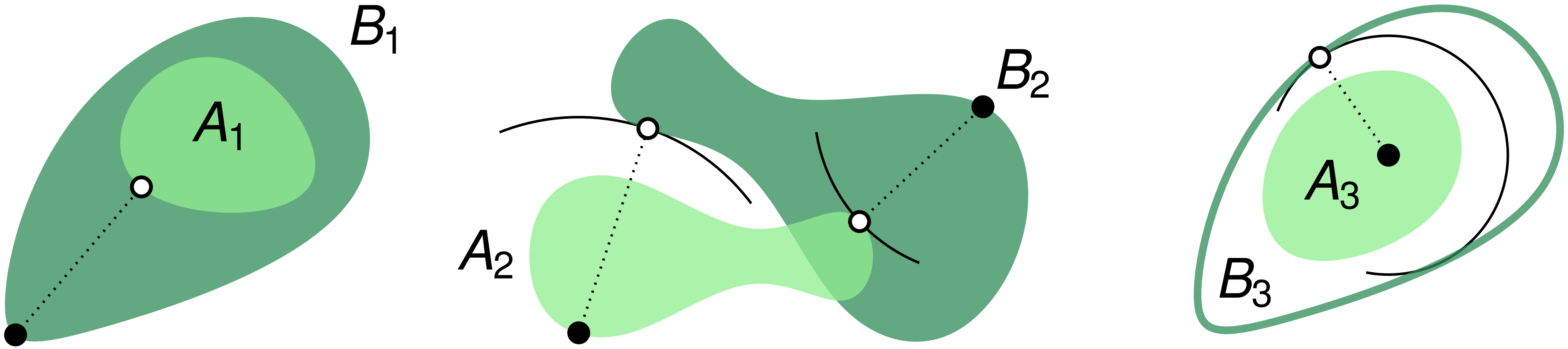
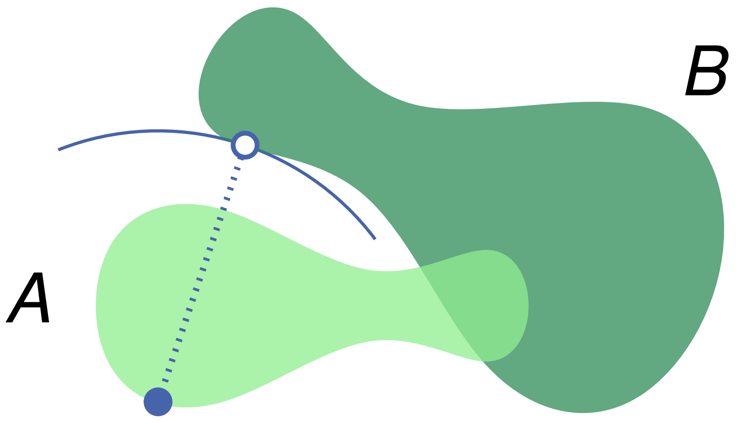
The Complexity of the Hausdorff Distance

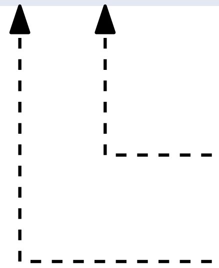
Paul Jungeblut, Linda Kleist, Till Miltzow



Hausdorff Distance: How similar are two sets?



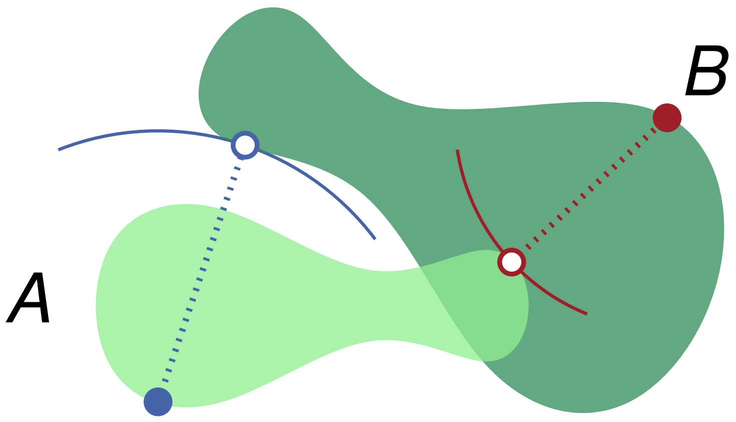
$$\vec{d}_{\mathrm{H}}(A, B) := \sup_{a \in A} \inf_{b \in B} \|a - b\|$$



Furthest point $a \in A$ from B.

How much to expand B until it contains A?

Hausdorff Distance: How similar are two sets?



$$\vec{d}_H(A, B) := \sup_{a \in A} \inf_{b \in B} ||a - b||$$

- Furthest point $a \in A$ from B.
- How much to expand B until it contains A?

Hausdorff Distance:

$$d_H(A, B) := \max\{\vec{d}_H(A, B), \vec{d}_H(B, A)\}$$

- Symmetry

Problem & Result

HAUSDORFF

Given:

- sets $A, B \subseteq \mathbb{R}^n$ (semi-algebraic)
- rational number $t \in \mathbb{Q}$

Question:

- Is $d_H(A, B) \leq t$?

Problem & Result

HAUSDORFF

Given:

- sets $A, B \subseteq \mathbb{R}^n$ (semi-algebraic)
- rational number $t \in \mathbb{Q}$

Question:

- Is $d_H(A, B) \leq t$?

Theorem:
HAUSDORFF is $\forall\exists_<\mathbb{R}$-complete.

