Computational Geometry Lecture

Voronoi Diagrams and Delaunay Triangulations

INSTITUT FÜR THEORETISCHE INFORMATIK · FAKULTÄT FÜR INFORMATIK

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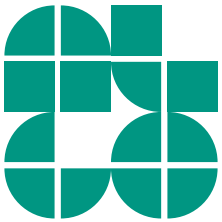


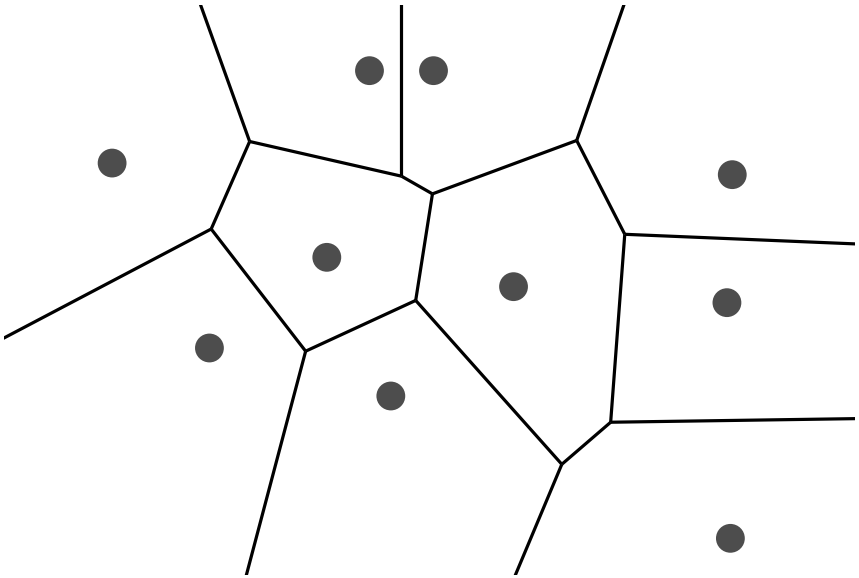
Two Mathematicians

Georgy Voronoi
(1868–1908)

Boris Delone
(1890–1980)

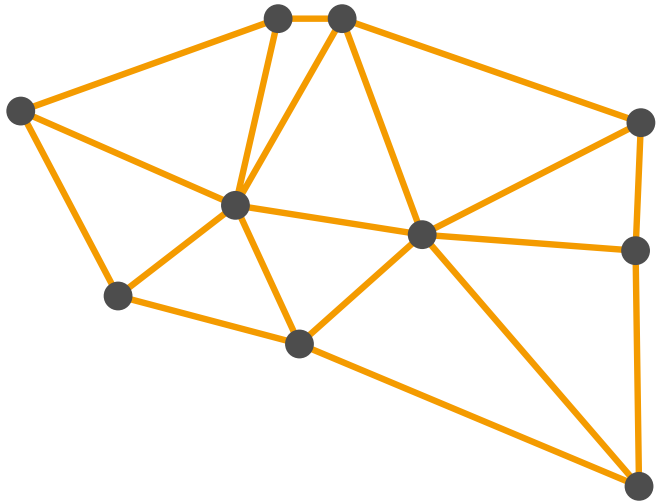
Two Mathematicians

Voronoi Diagram



Georgy Voronoi
(1868–1908)

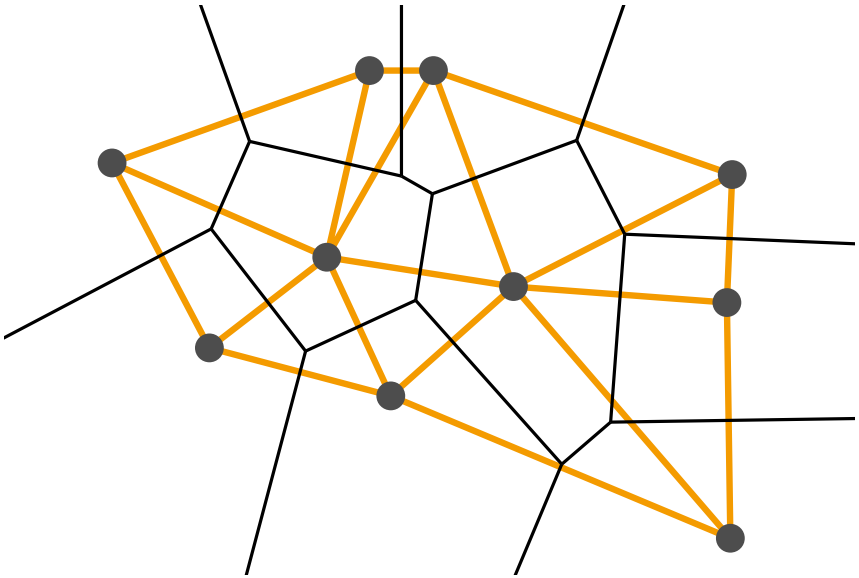
Delaunay Triangulation



Boris Delone
(1890–1980)

Two Mathematicians

Voronoi Diagram $\Leftrightarrow$ Delaunay Triangulation



Georgy Voronoi
(1868–1908)

Boris Delone
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Outline

1. Voronoi Diagrams and Delaunay Triangulations
   - Properties
   - Duality
2. Algorithms
   - Plane Sweep
   - Divide and Conquer
3. 3D Geometric Transformation

