Recognizing Unit Disk Graphs in Hyperbolic Geometry is $\exists\mathbb{R}$-Complete

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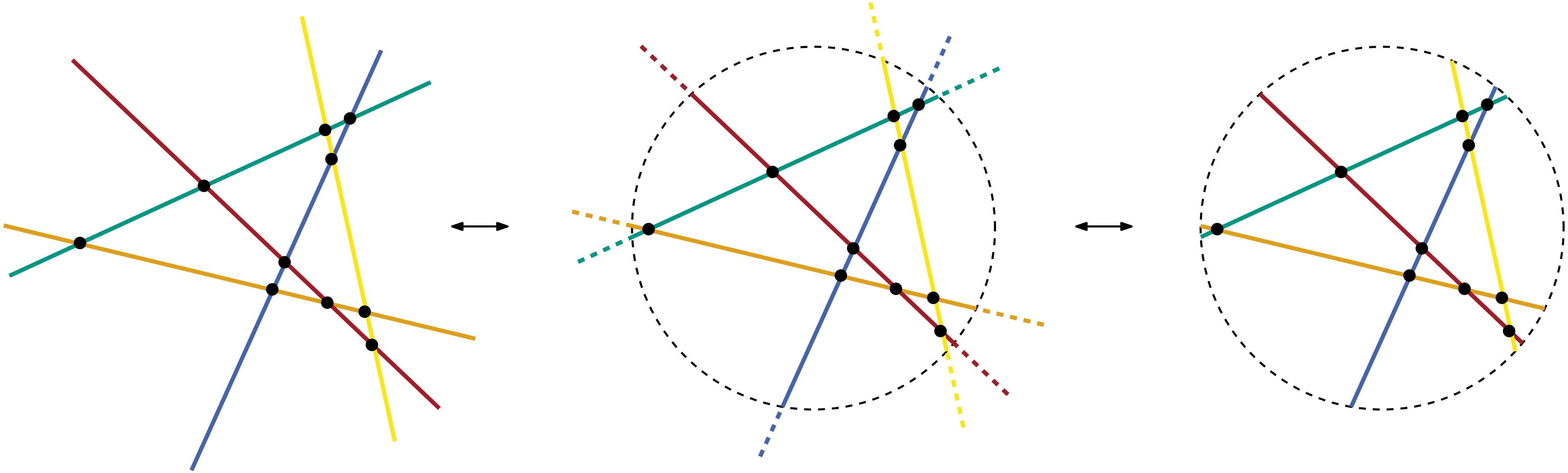
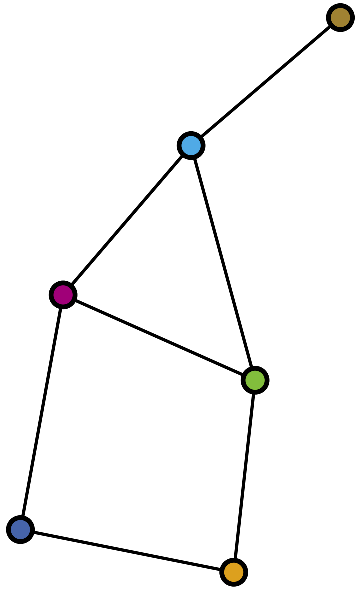
Nicholas Bieker, Thomas Bläsius, Emil Dohse, **Paul Jungeblut**



Unit Disk Graphs



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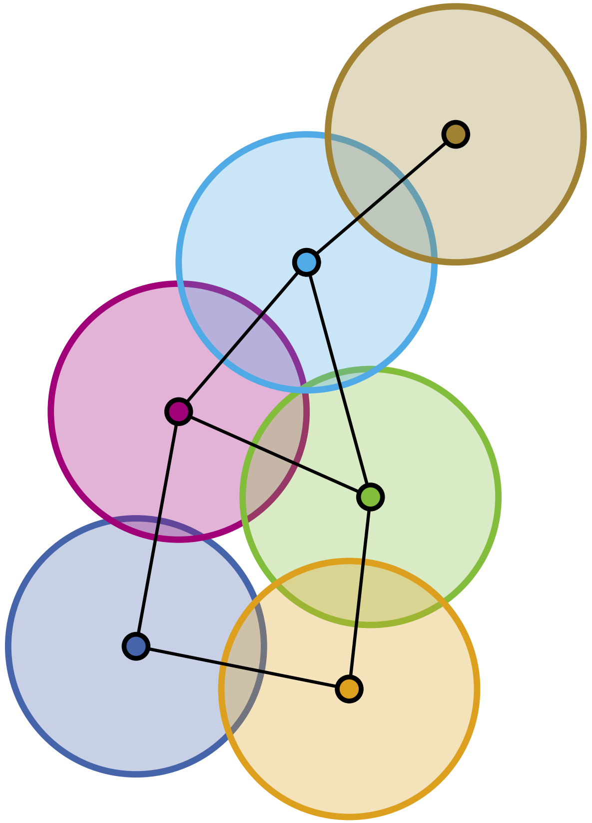
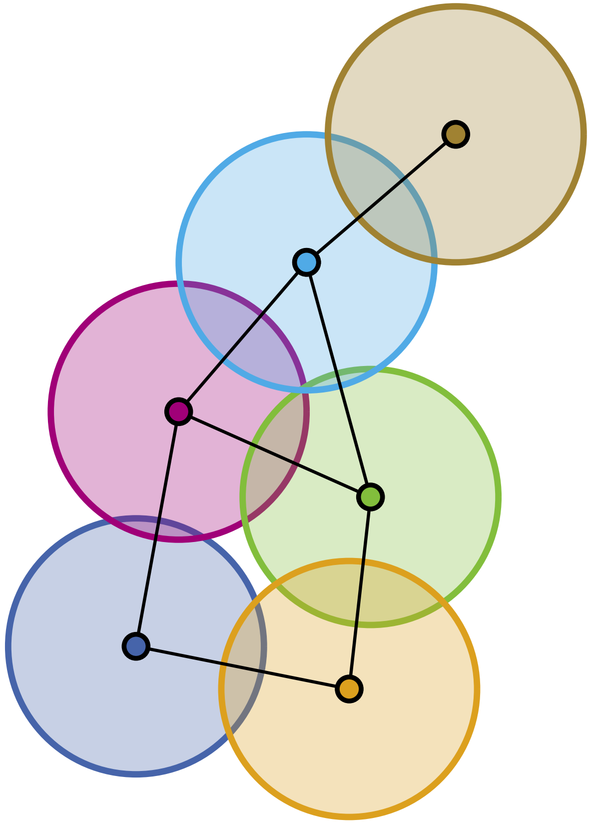