NP $\subseteq$ $\forall\exists_<\mathbb{R}$ $\subseteq$ PSPACE

Discussion

Complexity of A and B matters:

Polygons: $\in$ P

Semi-Algebraic: $\forall\exists_{<}\mathbb{R}$

[Alt et al. 1995]

Curved sets common in practice:

- Circles/Ellipses/...
- Cubic Splines

Discussion

Complexity of *A* and *B* matters:

Polygons: $\in$ P

[Alt et al. 1995]

Semi-Algebraic: $\forall\exists_{<}\mathbb{R}$

Curved sets common in practice:

- Circles/Ellipses/...
- Cubic Splines

Sophisticated algorithms:

- Gröbner Bases
  $\rightsquigarrow$ not always applicable
- Cylindrical Algebraic Decomposition
  $\rightsquigarrow$ impracticable

Discussion

Complexity of *A* and *B* matters:

Polygons: $\in$ P

[Alt et al. 1995]

Semi-Algebraic: $\forall\exists_{<}\mathbb{R}$

Curved sets common in practice:

- Circles/Ellipses/...
- Cubic Splines

Sophisticated algorithms:

- Gröbner Bases
  $\rightsquigarrow$ not always applicable
- Cylindrical Algebraic Decomposition
  $\rightsquigarrow$ impracticable

$\forall\exists_{<}\mathbb{R}$-Completeness:
Unlikely that better or custom algorithms for HAUSDORFF exist.

Semi-Algebraic Sets

$A, B \subseteq \mathbb{R}^n$ are described by logical formulas:

- polynomial equations and inequalities
- n free variabales

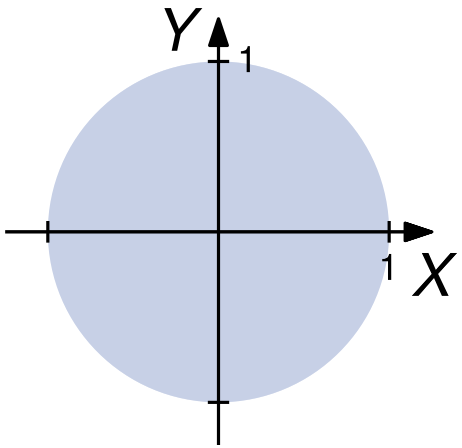
Semi-Algebraic Sets

$A, B \subseteq \mathbb{R}^n$ are described by logical formulas:

- polynomial equations and inequalities
- n free variabales

Examples:

$\varphi_1(X, Y) :\equiv X^2 + Y^2 \leq 1$



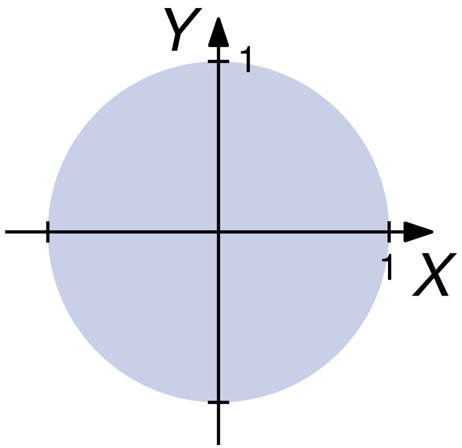
Semi-Algebraic Sets

$A, B \subseteq \mathbb{R}^n$ are described by logical formulas:

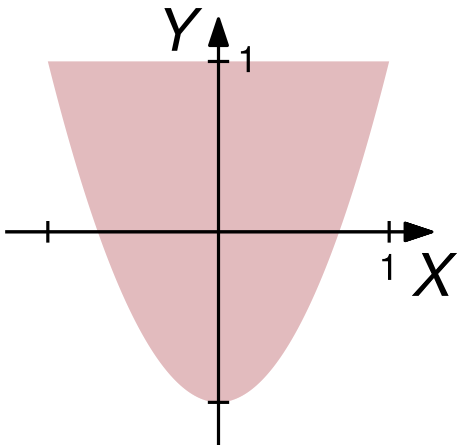
- polynomial equations and inequalities
- n free variabales

Examples:

$\varphi_1(X, Y) :\equiv X^2 + Y^2 \leq 1$



$\varphi_2(X, Y) :\equiv Y \geq 2X^2 - 1$



Semi-Algebraic Sets

$A, B \subseteq \mathbb{R}^n$ are described by logical formulas:

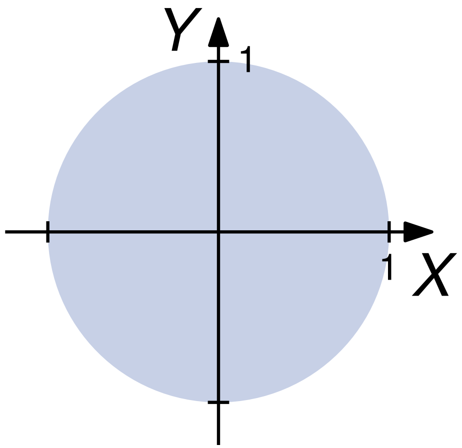
- polynomial equations and inequalities
- n free variabales

