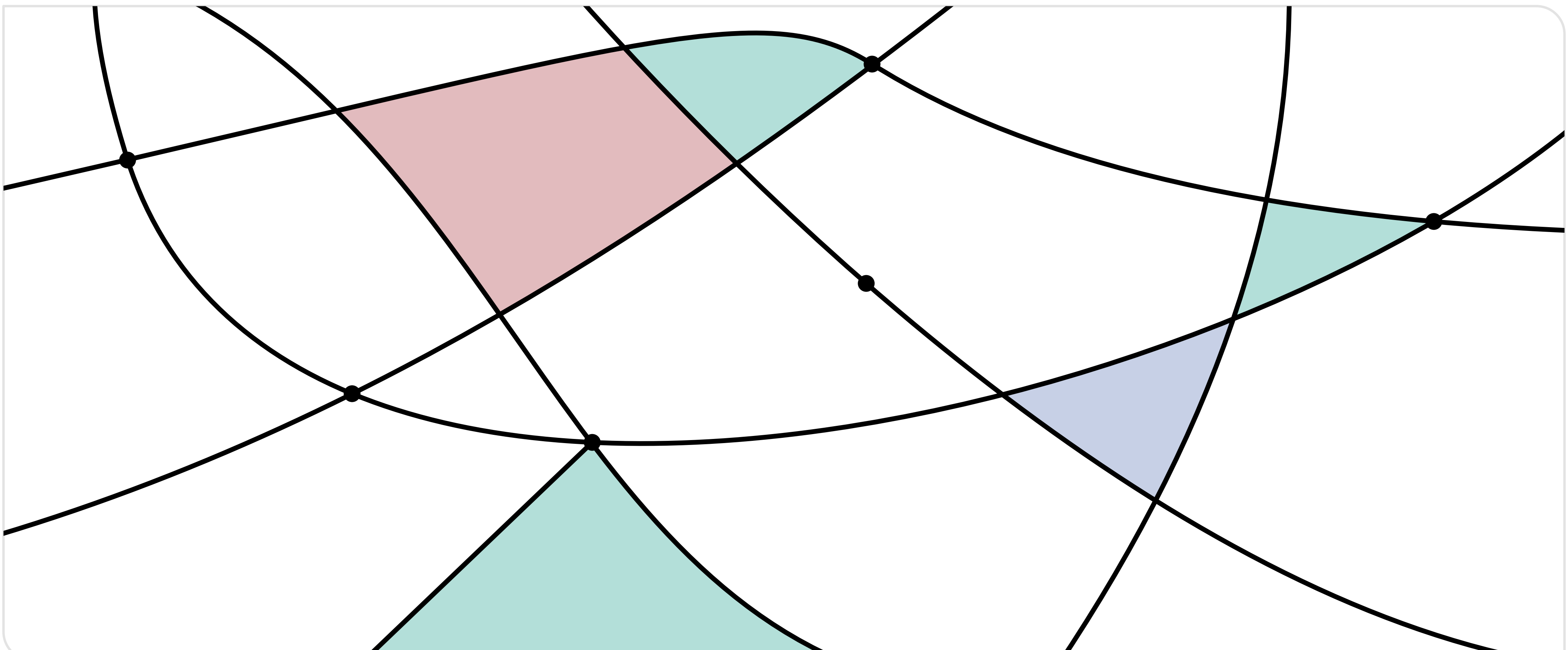
Karlsruher Institut für Technologie

Crossing Number of Simple 3-Plane Drawings

joint work with Michael Hoffmann, Ignaz Rutter, Torsten Ueckerdt



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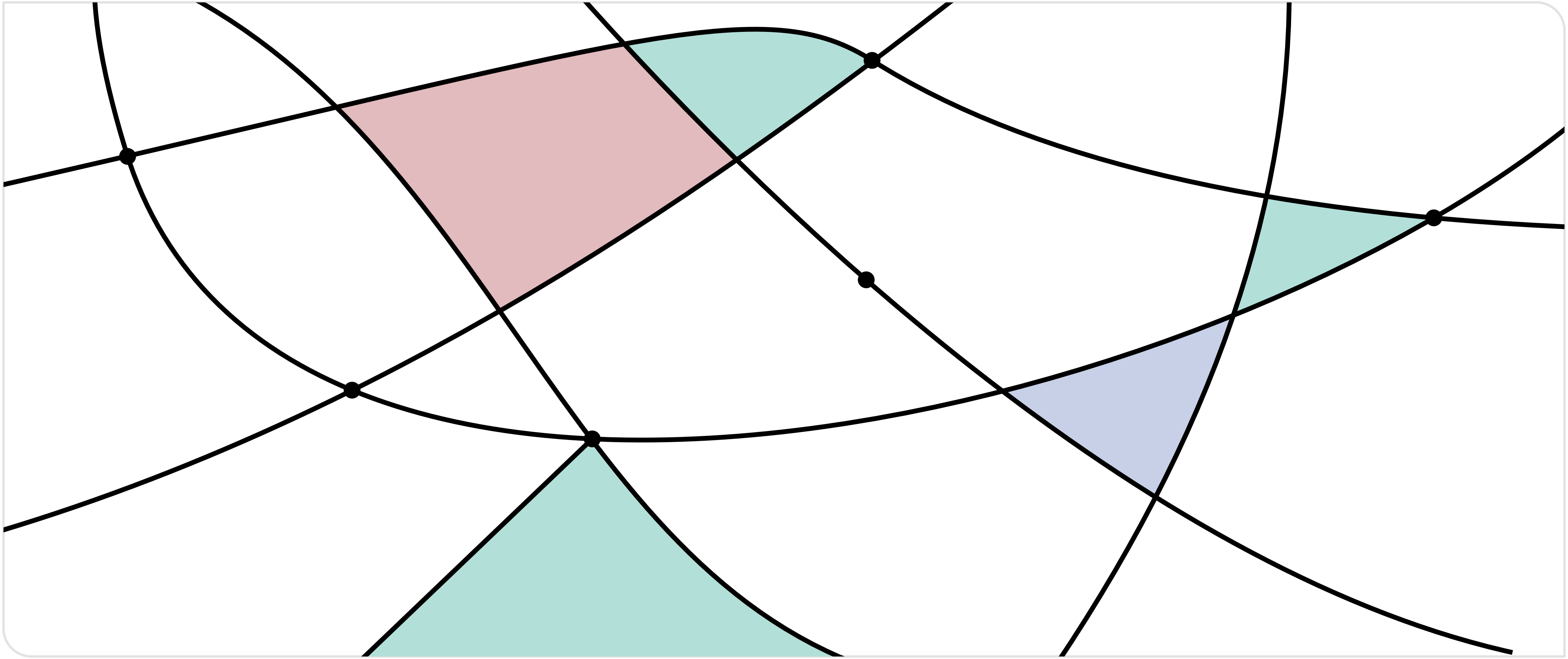
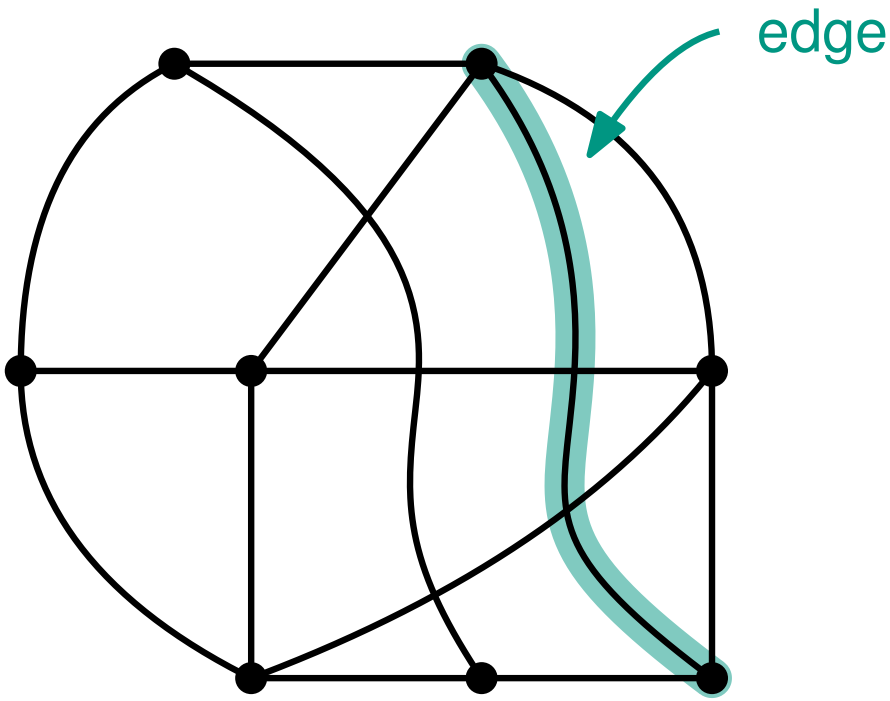
joint work with Michael Hoffmann, Ignaz Rutter, Torsten Ueckerdt



Simple Drawings



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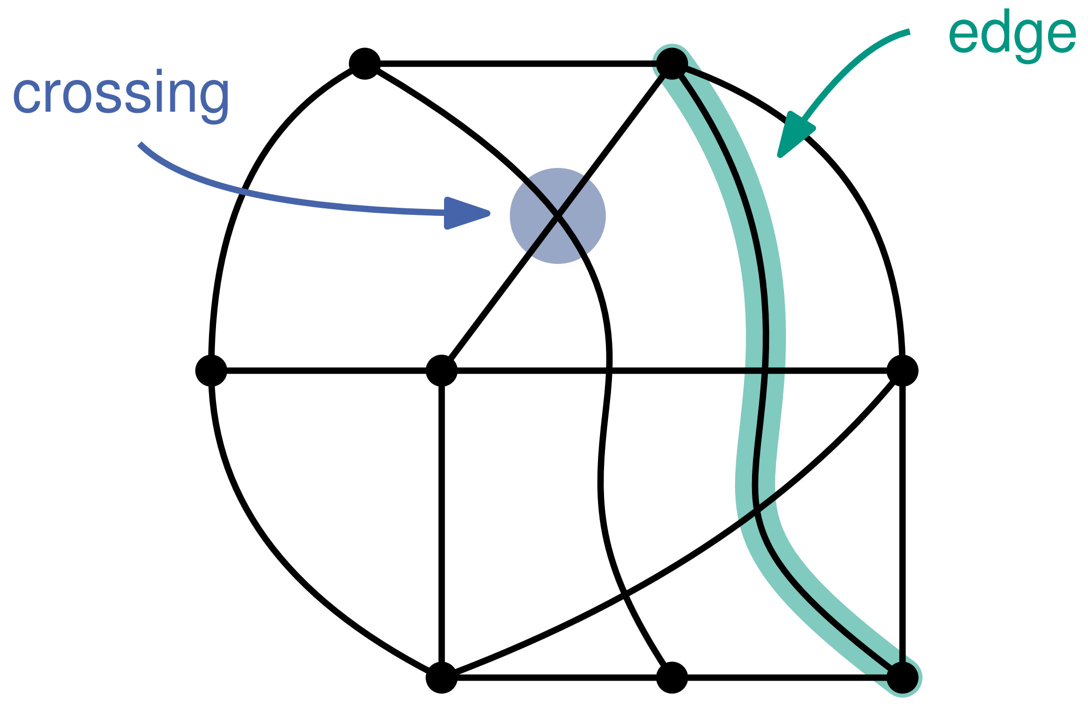

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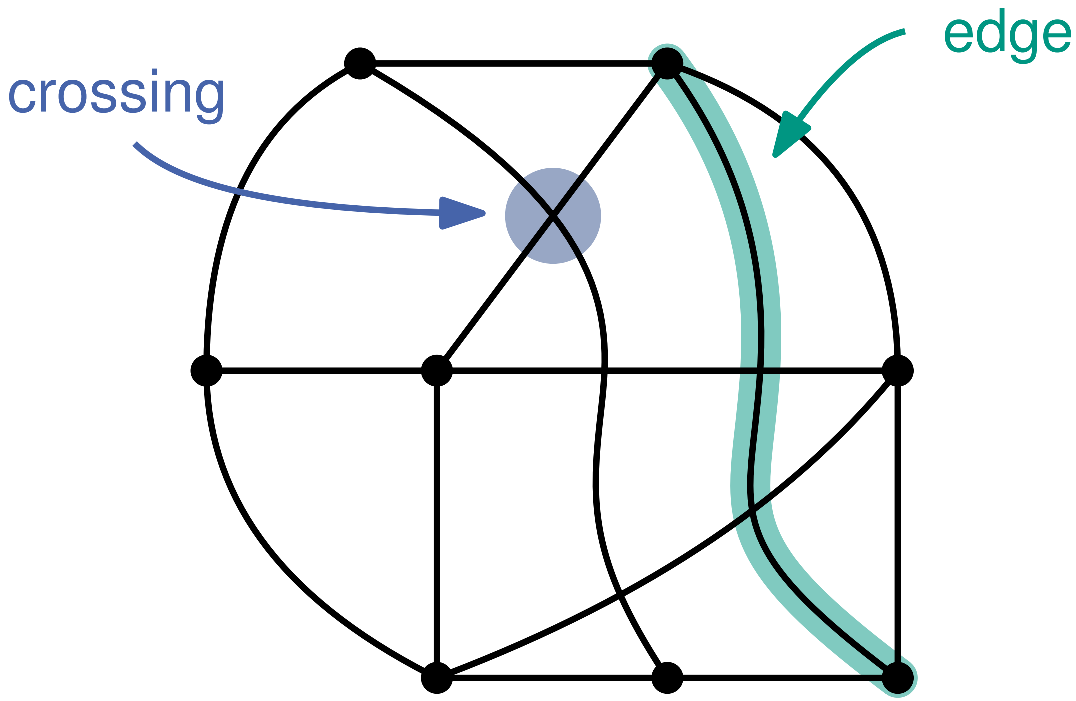

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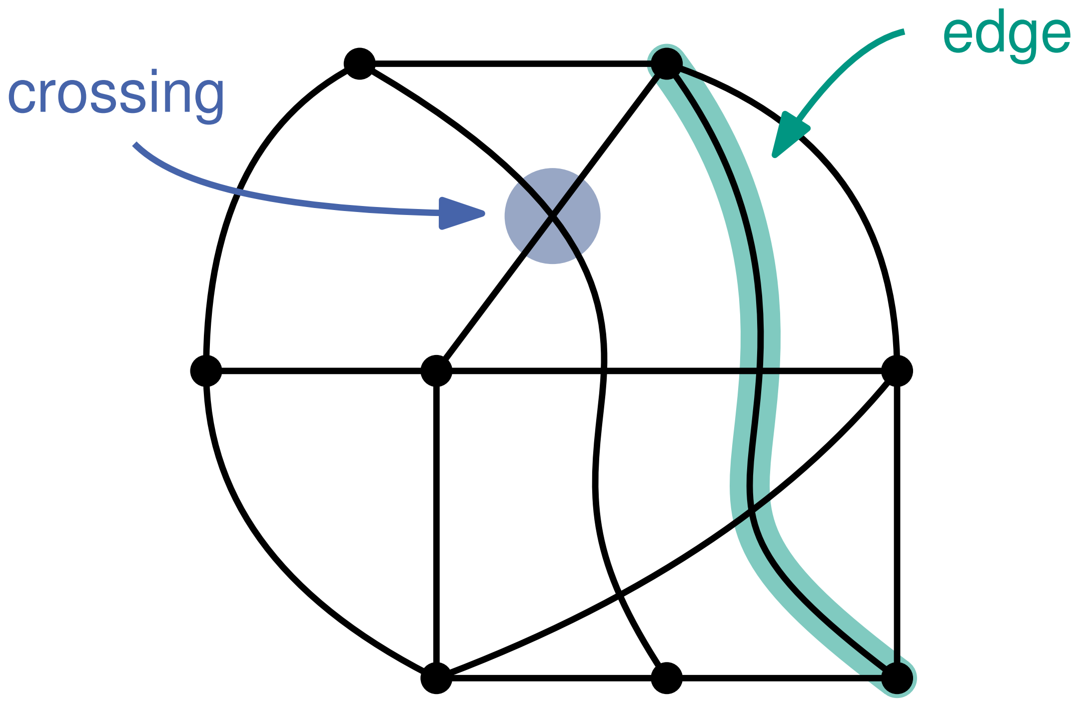

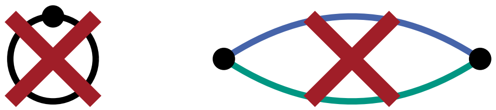

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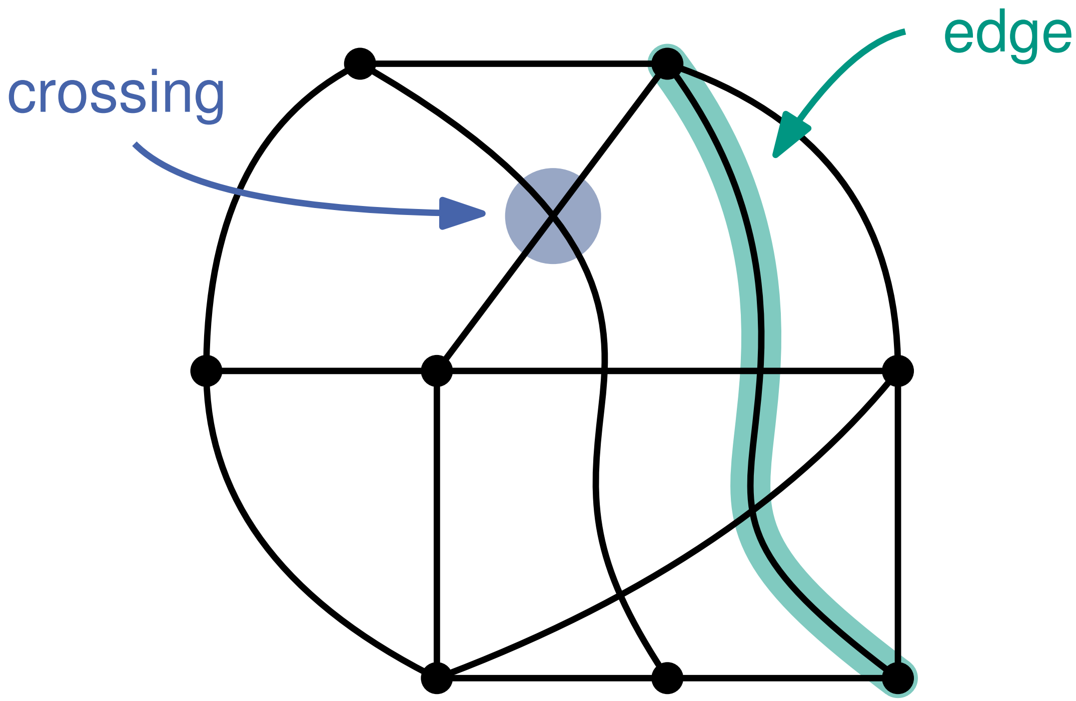

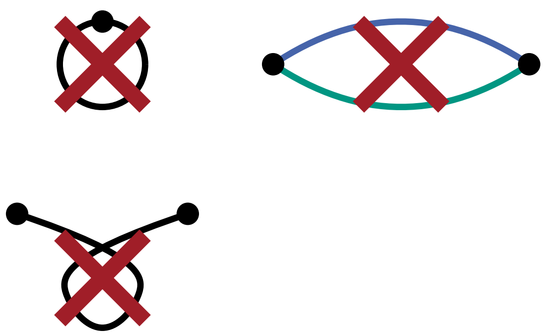

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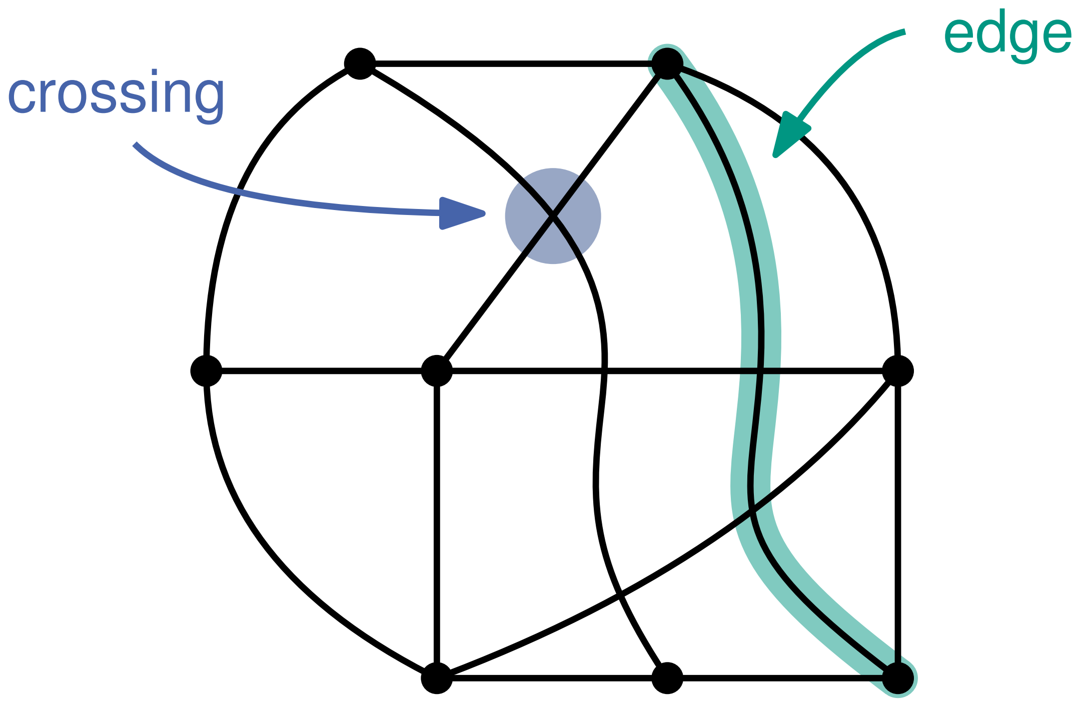

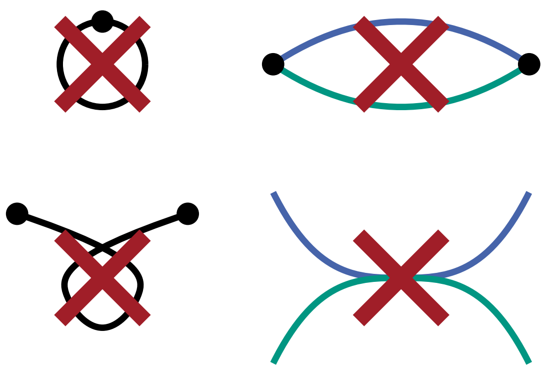

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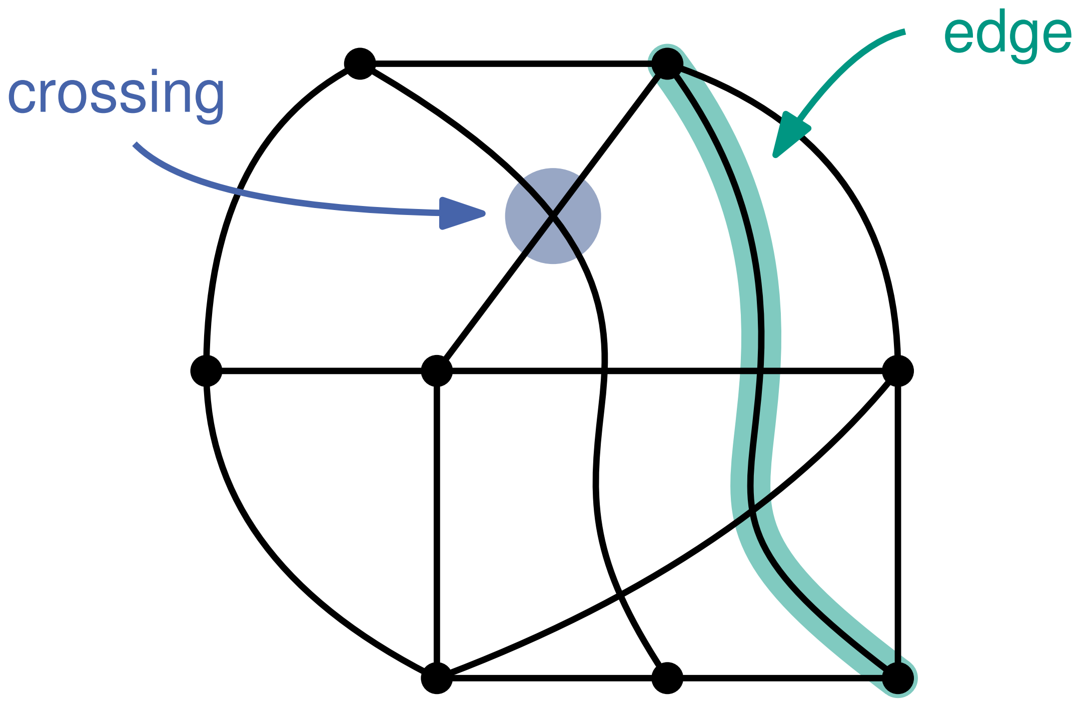

