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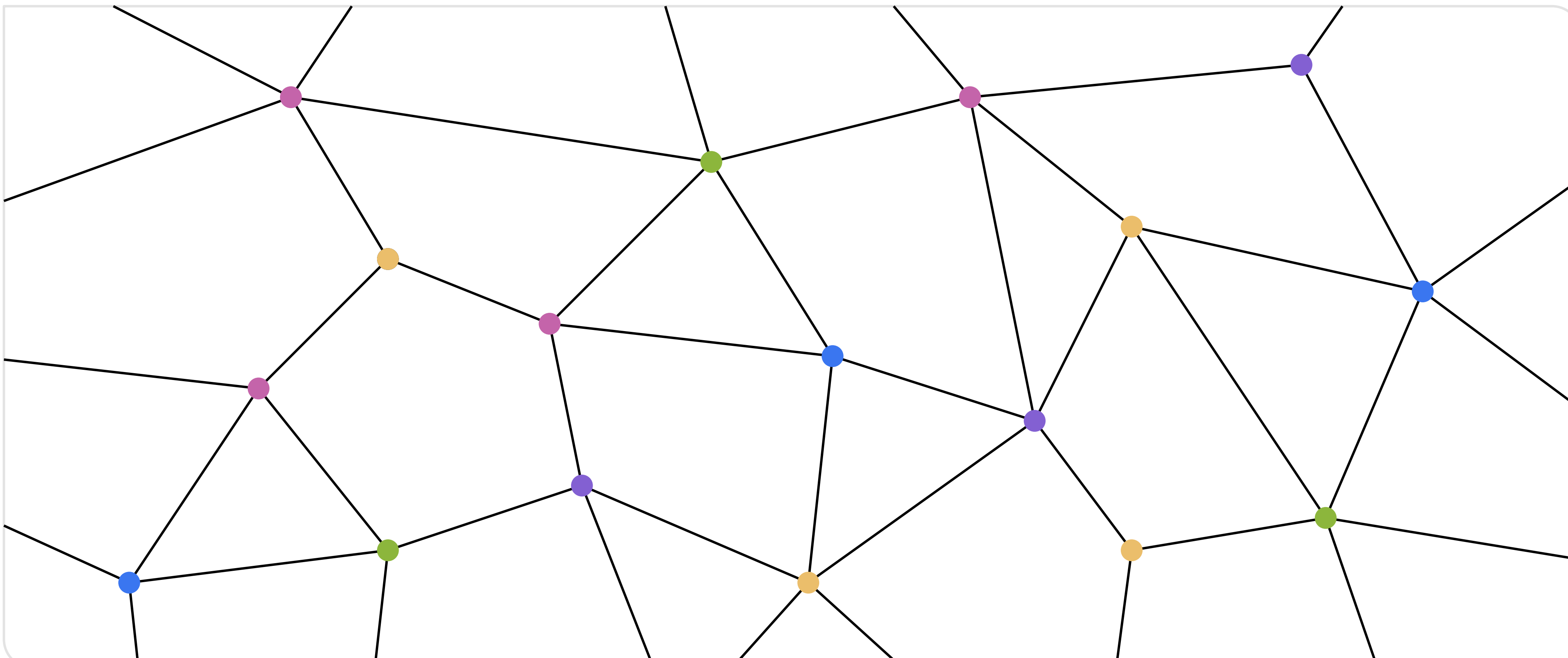
Strong odd coloring in minor-closed classes

joint work with Kolja Knauer, Fabian Klute, Irene Parada Juan Pablo Peña, and Torsten Ueckerdt



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Def: A vertex-coloring $\Phi\colon V(G) \to [k]$ of a graph G is **strong odd** if

- Φ is proper
- for each $i \in [k]$ and each $v \in V(G)$:

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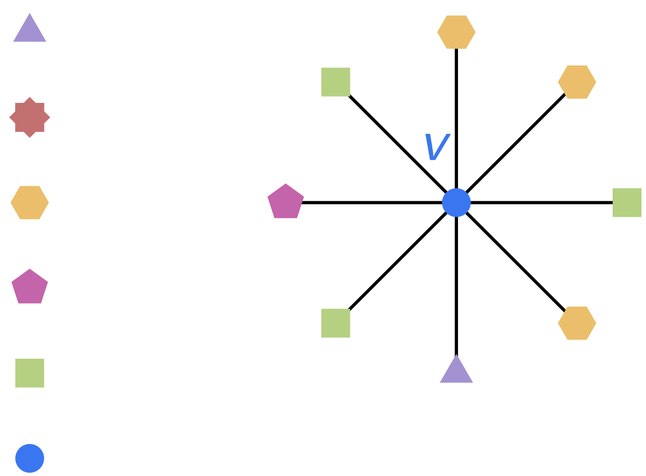

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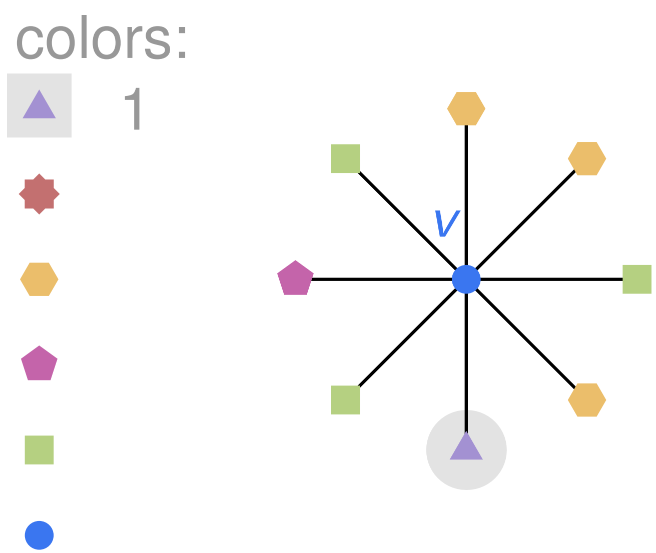

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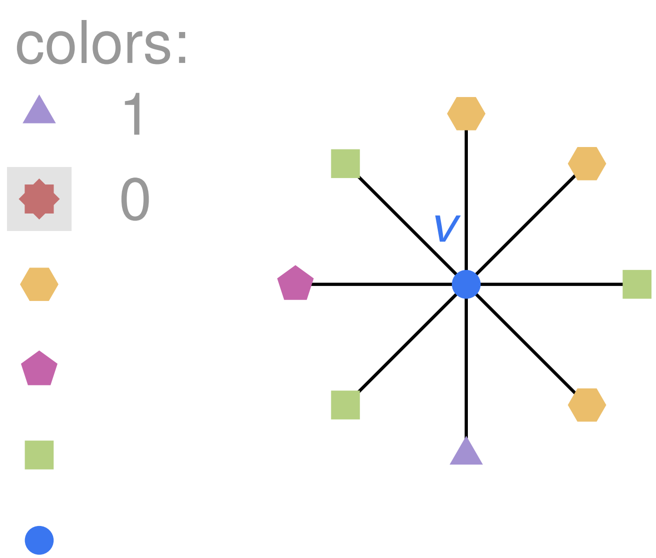

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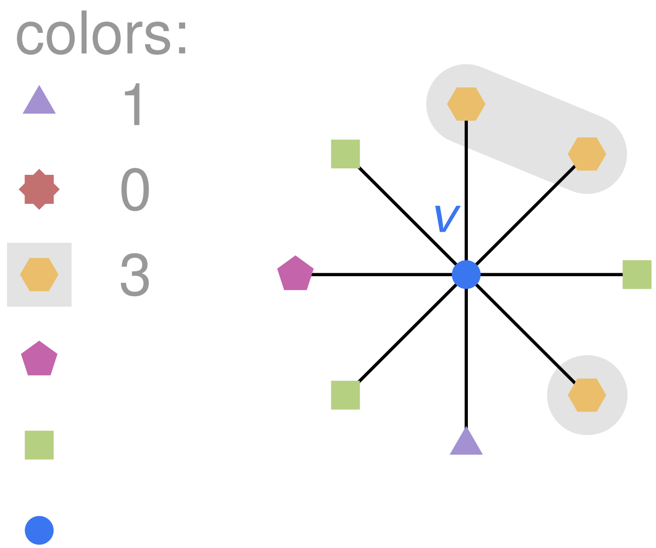

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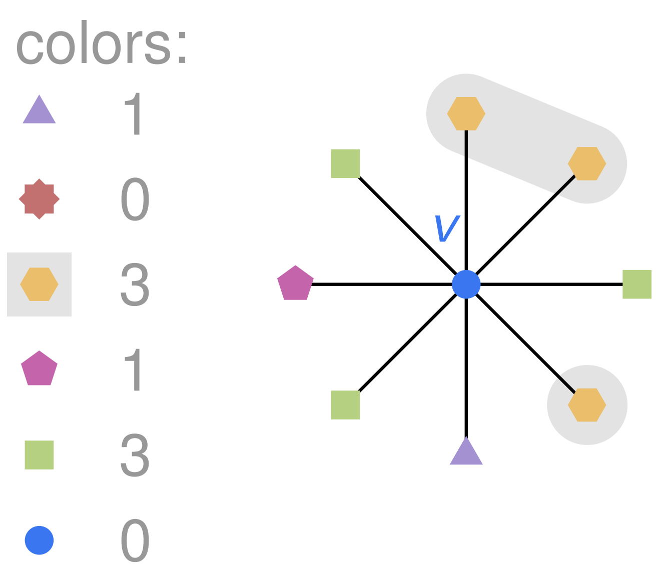

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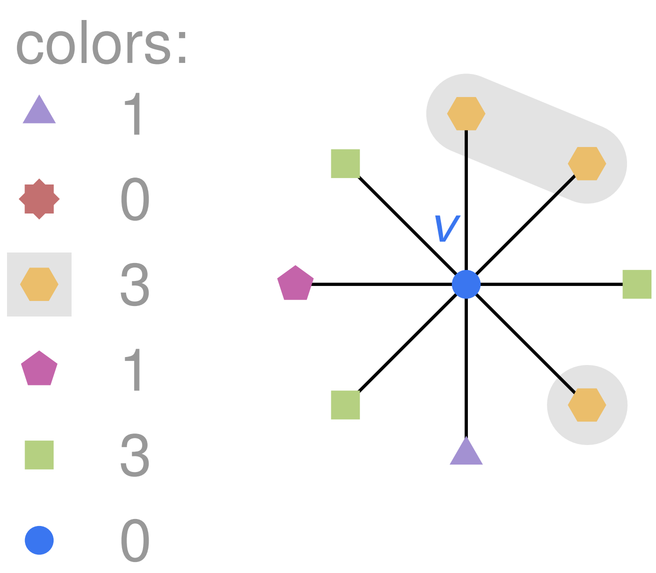

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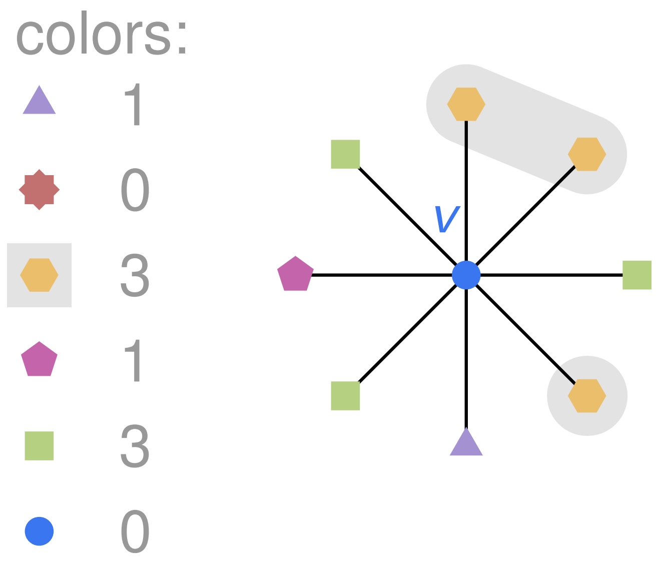
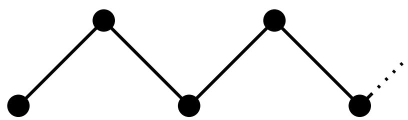


Example: Paths



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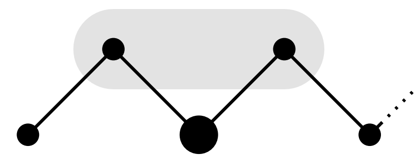

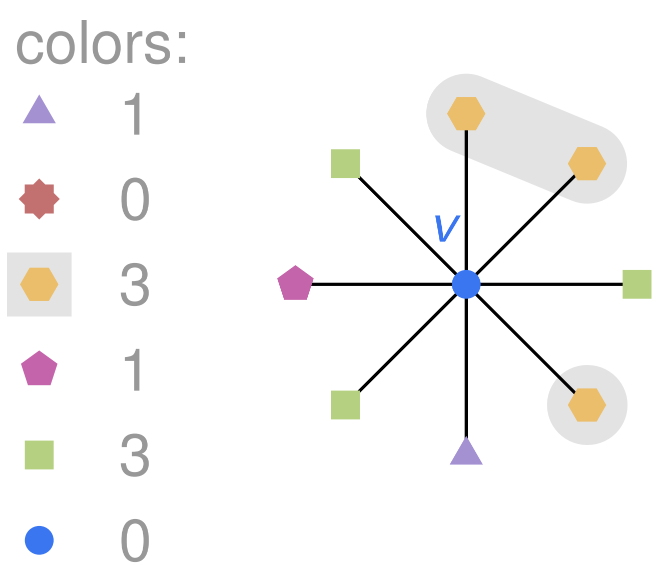

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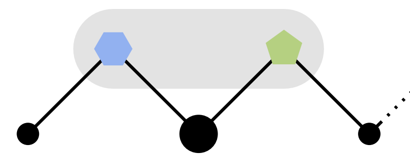

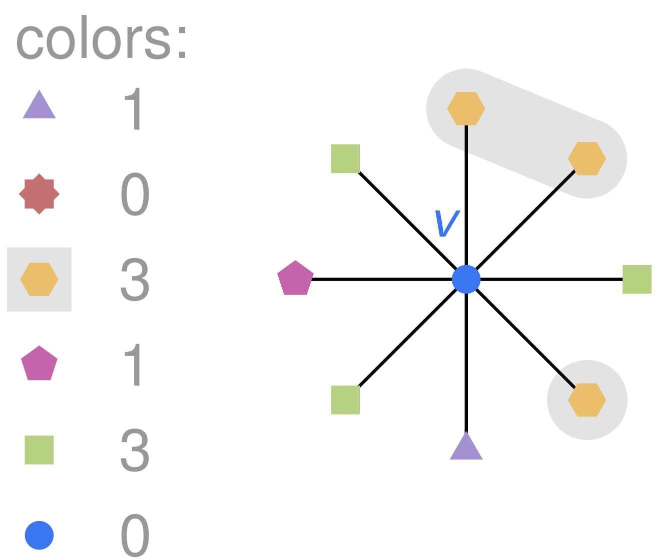

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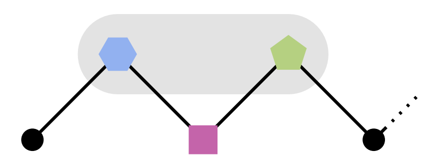

