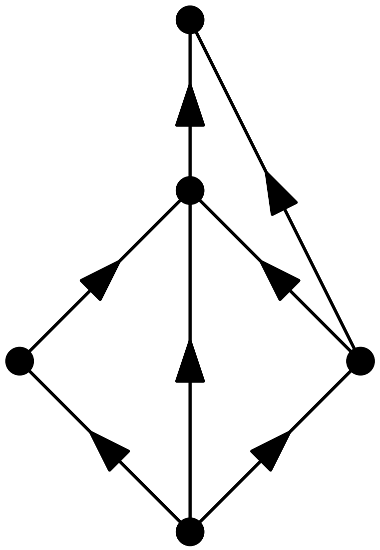
Karlsruhe Institute of Technology

A Sublinear Bound on the Page Number of Upward Planar Graphs

Mittagsseminar of the Discrete Math Group, TU Berlin · Oct 22, 2021

Paul Jungeblut, **Laura Merker**, Torsten Ueckerdt



upward planar graph

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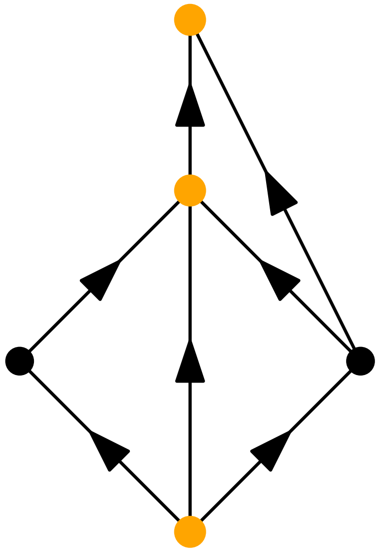

upward planar graph

height: max # of pairwise comparable vertices

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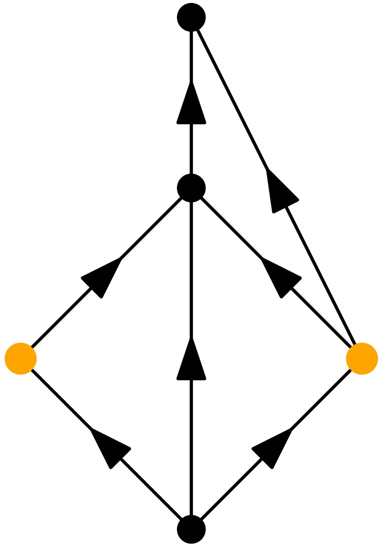

upward planar graph

height: max # of pairwise comparable vertices

width: max # of pairwise incomparable vertices

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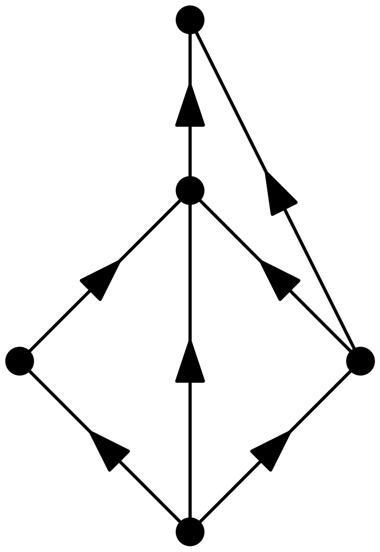


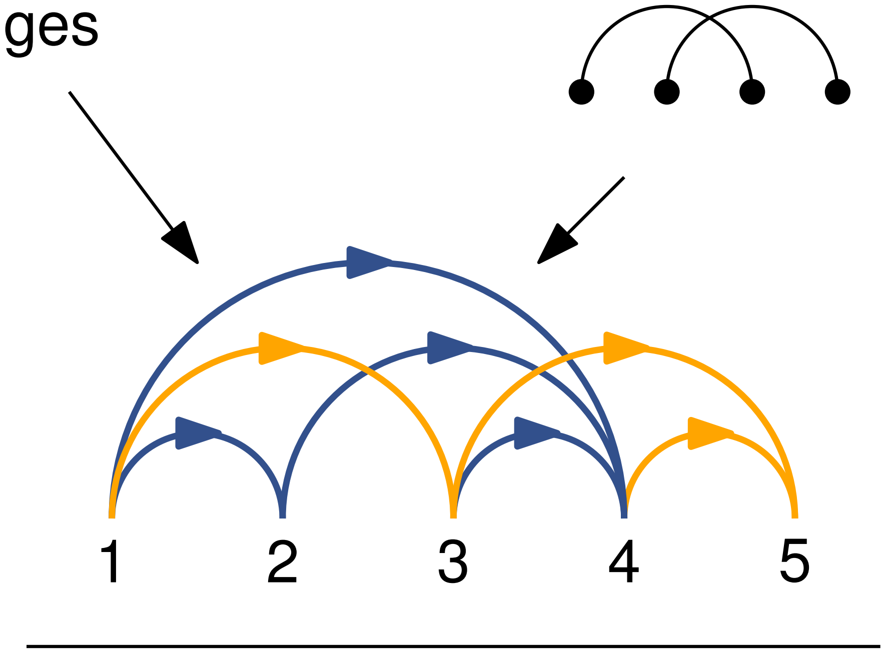
upward planar graph

height: max # of pairwise comparable vertices

width: max # of pairwise incomparable vertices

find a partition of the edges

no monochromatic



find topological vertex ordering

Open Problem

Is the page number of upward planar graphs bounded?