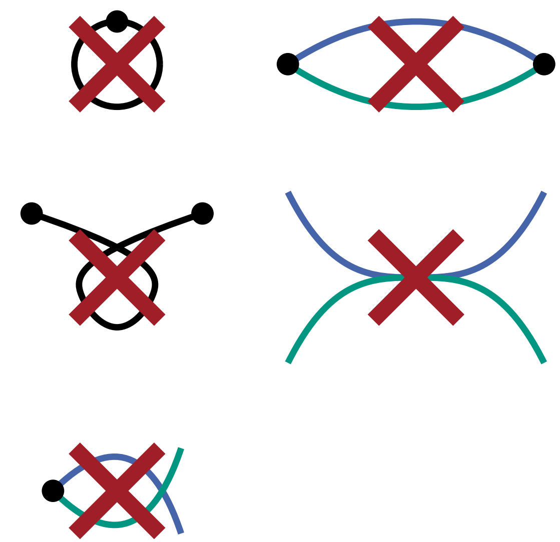

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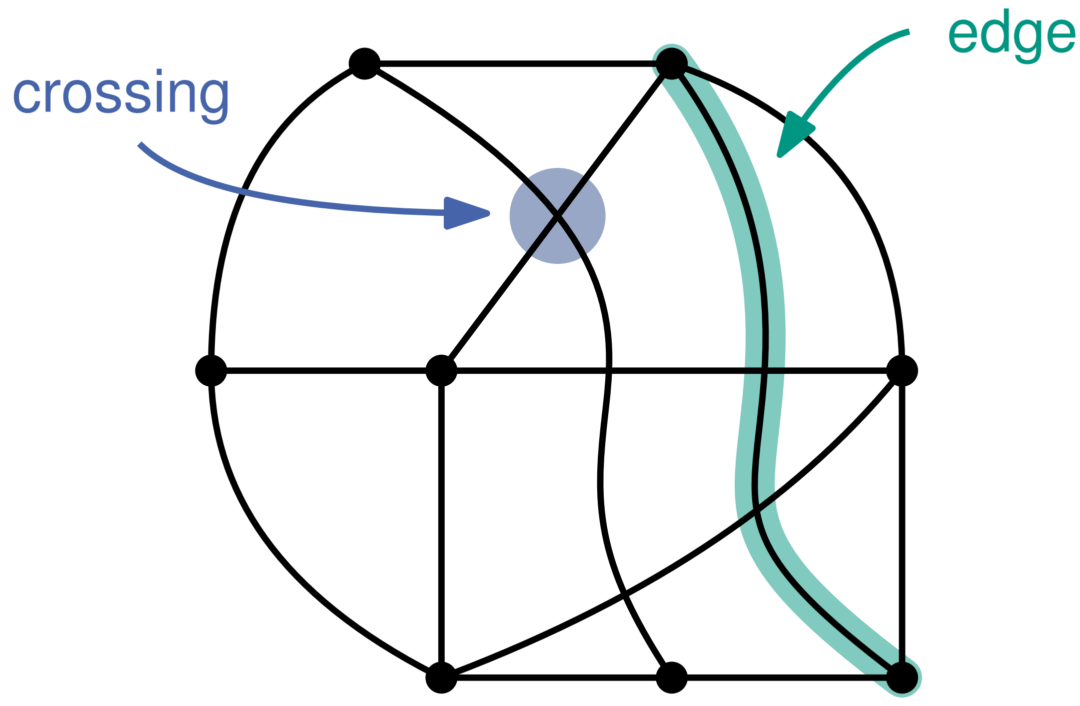

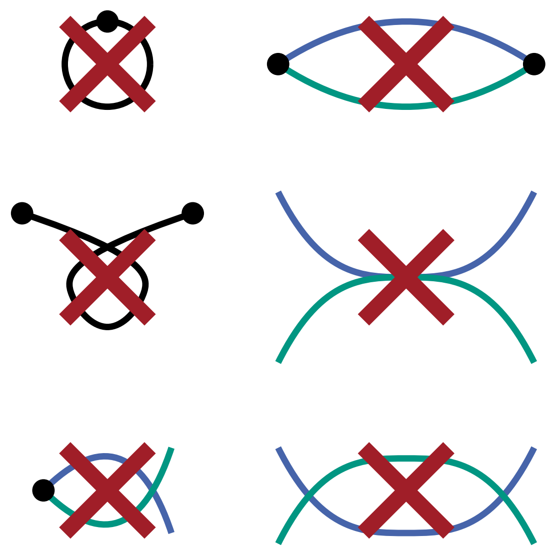

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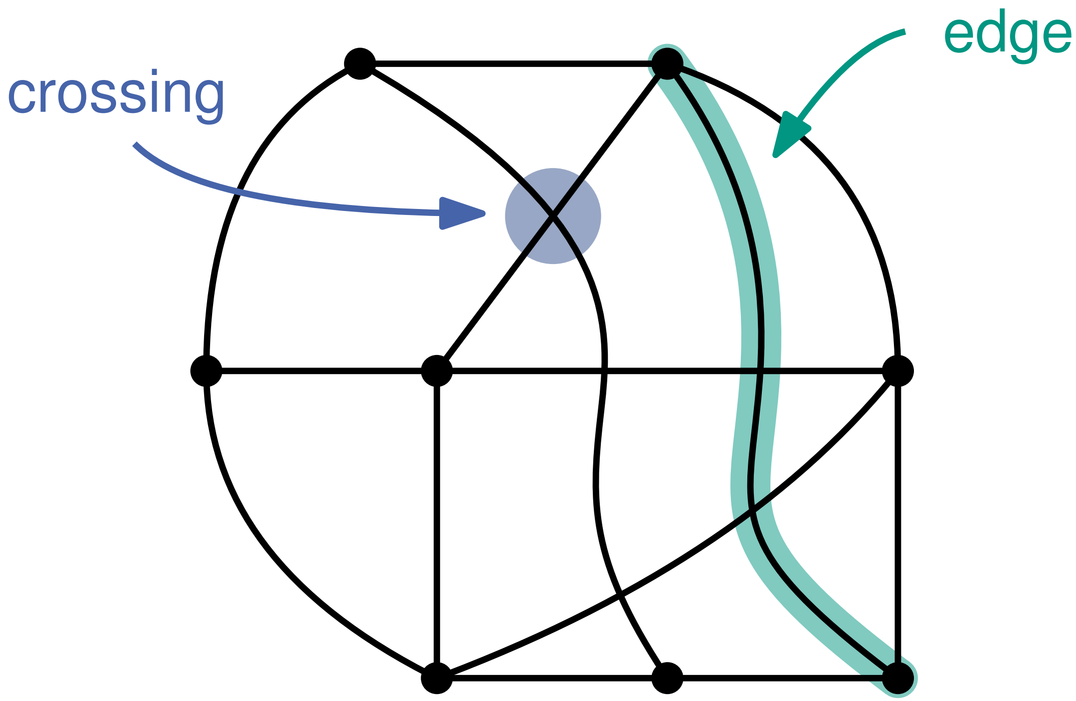

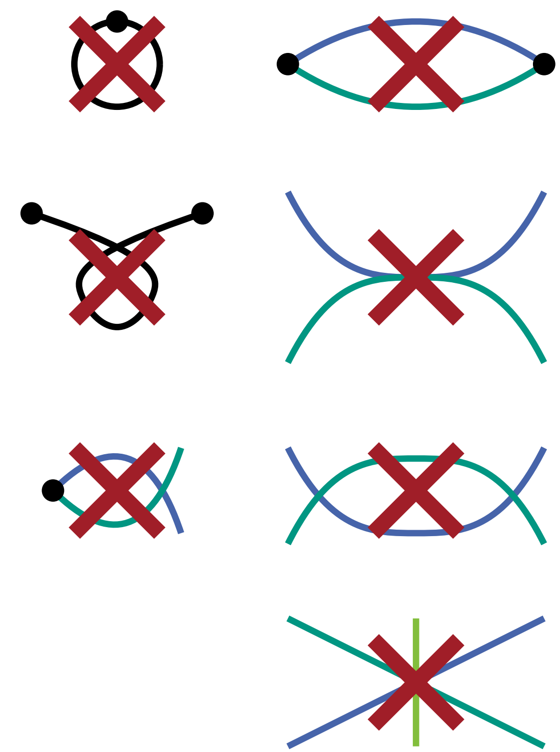

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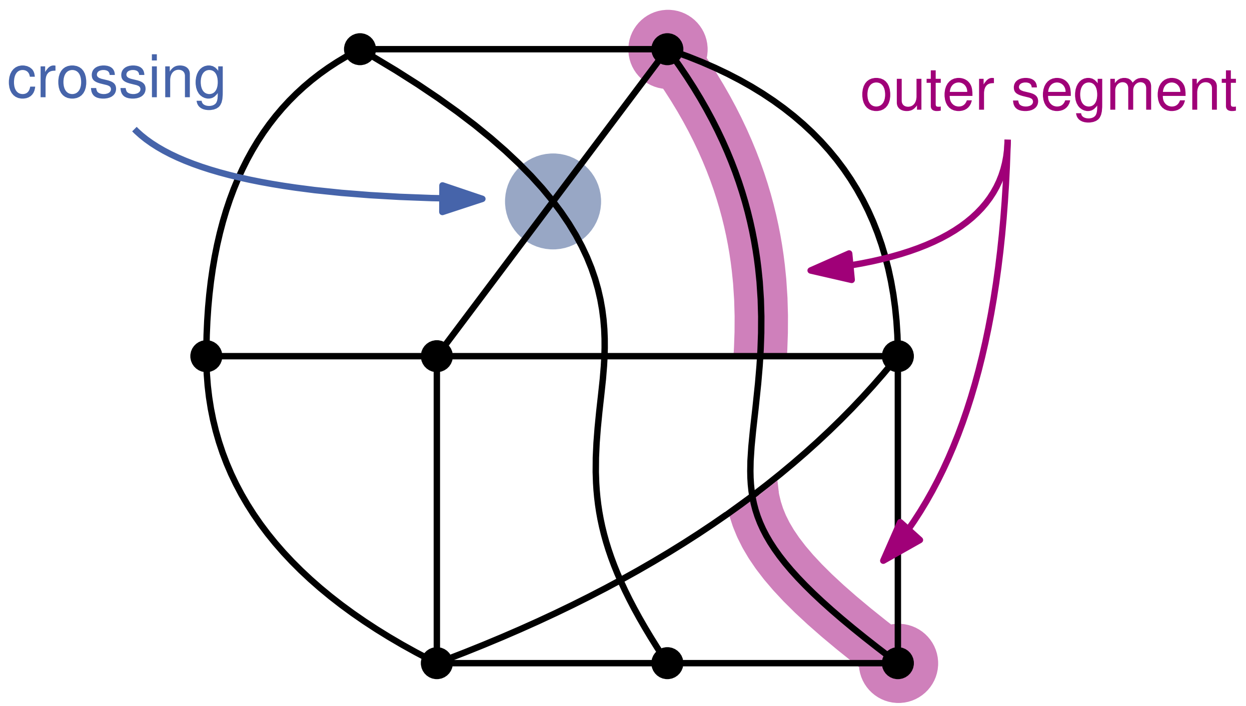

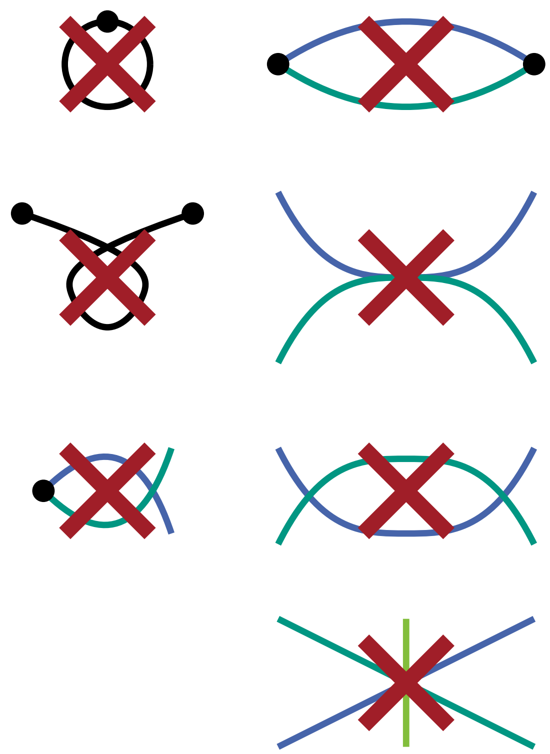

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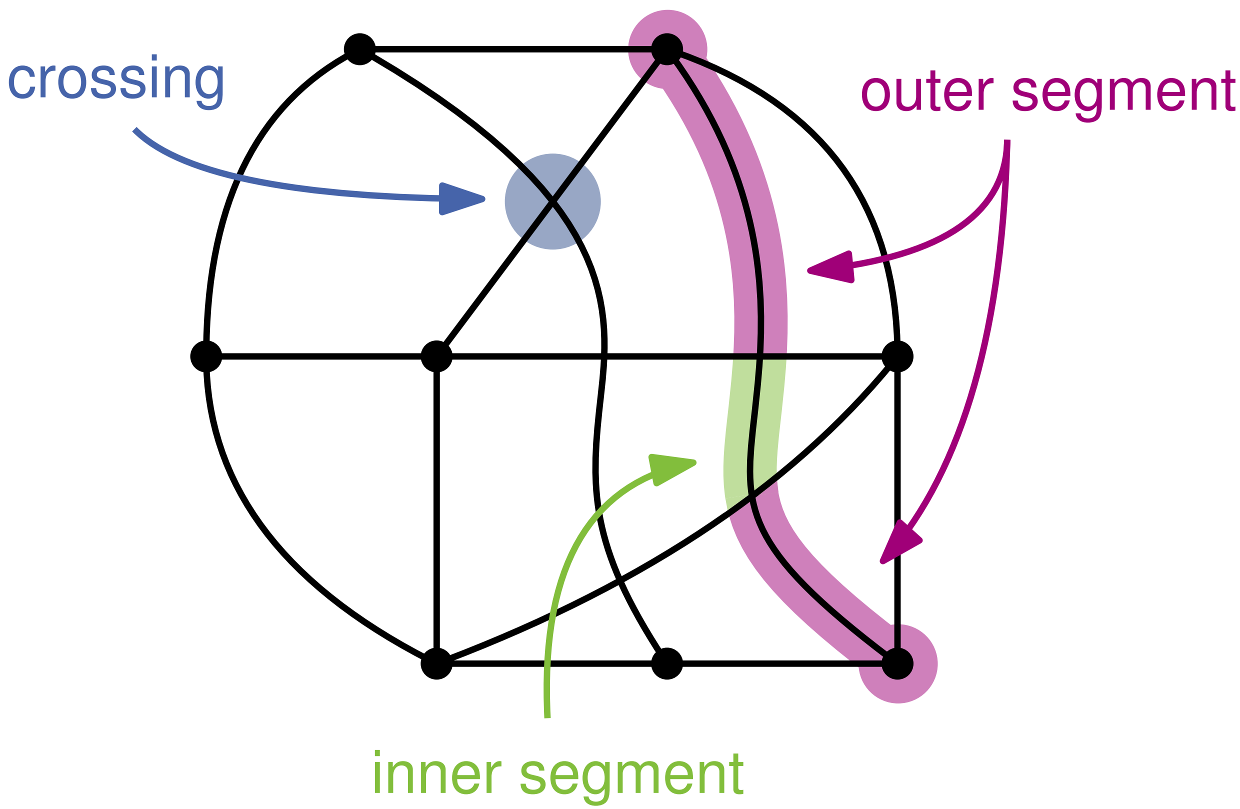

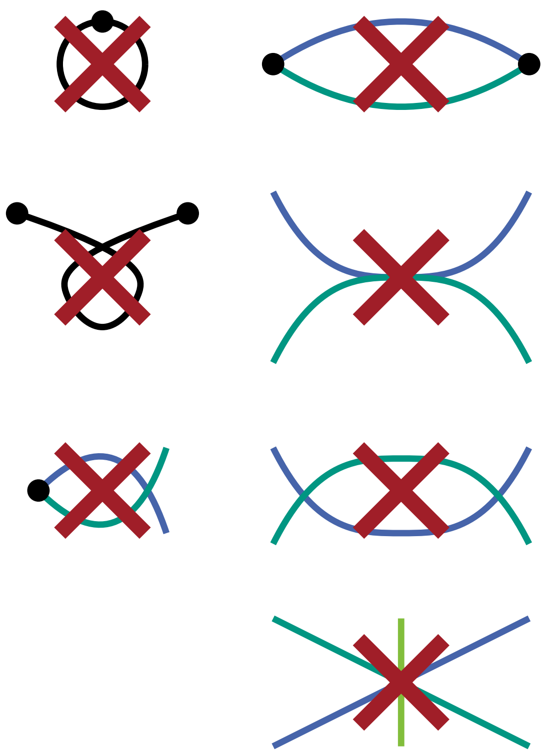

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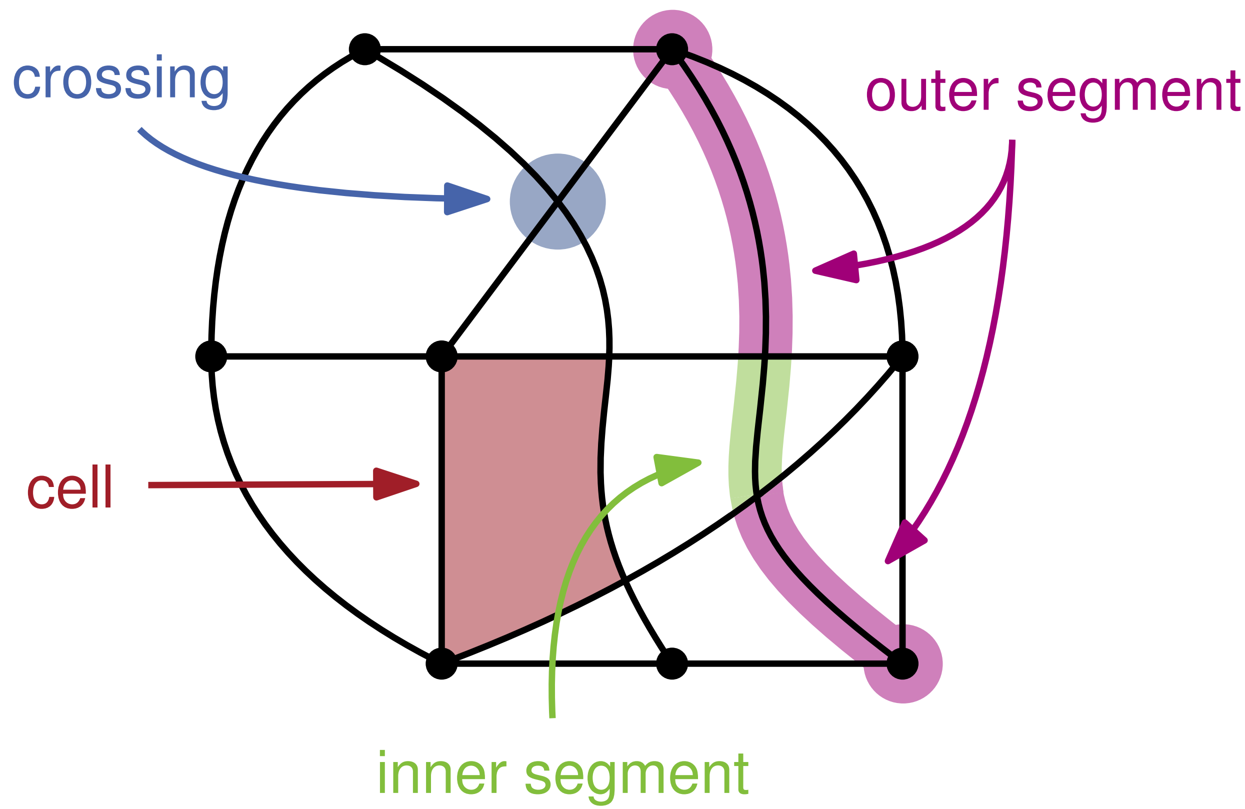
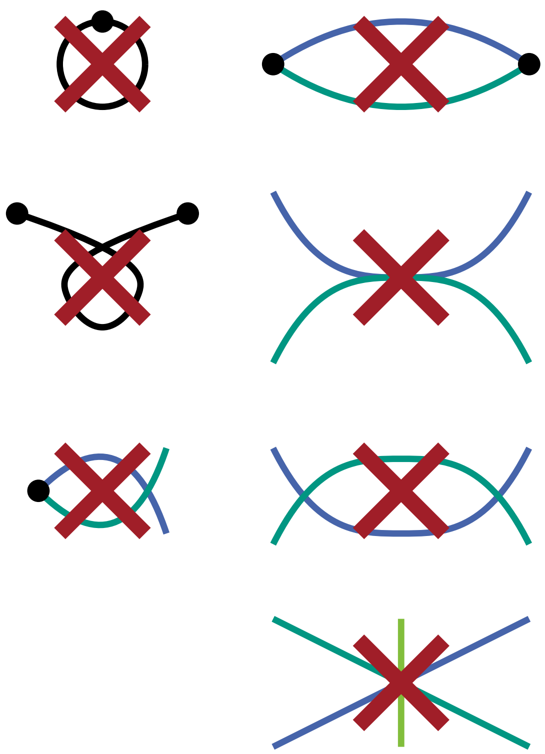
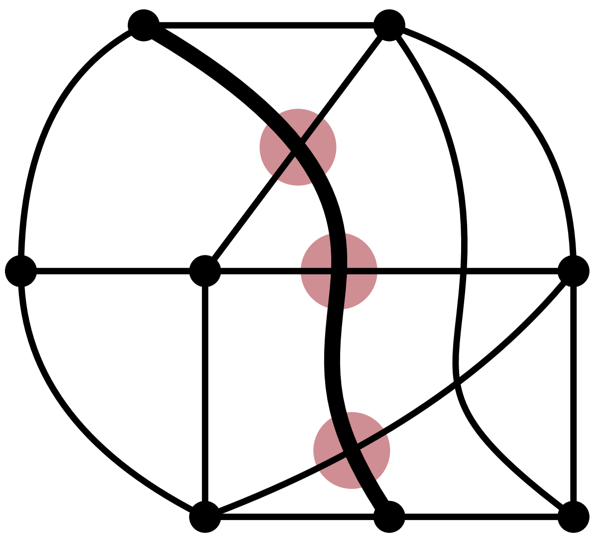