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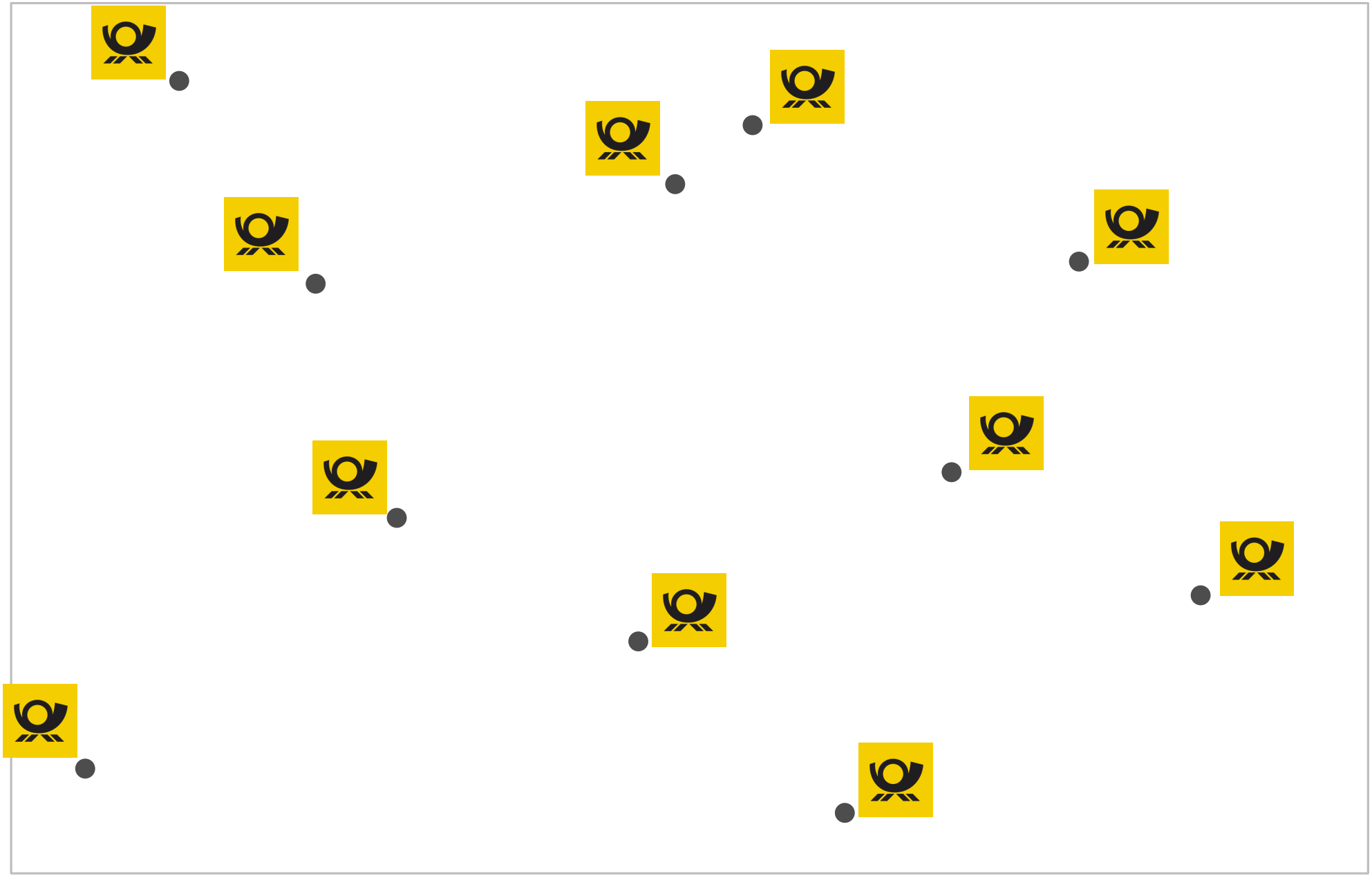
P: a set of n points in the plane