Definition:
G is a (Euclidean) unit disk graph if it is the intersection graph of unit disks in $\mathbb{R}^2$.

UDG: Class of unit disk graphs.

Unit Disk Graphs



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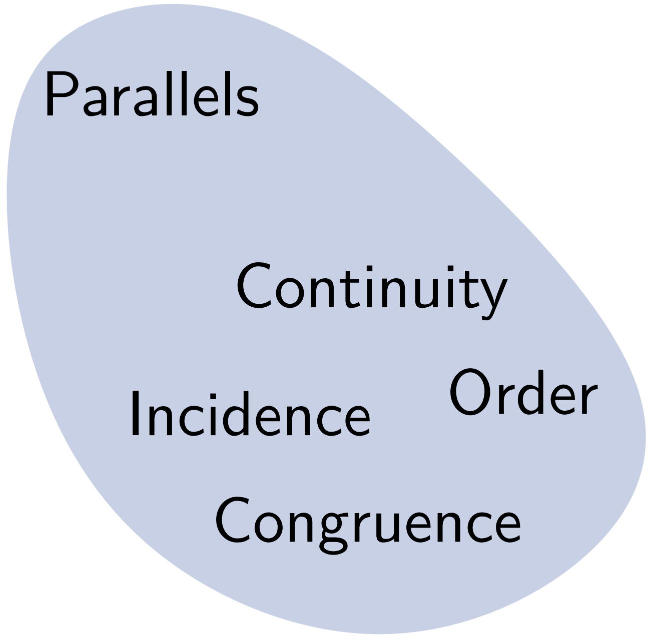
Why *unit* disk?

- all disks must have the same radius r
  $\rightsquigarrow$ *equally sized* disk graphs
- scaling the plane $\mathbb{R}^2$ allows to assume $r = 1$

Euclidean vs. Hyperbolic Geometry

Formalized by axiomatic systems:
(Euclid, Hilbert, . . .)

Euclidean plane $\mathbb{R}^2$



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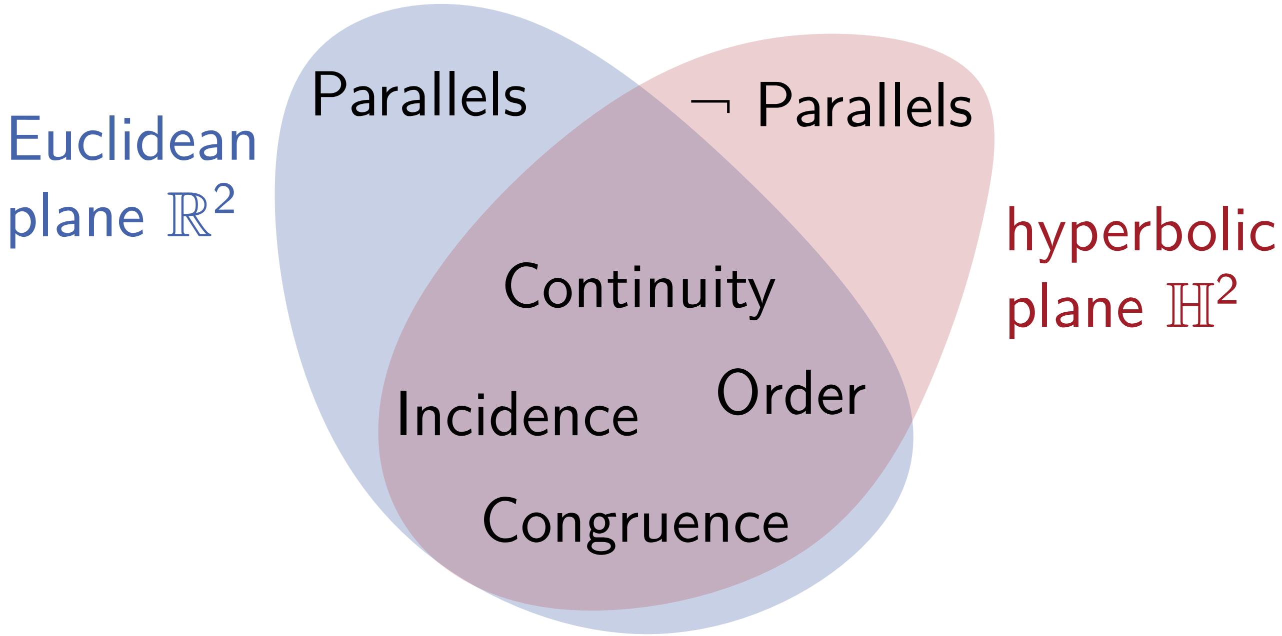

