Complexity of the Planar Slope Number Problem

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Main Reference

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On the Complexity of the Planar Slope Number Problem

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Abstract

The *planar slope number* of a planar graph G is defined as the minimum number of slopes that is required for a crossing-free straight-line drawing of G. We show that determining the planar slope number is hard in the *existential theory of the reals*. We discuss consequences for drawings that minimize the planar slope number.

Slope Number - Definition

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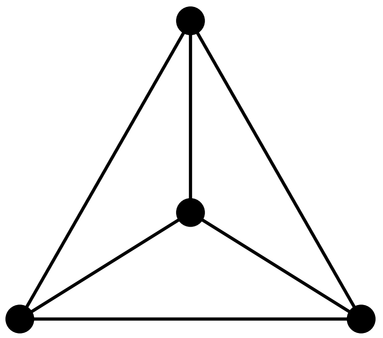
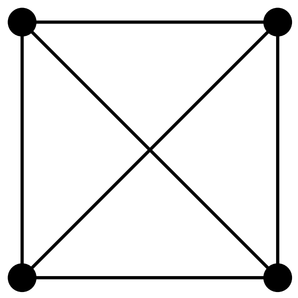
Slope Number

Given a graph G. The *slope number* is the minimum number of slopes needed in a straight-line drawing of G.



6 slopes



4 slopes

Planar Slope Number

$\rightsquigarrow$ only consider planar drawings

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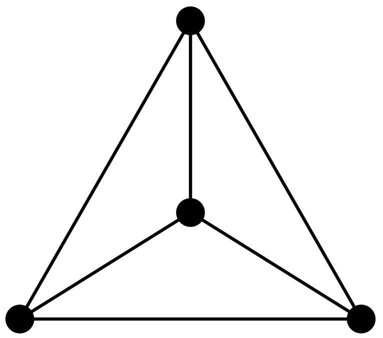

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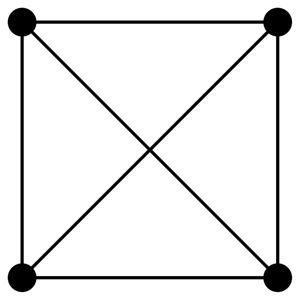

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Planar Slope Number

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Complete Graphs

- K_n requires exactly n slopes.
  Wade, Chu ’94

Slope Number - Definition

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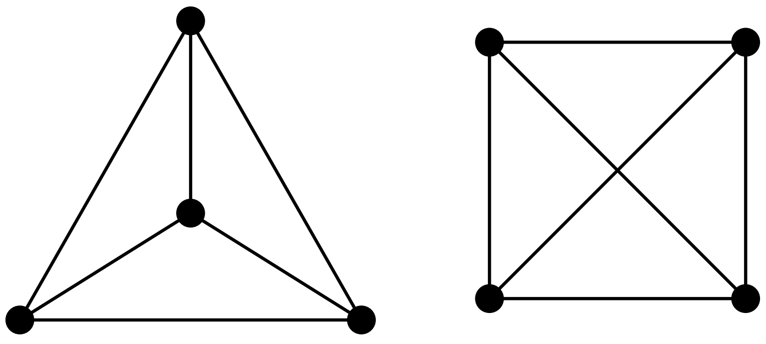

6 slopes   4 slopes

Planar Slope Number

$\rightsquigarrow$ only consider planar drawings

Maximum Degree Δ

- Lower bound: $\frac{\Delta}{2}$
  At most two edges incident to each vertex can have the same slope.
- $\Delta = 3$: 4 slopes are enough.
  Mukkamala, Szegedy ’09
- $\Delta = 5$: unbounded
  Barát, Matoušek, Wood ’06
  Pach, Pálvölgyi ’06
- $\Delta = 4$: unknown

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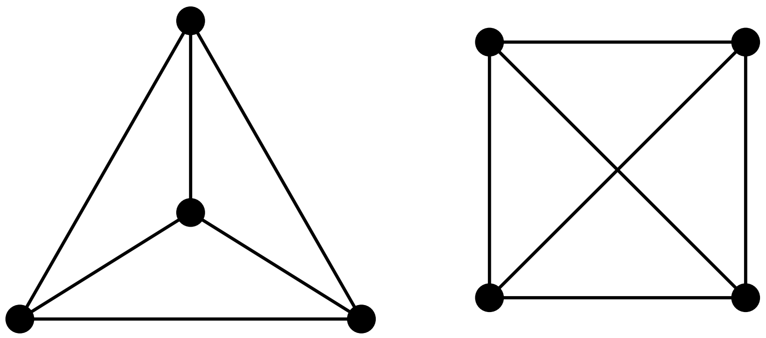

6 slopes 4 slopes

Planar Slope Number

$\rightsquigarrow$ only consider planar drawings

Planar Graphs

- Exponential in the maximum degree Δ.
  Keszegh, Pach, Pálvölgyi '11
- Decision problem $\exists\mathbb{R}$-complete.
  Hoffmann '17

Overview

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Theorem: (Hoffmann '17)
Deciding whether a graph G has planar slope number k is in NP for each fixed k.

Theorem: (Hoffmann '17)
Deciding whether the planar slope number is exactly $\frac{\Delta}{2}$ is $\exists\mathbb{R}$-complete.

Complexity Class $\exists\mathbb{R}$

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Definition: Existential Theory of the Reals

The *existential theory of the reals (ETR)* consists of all true sentences of the form

$$\exists X_1, \ldots \exists X_n : \Phi(X_1, \ldots, X_n)$$

where Φ is a quantifier free formular of polynomial (in)equalities with integer coefficients.

$$\exists X_1 \exists X_2 : X_1^2 + 3 \cdot X_2 = 7 \wedge X_1 > X_2$$

Solutions: $(X_1, X_2) = (2, 1), (-5, -6), \ldots$

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Solutions: $(X_1, X_2) = (2, 1), (-5, -6), \dots$

Definition: $\exists\mathbb{R}$

The class $\exists\mathbb{R}$ contains all decision problems that can be reduced to ETR in polynomial time.

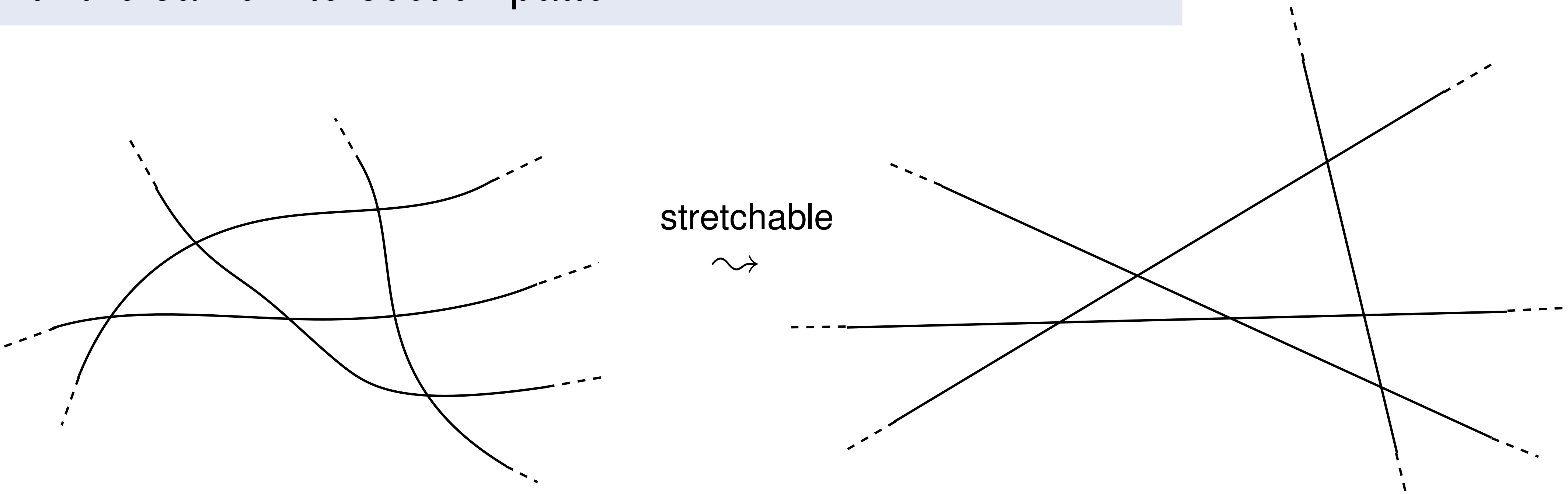
$$\text{NP} \subseteq \exists\mathbb{R} \subseteq \text{PSPACE}$$

Stretchability

Problem: Stretchability

Input: Arrangement of pseudolines L.

Question: Is L *stretchable*? I.e. is there a line arrangement with the same intersection pattern?



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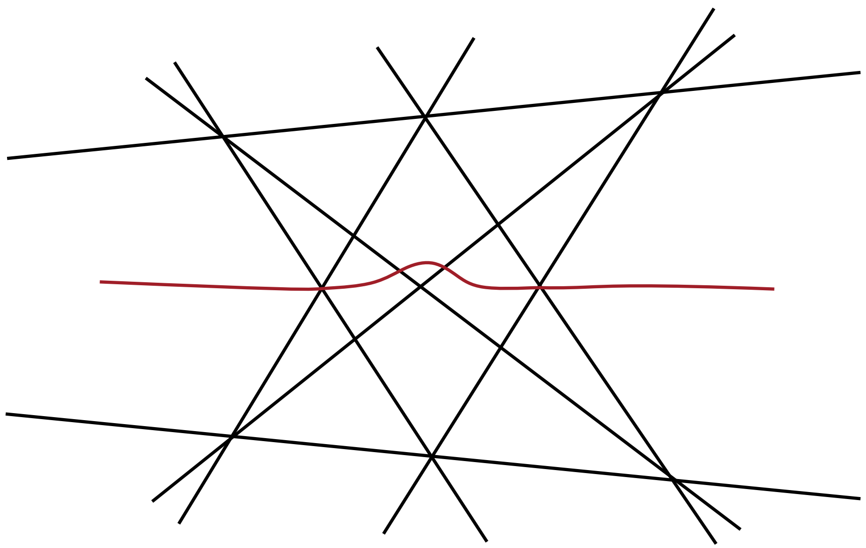

not stretchable
(by Pappu's Hexagon Theorem)

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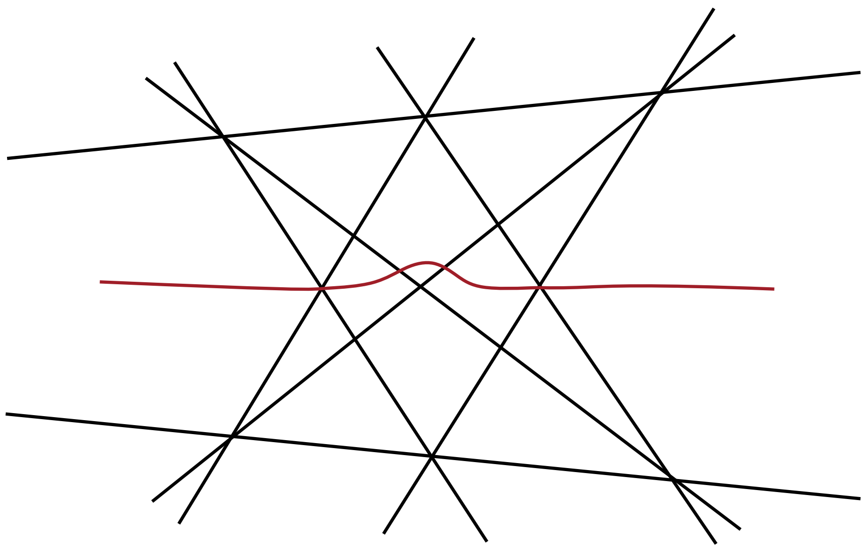

not stretchable
(by Pappu's Hexagon Theorem)

Theorem: (Mnëv '88)

STRETCHABILITY is $\exists\mathbb{R}$-complete.

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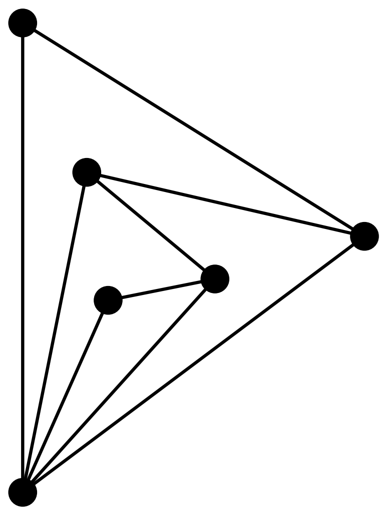
k-Slope Drawings - Fixed k

Theorem: (Hoffmann '17)

Deciding whether a graph G has a planar k-slope drawing is in NP for each fixed k.



