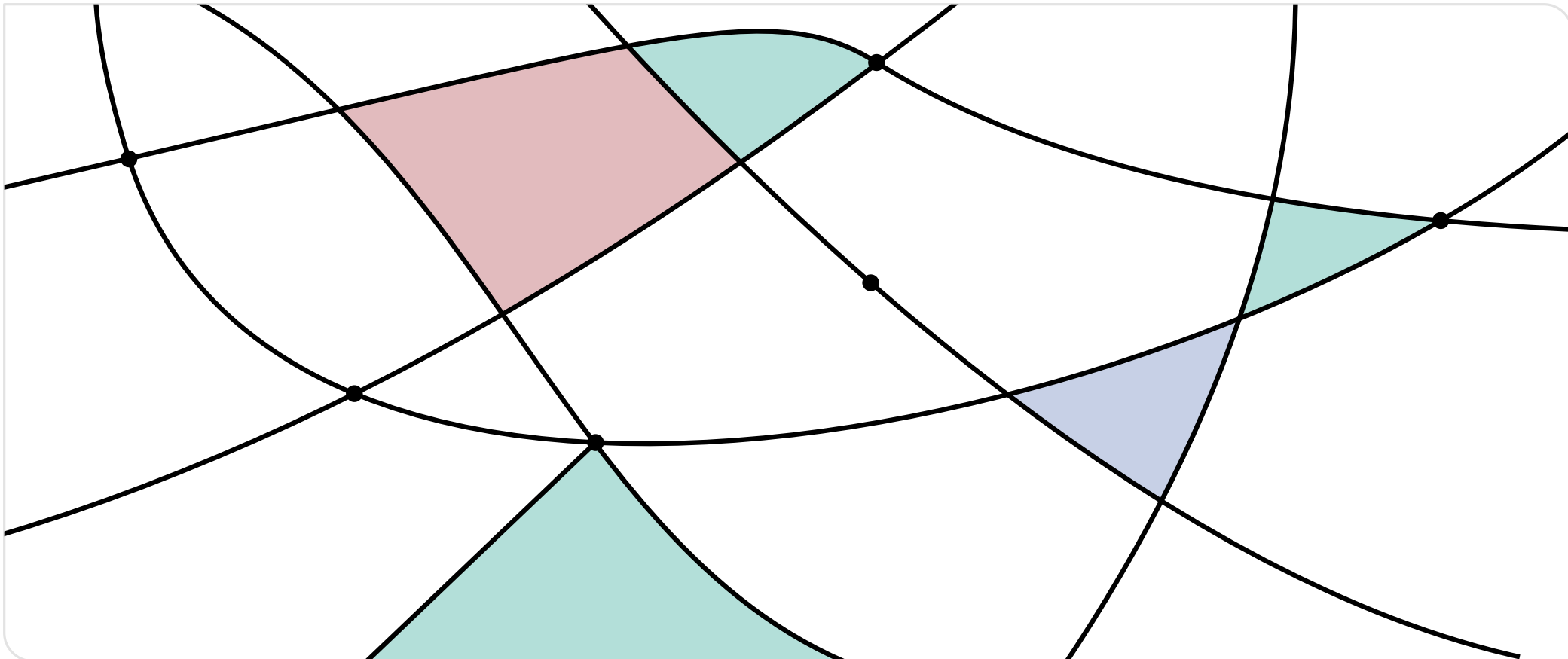


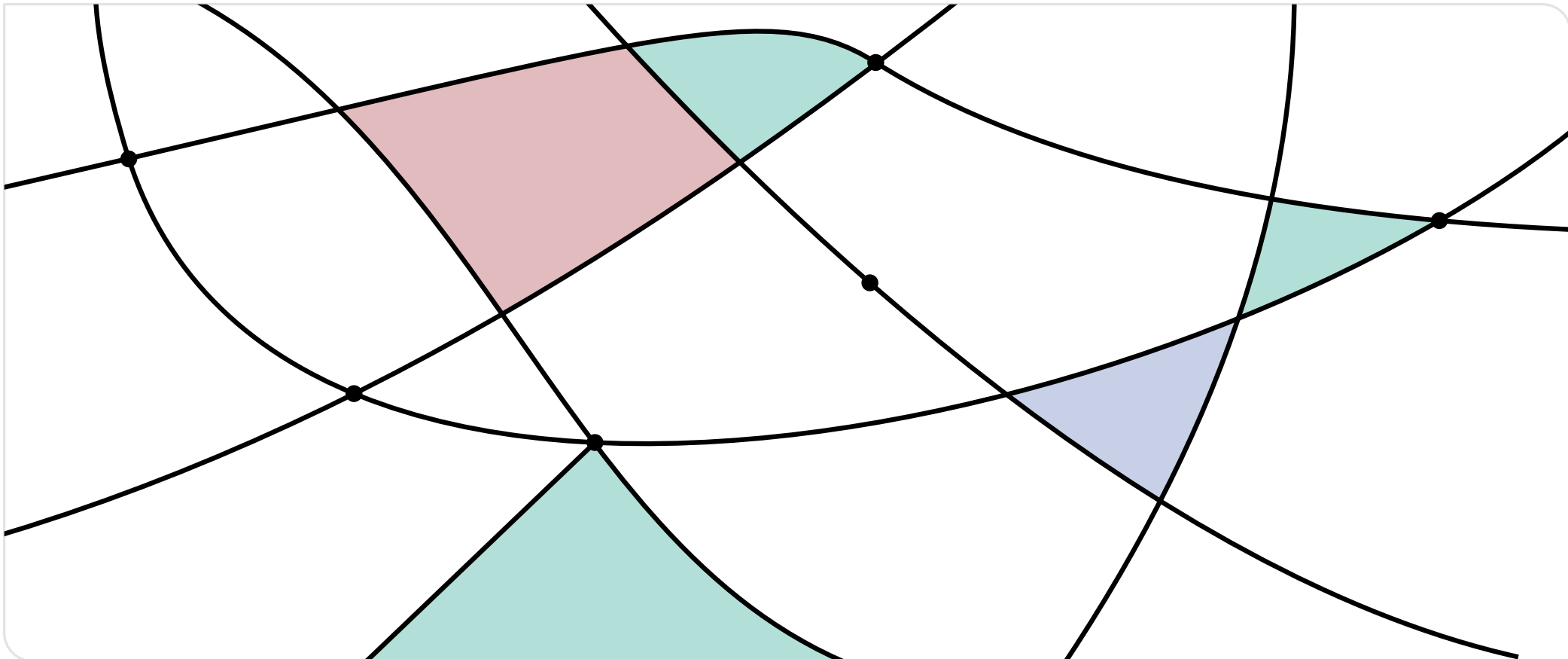
Crossing Number of Simple 3-Plane Drawings

joint work with Michael Hoffmann, Ignaz Rutter, Torsten Ueckerdt

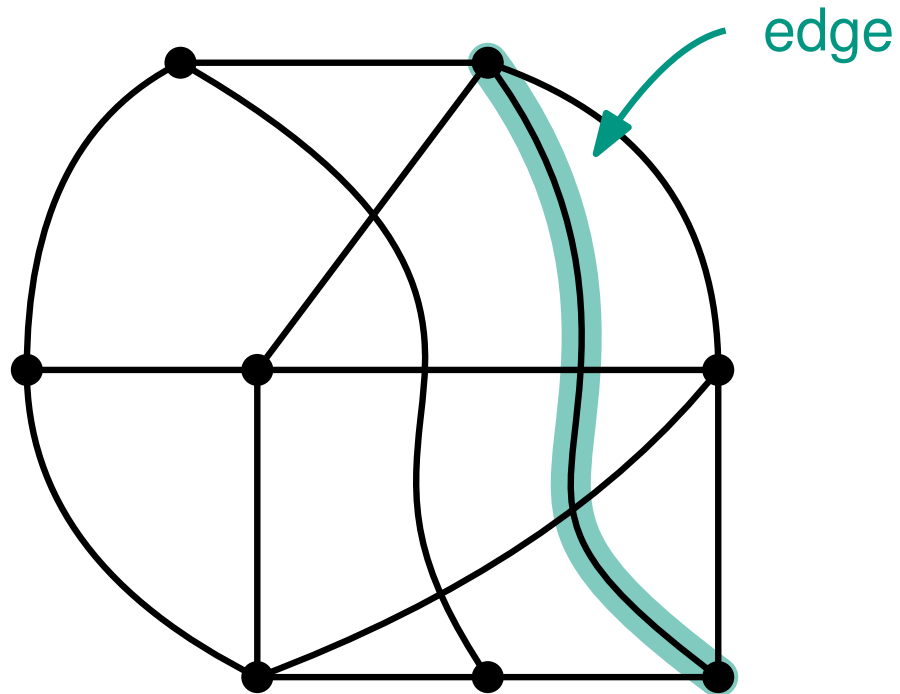


Crossing Number of Simple ~~3-Plane~~ Drawings 2-Plane

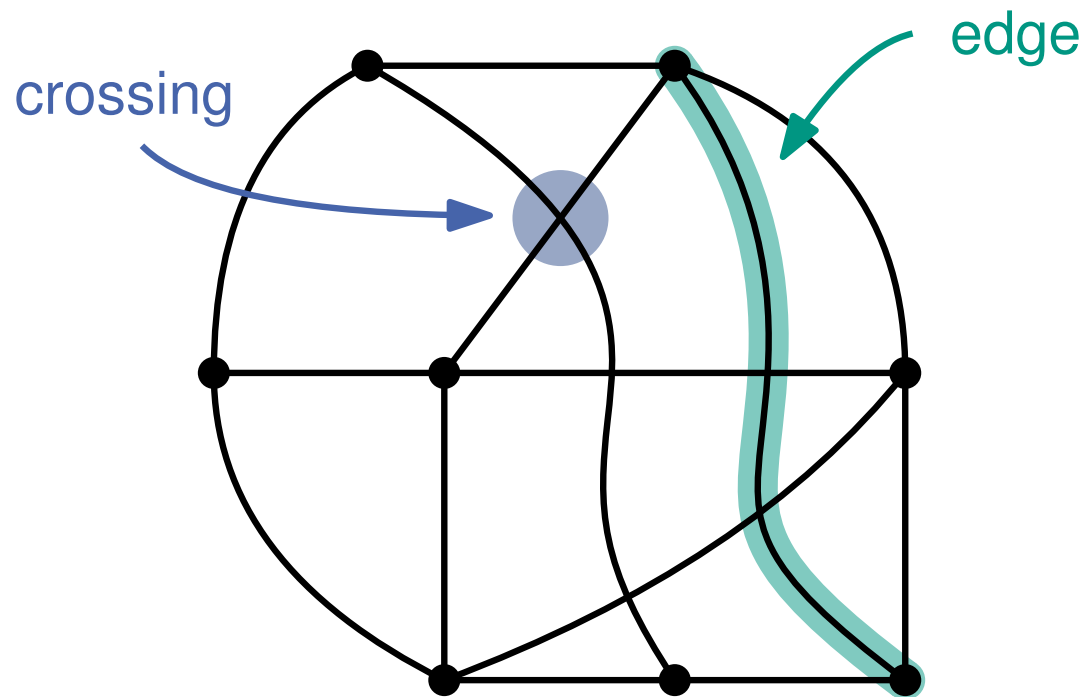
joint work with Michael Hoffmann, Ignaz Rutter, Torsten Ueckerdt



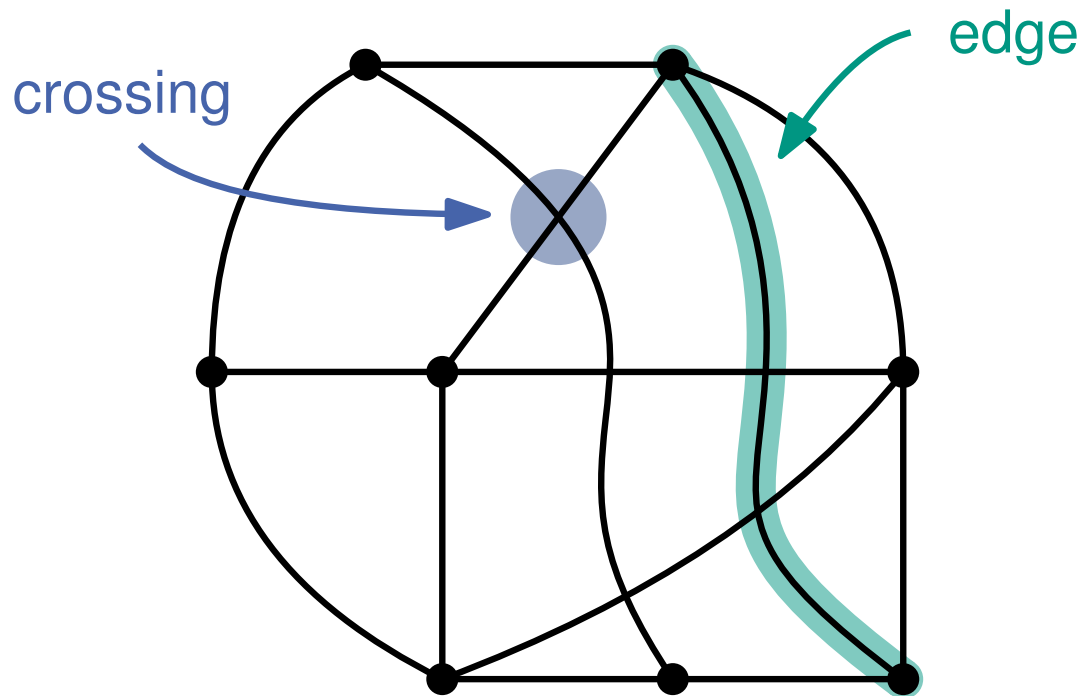
Simple Drawings



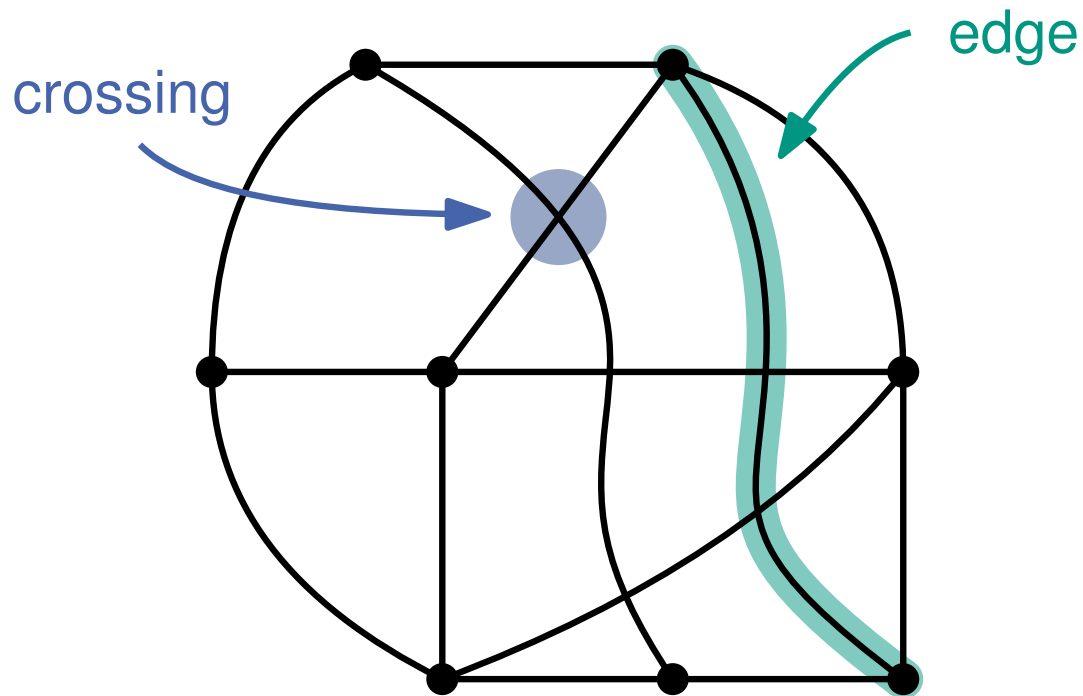
Simple Drawings



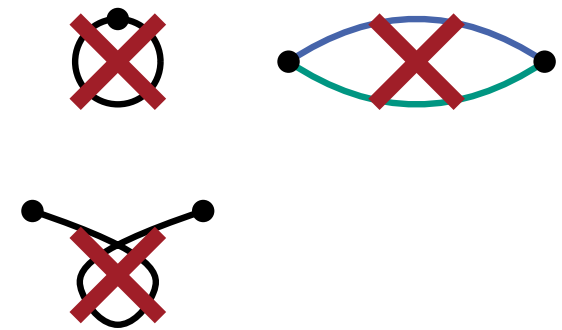
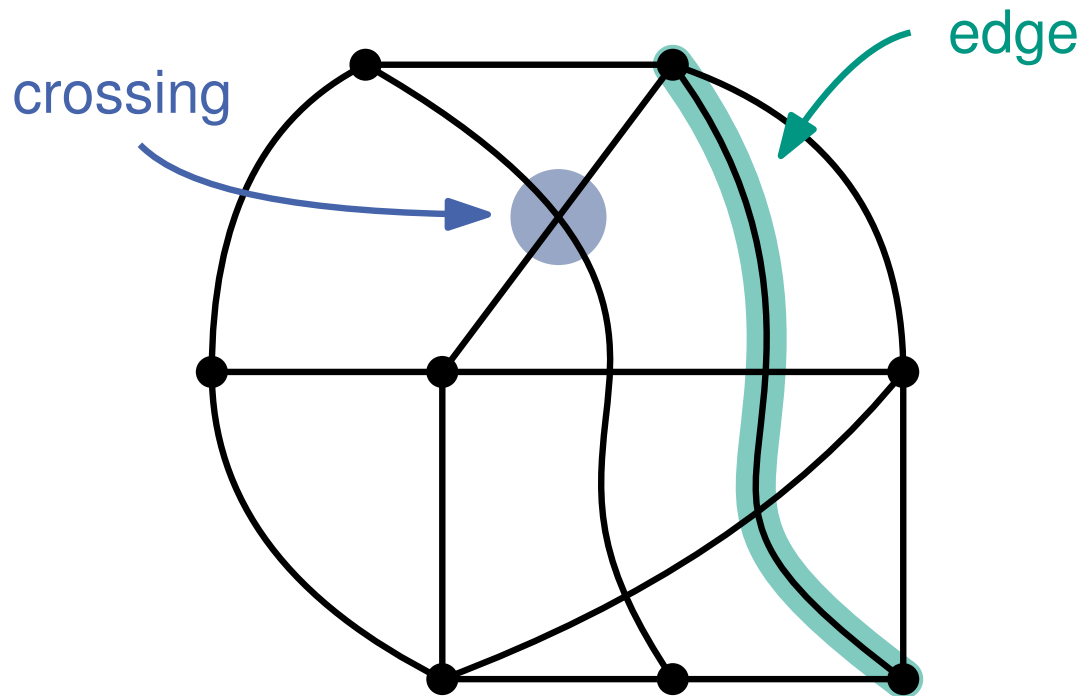
Simple Drawings



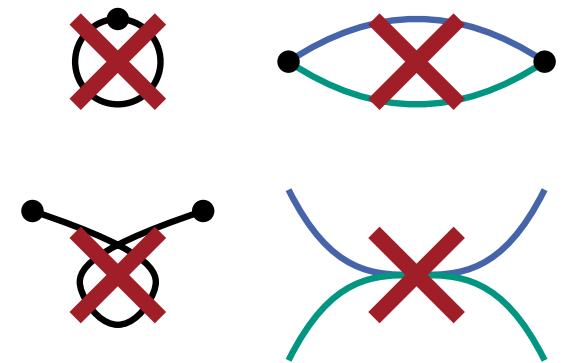
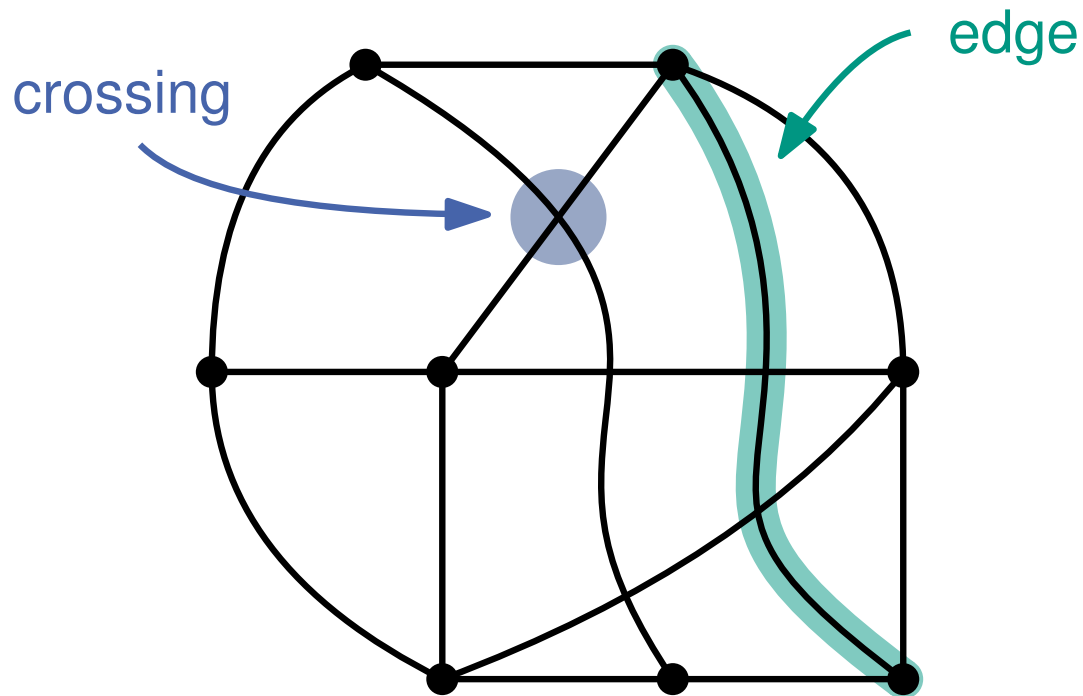
Simple Drawings