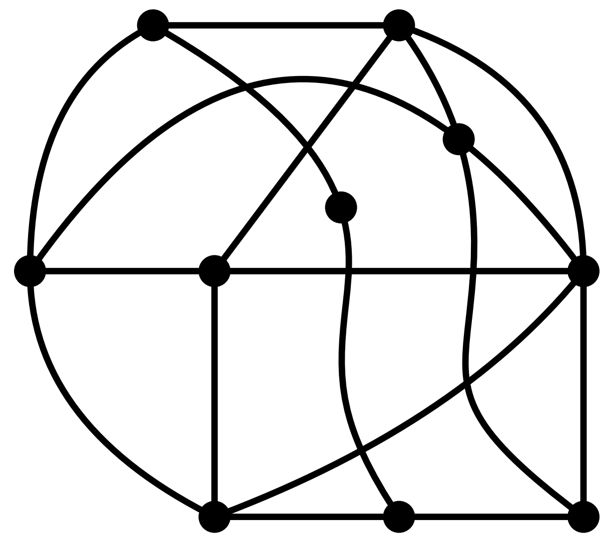
2-Plane Drawings

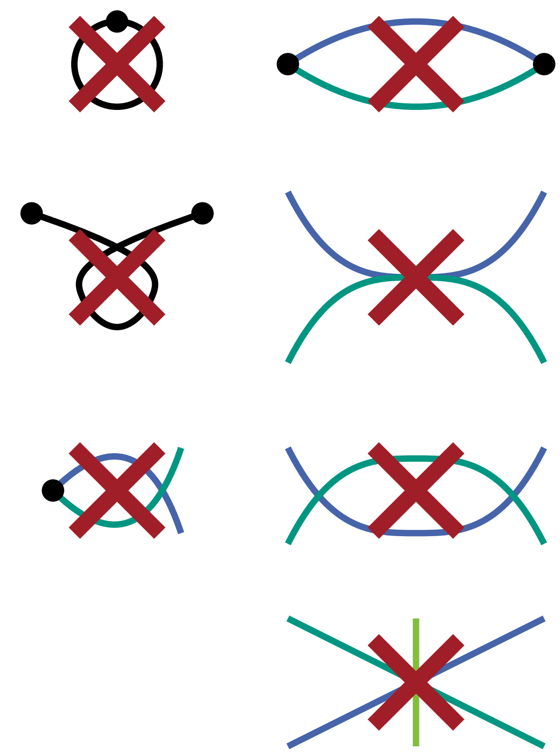
Def. A drawing is **2-plane** if every edge is crossed ≤ 2.

not 2-plane



2-plane





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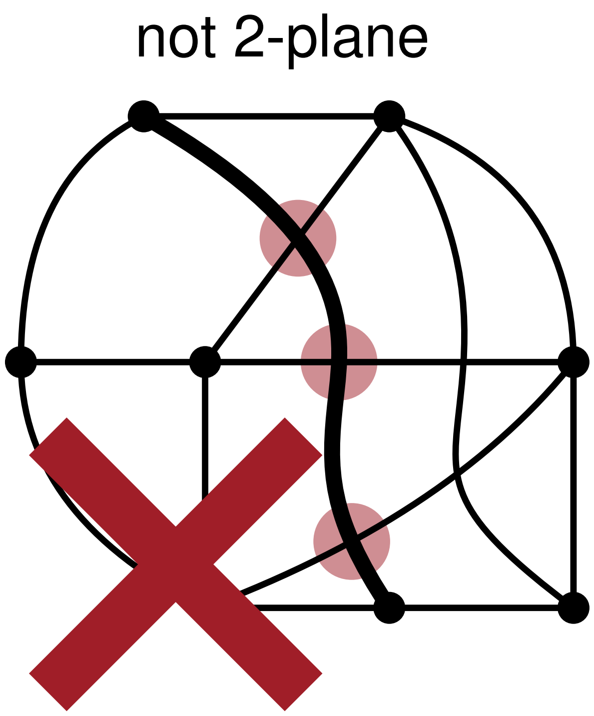

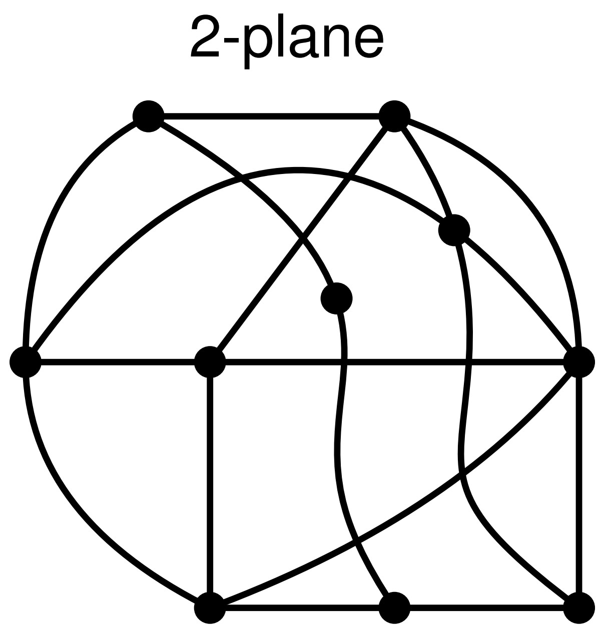

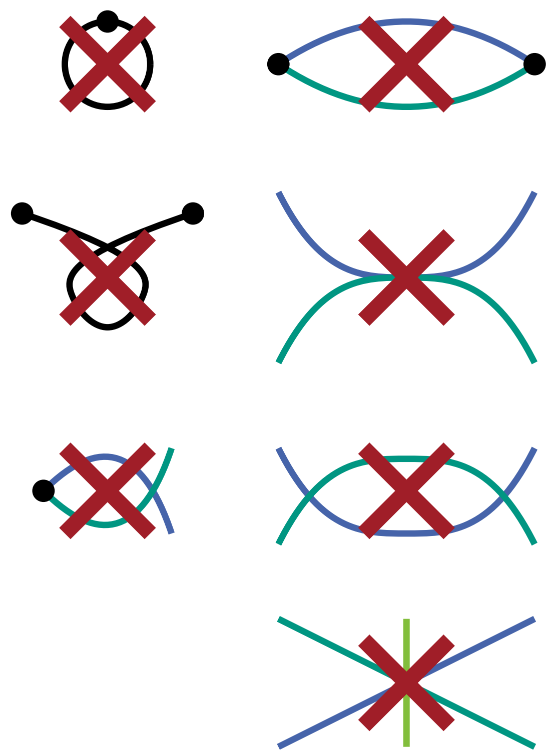

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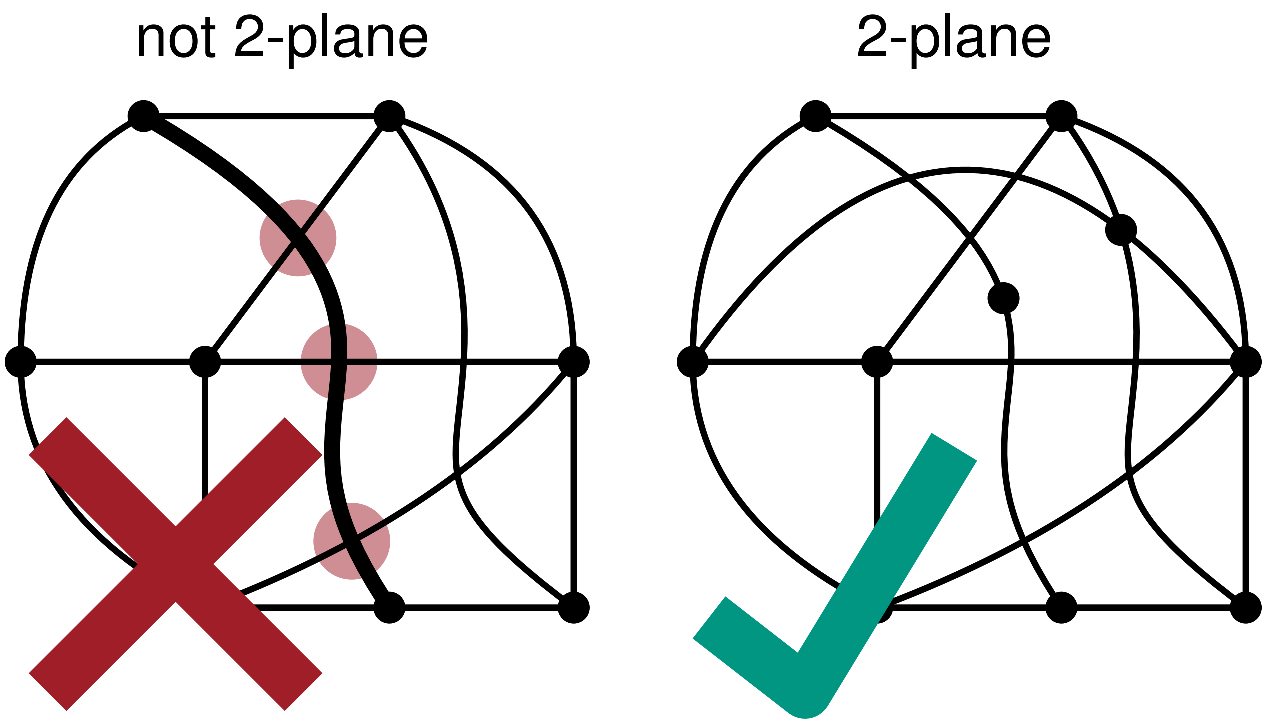
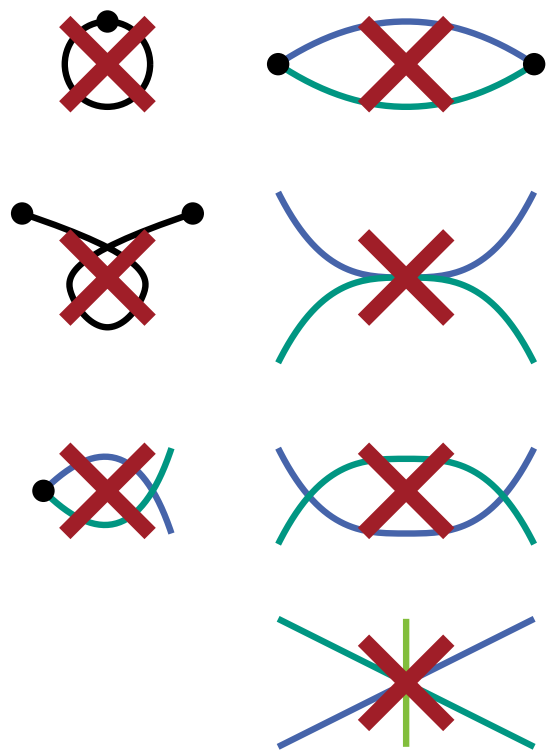
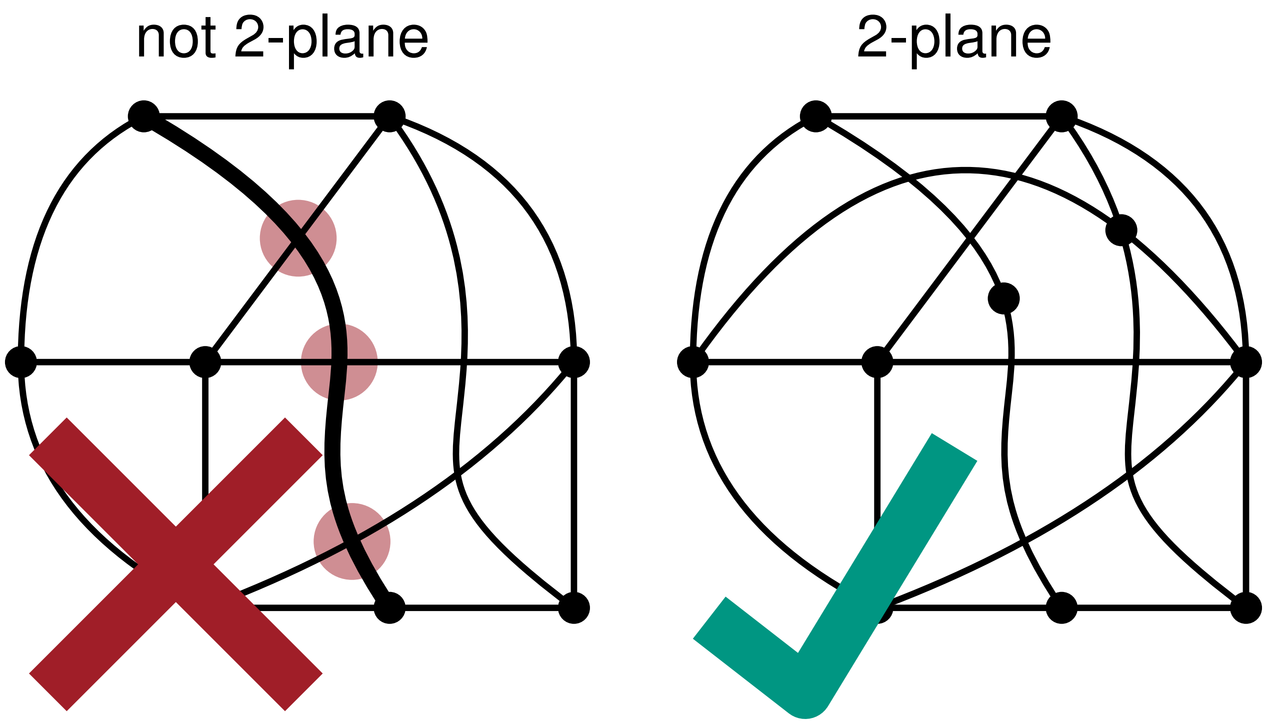


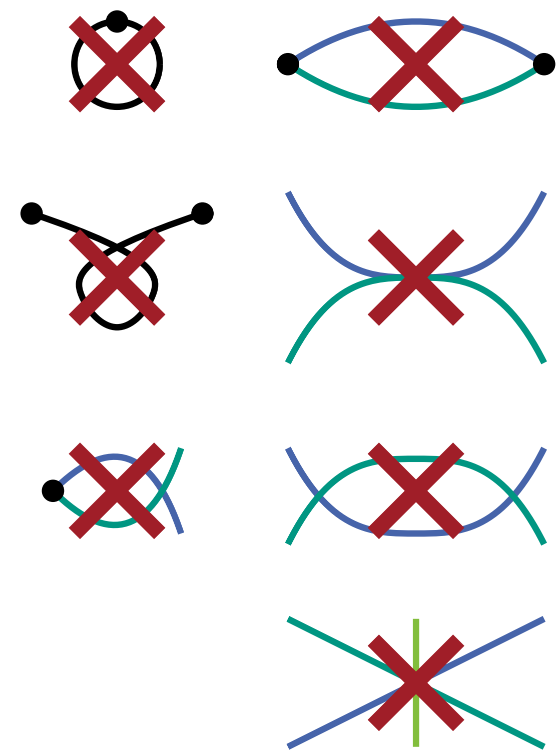


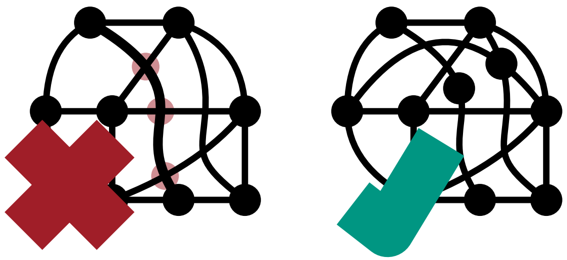
2-Plane Drawings

Def. A drawing is **2-plane** if every edge is crossed ≤ 2.

Question: How many crossings can a 2-plane drawing have?







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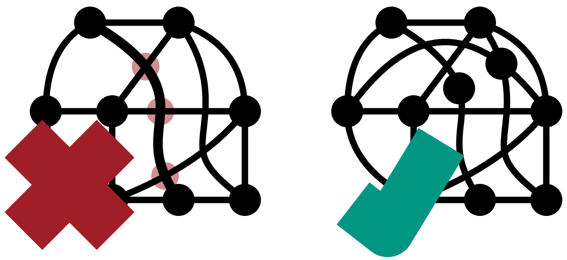

In terms of $m = |E|$:

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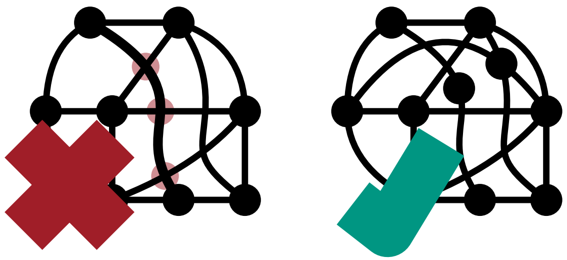
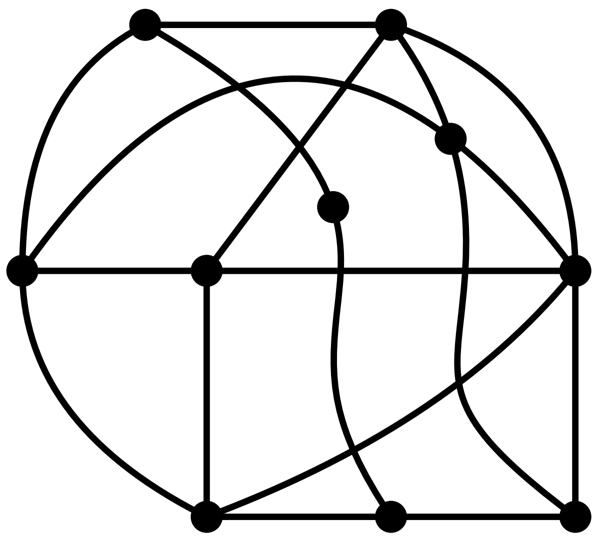


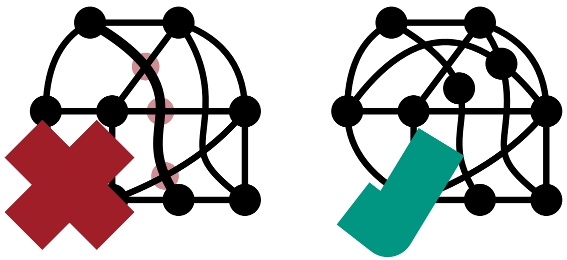
Question: How many crossings can a 2-plane drawing have?

In terms of $m = |E|$:

crossings

$$|\{(e, x) \mid x \in X, e \in E, x \text{ on } e\}|$$





Question: How many crossings can a 2-plane drawing have?

In terms of $m = |E|$:

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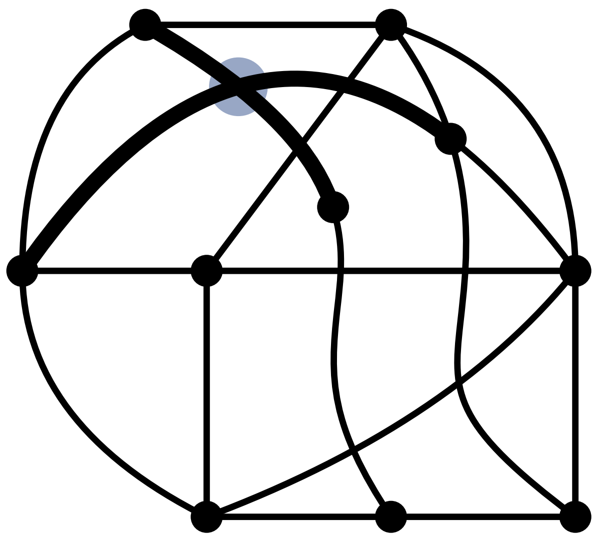
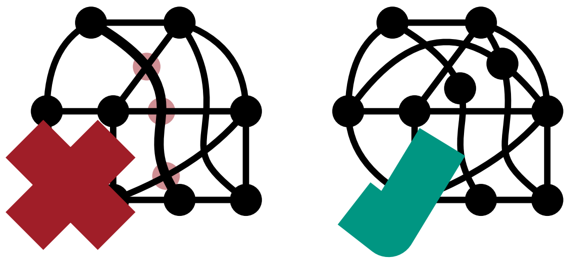
$$2|X| = |\{(e, x) \mid x \in X, e \in E, x \text{ on } e\}|$$





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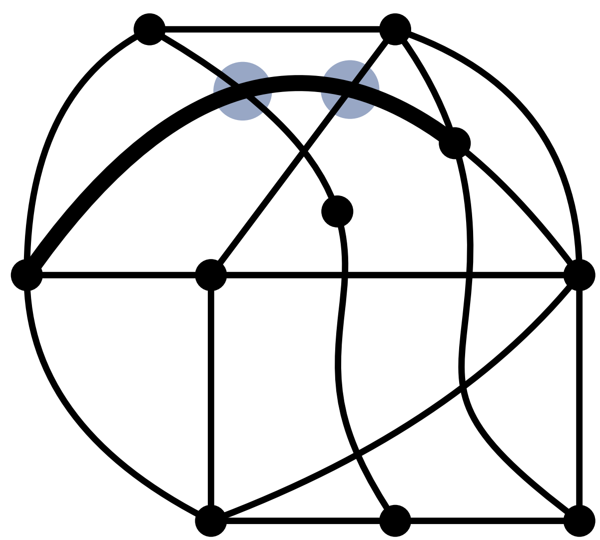

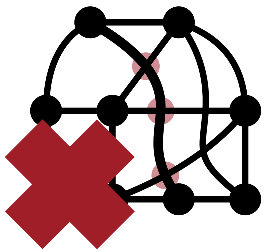

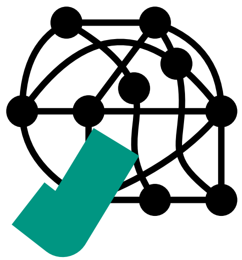

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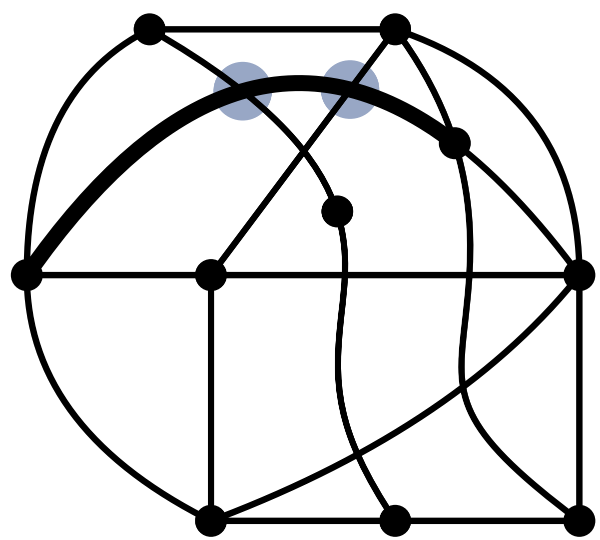
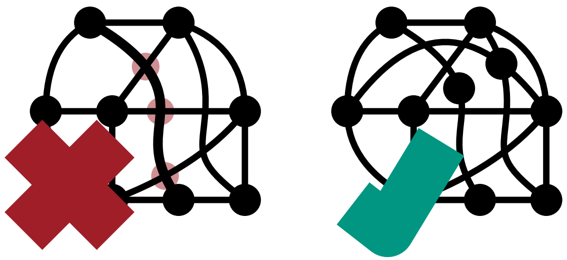
$$|X| \leq m$$