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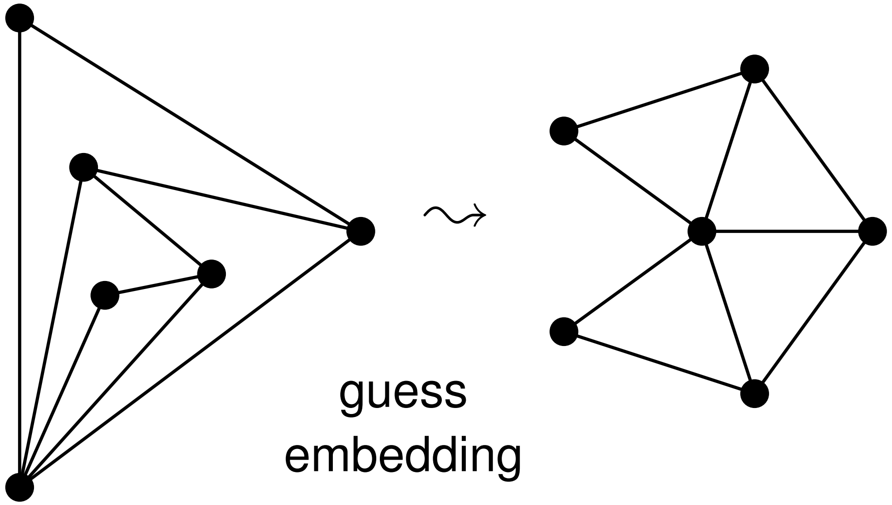

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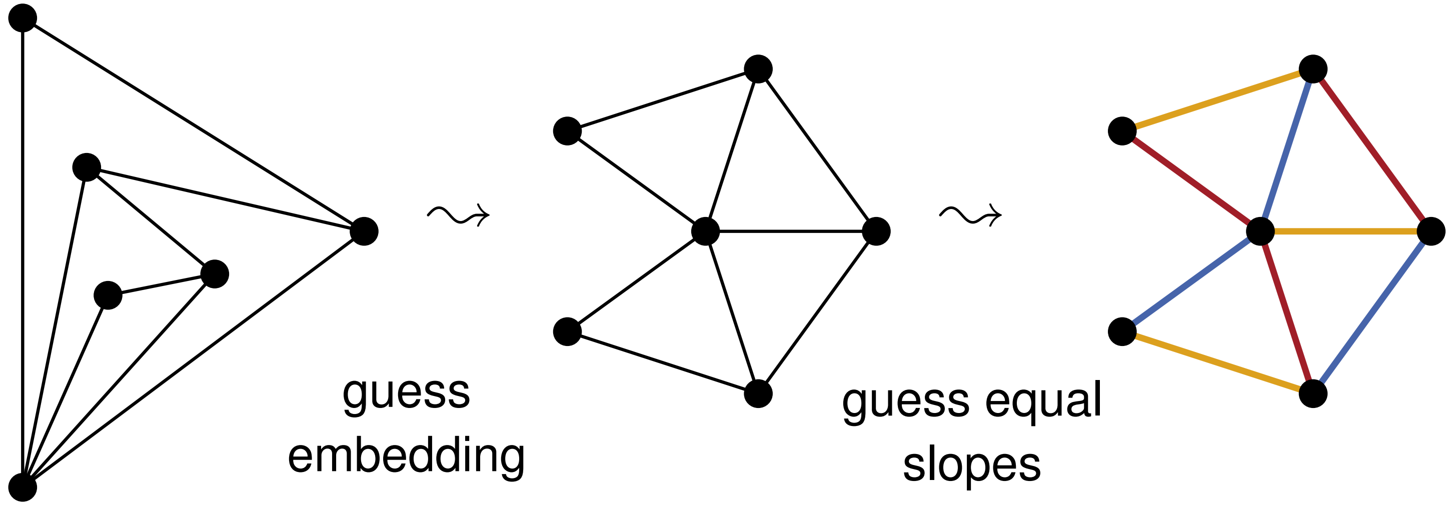

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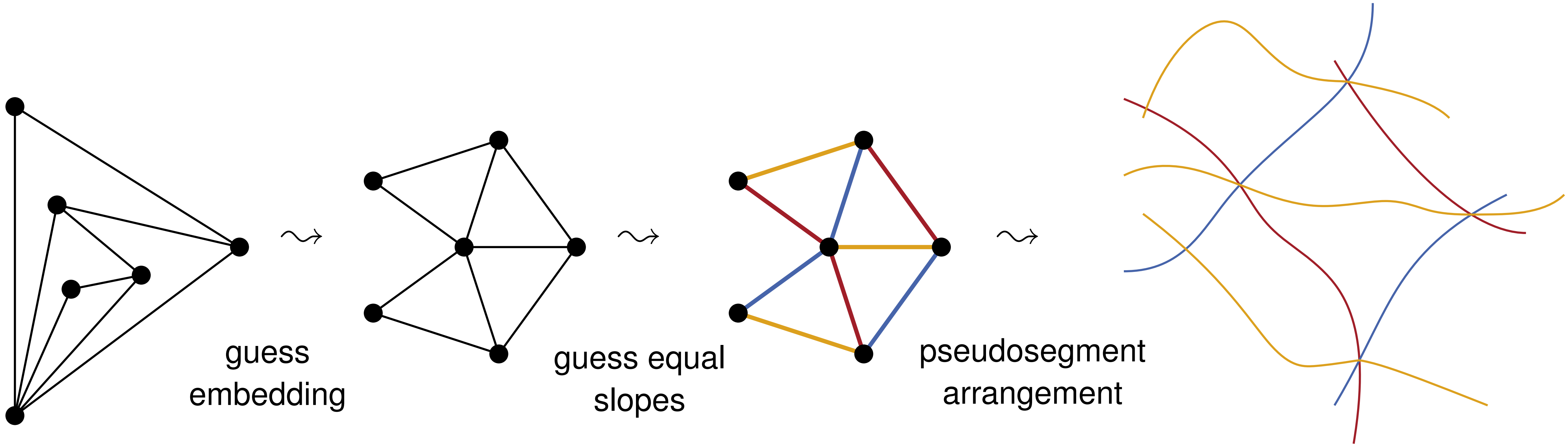

k-Slope Drawings - Fixed k

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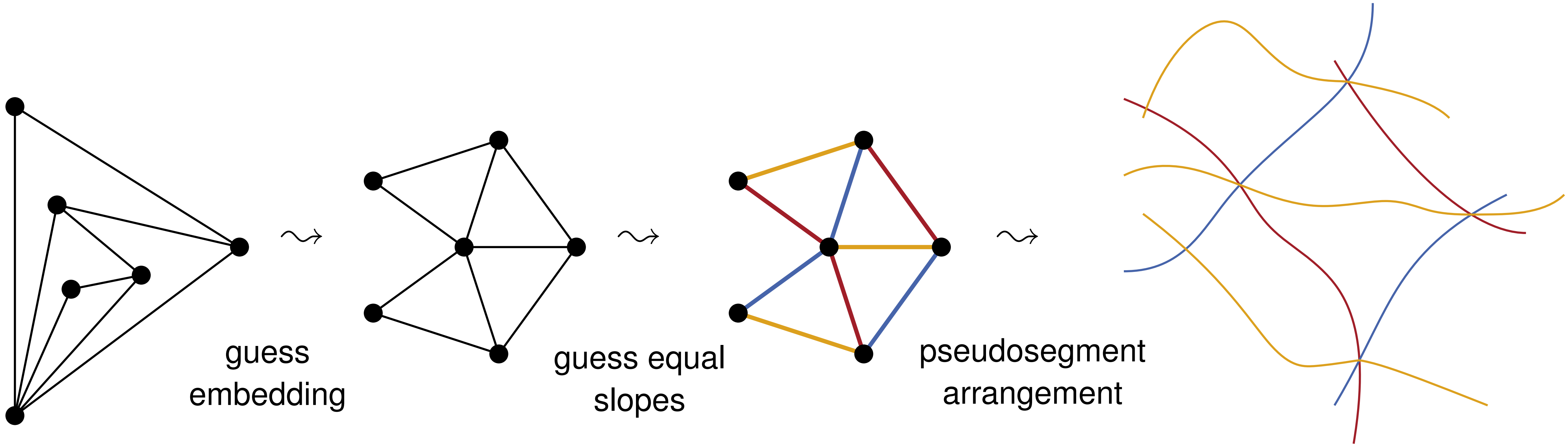
Theorem: (Hoffmann ’17)

Deciding whether a graph G has a planar k-slope drawing is in NP for each fixed k.

Theorem: (Kratochvíl, Matoušek ’94)

STRETCHABILITY of a pseudosegment arrangement with at most k slopes is in NP. (k fixed)



$k = \Delta/2$ Slopes - Reduction

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Theorem: (Hoffmann ’17)

Deciding whether the planar slope number is $\frac{\Delta}{2}$ is $\exists\mathbb{R}$-complete.

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$k = \Delta/2$ Slopes - Reduction

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Theorem: (Hoffmann ’17)

Deciding whether the planar slope number is $\frac{\Delta}{2}$ is $\exists\mathbb{R}$-complete.

$\exists\mathbb{R}$-Membership:

- Poly-time verification algorithm in the real RAM model.
  Erickson, Hoog, Miltzow 2020
- Given a drawing (coordinates of the vertices):
  Compute and count slopes.

$k = \Delta/2$ Slopes - Reduction

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Theorem: (Hoffmann ’17)

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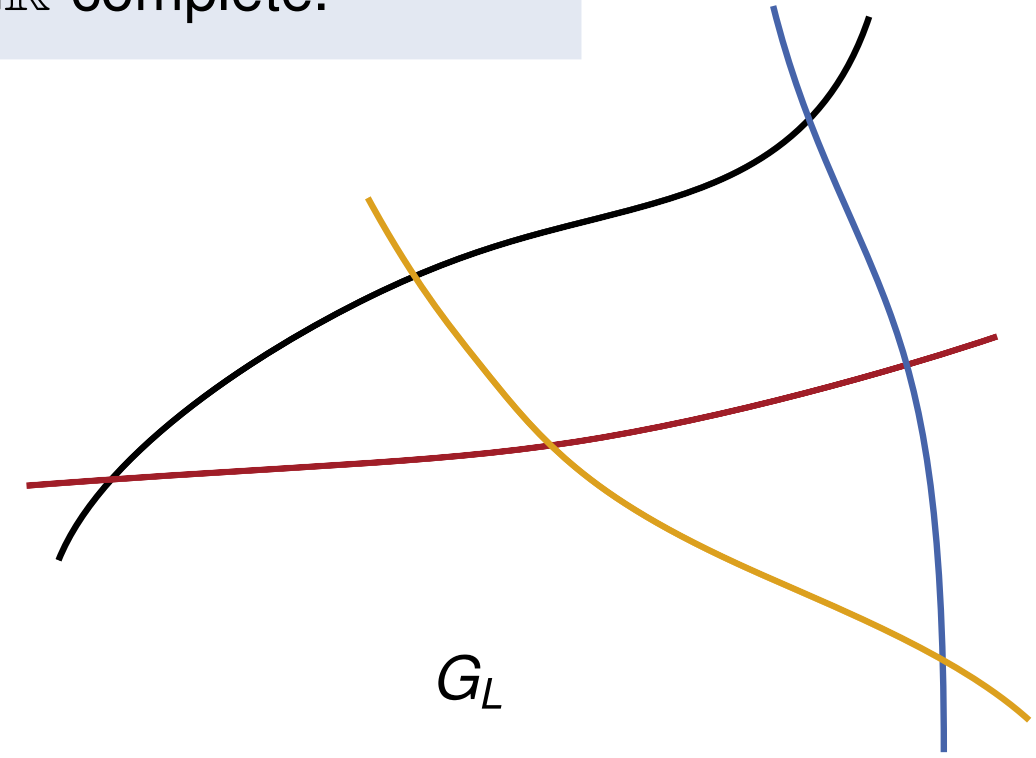
$\exists\mathbb{R}$-Hardness:

L: arrangement of n pseudolines



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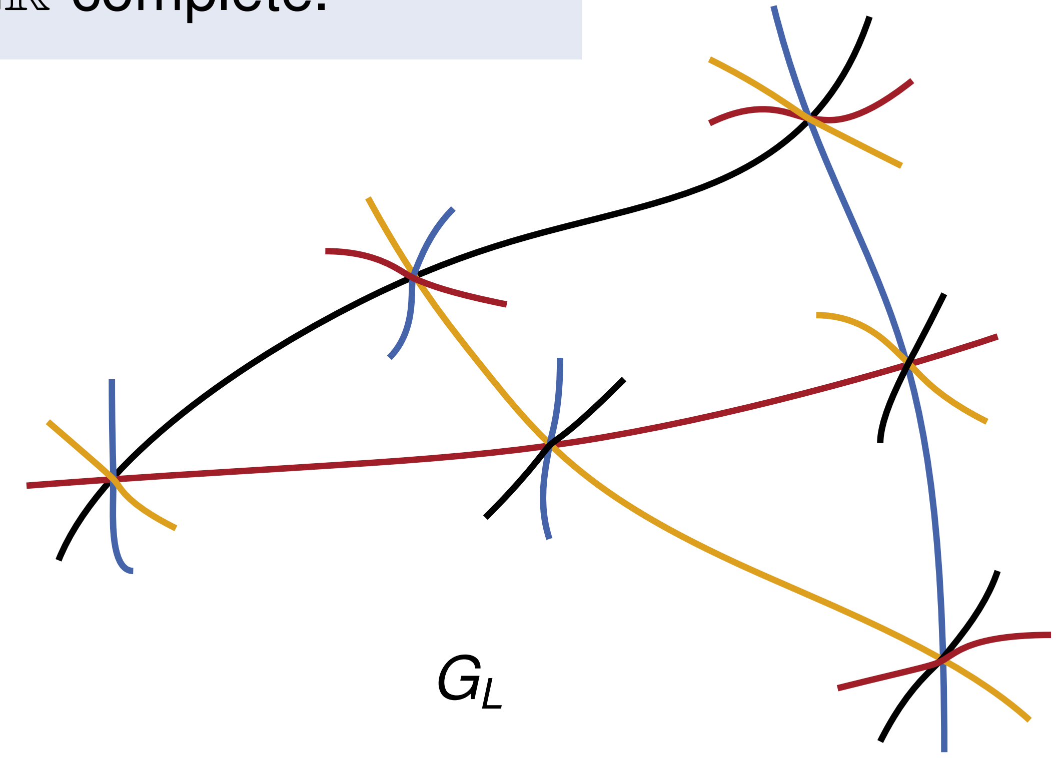

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L strechtable $\Rightarrow n = \frac{\Delta}{2}$ slopes

$\not\Leftarrow$ Only if we fix the planar embedding!

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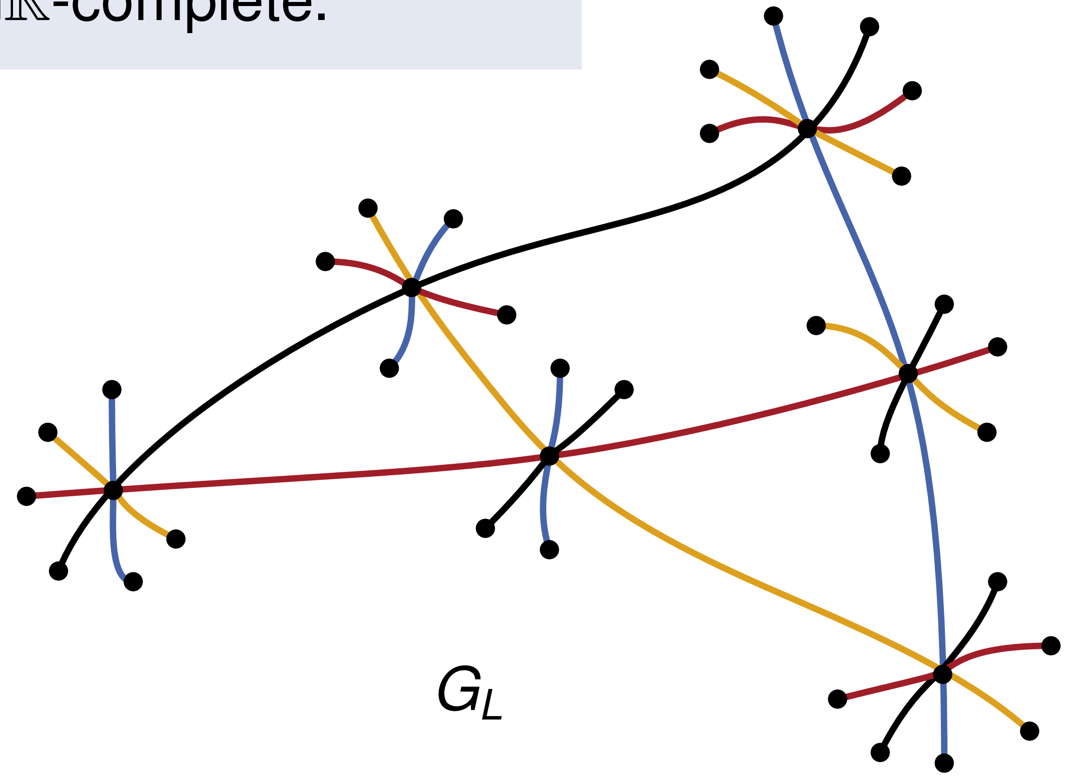

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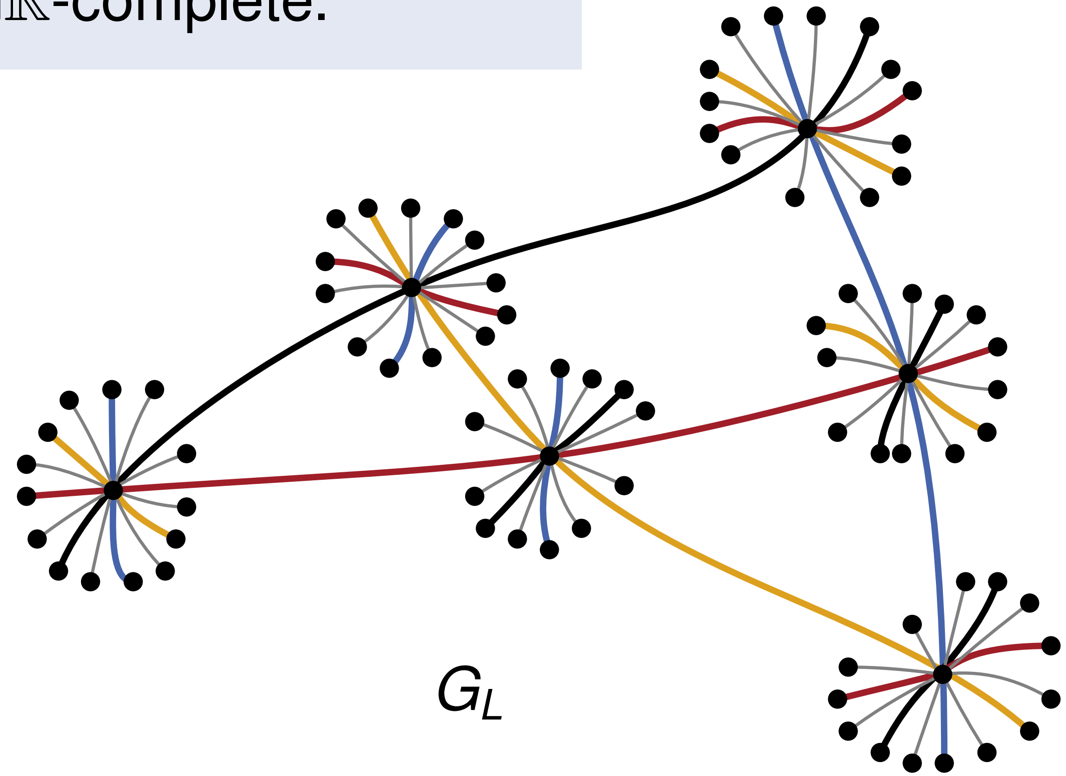

G_L

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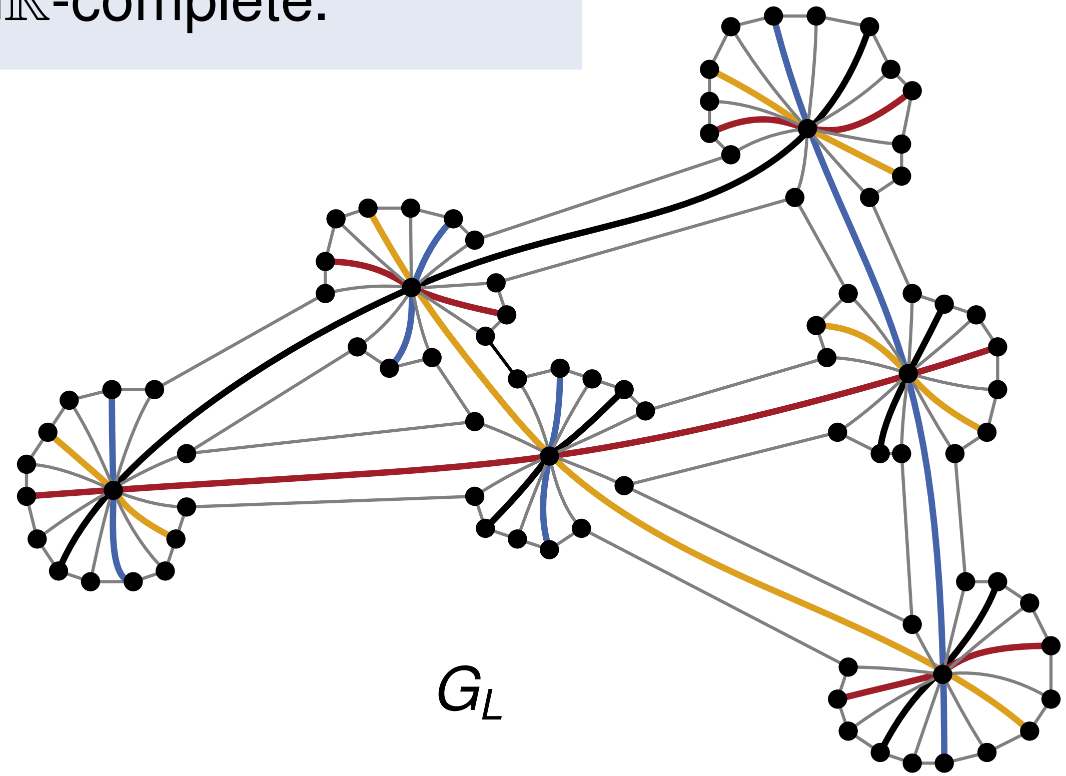

L strechtable $\Rightarrow n = \frac{\Delta}{2}$ slopes

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3-connected

$\Rightarrow$ unique embedding (Tutte)

L stretchable $\overset{!}{\Leftrightarrow}$ $\frac{\Delta}{2}$ slopes

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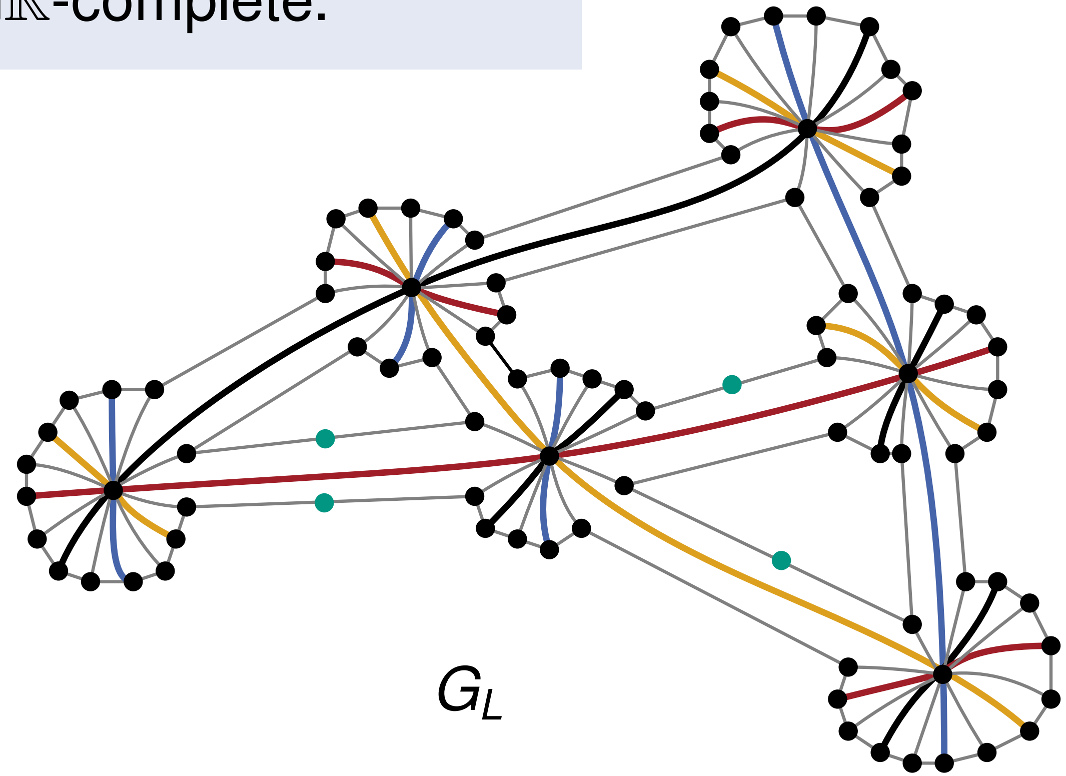