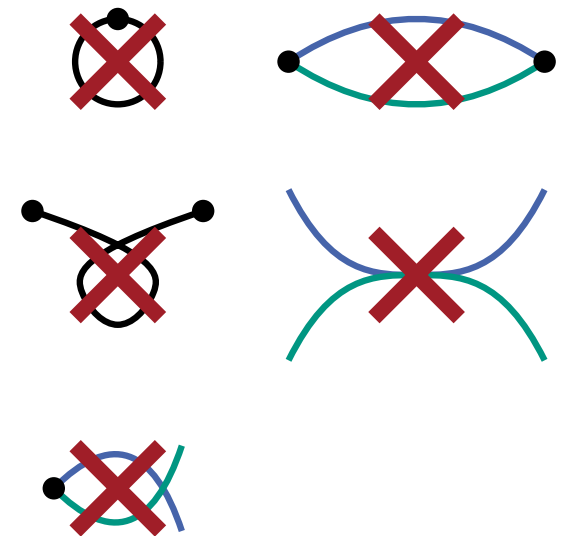
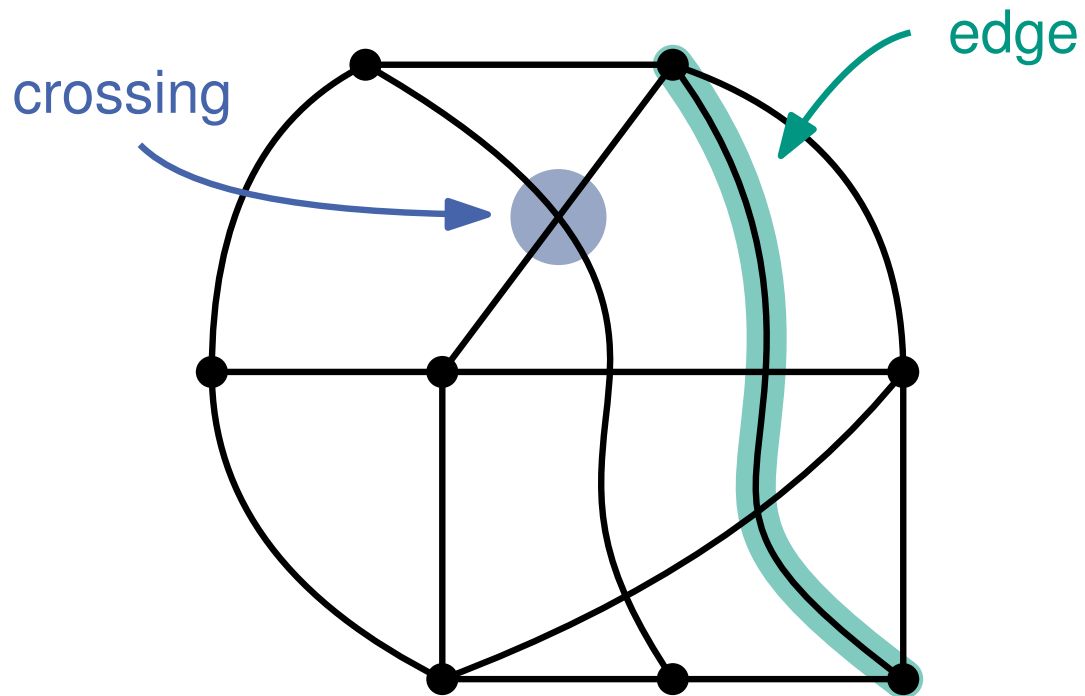
Simple Drawings



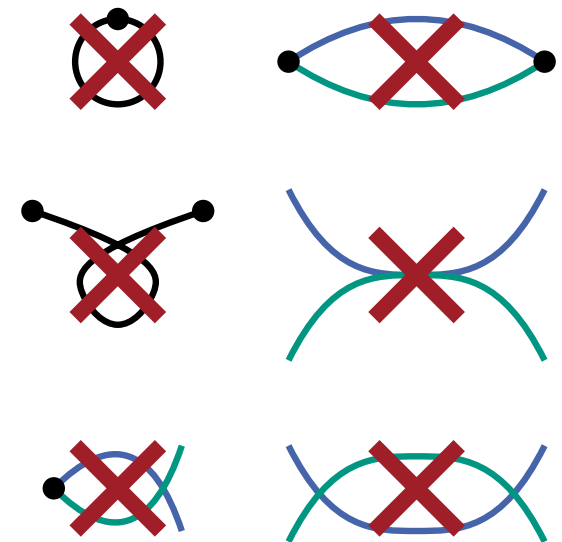
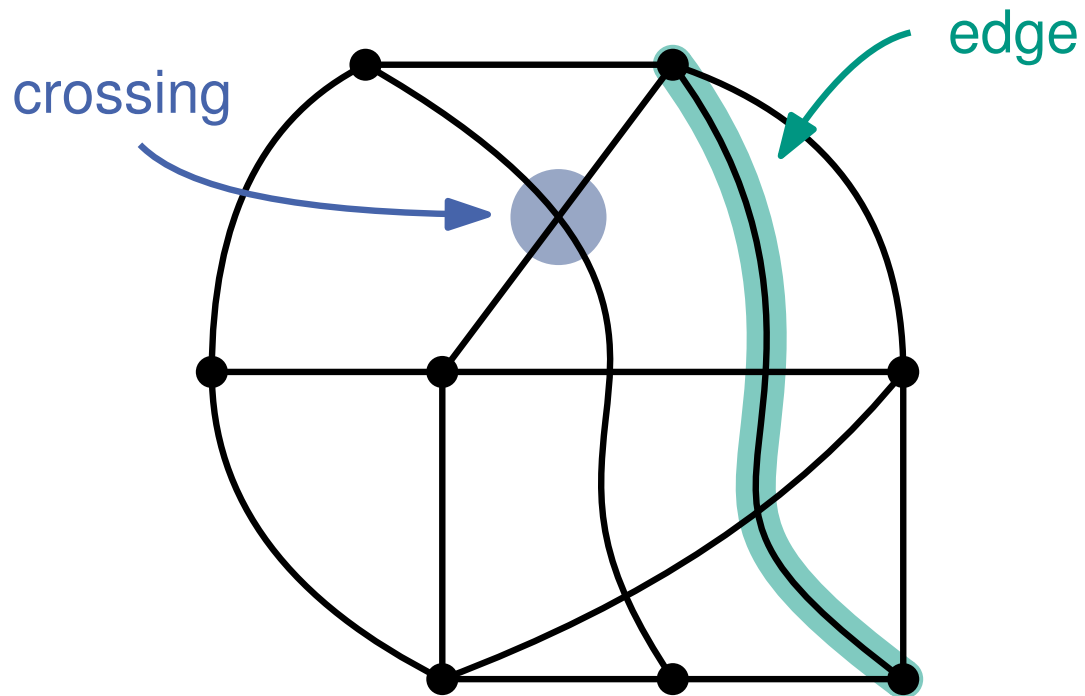
Simple Drawings



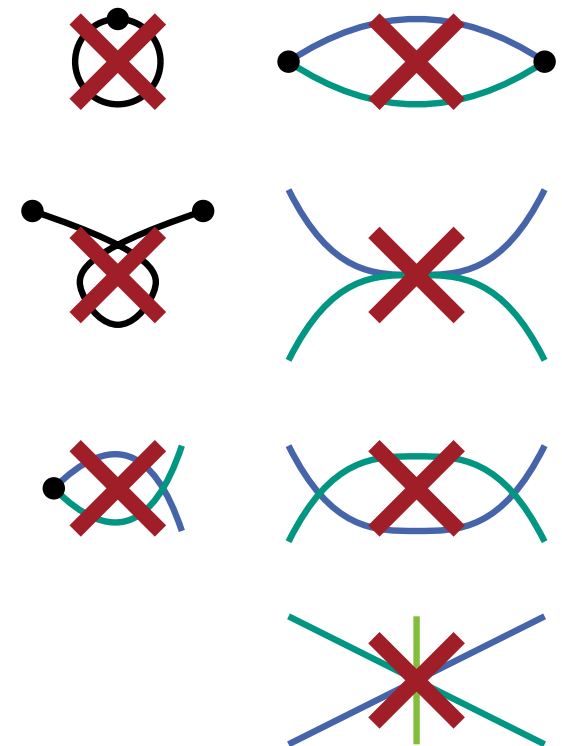
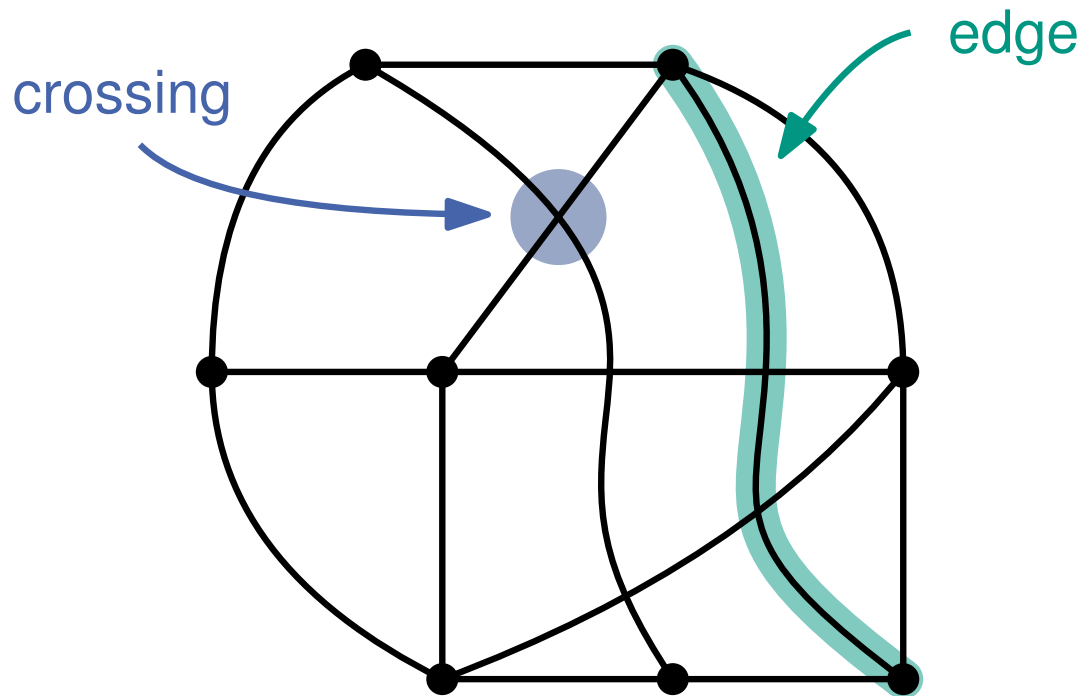
Simple Drawings



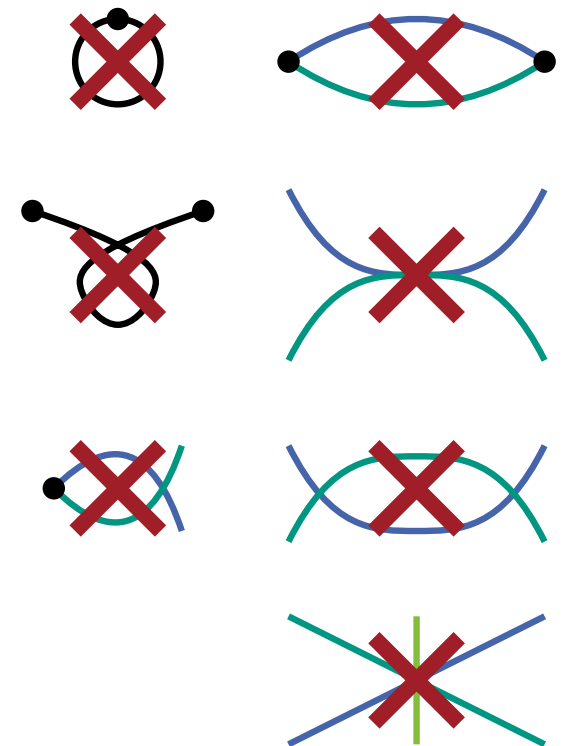
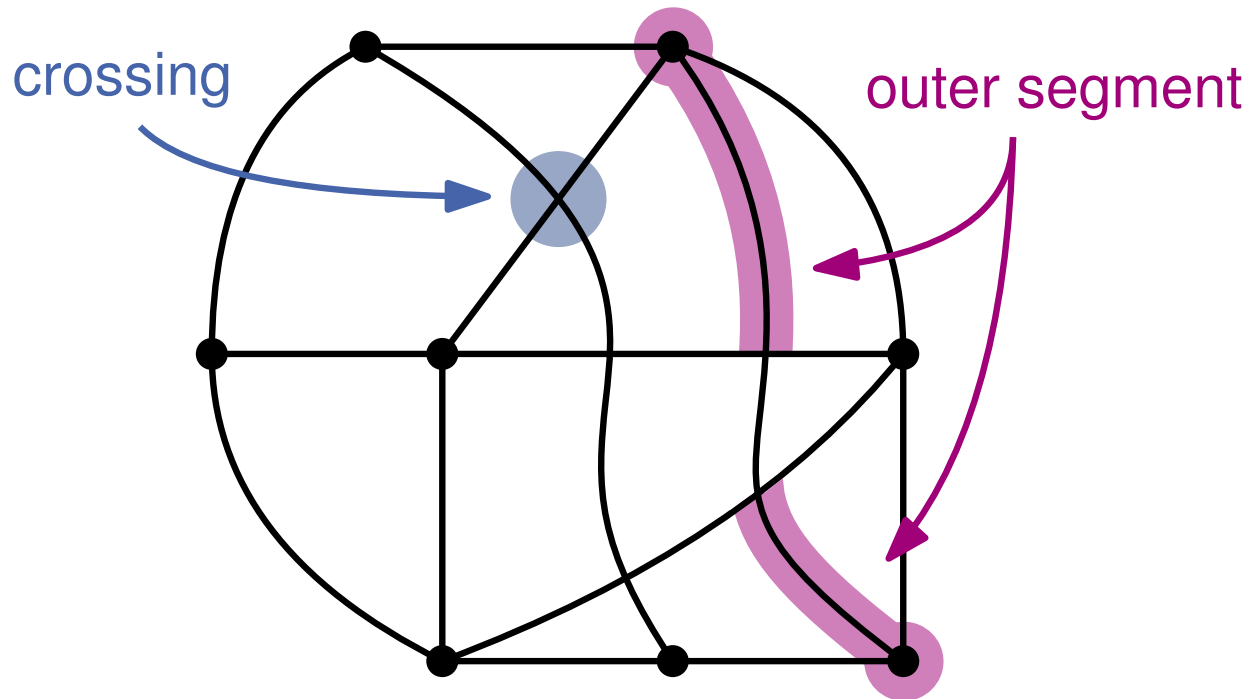
Simple Drawings



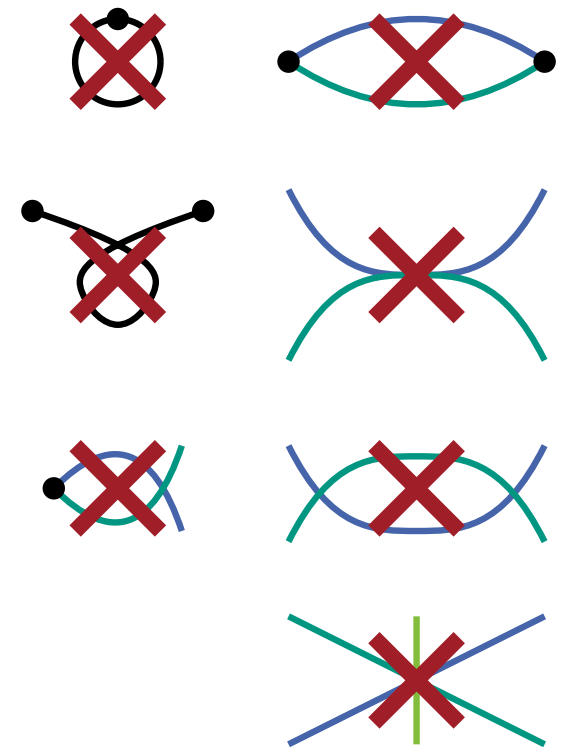
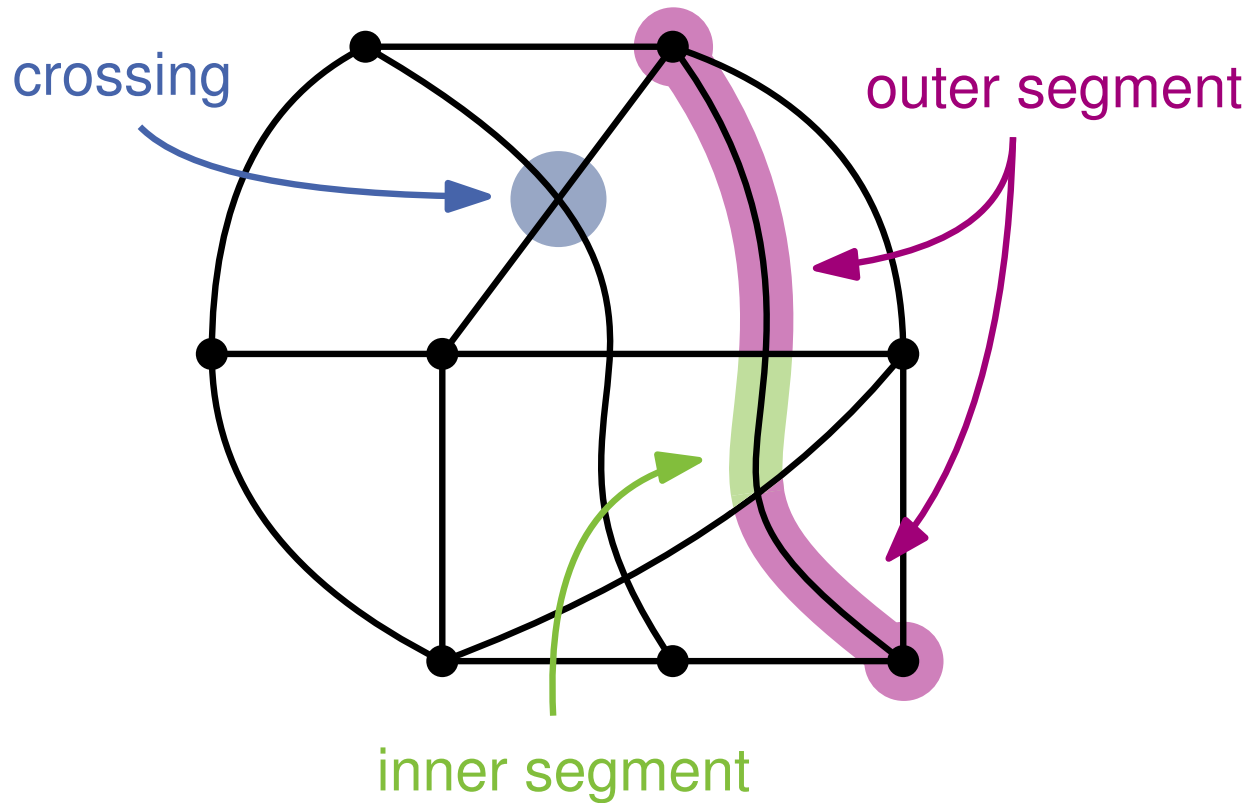
Simple Drawings



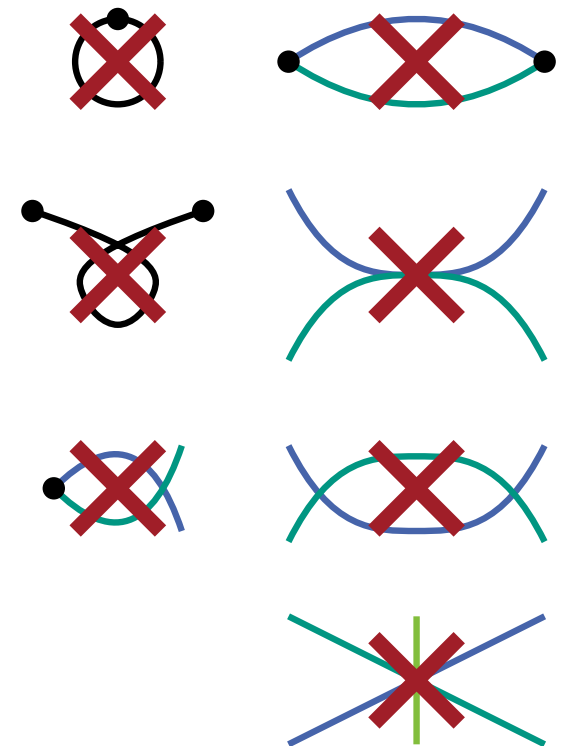
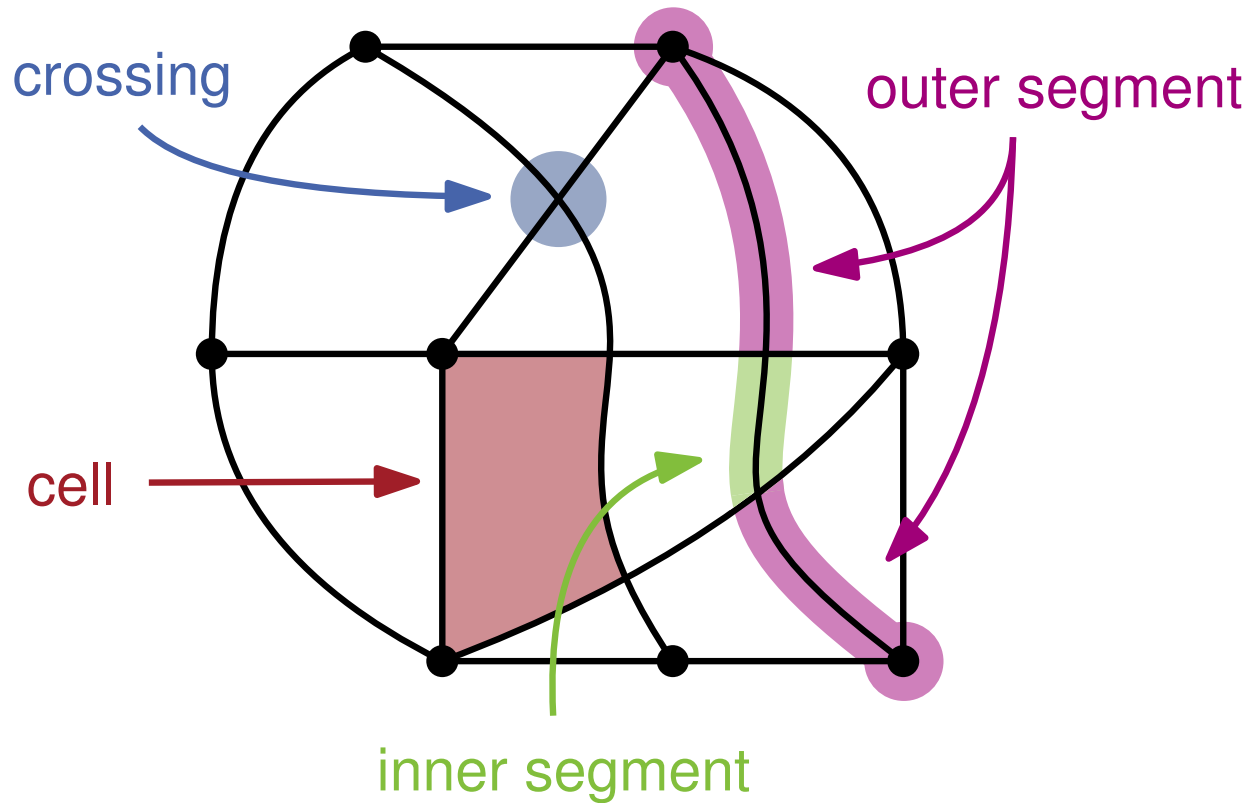
Simple Drawings