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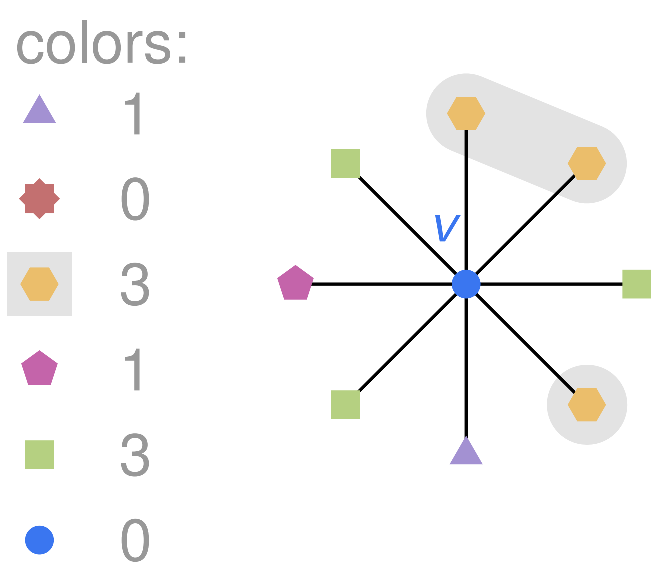

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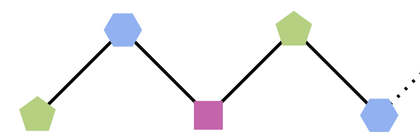

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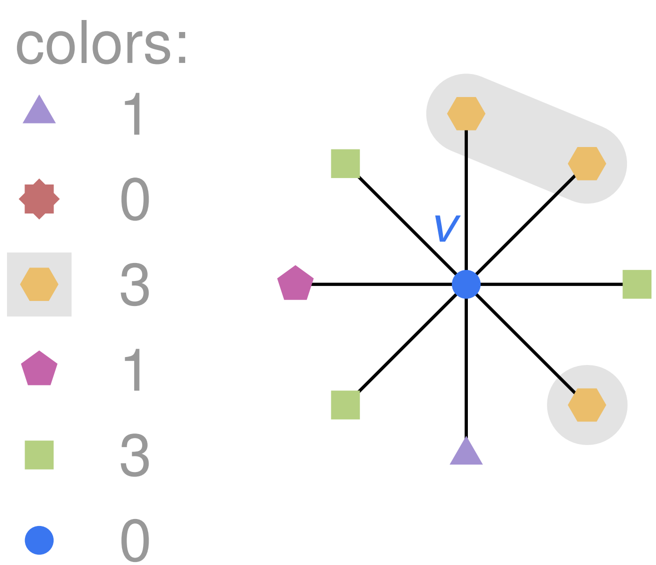

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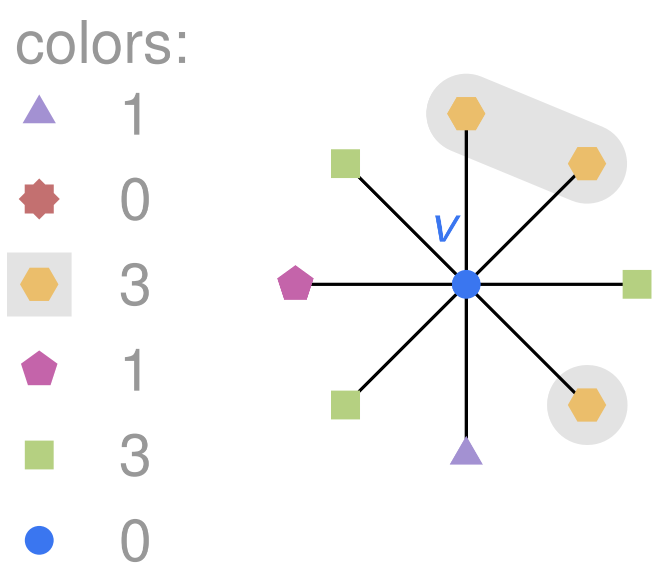

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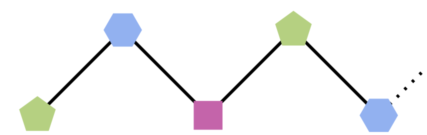

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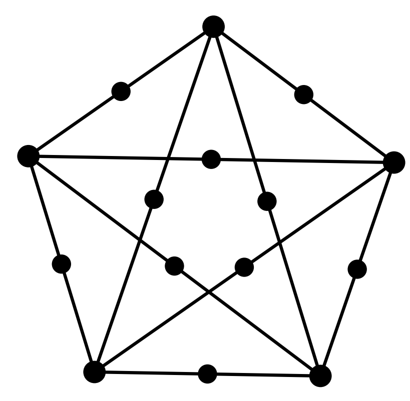

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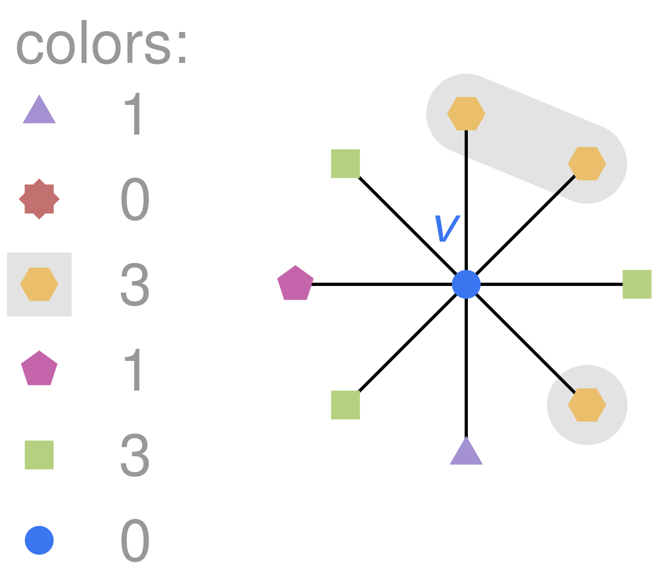

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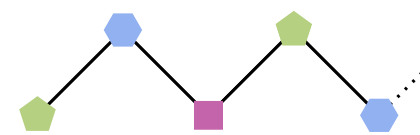

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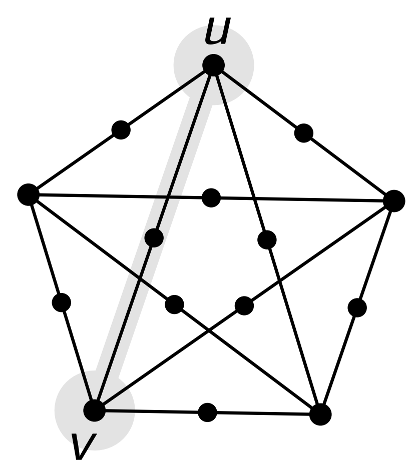

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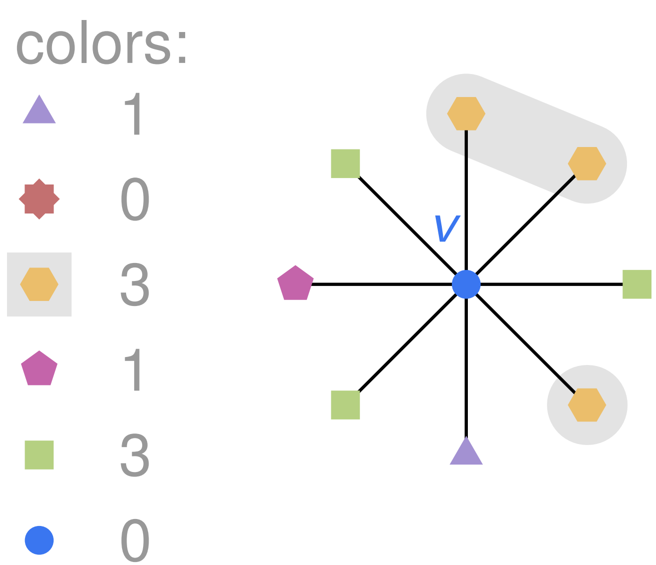

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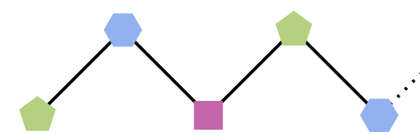

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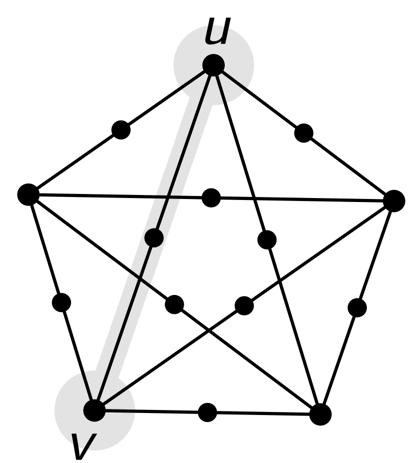

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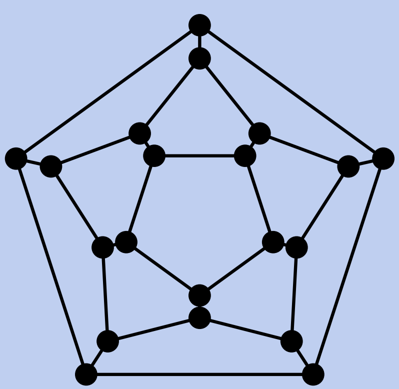

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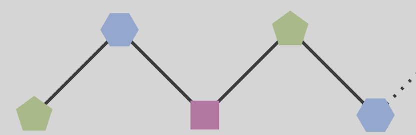

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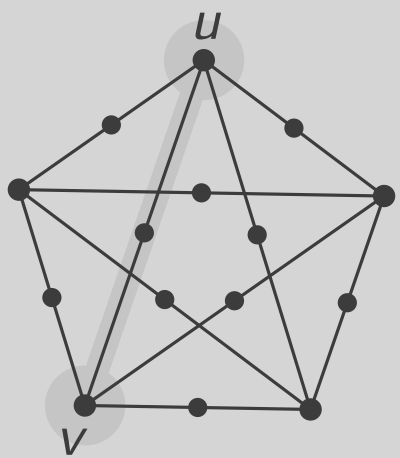

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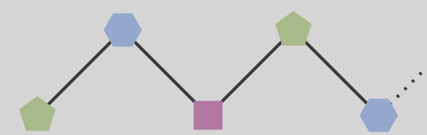

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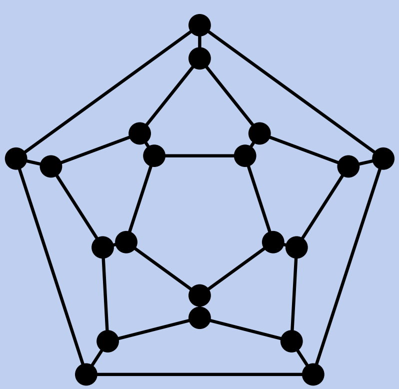

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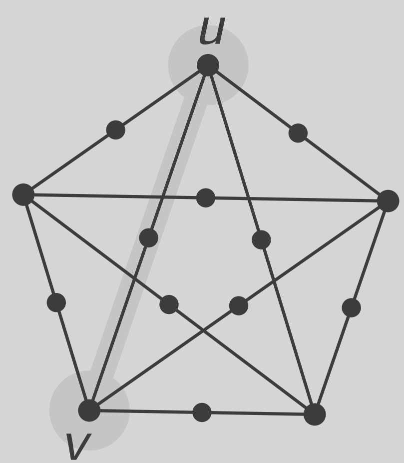

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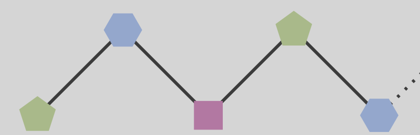

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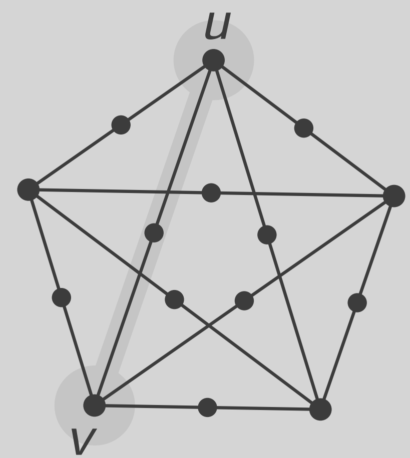

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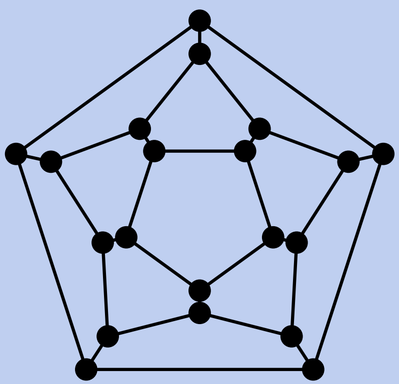

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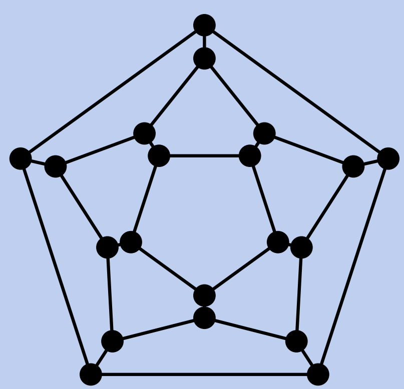

Exists $c \in \mathbb{R}$ such that $\chi_{so}(G) \leq c$ for all planar graphs?