$\varphi(X, Y) :\equiv$

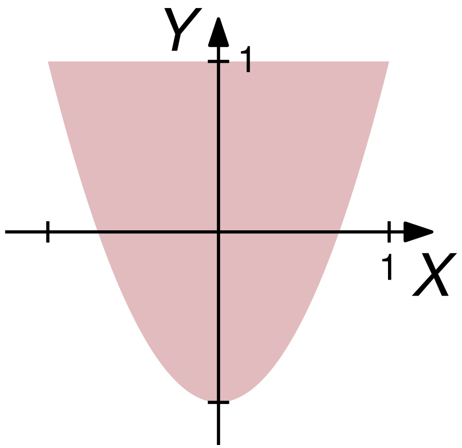
$\varphi_1(X, Y) \land \varphi_2(X, Y)$

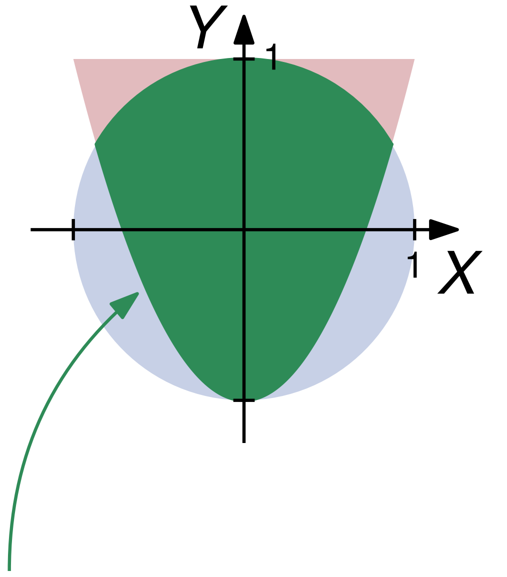
Examples:

$\varphi_1(X, Y) :\equiv X^2 + Y^2 \leq 1$



$\varphi_2(X, Y) :\equiv Y \geq 2X^2 - 1$

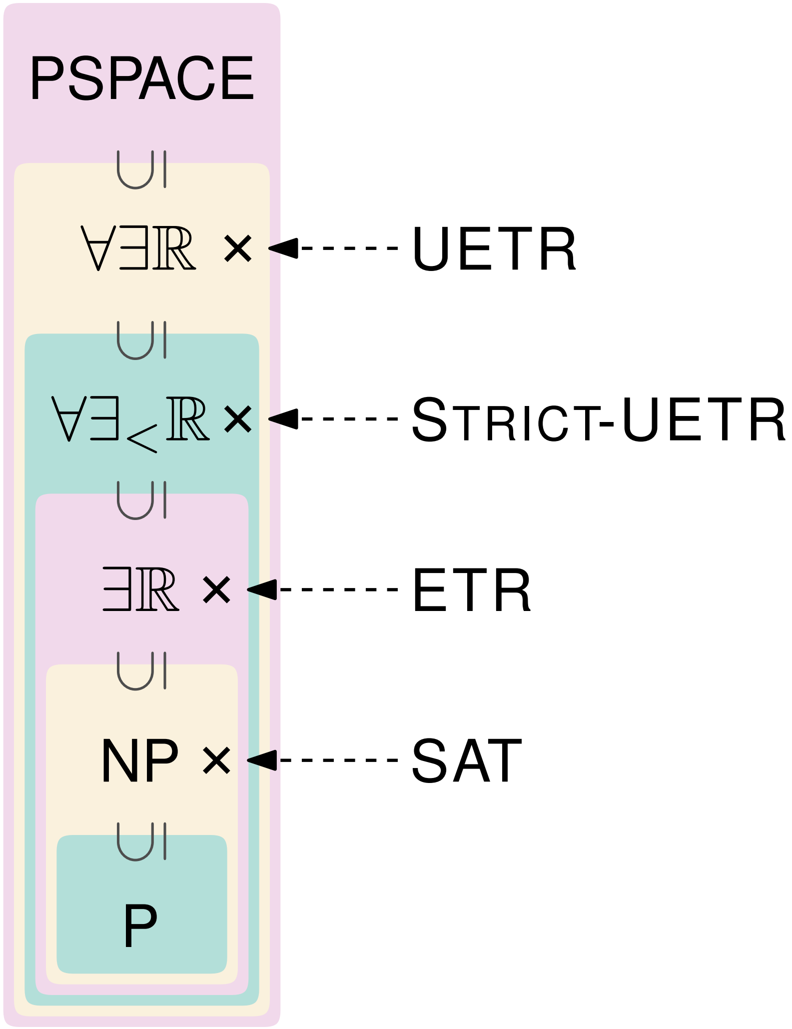




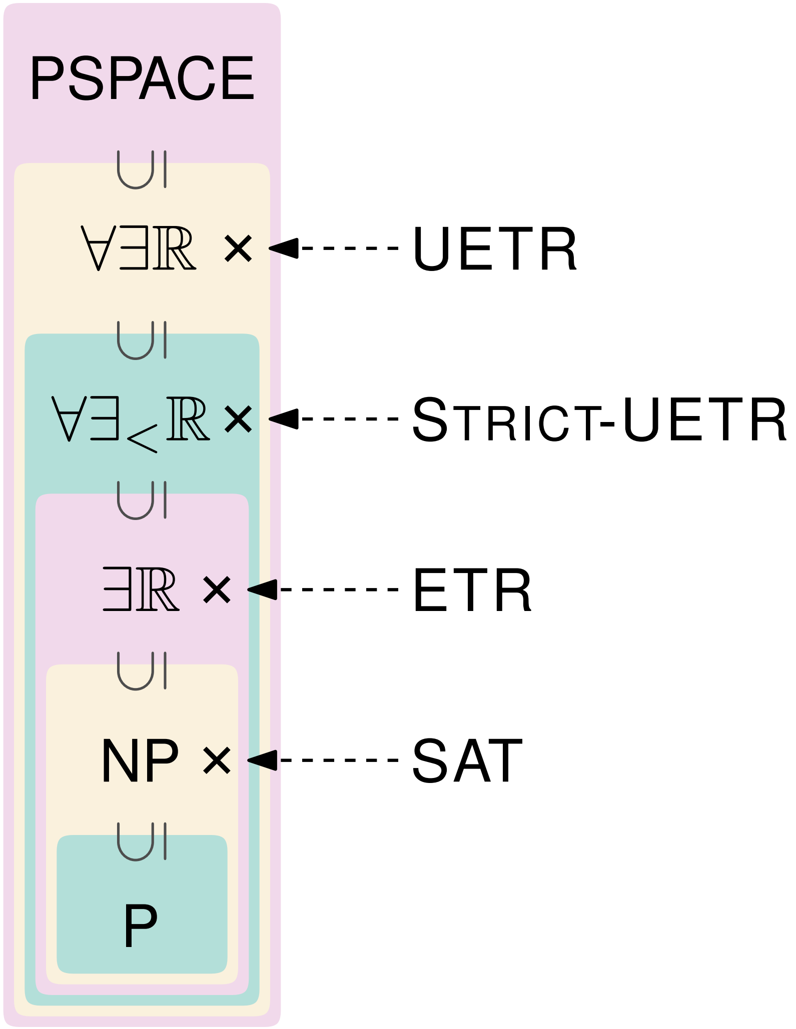
$\{(x, y) \in \mathbb{R}^2 \mid \varphi(x, y)\}$

Complexity Classes $\forall\exists\mathbb{R}$ and $\forall\exists_{<}\mathbb{R}$

Complexity Classes $\forall\exists\mathbb{R}$ and $\forall\exists_<\mathbb{R}$



Complexity Classes $\forall\exists\mathbb{R}$ and $\forall\exists_{<}\mathbb{R}$

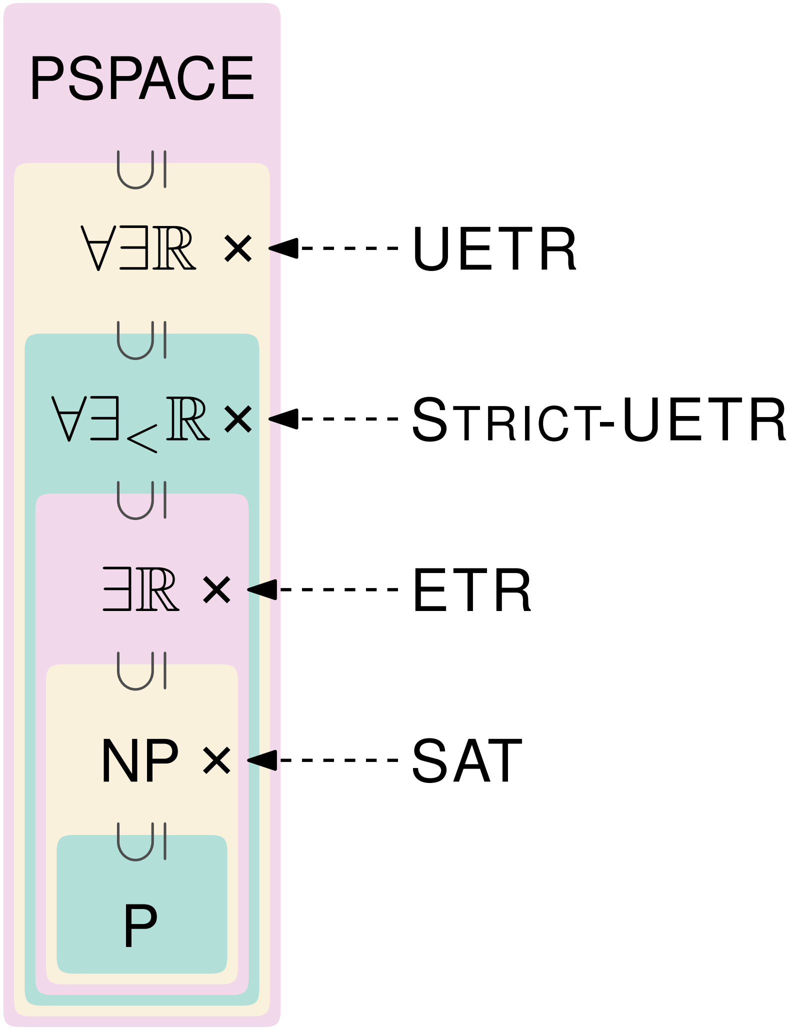


UETR (Universal Existential Theory of the Reals):
Given the following sentence, is it true?

$$\forall X \in \mathbb{R}^n \,.\, \exists Y \in \mathbb{R}^m \;:\; \varphi(X, Y)$$

STRICT-UETR: only strict inequalities ($<, \neq$) in φ

Complexity Classes $\forall\exists\mathbb{R}$ and $\forall\exists_{<}\mathbb{R}$



UETR (Universal Existential Theory of the Reals):
Given the following sentence, is it true?

$$\forall X \in \mathbb{R}^n \,.\, \exists Y \in \mathbb{R}^m \;:\; \varphi(X, Y)$$

Strict-UETR: only strict inequalities ($<$, $\neq$) in φ

Examples:

- $\forall X \in \mathbb{R} \,.\, \exists Y \in \mathbb{R} : XY < 0$
  (false: for $X = 0$ there is no Y)
- $\forall X \in \mathbb{R} \,.\, \exists Y \in \mathbb{R} : XY < 0 \vee X = 0$
  (true, but not strict)