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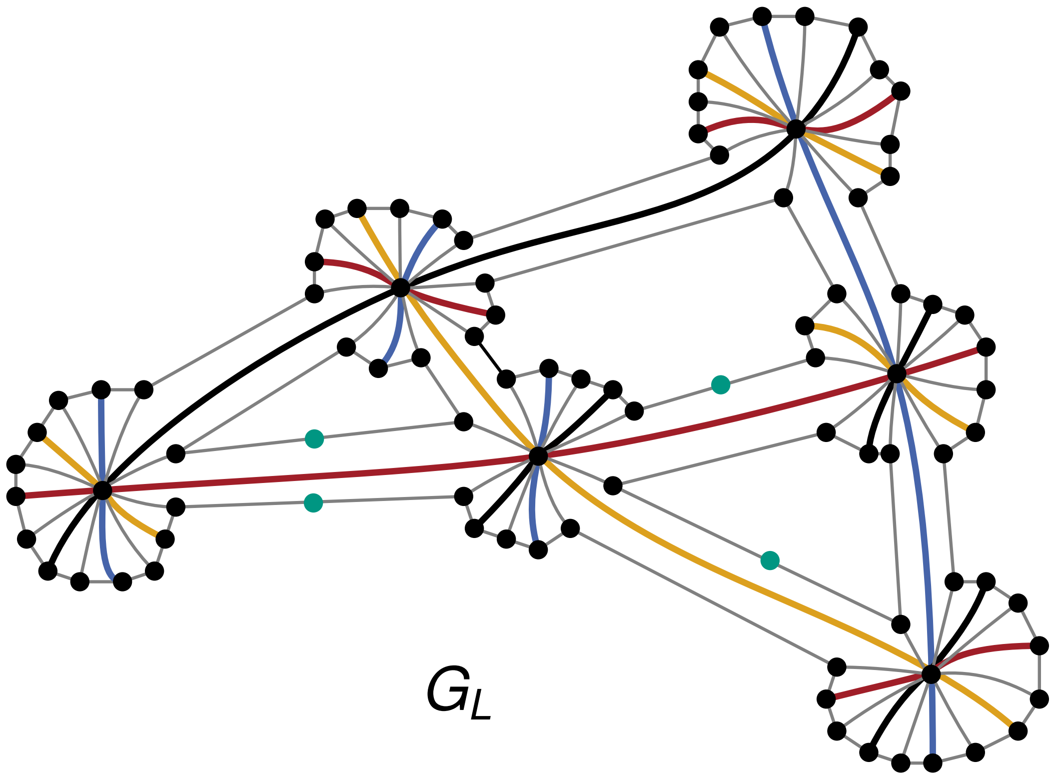
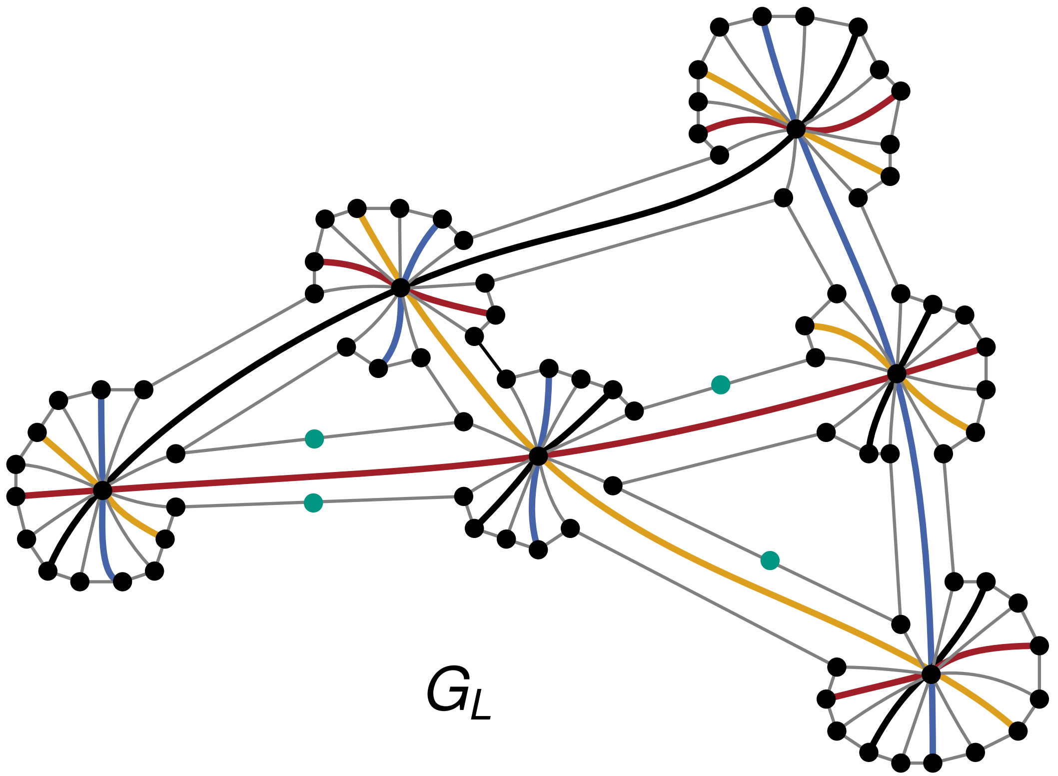
Drawing ⇒ Stretchable

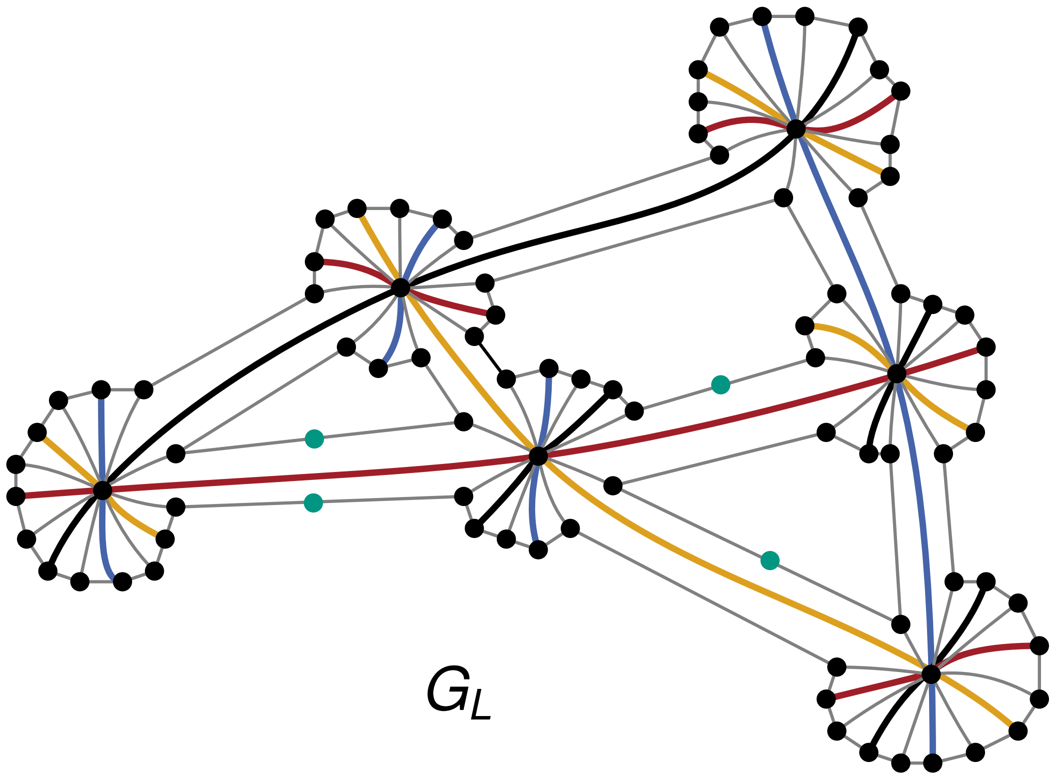


Drawing $\Rightarrow$ Stretchable



D: Straight-line drawing of G_L with $\frac{\Delta}{2}$ slopes.

Drawing $\Rightarrow$ Stretchable



D: Straight-line drawing of G_L with $\frac{\Delta}{2}$ slopes.

G_L (almost) 3-connected
$\Rightarrow$ same embedding as L

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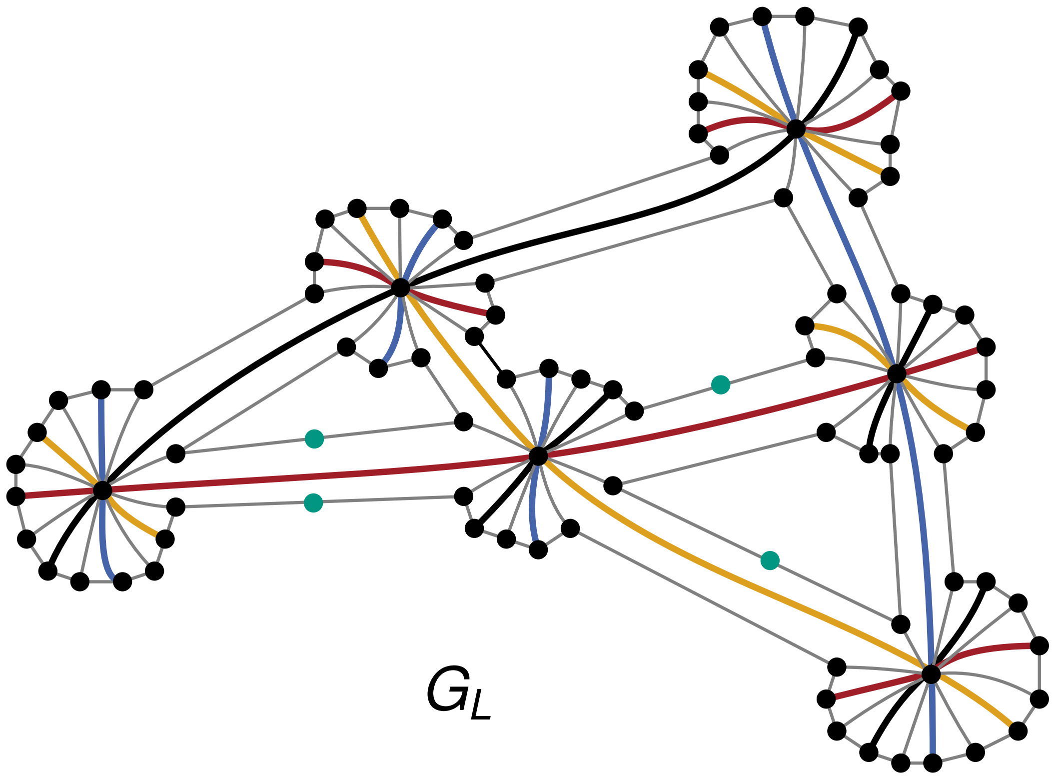
Drawing ⇒ Stretchable

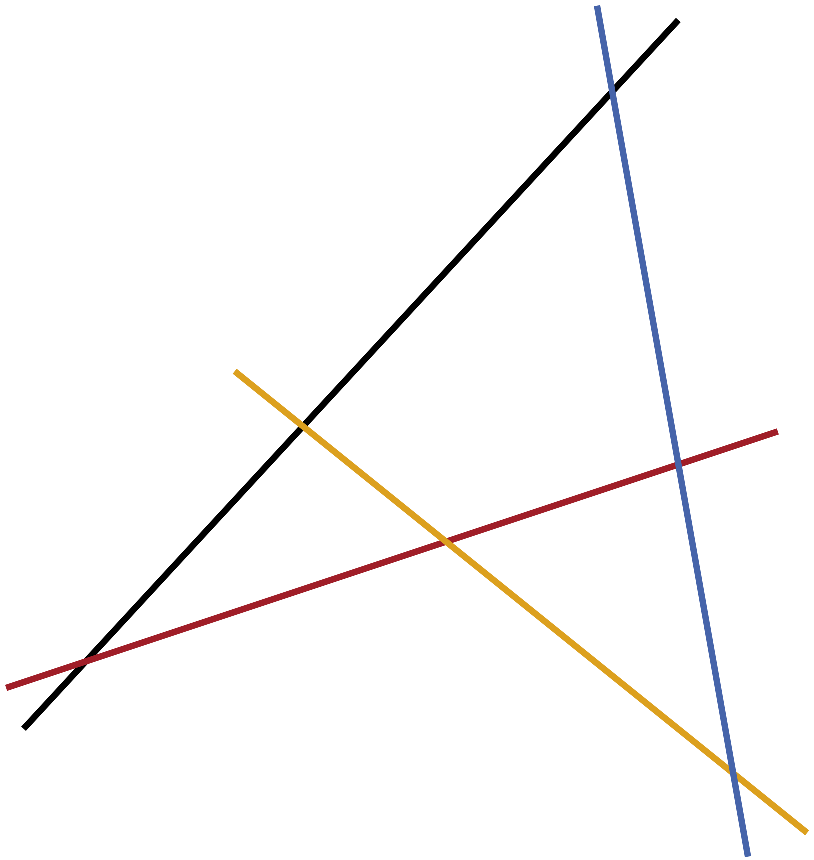


D: Straight-line drawing of G_L with $\frac{\Delta}{2}$ slopes.