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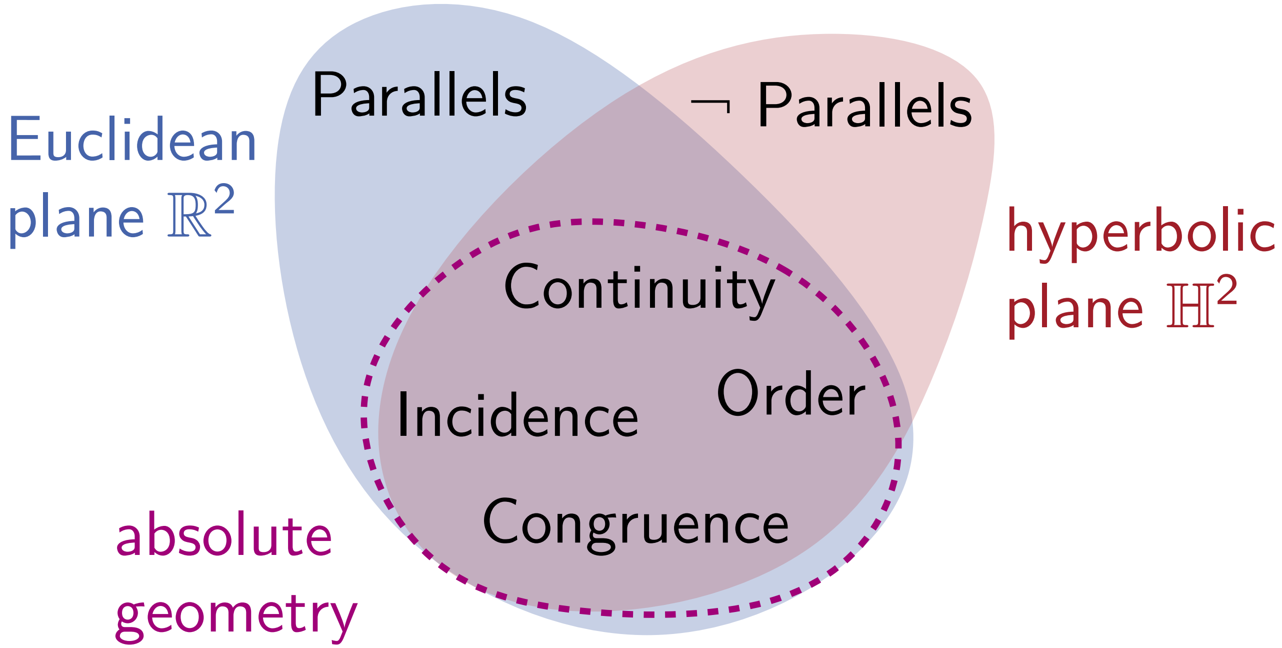

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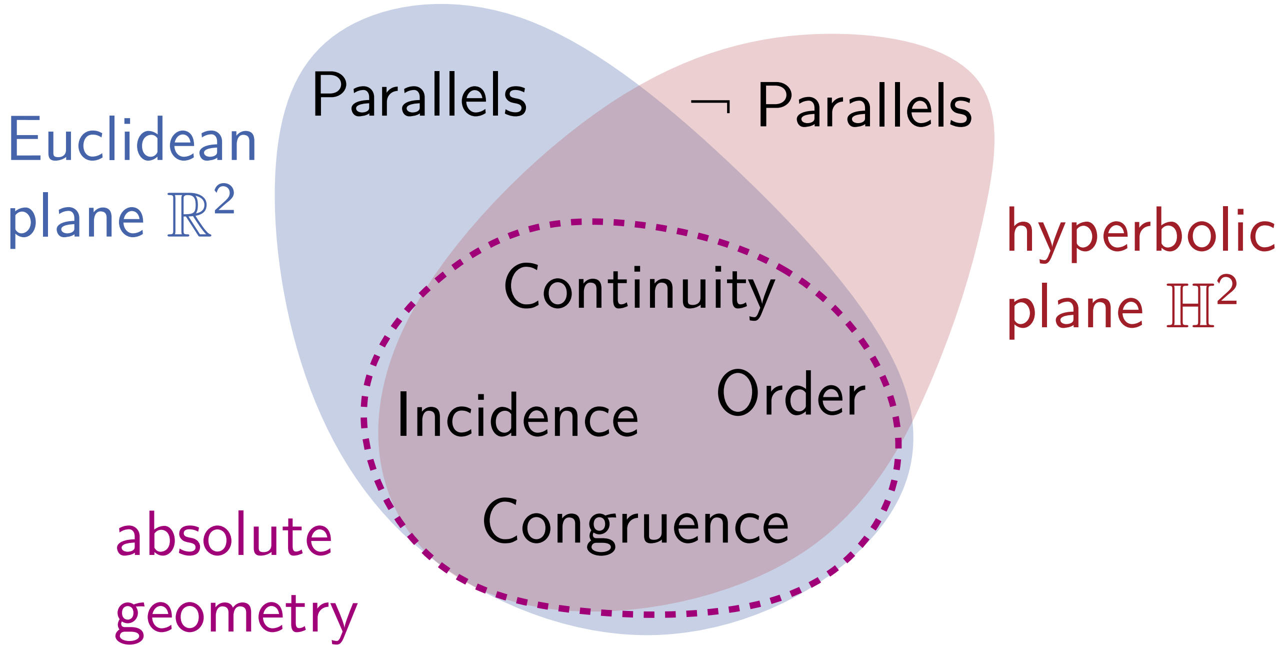

Models for $\mathbb{H}^2$:

- embedd $\mathbb{H}^2$ into $\mathbb{R}^d$
  $\rightsquigarrow$ allows to use human intuition for $\mathbb{R}^2$ in $\mathbb{H}^2$
- many different options:
  - Beltrami-Klein model
  - Poincaré model
  - Hyperboloid model

Hyperbolic Unit Disk Graphs

Definition:
A graph is a hyperbolic unit disk graph if it is the intersection graph of *equally sized* disks in $\mathbb{H}^2$.