$\forall\exists_{<}\mathbb{R}$-Hardness Reduction (Sketch)

Given: $\Phi :\equiv \forall X \in \mathbb{R}^n \,.\, \exists Y \in \mathbb{R}^m : \varphi(X, Y)$

Reduction: $A := \{x \in \mathbb{R}^n \mid \exists Y \in \mathbb{R}^m : \varphi(x, Y)\}$ ← all X for which there is a Y

$B := \mathbb{R}^n$

$\forall\exists_{<}\mathbb{R}$-Hardness Reduction (Sketch)

Given: $\Phi \;:\equiv\; \forall X \in \mathbb{R}^n \,.\, \exists Y \in \mathbb{R}^m : \varphi(X, Y)$

Reduction: $A \;:=\; \{x \in \mathbb{R}^n \mid \exists Y \in \mathbb{R}^m : \varphi(x, Y)\}$ ← all X for which there is a Y

$B \;:=\; \mathbb{R}^n$

Idea: Φ true $\rightsquigarrow$ $A = \mathbb{R}^n$ $\rightsquigarrow$ $d_{\mathrm{H}}(A, B) = 0$

Φ false $\rightsquigarrow$ $A \subsetneq \mathbb{R}^n$ $\rightsquigarrow$ $d_{\mathrm{H}}(A, B) > 0$

$\forall\exists_{<}\mathbb{R}$-Hardness Reduction (Sketch)

Given: $\Phi \;:\equiv\; \forall X \in \mathbb{R}^n \,.\, \exists Y \in \mathbb{R}^m : \varphi(X, Y)$

Reduction: $A \;:=\; \{x \in \mathbb{R}^n \mid \exists Y \in \mathbb{R}^m : \varphi(x, Y)\}$ ← all X for which there is a Y

$B \;:=\; \mathbb{R}^n$

Idea: Φ true $\rightsquigarrow$ $A = \mathbb{R}^n$ $\rightsquigarrow$ $d_{\mathrm{H}}(A, B) = 0$

Φ false $\rightsquigarrow$ $A \subsetneq \mathbb{R}^n$ $\rightsquigarrow$ $d_{\mathrm{H}}(A, B) > 0$

$d_{\mathrm{H}}(A, B) = 0$
$\iff$
A and B have the same closure

Problem 1:

single *counterexample* $\not\Rightarrow$ $d_{\mathrm{H}}(A, B) > 0$

$\forall\exists_{<}\mathbb{R}$-Hardness Reduction (Sketch)

Given: $\Phi \;:\equiv\; \forall X \in \mathbb{R}^n \,.\, \exists Y \in \mathbb{R}^m : \varphi(X, Y)$

Reduction: $A \;:=\; \{x \in \mathbb{R}^n \mid \exists Y \in \mathbb{R}^m : \varphi(x, Y)\}$ ← all X for which there is a Y

$B \;:=\; \mathbb{R}^n$

Idea: Φ true $\rightsquigarrow$ $A = \mathbb{R}^n$ $\rightsquigarrow$ $d_{\mathrm{H}}(A, B) = 0$

Φ false $\rightsquigarrow$ $A \subsetneq \mathbb{R}^n$ $\rightsquigarrow$ $d_{\mathrm{H}}(A, B) > 0$

$d_{\mathrm{H}}(A, B) = 0$ $\iff$ A and B have the same closure

---

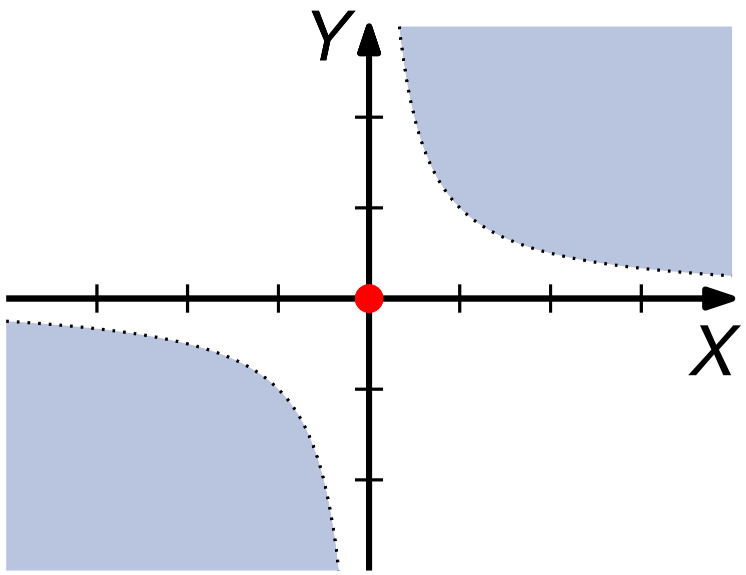
Problem 1:
single *counterexample* $\not\Rightarrow$ $d_{\mathrm{H}}(A, B) > 0$

Problem 2:
Definition of A contains an $\exists$.

Idea 1: Expanding Counterexamples

Idea: Make false sentences ”even falser”.