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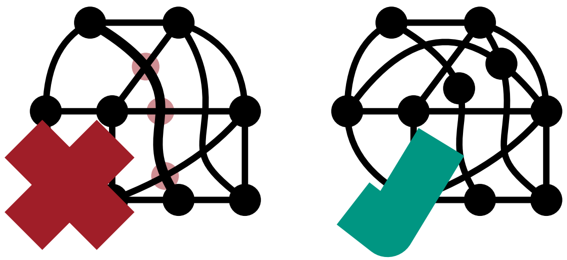

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crossings

$$|X| \leq m$$

Thm: $m \leq 5n - 10.$
(Pach, Tóth)

In terms of $n = |V|$:

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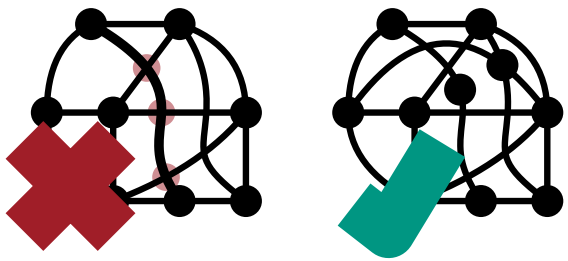

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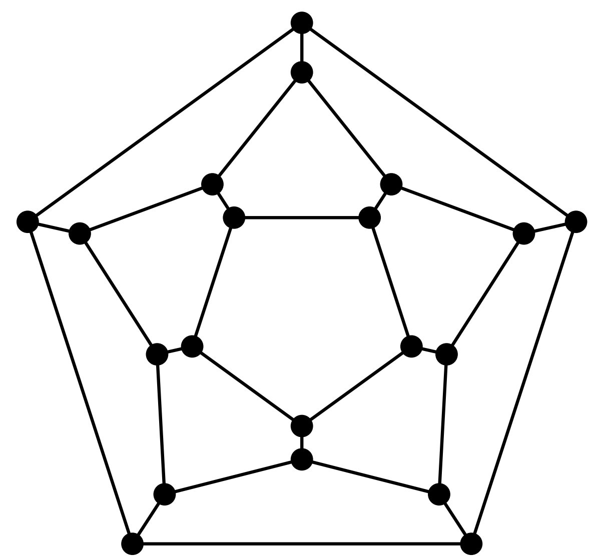

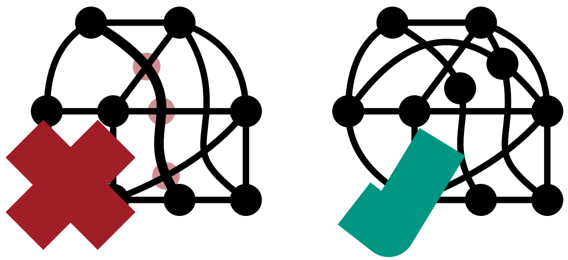

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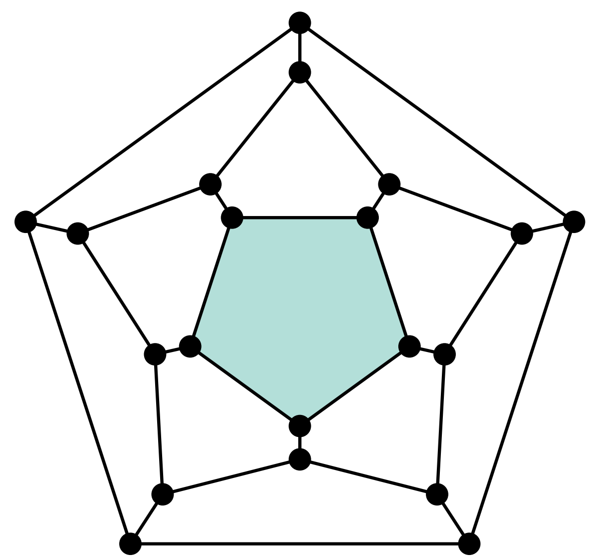

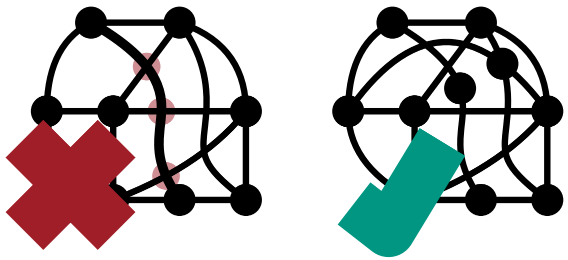

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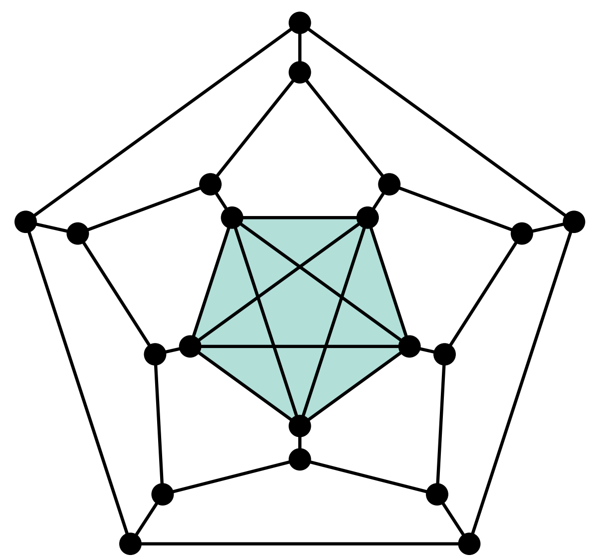

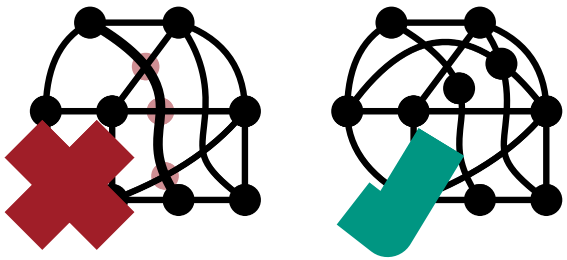

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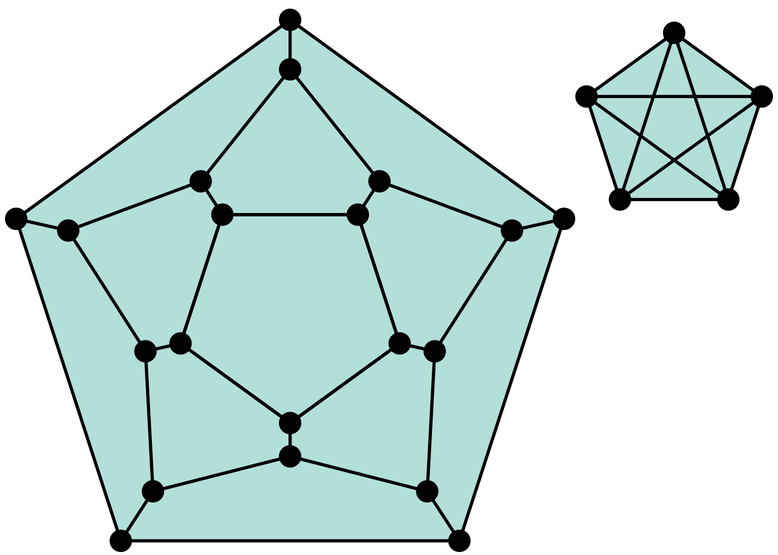

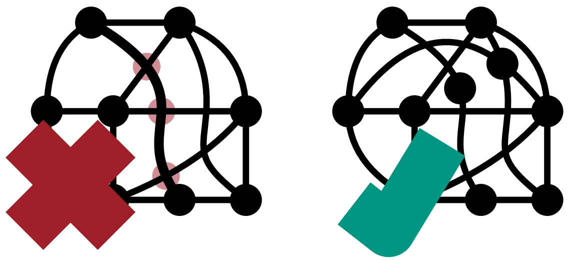

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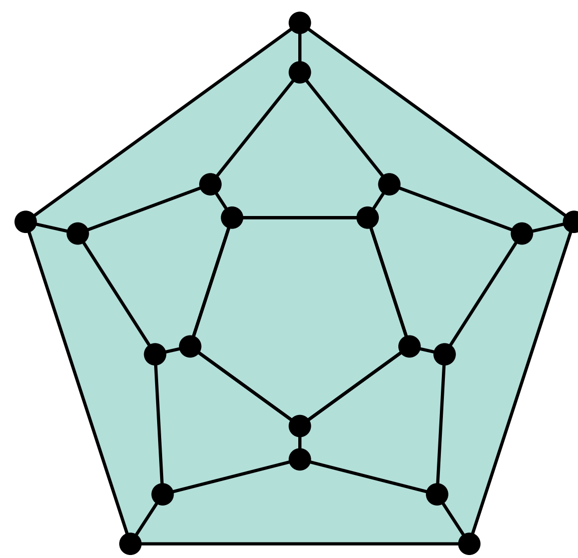
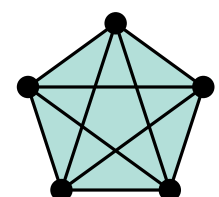
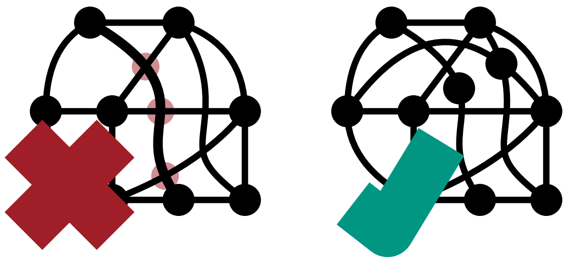




$$m = 5n - 10$$

$$|X| = \frac{10n - 4}{3}$$

$$\leq 3.\overline{3}n$$



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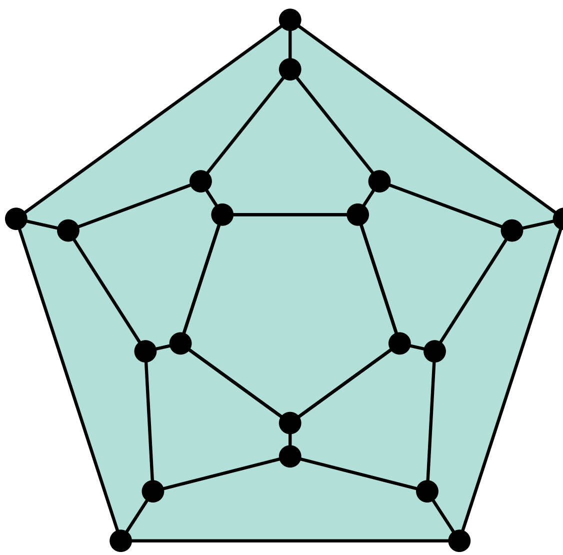
$$|X| \leq m$$

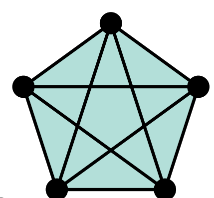
Thm: $m \leq 5n - 10.$
(Pach, Tóth)

In terms of $n = |V|$:

$$|X| \leq 5n - 10$$

Question:
Is $|X| \leq 3.\overline{3}n$ for every 2-plane drawing?





$$m = 5n - 10$$

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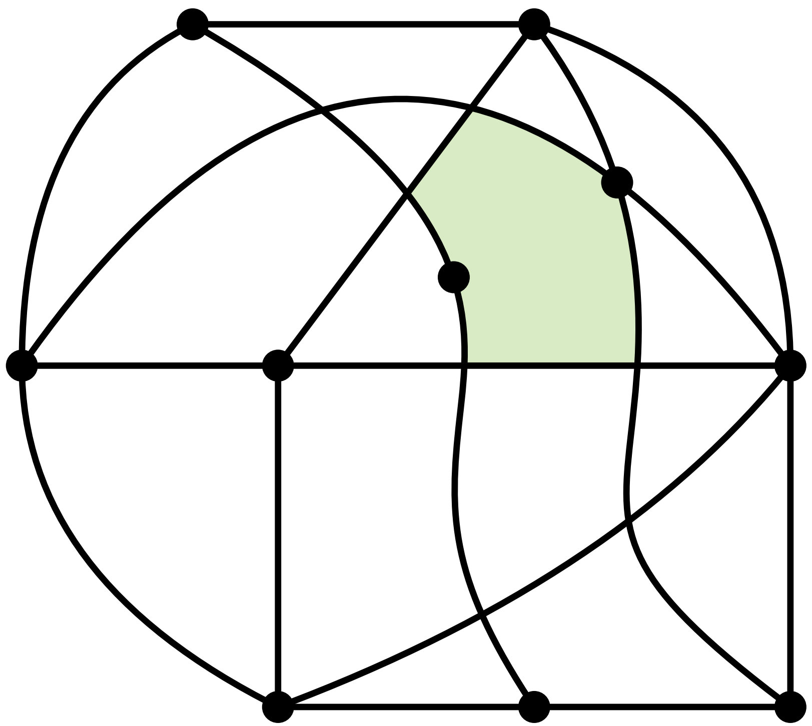
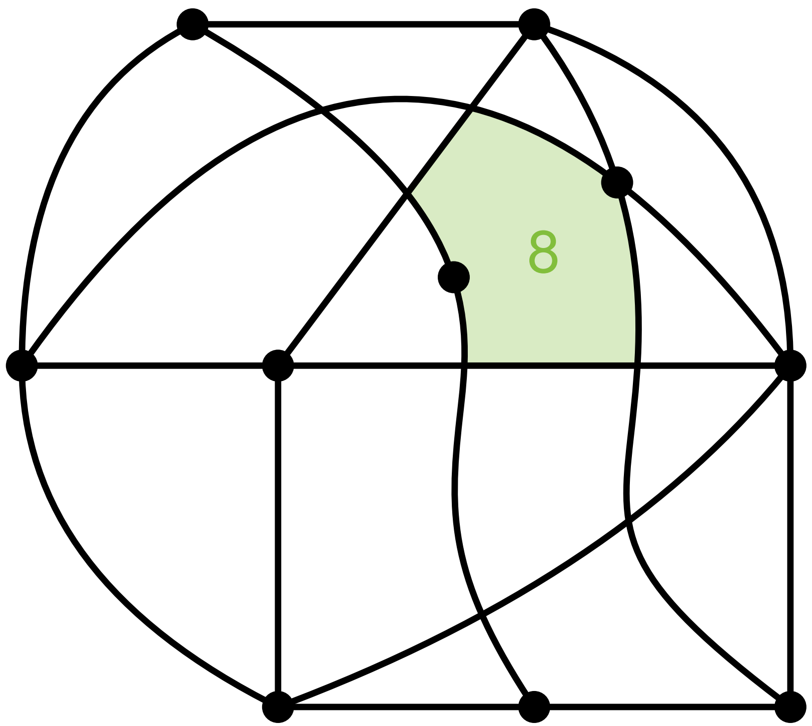
Sizes of Cells



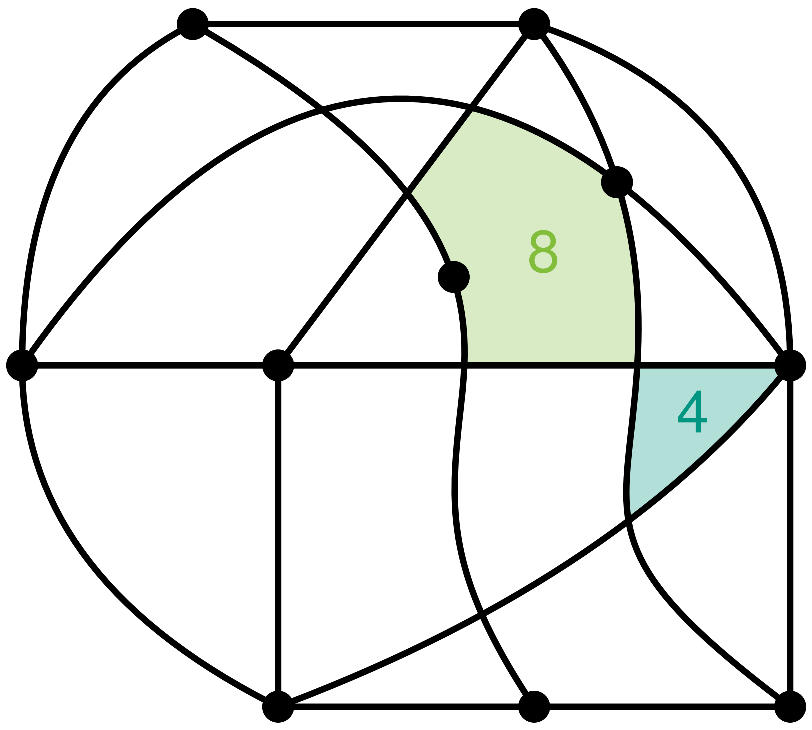
$\|c\| = \#\, \bullet\text{-inc} + \#\text{segment-inc}$