HUDG: Class of hyperbolic unit disk graphs.

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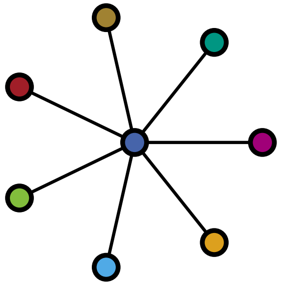
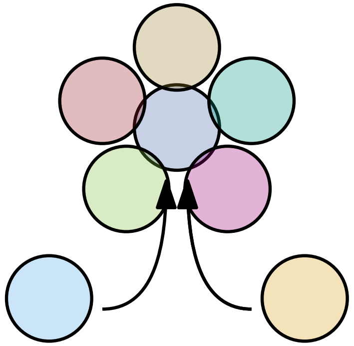
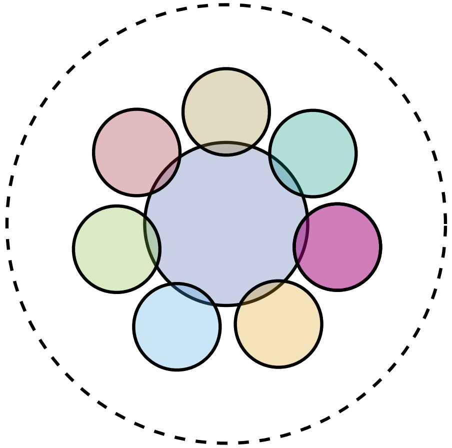


G: star with seven leaves



G $\notin$ UDG



G $\in$ HUDG

Poincaré disk model:

- $\mathbb{H}^2 \cong$ interior of a disk
- circles $\rightsquigarrow$ circles
- closer to the boundary: more distorted/compressed $\rightsquigarrow$ all circles have equal area

Our Results

Theorem:
Recognizing hyperbolic unit disk graphs is $\exists\mathbb{R}$-complete.

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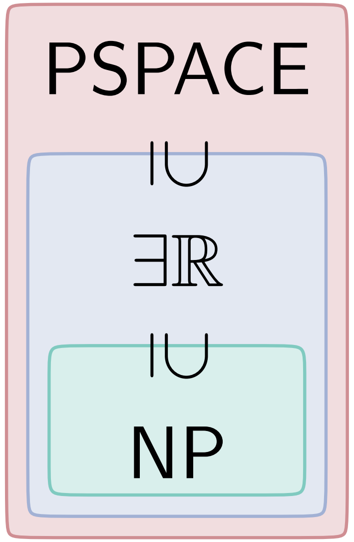
Complexity class $\exists\mathbb{R}$:
All problems reducible to the
existential theory of the reals (ETR).

Decide truth of formulas like:

$$\exists X_1, \ldots, X_n \in \mathbb{R}^n \; : \; X_1X_2 + 3X_3 = 10 \wedge X_2X_4 \leqslant 1$$

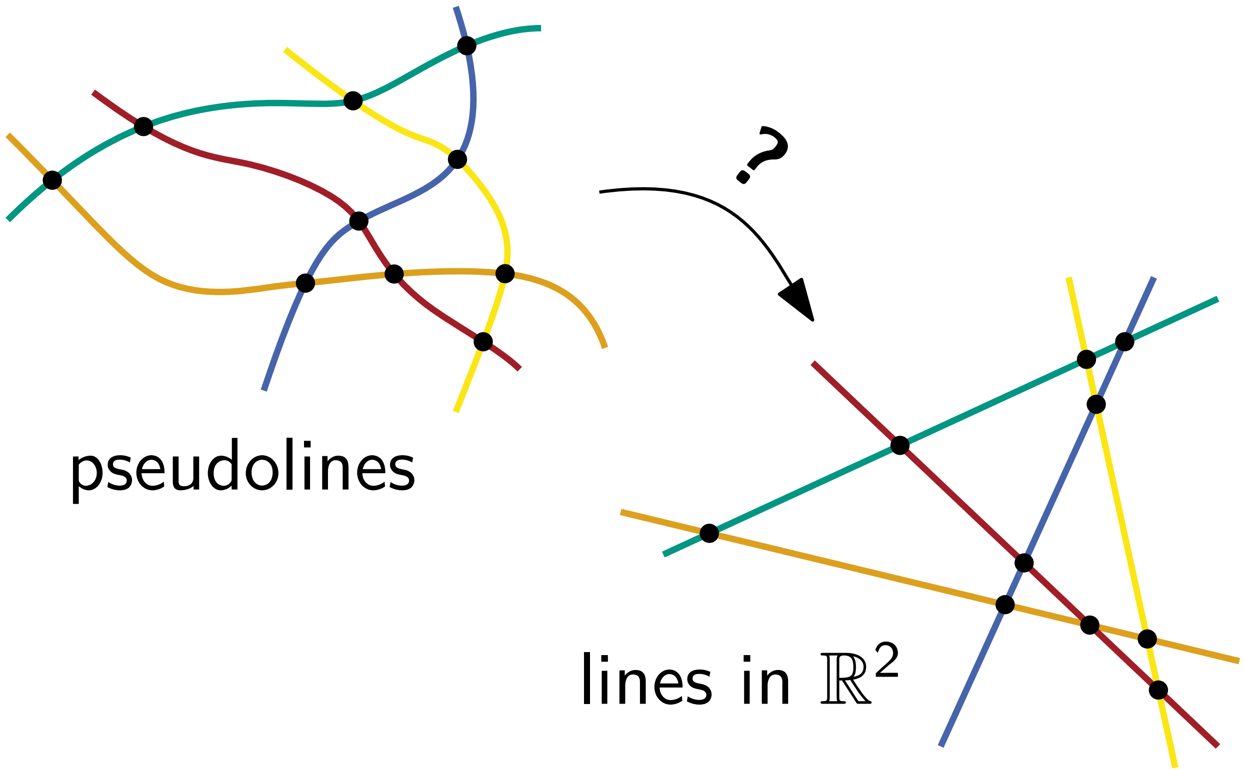
(polynomial systems of equations and inequalities)