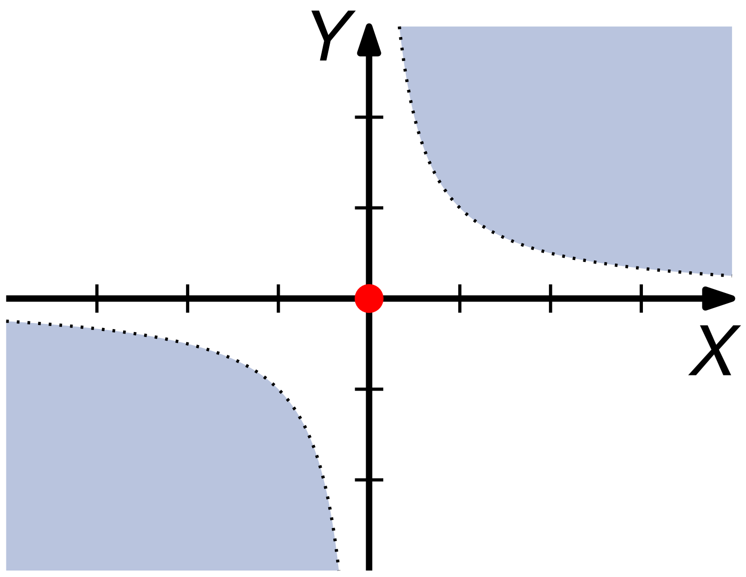
$$\forall X \in \mathbb{R} \,.\, \exists Y \in \mathbb{R} : XY > 1$$



Idea 1: Expanding Counterexamples

Idea: Make false sentences ”even falser”.

$$\forall X \in \mathbb{R} \,.\, \exists Y \in \mathbb{R} : XY > 1$$



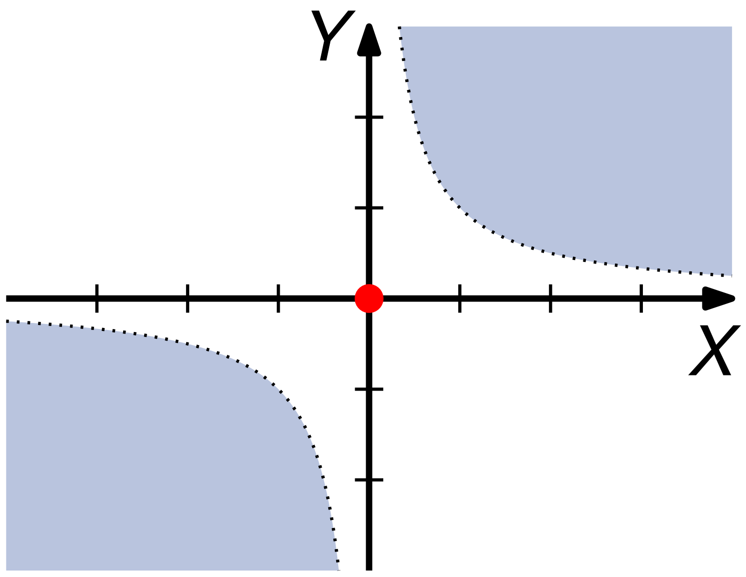
$A = \mathbb{R} \setminus \{0\} \qquad B = \mathbb{R}$

$\rightsquigarrow d_\mathrm{H}(A, B) = 0$

Idea 1: Expanding Counterexamples

Idea: Make false sentences ”even falser”.

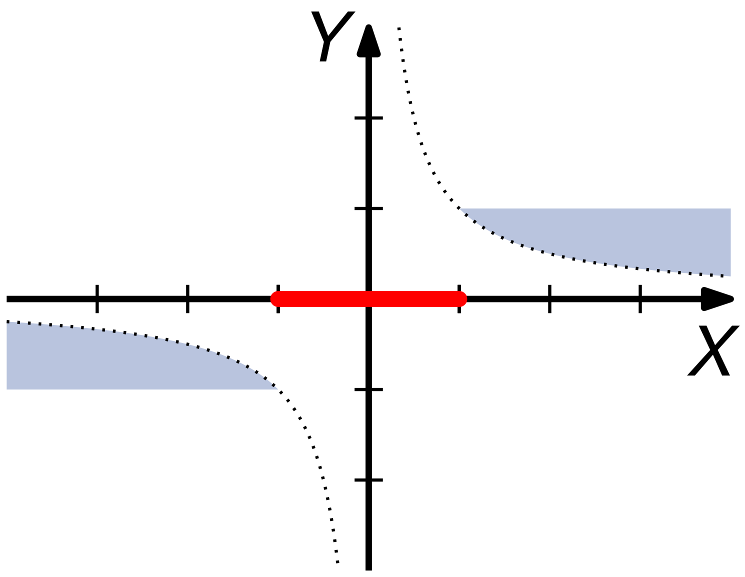
$$\forall X \in \mathbb{R} \,.\, \exists Y \in \mathbb{R} : XY > 1$$