Open for:

- upward planar graphs
- planar posets
- planar lattices
- upward outerplanar graphs
- upward planar 2-trees

Bounded for:

- single-source outerplanar graphs (Bhore et al. 2021)
- upward outerpaths (Nöllenburg, Pupyrev 2021)
- upward planar 3-trees (Frati et al. 2013)
- upward planar graphs whose 4-connected components have bounded page number (Frati et al. 2013)

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Lower Bound

There is an upward planar graph G with $\mathrm{pn}(G) \geqslant 4$ (Hung 1993).

Theorem

There is an upward planar graph G with $\mathrm{pn}(G) \geqslant 5$.

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Upper Bounds for Upward Planar Graphs

Frati et al. (2013): $O(h \log n)$

Theorem

$\mathrm{pn}(G) \leqslant O(h \log h)$
for $h = \mathrm{height}(G)$

Theorem

$\mathrm{pn}(G) \leqslant 8 \cdot \mathrm{width}(G)$

Trivial: $O(n)$

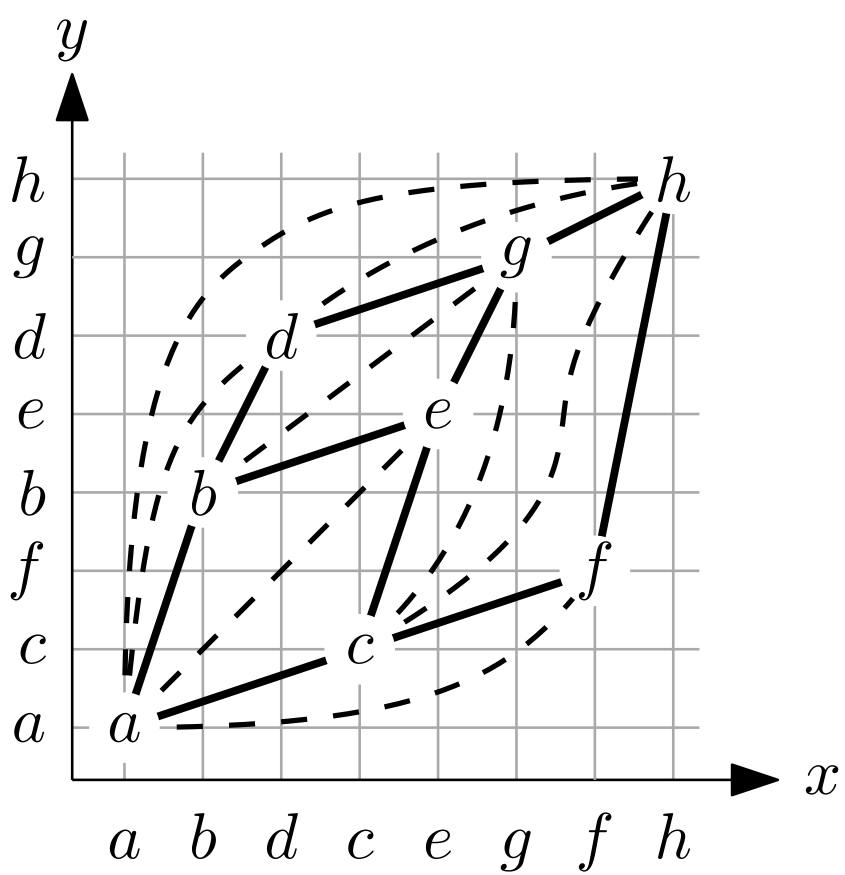
Theorem

The page number of n-vertex upward planar graphs is $O(n^{2/3} \log(n)^{2/3})$.

Height

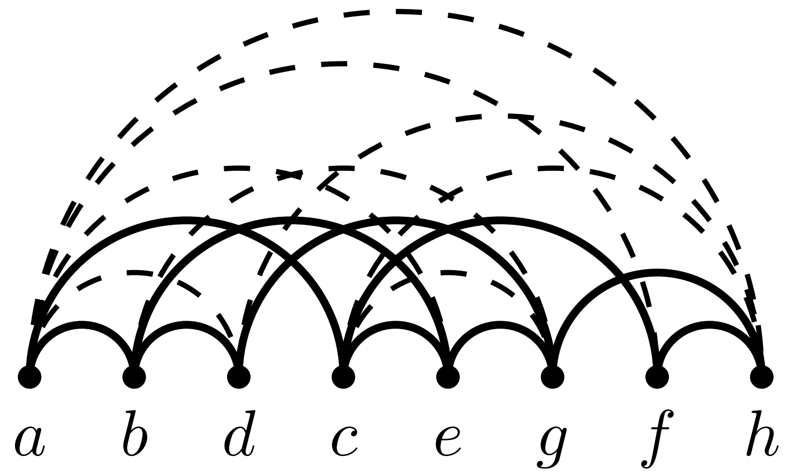
= # vertices in a longest path

dominance drawing



Theorem

pn(G) $\leqslant O(h \log h)$ for h = height(G)