∃ℝ-Hardness

Simple Stretchability

- every two lines intersect
- no more than two lines intersect in any point



pseudolines

lines in $\mathbb{R}^2$

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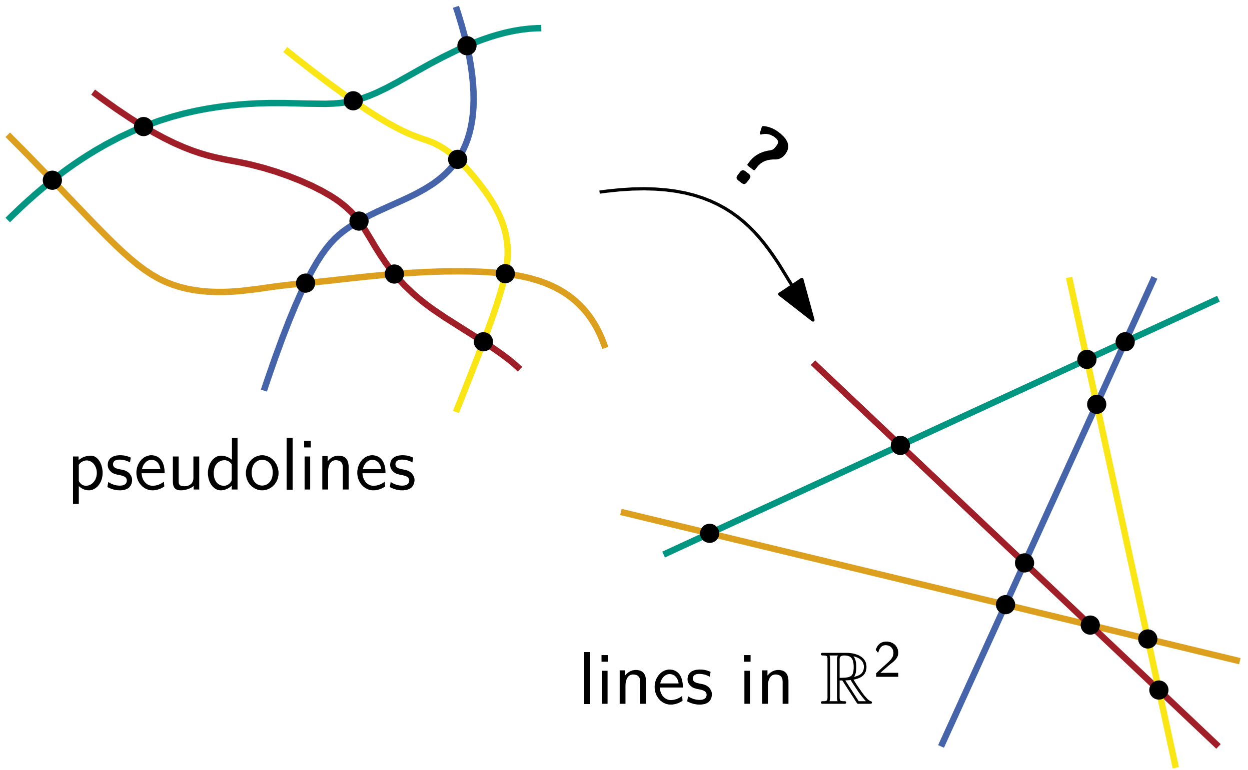