$A = \mathbb{R} \setminus \{0\} \quad B = \mathbb{R}$

$\rightsquigarrow d_H(A, B) = 0$

$$\forall X \in \mathbb{R} \,.\, \exists Y \in [-1, 1] : XY > 1$$

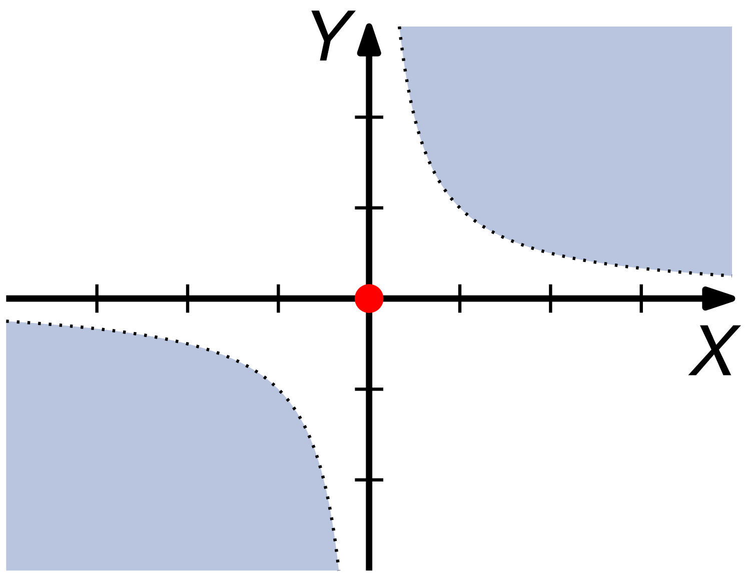


Paul Jungeblut, Linda Kleist, Till Miltzow

Idea 1: Expanding Counterexamples

Idea: Make false sentences "even falser".

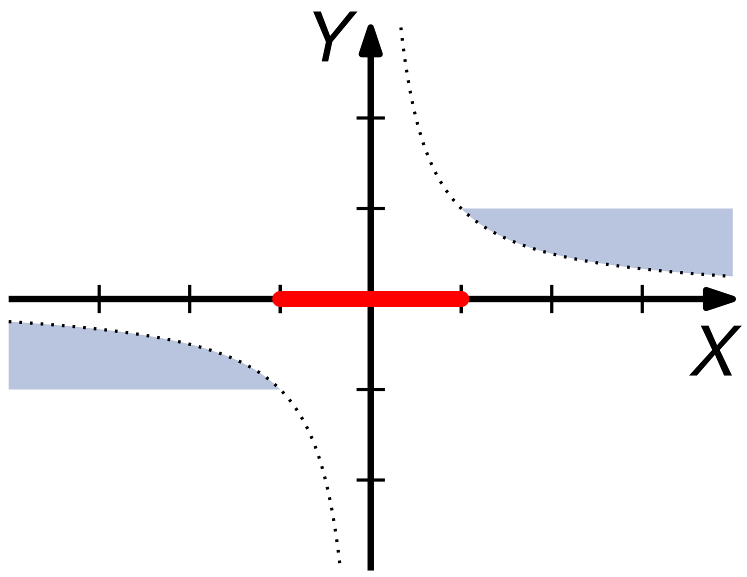
$\forall X \in \mathbb{R} \,.\, \exists Y \in \mathbb{R} : XY > 1$



$A = \mathbb{R} \setminus \{0\} \quad B = \mathbb{R}$

$\rightsquigarrow d_{\mathrm{H}}(A, B) = 0$

$\forall X \in \mathbb{R} \,.\, \exists Y \in [-1, 1] : XY > 1$



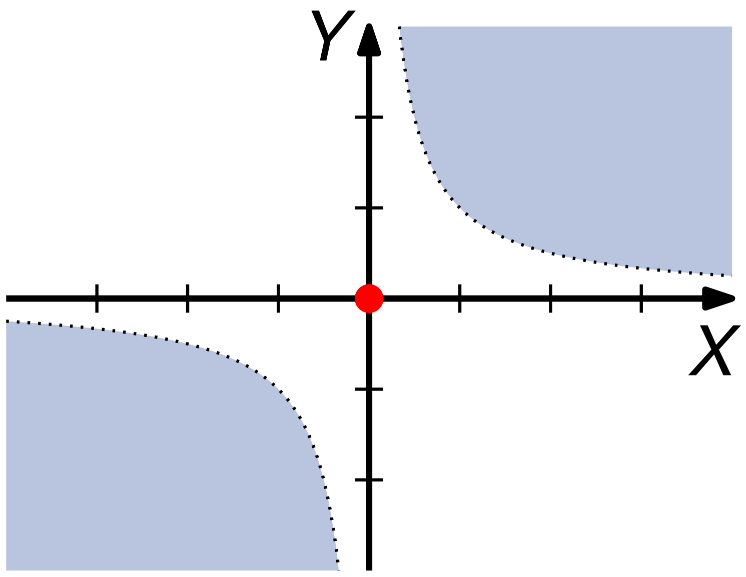
$A = \mathbb{R} \setminus [-1, 1] \quad B = \mathbb{R}$

$\rightsquigarrow d_{\mathrm{H}}(A, B) = 1$

Idea 1: Expanding Counterexamples

Idea: Make false sentences ”even falser”.

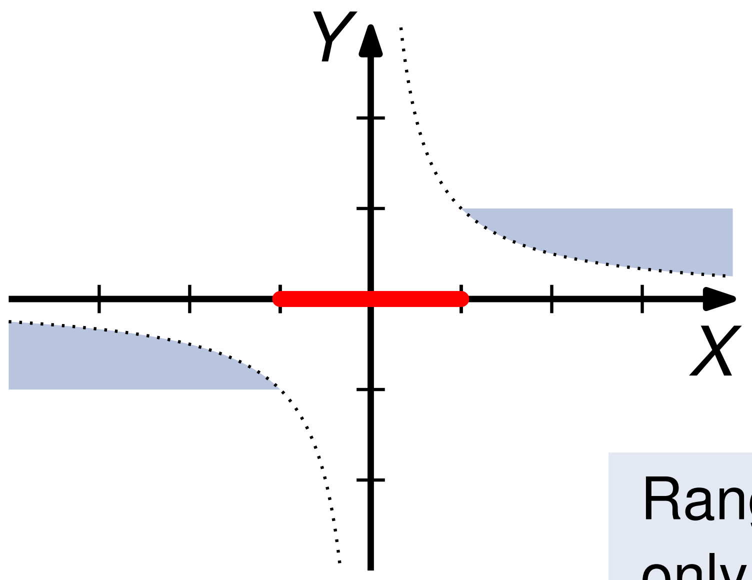
$\forall X \in \mathbb{R} \,.\, \exists Y \in \mathbb{R} : XY > 1$