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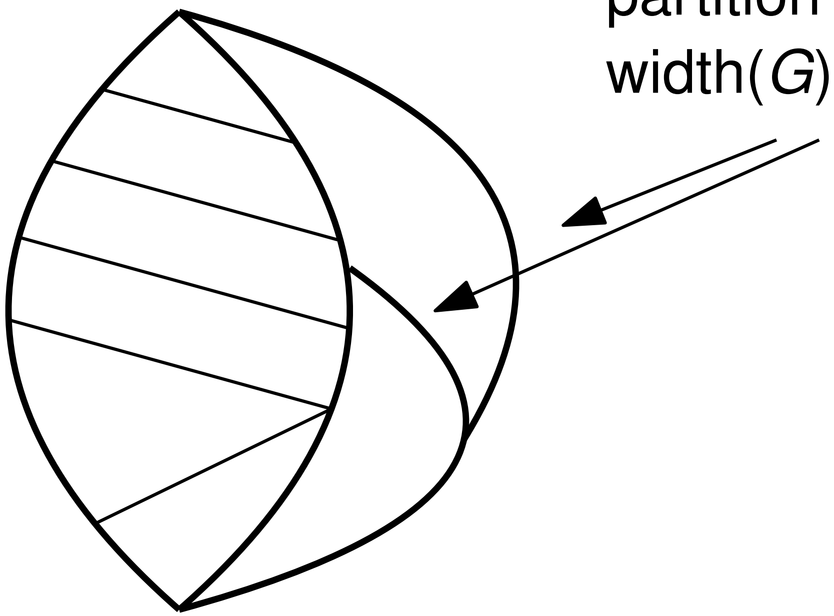
Width