Theorem: (Mnëv 1988)
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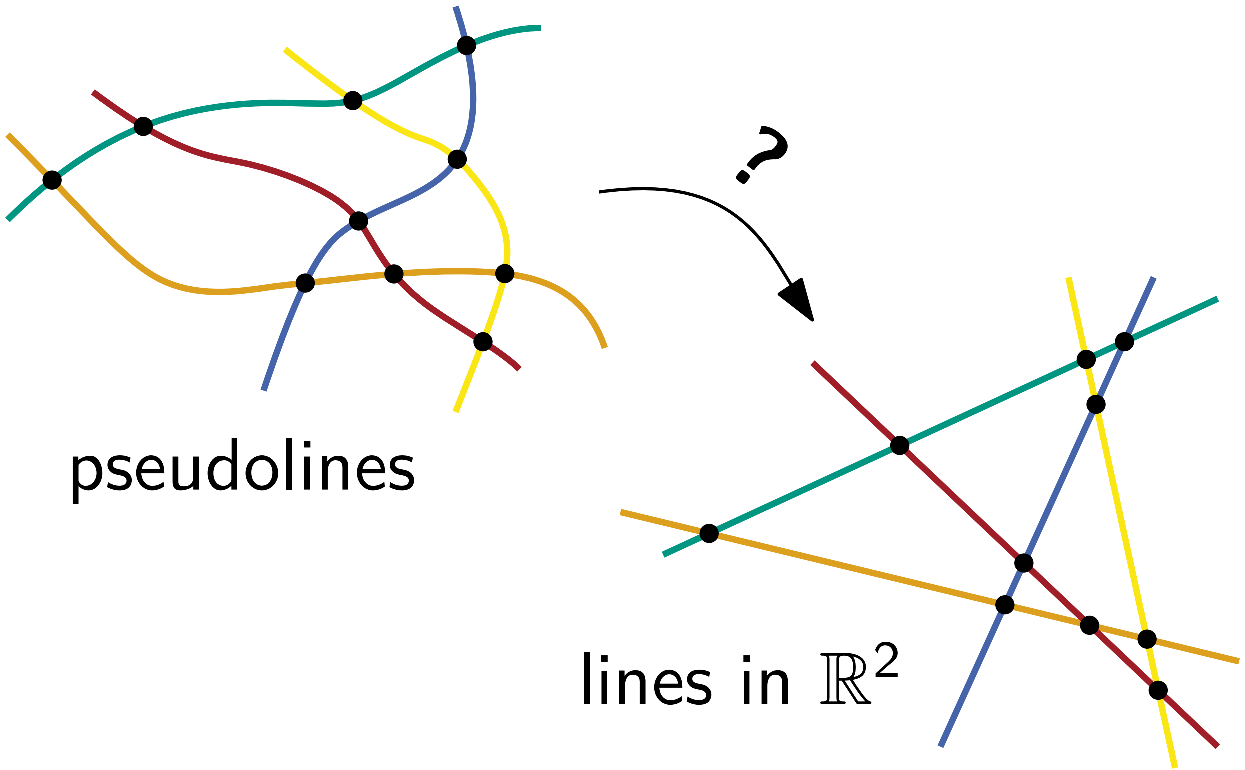

Theorem: (Mnëv 1988)
Simple Stretchability is ∃ℝ-complete.

Theorem: (McDiarmid, Müller 2010)
Simple Stretchability can be reduced to recognizing UDGs.

$\exists\mathbb{R}$-Hardness

D - Instance of Simple Stretchability

$\mathrm{G_D}$ - Graph constructed from D following McDiarmid and Müller

[McDiarmid, Müller 2010]

D stretchable in $\mathbb{R}^2$ $\iff$ $\mathrm{G_D} \in$ UDG

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$\Downarrow$ [BFKS 2023]

$\mathrm{G_D} \in$ HUDG

Theorem: (Bläsius, Friedrich, Katzmann, Stephan 2023)
It holds that UDG $\subseteq$ HUDG.

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same argument:
uses only *absolute geometry*

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$\Updownarrow$ **next slide** $\Downarrow$ [BFKS 2023]

D stretchable in $\mathbb{H}^2$ $\Longleftarrow$ $G_D \in$ HUDG

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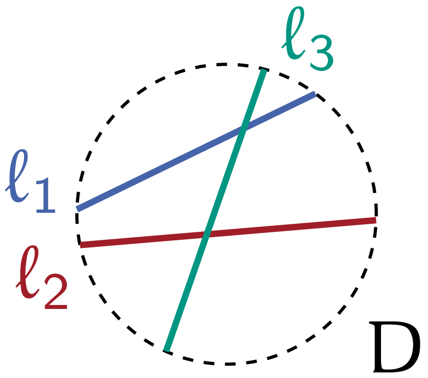
Simple Stretchability

Beltrami-Klein model:

- $\mathbb{H}^2 \cong$ interior of disk D
- hyperbolic lines $\cong$ chords of D



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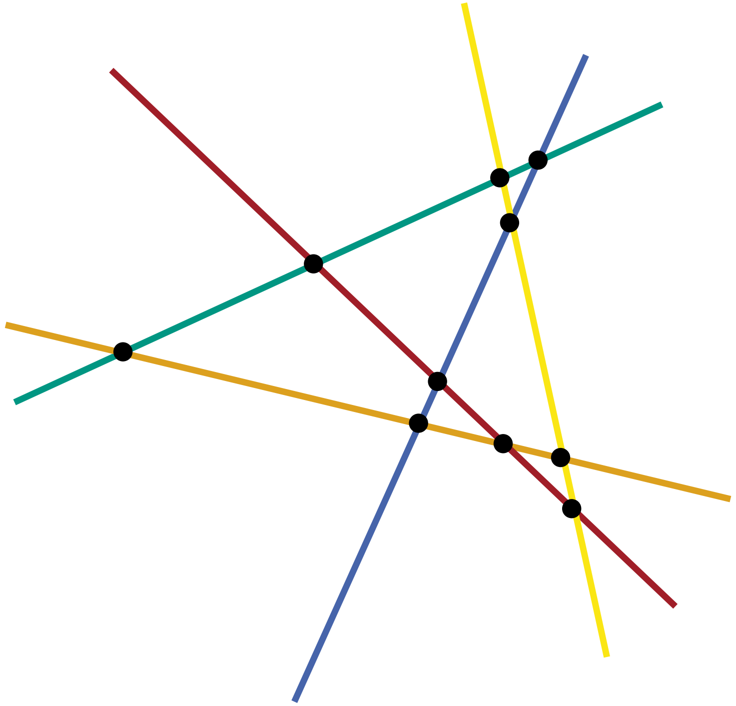