The Post Office Problem



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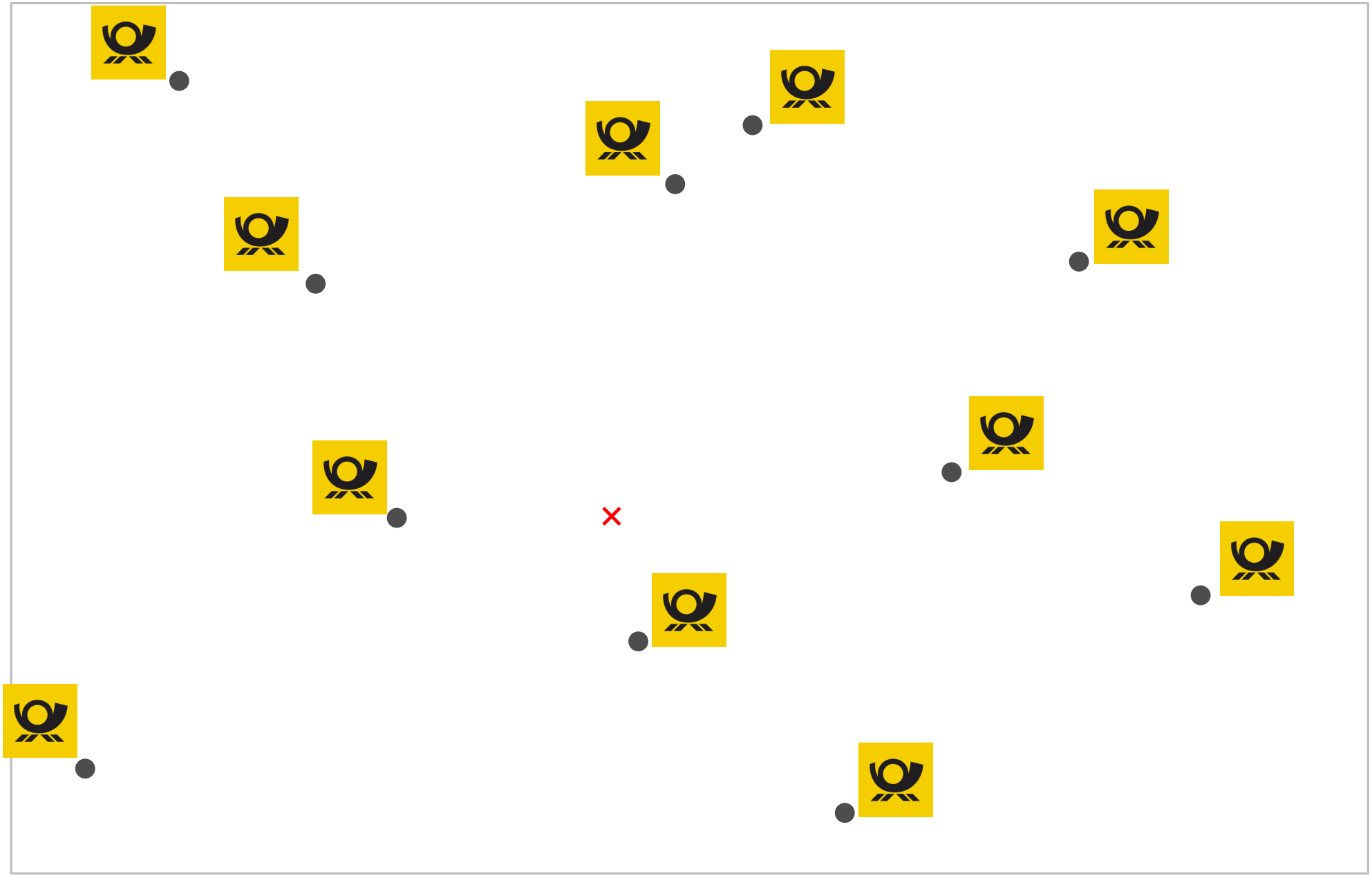

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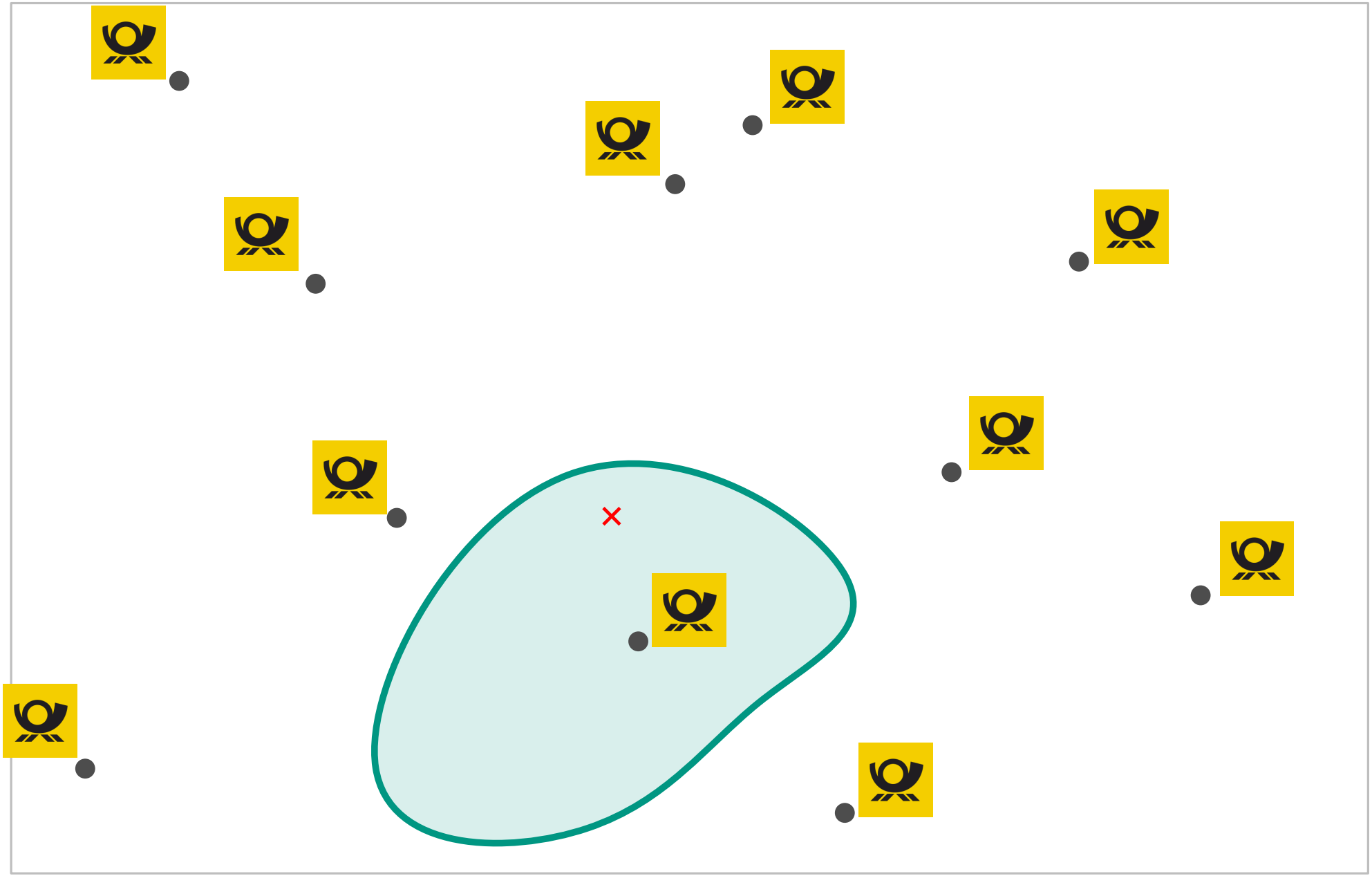