Simple Drawings



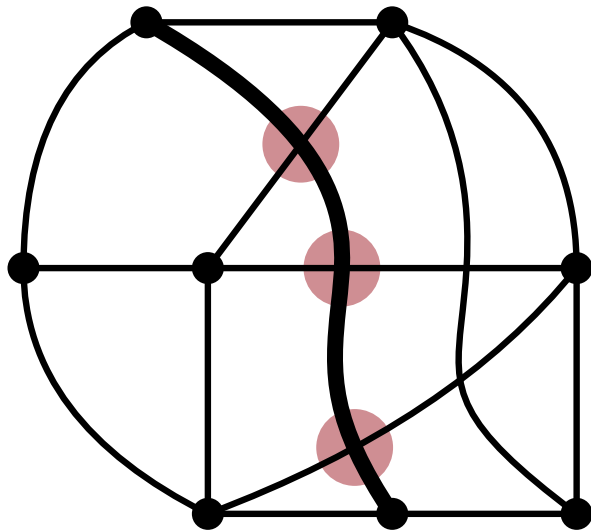
Simple Drawings



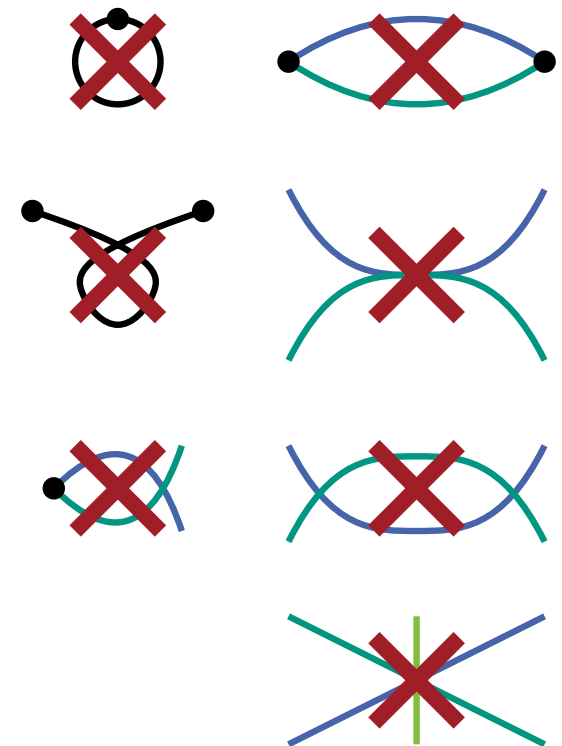
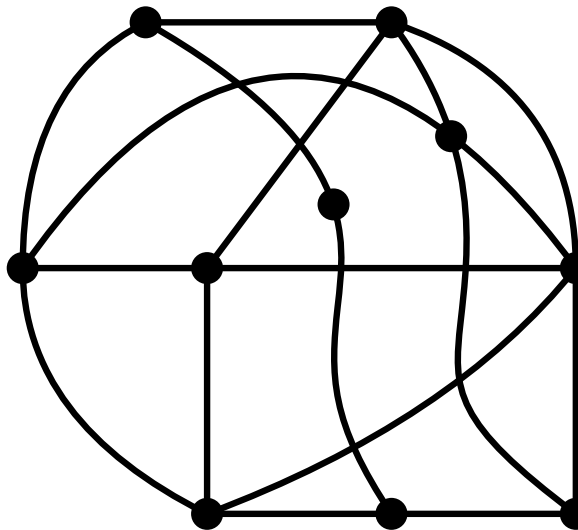
2-Plane Drawings

Def. A drawing is **2-plane** if every edge is crossed ≤ 2 .

not 2-plane



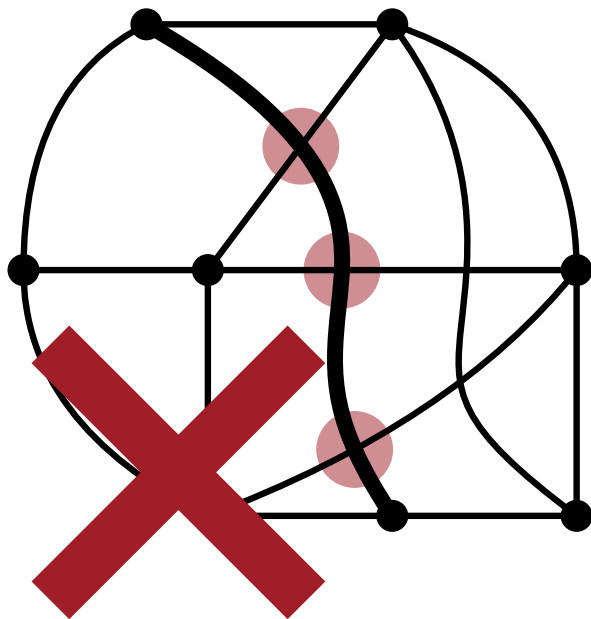
2-plane



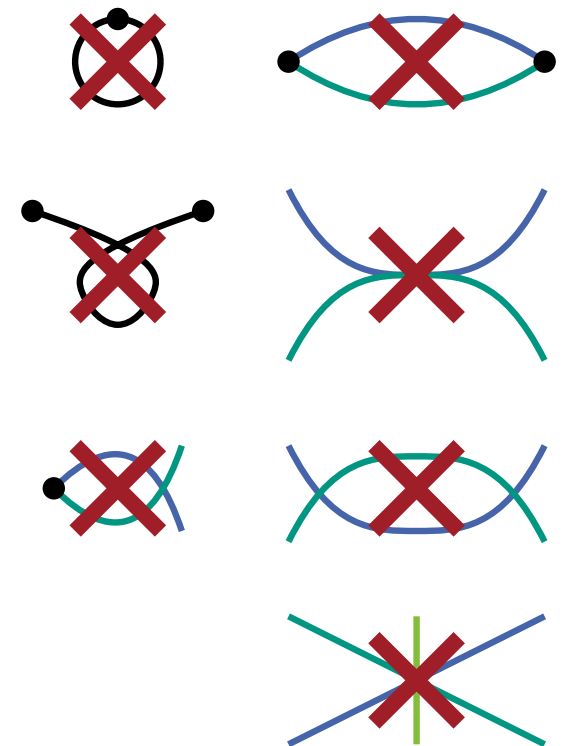
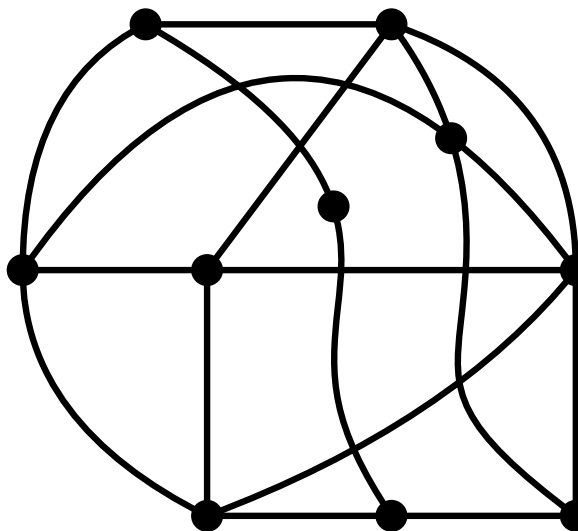
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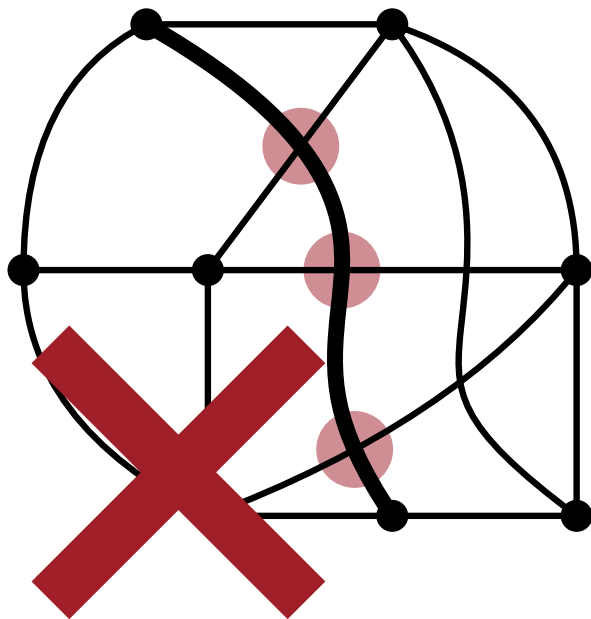
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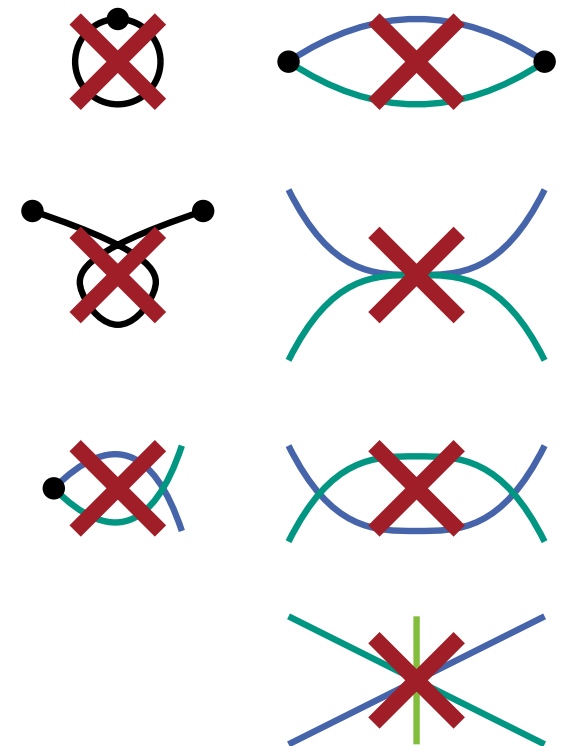
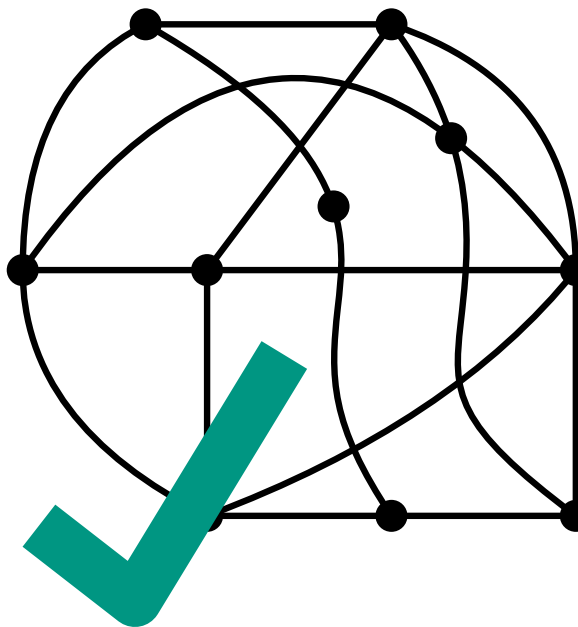
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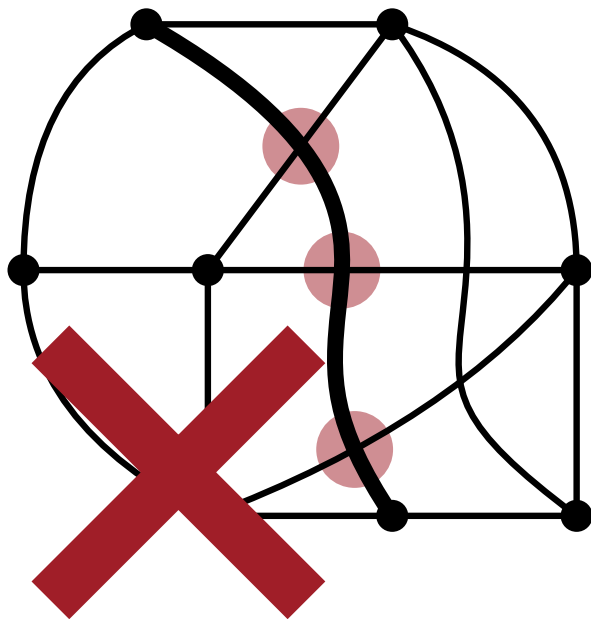


2-Plane Drawings

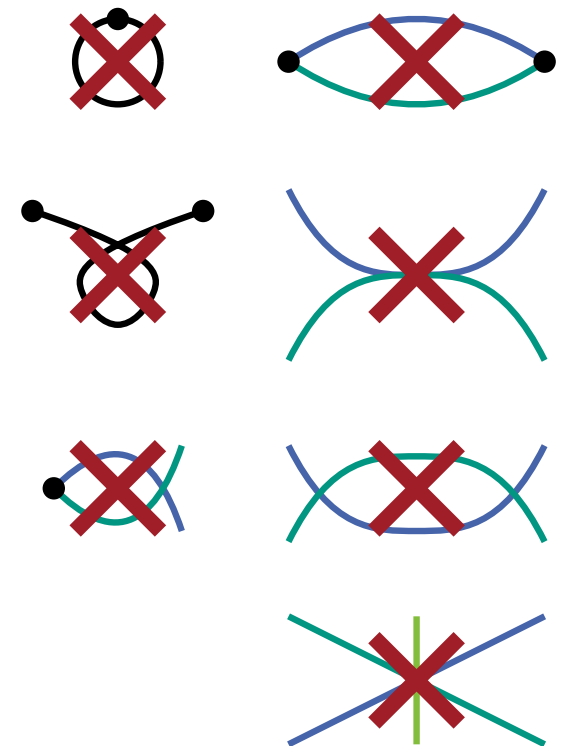
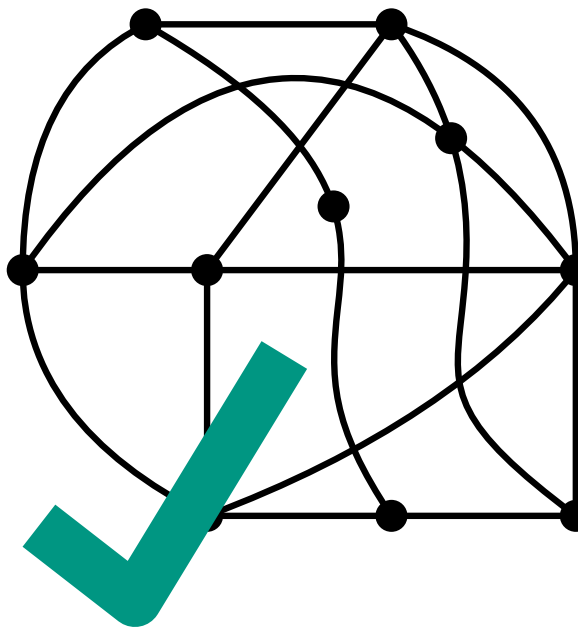
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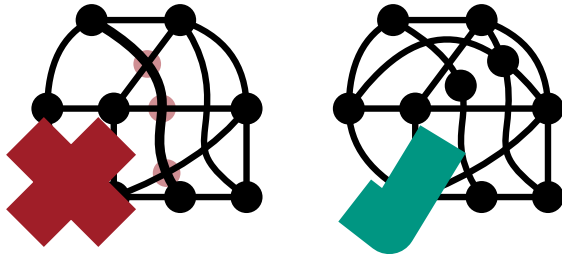
Question: How many crossings can a 2-plane drawing have?

not 2-plane

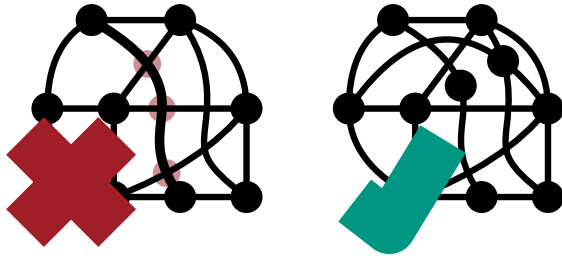


2-plane



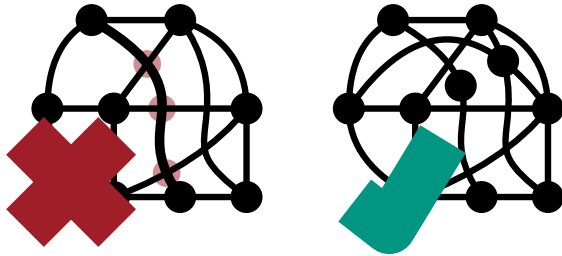


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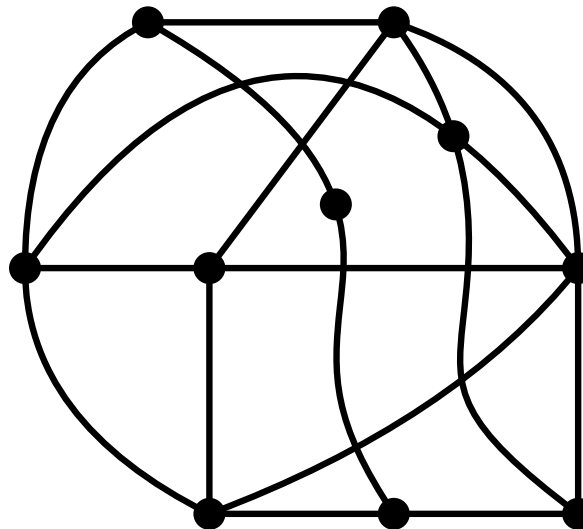
In terms of $m = |E|$:

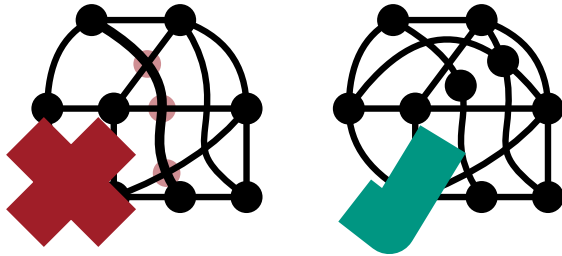
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In terms of $m = |E|$: crossings
 $|\{(e, x) \mid x \in X, e \in E, x \text{ on } e\}|$

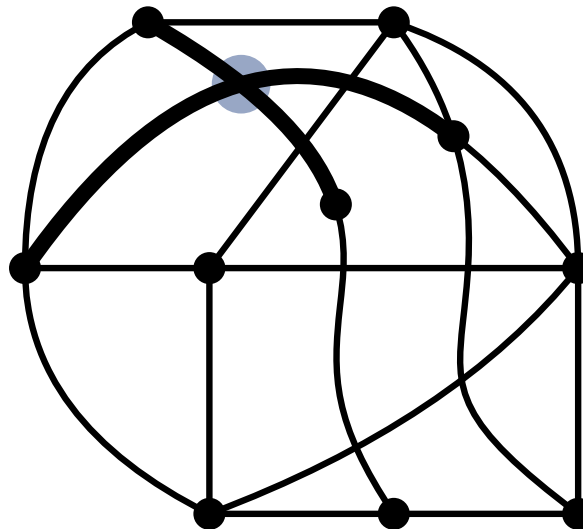


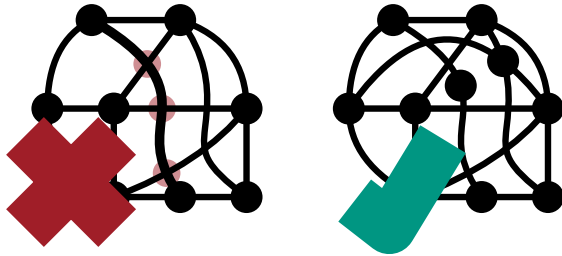


Question: How many crossings can a 2-plane drawing have?

In terms of $m = |E|$: crossings

$$2|X| = |\{(e, x) \mid x \in X, e \in E, x \text{ on } e\}|$$

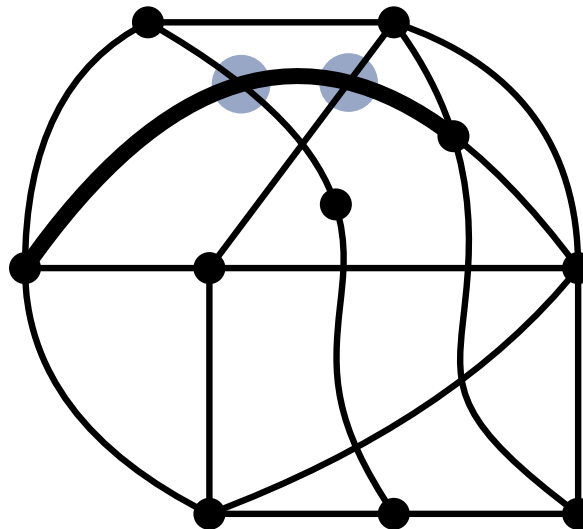


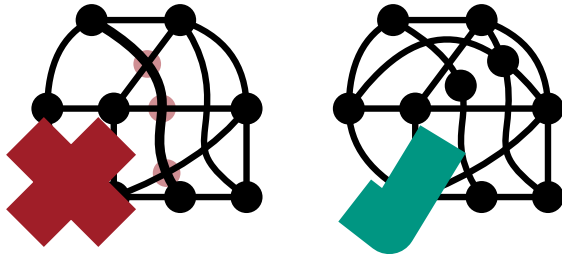


Question: How many crossings can a 2-plane drawing have?