G_L (almost) 3-connected
$\Rightarrow$ same embedding as L

for $v \in V$ with $\deg(v) = \Delta$:
$\Rightarrow$ opposite edges have equal slopes
$\Rightarrow$ L is stretched

Stretchable ⇒ Drawing



Stretchable ⇒ Drawing

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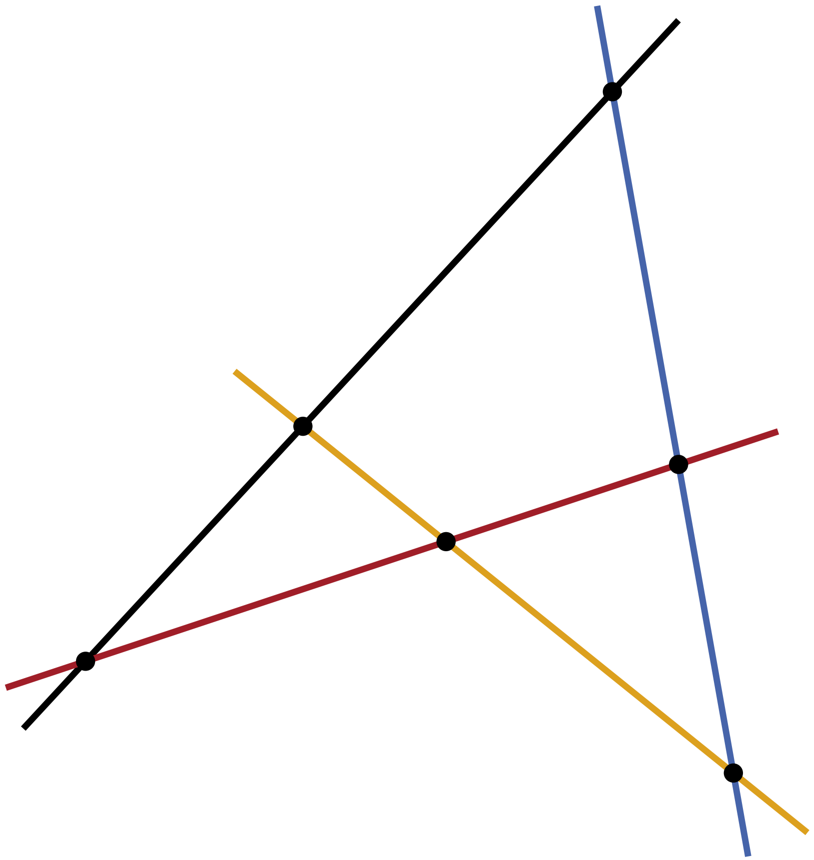

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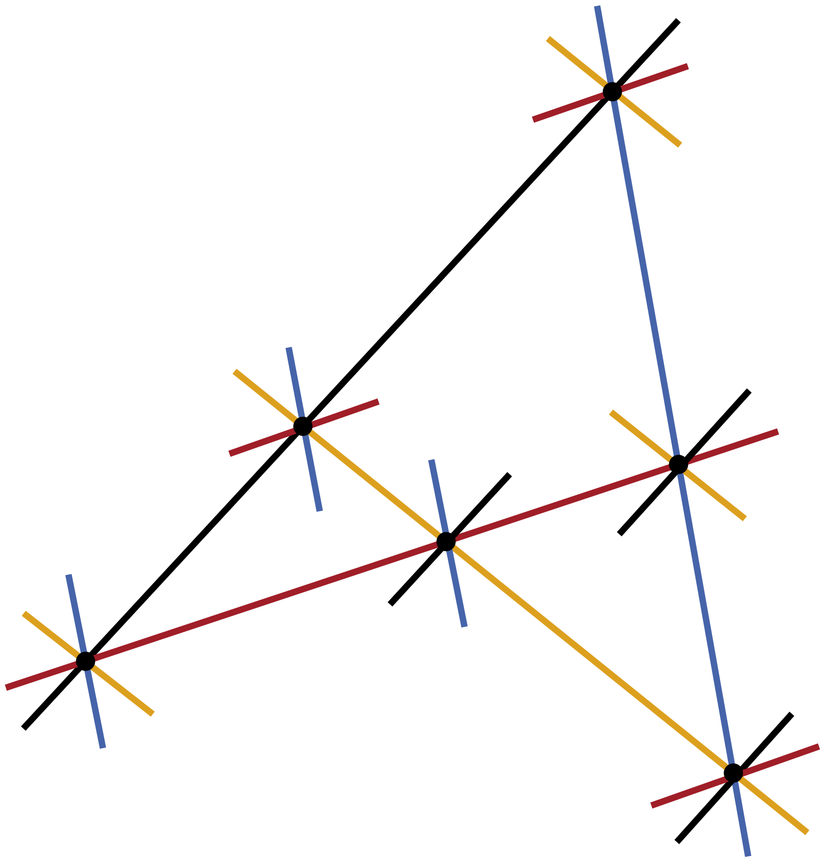
Stretchable ⇒ Drawing



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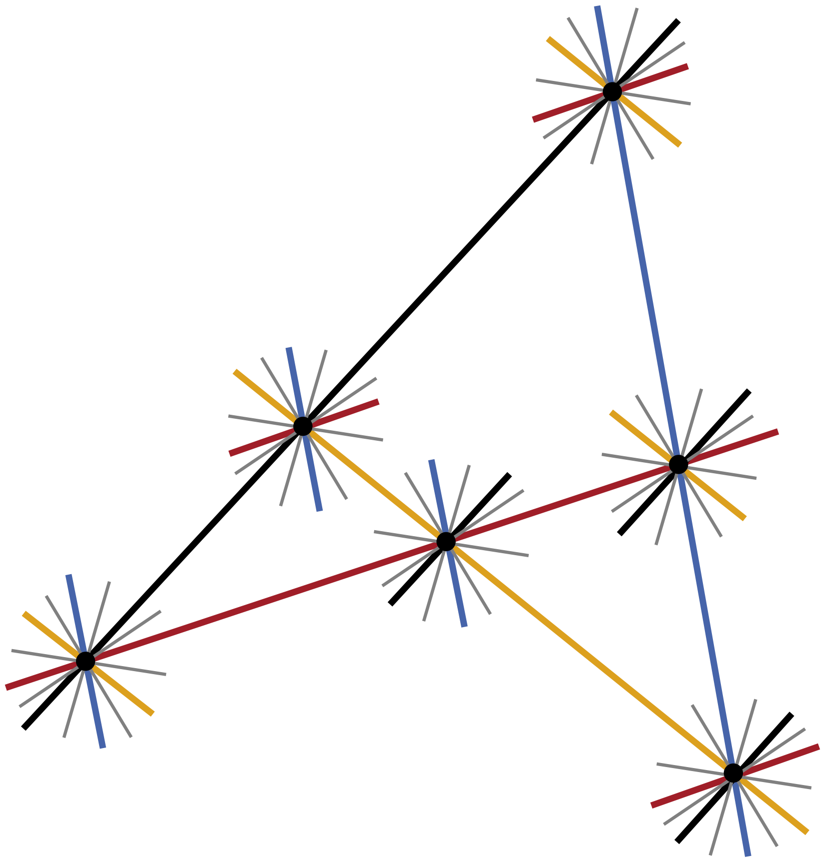

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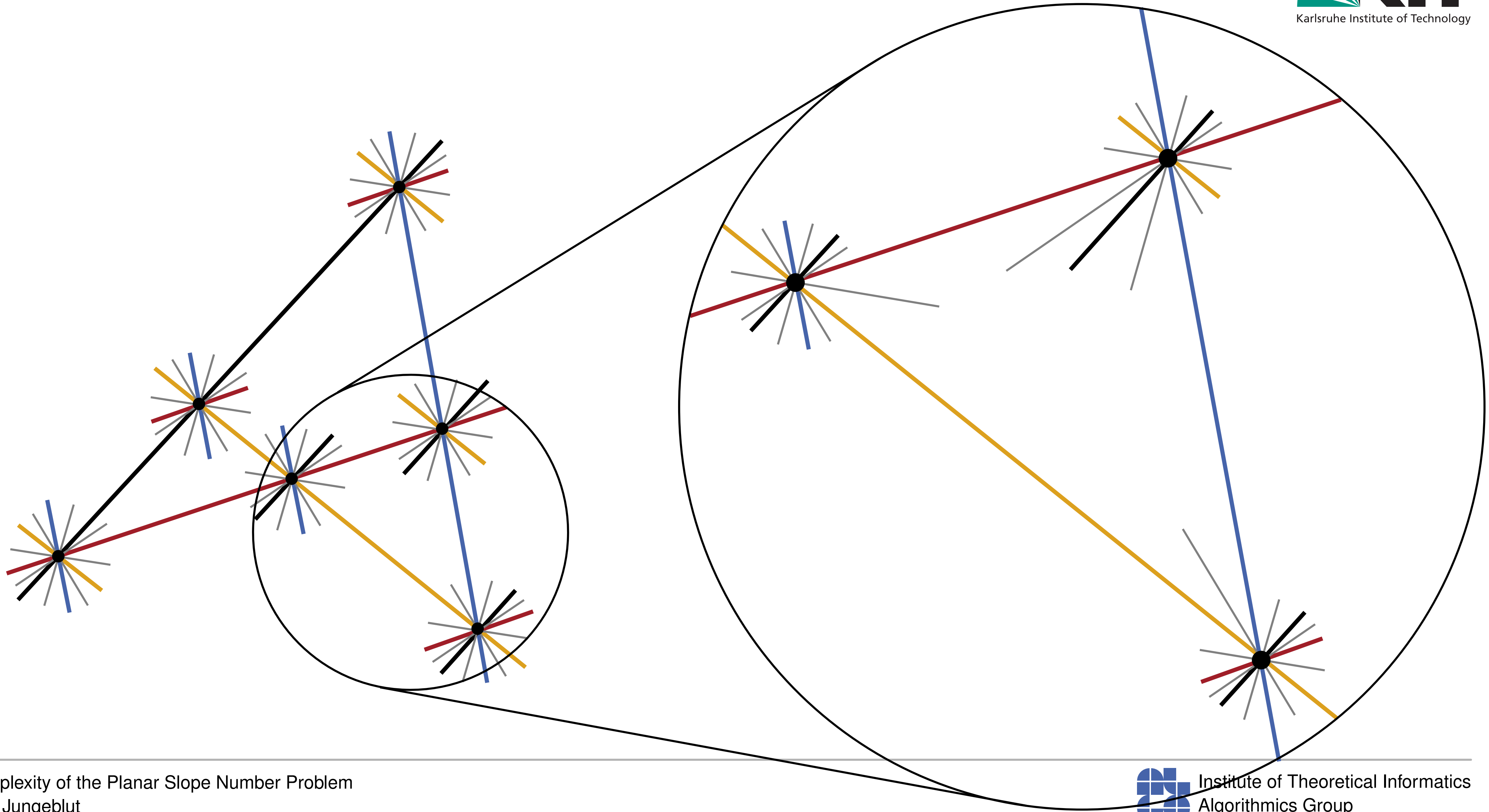