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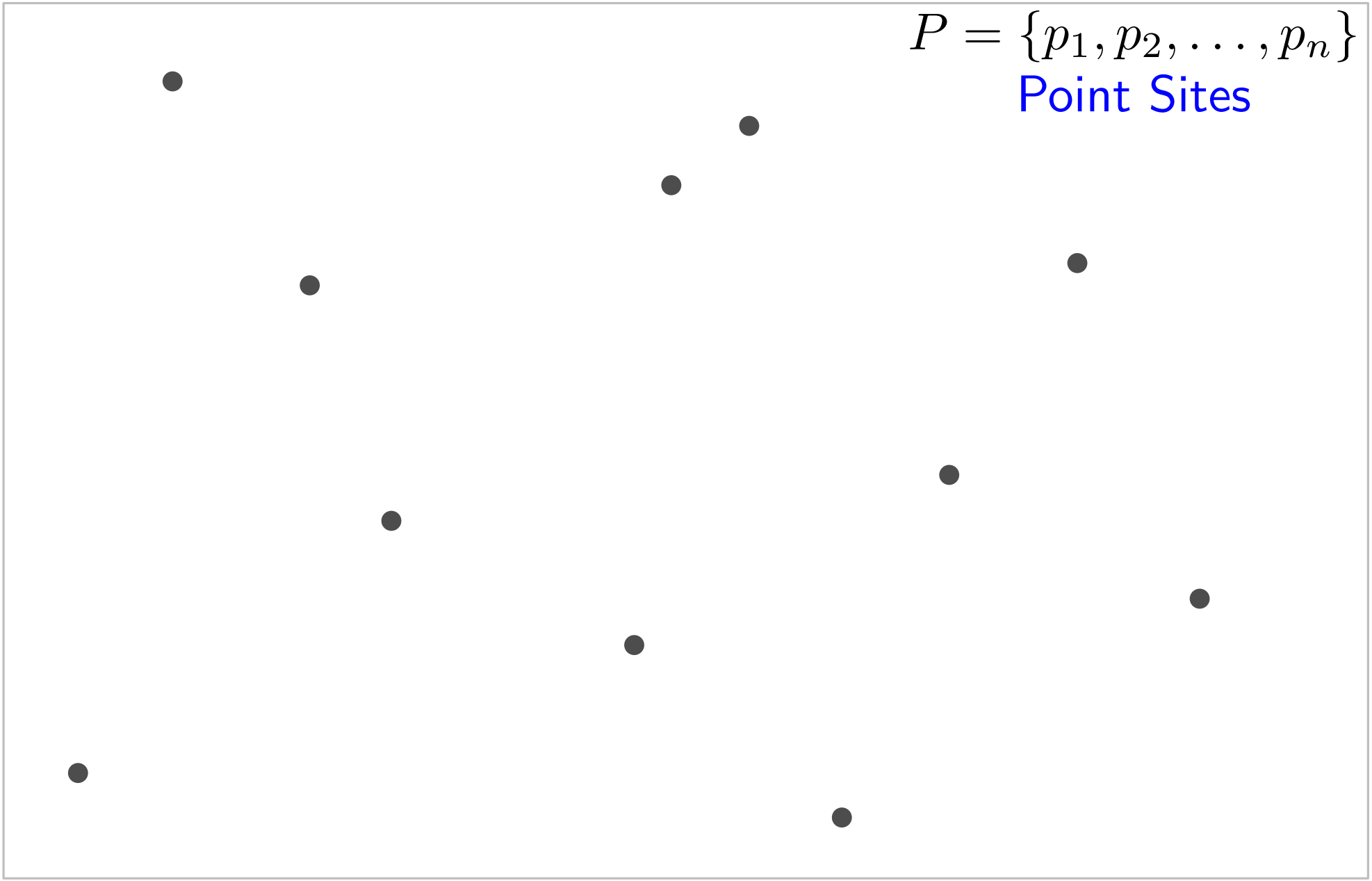