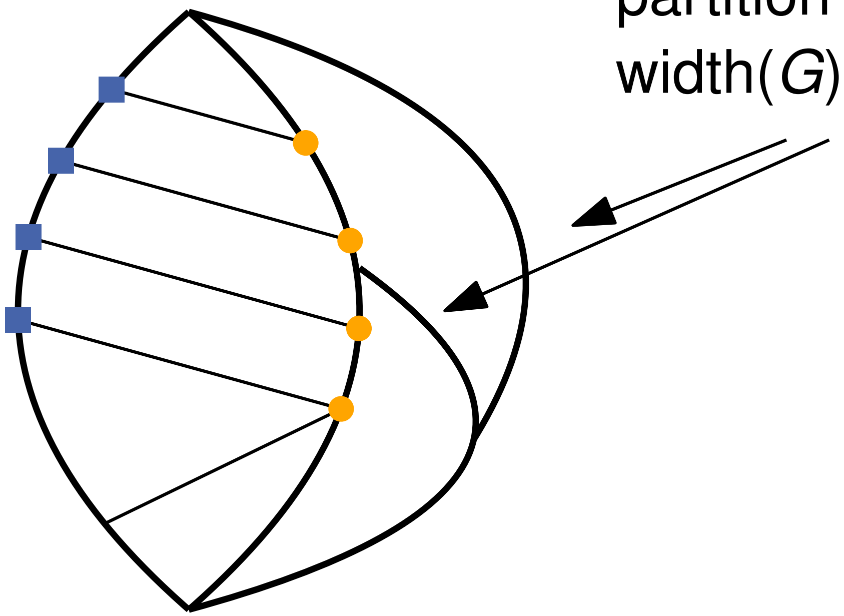
= max # pairwise incomparable vertices



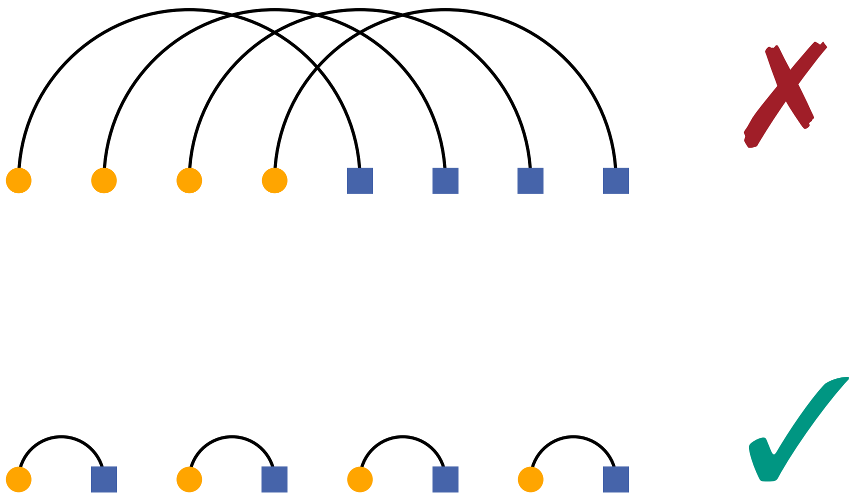
partition vertices into
width(G) paths

Width

= max # pairwise incomparable vertices



partition vertices into
width(G) paths



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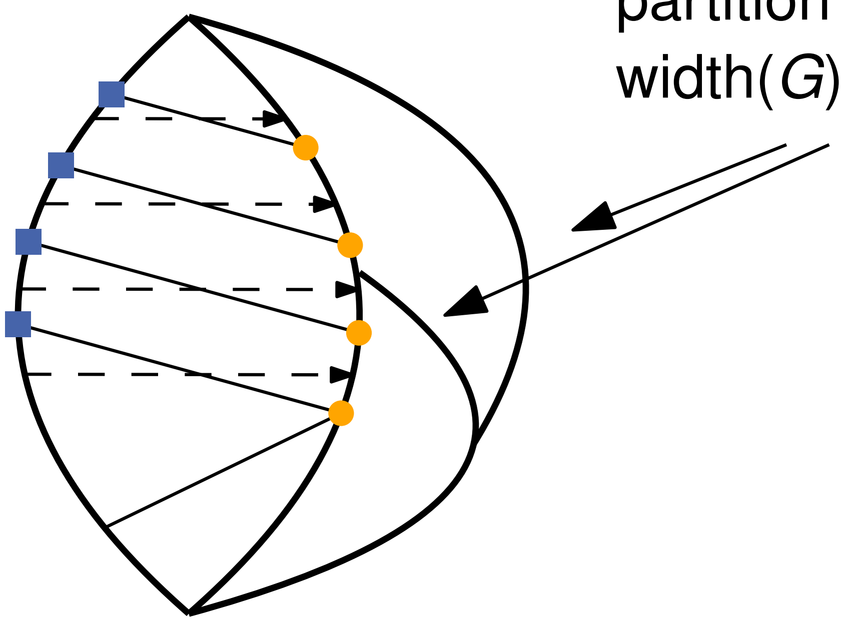
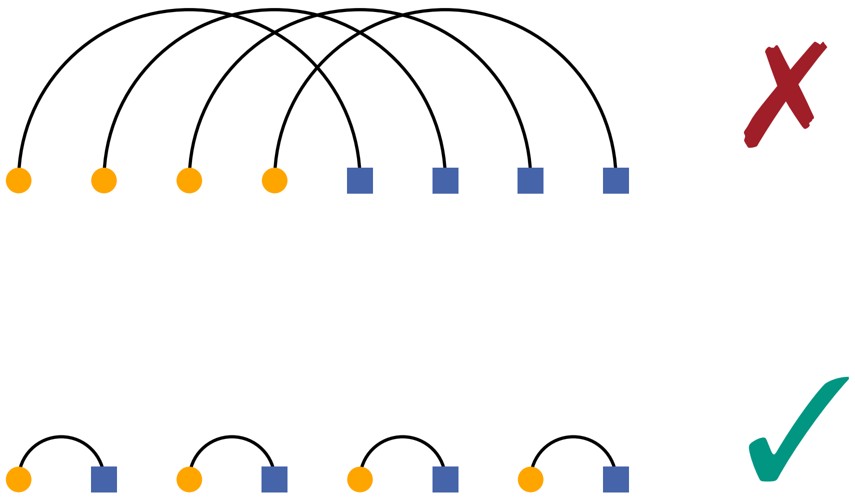
Width

= max # pairwise incomparable vertices



partition vertices into width(G) paths



Theorem

$8 \cdot \text{width}(G)$ pages suffice with any topological vertex ordering

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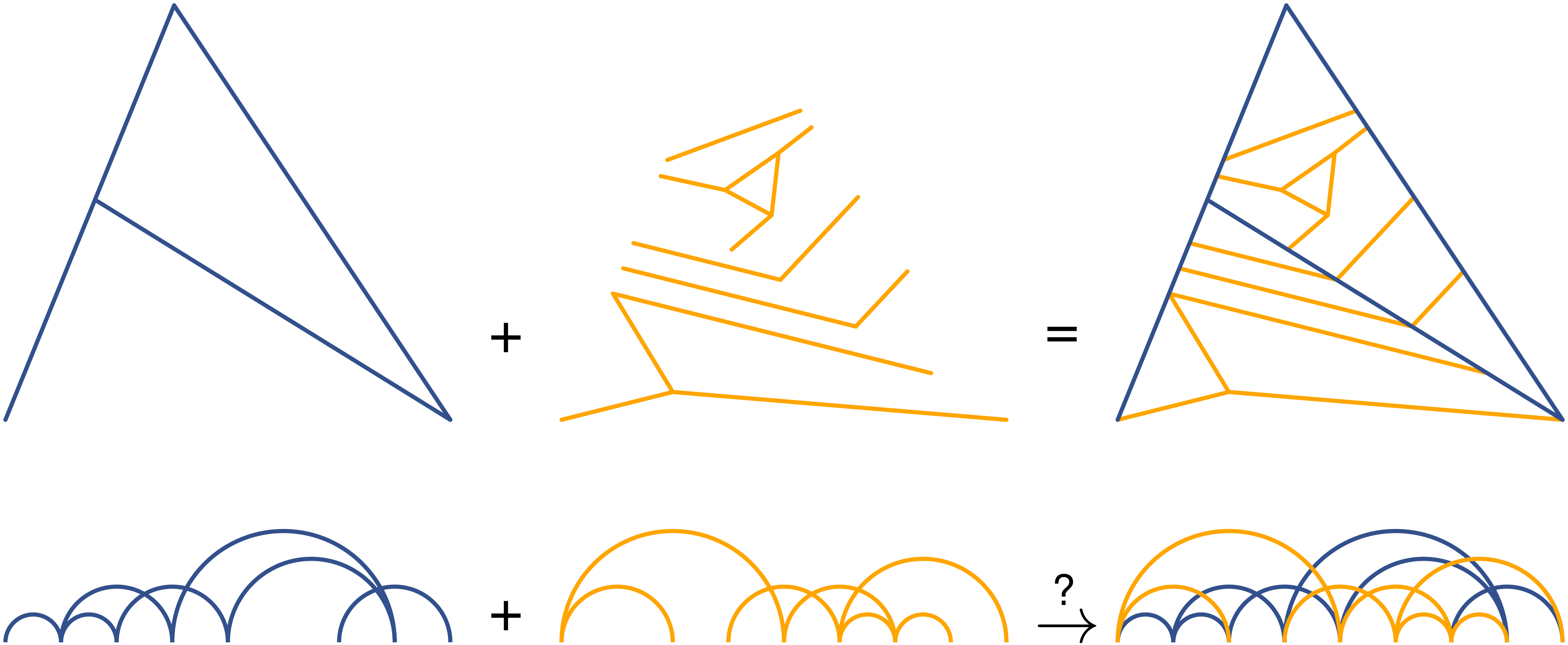