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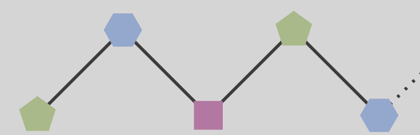

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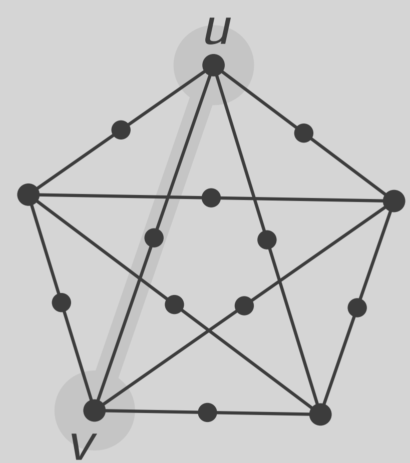

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Product Structure

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Theorem [J. ACM 2020] Every planar graph is a subgraph of $P_n \boxtimes H$ for

- some path P_n
- some graph H with $\mathrm{tw}(H) \leq 8$

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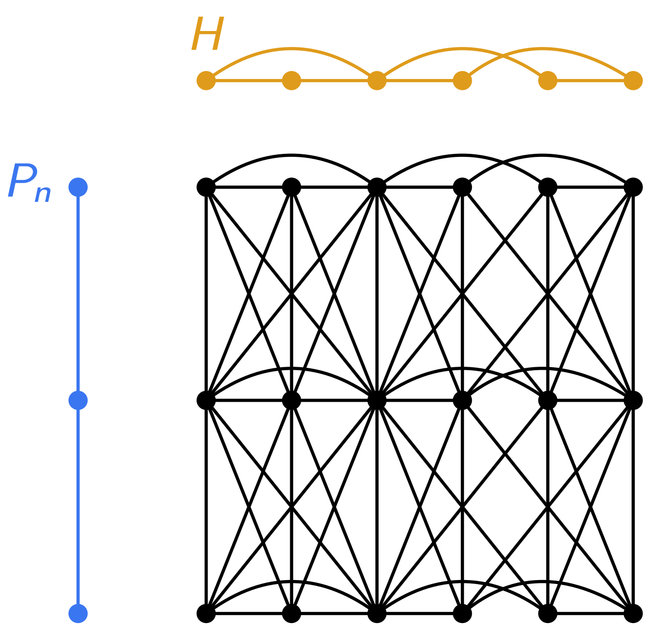

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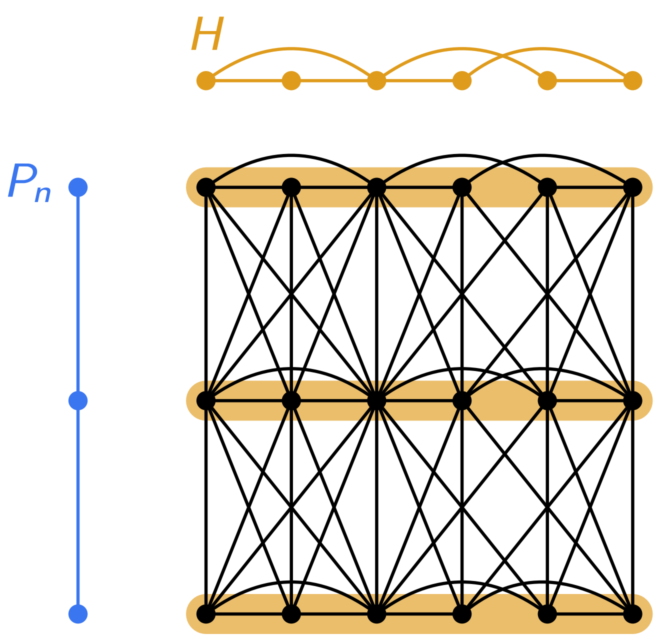

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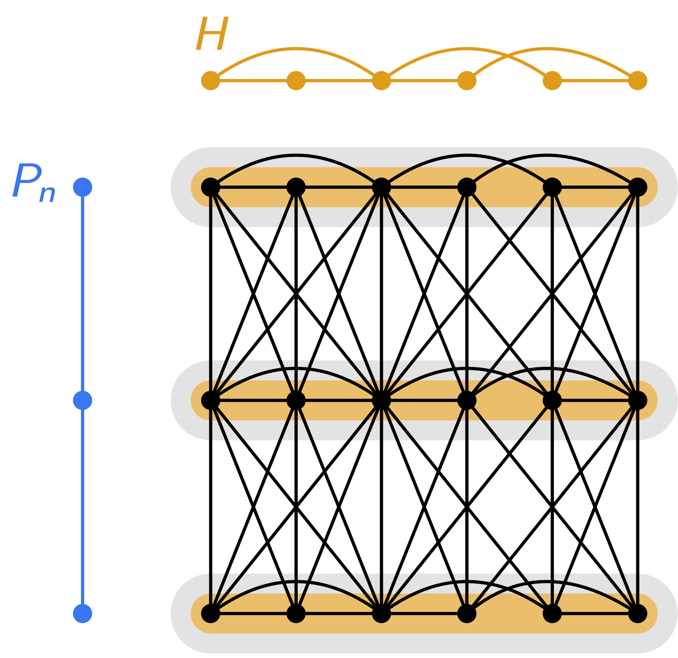

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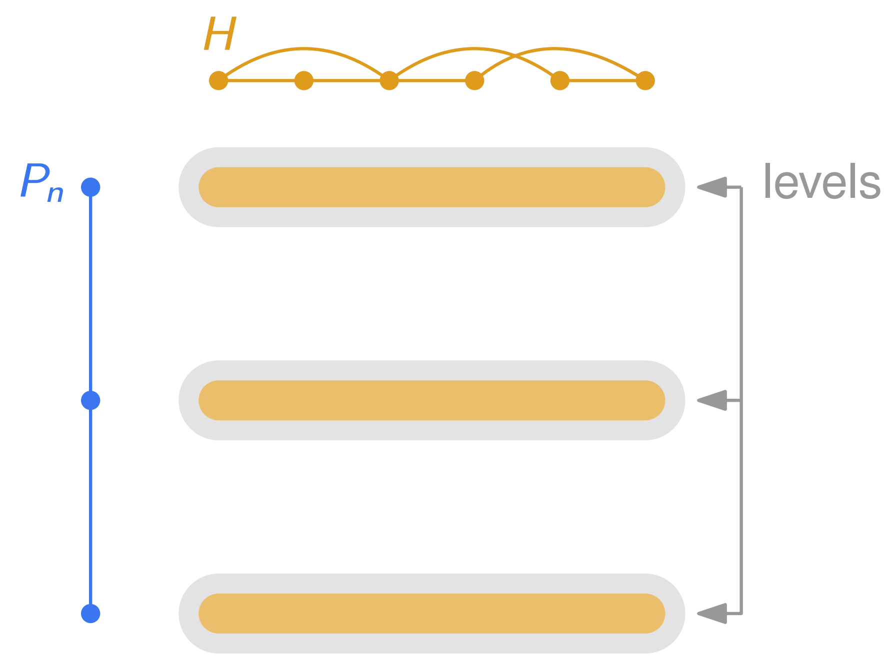

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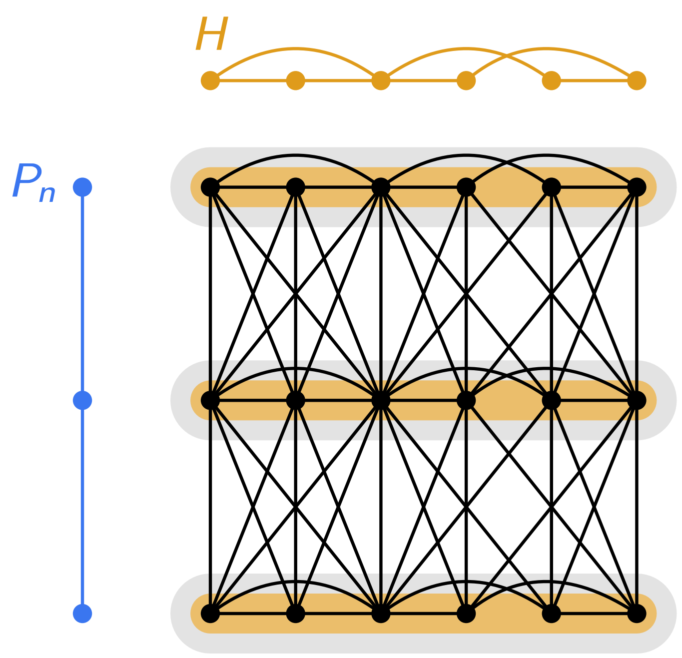

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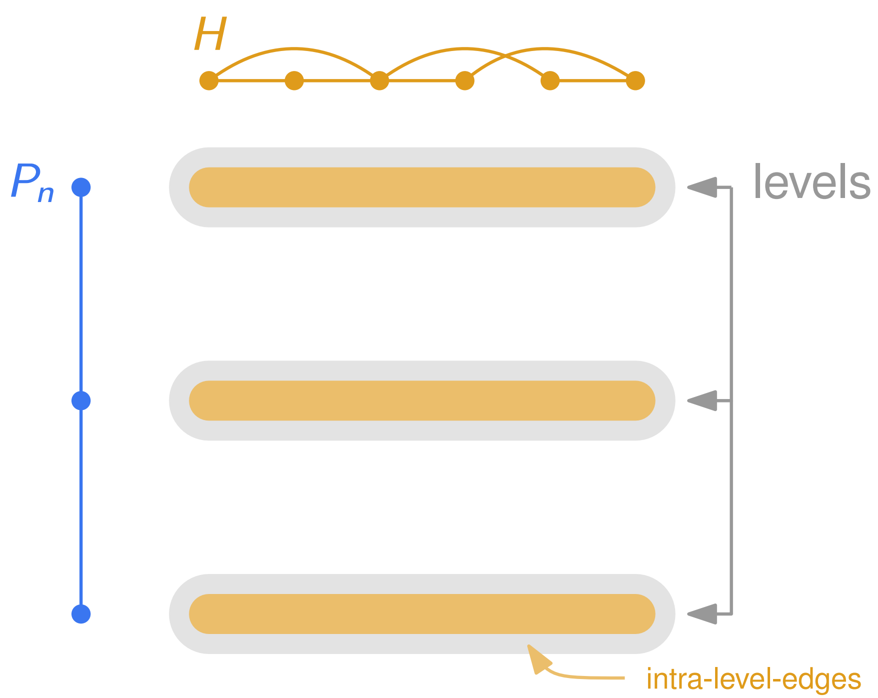

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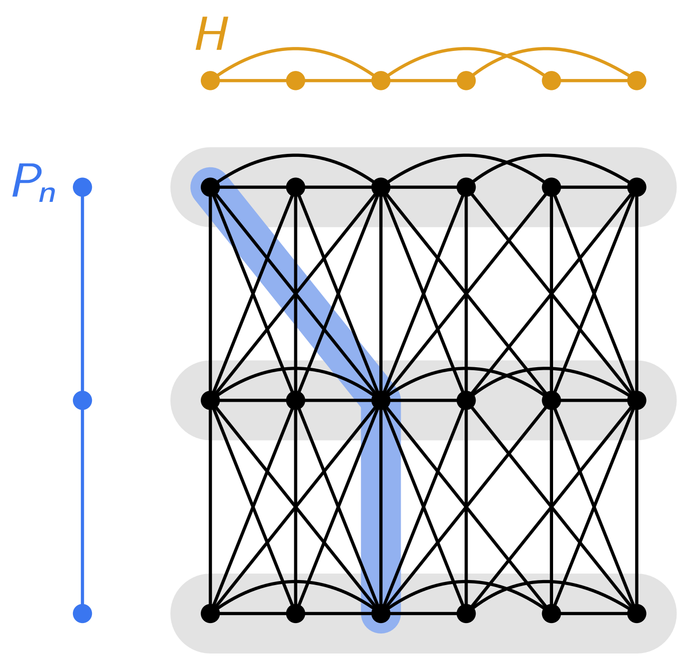

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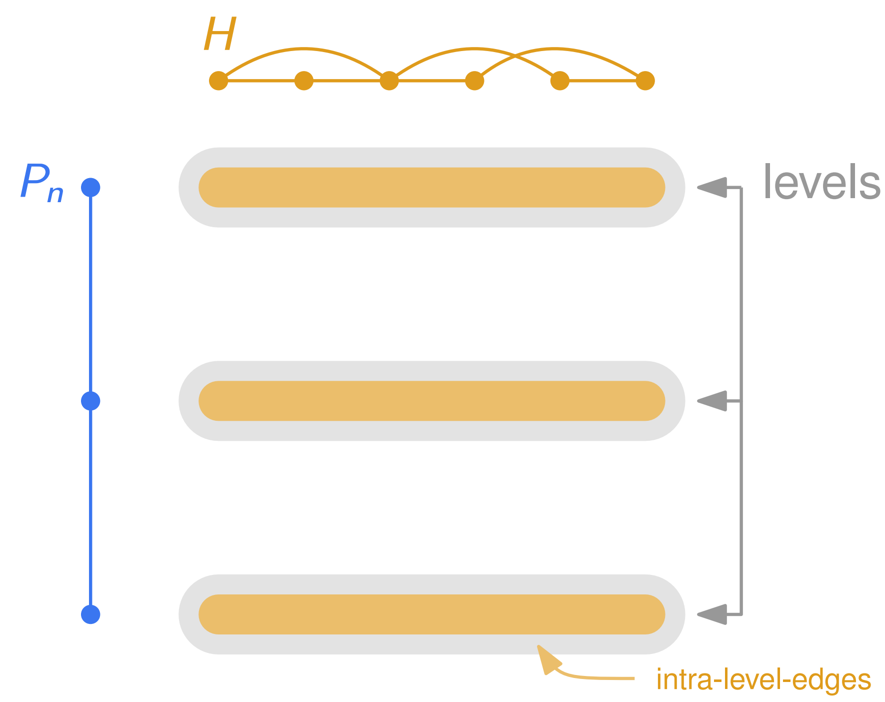


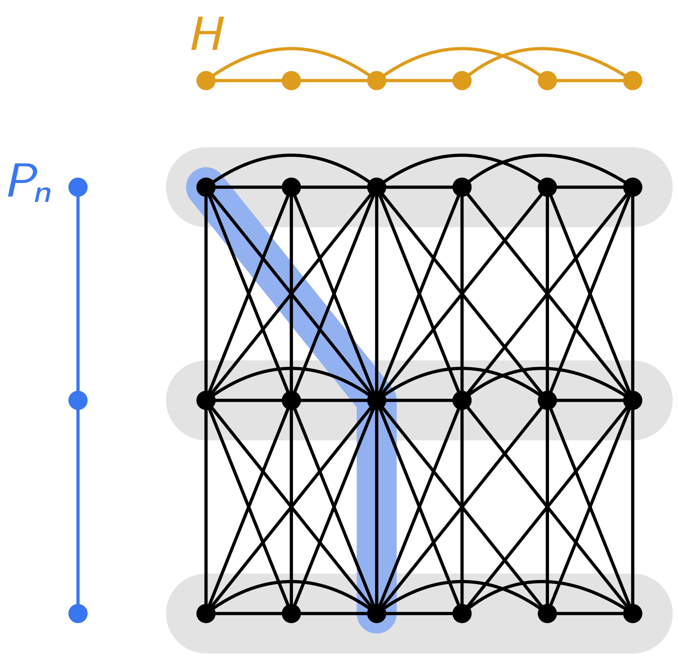
Product Structure

Theorem [J. ACM 2020] Every planar graph is a subgraph of $P_n \boxtimes H$ for

- some path P_n
- some graph H with $\mathrm{tw}(H) \leq 8$

treewidth

Example



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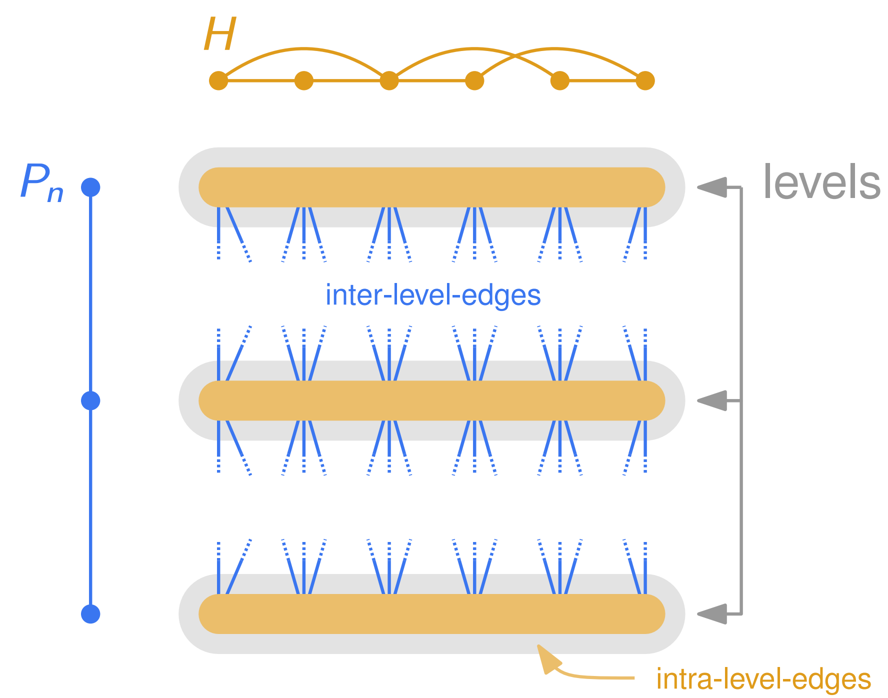

Idea: 1) Bound $\chi_{so}(P_n \boxtimes H)$ in $\chi_{so}(H)$

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Question: Is $\chi_{so}(G) \leq \chi_{so}(H)$ if $G \subseteq H$?