Sizes of Cells



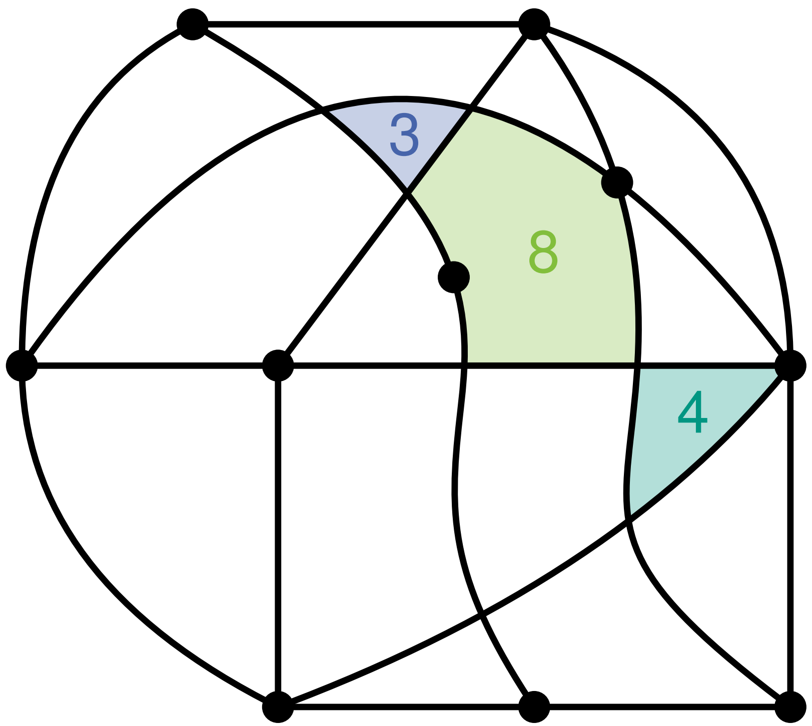
$\|c\| = \#$ ●-inc $+$ #segment-inc

Sizes of Cells



$\|c\| = \#\bullet$-inc $+$ #segment-inc

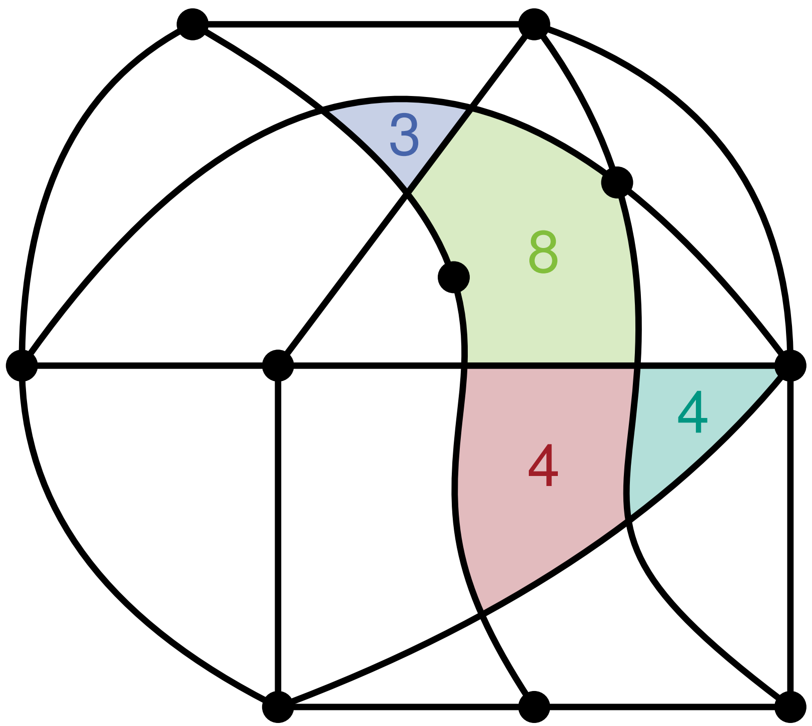
Sizes of Cells



$\|c\| = \#$ ●-inc $+$ #segment-inc

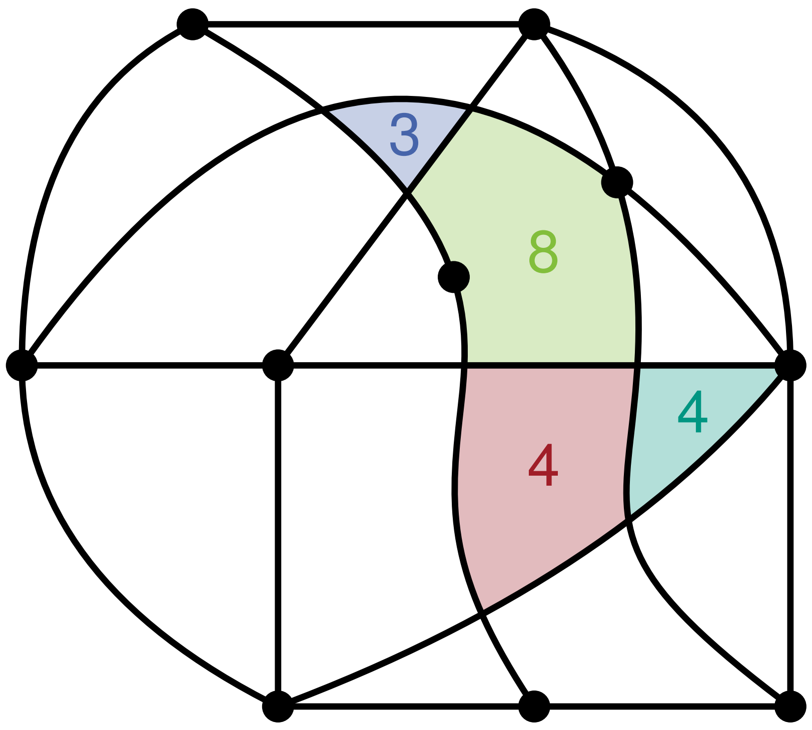
Sizes of Cells

Karlsruher Institut für Technologie



$\|c\| = \#$ ●-inc $+$ $\#$segment-inc

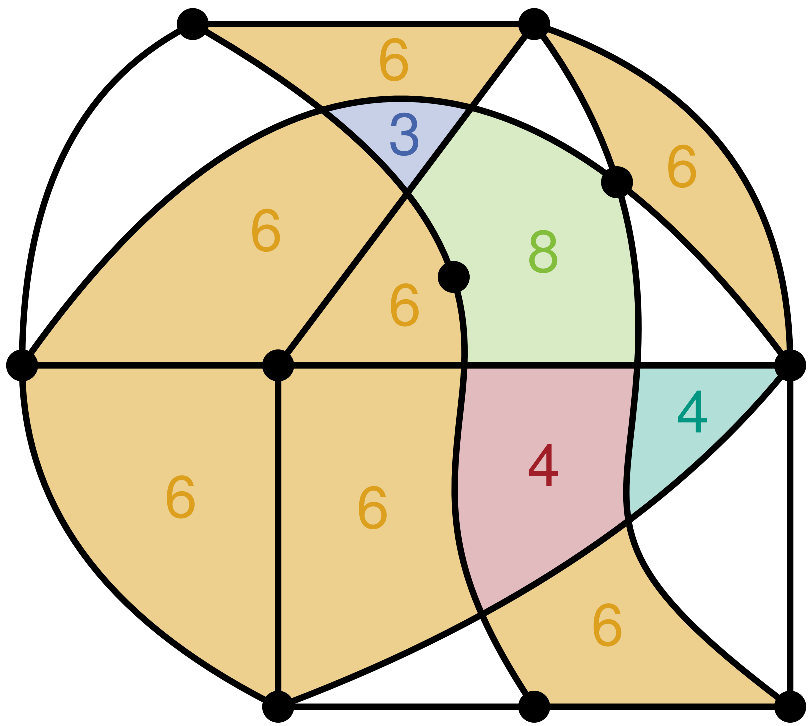
Sizes of Cells



$\|c\| = \#\bullet\text{-inc} + \#\text{segment-inc}$

$\mathcal{C}_i = \{\text{cell } c \colon \|c\| = i\}$

Sizes of Cells

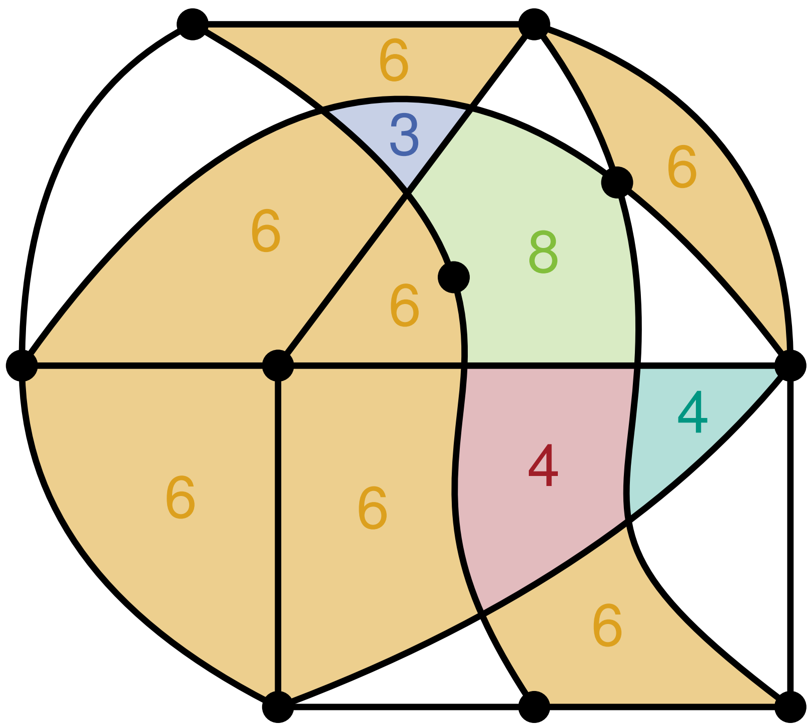


$\|c\| = \#\,\bullet\text{-inc} + \#\text{segment-inc}$

$\mathcal{C}_i = \{\text{cell } c : \|c\| = i\}$

$\mathcal{C}_6$

Sizes of Cells



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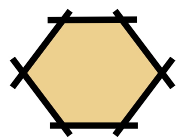
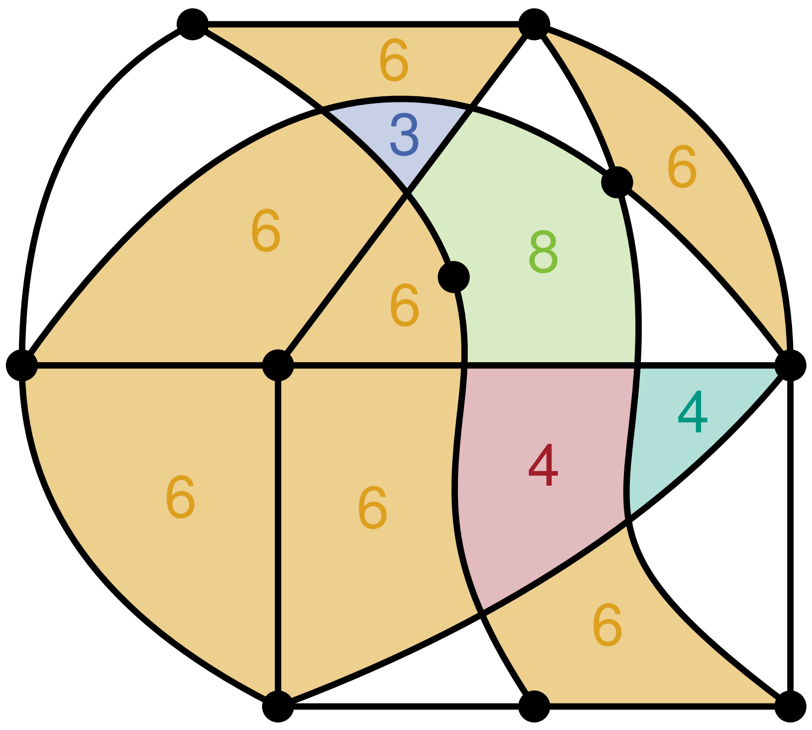
$\mathcal{C}_6$ 

Sizes of Cells



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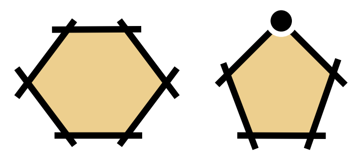
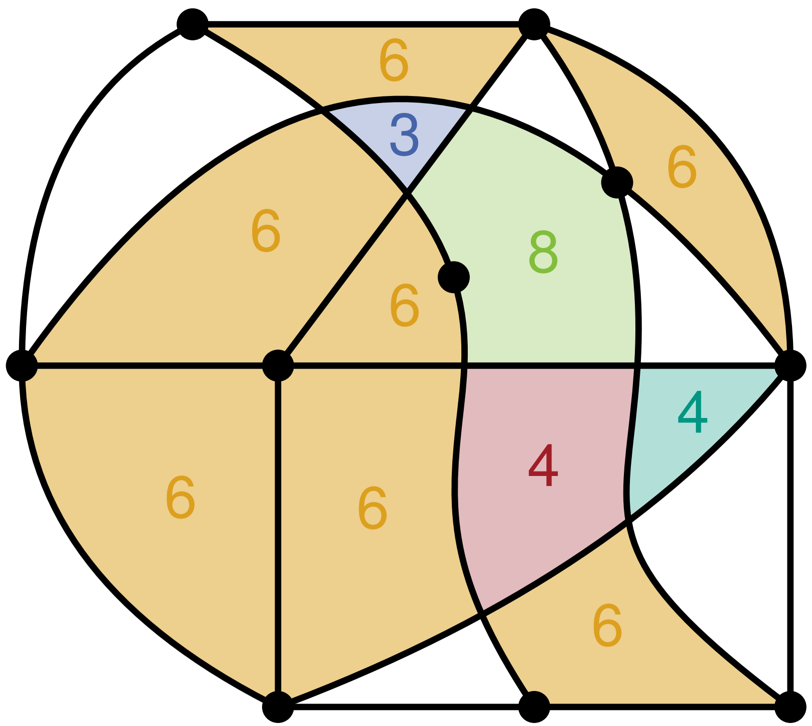
$\mathcal{C}_6$ 

Sizes of Cells



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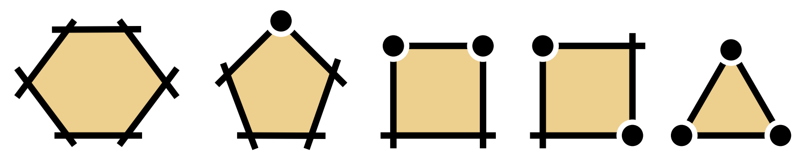
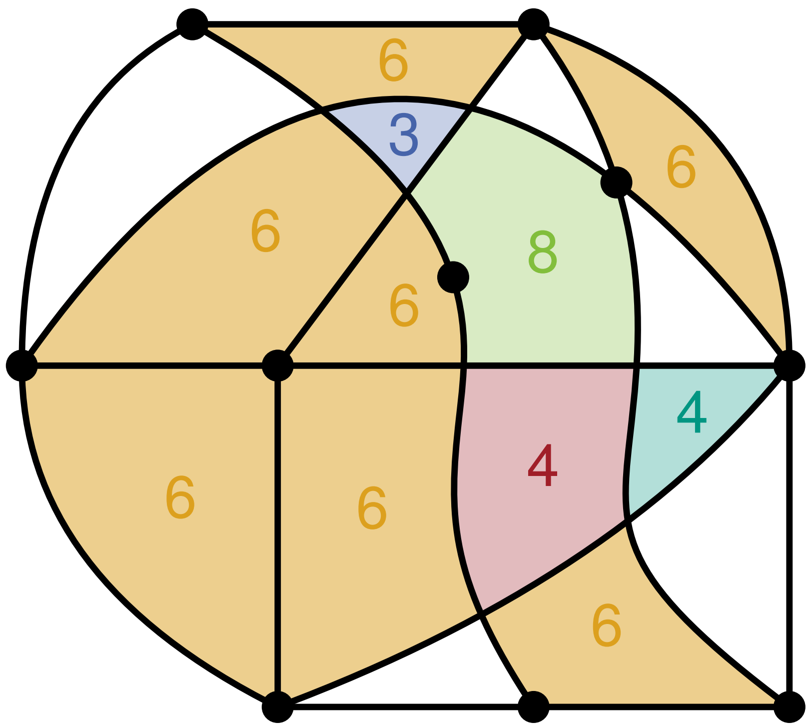


Sizes of Cells



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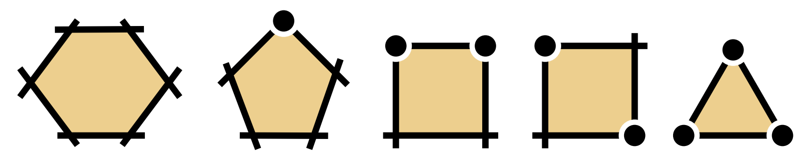
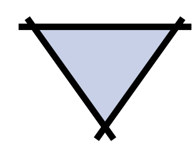
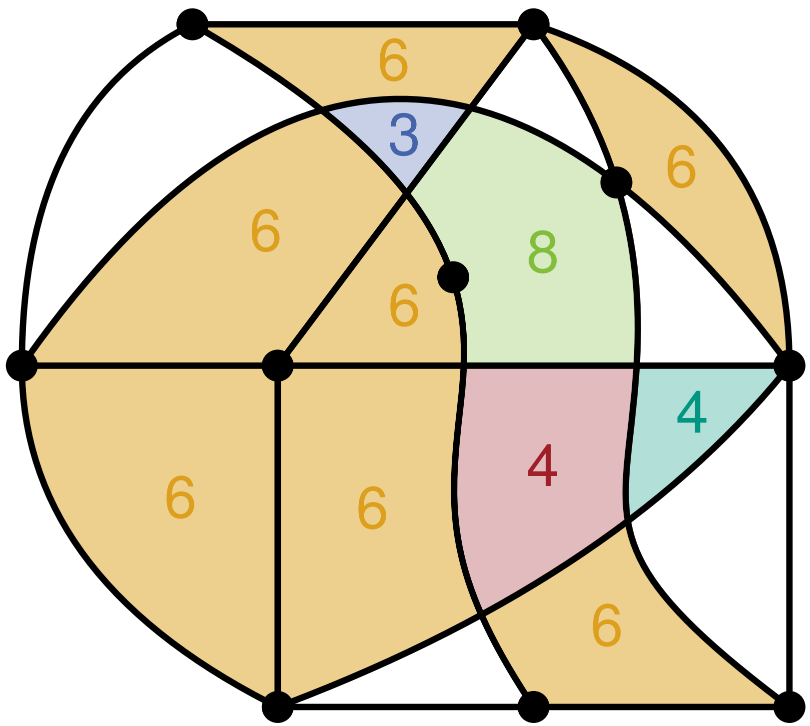


$\mathcal{C}_3$



Sizes of Cells



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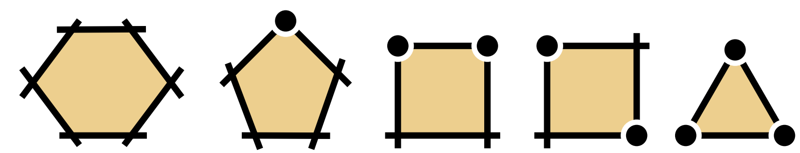
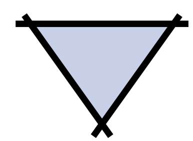
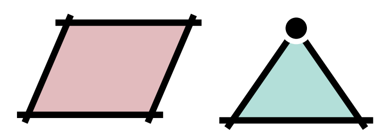


$\mathcal{C}_3$



$\mathcal{C}_4$

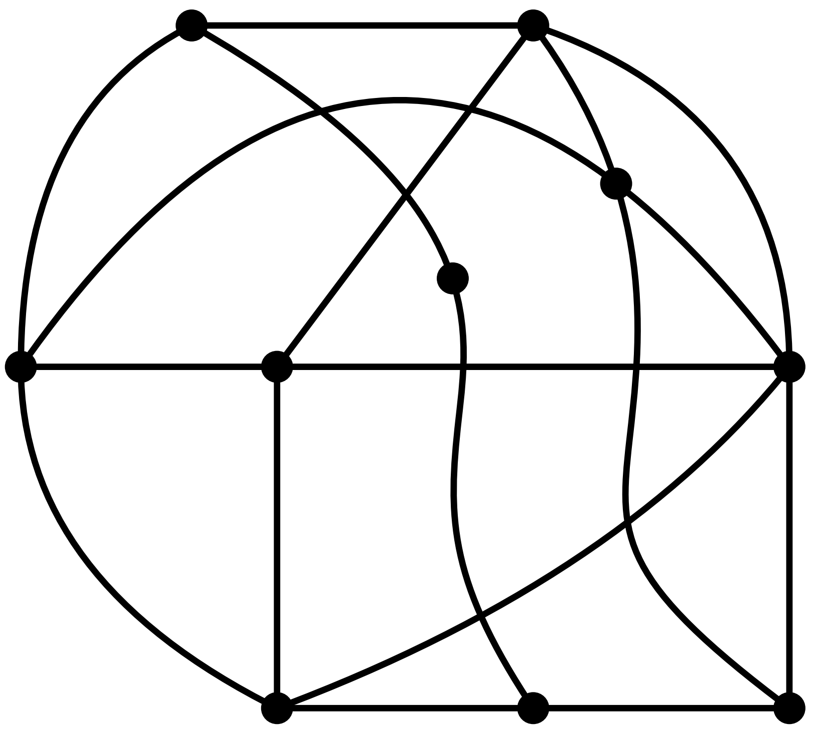


Density Formula

Lemma: For every $t \in \mathbb{R}$ and every connected drawing of a graph $G = (V, E)$ with $|E| \geq 1$, we have

$$|X| = t \cdot (|V| - 2) + \sum_{i=1}^{\infty} \left(t + \frac{1-t}{4} i\right) |\mathcal{C}_i| - |E|$$

(Kaufmann, Klemz, Knorr, Reddy, Schröder, Ueckerdt 2024)



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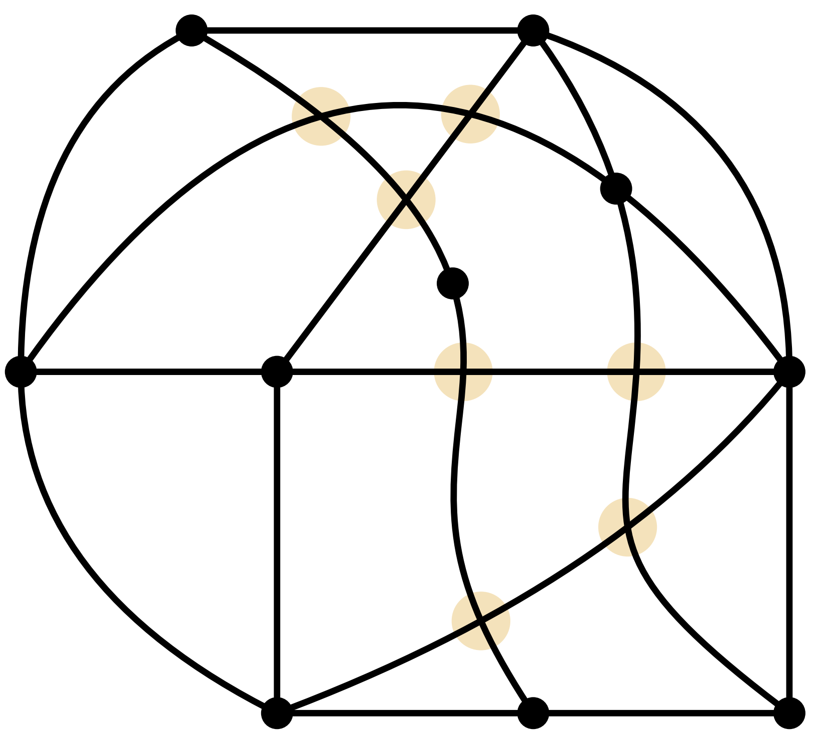

$7 =$

Density Formula

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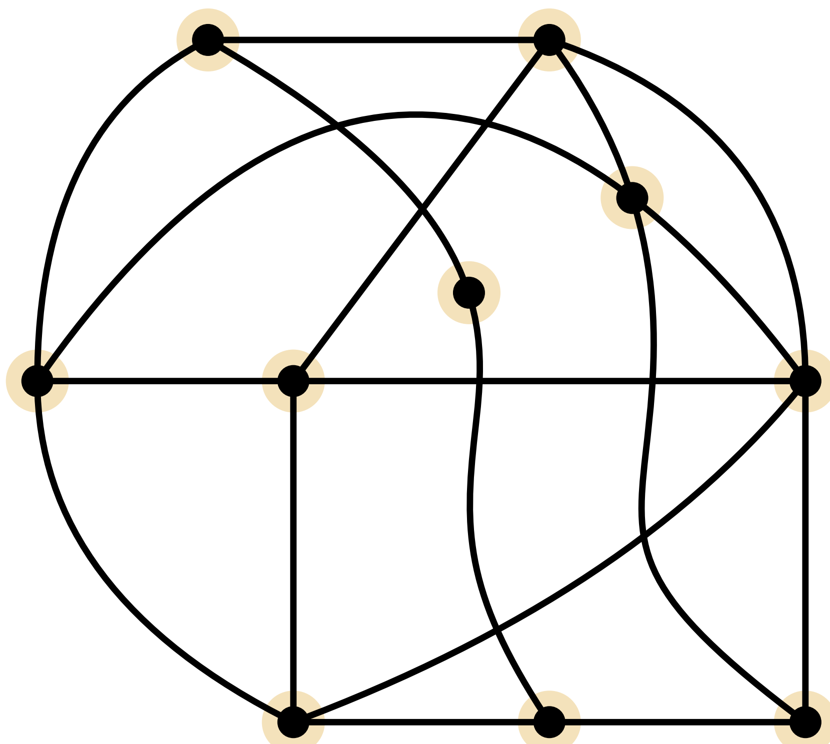

$$7 = t \cdot (10 - 2)$$

Density Formula

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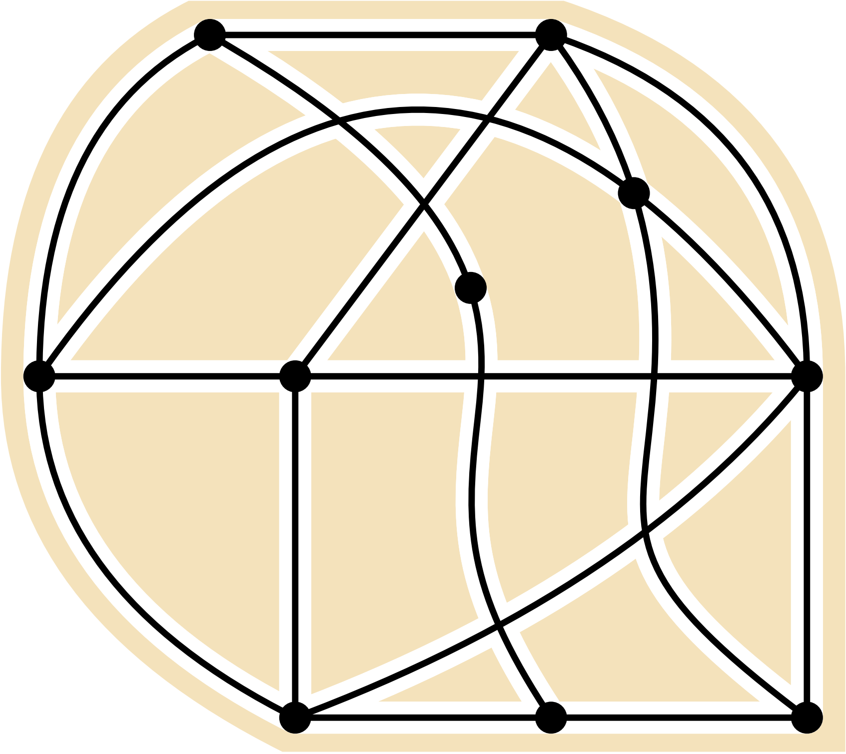

$$7 = t \cdot (10 - 2) + \sum_{i=1}^{\infty} w_{i,t} |\mathcal{C}_i|$$

Density Formula

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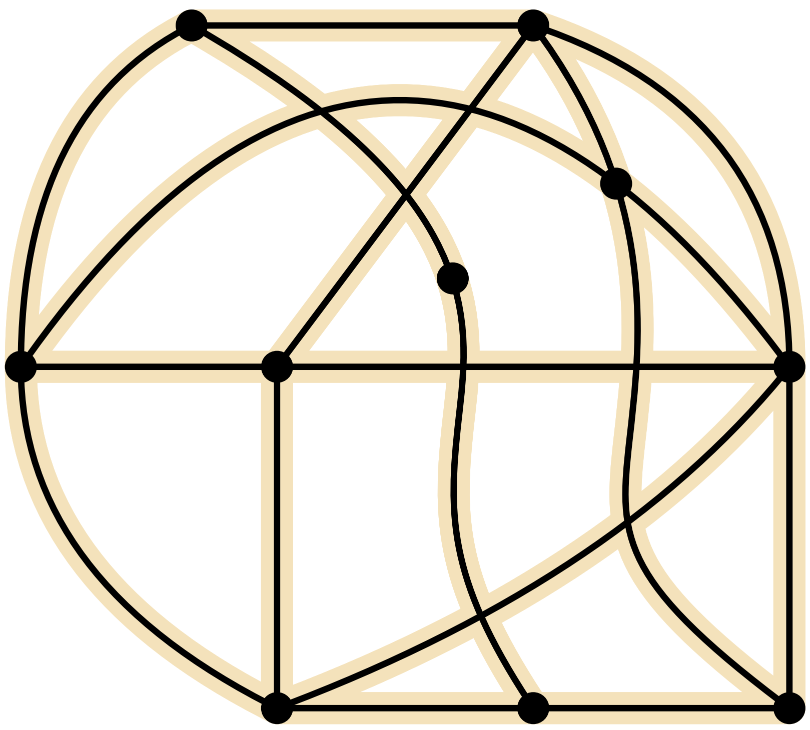