$A = \mathbb{R} \setminus \{0\} \quad B = \mathbb{R}$

$\rightsquigarrow d_H(A, B) = 0$

$\forall X \in \mathbb{R} \,.\, \exists Y \in [-1, 1] : XY > 1$



$A = \mathbb{R} \setminus [-1, 1] \quad B = \mathbb{R}$

$\rightsquigarrow d_H(A, B) = 1$

Range restrictions only possible for STRICT-UETR.
$\rightsquigarrow \forall\exists_{<}\mathbb{R}$-hardness

Open Problems

Problem 1:

$$\forall\exists_{<}\mathbb{R} \stackrel{?}{=} \forall\exists\mathbb{R}$$

Are strict formulas really easier?

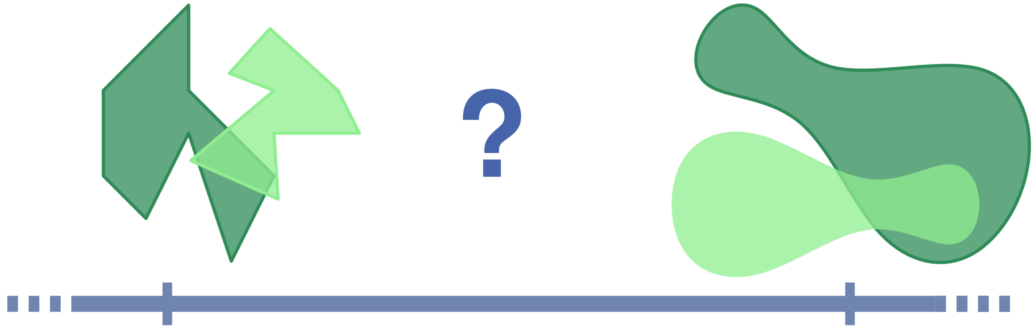
Open Problems

Problem 1:

$$\forall\exists_{<}\mathbb{R} \stackrel{?}{=} \forall\exists\mathbb{R}$$

Are strict formulas really easier?

Problem 2:



Polygons: $\in$ P

[Alt et al. 1995]

Semi-Algebraic: $\forall\exists_{<}\mathbb{R}$

Where is the tipping point in the complexity of HAUSDORFF?

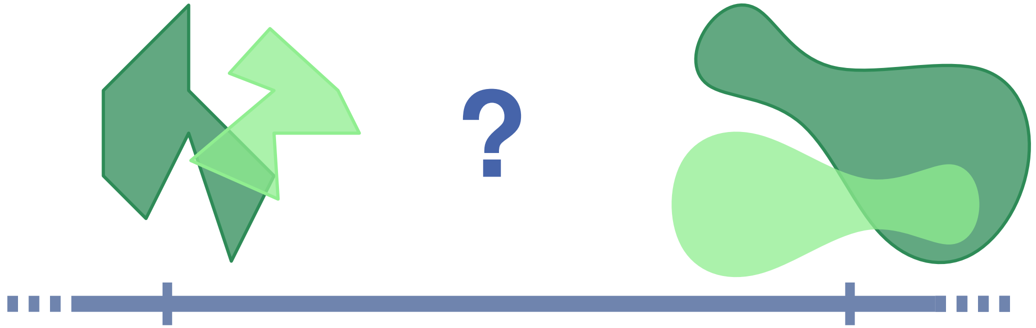
Open Problems

Problem 1:

$$\forall\exists_{<}\mathbb{R} \stackrel{?}{=} \forall\exists\mathbb{R}$$

Are strict formulas really easier?

Problem 2:



Polygons: $\in$ P

[Alt et al. 1995]

Semi-Algebraic: $\forall\exists_{<}\mathbb{R}$

Where is the tipping point in the complexity of HAUSDORFF?

Questions?

Thank you!

Problem 2: Increase Dimension

Problem: Definition of $A := \{x \in \mathbb{R}^n \mid \exists Y \in \mathbb{R}^m : \varphi(x, Y)\}$ contains an $\exists$.

Problem 2: Increase Dimension

Problem: Definition of $A := \{x \in \mathbb{R}^n \mid \exists Y \in \mathbb{R}^m : \varphi(x, Y)\}$ contains an $\exists$.

Idea: Include Y variables into A and B:

$$A := \{(x, y) \in \mathbb{R}^n \times [-\varepsilon, \varepsilon]^m \mid \varphi(x, y)\}$$

$$B := \mathbb{R}^n \times \{0\}^m$$

tiny, tiny ε

Problem 2: Increase Dimension

Problem: Definition of $A := \{x \in \mathbb{R}^n \mid \exists Y \in \mathbb{R}^m : \varphi(x, Y)\}$ contains an $\exists$.

Idea: Include Y variables into A and B:

$$A := \{(x, y) \in \mathbb{R}^n \times [-\varepsilon, \varepsilon]^m \mid \varphi(x, y)\}$$

$$B := \mathbb{R}^n \times \{0\}^m$$

tiny, tiny ε

Details are tricky:

- Even for true formulas Φ we now have $d_{\mathrm{H}}(A, B) > 0$.
- There is a threshold t such that:
  $d_{\mathrm{H}}(A, B) \leq t \iff \Phi$ is true