Recall: strong odd if proper and for each $i \in [k]$ and each $v \in V(G)$:
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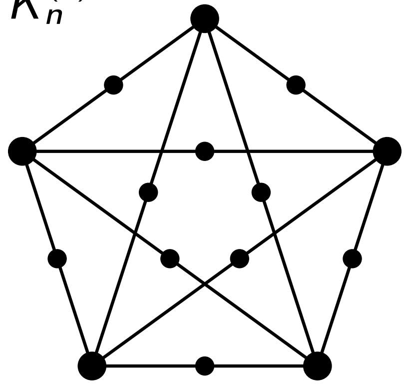
$G = K_n^{(1)}$



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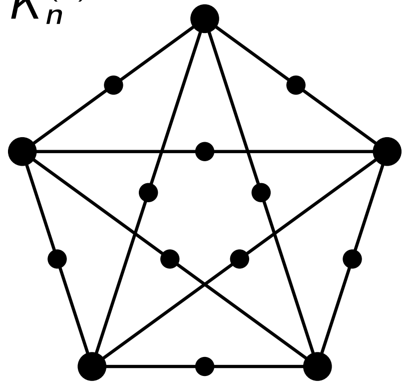

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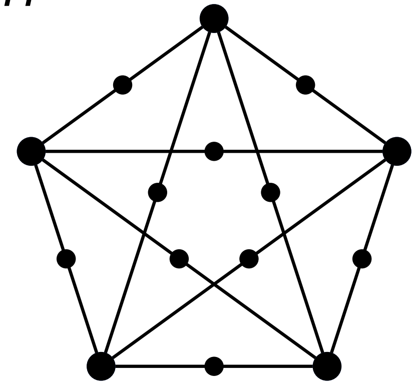

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coloring is proper

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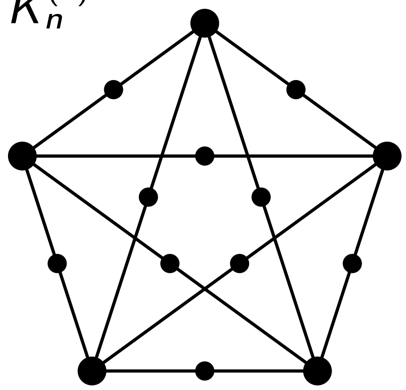

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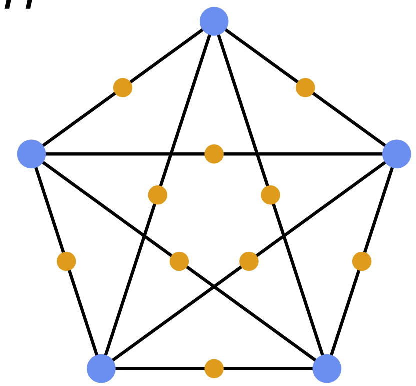

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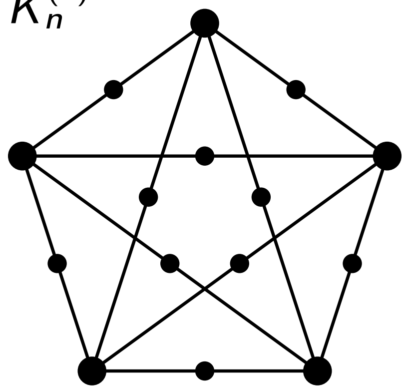

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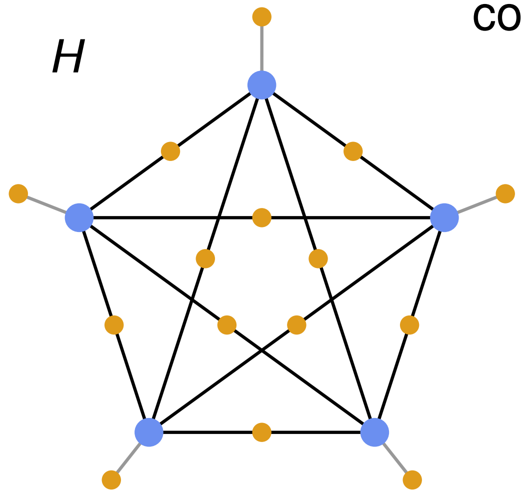

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 Miriam Goetze, Kolja Knauer, Fabian Klute, Irene Parada Juan Pablo Peña, and Torsten Ueckerdt – Strong odd coloring in minor closed classes

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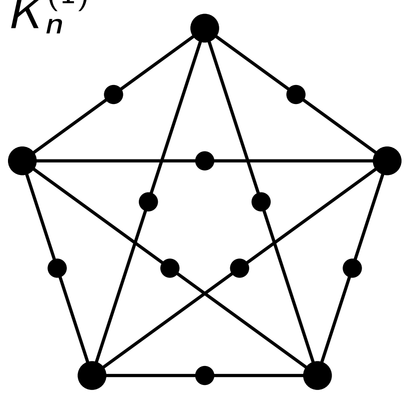

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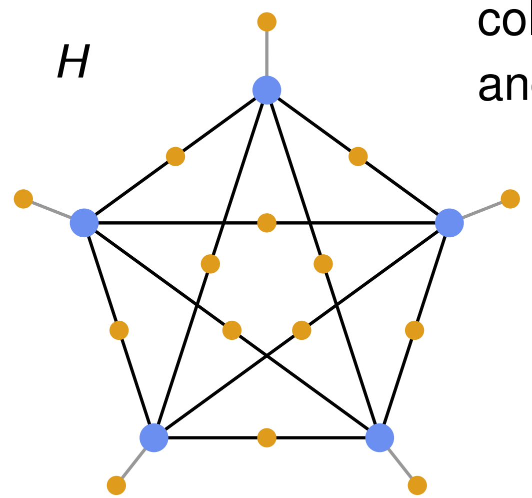

coloring is proper and strong odd for:

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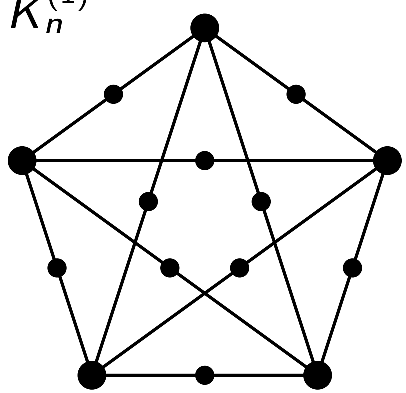

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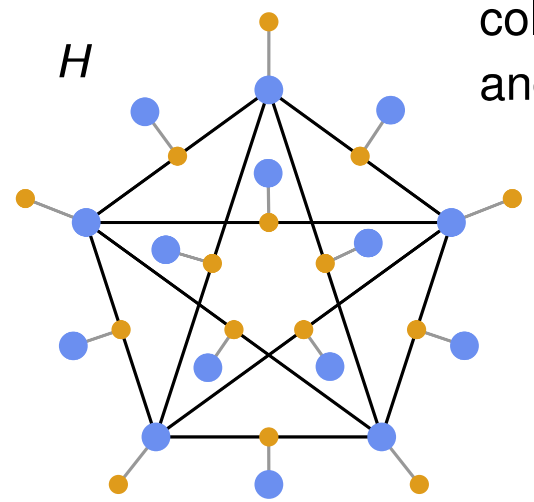

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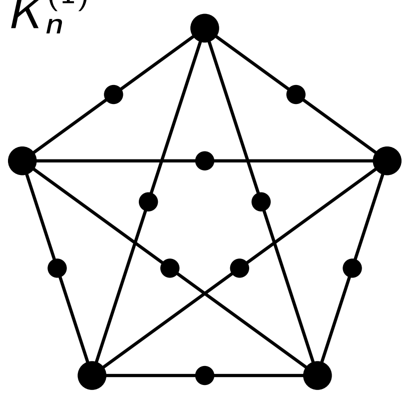

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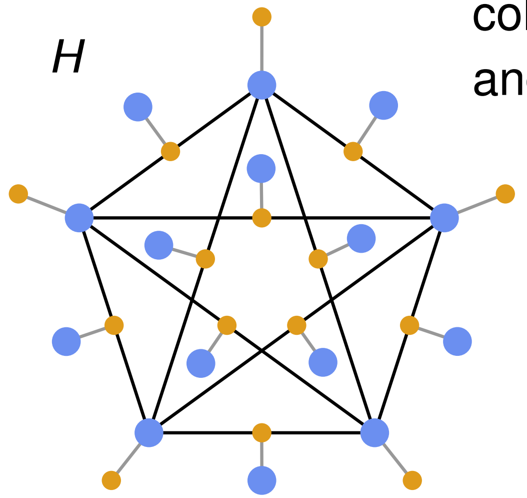

$\chi_{\text{so}}(H) = 2$.

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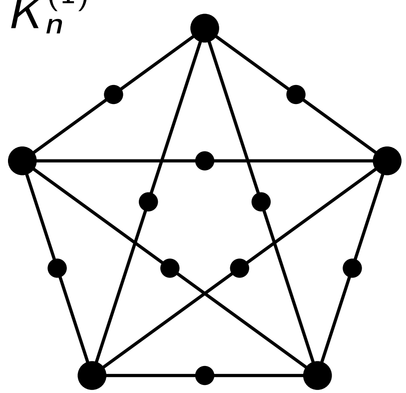
Question: Is $\chi_{so}(G) \leq \chi_{so}(H)$ if $G \subseteq H$? **No!**

$G = K_n^{(1)}$



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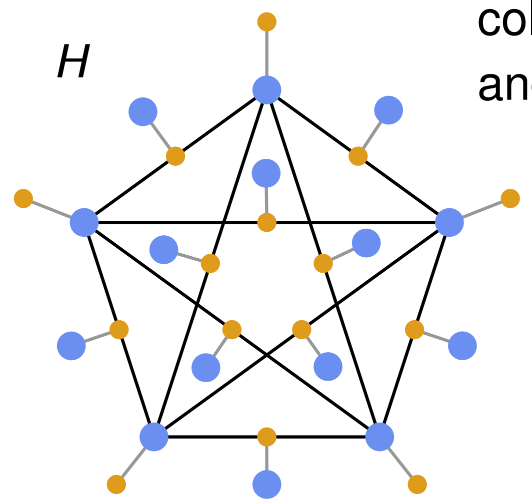

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restriction to G is strong odd

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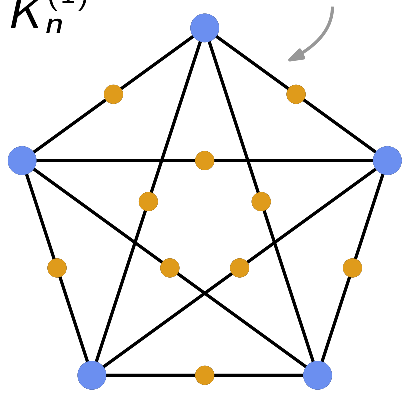
$G = K_n^{(1)}$

not strong odd on G



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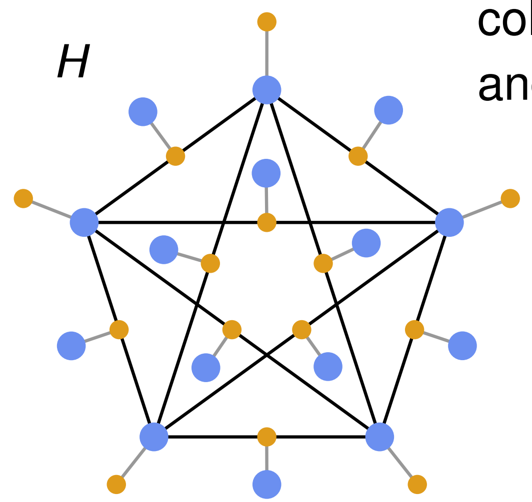

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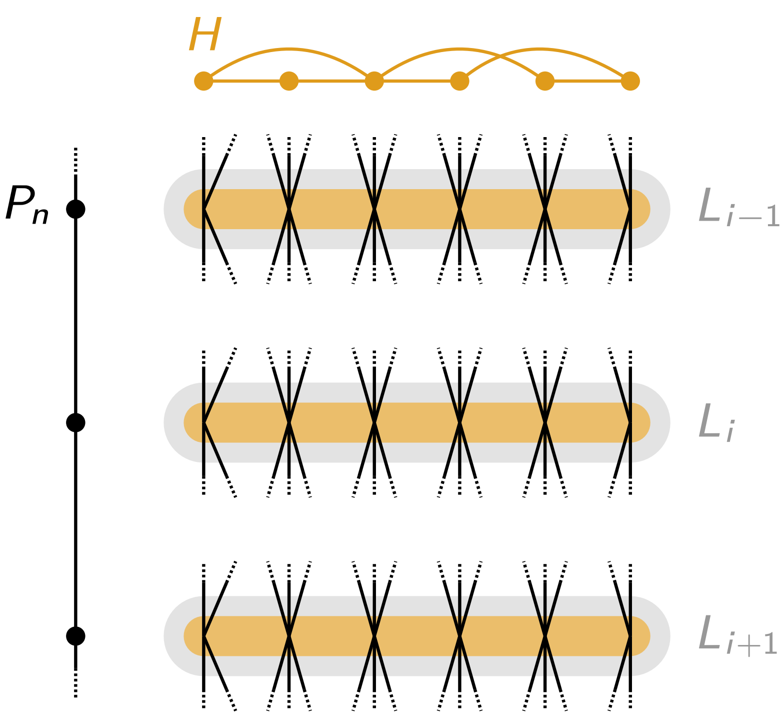