In terms of $m = |E|$: crossings

$$2|X| = |\{(e, x) \mid x \in X, e \in E, x \text{ on } e\}| \leq 2m$$



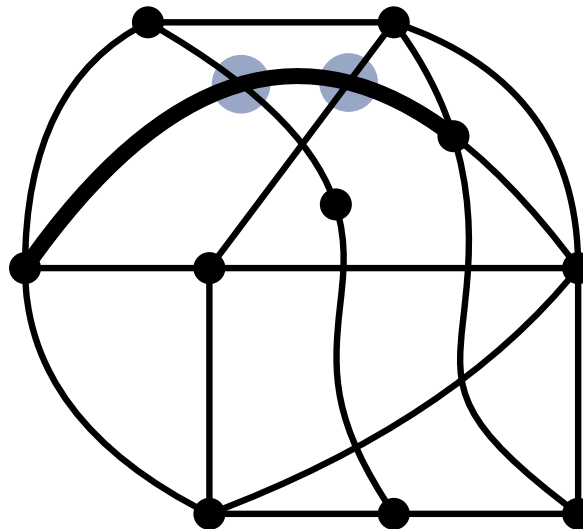


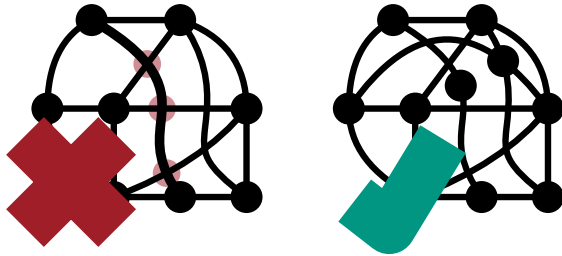
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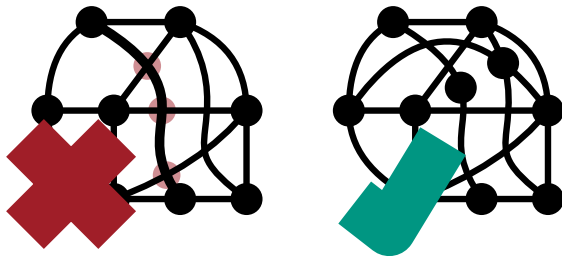
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In terms of $n = |V|$:



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In terms of $m = |E|$: crossings

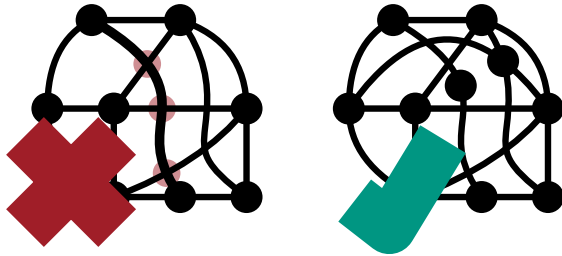
$$2|X| = |\{(e, x) \mid x \in X, e \in E, x \text{ on } e\}| \leq 2m$$

$$|X| \leq m$$

Thm: $m \leq 5n - 10$.
(Pach, Tóth)

In terms of $n = |V|$:

$$|X| \leq 5n - 10$$



Question: How many crossings can a 2-plane drawing have?

In terms of $m = |E|$: crossings

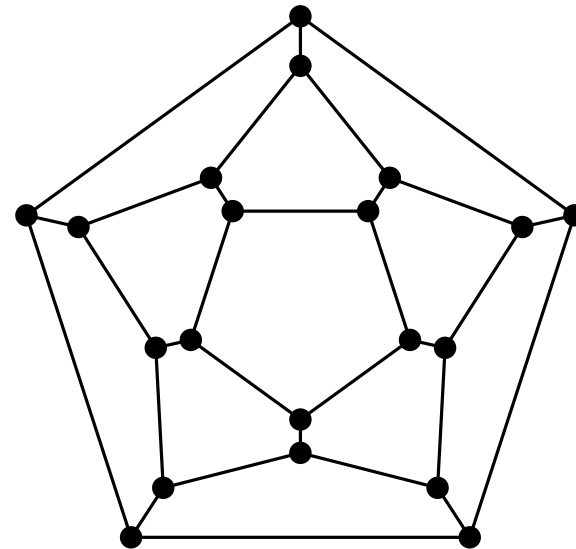
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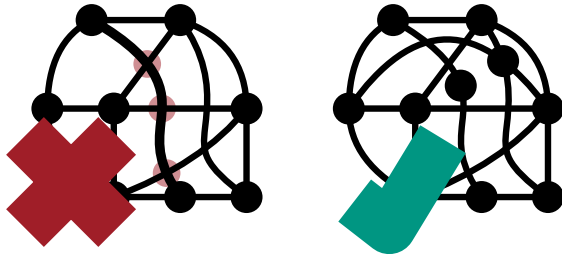
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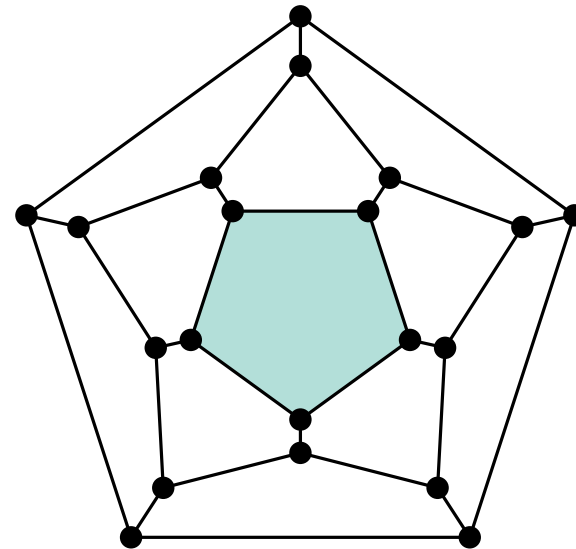
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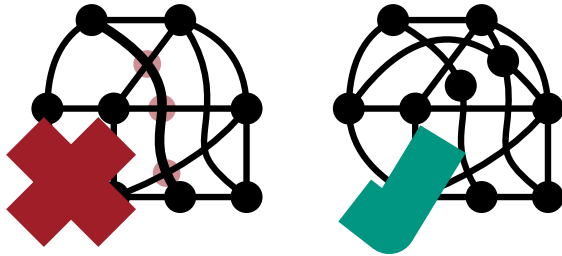
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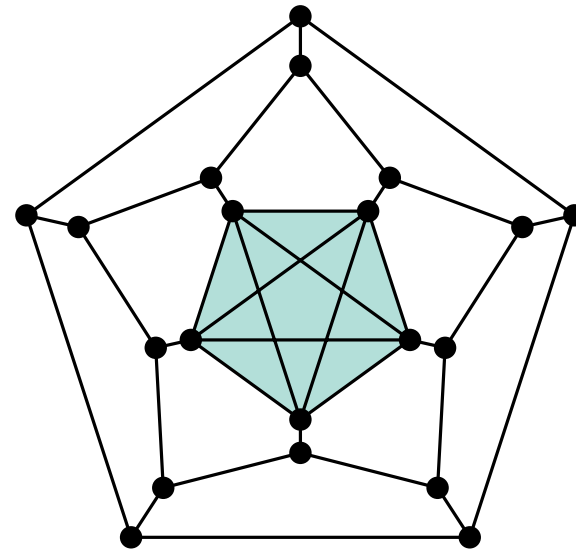
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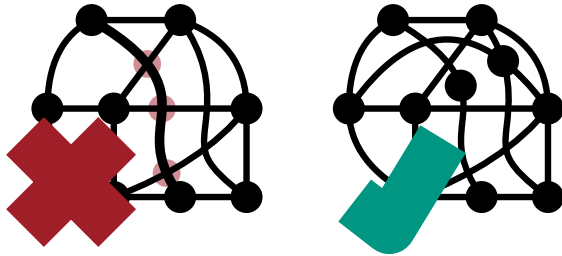
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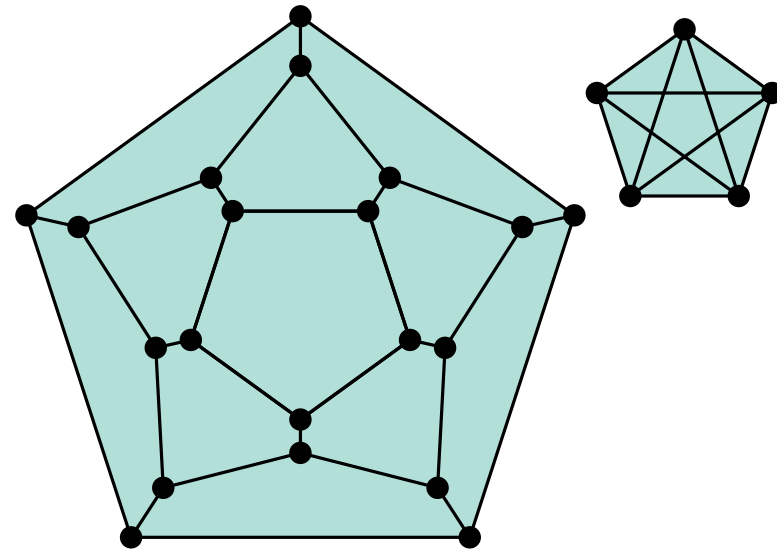
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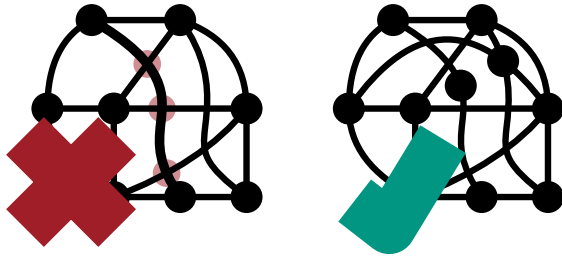
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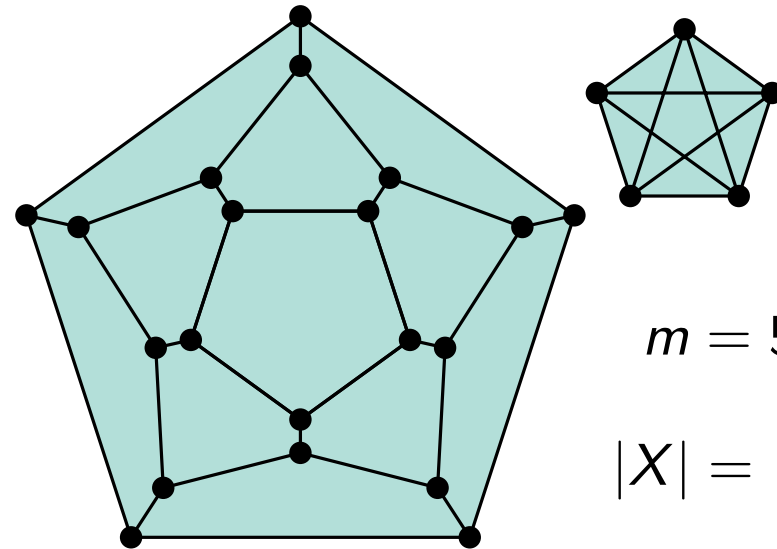
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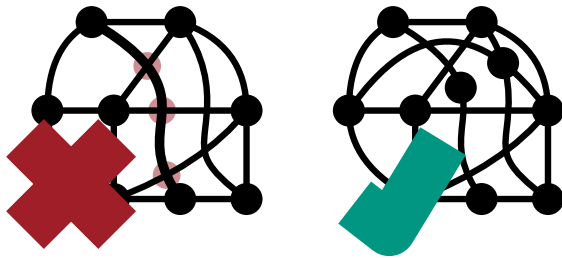
In terms of $n = |V|$:

$$|X| \leq 5n - 10$$



$$m = 5n - 10$$

$$|X| = \frac{10n - 4}{3} \leq 3.\bar{3}n$$



Question: How many crossings can a 2-plane drawing have?

In terms of $m = |E|$: crossings

$$2|X| = |\{(e, x) \mid x \in X, e \in E, x \text{ on } e\}| \leq 2m$$

$$|X| \leq m$$

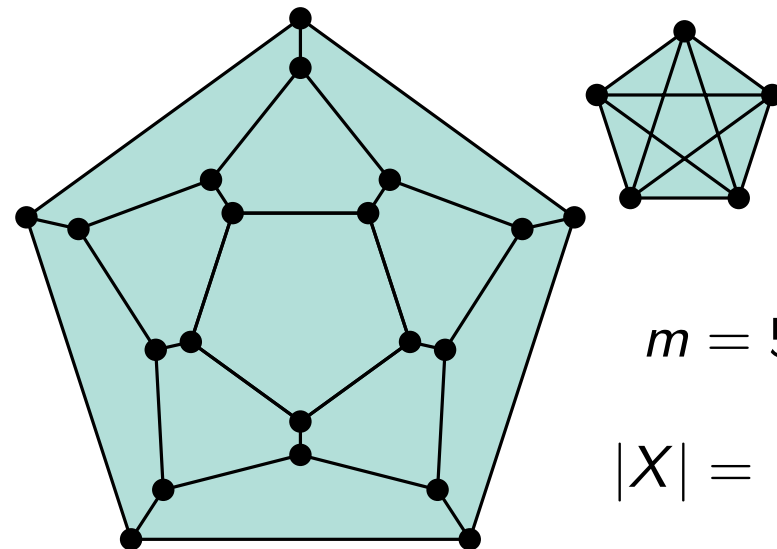
Thm: $m \leq 5n - 10$.
(Pach, Tóth)

In terms of $n = |V|$:

$$|X| \leq 5n - 10$$

Question:

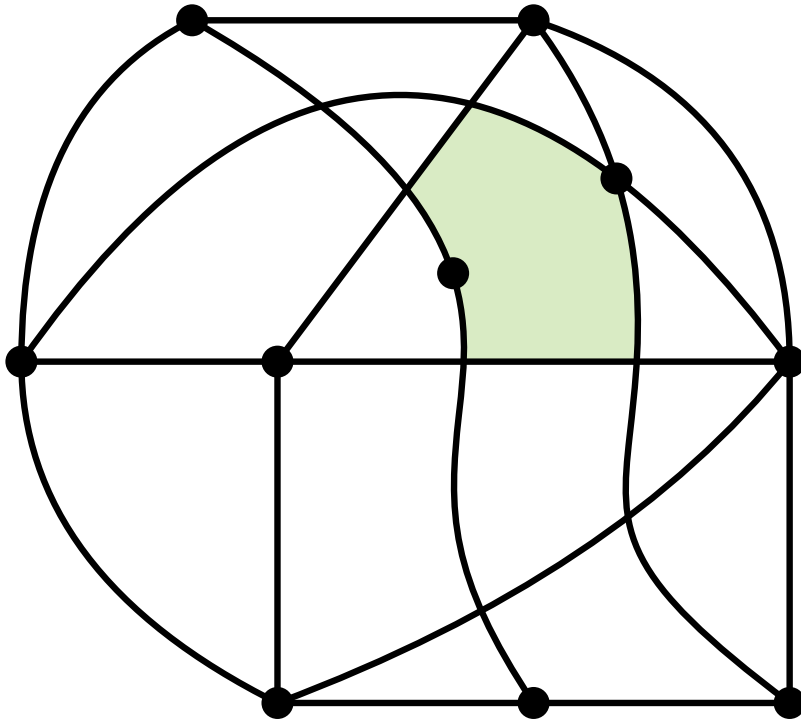
Is $|X| \leq 3.\bar{3}n$ for every 2-plane drawing?



$$m = 5n - 10$$

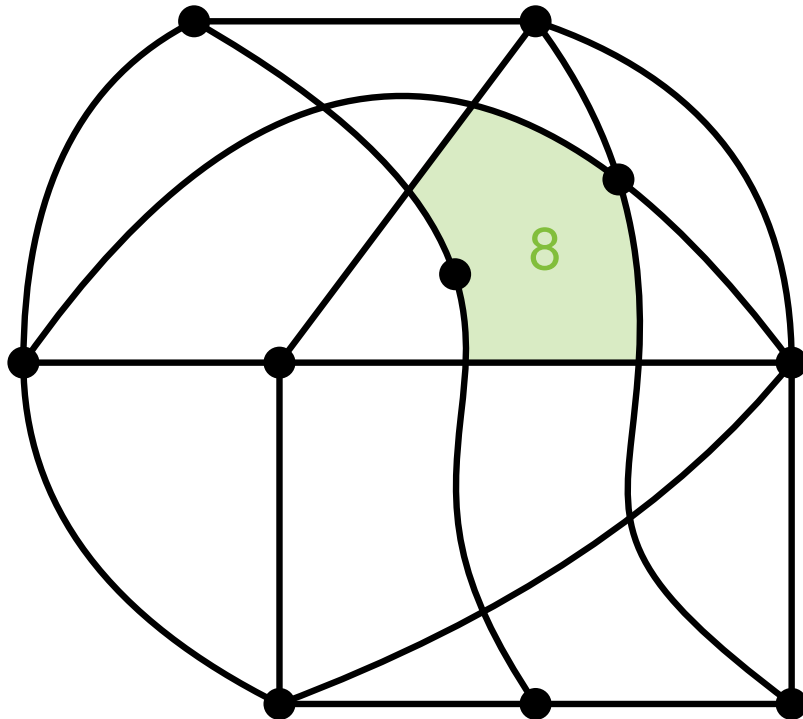
$$|X| = \frac{10n - 4}{3} \leq 3.\bar{3}n$$

Sizes of Cells



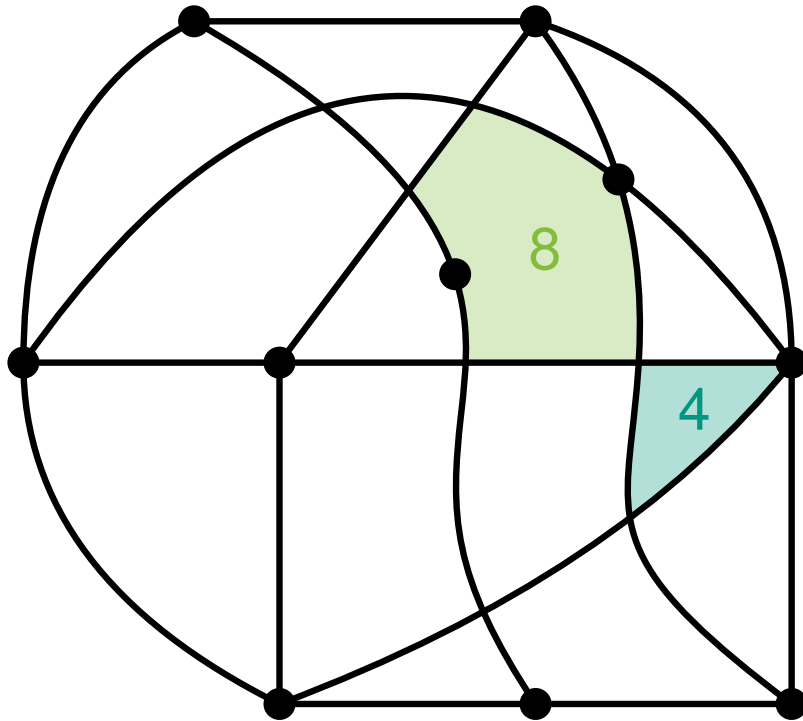
$$\|c\| = \# \bullet\text{-inc} + \#\text{segment-inc}$$

Sizes of Cells



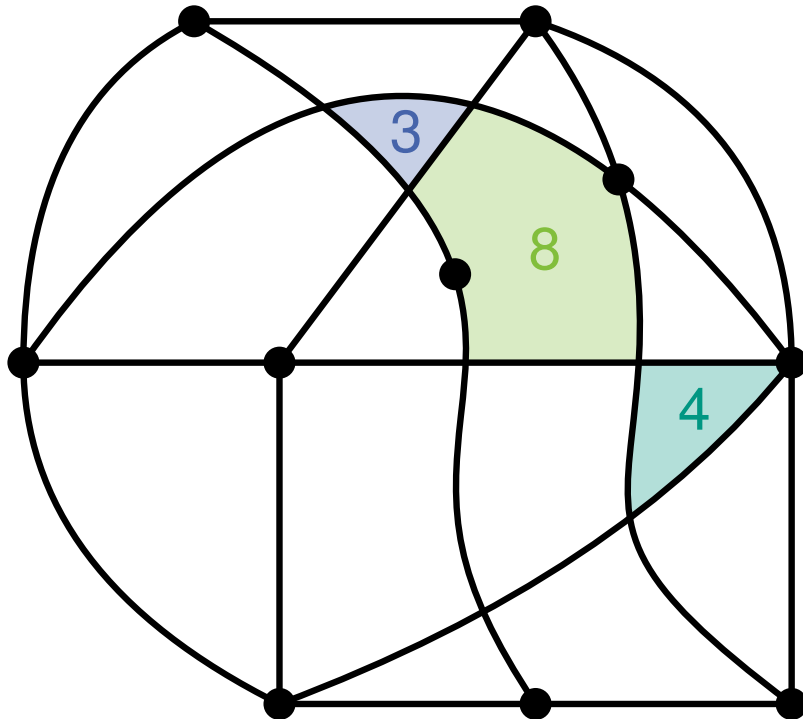
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Sizes of Cells



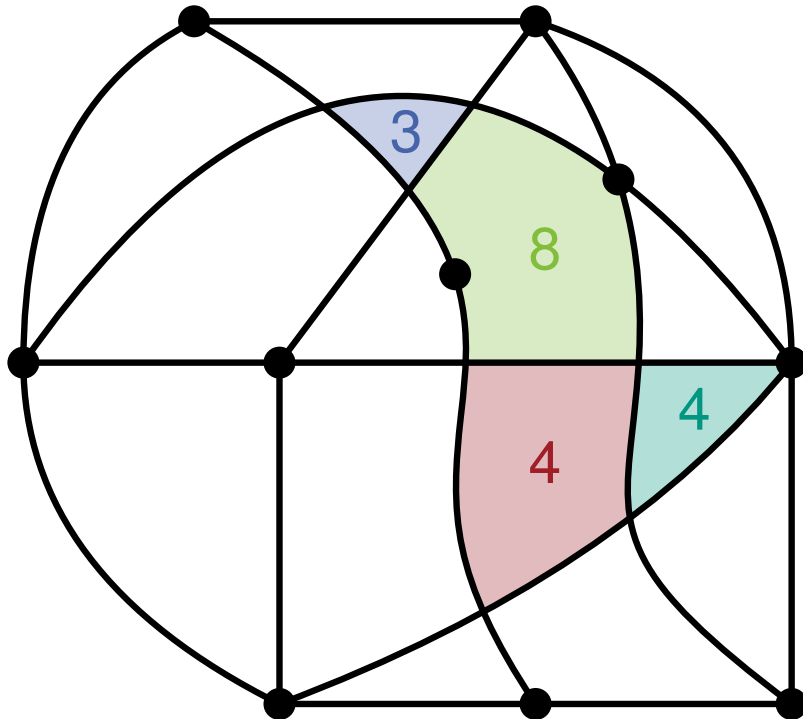
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Sizes of Cells