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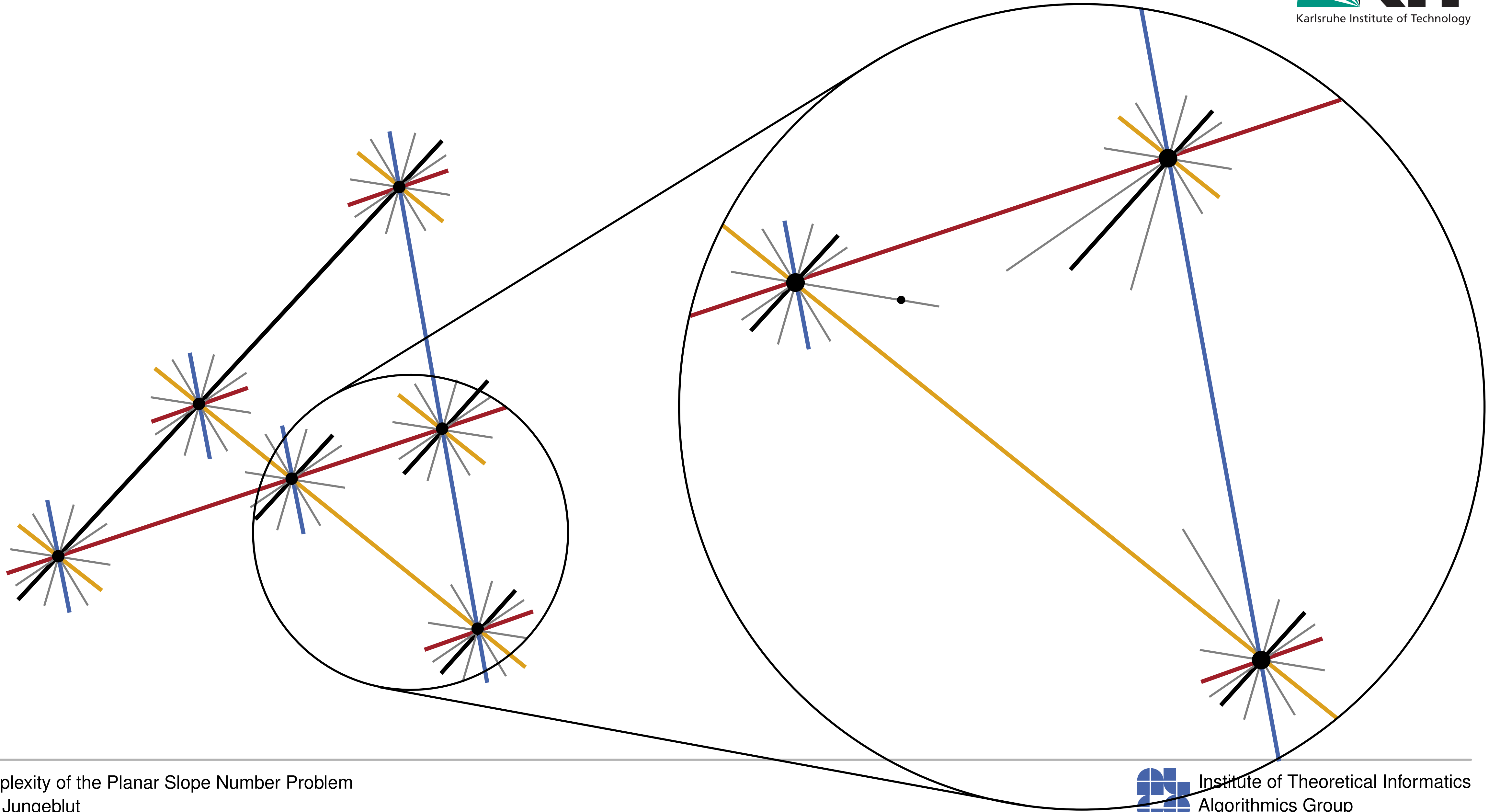

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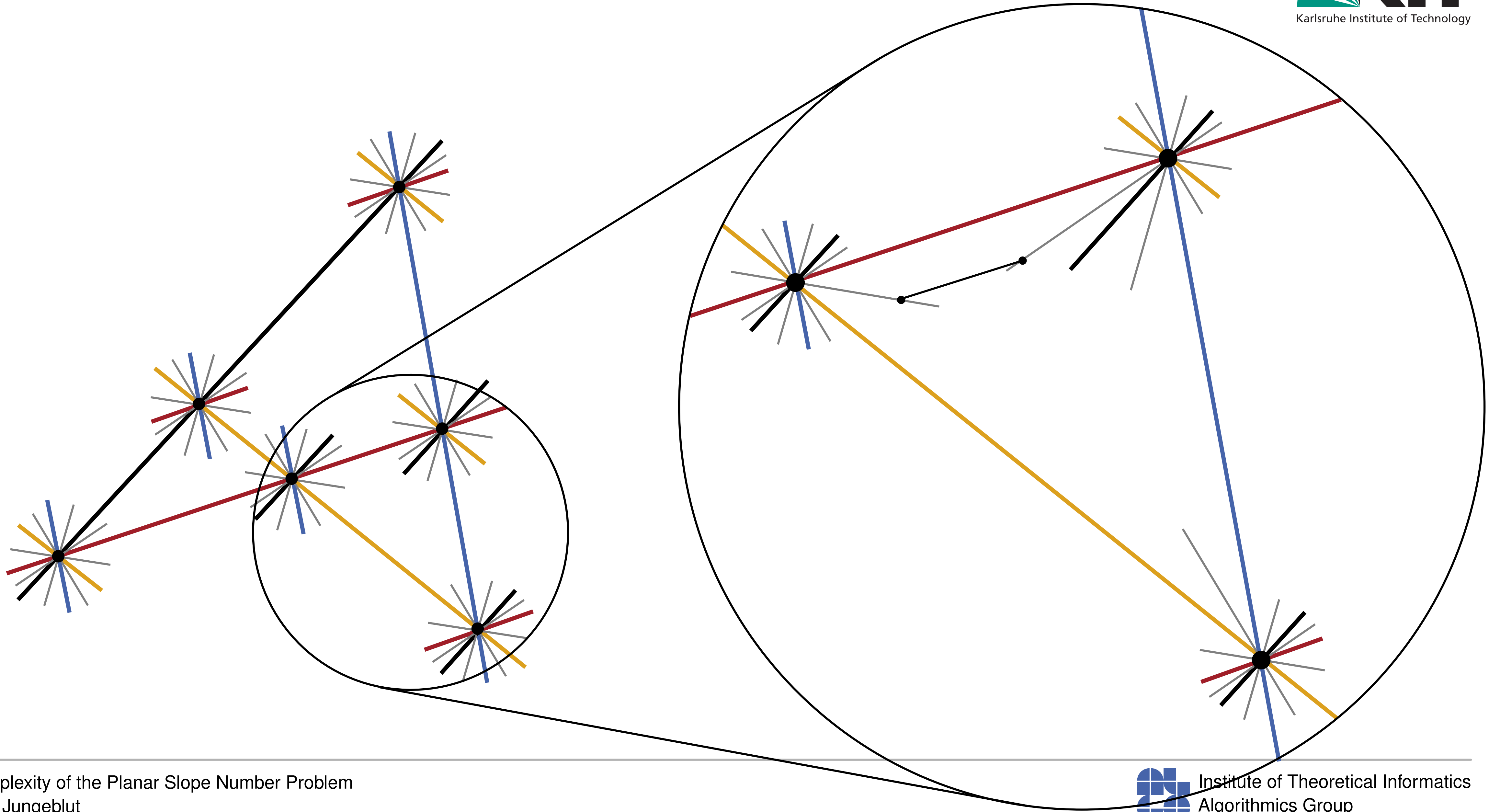

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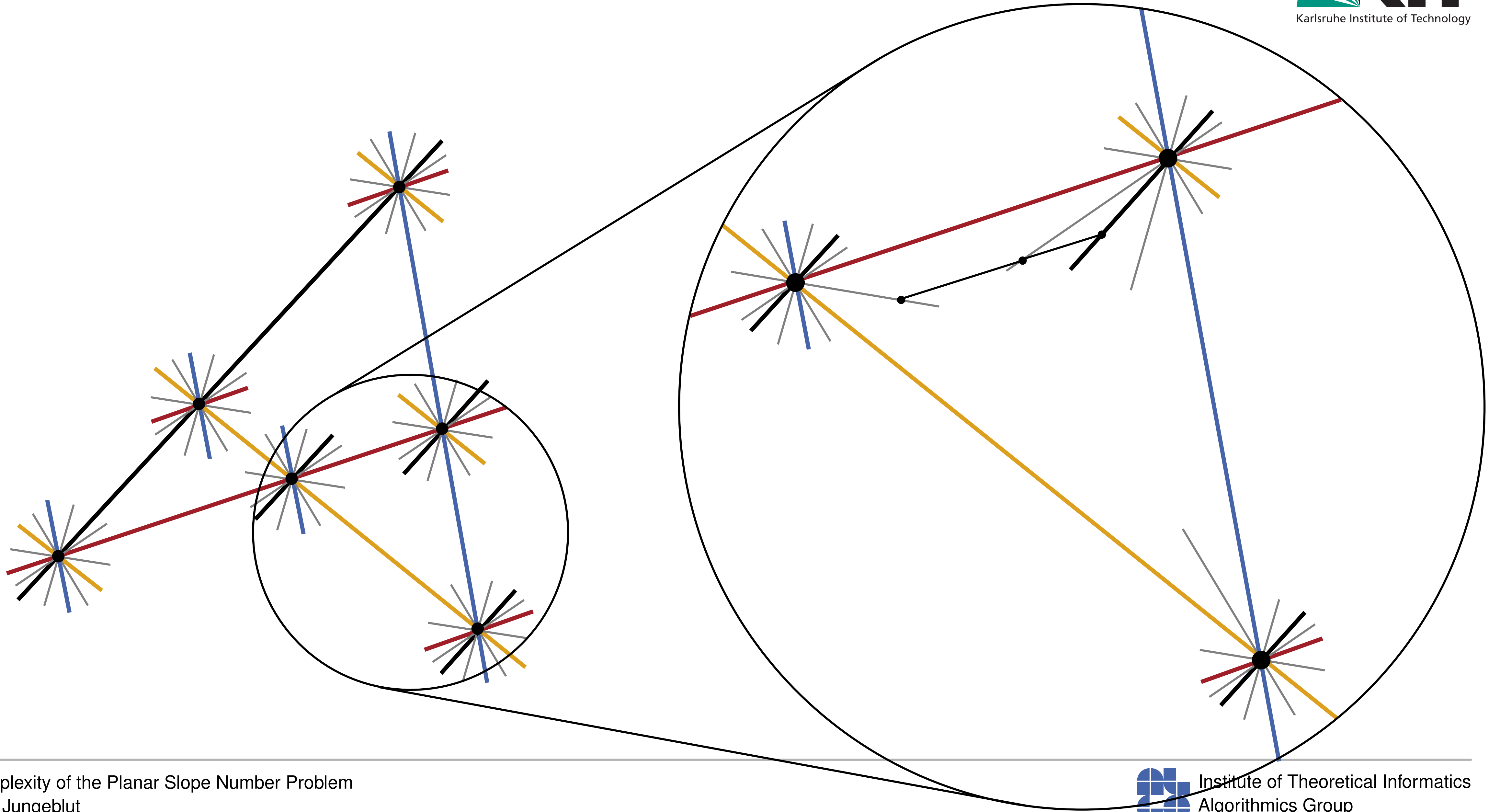