small width small height small page number

Given: upward planar graph G and vertices $L \subseteq V(G)$ of long paths

- $8 \cdot \text{width}(L)$ pages for $G[L]$ + subdivided edges
- with any topological ordering of G', where $V(G') \supseteq V(G)$

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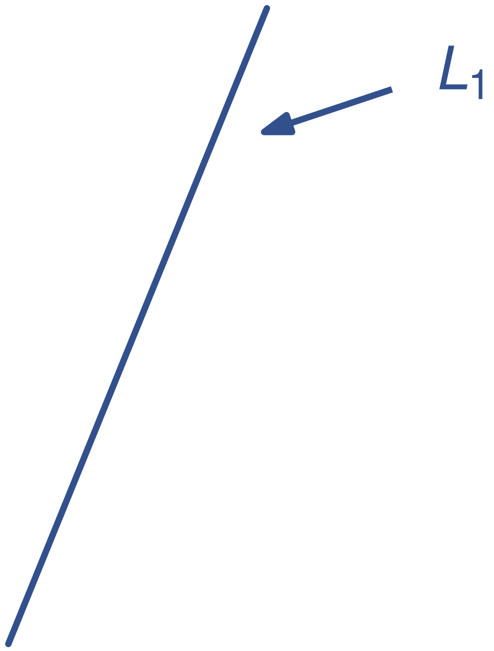
Given: upward planar graph G and vertices $L \subseteq V(G)$ of long paths

- $8 \cdot \text{width}(L)$ pages for $G[L]$ + subdivided edges
- with any topological ordering of G', where $V(G') \supseteq V(G)$



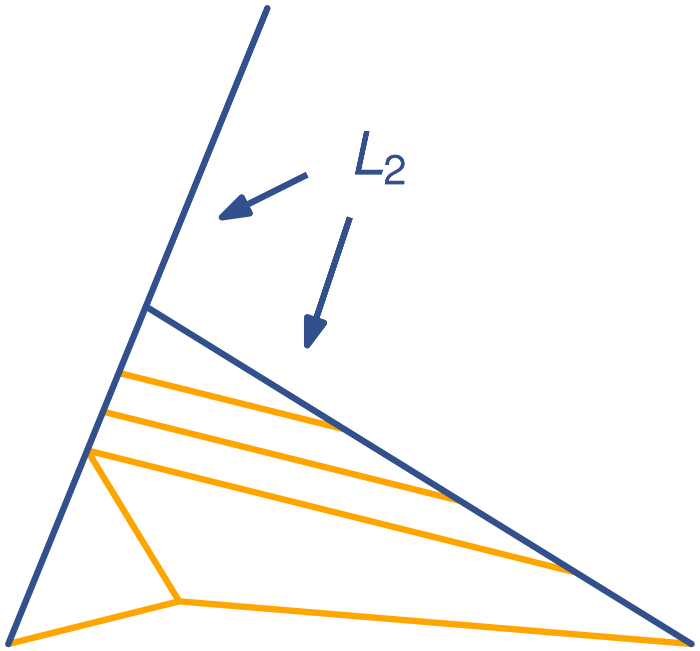
Choose a long path ($\geqslant \ell = n^{2/3} / \log(n)^{1/3}$ uncovered vertices)

$\text{width}(L_1) \leqslant 1$

$8 \cdot \text{width}(L_1)$

Given: upward planar graph G and vertices $L \subseteq V(G)$ of long paths

- $8 \cdot \text{width}(L)$ pages for $G[L]$ + subdivided edges
- with any topological ordering of G', where $V(G') \supseteq V(G)$