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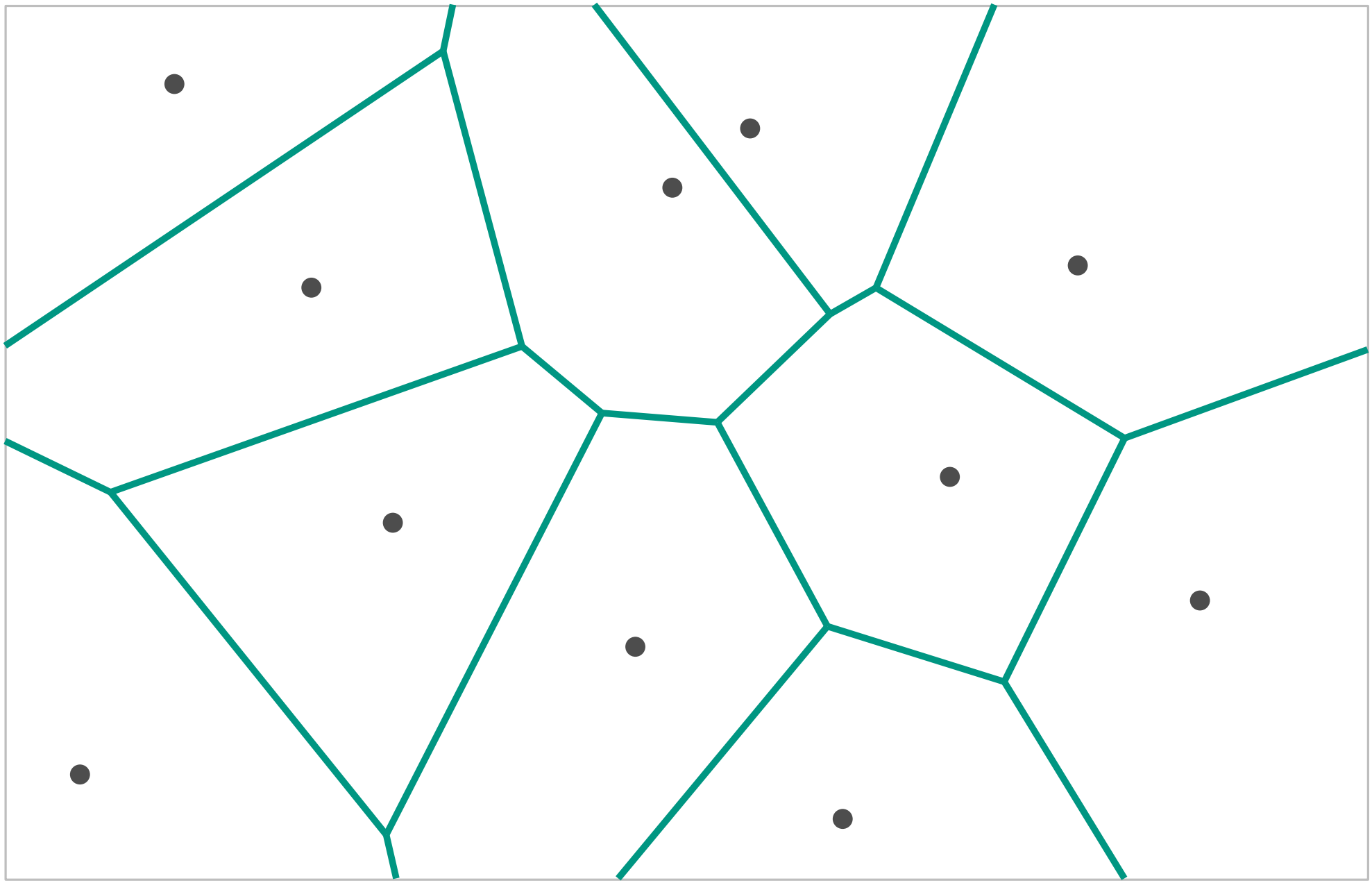

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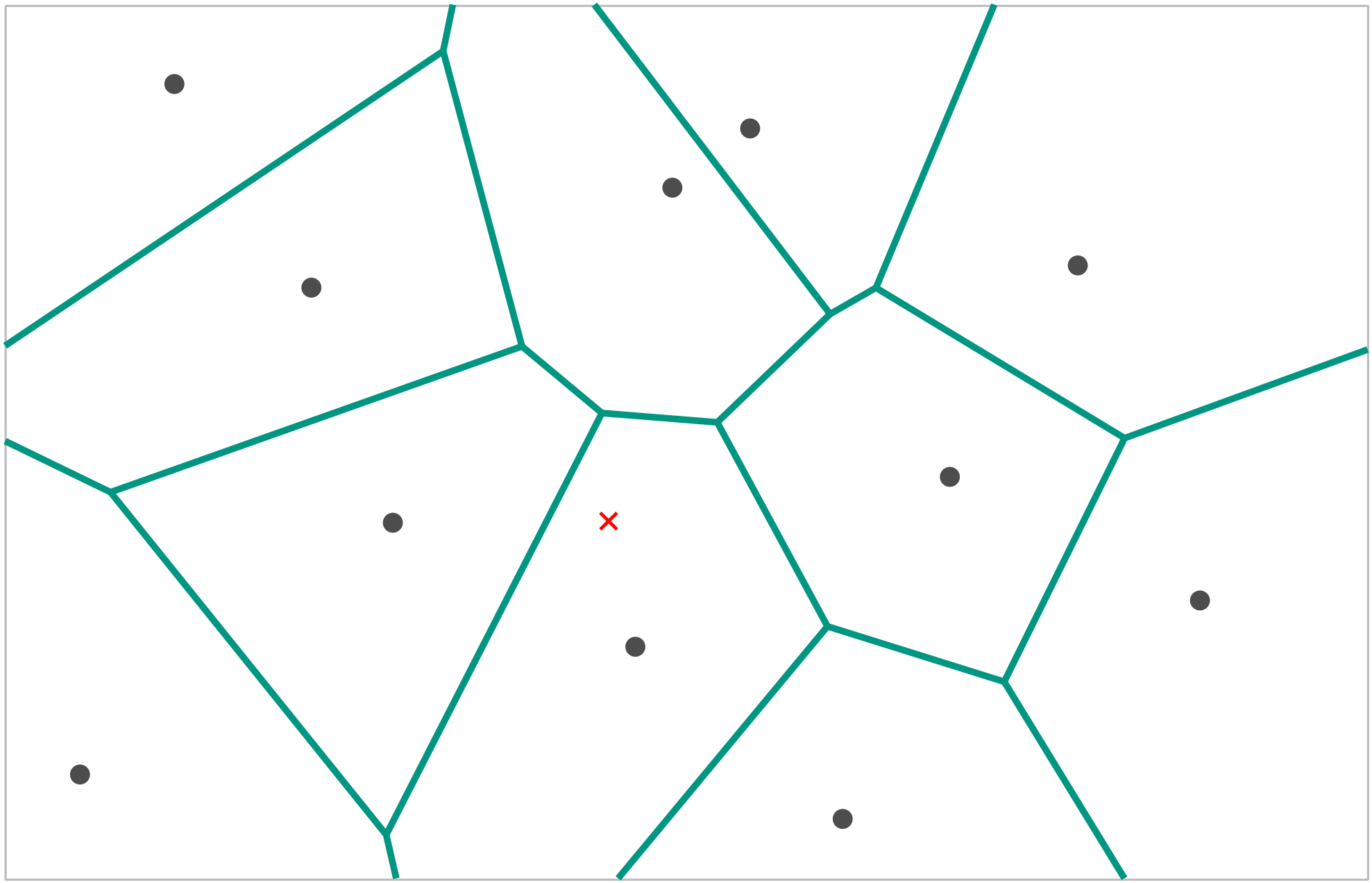