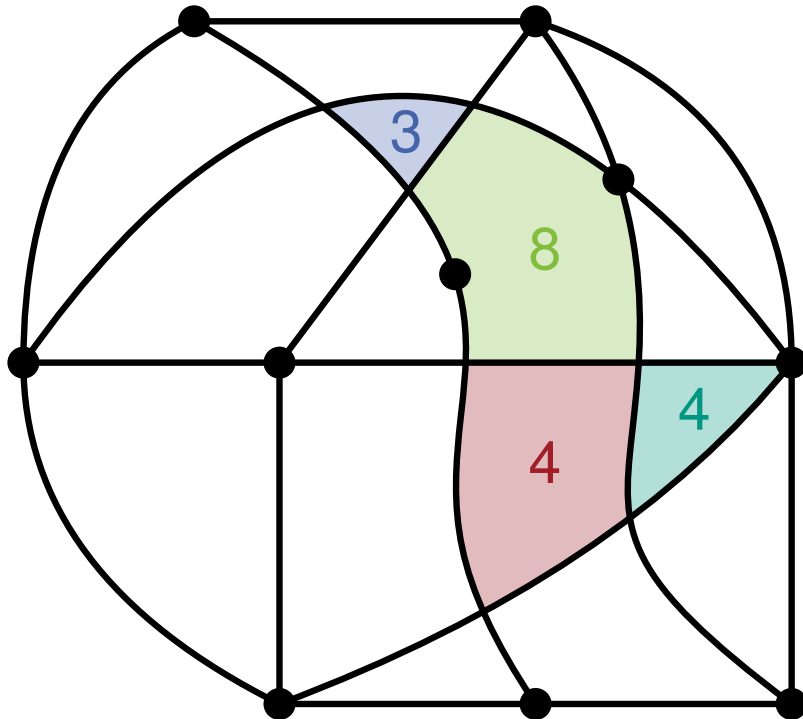
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Sizes of Cells



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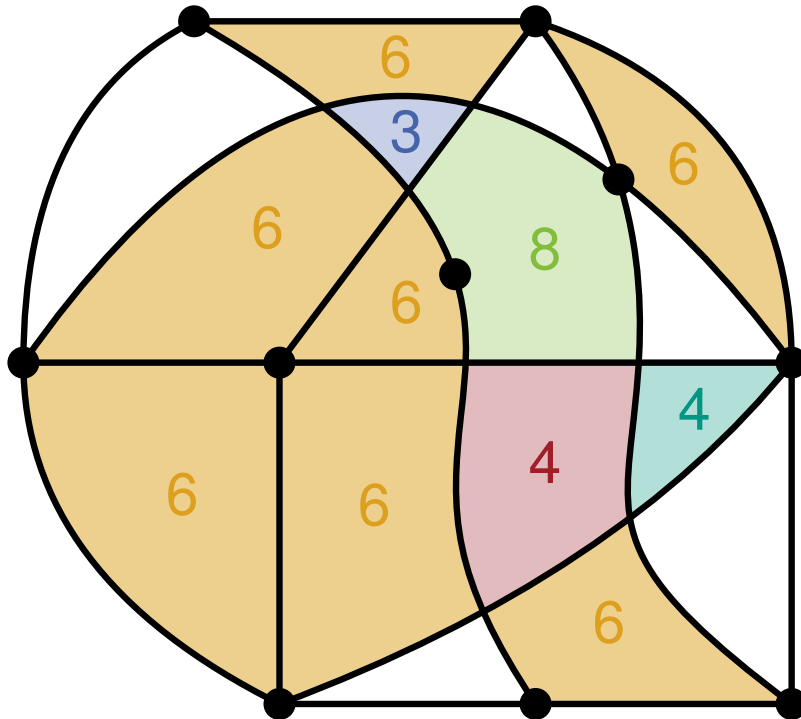
Sizes of Cells



$$\|c\| = \# \bullet\text{-inc} + \#\text{segment-inc}$$

$$\mathcal{C}_i = \{\text{cell } c : \|c\| = i\}$$

Sizes of Cells

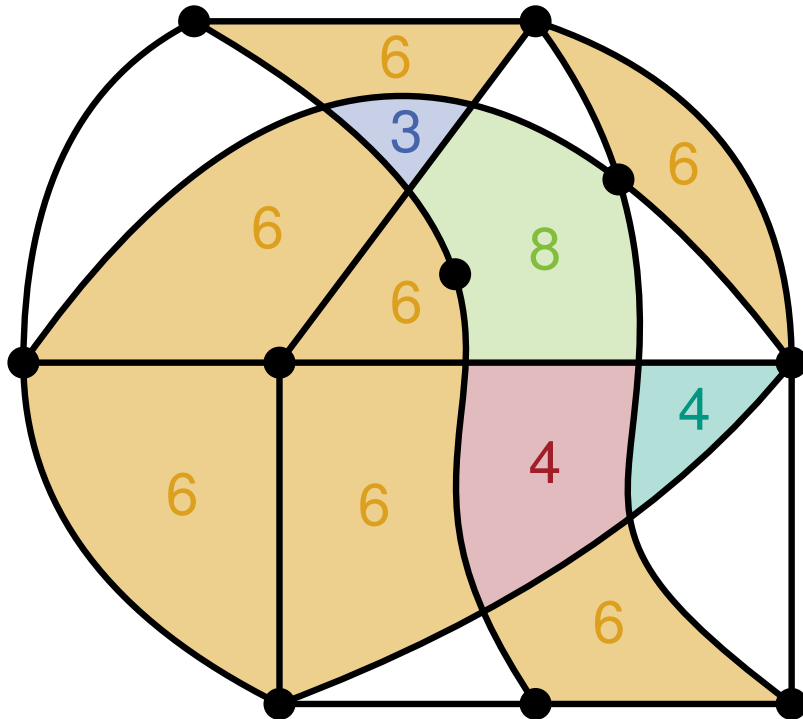


$$\|c\| = \# \bullet\text{-inc} + \#\text{segment-inc}$$

$$\mathcal{C}_i = \{\text{cell } c : \|c\| = i\}$$

$\mathcal{C}_6$

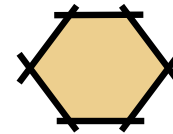
Sizes of Cells



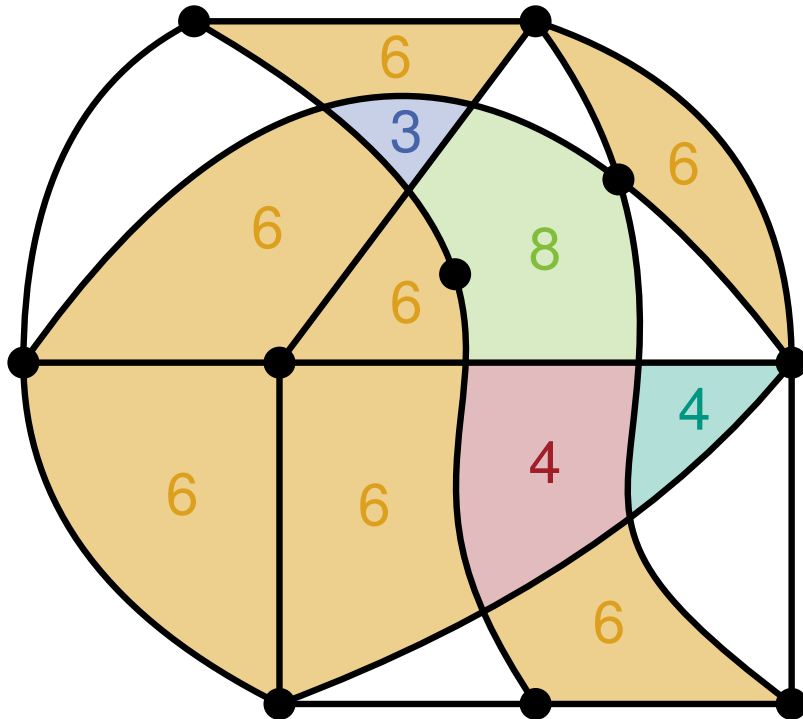
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$\mathcal{C}_6$



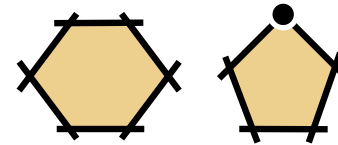
Sizes of Cells



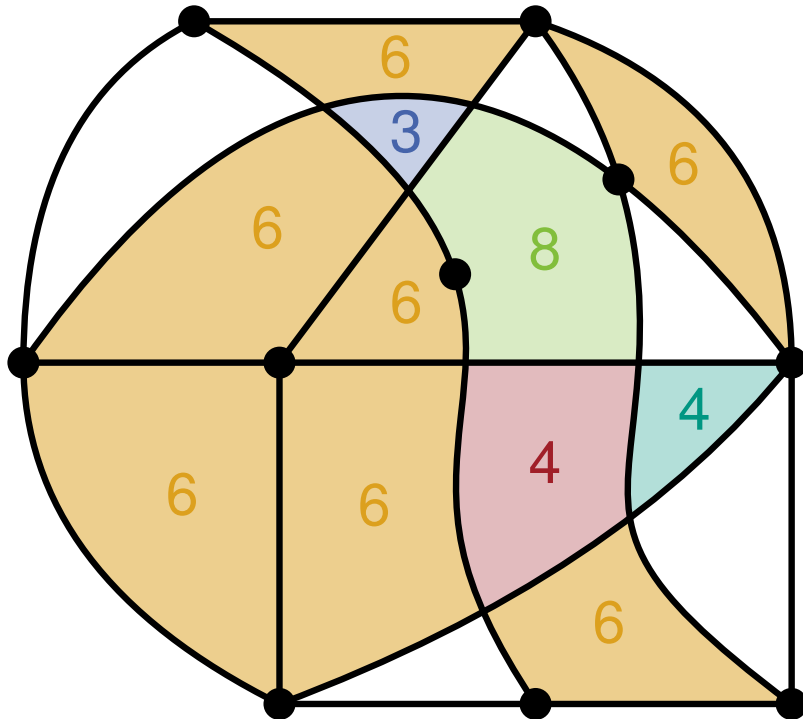
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$\mathcal{C}_6$



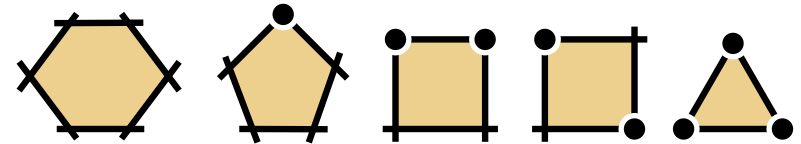
Sizes of Cells



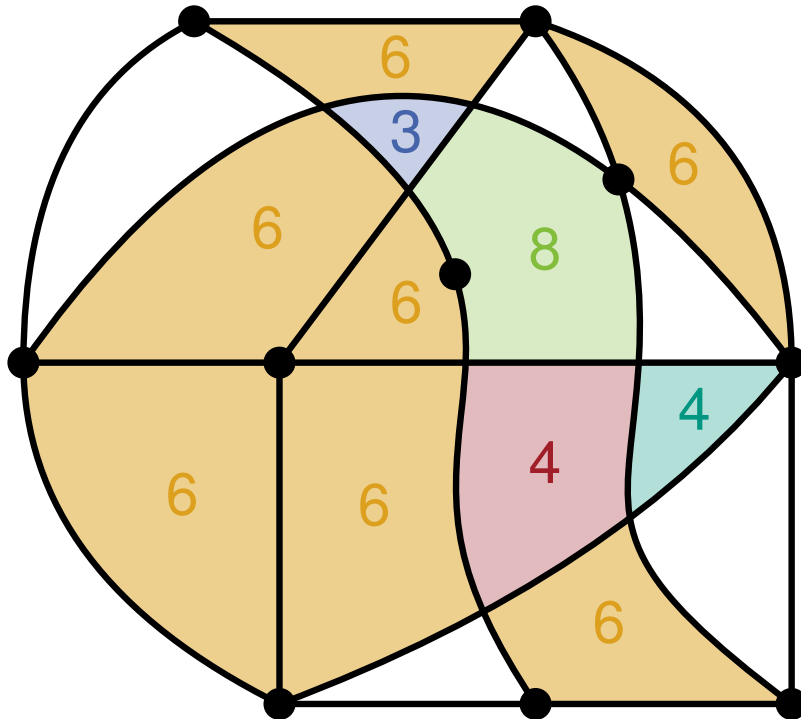
$$\|c\| = \# \bullet\text{-inc} + \#\text{segment-inc}$$

$$\mathcal{C}_i = \{\text{cell } c : \|c\| = i\}$$

$\mathcal{C}_6$

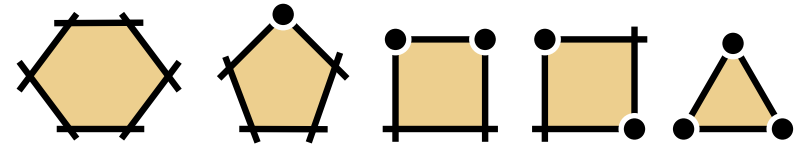
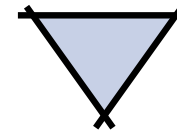


Sizes of Cells

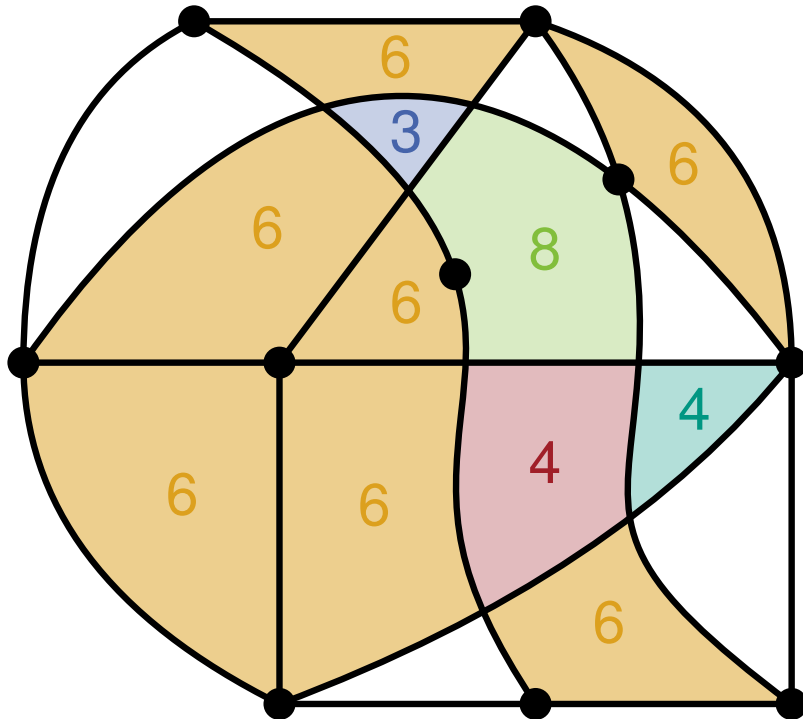


$$\|c\| = \# \bullet\text{-inc} + \#\text{segment-inc}$$

$$\mathcal{C}_i = \{\text{cell } c : \|c\| = i\}$$

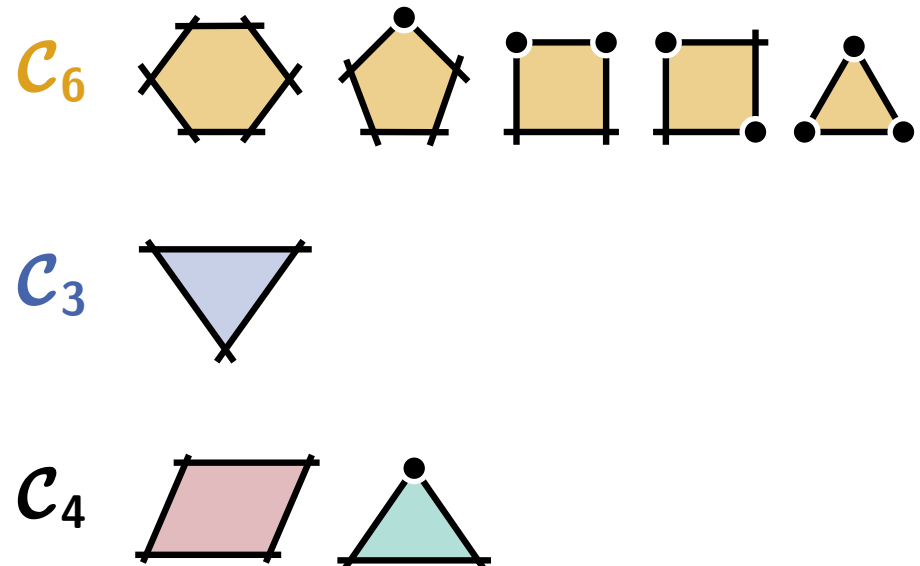
 $\mathcal{C}_6$

 $\mathcal{C}_3$


Sizes of Cells



$$\|c\| = \# \bullet\text{-inc} + \#\text{segment-inc}$$

$$\mathcal{C}_i = \{\text{cell } c : \|c\| = i\}$$

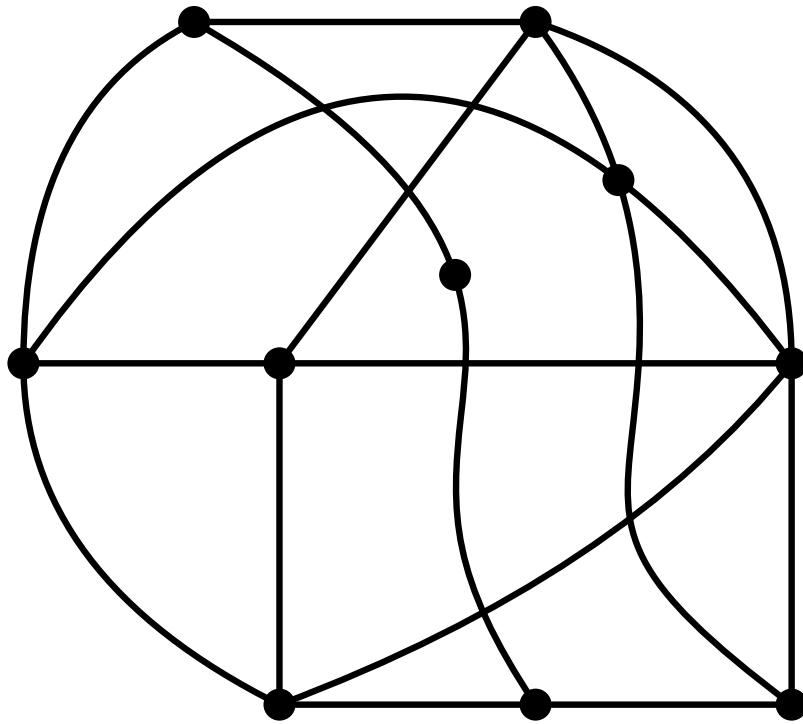


Density Formula

Lemma: For every $t \in \mathbb{R}$ and every connected drawing of a graph $G = (V, E)$ with $|E| \geq 1$, we have

$$|X| = t \cdot (|V| - 2) + \sum_{i=1}^{\infty} \left(t + \frac{1-t}{4} i \right) |C_i| - |E|$$

(Kaufmann, Klemz, Knorr, Reddy, Schröder, Ueckerdt 2024)

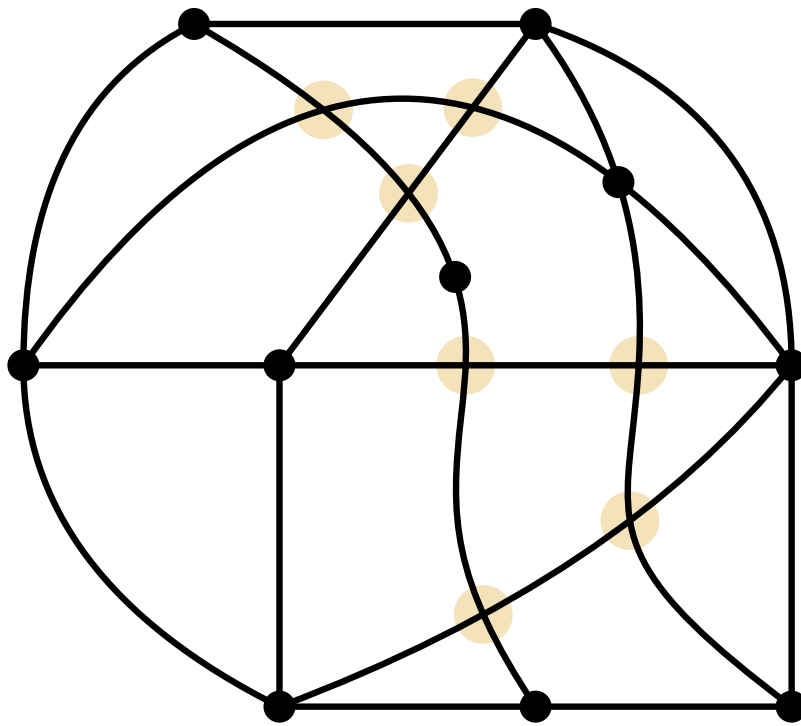


Density Formula

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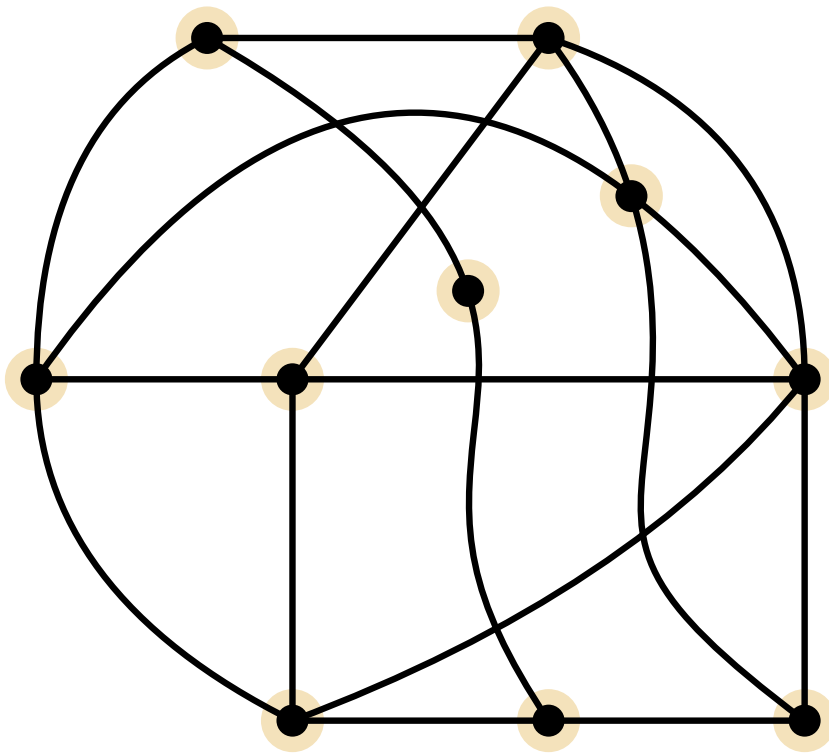
7 =