$$7 = t \cdot (10 - 2) + \sum_{i=1}^{\infty} w_{i,t} |\mathcal{C}_i| - 18$$

Density Formula

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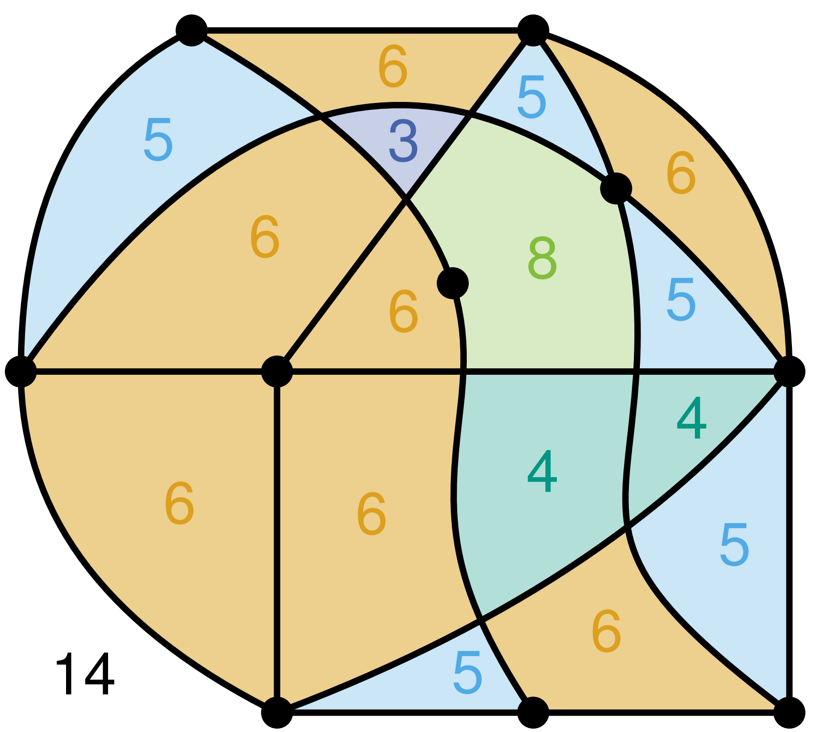

$$7 = t \cdot (10 - 2) + \sum_{i=1}^{\infty} w_{i,t} |\mathcal{C}_i| - 18$$

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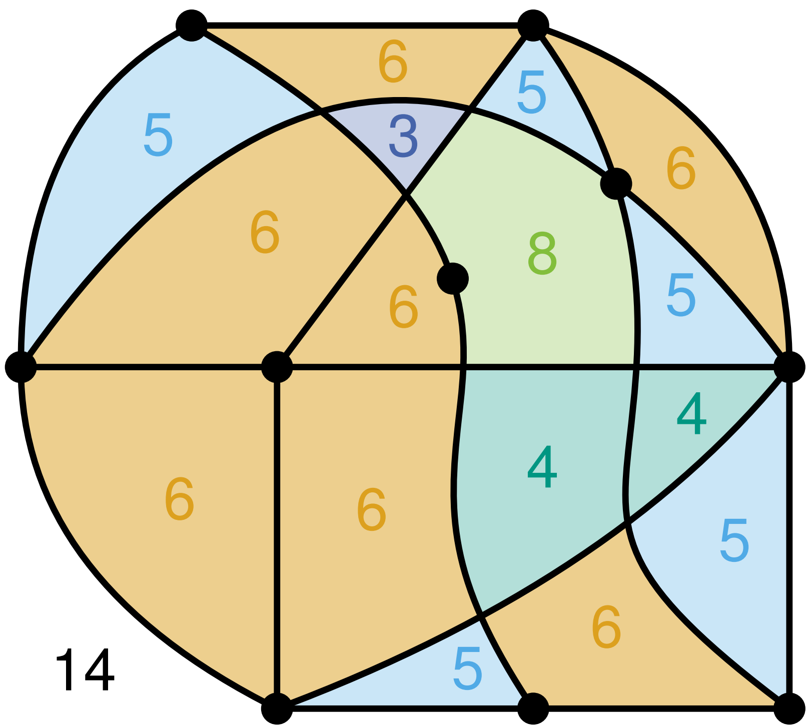

$\boldsymbol{t = 5}$ $\quad |V| \geq 3$

$$|X| = 5(|V| - 2) + \sum_{i=3}^{\infty} (5 - i) |\mathcal{C}_i| - |E|$$

$$7 = t \cdot (10 - 2) + \sum_{i=1}^{\infty} w_{i,t} |\mathcal{C}_i| - 18$$

Density Formula

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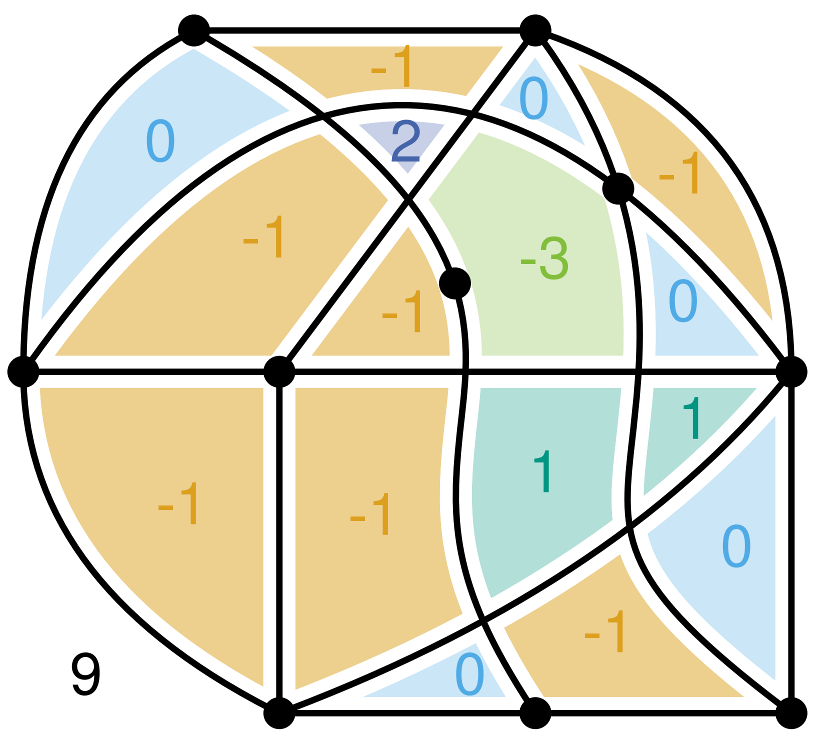

$\boldsymbol{t = 5}$ $\quad |\boldsymbol{V}| \geq \mathbf{3}$

$$|X| = 5(|V| - 2) + \sum_{i=3}^{\infty} (5 - i)\, |\mathcal{C}_i| - |E|$$

Density Formula

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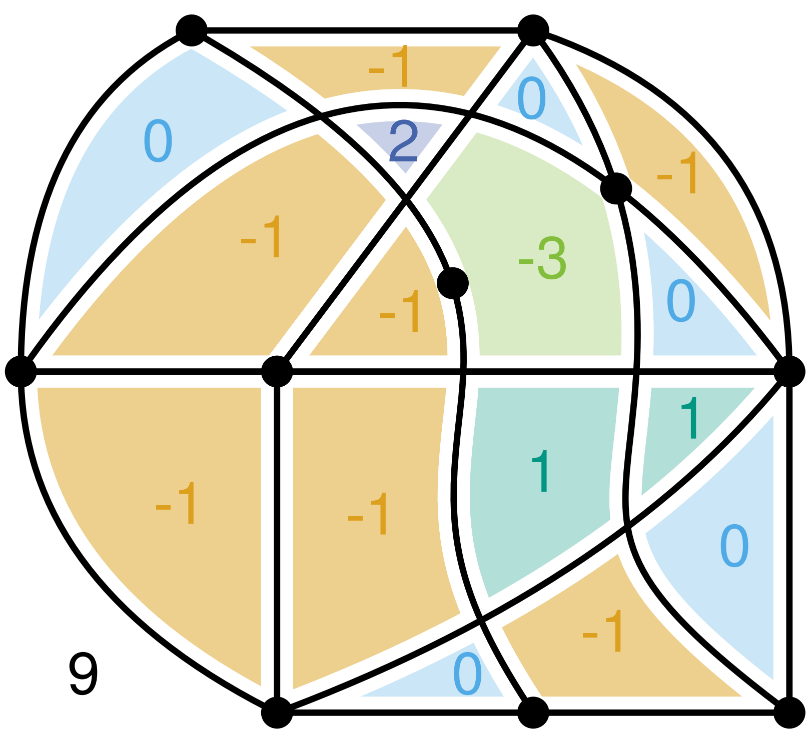


$\boldsymbol{t = 5}$ $|V| \geq 3$

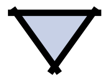
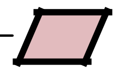
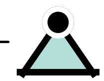
$$|X| = 5(|V| - 2) + \sum_{i=3}^{\infty} (5 - i) |\mathcal{C}_i| - |E|$$

$$|X| = 5(|V| - 2) + 2 \triangledown + \square + \triangle - \sum_{i=6}^{\infty} (i - 5)|\mathcal{C}_i| - |E|$$

First Attempt

Question: Is $|X| \leq \frac{10}{3}n$ for every 2-plane drawing?

Obs: May assume no plane edges.

$|X| = 5(n-2) + 2$  $+$  $+$  $- \sum_{i=6}^{\infty}(i-5)|\mathcal{C}_i| - m$

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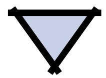
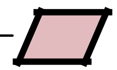
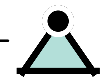
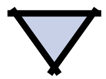
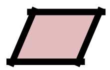
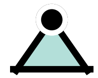
$|X| = 5(n-2) + 2$  $+$  $+$  $- \sum_{i=6}^{\infty}(i-5)|\mathcal{C}_i| - m$

Idea: bound , ,  in terms of $|X|$, m and $\sum_{i=6}^{\infty}(i-5)|\mathcal{C}_i|$

First Attempt

Question: Is $|X| \leq \frac{10}{3}n$ for every 2-plane drawing?

Obs: May assume no plane edges.

$$|X| = 5(n-2) + 2\,\triangledown + ▱ + \dot{\triangle} - \sum_{i=6}^{\infty}(i-5)|\mathcal{C}_i| - m$$

Idea: bound $\triangledown$, ▱, $\dot{\triangle}$ in terms of $|X|$, m and $\sum_{i=6}^{\infty}(i-5)|\mathcal{C}_i|$

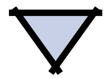
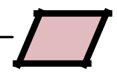
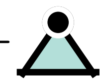
Observation:

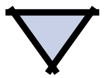
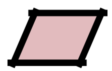
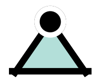
$$|X| \geq 3\,\triangledown$$

First Attempt

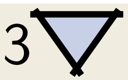
Question: Is $|X| \leq \frac{10}{3}n$ for every 2-plane drawing?

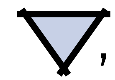
Obs: May assume no plane edges.

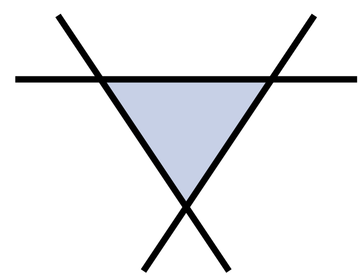
$|X| = 5(n-2) + 2$  $+$  $+$  $- \sum_{i=6}^{\infty}(i-5)|\mathcal{C}_i| - m$

Idea: bound , ,  in terms of $|X|$, m and $\sum_{i=6}^{\infty}(i-5)|\mathcal{C}_i|$

Observation:

$|X| \geq 3$ 

$|\{(x, c) \mid x \in X, c$ is , x on c$\}|$



First Attempt

Question: Is $|X| \leq \frac{10}{3}n$ for every 2-plane drawing?

Obs: May assume no plane edges.

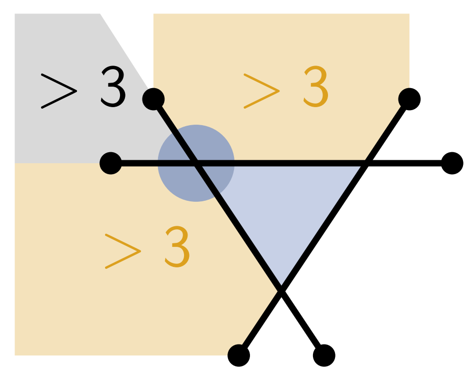
$$|X| = 5(n-2) + 2\,\bigtriangledown + \square + \triangle - \sum_{i=6}^{\infty}(i-5)|\mathcal{C}_i| - m$$

Idea: bound $\bigtriangledown$, $\square$, $\triangle$ in terms of $|X|$, m and $\sum_{i=6}^{\infty}(i-5)|\mathcal{C}_i|$

Observation:

$|X| \geq 3\,\bigtriangledown$

$|\{(x, c) \mid x \in X, c$ is $\bigtriangledown$, x on c$\}|$



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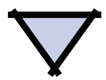
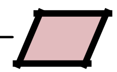
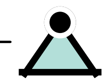
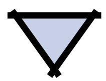
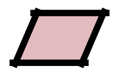
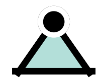
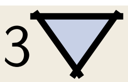
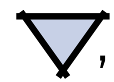
Obs: May assume no plane edges.

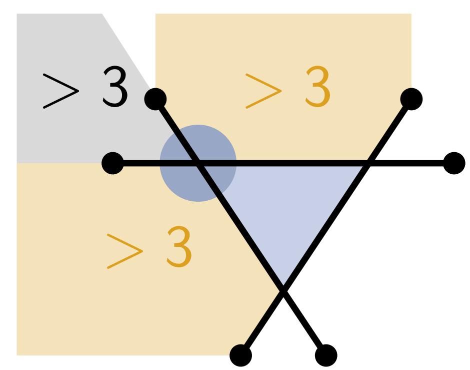
$|X| = 5(n-2) + 2$  $+$  $+$  $- \sum_{i=6}^{\infty}(i-5)|\mathcal{C}_i| - m$

Idea: bound , ,  in terms of $|X|$, m and $\sum_{i=6}^{\infty}(i-5)|\mathcal{C}_i|$

Observation:

$|X| \geq 3$ 

$|X| \geq |\{(x, c) \mid x \in X, c$ is , x on c$\}|$



First Attempt

Question: Is $|X| \leq \frac{10}{3}n$ for every 2-plane drawing?

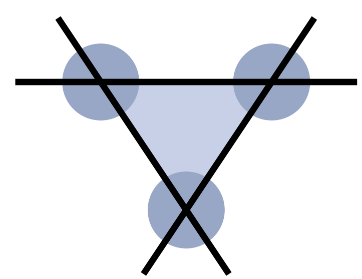
Obs: May assume no plane edges.

$$|X| = 5(n-2) + 2\,\triangledown + ▱ + \dot{\triangle} - \sum_{i=6}^{\infty}(i-5)|\mathcal{C}_i| - m$$

Idea: bound $\triangledown$, ▱, $\dot{\triangle}$ in terms of $|X|$, m and $\sum_{i=6}^{\infty}(i-5)|\mathcal{C}_i|$

Observation:

$|X| \geq 3\,\triangledown$ $\qquad |X| \geq |\{(x, c) \mid x \in X, c$ is $\triangledown$, x on c$\}| = 3\,\triangledown$



First Attempt

Question: Is $|X| \leq \frac{10}{3}n$ for every 2-plane drawing?

Obs: May assume no plane edges.

$$|X| = 5(n-2) + 2\,\triangledown + \square + \triangle - \sum_{i=6}^{\infty}(i-5)|\mathcal{C}_i| - m$$

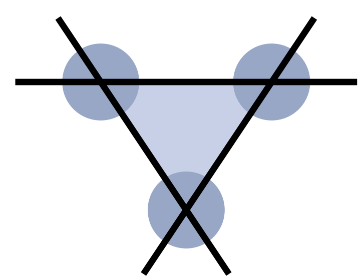
Idea: bound $\triangledown$, $\square$, $\triangle$ in terms of $|X|$, m and $\sum_{i=6}^{\infty}(i-5)|\mathcal{C}_i|$

Observation:

$|X| \geq 3\,\triangledown \qquad |X| \geq |\{(x,c) \mid x \in X, c$ is $\triangledown$, x on c$\}| = 3\,\triangledown$

$|X| \geq \square$

$|X| \geq \triangle$



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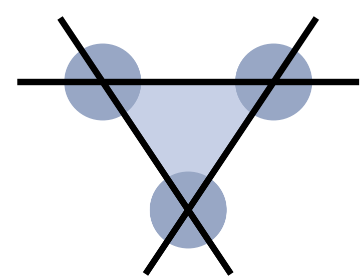
Observation:

$|X| \geq 3\triangledown$ $\quad |X| \geq |\{(x,c) \mid x \in X, c$ is $\triangledown$, x on c$\}| = 3\triangledown$

$|X| \geq \square$

$|X| \geq \triangle$

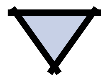
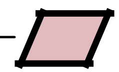
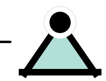


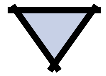
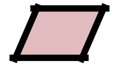
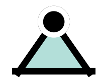
Recall: $|X| \leq m$

First Attempt

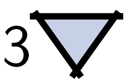
Question: Is $|X| \leq \frac{10}{3}n$ for every 2-plane drawing?

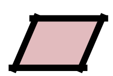
Obs: May assume no plane edges.

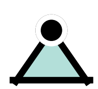
$|X| = 5(n-2)$ $+2$  $+$  $+$  $- \sum_{i=6}^{\infty}(i-5)|\mathcal{C}_i| - m \qquad (\star)$

Idea: bound , ,  in terms of $|X|$, m and $\sum_{i=6}^{\infty}(i-5)|\mathcal{C}_i|$

Observation:

$|X| \geq 3$ 

$|X| \geq$ 

$|X| \geq$ 

$|X| \leq m$

$(\star)$

First Attempt

Question: Is $|X| \leq \frac{10}{3}n$ for every 2-plane drawing?

Obs: May assume no plane edges.

$$|X| = 5(n-2) + 2\triangledown + \square + \triangle \overbrace{- \sum_{i=6}^{\infty}(i-5)|\mathcal{C}_i|}^{\leq 0} - m \qquad (\star)$$

Idea: bound $\triangledown$, $\square$, $\triangle$ in terms of $|X|$, m and $\sum_{i=6}^{\infty}(i-5)|\mathcal{C}_i|$

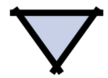
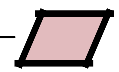
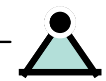
Observation:

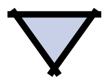
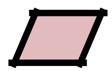
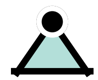
$$\begin{cases} |X| \geq 3\triangledown \\ |X| \geq \square \\ |X| \geq \triangle \\ |X| \leq m \\ (\star) \end{cases}$$

First Attempt

Question: Is $|X| \leq \frac{10}{3}n$ for every 2-plane drawing?

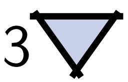
Obs: May assume no plane edges.

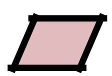
$|X| = 5(n-2) + 2$  $+$  $+$  $\overbrace{- \sum_{i=6}^{\infty}(i-5)|\mathcal{C}_i|}^{\leq 0} - m \qquad (\star)$

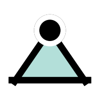
Idea: bound , ,  in terms of $|X|$, m and $\sum_{i=6}^{\infty}(i-5)|\mathcal{C}_i|$

Observation:

LP $\left\{\right.$

$|X| \geq 3$ 

$|X| \geq$ 

$|X| \geq$ 

$|X| \leq m$

$(\star)$

Maximize $|X|$ in LP

First Attempt

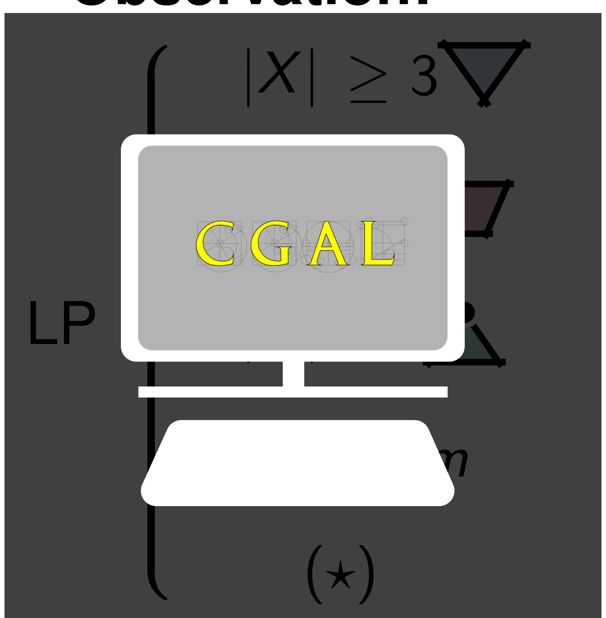
Question: Is $|X| \leq \frac{10}{3}n$ for every 2-plane drawing?

Obs: May assume no plane edges.

$$|X| = 5(n-2) + 2\,\triangledown + \square + \triangle^{\bullet} \underbrace{-\sum_{i=6}^{\infty}(i-5)|\mathcal{C}_i|}^{\leq 0} - m \qquad (\star)$$

Idea: bound $\triangledown$, $\square$, $\triangle^{\bullet}$ in terms of $|X|$, m and $\sum_{i=6}^{\infty}(i-5)|\mathcal{C}_i|$

Observation:



Maximize $|X|$ in LP

First Attempt

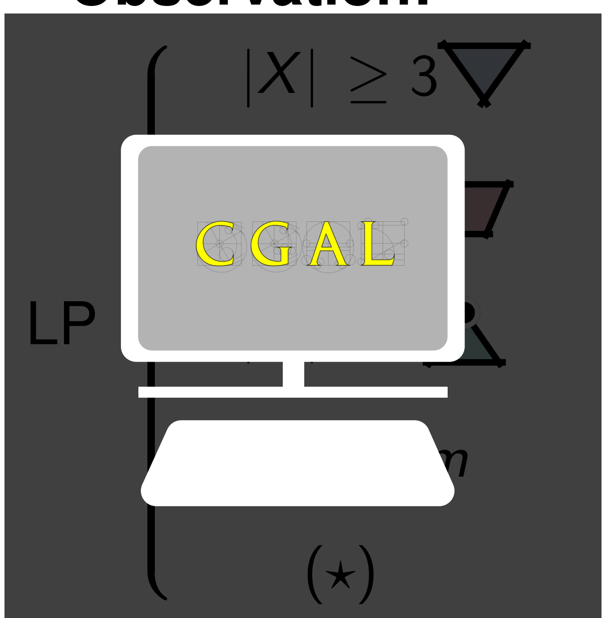
Question: Is $|X| \leq \frac{10}{3} n$ for every 2-plane drawing?

Obs: May assume no plane edges.