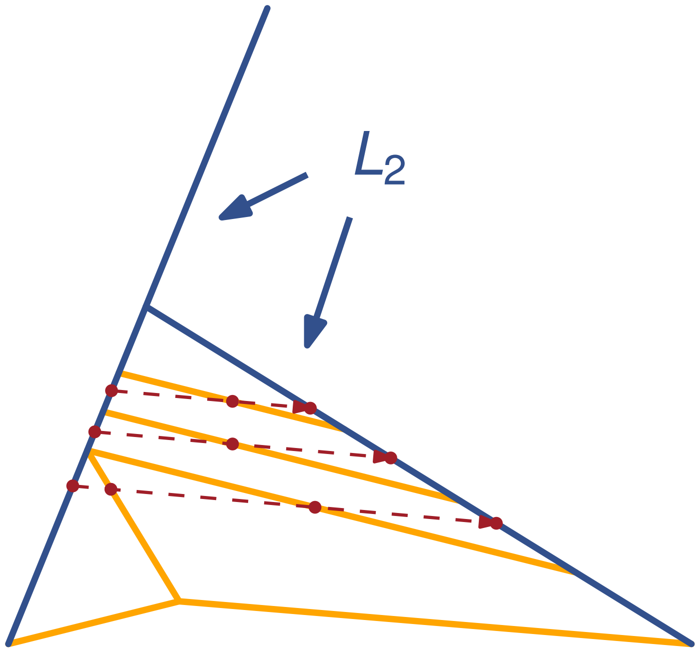
Choose a long path ($\geqslant \ell = n^{2/3} / \log(n)^{1/3}$ uncovered vertices)

$\text{width}(L_2) \leqslant 2$

$8 \cdot \text{width}(L_1) + 8 \cdot \text{width}(L_2)$

Given: upward planar graph G and vertices $L \subseteq V(G)$ of long paths

- $8 \cdot \text{width}(L)$ pages for $G[L]$ + subdivided edges
- with any topological ordering of G', where $V(G') \supseteq V(G)$



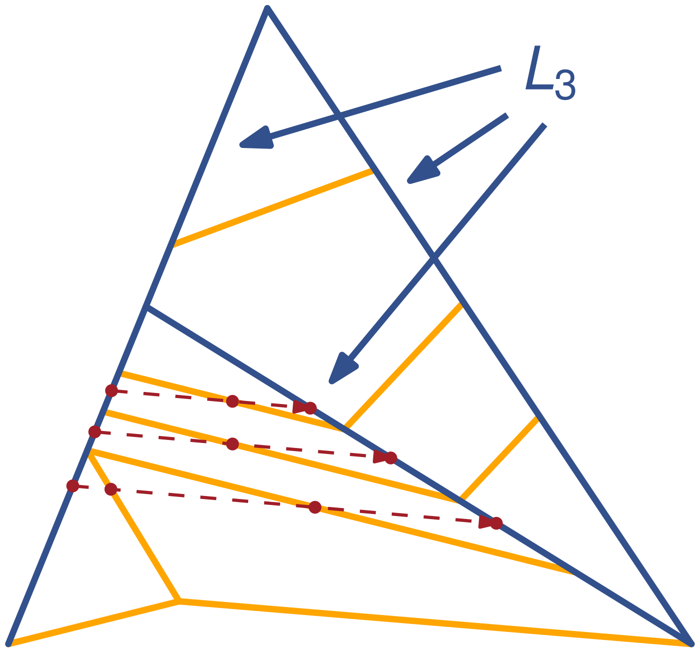
Choose a long path ($\geqslant \ell = n^{2/3} / \log(n)^{1/3}$ uncovered vertices)

$\text{width}(L_2) \leqslant 2$

$$8 \cdot \text{width}(L_1) \; + \; 8 \cdot \text{width}(L_2)$$

Given: upward planar graph G and vertices $L \subseteq V(G)$ of long paths

- $8 \cdot \text{width}(L)$ pages for $G[L]$ + subdivided edges
- with any topological ordering of G', where $V(G') \supseteq V(G)$



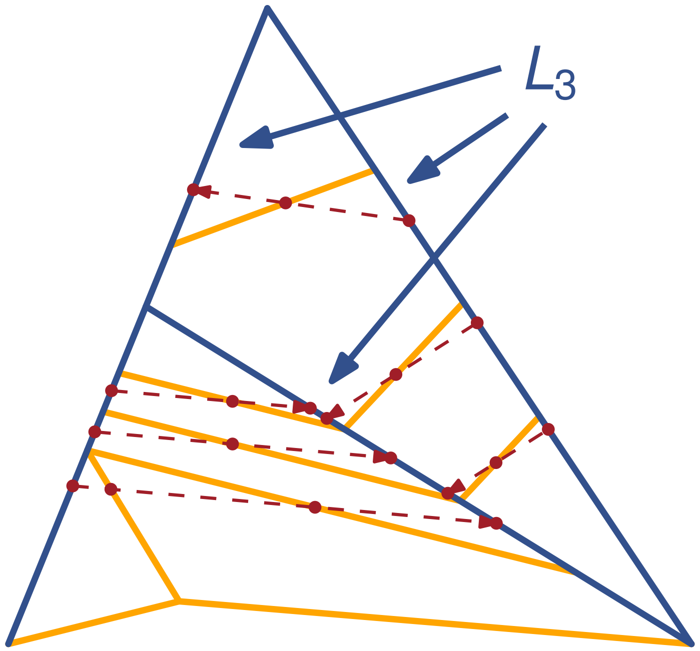
Choose a long path ($\geqslant \ell = n^{2/3} / \log(n)^{1/3}$ uncovered vertices)

$\text{width}(L_3) \leqslant 3$

$8 \cdot \text{width}(L_1) \; + \; 8 \cdot \text{width}(L_2) \; + \; 8 \cdot \text{width}(L_3)$