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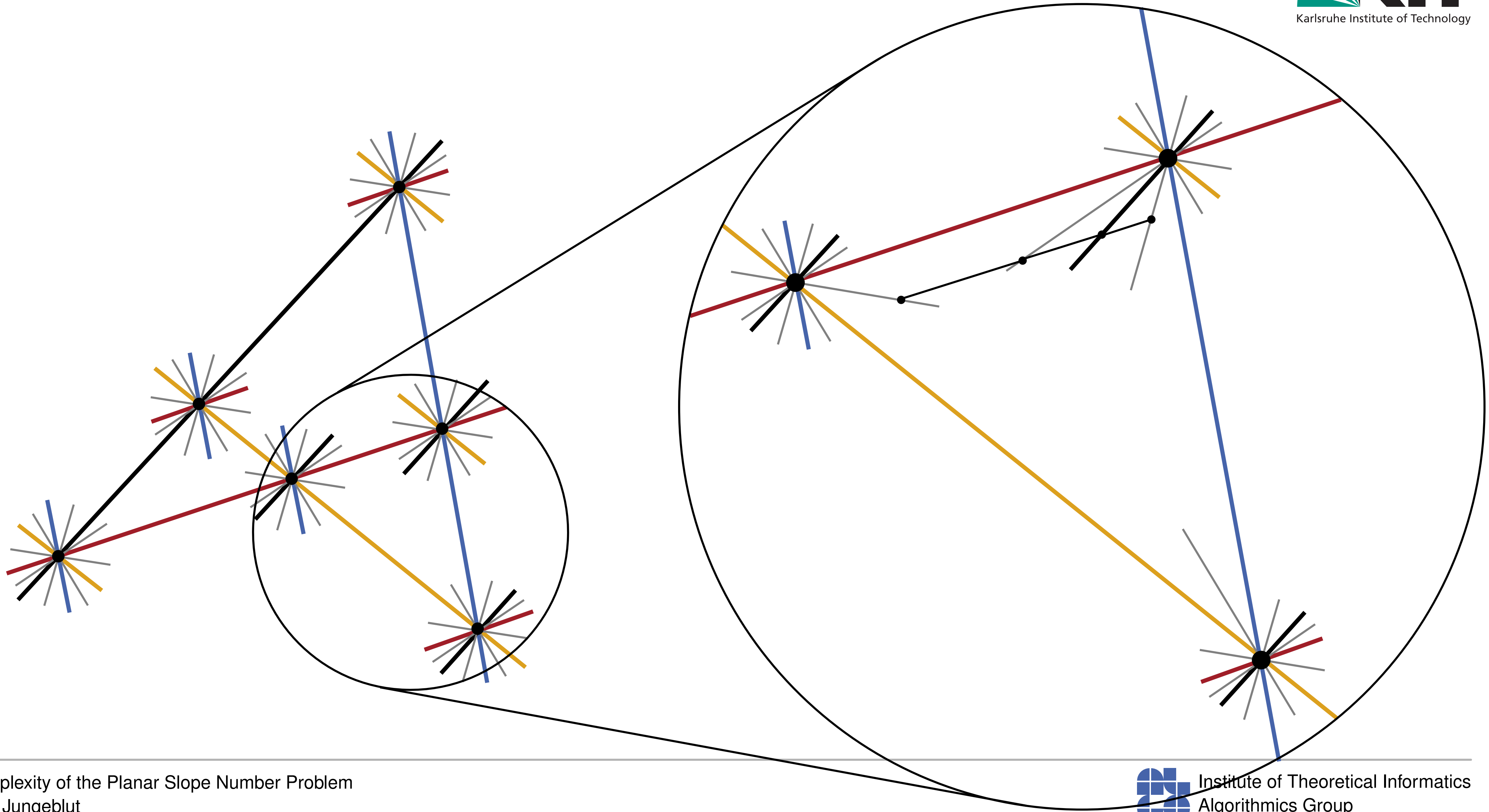

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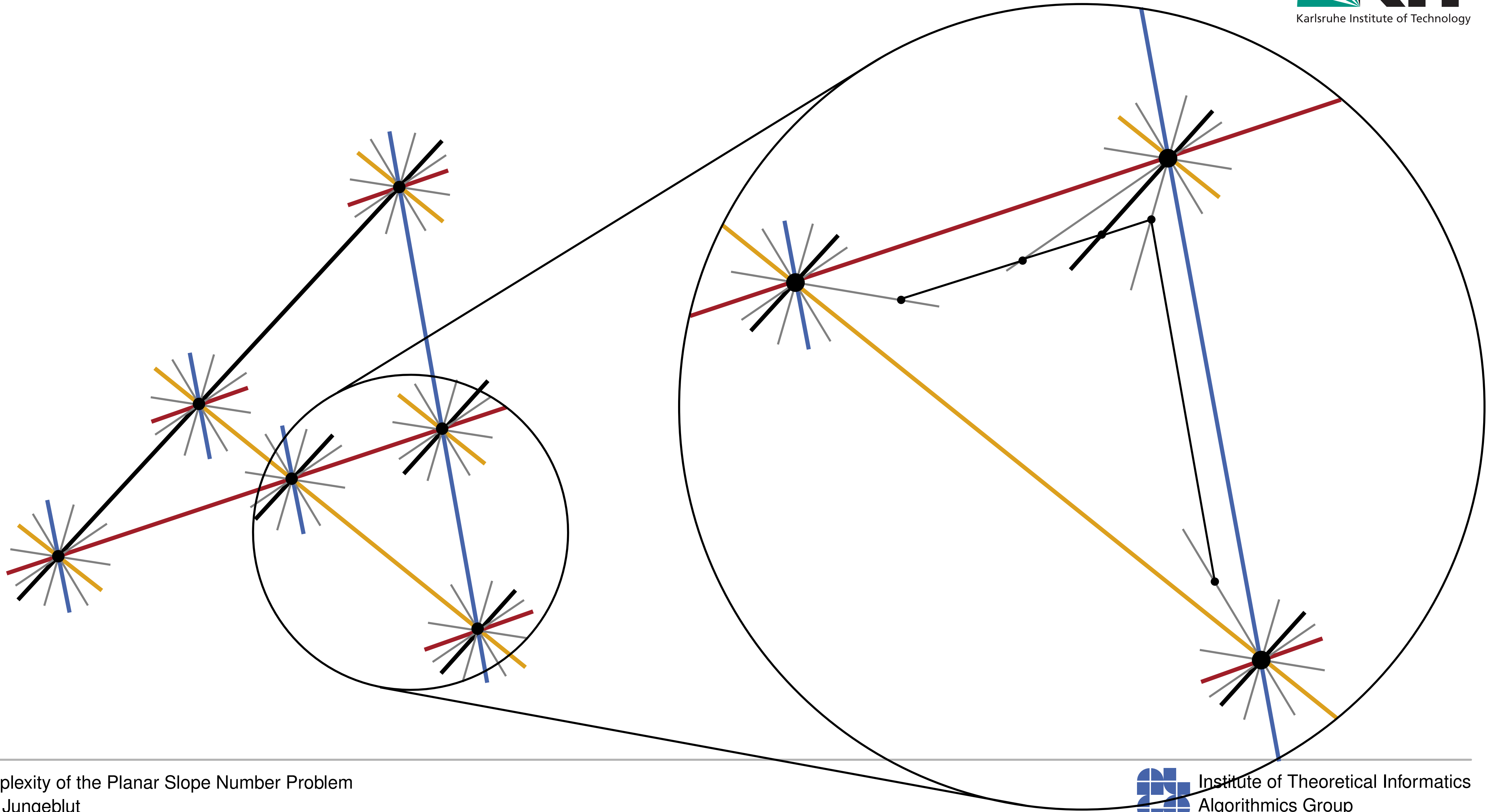

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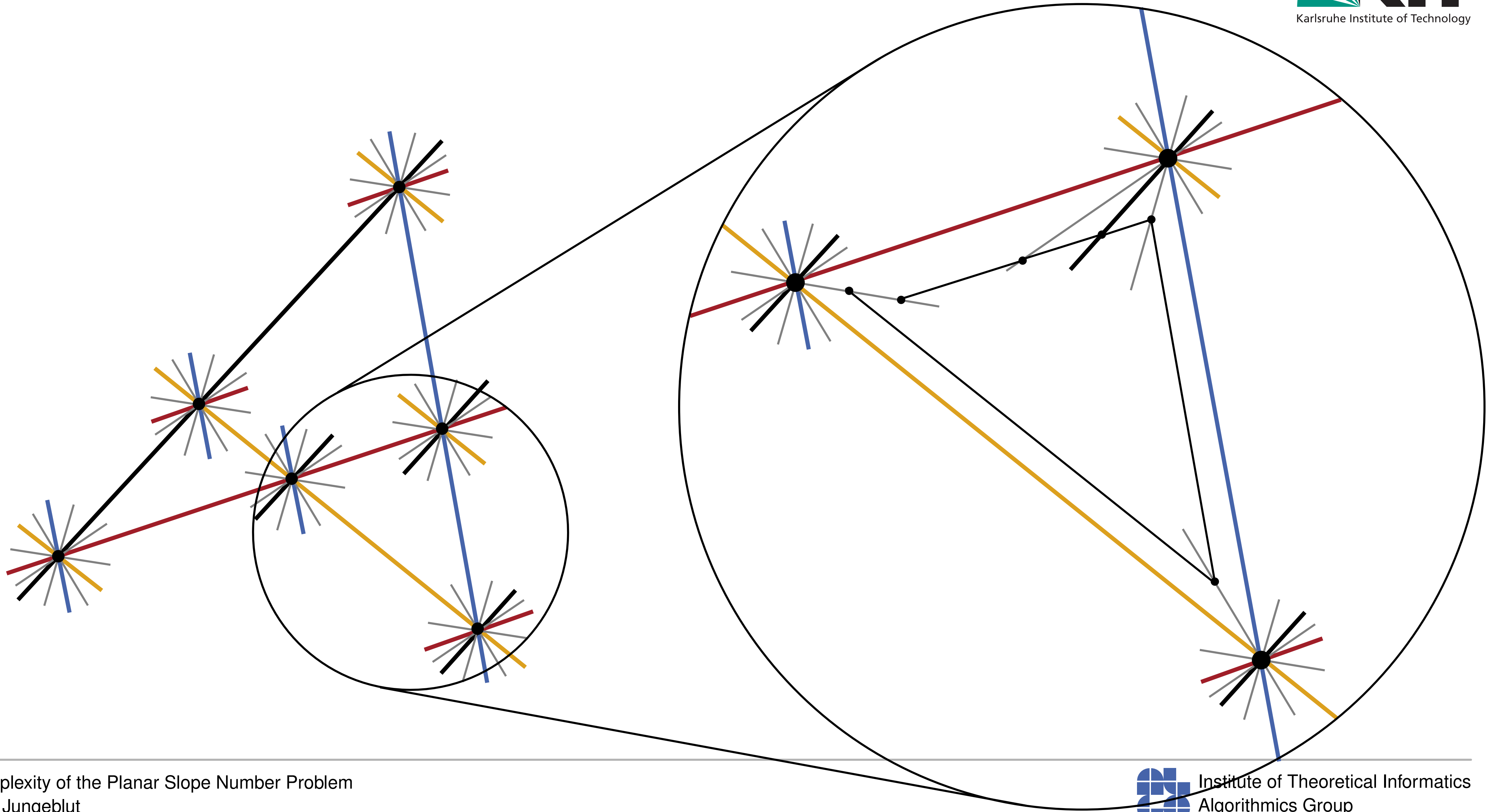

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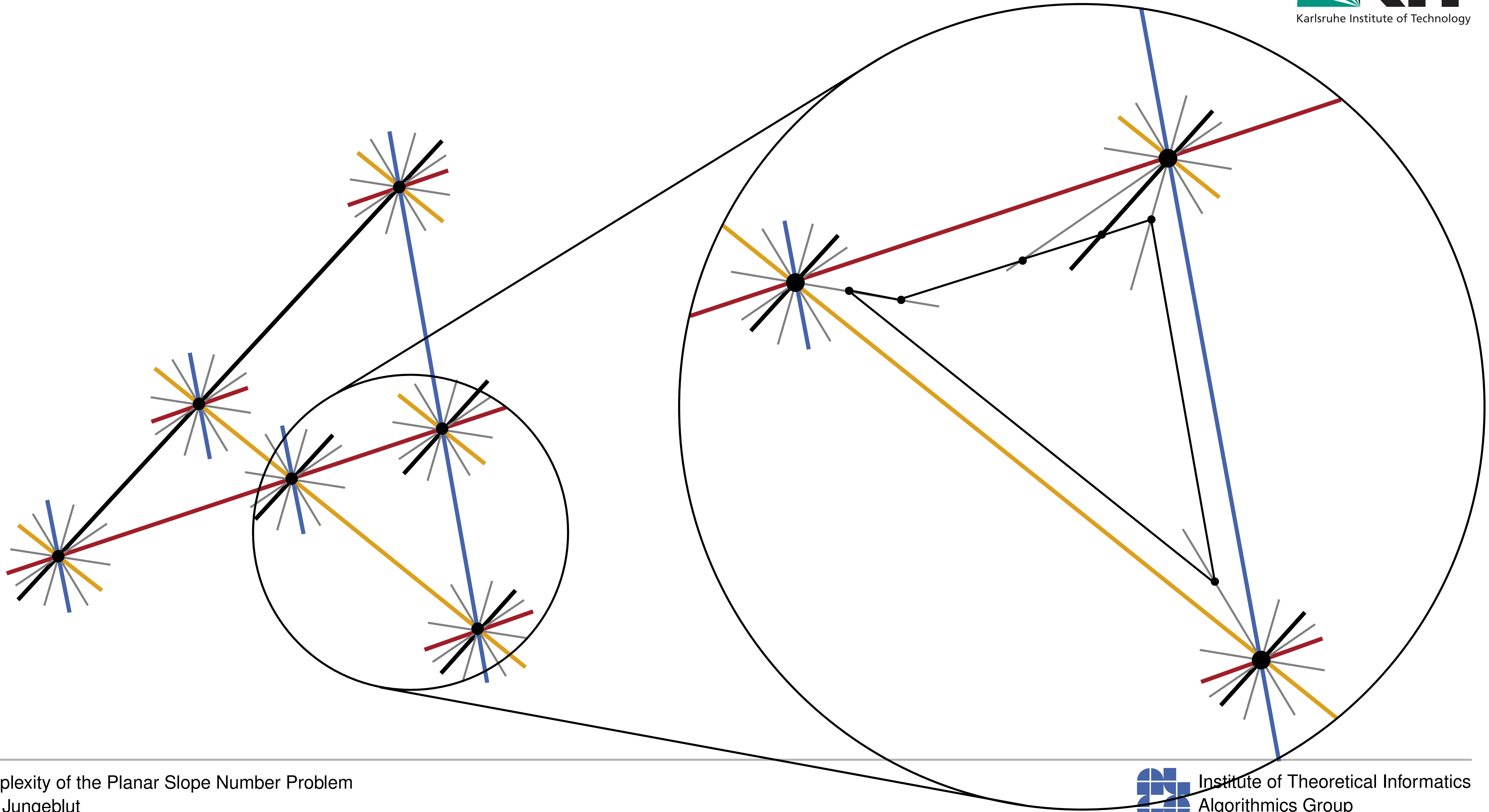