Density Formula

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(Kaufmann, Klemz, Knorr, Reddy, Schröder, Ueckerdt 2024)



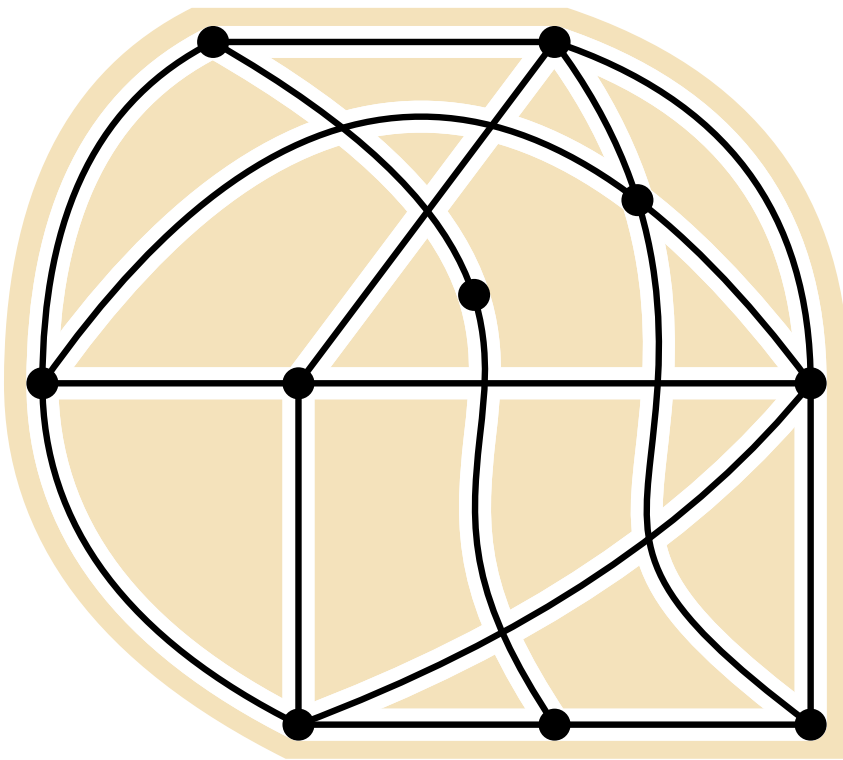
$$7 = t \cdot (10 - 2)$$

Density Formula

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$$|X| = t \cdot (|V| - 2) + \sum_{i=1}^{\infty} \left(t + \frac{1-t}{4} i \right) |C_i| - |E|$$

(Kaufmann, Klemz, Knorr, Reddy, Schröder, Ueckerdt 2024)



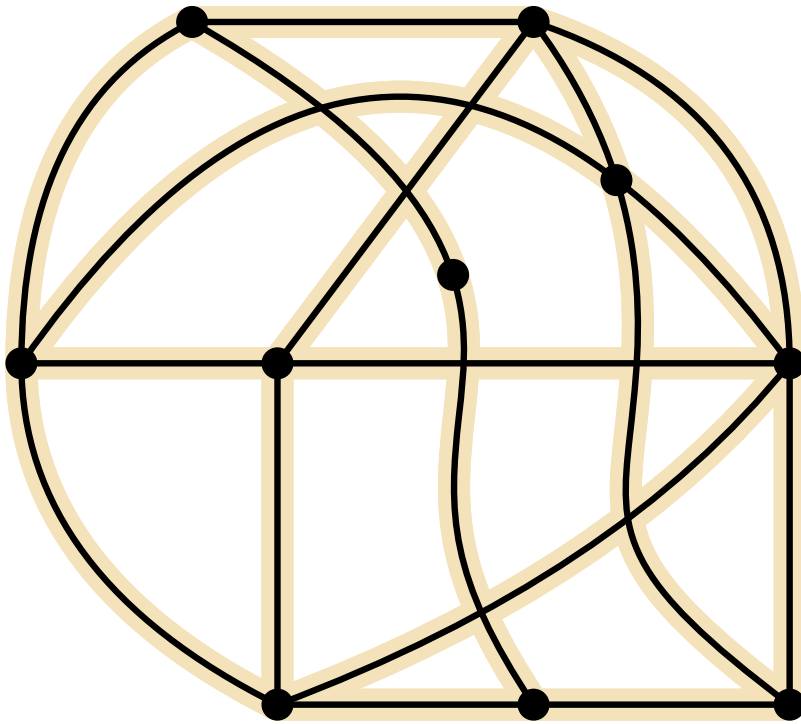
$$7 = t \cdot (10 - 2) + \sum_{i=1}^{\infty} w_{i,t} |C_i|$$

Density Formula

Lemma: For every $t \in \mathbb{R}$ and every connected drawing of a graph $G = (V, E)$ with $|E| \geq 1$, we have

$$|X| = t \cdot (|V| - 2) + \sum_{i=1}^{\infty} \left(t + \frac{1-t}{4} i \right) |C_i| - |E|$$

(Kaufmann, Klemz, Knorr, Reddy, Schröder, Ueckerdt 2024)



$$7 = t \cdot (10 - 2) + \sum_{i=1}^{\infty} w_{i,t} |C_i| - 18$$

$$|X| = t \cdot (|V| - 2) + \sum_{i=1}^{\infty} \left(t + \frac{1-t}{4} i \right) |\mathcal{C}_i| - |E|$$

A complex geometric diagram featuring a circle divided into multiple regions by several intersecting arcs. The regions are colored in four distinct colors: light blue, light orange, light green, and light teal. Each region contains a numerical value:

- Light Blue Regions (7 total):** Values include 5, 5, 5, 5, 5, 5, and 5.
- Light Orange Regions (7 total):** Values include 6, 6, 6, 6, 6, 6, and 6.
- Light Green Region (1 total):** Value is 8.
- Light Teal Regions (2 total):** Values are 4 and 4.
- Other Labels:** A small purple-shaded triangular region at the top center contains the value 3. A large black number 14 is positioned outside the bottom-left boundary of the circle.

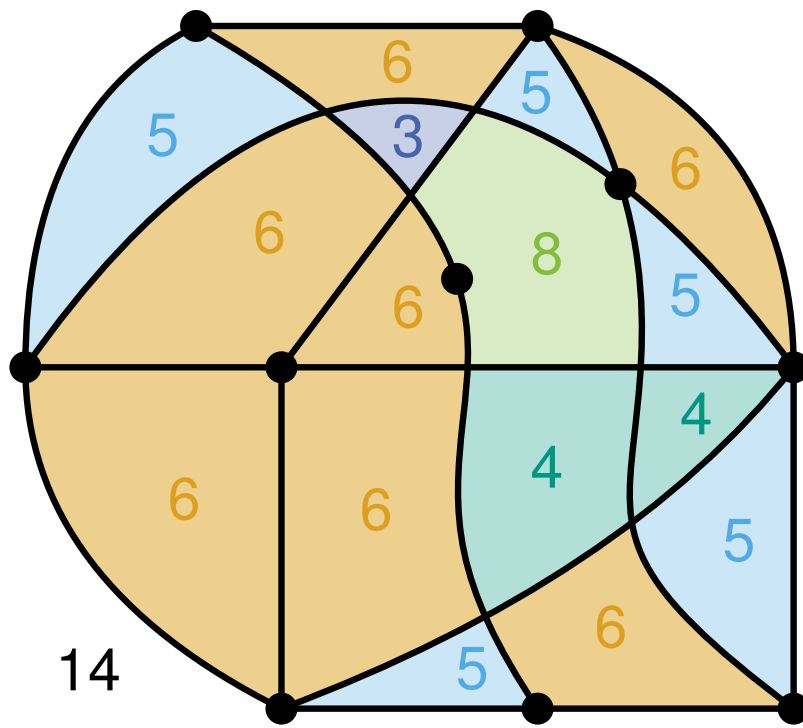
$$7 = t \cdot (10 - 2) + \sum_{i=1}^{\infty} w_{i,t} |\mathcal{C}_i| - 18$$

Density Formula

Lemma: For every $t \in \mathbb{R}$ and every connected drawing of a graph $G = (V, E)$ with $|E| \geq 1$, we have

$$|X| = t \cdot (|V| - 2) + \sum_{i=1}^{\infty} \left(t + \frac{1-t}{4} i \right) |C_i| - |E|$$

(Kaufmann, Klemz, Knorr, Reddy, Schröder, Ueckerdt 2024)



$$t = 5 \quad |V| \geq 3$$

$$|X| = 5(|V| - 2) + \sum_{i=3}^{\infty} (5 - i) |C_i| - |E|$$

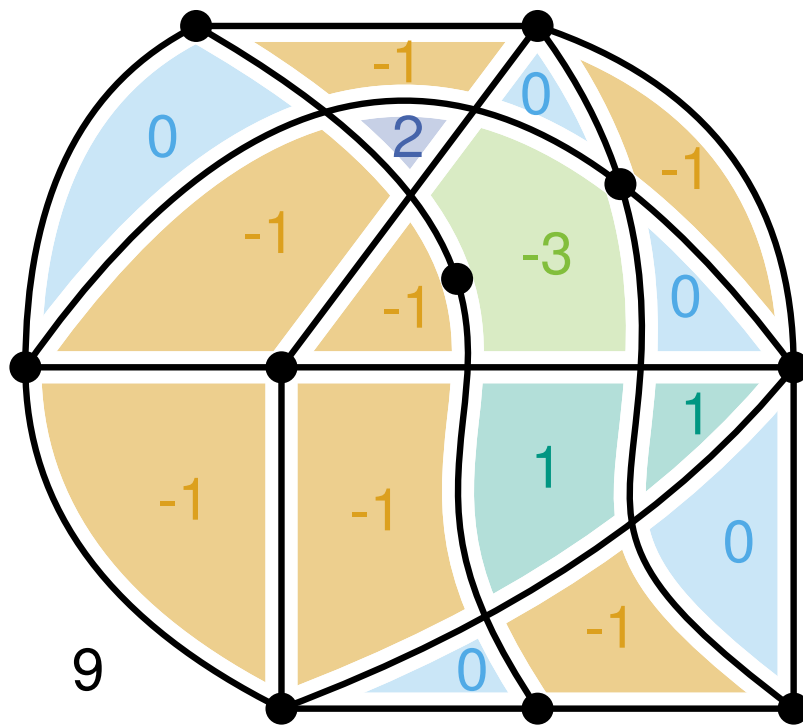
$$7 = t \cdot (10 - 2) + \sum_{i=1}^{\infty} w_{i,t} |C_i| - 18$$

Density Formula

Lemma: For every $t \in \mathbb{R}$ and every connected drawing of a graph $G = (V, E)$ with $|E| \geq 1$, we have

$$|X| = t \cdot (|V| - 2) + \sum_{i=1}^{\infty} \left(t + \frac{1-t}{4}i\right) |C_i| - |E|$$

(Kaufmann, Klemz, Knorr, Reddy, Schröder, Ueckerdt 2024)



$$t = 5$$

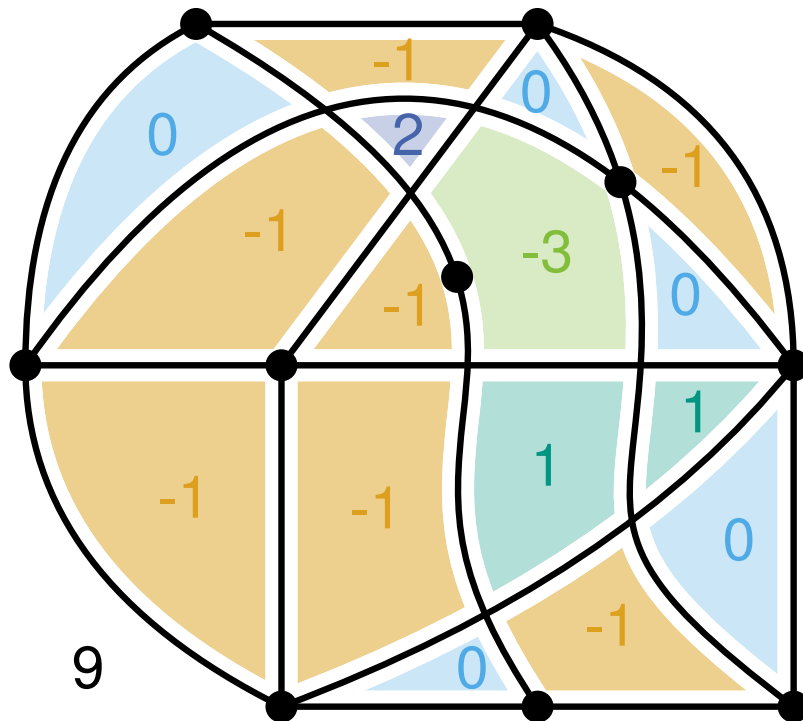
$$|V| \geq 3$$

$$|X| = 5(|V| - 2) + \sum_{i=3}^{\infty} (5 - i) |C_i| - |E|$$

$$|X| = t \cdot (|V| - 2) + \sum_{i=1}^{\infty} \left(t + \frac{1-t}{4}i\right) |\mathcal{C}_i| - |E|$$
 $t = 5 \quad |V| \geq 3$

$$|X| = 5(|V| - 2) + \sum_{i=3}^{\infty} (5 - i) |\mathcal{C}_i| - |E|$$

$$|X| = 5(|V| - 2) + 2 \nabla + \text{parallelogram} + \text{triangle with dot} \\ - \sum_{i=6}^{\infty} (i - 5) |\mathcal{C}_i| - |E|$$



First Attempt

Question: Is $|X| \leq \frac{10}{3}n$ for every 2-plane drawing?

Obs: May assume no plane edges.

$$|X| = 5(n - 2) + 2 \triangle + \square + \triangle \circ - \sum_{i=6}^{\infty} (i - 5) |\mathcal{C}_i| - m$$

First Attempt

Question: Is $|X| \leq \frac{10}{3}n$ for every 2-plane drawing?

Obs: May assume no plane edges.

$$|X| = 5(n - 2) + 2 \nabla + \text{red parallelogram} + \text{green triangle with dot} - \sum_{i=6}^{\infty} (i - 5) |\mathcal{C}_i| - m$$

Idea: bound ∇ , ,  in terms of $|X|$, m and $\sum_{i=6}^{\infty} (i - 5) |\mathcal{C}_i|$

First Attempt

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$$|X| = 5(n - 2) + 2 \nabla + \text{red parallelogram} + \text{triangle with dot} - \sum_{i=6}^{\infty} (i - 5) |\mathcal{C}_i| - m$$

Idea: bound ∇ , red parallelogram , triangle with dot in terms of $|X|$, m and $\sum_{i=6}^{\infty} (i - 5) |\mathcal{C}_i|$

Observation:

$$|X| \geq 3 \nabla$$

First Attempt

Question: Is $|X| \leq \frac{10}{3}n$ for every 2-plane drawing?

Obs: May assume no plane edges.

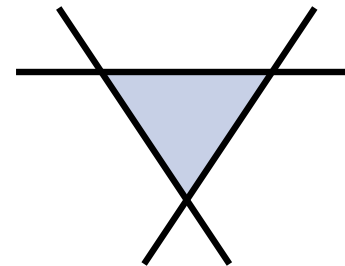
$$|X| = 5(n - 2) + 2 \nabla + \text{red parallelogram} + \text{green triangle with dot} - \sum_{i=6}^{\infty} (i - 5) |\mathcal{C}_i| - m$$

Idea: bound ∇ , ,  in terms of $|X|$, m and $\sum_{i=6}^{\infty} (i - 5) |\mathcal{C}_i|$

Observation:

$$|X| \geq 3 \nabla$$

$$|\{(x, c) \mid x \in X, c \text{ is } \nabla, x \text{ on } c\}|$$



First Attempt

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Obs: May assume no plane edges.

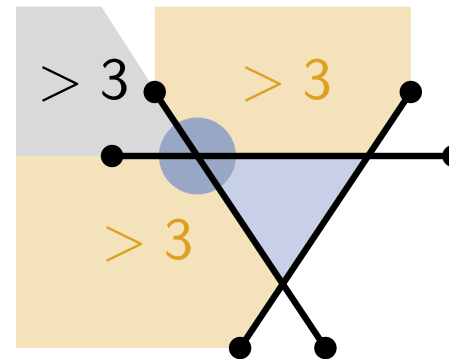
$$|X| = 5(n - 2) + 2 \nabla + \text{red parallelogram} + \text{green triangle with dot} - \sum_{i=6}^{\infty} (i - 5) |\mathcal{C}_i| - m$$

Idea: bound ∇ , red parallelogram, green triangle with dot in terms of $|X|$, m and $\sum_{i=6}^{\infty} (i - 5) |\mathcal{C}_i|$

Observation:

$$|X| \geq 3 \nabla$$

$$|\{(x, c) \mid x \in X, c \text{ is } \nabla, x \text{ on } c\}|$$



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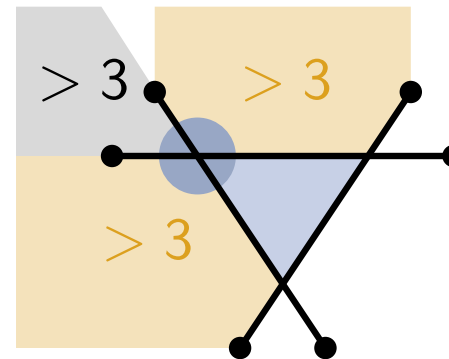
$$|X| = 5(n - 2) + 2 \nabla + \text{red parallelogram} + \text{green triangle with dot} - \sum_{i=6}^{\infty} (i - 5) |\mathcal{C}_i| - m$$

Idea: bound ∇ , ,  in terms of $|X|$, m and $\sum_{i=6}^{\infty} (i - 5) |\mathcal{C}_i|$

Observation:

$$|X| \geq 3 \nabla$$

$$|X| \geq |\{(x, c) \mid x \in X, c \text{ is } \nabla, x \text{ on } c\}|$$



First Attempt

Question: Is $|X| \leq \frac{10}{3}n$ for every 2-plane drawing?

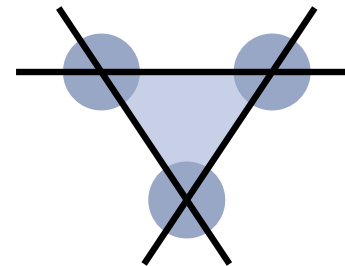
Obs: May assume no plane edges.

$$|X| = 5(n - 2) + 2 \nabla + \text{red parallelogram} + \text{green triangle with dot} - \sum_{i=6}^{\infty} (i - 5) |\mathcal{C}_i| - m$$

Idea: bound ∇ , red parallelogram, green triangle with dot in terms of $|X|$, m and $\sum_{i=6}^{\infty} (i - 5) |\mathcal{C}_i|$

Observation:

$$|X| \geq 3 \nabla \quad |X| \geq |\{(x, c) \mid x \in X, c \text{ is } \nabla, x \text{ on } c\}| = 3 \nabla$$



First Attempt

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Obs: May assume no plane edges.

$$|X| = 5(n - 2) + 2 \nabla + \text{red parallelogram} + \text{green triangle with dot} - \sum_{i=6}^{\infty} (i - 5) |\mathcal{C}_i| - m$$

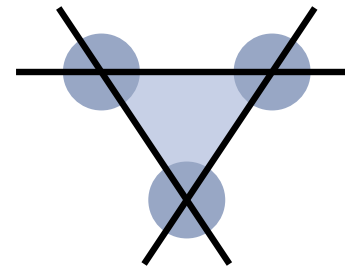
Idea: bound ∇ , red parallelogram , $\text{green triangle with dot}$ in terms of $|X|$, m and $\sum_{i=6}^{\infty} (i - 5) |\mathcal{C}_i|$

Observation:

$$|X| \geq 3 \nabla \quad |X| \geq |\{(x, c) \mid x \in X, c \text{ is } \nabla, x \text{ on } c\}| = 3 \nabla$$

$$|X| \geq \text{red parallelogram}$$

$$|X| \geq \text{green triangle with dot}$$



First Attempt

Question: Is $|X| \leq \frac{10}{3}n$ for every 2-plane drawing?

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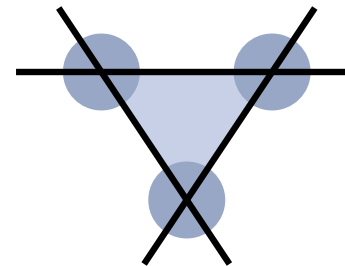
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Observation:

$$|X| \geq 3 \nabla \quad |X| \geq |\{(x, c) \mid x \in X, c \text{ is } \nabla, x \text{ on } c\}| = 3 \nabla$$

$$|X| \geq \text{red parallelogram}$$

$$|X| \geq \text{green triangle with dot}$$



Recall: $|X| \leq m$

First Attempt

Question: Is $|X| \leq \frac{10}{3}n$ for every 2-plane drawing?