Suppose we can color H with few colors

Idea: Color single layer L_i such that:

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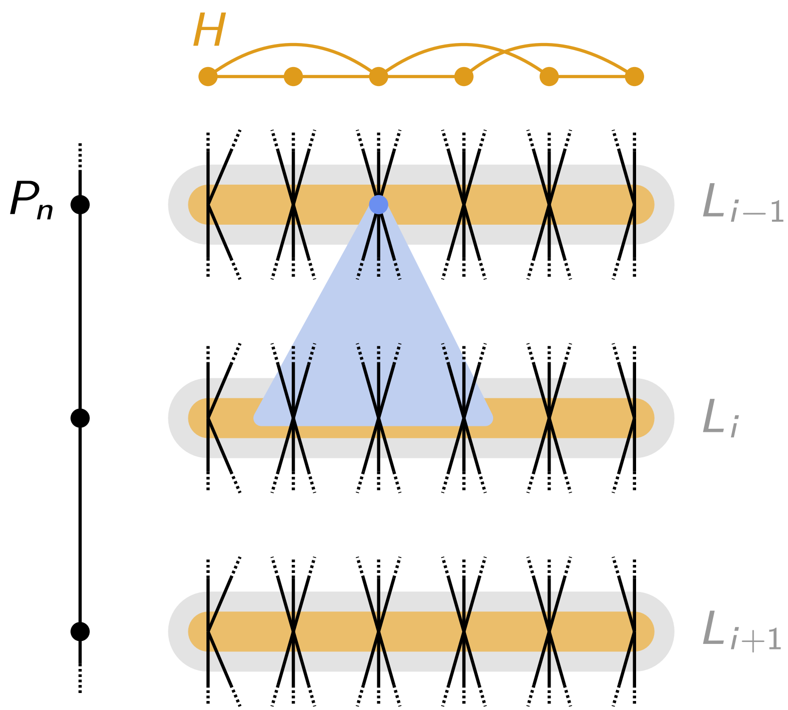

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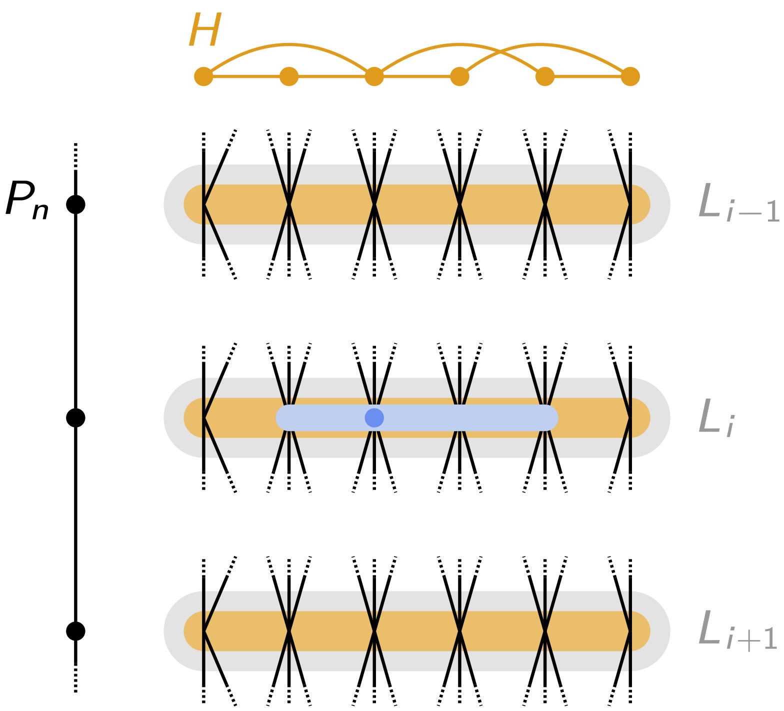

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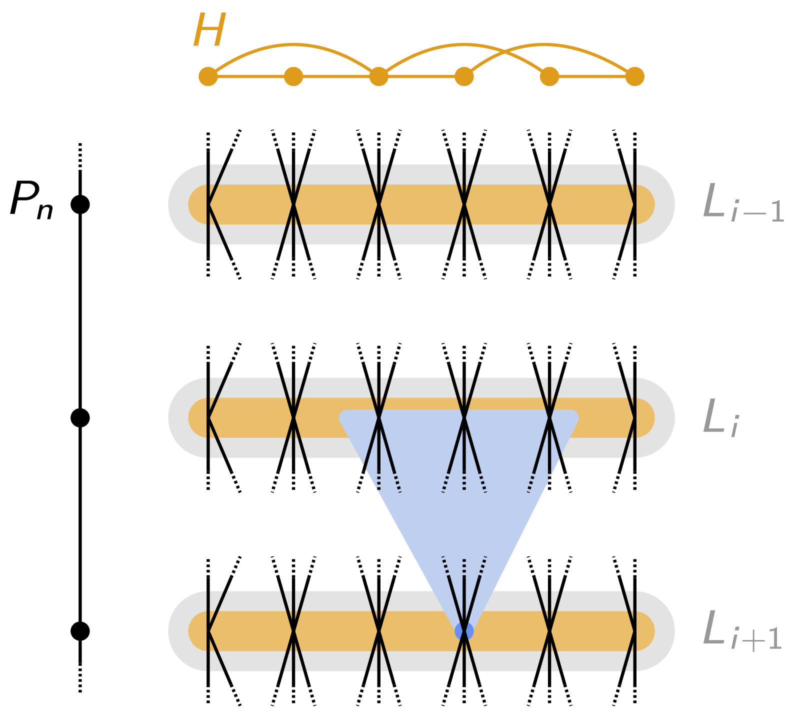

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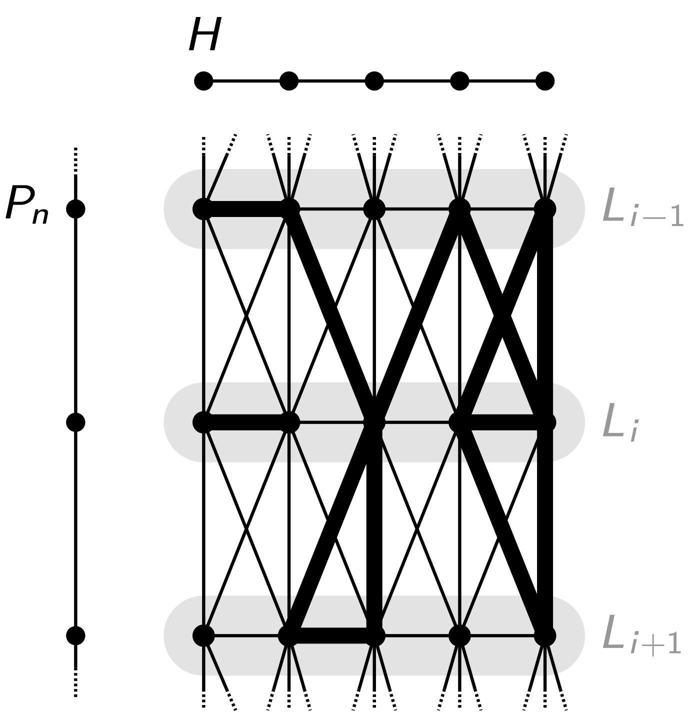

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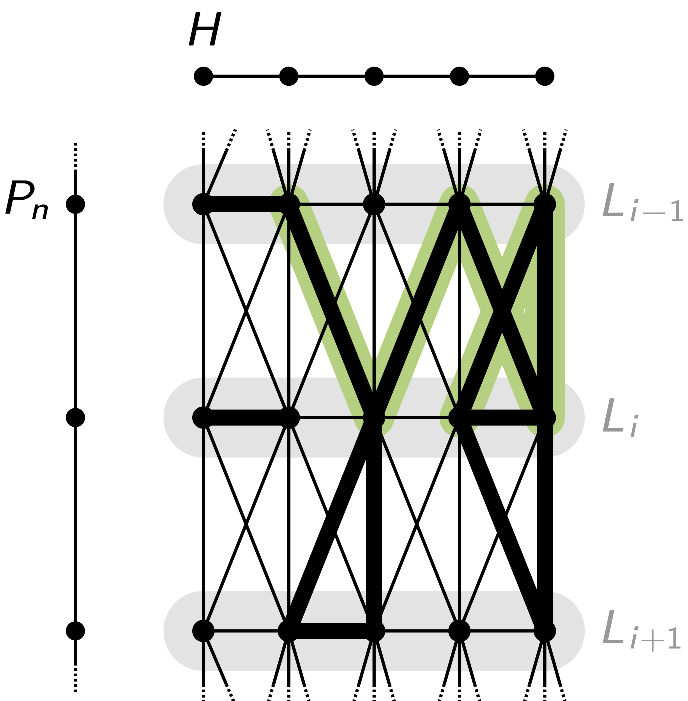

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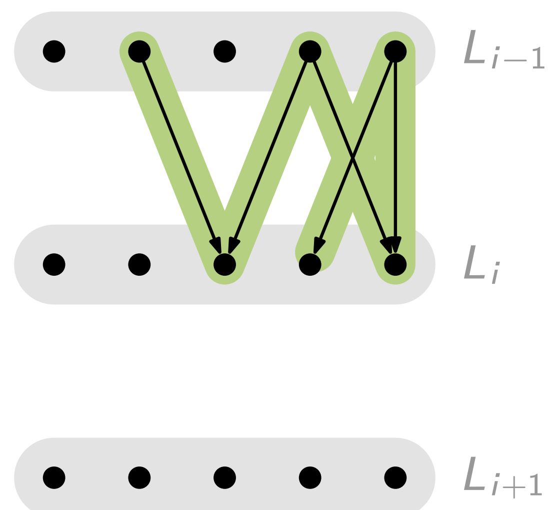

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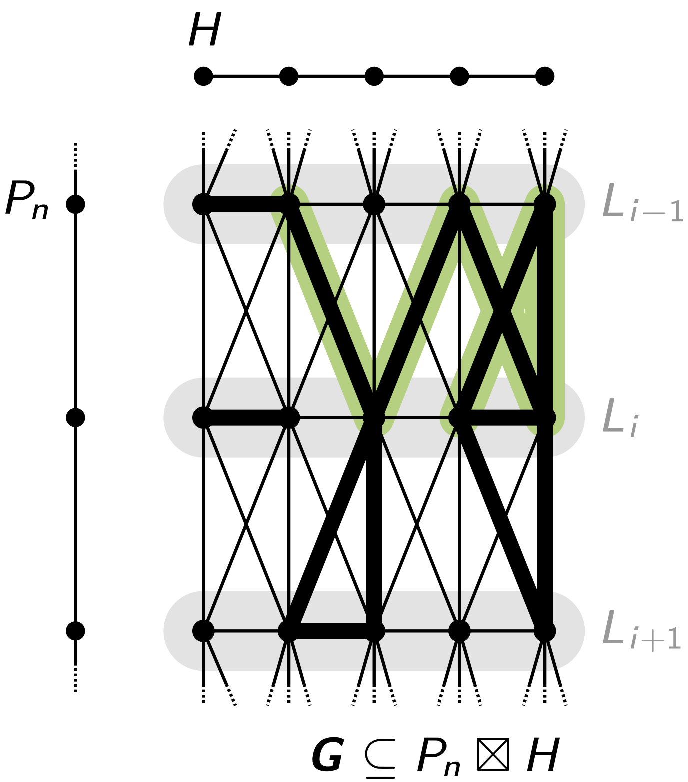

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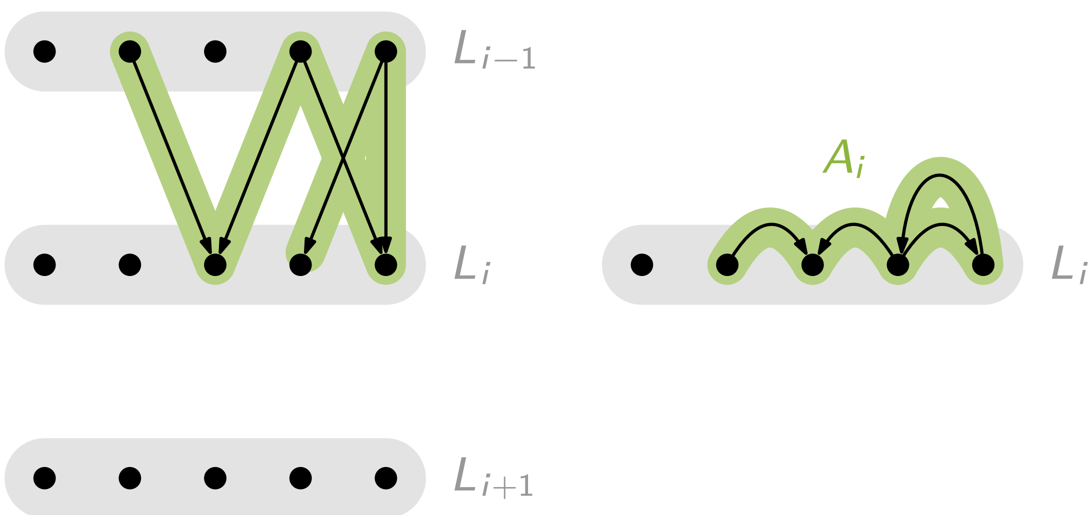

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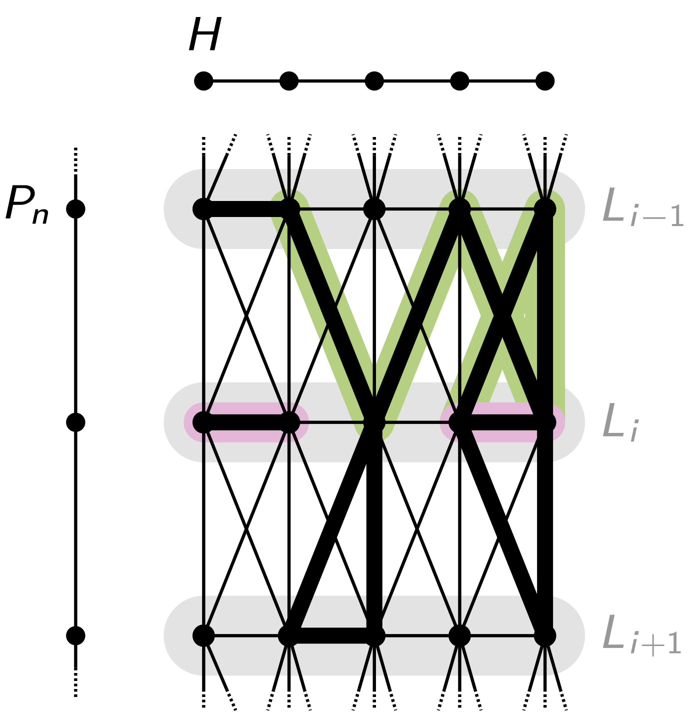