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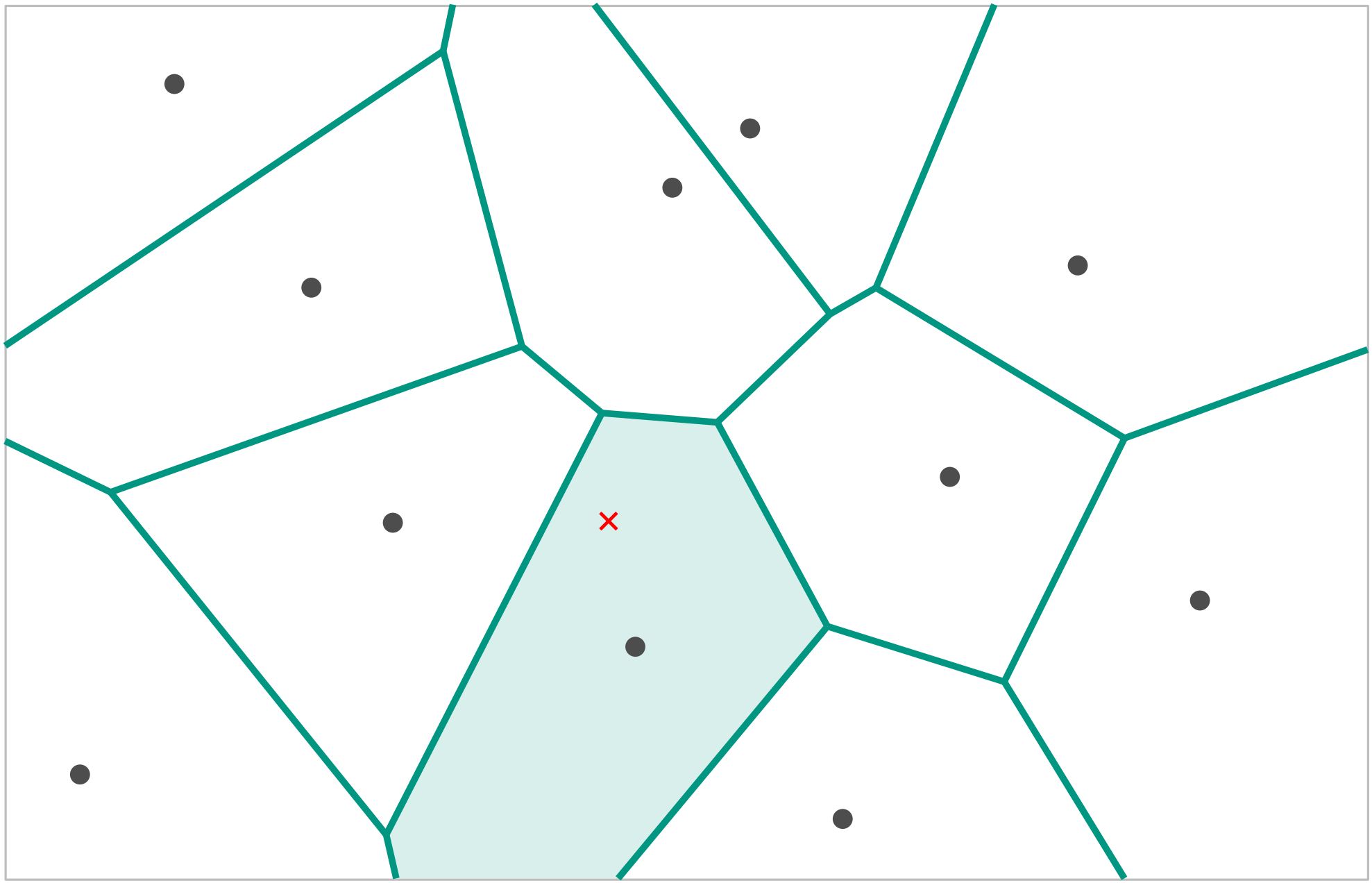