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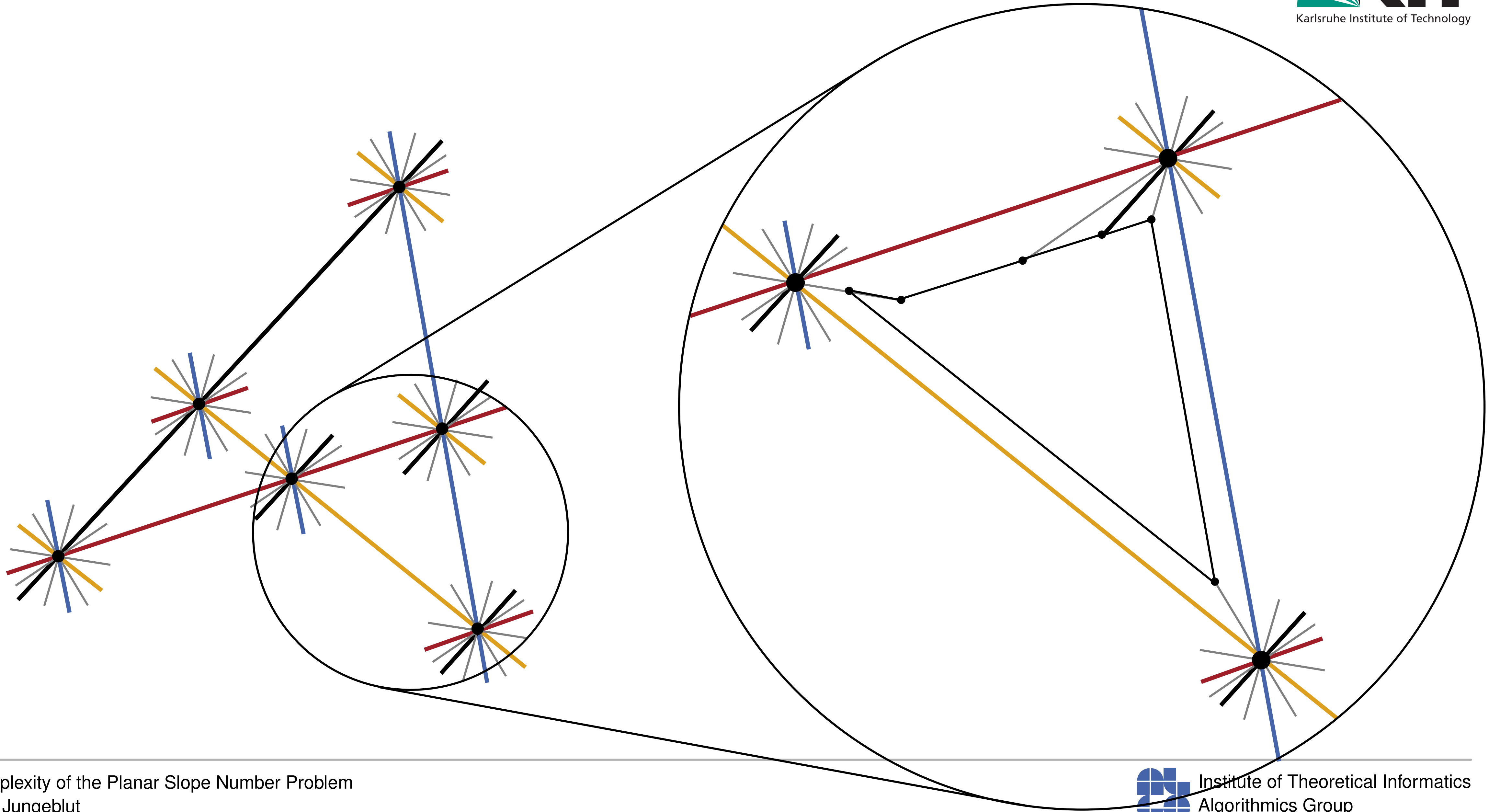

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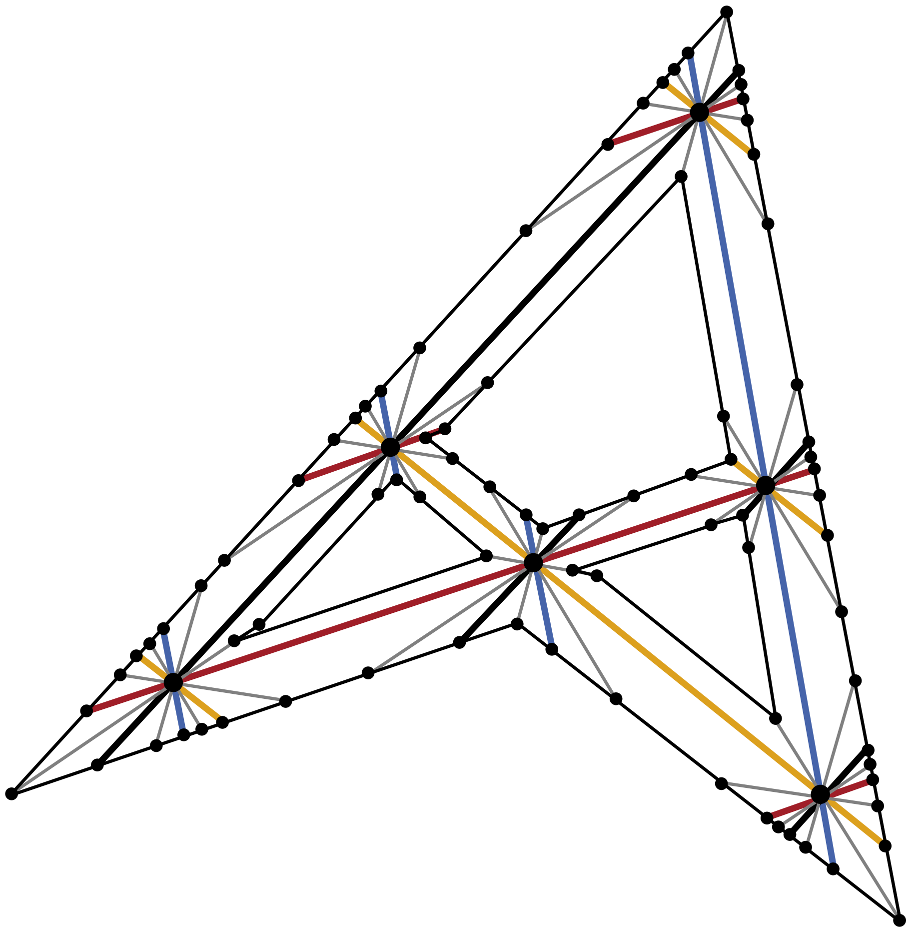

straight-line drawing with
exactly $\frac{\Delta}{2}$ slopes

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Open Problems

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Non-planar Graphs:
What is the complexity of the slope number problem for non-planar graphs?

Directed Graphs:
What is the (planar) slope number of directed/upward planar graphs?

Bounded Degree:
What is the slope number of graphs with $\Delta = 4$?

Thank you very much!

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Stretchability with k Slopes, k fixed

Question: Can we realize a pseudoline arragnement $L = \{\ell_1, \ldots, \ell_n\}$ with k slopes?

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Line ℓ_i: $y_i = \underbrace{a_i}_{\text{unknown slope}} \cdot x + \underbrace{b_i}_{\text{unknown offset}}$

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Stretchability with k Slopes, k fixed

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Question: Can we realize a pseudoline arrangement $L = \{\ell_1, \dots, \ell_n\}$ with k slopes?

Line ℓ_i: $y_i = a_i \cdot x + b_i$

unknown offset

unknown slope

From the combinatorial description of L we get:

- Which lines should be parallel.
- Ordering of the slopes.
- Ordering of the intersections.

Stretchability with k Slopes, k fixed

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b_i: unknown offset

a_i: unknown slope

From the combinatorial description of L we get:

- Which lines should be parallel. $\ell_i \parallel \ell_j \rightsquigarrow$ identify(a_i, a_j)
- Ordering of the slopes.
- Ordering of the intersections.

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Stretchability with k Slopes, k fixed

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Question: Can we realize a pseudoline arragnement $L = \{\ell_1, \dots, \ell_n\}$ with k slopes?

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Stretchability with k Slopes, k fixed

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b_i — unknown offset

a_i — unknown slope

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- Ordering of the intersections.

On pseudoline ℓ_p:
Intersection
with ℓ_q left of ℓ_p. $\rightsquigarrow$ $\frac{b_q - b_p}{a_p - a_q} < \frac{b_r - b_p}{a_p - a_r}$ $\rightsquigarrow$ polynomial inequality

(Because we know the signs of the denominators.)

Stretchability with k Slopes, k fixed

Write system of inequalities (fixing the order of intersections) in matrix form:

$$A \cdot \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} > 0$$

Matrix where each entry
is a linear polynomial
in $a_1, \ldots, a_k$.

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Stretchability with k Slopes, k fixed

Write system of inequalities (fixing the order of intersections) in matrix form:

$$A \cdot \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \cancel{> 0} \geq \varepsilon \cdot 1 \quad \text{for some } \varepsilon > 0$$

Matrix where each entry is a linear polynomial in $a_1, \ldots, a_k$.

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Stretchability with k Slopes, k fixed

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Write system of inequalities (fixing the order of intersections) in matrix form:

$$A \cdot \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \cancel{> 0} \geq \varepsilon \cdot 1 \quad \text{for some } \varepsilon > 0$$

Matrix where each entry is a linear polynomial in $a_1, \ldots, a_k$.

Components of b corresponding to colums of C.

If system has a solution, it has a *basic solution*:
Regular square submatrix C of A, such that $C \cdot b^{-} = \varepsilon \cdot 1$.

Stretchability with k Slopes, k fixed

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$$b_i^- = \frac{\det A_i}{\det A}$$

where A_i is A with the i-th colum replaced by $\varepsilon \cdot 1$.

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$\det A = p_0(a_1, \ldots, a_k, \varepsilon)$ w.l.o.g. positive at solution
$\det A_i = p_i(a_1, \ldots, a_k, \varepsilon)$

Stretchability with k Slopes, k fixed

Summarizing:

$A \cdot b > 0$ solvable $\Leftrightarrow$ $\exists$ polynomials $p_0(a_1, \ldots, a_k, \varepsilon), p_1(a_1, \ldots, a_k, \varepsilon) \ldots, p_n(a_1, \ldots, a_k, \varepsilon)$

with $\{p_0, p_1, \ldots, p_n\}$ bounded by a fixed polynomial in n

and

real numbers $\varepsilon > 0$, $a_1, \ldots, a_k$ such that

- $p_0(a_1, \ldots, a_k, \varepsilon) > 0$
- b with $b_i = \frac{p_i(a_1,\ldots,a_k,\varepsilon)}{p_0(a_1,\ldots,a_k,\varepsilon)}$ is a solution

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Stretchability with k Slopes, k fixed

Summarizing:

$$A \cdot b > 0 \text{ solvable} \Leftrightarrow \underbrace{\begin{array}{l} \exists \text{ polynomials } p_0(a_1, \dots, a_k, \varepsilon), p_1(a_1, \dots, a_k, \varepsilon) \dots, p_n(a_1, \dots, a_k, \varepsilon) \\ \quad \text{with } \{p_0, p_1, \dots, p_n\} \text{ bounded by a fixed polynomial in } n \\ \quad \text{and} \\ \quad \text{real numbers } \varepsilon > 0,\ a_1, \dots, a_k \text{ such that} \\ \quad\quad - \ p_0(a_1, \dots, a_k, \varepsilon) > 0 \\ \quad\quad - \ b \text{ with } b_i = \frac{p_i(a_1,\dots,a_k,\varepsilon)}{p_0(a_1,\dots,a_k,\varepsilon)} \text{ is a solution} \end{array}}_{\text{guess non-deterministicly}}$$

$\rightsquigarrow$

Now $A \cdot b > 0$ is a system of polynomial inequalities with a fixed number of variables. Can be solved in polynomial time.

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