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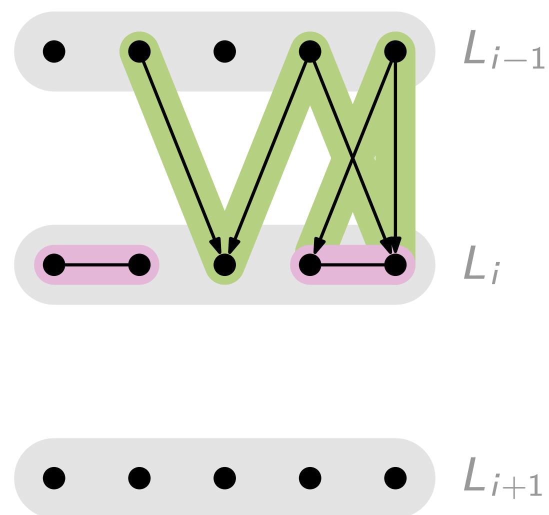

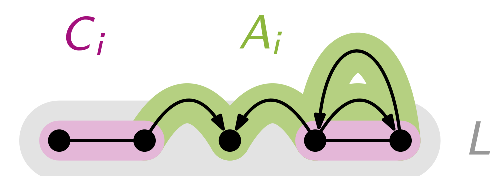

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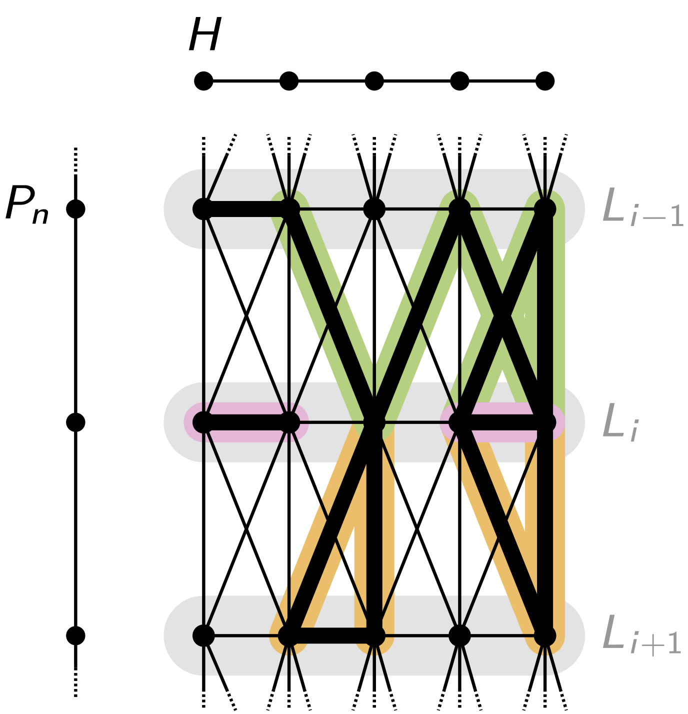

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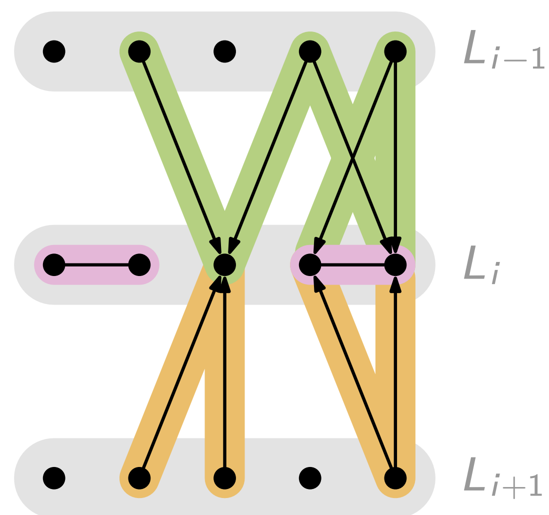

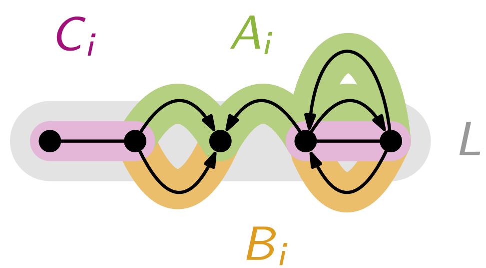

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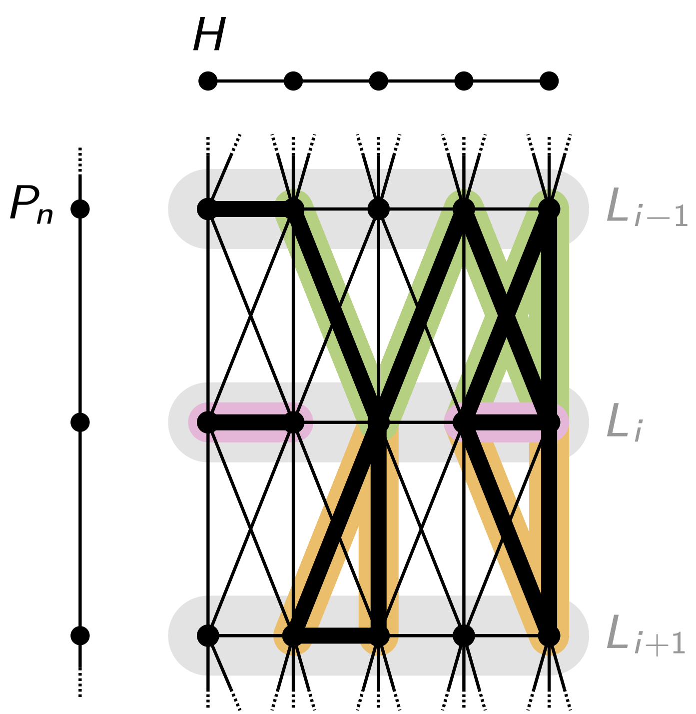

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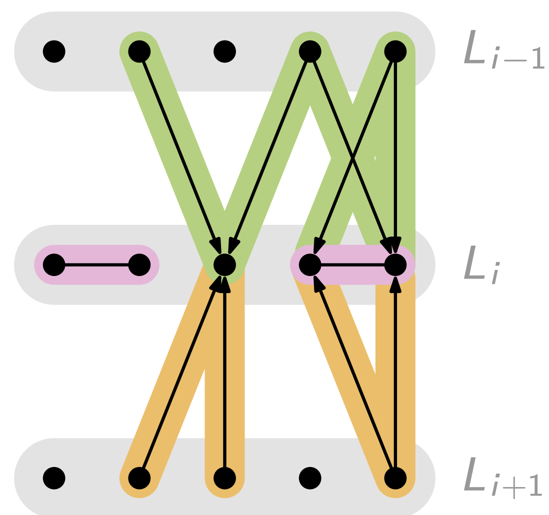

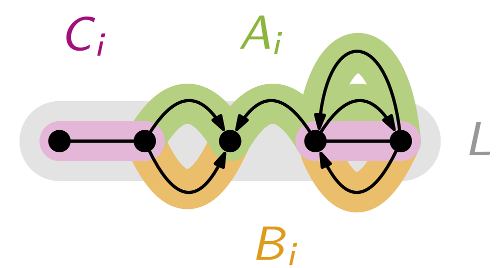

Lemma:
$C_i, A_i, B_i \subseteq H$

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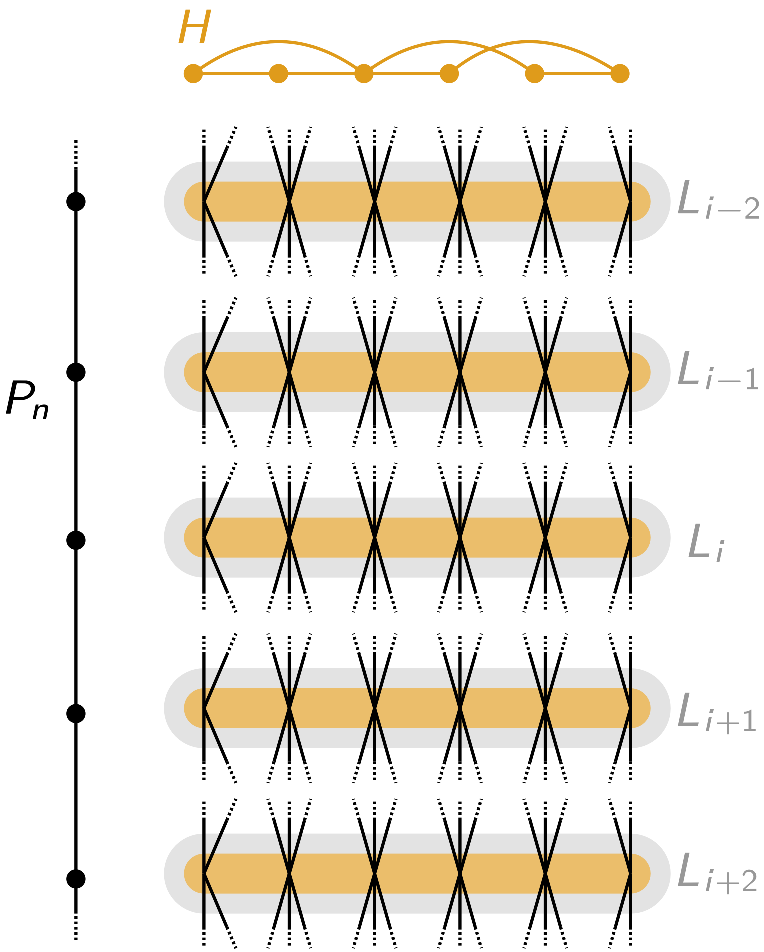

Theorem: For every k we have $\chi_{so}(G) \leq 3c_k$ for every G with $G \subseteq P_n \boxtimes H$ for some H and P_n.

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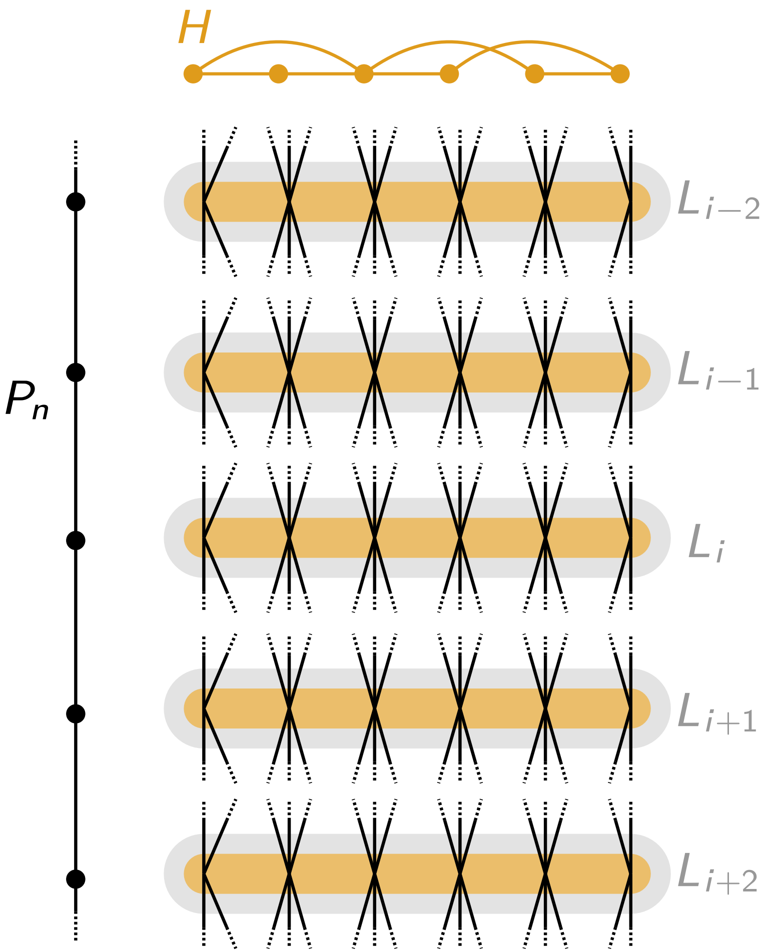

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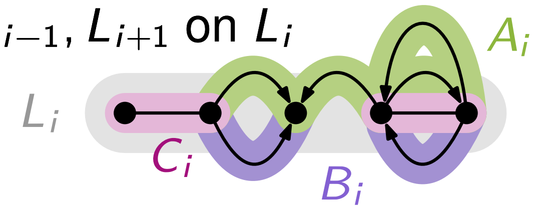

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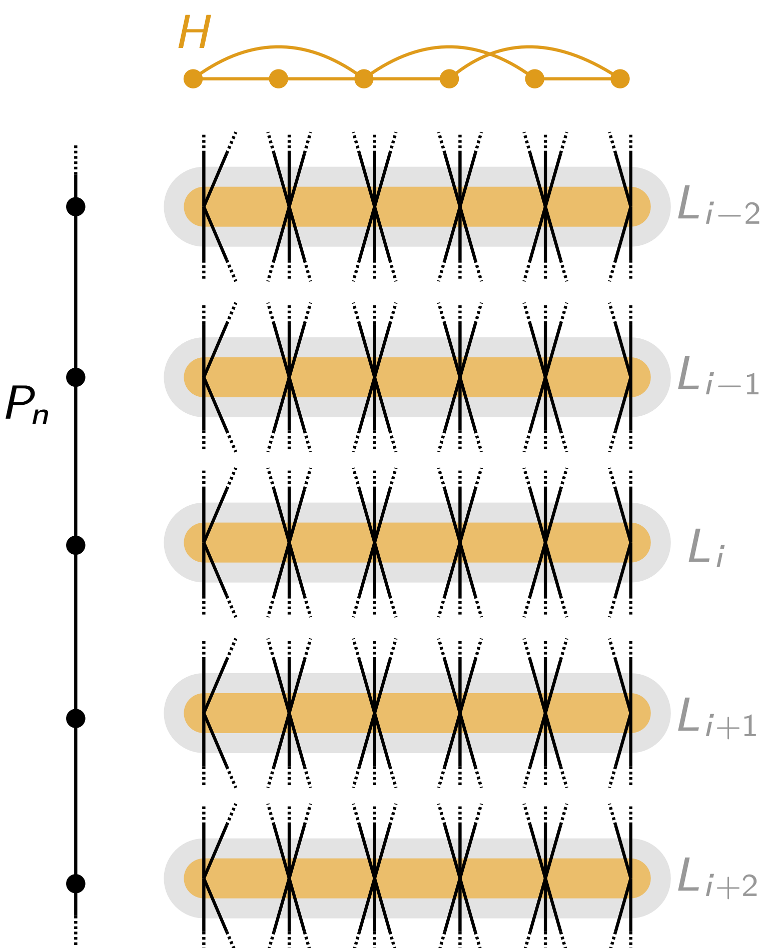