Obs: May assume no plane edges.

$$|X| = 5(n - 2) + 2 \nabla + \text{red parallelogram} + \text{green triangle with dot} - \sum_{i=6}^{\infty} (i - 5) |\mathcal{C}_i| - m \quad (\star)$$

Idea: bound ∇ , red parallelogram , $\text{green triangle with dot}$ in terms of $|X|$, m and $\sum_{i=6}^{\infty} (i - 5) |\mathcal{C}_i|$

Observation:

$$\left\{ \begin{array}{l} |X| \geq 3 \nabla \\ |X| \geq \text{red parallelogram} \\ |X| \geq \text{green triangle with dot} \\ |X| \leq m \\ (\star) \end{array} \right.$$

First Attempt

Question: Is $|X| \leq \frac{10}{3}n$ for every 2-plane drawing?

Obs: May assume no plane edges.

$$|X| = 5(n - 2) + 2 \nabla + \text{red parallelogram} + \text{green triangle with dot} - \overbrace{\sum_{i=6}^{\infty} (i - 5) |\mathcal{C}_i|}^{\leq 0} - m \quad (\star)$$

Idea: bound ∇ , red parallelogram , $\text{green triangle with dot}$ in terms of $|X|$, m and $\sum_{i=6}^{\infty} (i - 5) |\mathcal{C}_i|$

Observation:

$$\left\{ \begin{array}{l} |X| \geq 3 \nabla \\ |X| \geq \text{red parallelogram} \\ |X| \geq \text{green triangle with dot} \\ |X| \leq m \end{array} \right. \quad (\star)$$

First Attempt

Question: Is $|X| \leq \frac{10}{3}n$ for every 2-plane drawing?

Obs: May assume no plane edges.

$$|X| = 5(n - 2) + 2 \nabla + \text{red parallelogram} + \text{green triangle with dot} - \overbrace{\sum_{i=6}^{\infty} (i - 5) |\mathcal{C}_i|}^{\leq 0} - m \quad (\star)$$

Idea: bound ∇ , red parallelogram , $\text{green triangle with dot}$ in terms of $|X|$, m and $\sum_{i=6}^{\infty} (i - 5) |\mathcal{C}_i|$

Observation:

$$\text{LP} \left\{ \begin{array}{l} |X| \geq 3 \nabla \\ |X| \geq \text{red parallelogram} \\ |X| \geq \text{green triangle with dot} \\ |X| \leq m \\ (\star) \end{array} \right.$$

Maximize $|X|$ in LP

First Attempt

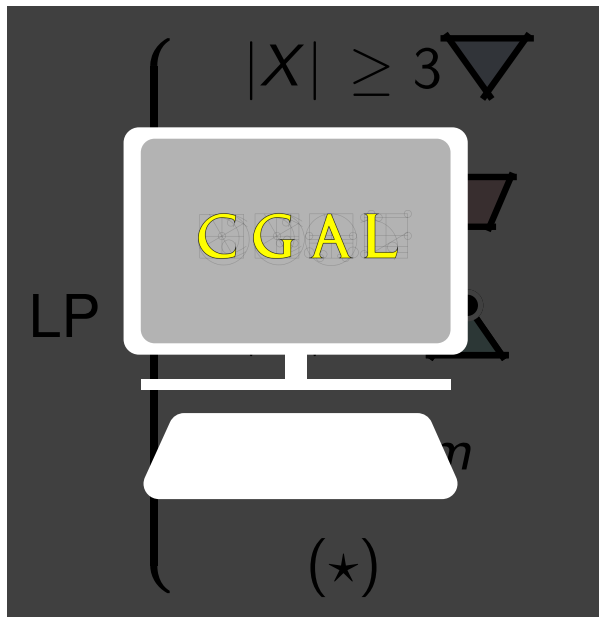
Question: Is $|X| \leq \frac{10}{3}n$ for every 2-plane drawing?

Obs: May assume no plane edges.

$$|X| = 5(n - 2) + 2 \nabla + \text{red parallelogram} + \text{green triangle} - \overbrace{\sum_{i=6}^{\infty} (i - 5) |\mathcal{C}_i|}^{\leq 0} - m \quad (\star)$$

Idea: bound ∇ , red parallelogram , green triangle in terms of $|X|$, m and $\sum_{i=6}^{\infty} (i - 5) |\mathcal{C}_i|$

Observation:



Maximize $|X|$ in LP

First Attempt

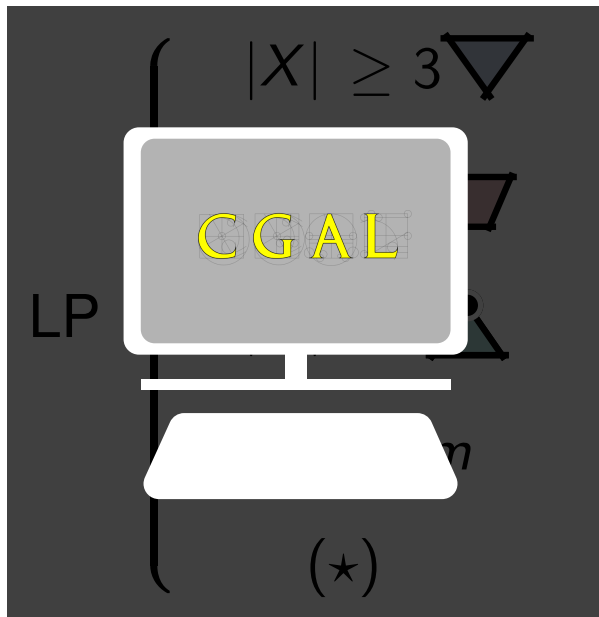
Question: Is $|X| \leq \frac{10}{3}n$ for every 2-plane drawing?

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$$|X| = 5(n - 2) + 2 \nabla + \text{red parallelogram} + \text{green triangle} - \overbrace{\sum_{i=6}^{\infty} (i - 5) |C_i|}^{\leq 0} - m \quad (\star)$$

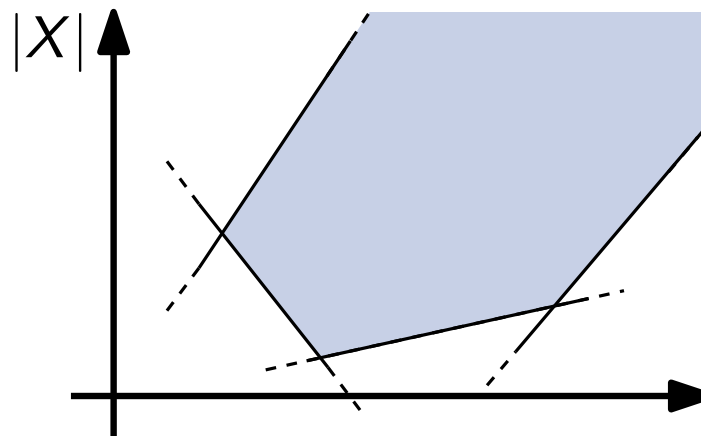
Idea: bound ∇ , red parallelogram , green triangle in terms of $|X|$, m and $\sum_{i=6}^{\infty} (i - 5) |C_i|$

Observation:



Maximize $|X|$ in LP

$|X|$ unbounded



First Attempt

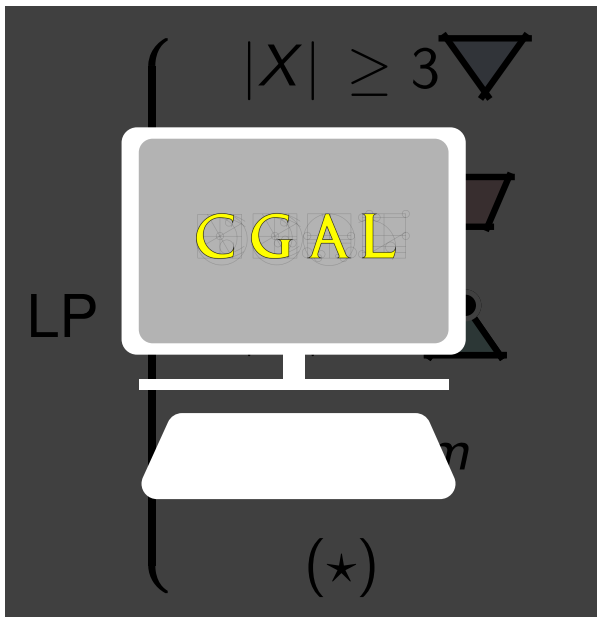
Question: Is $|X| \leq \frac{10}{3}n$ for every 2-plane drawing?

Obs: May assume no plane edges.

$$|X| = 5(n - 2) + 2 \nabla + \text{parallelogram} + \triangle \quad \overbrace{- \sum_{i=6}^{\infty} (i - 5) |C_i|}^{\leq 0} - m \quad (\star)$$

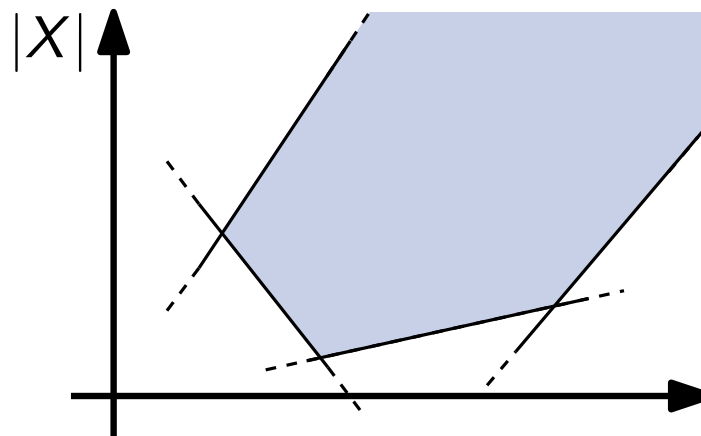
Idea: bound ∇ , parallelogram , $\triangle$ in terms of $|X|$, m and $\sum_{i=6}^{\infty} (i - 5) |C_i|$

Observation:



Maximize $|X|$ in LP

$|X|$ unbounded



Idea: Need to take cells of size ≥ 6 into account

First Attempt

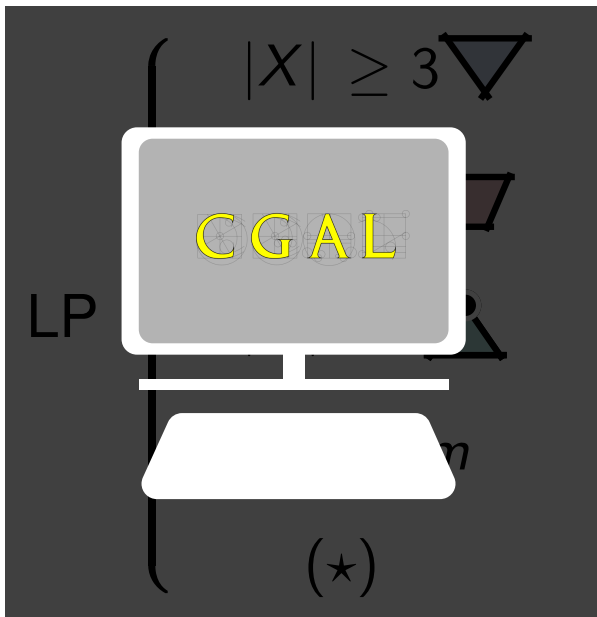
Question: Is $|X| \leq \frac{10}{3}n$ for every 2-plane drawing?

Obs: May assume no plane edges.

$$|X| = 5(n - 2) + 2 \nabla + \text{parallelogram} + \triangle \quad \overbrace{- \sum_{i=6}^{\infty} (i - 5) |C_i|}^{\leq 0} - m \quad (\star)$$

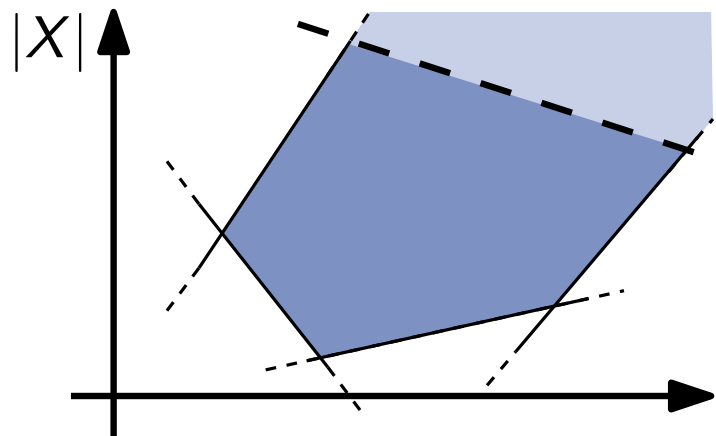
Idea: bound ∇ , parallelogram , $\triangle$ in terms of $|X|$, m and $\sum_{i=6}^{\infty} (i - 5) |C_i|$

Observation:



Maximize $|X|$ in LP

$|X|$ unbounded



Idea: Need to take cells of size ≥ 6 into account

Large Cells

Question: Number of large cells?

Large Cells

Question: Number of large cells?

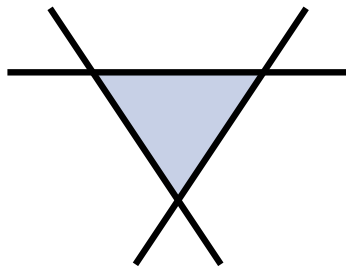
$$3 \nabla \leq \underbrace{\nabla \mid \bigcirc}_{\substack{\text{\# of segments shared} \\ \text{by } \nabla \text{ and } \bigcirc}} \quad \text{a cell of size } \geq 6$$

Large Cells

Question: Number of large cells?

$$3 \nabla \leq \underbrace{\nabla \mid \bigcirc}_{\text{\# of segments shared by } \nabla \text{ and } \bigcirc} \quad \text{a cell of size } \geq 6$$

Proof:

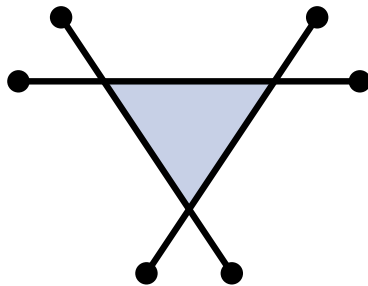


Large Cells

Question: Number of large cells?

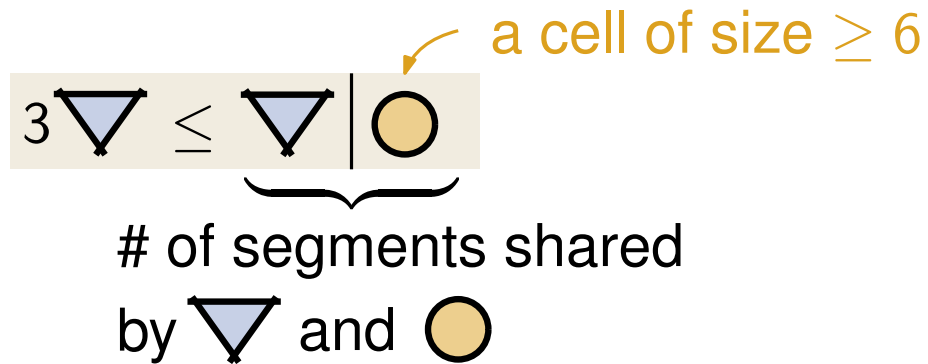
$$3 \nabla \leq \underbrace{\nabla \mid \bigcirc}_{\text{\# of segments shared by } \nabla \text{ and } \bigcirc} \quad \text{a cell of size } \geq 6$$

Proof:

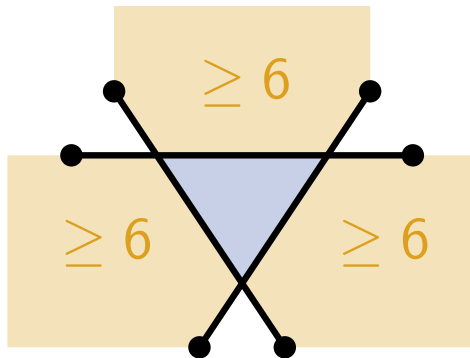


Large Cells

Question: Number of large cells?



Proof:



Large Cells

Question: Number of large cells?

Recall: May assume no plane edges.

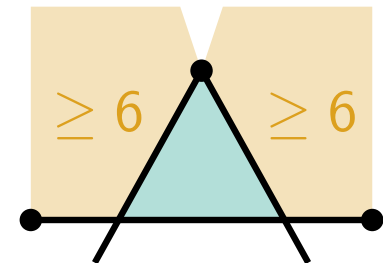
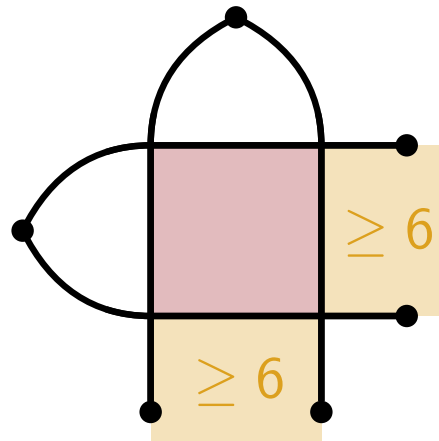
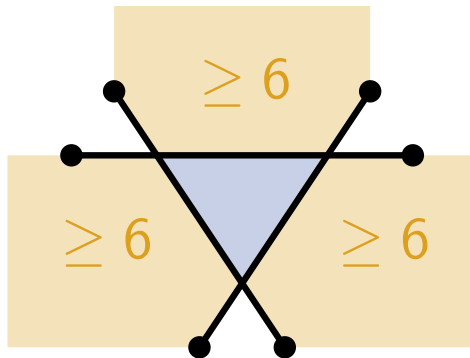
$3 \nabla \leq \underbrace{\nabla | \bigcirc}_{\text{a cell of size } \geq 6}$

of segments shared
by ∇ and $\bigcirc$

$$2 \text{ (parallelogram) } \leq \text{ (parallelogram) } | \bigcirc$$

$$2 \text{ (triangle with dot) } \leq \text{ (triangle with dot) } | \bigcirc$$

Proof:



Large Cells

Question: Number of large cells?

Want: $? \leq \sum_{i \geq 6} (i - 5) |C_i|$

Large Cells

Question: Number of large cells?

Want: $? \leq \sum_{i \geq 6} (i - 5) |C_i|$

$$\nabla | \bigcirc + \text{parallelogram} | \bigcirc + \text{triangle with dot} | \bigcirc \leq \sum_{i \geq 6} i \cdot |C_i|$$

Large Cells

Question: Number of large cells?

Want: $? \leq \sum_{i \geq 6} (i - 5) |C_i|$

$$\nabla | \bigcirc + \text{parallelogram} | \bigcirc + \triangle | \bigcirc \leq \sum_{i \geq 6} i \cdot |C_i|$$

Idea: count # of segments on large cells

$\geq$ # segments

Large Cells

Question: Number of large cells?

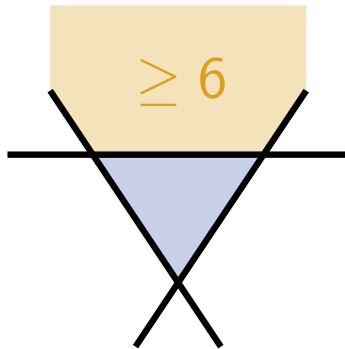
Want: $? \leq \sum_{i \geq 6} (i - 5) |C_i|$

$$\nabla | \bigcirc + \text{parallelogram} | \bigcirc + \triangle | \bigcirc \leq \sum_{i \geq 6} i \cdot |C_i|$$

Idea: count # of segments on large cells

$\geq$ # segments

Proof:



Large Cells

Question: Number of large cells?

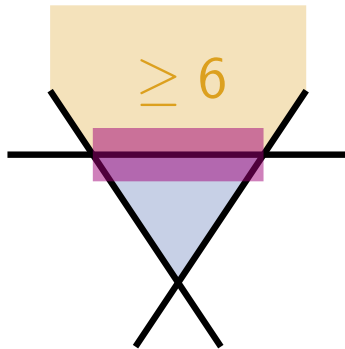
Want: $? \leq \sum_{i \geq 6} (i - 5) |C_i|$

$$\nabla | \bigcirc + \text{parallelogram} | \bigcirc + \triangle | \bigcirc \leq \sum_{i \geq 6} i \cdot |C_i|$$

Idea: count # of segments on large cells

$\geq$ # segments

Proof:



Large Cells

Question: Number of large cells?

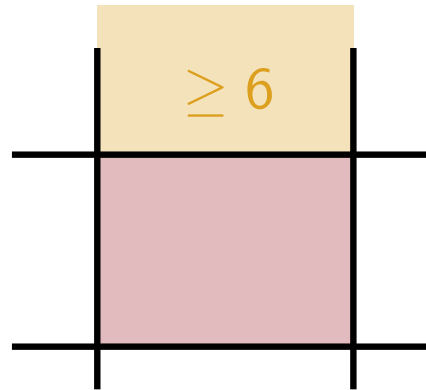
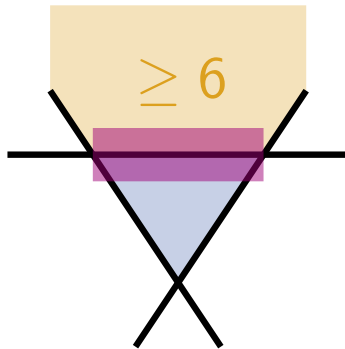
Want: $? \leq \sum_{i \geq 6} (i - 5) |C_i|$

$$\nabla | \bigcirc + \text{parallelogram} | \bigcirc + \triangle | \bigcirc \leq \sum_{i \geq 6} i \cdot |C_i|$$

Idea: count # of segments on large cells

$\geq$ # segments

Proof:



Large Cells

Question: Number of large cells?

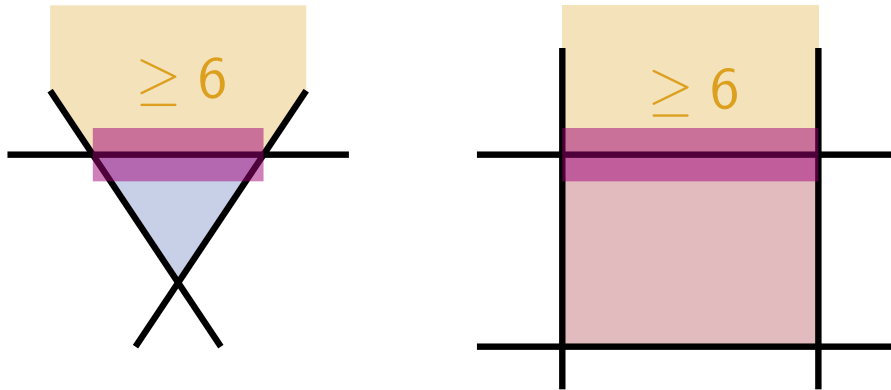
Want: $? \leq \sum_{i \geq 6} (i - 5) |C_i|$

$$\nabla | \bigcirc + \text{parallelogram} | \bigcirc + \triangle | \bigcirc \leq \sum_{i \geq 6} i \cdot |C_i|$$

Idea: count # of segments on large cells

$\geq$ # segments

Proof:



Large Cells

Question: Number of large cells?