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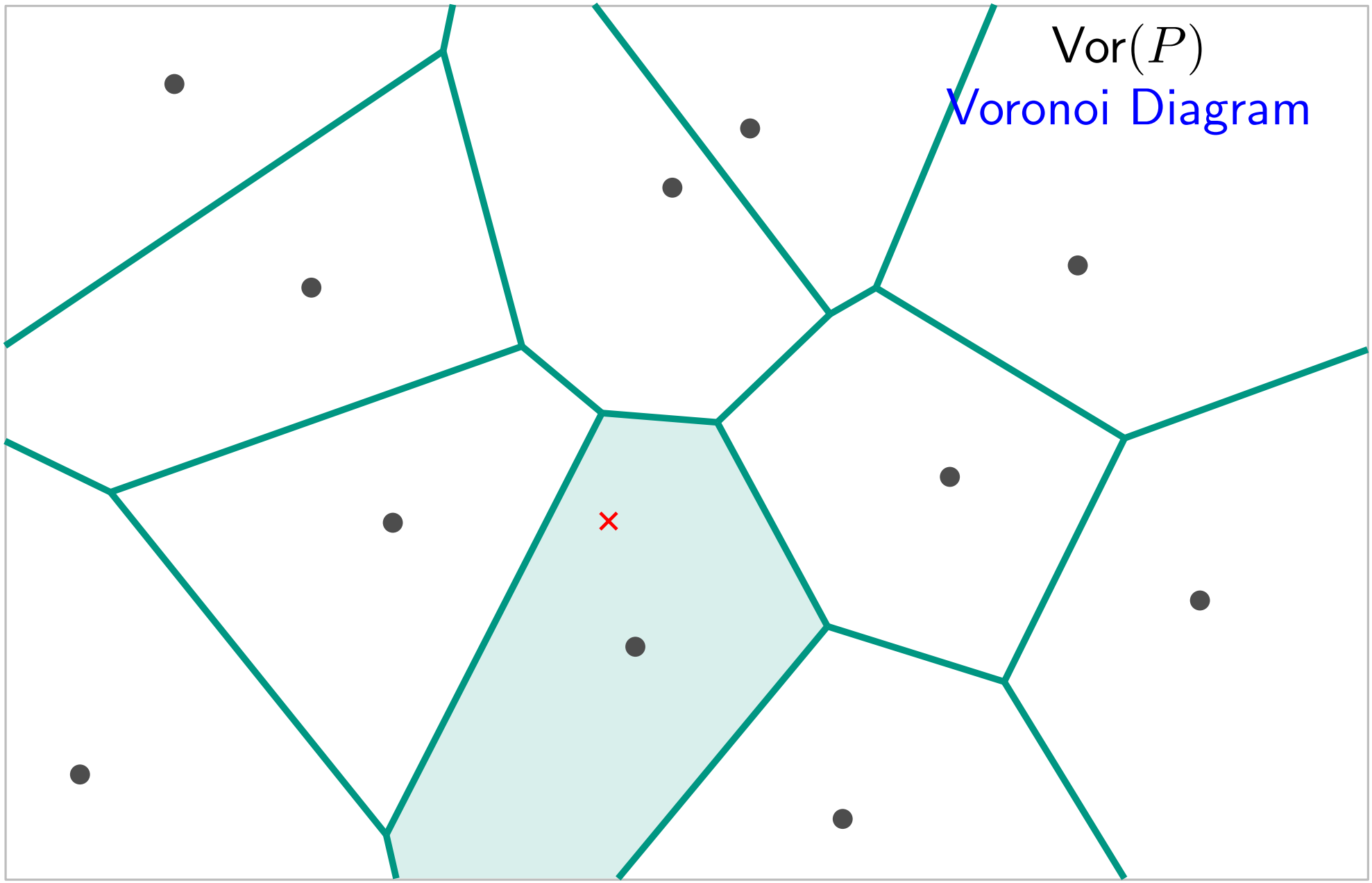