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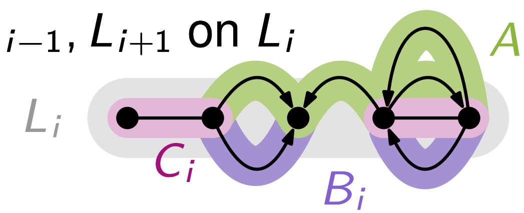

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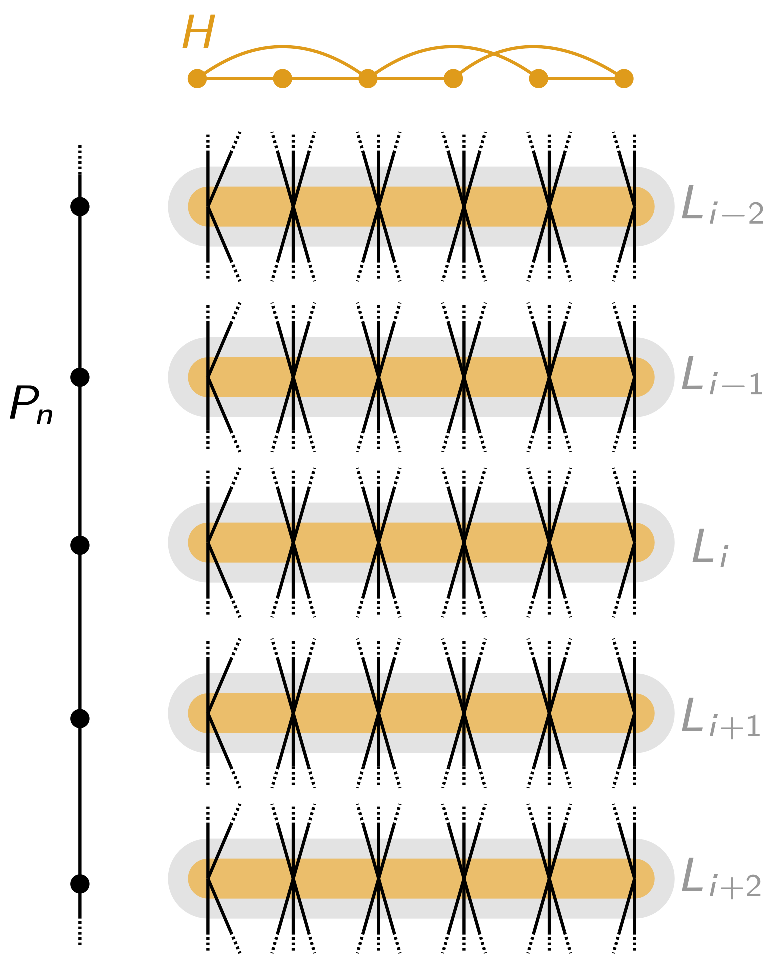

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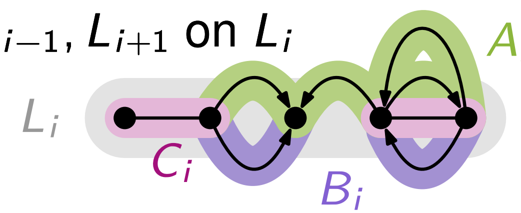

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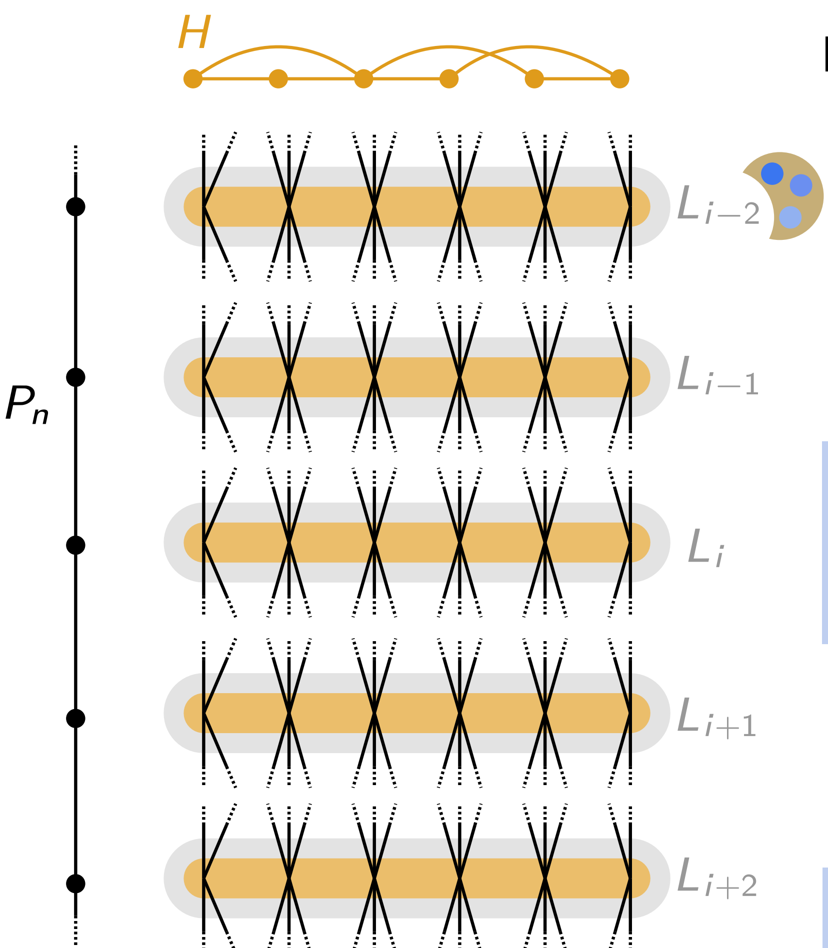

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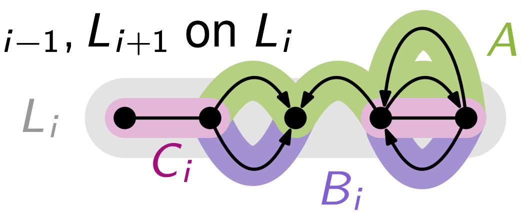

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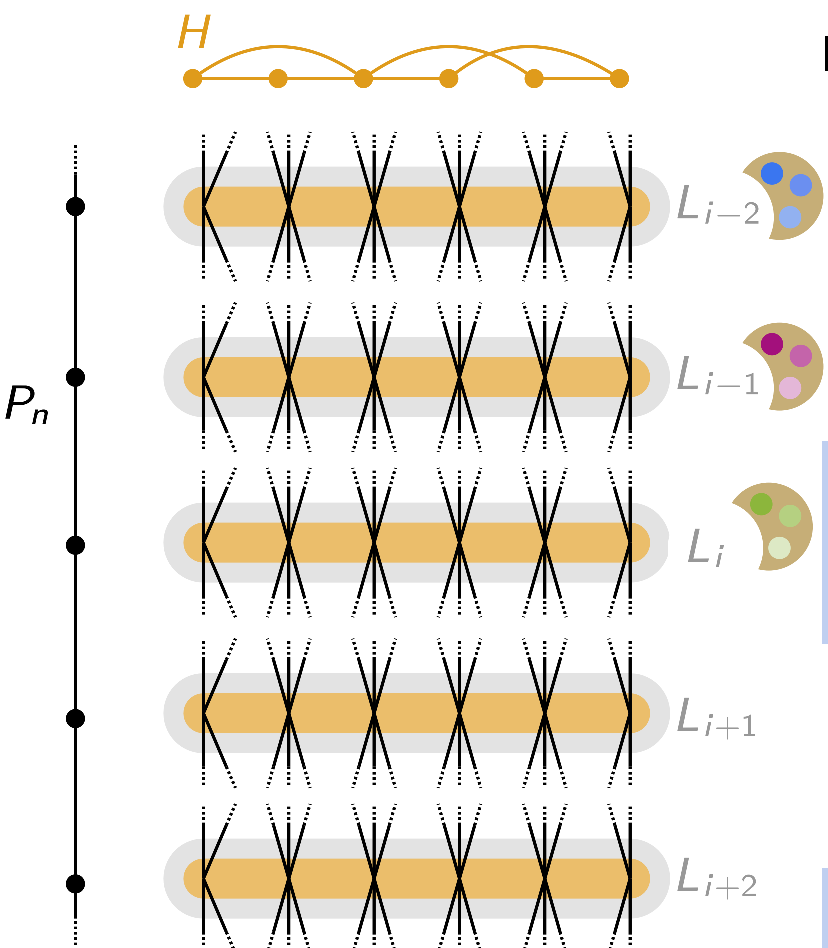

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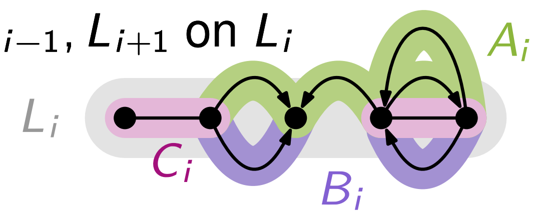

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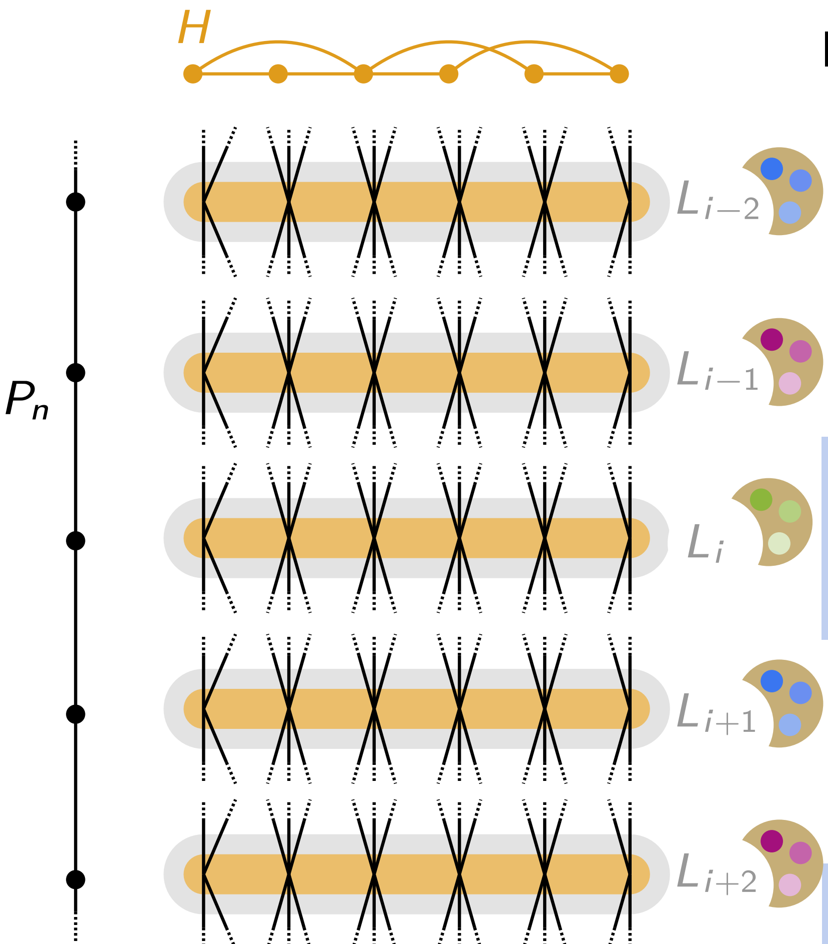

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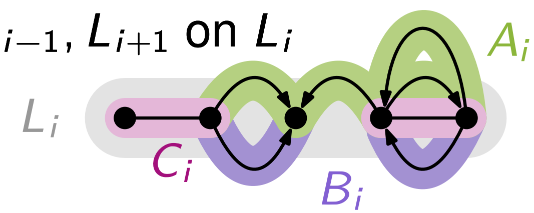

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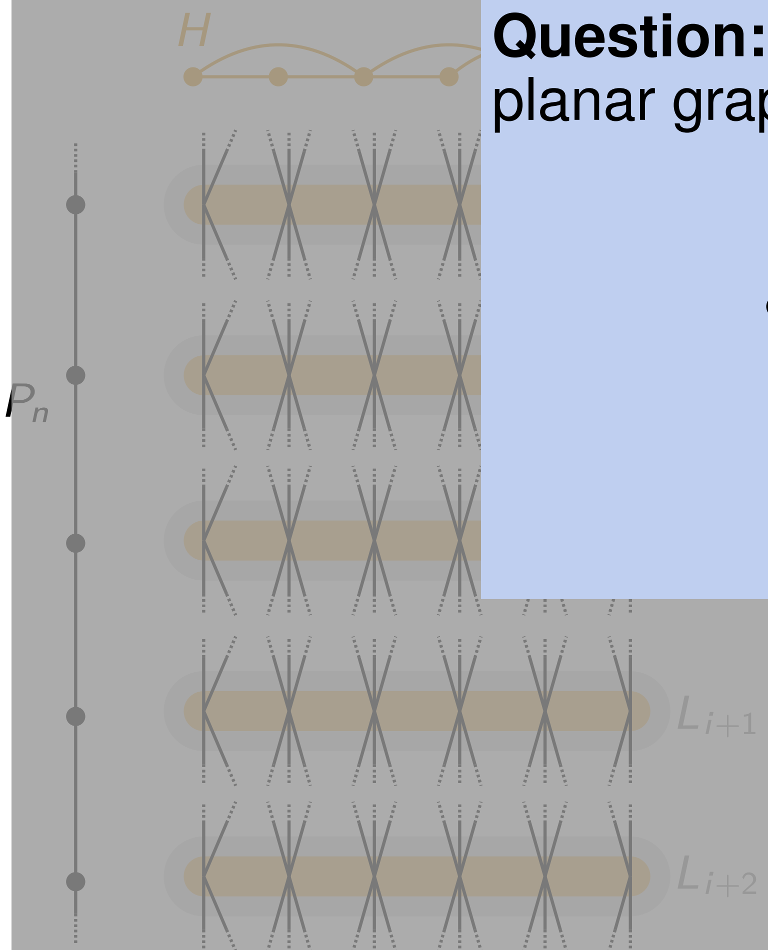
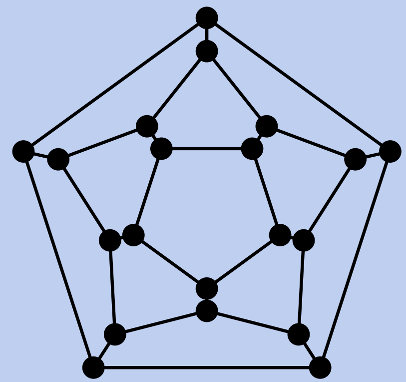


Question: Is χ_{so} bounded for all planar graphs?