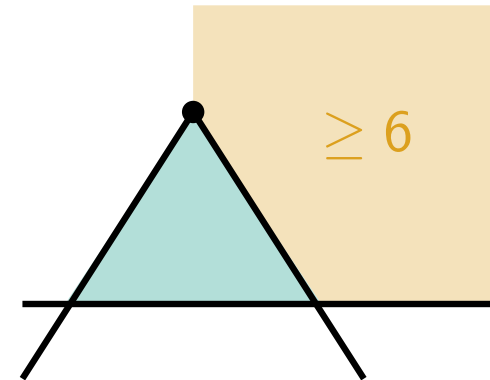
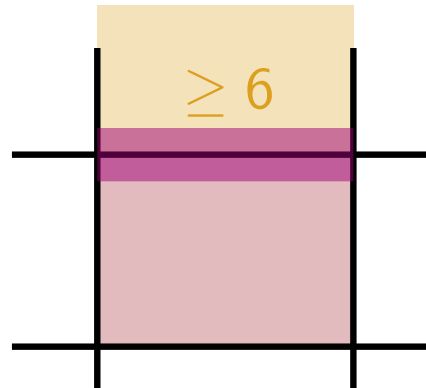
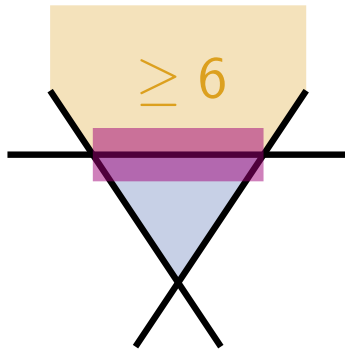
Want: $? \leq \sum_{i \geq 6} (i - 5) |C_i|$

$$\nabla | \bigcirc + \text{parallelogram} | \bigcirc + \triangle | \bigcirc \leq \sum_{i \geq 6} i \cdot |C_i|$$

Idea: count # of segments on large cells

$\geq$ # segments

Proof:



Large Cells

Question: Number of large cells?

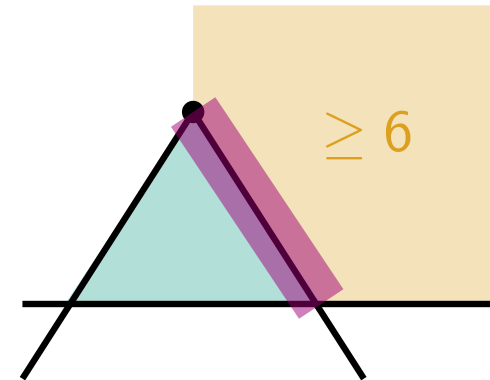
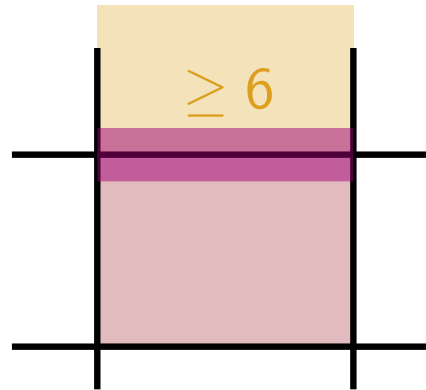
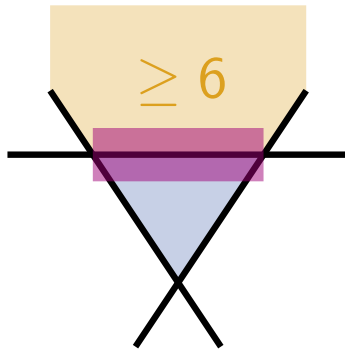
Want: $? \leq \sum_{i \geq 6} (i - 5) |C_i|$

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Idea: count # of segments on large cells

$\uparrow \geq \# \text{ segments}$

Proof:



Large Cells

Question: Number of large cells?

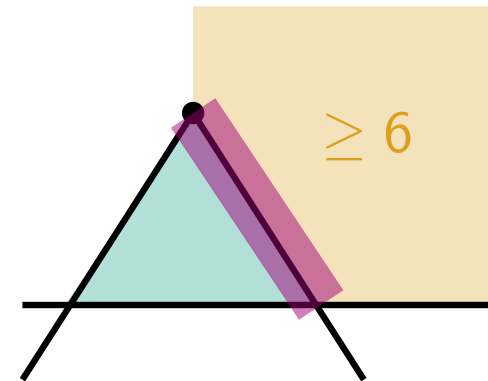
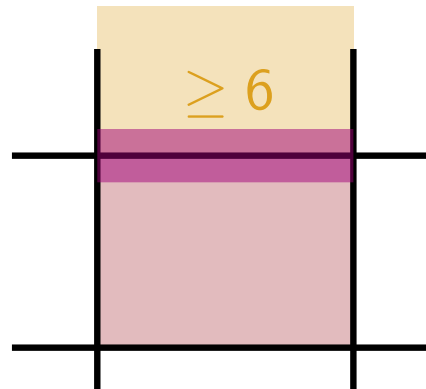
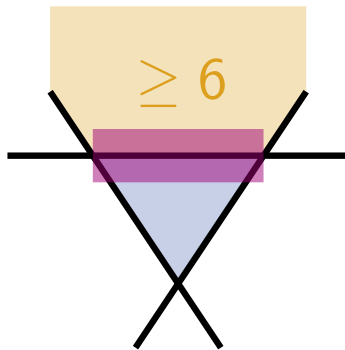
Want: $? \leq \sum_{i \geq 6} (i - 5) |C_i|$

$$\nabla | \bigcirc + \text{parallelogram} | \bigcirc + \triangle | \bigcirc \leq \sum_{i \geq 6} i \cdot |C_i|$$

Idea: count # of segments on large cells

$\uparrow \geq \# \text{ segments}$

Proof:



Obs: $\frac{1}{6} \sum_{i \geq 6} i \cdot |C_i| \leq \sum_{i \geq 6} (i - 5) |C_i|$

Large Cells

Question: Number of large cells?

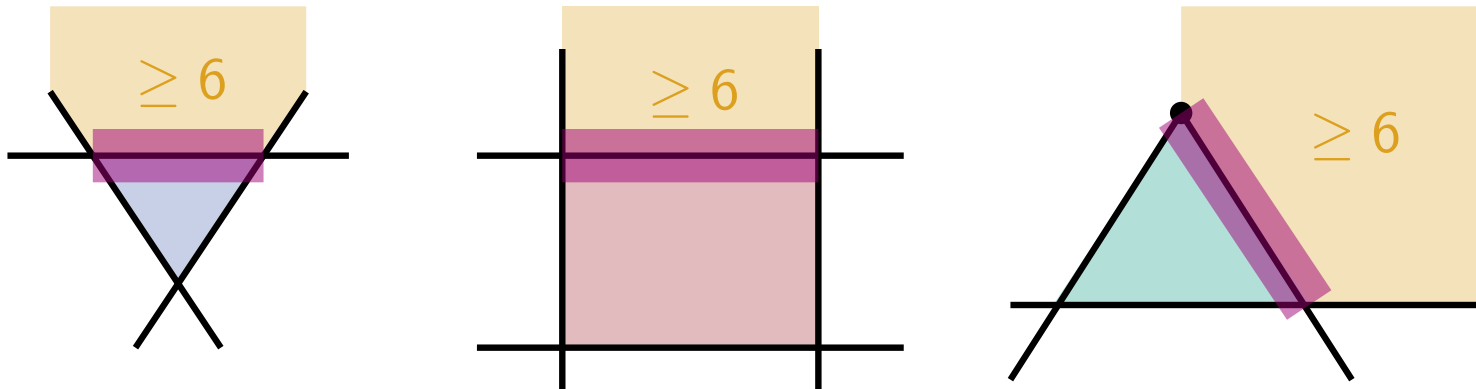
Want: $? \leq \sum_{i \geq 6} (i - 5) |C_i|$

$$\nabla | \bigcirc + \text{parallelogram} | \bigcirc + \triangle | \bigcirc \leq \sum_{i \geq 6} i \cdot |C_i|$$

Idea: count # of segments on large cells

$\uparrow \geq \# \text{ segments}$

Proof:



Obs: $\frac{1}{6} \sum_{i \geq 6} i \cdot |C_i| \leq \sum_{i \geq 6} (i - 5) |C_i|$

$\rightsquigarrow \nabla | \bigcirc + \text{parallelogram} | \bigcirc + \triangle | \bigcirc \leq 6 \cdot \sum_{i \geq 6} (i - 5) \cdot |C_i|$

Second Attempt

Inequations:

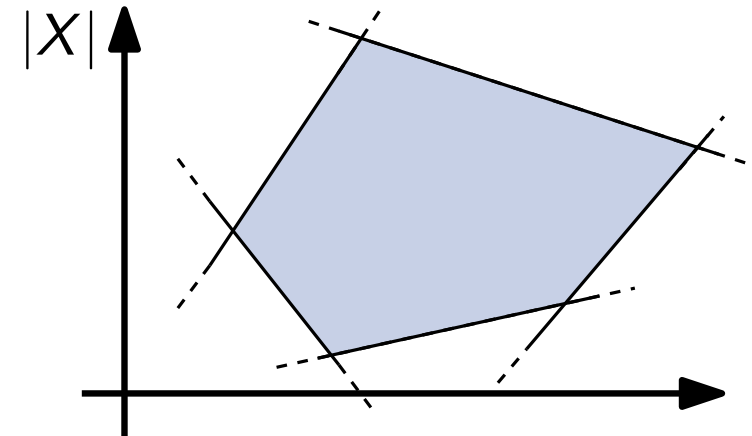
$$\left\{ \begin{array}{ll}
 3 \nabla \leq \nabla | \bigcirc & |X| \geq 3 \nabla \\
 2 \triangle \leq \triangle | \bigcirc & |X| \geq \text{parallelogram} \\
 2 \text{parallelogram} \leq \text{parallelogram} | \bigcirc & |X| \geq \triangle \\
 & |X| \leq m \\
 \\
 \nabla | \bigcirc + \text{parallelogram} | \bigcirc + \triangle | \bigcirc \leq 6 \cdot \sum_{i \geq 6} (i - 5) |C_i| \\
 |X| = 5(n - 2) + 2 \nabla + \text{parallelogram} + \triangle - \sum_{i=6}^{\infty} (i - 5) |C_i| - m
 \end{array} \right.$$

Second Attempt

Inequations:

$$\left\{ \begin{array}{ll} 3 \nabla \leq \nabla \mid \bigcirc & |X| \geq 3 \nabla \\ 2 \blacktriangle \leq \blacktriangle \mid \bigcirc & |X| \geq \text{parallelogram} \\ 2 \text{parallelogram} \leq \text{parallelogram} \mid \bigcirc & |X| \geq \blacktriangle \\ & |X| \leq m \\ \\ \nabla \mid \bigcirc + \text{parallelogram} \mid \bigcirc + \blacktriangle \mid \bigcirc \leq 6 \cdot \sum_{i \geq 6} (i - 5) |C_i| \\ |X| = 5(n - 2) + 2 \nabla + \text{parallelogram} + \blacktriangle - \sum_{i=6}^{\infty} (i - 5) |C_i| - m \end{array} \right.$$

Visualization:



Maximize $|X|$ in LP

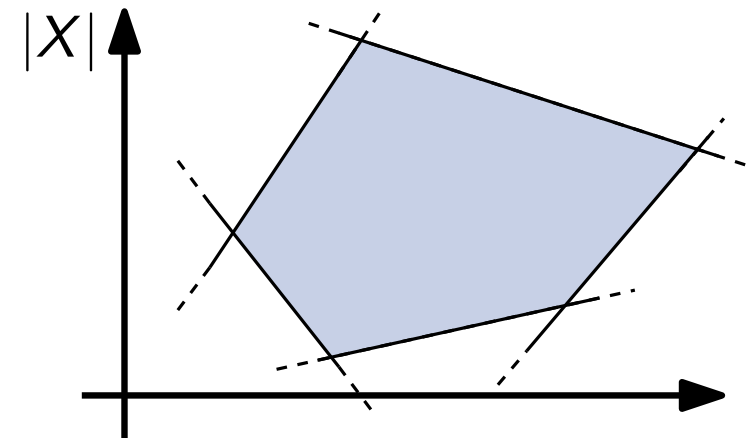
Want: $|X| \leq 3.\bar{3}n$

Second Attempt

Inequations:

$$\left\{ \begin{array}{ll} 3 \nabla \leq \nabla | \bigcirc & |X| \geq 3 \nabla \\ 2 \blacktriangle \leq \blacktriangle | \bigcirc & |X| \geq \text{parallelogram} \\ 2 \text{parallelogram} \leq \text{parallelogram} | \bigcirc & |X| \geq \blacktriangle \\ & |X| \leq m \\ \\ \nabla | \bigcirc + \text{parallelogram} | \bigcirc + \blacktriangle | \bigcirc \leq 6 \cdot \sum_{i \geq 6} (i - 5) |C_i| \\ |X| = 5(n - 2) + 2 \nabla + \text{parallelogram} + \blacktriangle - \sum_{i=6}^{\infty} (i - 5) |C_i| - m \end{array} \right.$$

Visualization:



➡ **Idea:** Maximize $|X|$ in LP

Obtain: $|X| \leq 30n$

Want: $|X| \leq 3.\bar{3}n$

Second Attempt

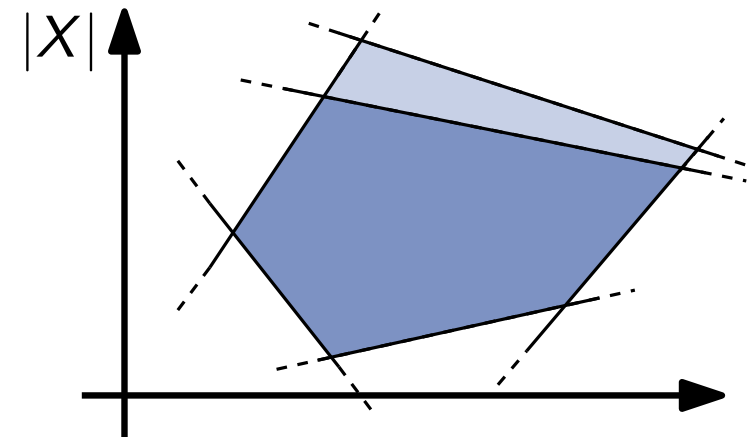
Inequations:

$$\left\{ \begin{array}{ll} 3 \nabla \leq \nabla \mid \bigcirc & |X| \geq 3 \nabla \\ 2 \blacktriangle \leq \blacktriangle \mid \bigcirc & |X| \geq \text{parallelogram} \\ 2 \text{parallelogram} \leq \text{parallelogram} \mid \bigcirc & |X| \geq \blacktriangle \\ & |X| \leq m \end{array} \right.$$

$$4 \nabla \mid \bigcirc + 4 \text{parallelogram} \mid \bigcirc + 1.5 \blacktriangle \mid \bigcirc \leq 6 \cdot \sum_{i \geq 6} (i - 5) |C_i|$$

$$|X| = 5(n - 2) + 2 \nabla + \text{parallelogram} + \blacktriangle - \sum_{i=6}^{\infty} (i - 5) |C_i| - m$$

Visualization:



➡ **Idea:** Maximize $|X|$ in LP

Obtain: $|X| \leq 3\bar{3}n$ ~~$3\bar{3}n$~~

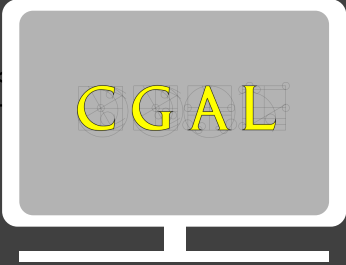
Want: $|X| \leq 3\bar{3}n$ Maximize $|E|$ in LP

Obtain: $|E| \leq 5n$

Second Attempt

Inequations:

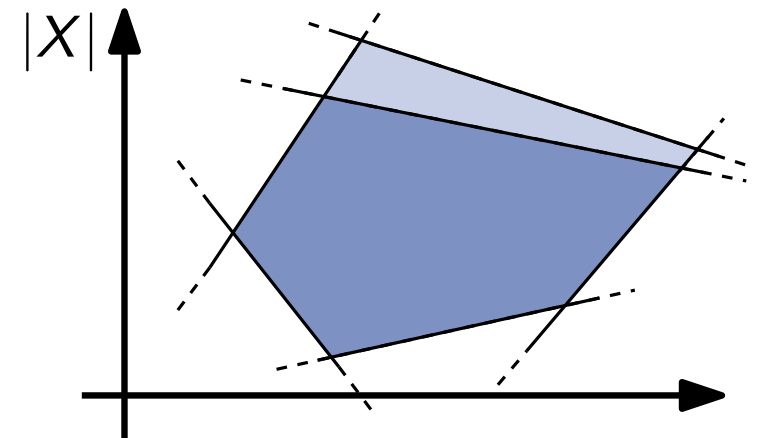
$$\left\{ \begin{array}{ll} 3 \nabla \leq \nabla | \bigcirc & |X| \geq 3 \nabla \\ 2 \triangle \leq \triangle | \bigcirc & |X| \geq \text{parallelogram} \\ 2 \text{parallelogram} \leq \text{parallelogram} | \bigcirc & |X| \geq \triangle \\ & |X| \leq m \end{array} \right.$$



$$4 \nabla | \bigcirc + \text{parallelogram} | \bigcirc + 1.5 \triangle | \bigcirc \leq 6 \cdot \sum_{i \geq 6} (i - 5) |C_i|$$

$$|X| = 5(n - 2) + 2 \nabla + \text{parallelogram} + \triangle - \sum_{i=6}^{\infty} (i - 5) |C_i| - m$$

Visualization:



~ Idea: Maximize $|X|$ in LP

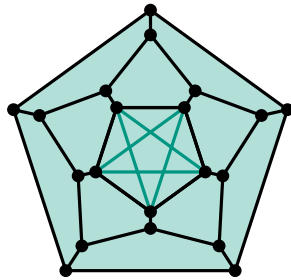
Obtain: $|X| \leq 3\bar{3}n$ ~~$3\bar{3}n$~~

Want: $|X| \leq 3\bar{3}n$ Maximize $|E|$ in LP

Obtain: $|E| \leq 5n$

Overview

Results



2-plane

m

$$\leq 5n$$

(Pach, Tóth 1997)

$|X|$

$$\leq 3.\bar{3}n$$

(Bekos et al. 2024)

Overview

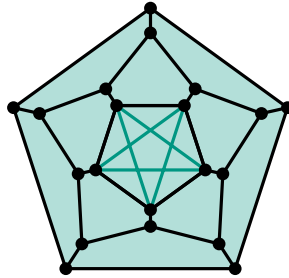
Technique

$$\text{LP} \left\{ \begin{array}{l} 2 \text{ (triangle)} \leq \text{ (triangle)} | \text{ (circle)} \\ 2 \text{ (parallelogram)} \leq \text{ (parallelogram)} | \text{ (circle)} \\ \dots \\ \text{Density Formula} \end{array} \right.$$

Maximize $|X|$ in LP

→ Bound on $|X|$

Results



2-plane

m

$\leq 5n$

(Pach, Tóth 1997)

$|X|$

$\leq 3.3n$

(Bekos et al. 2024)

Overview

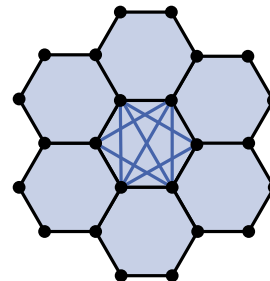
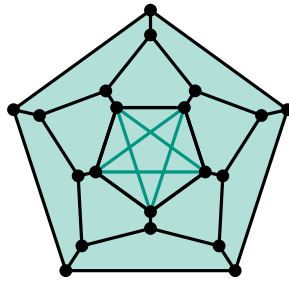
Technique

$$\text{LP} \begin{cases} 2 \text{ (triangle) } \leq \text{ (triangle) } | \text{ (circle) } \\ 2 \text{ (parallelogram) } \leq \text{ (parallelogram) } | \text{ (circle) } \\ \dots \\ \text{Density Formula} \end{cases}$$

Maximize $|X|$ in LP

→ Bound on $|X|$

Results



2-plane

3-plane

m

$\leq 5n$

(Pach, Tóth 1997)

$\leq 5.5n$

(Pach, Radoicic, Tardos, Tóth 2006)

$|X|$

$\leq 3.3n$

(Bekos et al. 2024)

$\leq 5.5n$

Overview

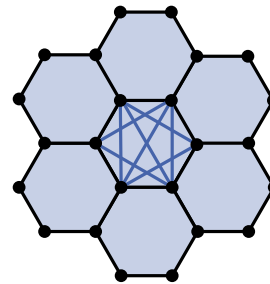
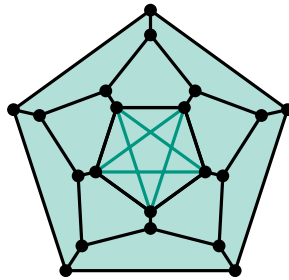
Technique

$$\text{LP} \begin{cases} 2 \text{ (triangle) } \leq \text{ (triangle) } | \text{ (circle) } \\ 2 \text{ (square) } \leq \text{ (square) } | \text{ (circle) } \\ \dots \\ \text{Density Formula} \end{cases}$$

Maximize $|X|$ in LP

→ Bound on $|X|$

Results



	2-plane	3-plane	4-plane
m	$\leq 5n$ (Pach, Tóth 1997)	$\leq 5.5n$ (Pach, Radoicic, Tardos, Tóth 2006)	$\leq 6n$ (Ackerman 2015)
$ X $	$\leq 3.3n$ (Bekos et al. 2024)	$\leq 5.5n$	?