Given: upward planar graph G and vertices $L \subseteq V(G)$ of long paths

- $8 \cdot \text{width}(L)$ pages for $G[L]$ + subdivided edges
- with any topological ordering of G', where $V(G') \supseteq V(G)$



Choose a long path ($\geqslant \ell = n^{2/3} / \log(n)^{1/3}$ uncovered vertices)

$\text{width}(L_3) \leqslant 3$

$$8 \cdot \text{width}(L_1) \; + \; 8 \cdot \text{width}(L_2) \; + \; 8 \cdot \text{width}(L_3)$$

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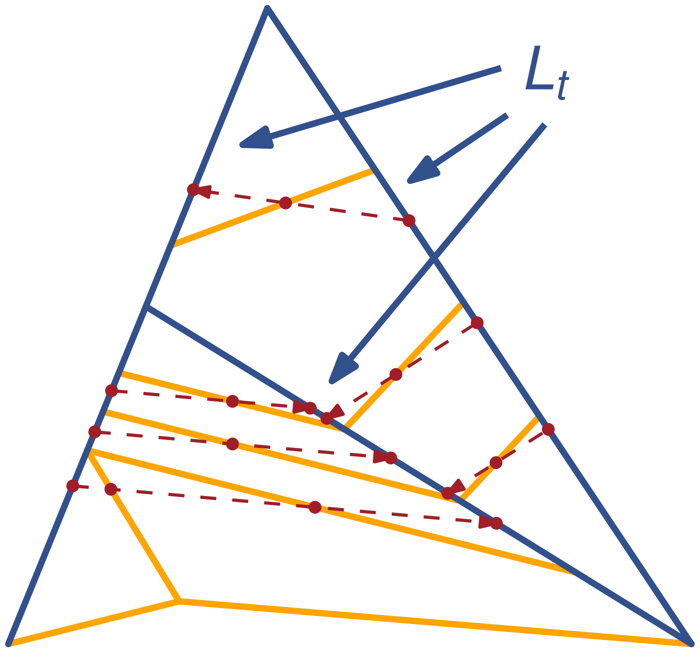
Given: upward planar graph G and vertices $L \subseteq V(G)$ of long paths

- $8 \cdot \text{width}(L)$ pages for $G[L]$ + subdivided edges
- with any topological ordering of G', where $V(G') \supseteq V(G)$



Choose a long path ($\geqslant \ell = n^{2/3} / \log(n)^{1/3}$ uncovered vertices)

$\text{width}(L_t) \leqslant t \leqslant n/\ell$

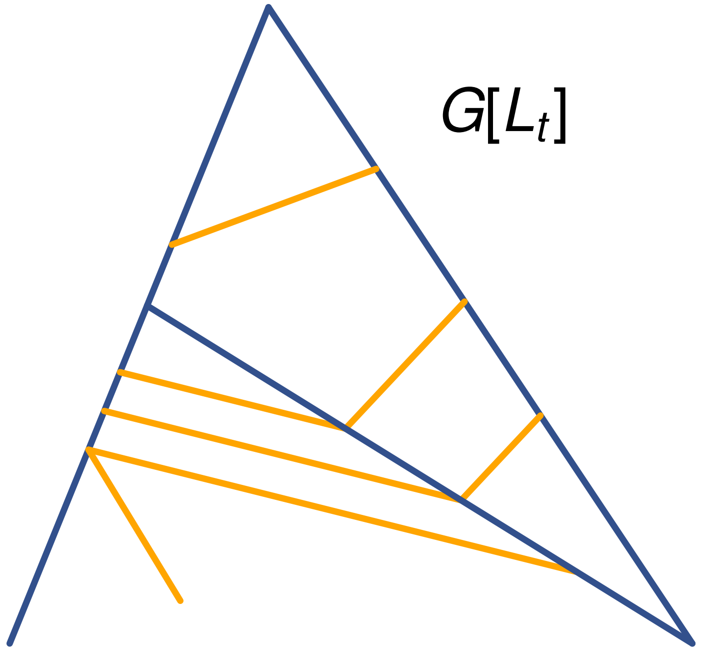
$8 \cdot \text{width}(L_1) + 8 \cdot \text{width}(L_2) + 8 \cdot \text{width}(L_3) + \cdots$