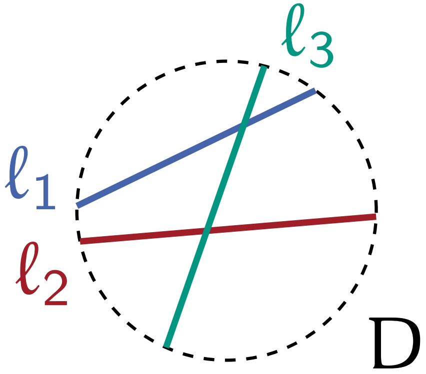

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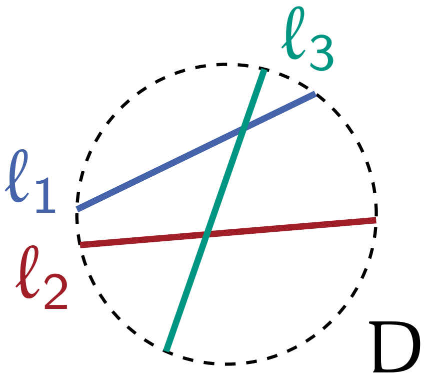

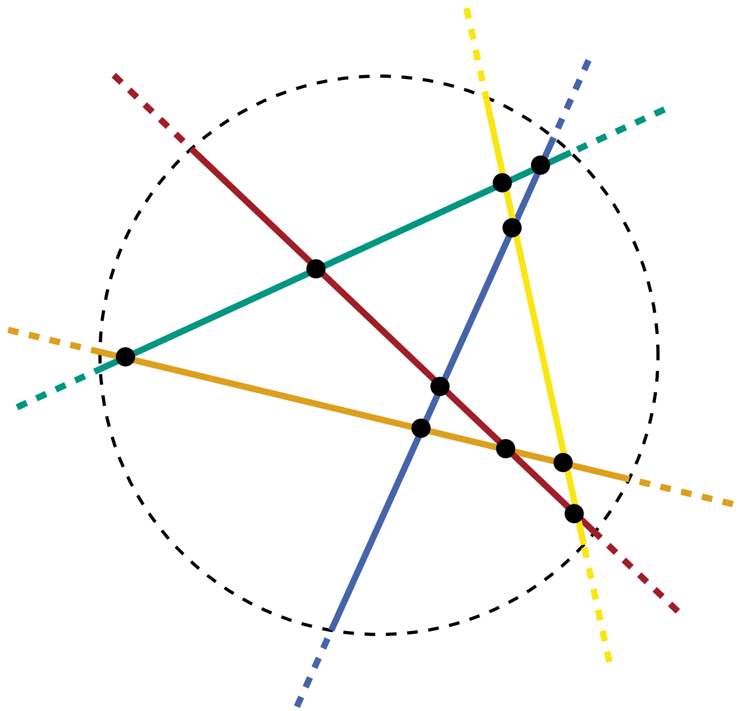

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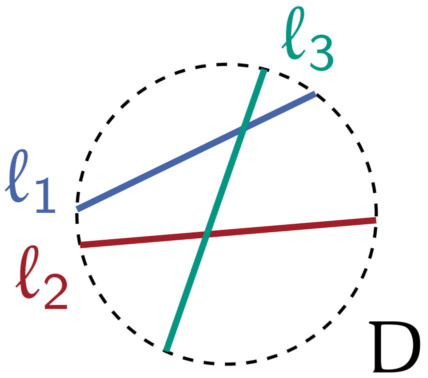

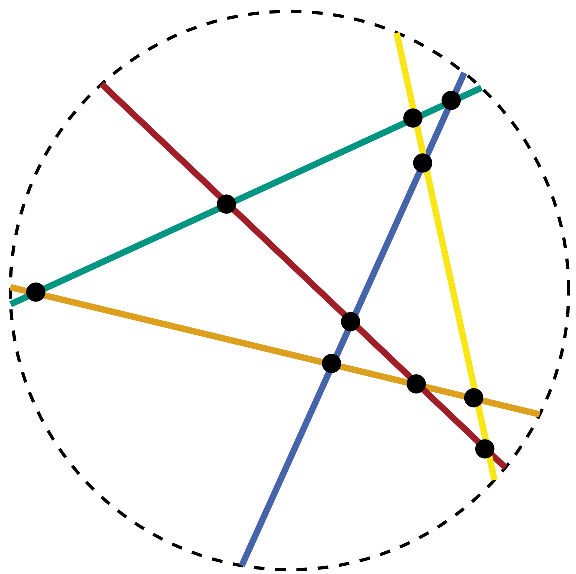

Theorem:
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$\exists\mathbb{R}$-Membership

Idea: Given coordinates, verify that all neighbors are closer to each other than all non-neighbors. (in polynomial time on a real RAM machine)

Problem: Involves computing distances in $\mathbb{H}^2$:
requires hyperbolic functions $\rightsquigarrow$ not computable on a real RAM