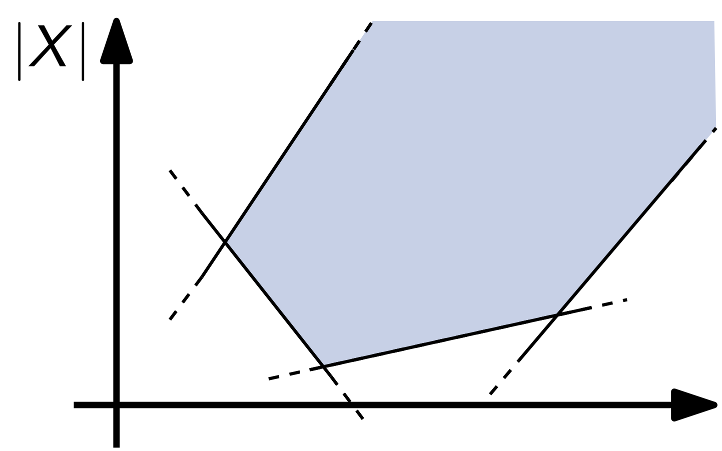
$$|X| = 5(n-2) + 2 \;+\; + \;\underbrace{- \textstyle\sum_{i=6}^{\infty}(i-5)|\mathcal{C}_i|}_{\leq 0} - m \qquad (\star)$$

Idea: bound , , in terms of $|X|$, m and $\sum_{i=6}^{\infty}(i-5)|\mathcal{C}_i|$

Observation:



Maximize $|X|$ in LP



$|X|$ unbounded

First Attempt

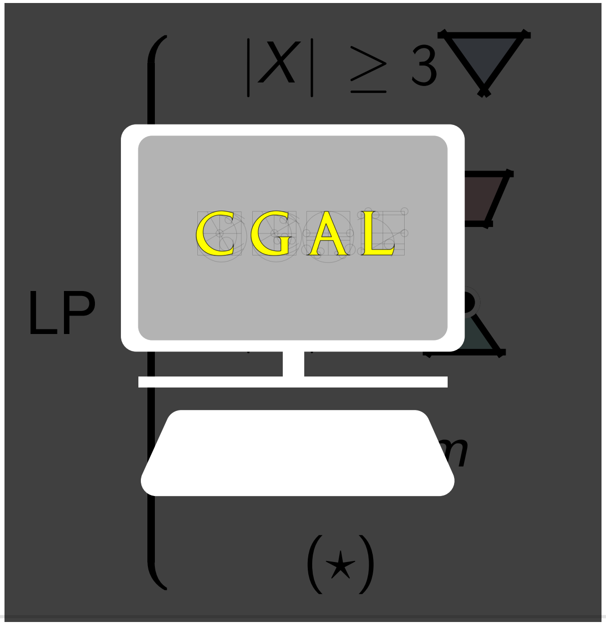
Question: Is $|X| \leq \frac{10}{3}n$ for every 2-plane drawing?

Obs: May assume no plane edges.

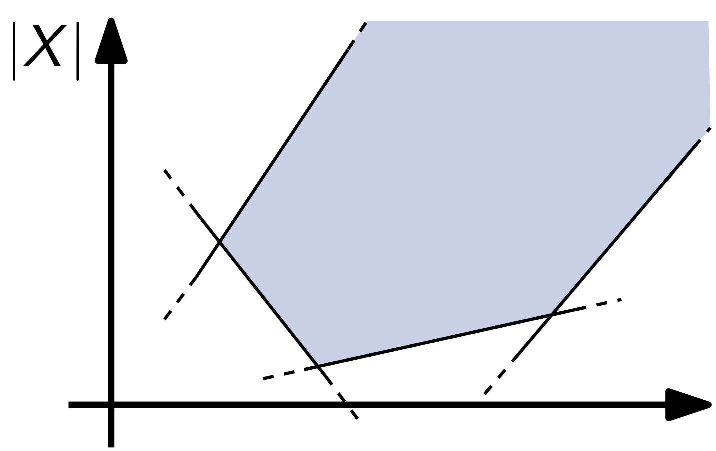
$$|X| = 5(n-2) + 2\bigtriangledown + ▱ + \triangle \underbrace{- \sum_{i=6}^{\infty}(i-5)|\mathcal{C}_i|}_{\leq 0} - m \qquad (\star)$$

Idea: bound $\bigtriangledown$, ▱, $\triangle$ in terms of $|X|$, m and $\sum_{i=6}^{\infty}(i-5)|\mathcal{C}_i|$

Observation:



Maximize $|X|$ in LP



$|X|$ unbounded

Idea: Need to take cells of size ≥ 6 into account

First Attempt

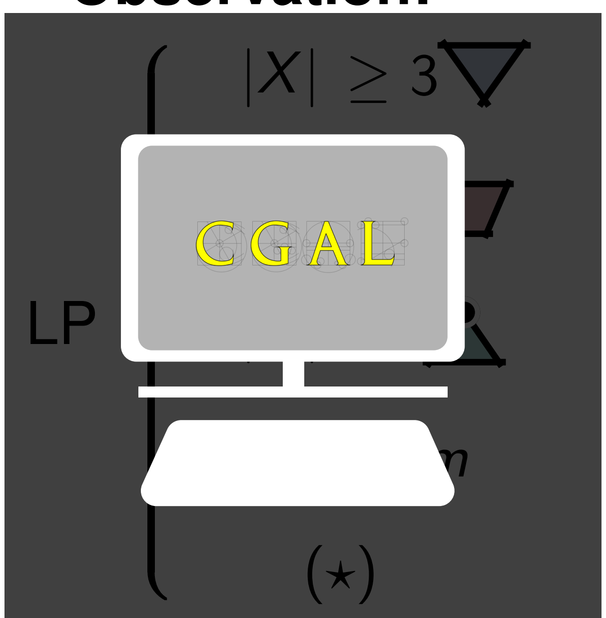
Question: Is $|X| \leq \frac{10}{3}n$ for every 2-plane drawing?

Obs: May assume no plane edges.

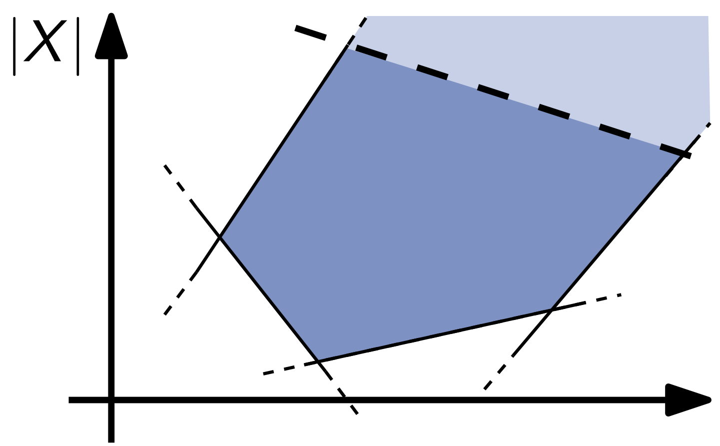
$$|X| = 5(n-2) + 2\,\triangledown + \square + \triangle \underbrace{- \textstyle\sum_{i=6}^{\infty}(i-5)|\mathcal{C}_i|}_{\leq 0} - m \qquad (\star)$$

Idea: bound $\triangledown$, $\square$, $\triangle$ in terms of $|X|$, m and $\sum_{i=6}^{\infty}(i-5)|\mathcal{C}_i|$

Observation:



Maximize $|X|$ in LP



$|X|$ unbounded

Idea: Need to take cells of size ≥ 6 into account

Large Cells

Question: Number of large cells?

Large Cells

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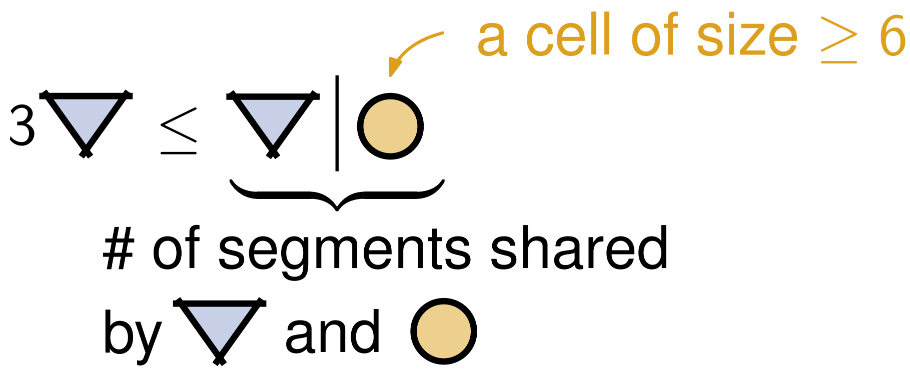
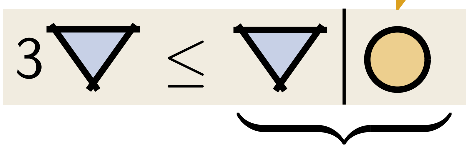


Large Cells

Question: Number of large cells?

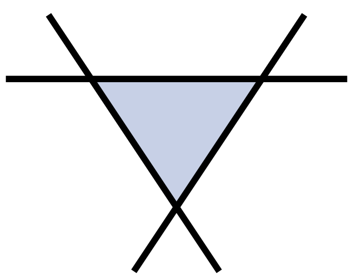
a cell of size ≥ 6



\# of segments shared

by and

Proof:



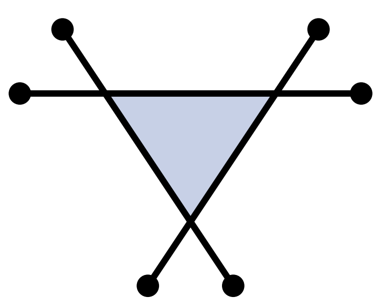
Large Cells

Question: Number of large cells?

a cell of size ≥ 6

$$3\,\triangledown \leq \underbrace{\triangledown \mid \bigcirc}_{\text{\# of segments shared by } \triangledown \text{ and } \bigcirc}$$

Proof:



Large Cells

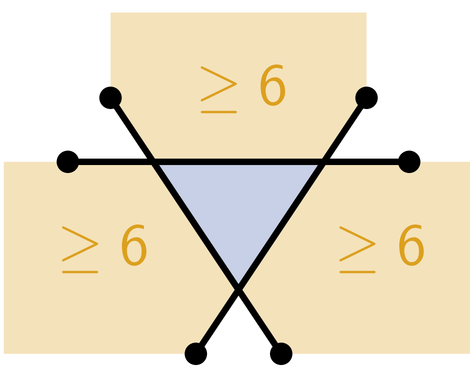
Question: Number of large cells?

a cell of size ≥ 6

$$3\,\triangledown \leq \underbrace{\triangledown \mid \bigcirc}$$

\# of segments shared by $\triangledown$ and $\bigcirc$

Proof:

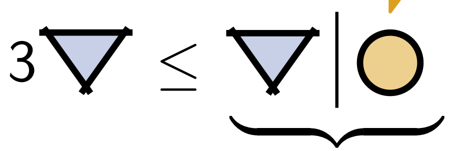


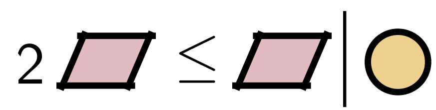
Large Cells

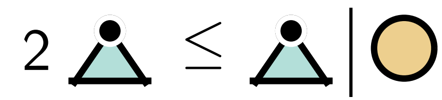
Question: Number of large cells?

Recall: May assume no plane edges.

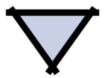
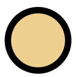
a cell of size ≥ 6



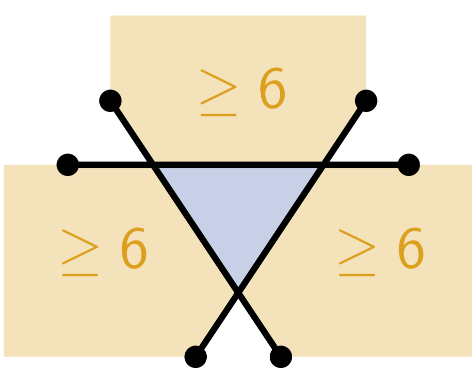


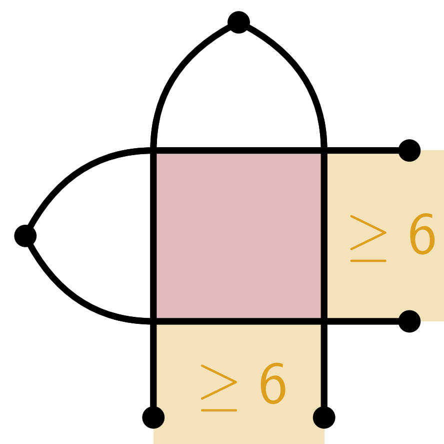


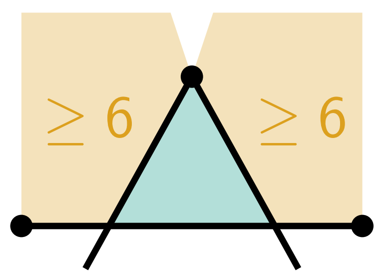
# of segments shared

by  and 

Proof:







Large Cells

Question: Number of large cells?

Want: $? \leq \sum_{i \geq 6} (i-5)|\mathcal{C}_i|$

Large Cells

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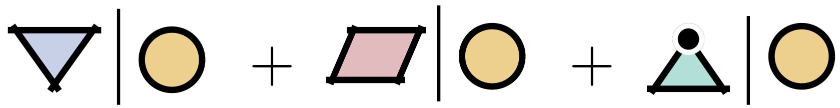
Want: $? \leq \sum_{i\geq 6}(i-5)|\mathcal{C}_i|$

 $\leq \sum_{i\geq 6} i \cdot |\mathcal{C}_i|$

Large Cells

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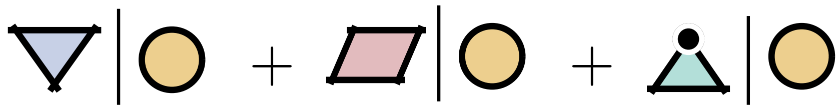


$\leq \sum_{i \geq 6} i \cdot |\mathcal{C}_i|$

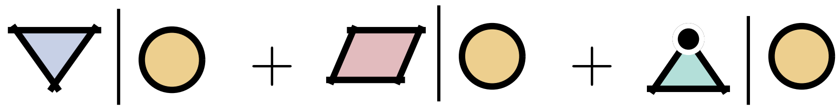
$\geq$ # segments

Idea: count # of segments on large cells

Large Cells

Question: Number of large cells?

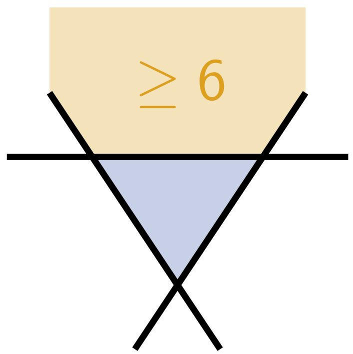
Want: $? \leq \sum_{i \geq 6} (i-5)|\mathcal{C}_i|$

 $\leq \sum_{i \geq 6} i \cdot |\mathcal{C}_i|$

$\geq$ # segments

Idea: count # of segments on large cells

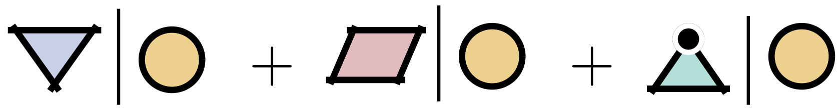
Proof:



Large Cells

Question: Number of large cells?

Want: $? \leq \sum_{i\geq 6}(i-5)|\mathcal{C}_i|$

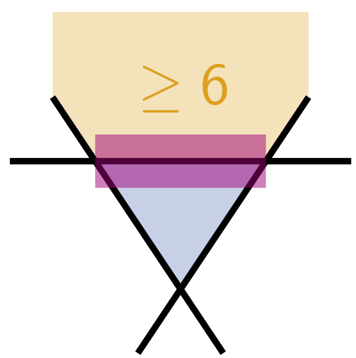


$\leq \sum_{i\geq 6} i \cdot |\mathcal{C}_i|$

$\geq$ # segments

Idea: count # of segments on large cells

Proof:



Large Cells

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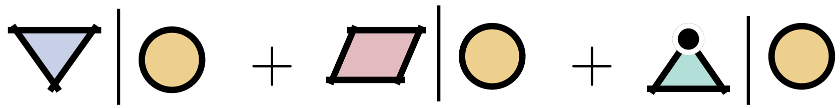 $\leq \sum_{i \geq 6} i \cdot |\mathcal{C}_i|$

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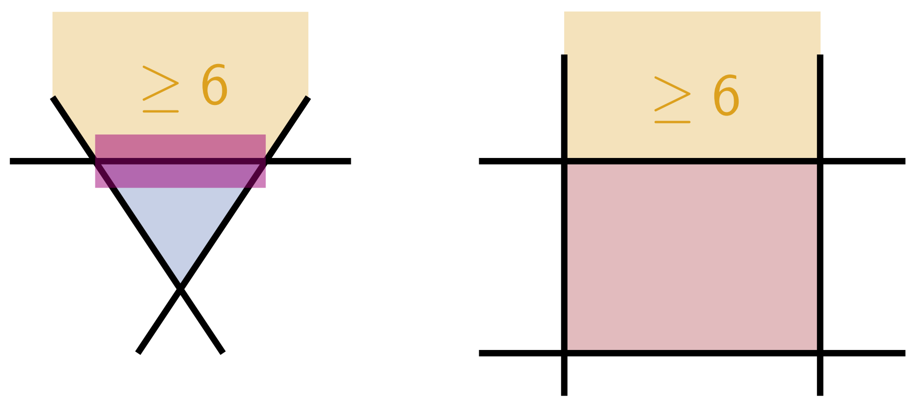

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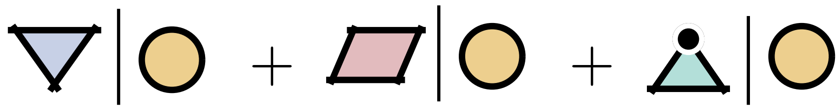 $\leq \sum_{i \geq 6} i \cdot |\mathcal{C}_i|$

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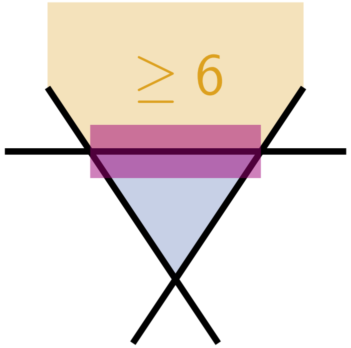

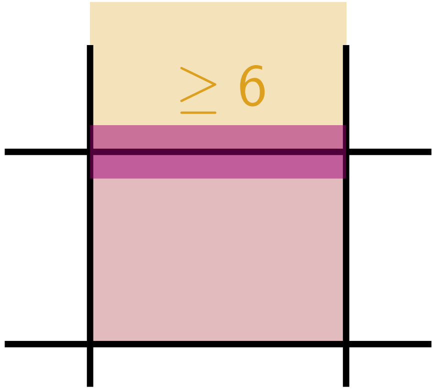

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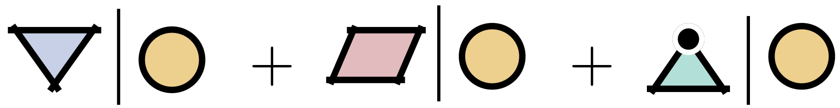 $\leq \sum_{i \geq 6} i \cdot |\mathcal{C}_i|$

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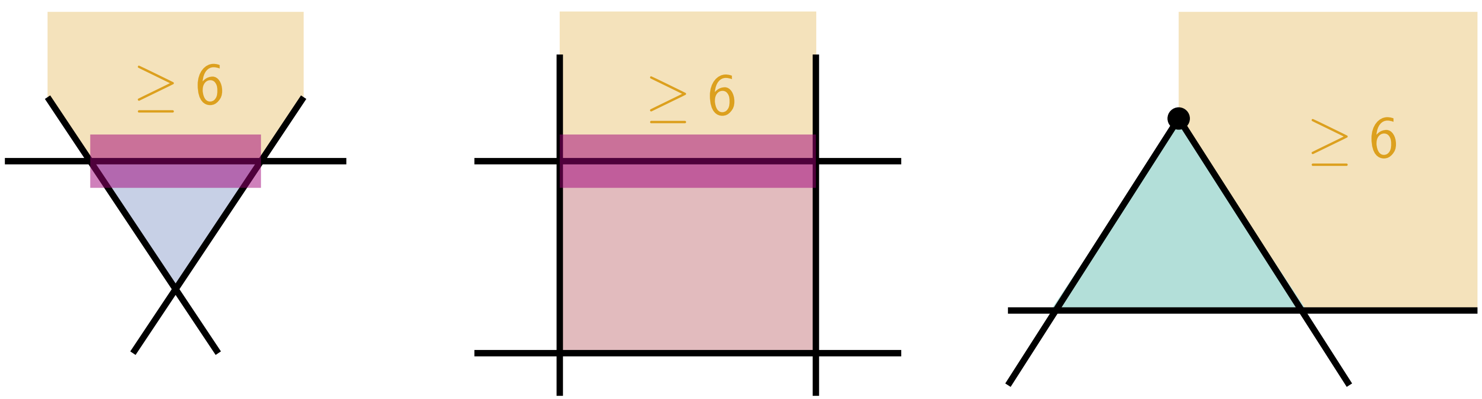

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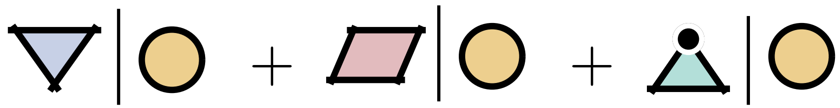

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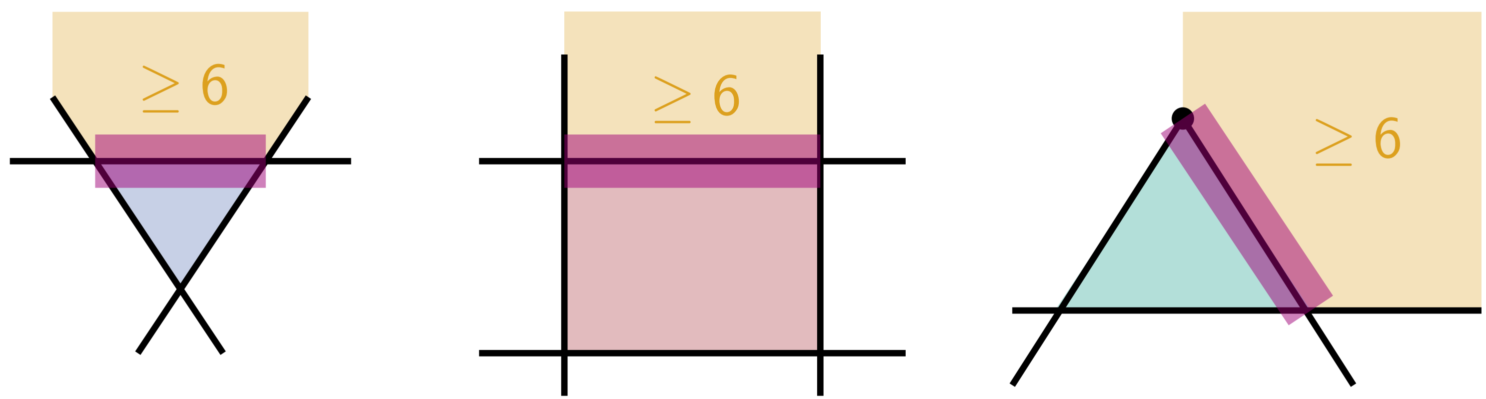
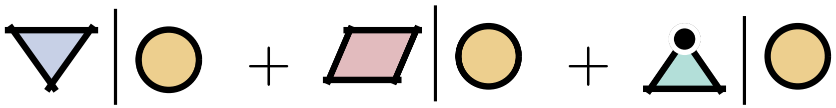


Large Cells

Question: Number of large cells?