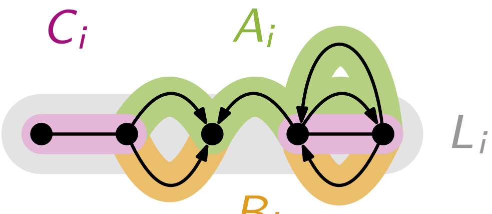


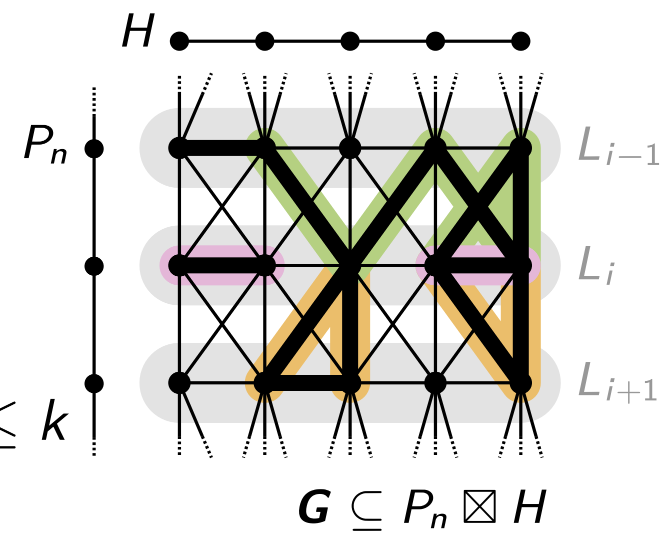
Yes!

Theorem: For every k we have $\chi_{so}(G) \leq 3c_k$ for every G with $G \subseteq P_n \boxtimes H$ for some H and P_n.

Overview

Technique:

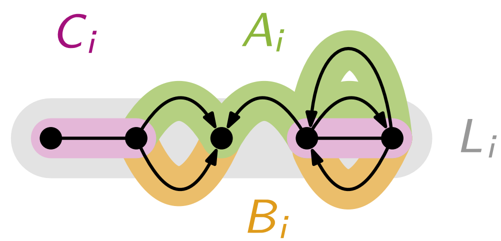


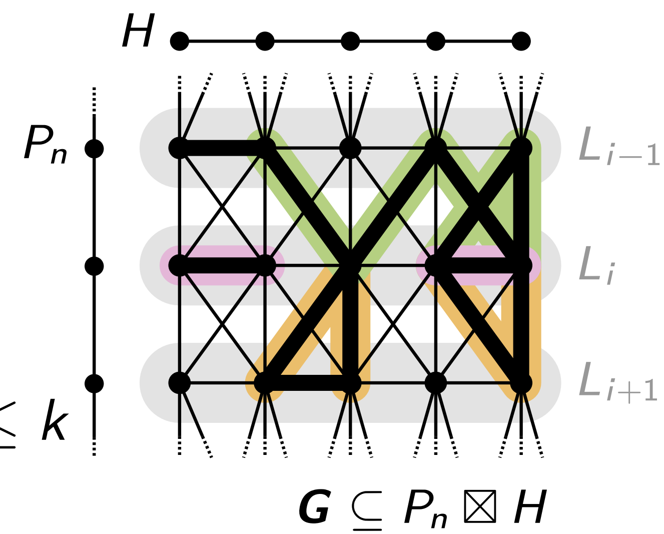


- product structure: $P_n \boxtimes H$ where $\mathrm{tw}(H) \leq k$
- strong odd for directed subgraphs

Overview

Technique:





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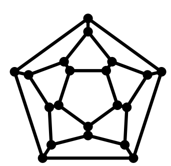
Results:



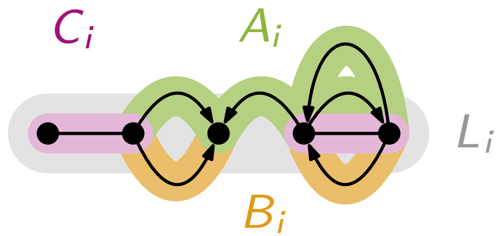
planar

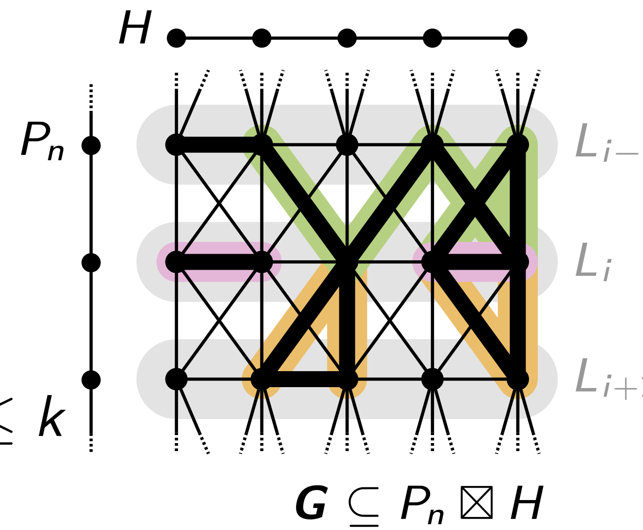
χ_{so} bounded?

✓ [CPST, 2024]

Overview

Technique:





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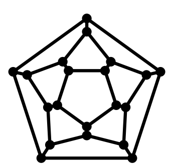
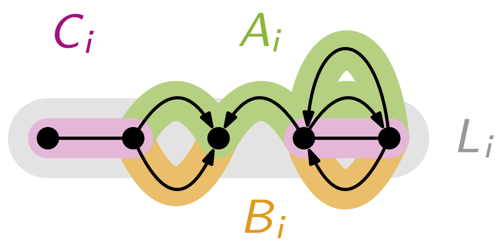
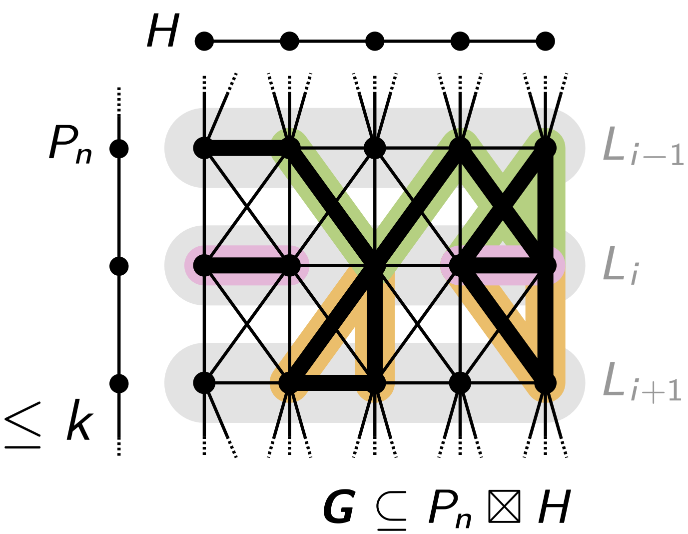


	planar	bounded row treewidth
χ_{so} bounded?	✓ [CPST, 2024]	✓

Overview

Technique:





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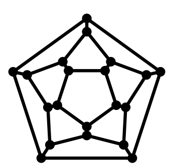
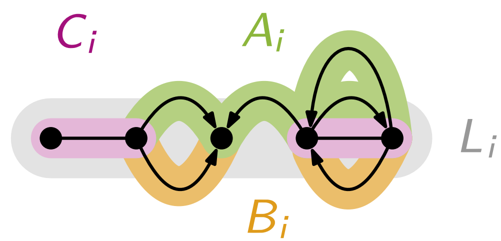
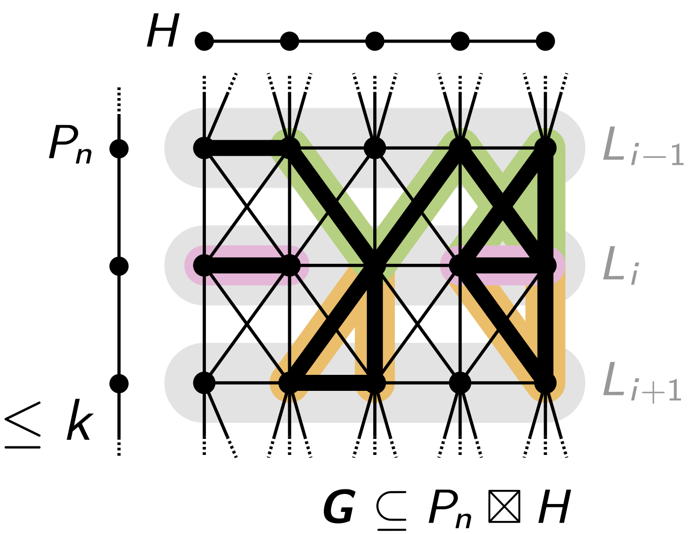


	planar	bounded row treewidth	proper minor-closed
χ_{so} bounded?	✓ [CPST, 2024]	✓	✓

Overview

Technique:





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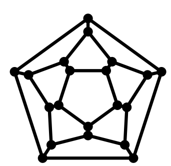

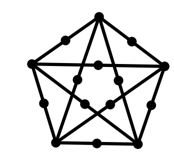

	planar	bounded row treewidth	proper minor-closed		all graphs
χ_{so} bounded?	✓ [CPST, 2024]	✓	✓		✗

Overview

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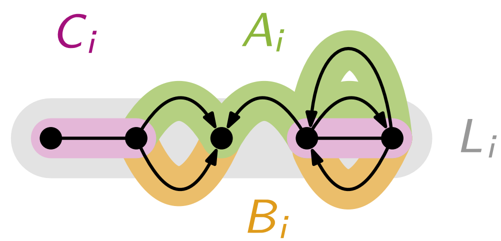

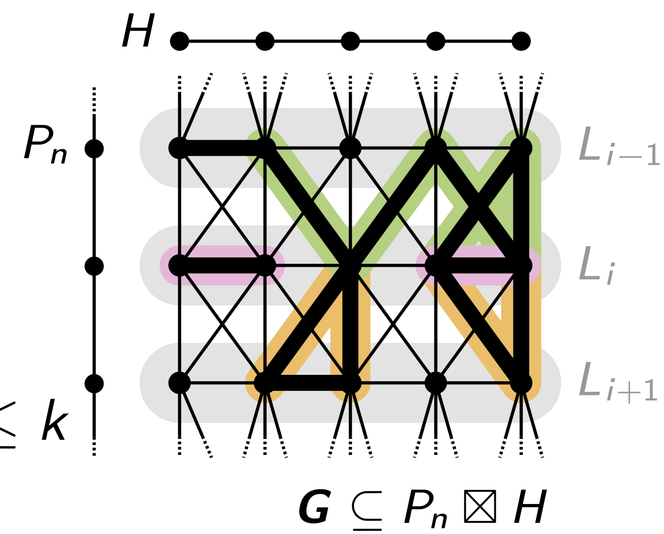

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Results:

	planar	bounded row treewidth	proper minor-closed	bounded expansion	all graphs
χ_{so} bounded?	✓ [CPST, 2024]	✓	✓	?	✗

Overview

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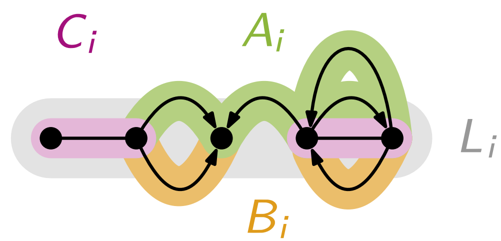

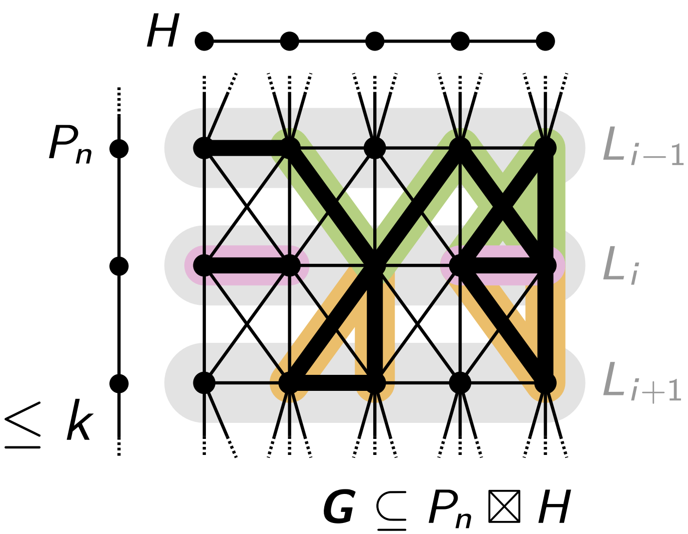

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χ_{so} bounded?	✓ [CPST, 2024]	✓	✓	✓ [Pilipczuk, 2025]	✗

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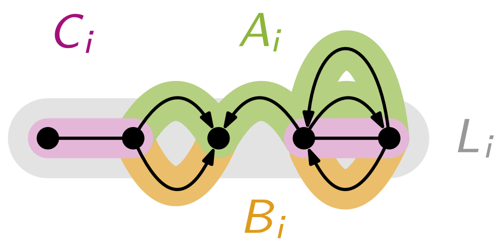

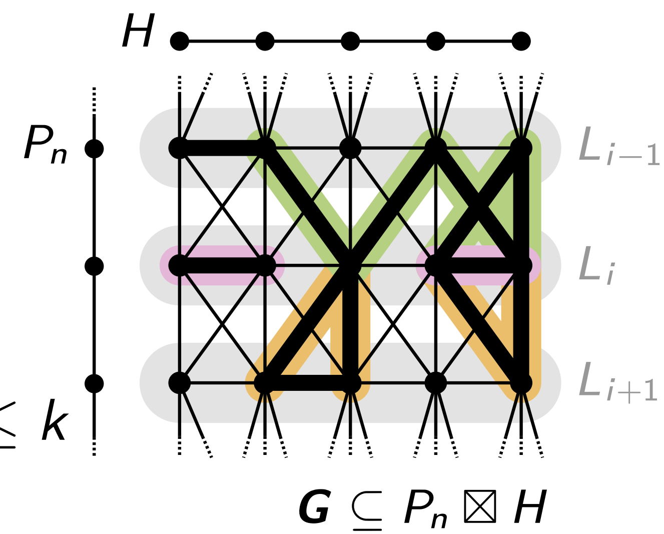

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Results:

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Thanks!