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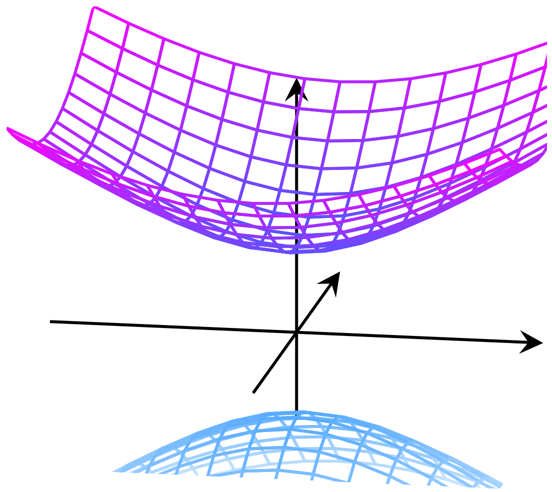
Hyperboloid model:

- $\mathbb{H}^2 \cong$ points in $\mathbb{R}^3$ with $z^2 - x^2 - y^2 = 1$ and $z > 0$
- $d\big((x_1, y_1, z_1), (x_2, y_2, z_2)\big) = \operatorname{arcosh}(z_1 z_2 - x_1 x_2 - y_1 y_2)$



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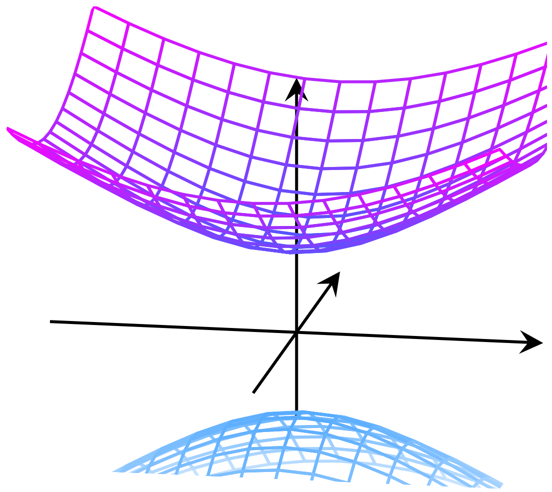
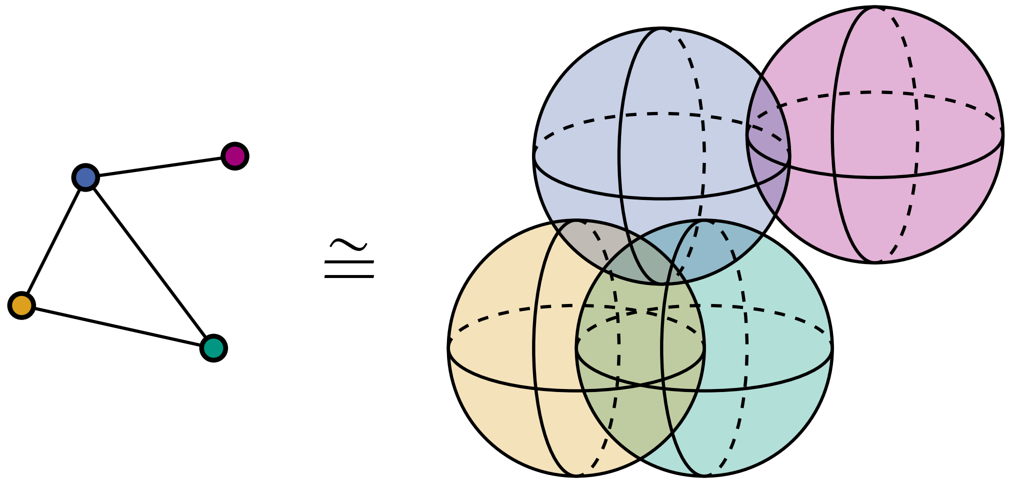
monotone function — polynomial



Open Problems

Problem 1:

Generalize to higher dimensions:



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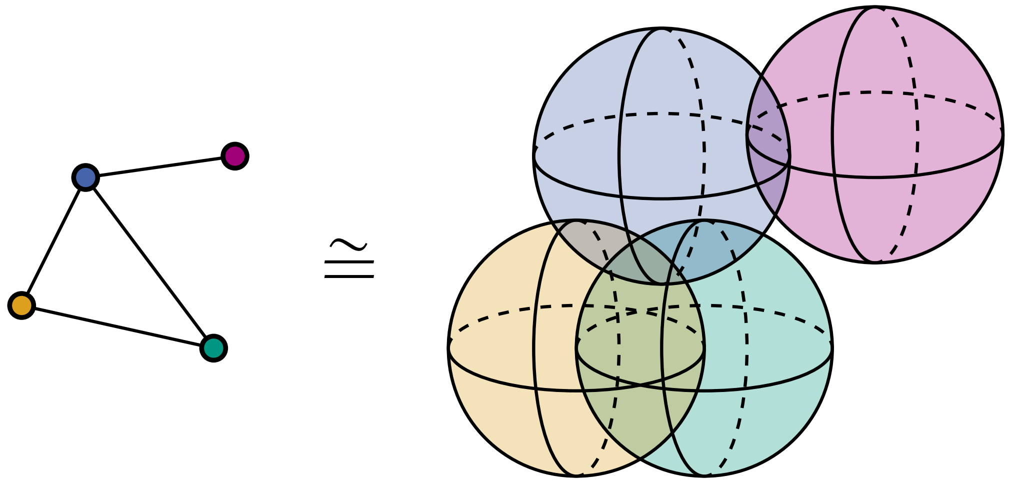

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Problem 2:

Use reduction as a framework for more problems:

(Unit) Segment Graphs

Linkage Realization

RAC-Drawings

⋮