Want: $? \leq \sum_{i\geq 6}(i-5)|\mathcal{C}_i|$

 $\leq \sum_{i\geq 6} i \cdot |\mathcal{C}_i|$

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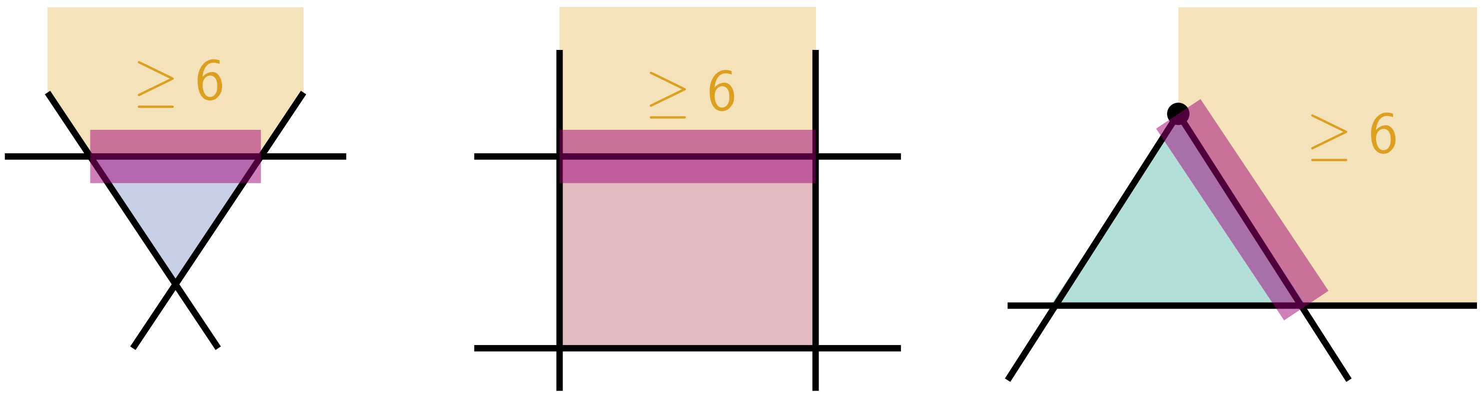
Idea: count # of segments on large cells

Proof:

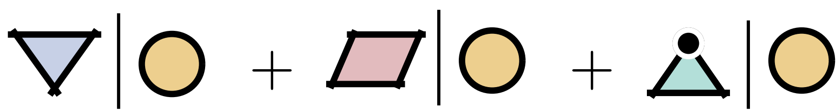


Obs: $\frac{1}{6}\sum_{i\geq 6} i \cdot |\mathcal{C}_i| \leq \sum_{i\geq 6}(i-5)|\mathcal{C}_i|$

Large Cells

Question: Number of large cells?

Want: $? \leq \sum_{i\geq 6}(i-5)|\mathcal{C}_i|$

 $\leq \sum_{i\geq 6} i \cdot |\mathcal{C}_i|$

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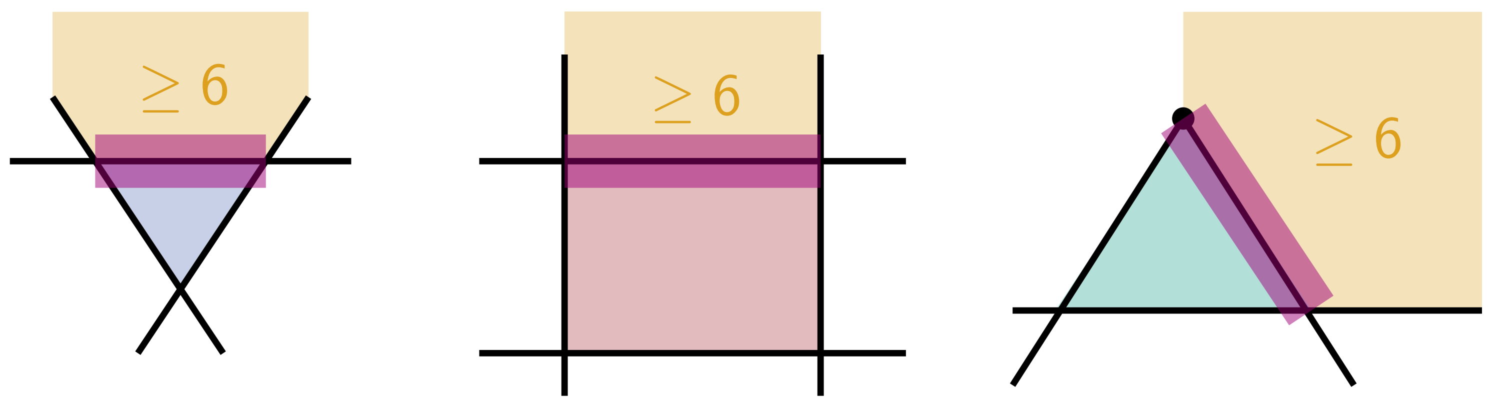

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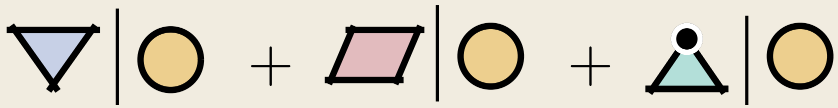
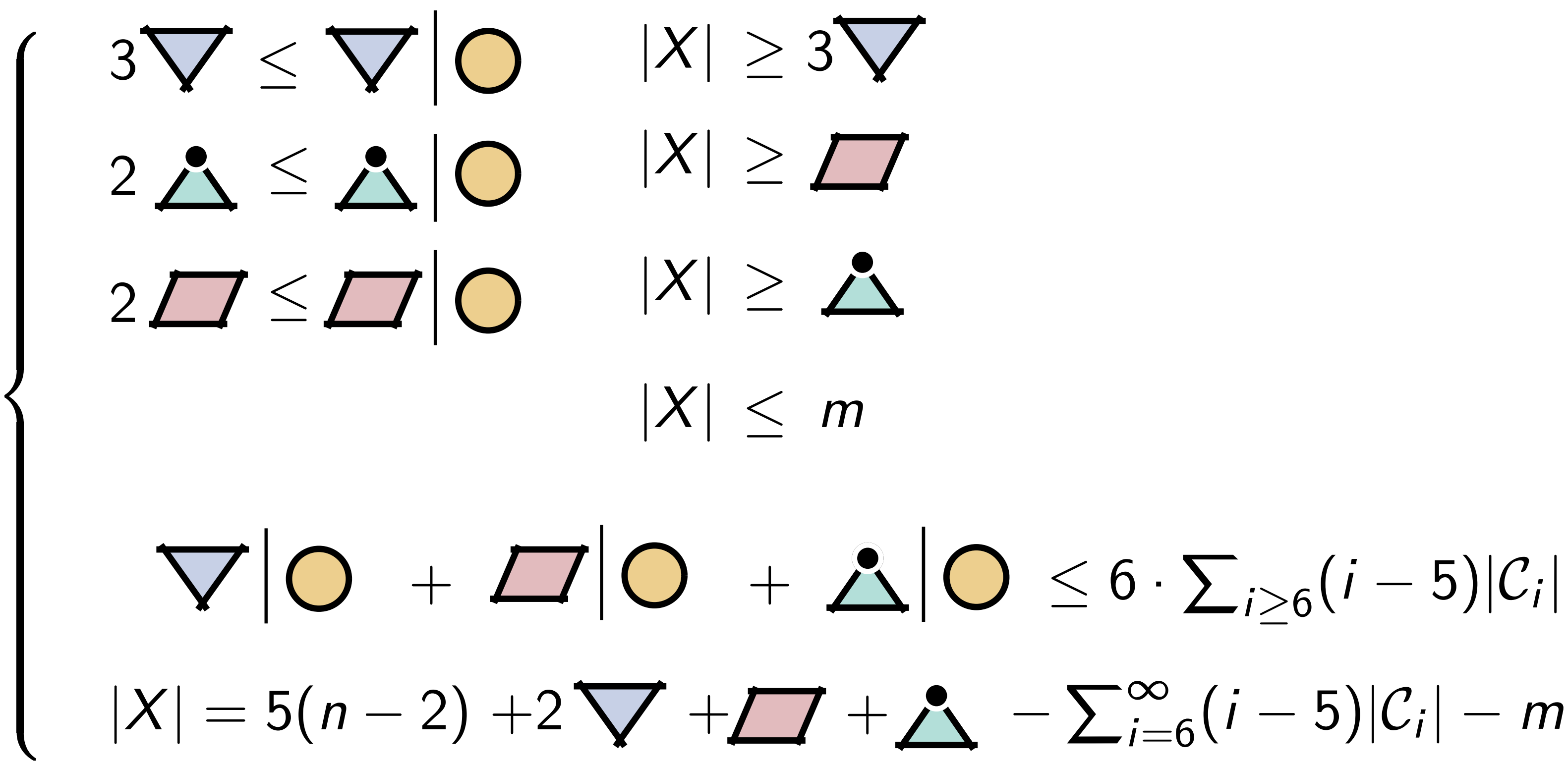
$\rightsquigarrow$  $\leq 6 \cdot \sum_{i\geq 6}(i-5) \cdot |\mathcal{C}_i|$

Second Attempt

Inequations:

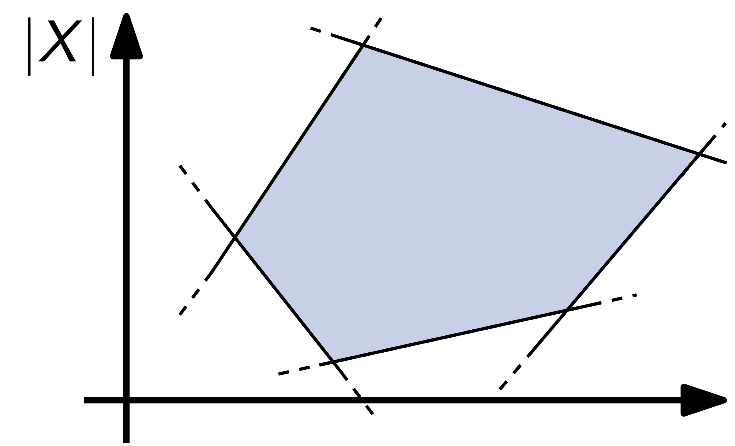


Second Attempt

Inequations:

$$\begin{cases} 3\,\triangledown \le \triangledown \,|\, \bigcirc & |X| \ge 3\,\triangledown \\ 2\,\dot{\triangle} \le \dot{\triangle} \,|\, \bigcirc & |X| \ge \lozenge \\ 2\,\lozenge \le \lozenge \,|\, \bigcirc & |X| \ge \dot{\triangle} \\ & |X| \le m \\ \triangledown \,|\, \bigcirc + \lozenge \,|\, \bigcirc + \dot{\triangle} \,|\, \bigcirc \le 6 \cdot \sum_{i \ge 6} (i-5)|\mathcal{C}_i| & \\ |X| = 5(n-2) + 2\,\triangledown + \lozenge + \dot{\triangle} - \sum_{i=6}^{\infty} (i-5)|\mathcal{C}_i| - m & \end{cases}$$

Visualization:



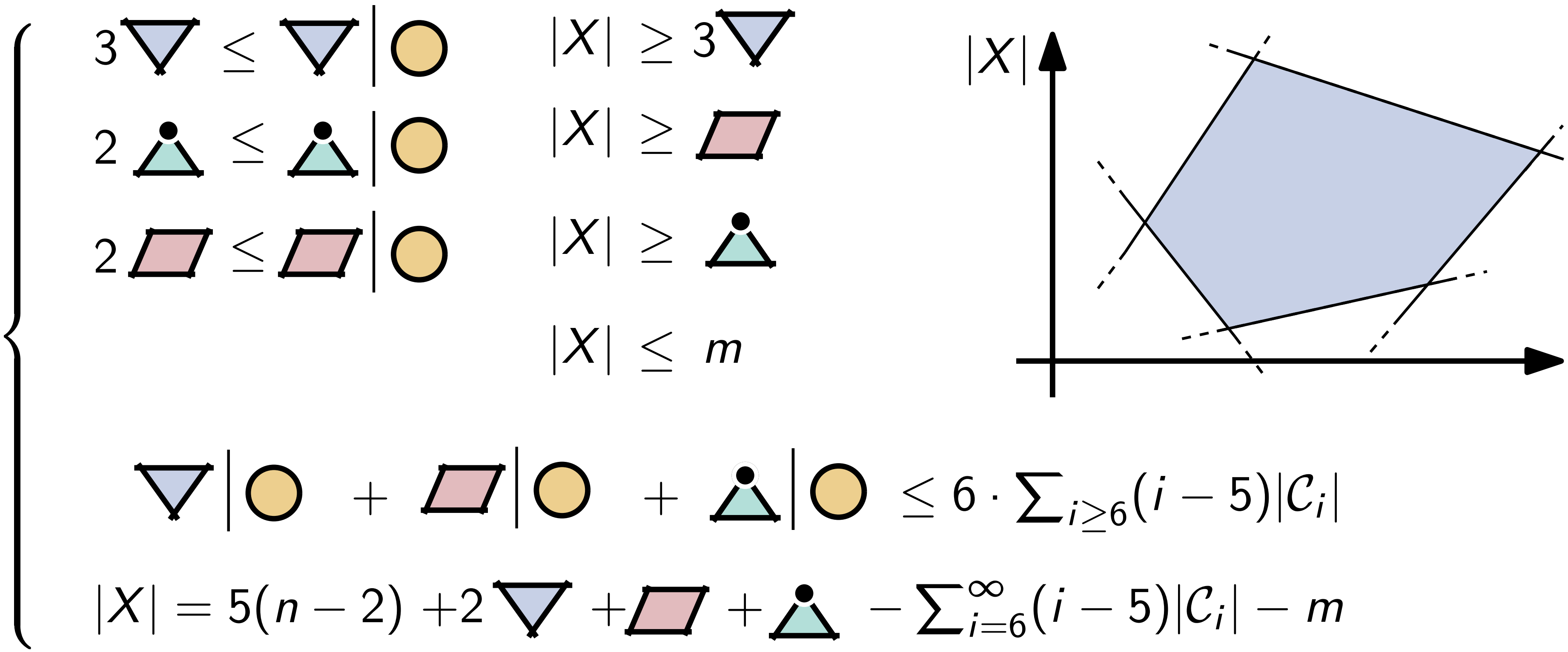
Maximize $|X|$ in LP

Want: $|X| \le 3.\overline{3}n$

Second Attempt

Inequations:

Visualization:



Idea: Maximize $|X|$ in LP

Obtain: $|X| \leq 30n$

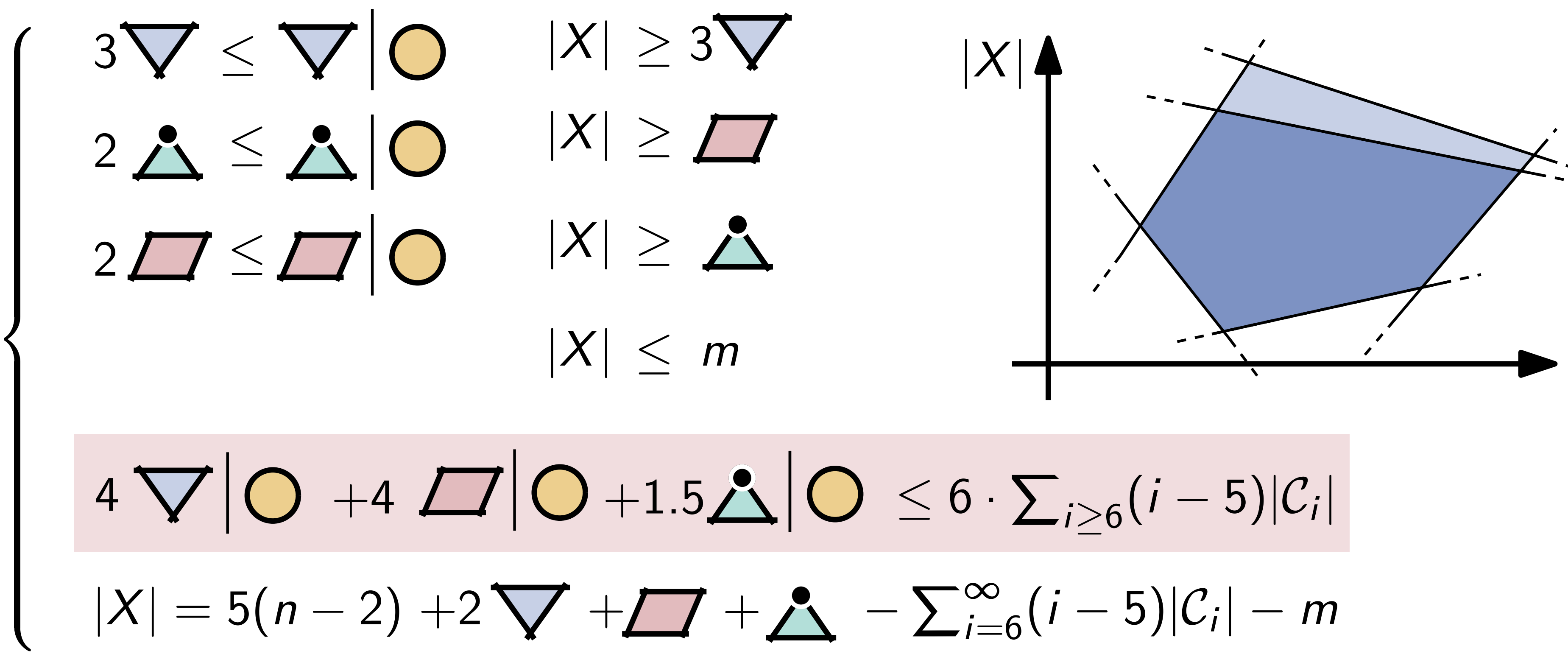
Want: $|X| \leq 3.\overline{3}n$

Second Attempt

Karlsruher Institut für Technologie

Inequations:

Visualization:



Idea: Maximize $|X|$ in LP

Obtain: $|X| \leq \cancel{30n}\ 3.\overline{3}n$

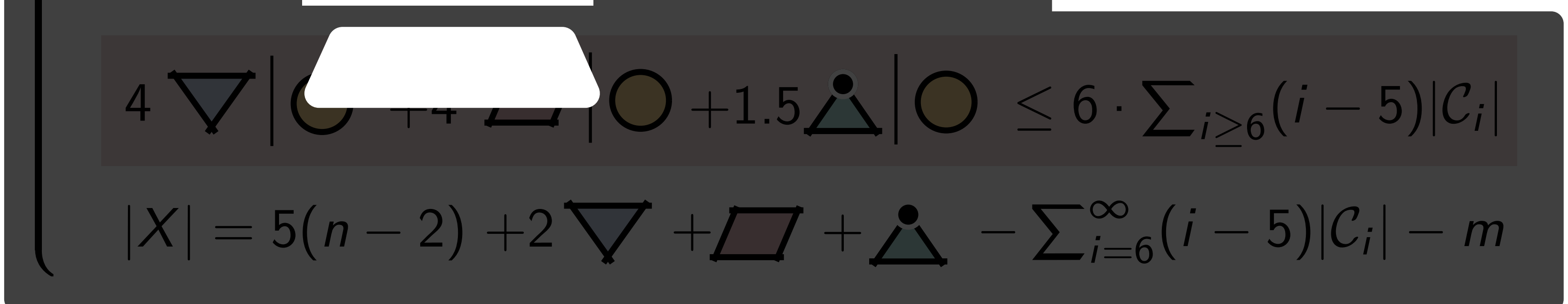
Want: $|X| \leq 3.\overline{3}n$

Maximize $|E|$ in LP

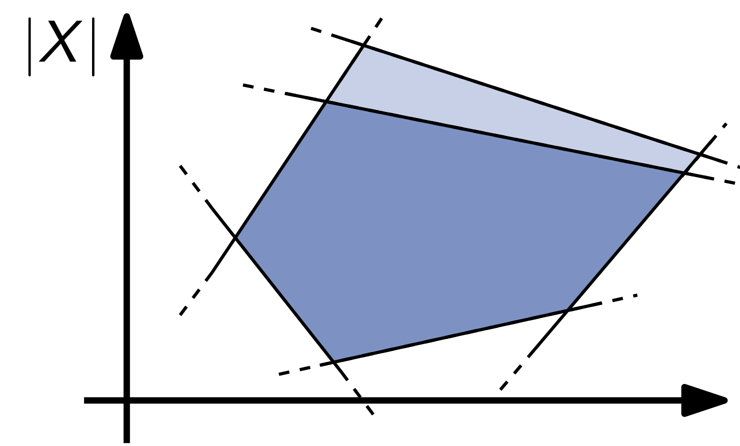
Obtain: $|E| \leq 5n$

Second Attempt

Inequations:



Visualization:

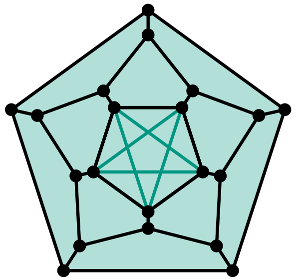


⇝ **Idea:** Maximize $|X|$ in LP **Obtain:** $|X| \leq$ ~~$3.\overline{3}n$~~ $3.\overline{3}n$

Want: $|X| \leq 3.\overline{3}n$ Maximize $|E|$ in LP **Obtain:** $|E| \leq 5n$

Overview

Results



2-plane

m	$\leq 5n$ (Pach, Tóth 1997)
$\lvert X \rvert$	$\leq 3.\overline{3}n$ (Bekos et al. 2024)

Overview

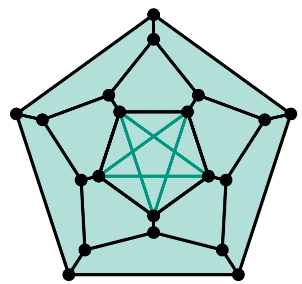
Technique

LP
- $2 \leq |$
- $2 \leq |$
- $\dots$
- Density Formula

Maximize $|X|$ in LP

$\rightsquigarrow$ Bound on $|X|$

Results



	2-plane
m	$\leq 5n$ (Pach, Tóth 1997)
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Overview

Karlsruher Institut für Technologie

Technique

LP: $2 \leq |$
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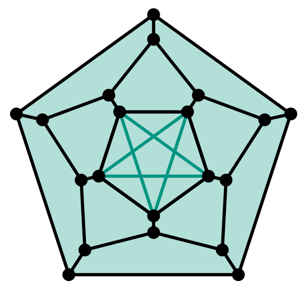
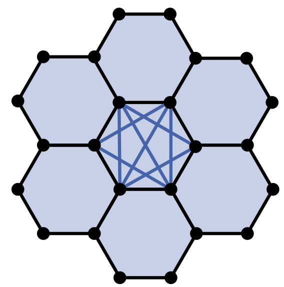
$\dots$
Density Formula

Maximize $|X|$ in LP

Bound on $|X|$

Results

	2-plane	3-plane
m	$\leq 5n$ (Pach, Tóth 1997)	$\leq 5.5n$ (Pach, Radoicic, Tardos, Tóth 2006)
$\lvert X \rvert$	$\leq 3.\overline{3}n$ (Bekos et al. 2024)	$\leq 5.5n$

Overview

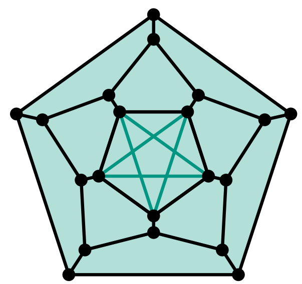
Technique

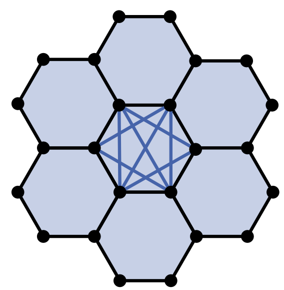
LP
- $2 \ldots \leq \ldots |$
- $2 \ldots \leq \ldots |$
- $\ldots$
- Density Formula

Maximize $|X|$ in LP

Bound on $|X|$

Results





	2-plane	3-plane	4-plane
m	$\leq 5n$ (Pach, Tóth 1997)	$\leq 5.5n$ (Pach, Radoicic, Tardos, Tóth 2006)	$\leq 6n$ (Ackerman 2015)
$\lvert X\rvert$	$\leq 3.\overline{3}n$ (Bekos et al. 2024)	$\leq 5.5n$	?