$= 8 + 8 \cdot 2 + 8 \cdot 3 + \cdots = O(n^{2/3} \log(n)^{2/3})$ pages

Embedded: $G[L_t]$, where t = # of long paths

+ all subdivided edges

Already embedded: small width



Recall:
Any topological ordering yields
$O(n^{2/3} \log(n)^{2/3})$ pages

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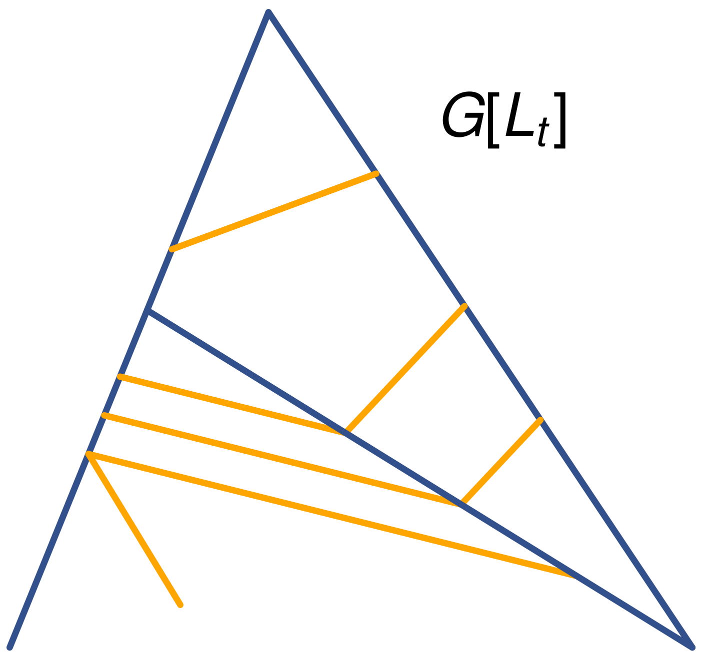
Already embedded: small width



Left to embed: small height

Recall:
Any topological ordering yields $O(n^{2/3} \log(n)^{2/3})$ pages

Observe:
Each edge has at least one endpoint in $S = V - L_t$

Lemma (without proof)

- Given: Upward planar graph G, $S \subseteq V(G)$
- $O(\text{height}(S) \cdot \log(n))$ pages suffice for edges with at least one endpoint in S

Left to embed: small height

Observe:
Each edge has at least one endpoint in $S = V - L_t$

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Lemma (without proof)

- Given: Upward planar graph G, $S \subseteq V(G)$
- $O(\text{height}(S) \cdot \log(n))$ pages suffice for edges with at least one endpoint in S

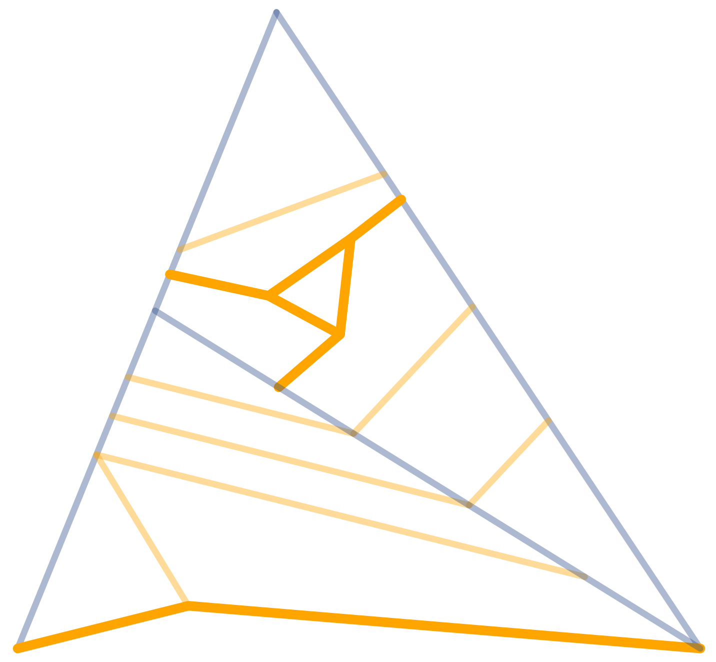
$\Downarrow$

Page number:

$O(\text{height}(V - L_t) \cdot \log(n))$
$= O(\ell \cdot \log(n))$
$= O(n^{2/3} \log(n)^{2/3})$

$(\ell = n^{2/3} / \log(n)^{1/3})$

Left to embed: small height



Observe:
Each edge has at least one endpoint in $S = V - L_t$

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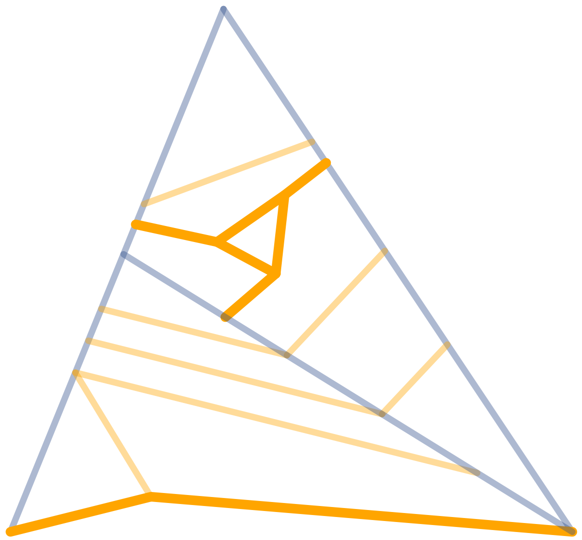
Small width

For any topological ordering $O(n^{2/3} \log(n)^{2/3})$ pages suffice

Small height



Lemma constructs ordering s.t. $O(n^{2/3} \log(n)^{2/3})$ pages suffice for remaining edges

Theorem

The page number of n-vertex upward planar graphs is $O(n^{2/3} \log(n)^{2/3})$.

Open Problems

Open Problem

Is the page number of upward planar graphs bounded?

What is the maximum page number of

- upward planar graphs,
- planar posets,
- acyclic outerplanar graphs / acyclic 2-trees?

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