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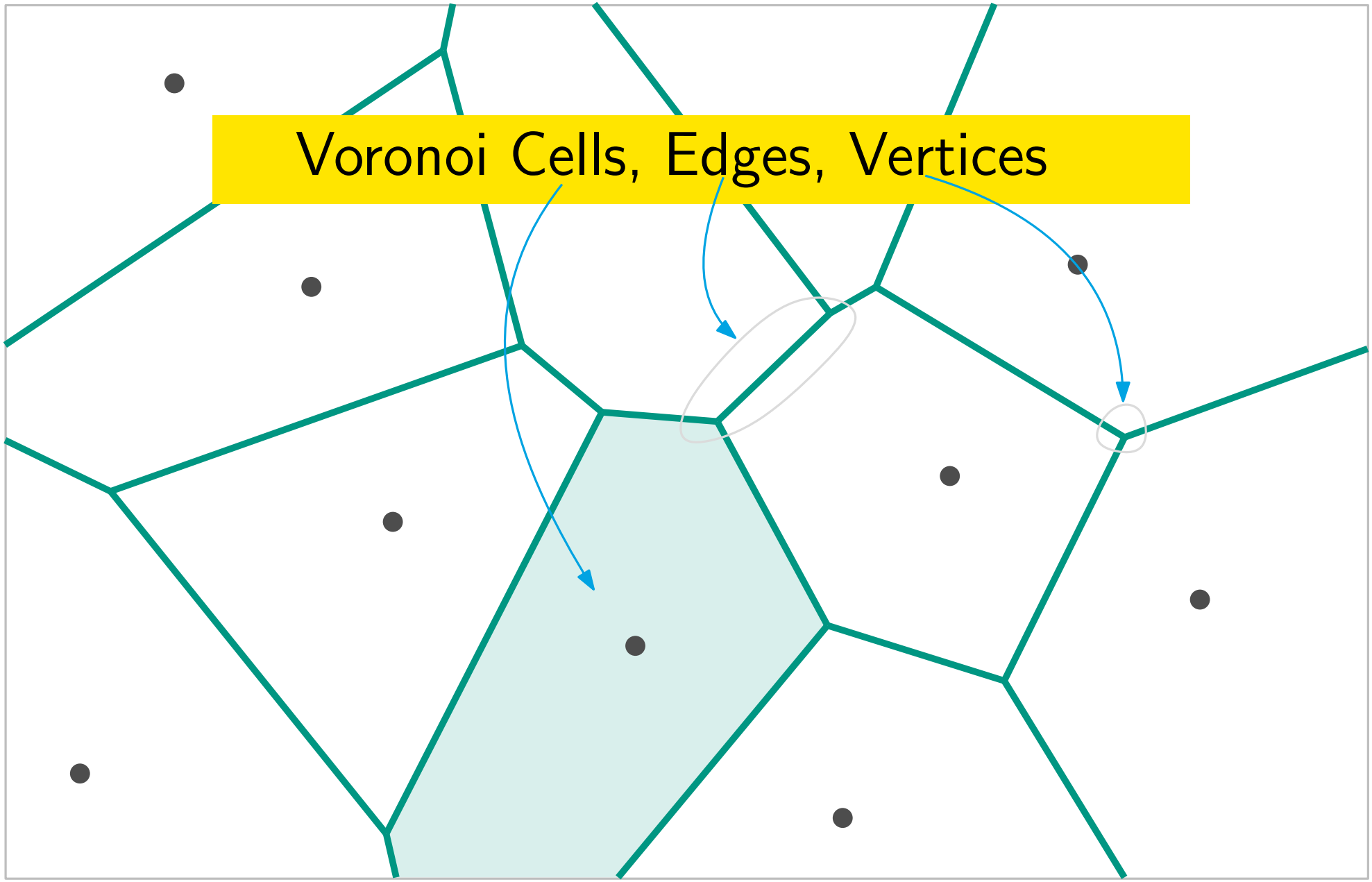
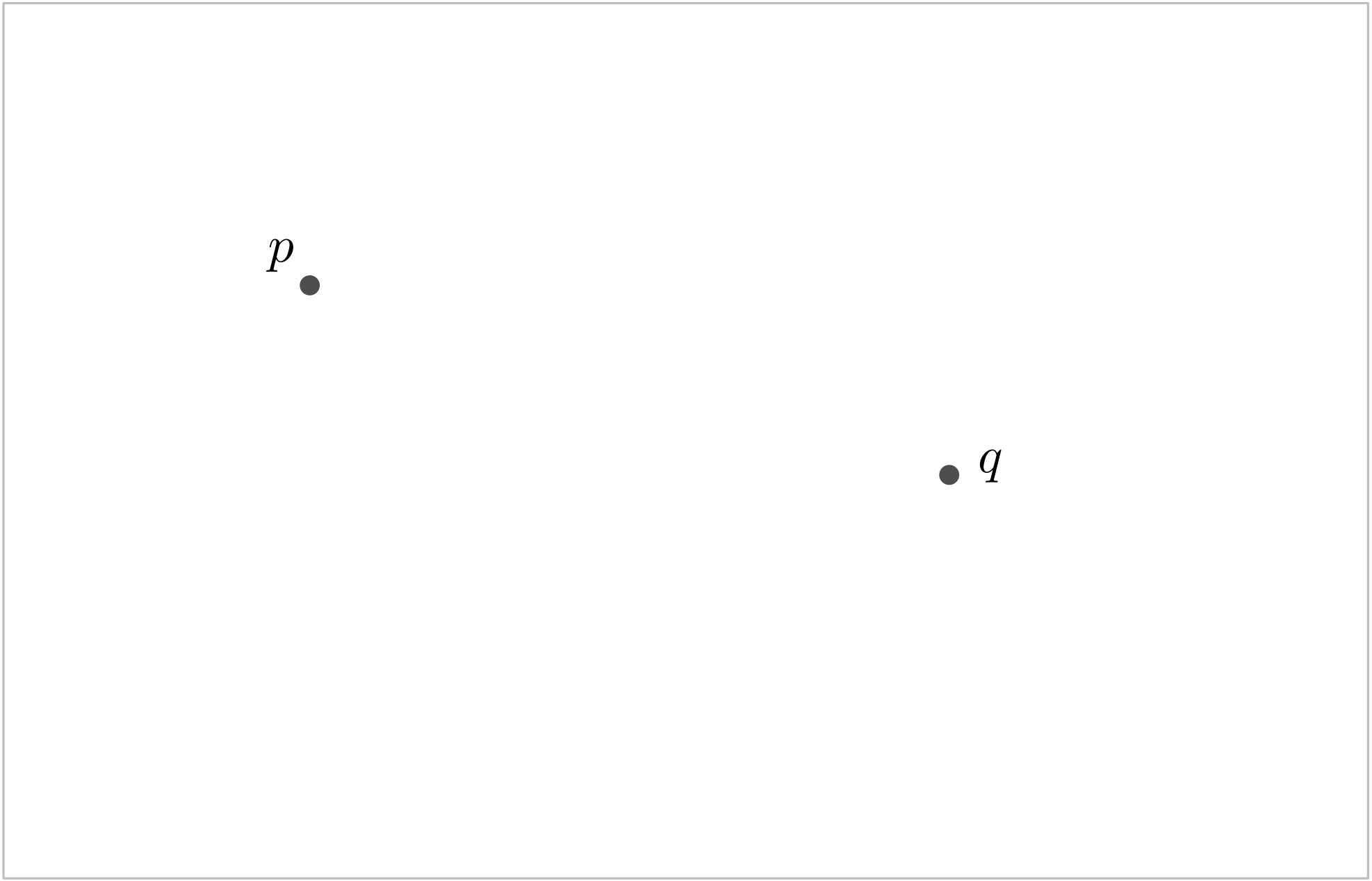


Bisector and Half-Plane



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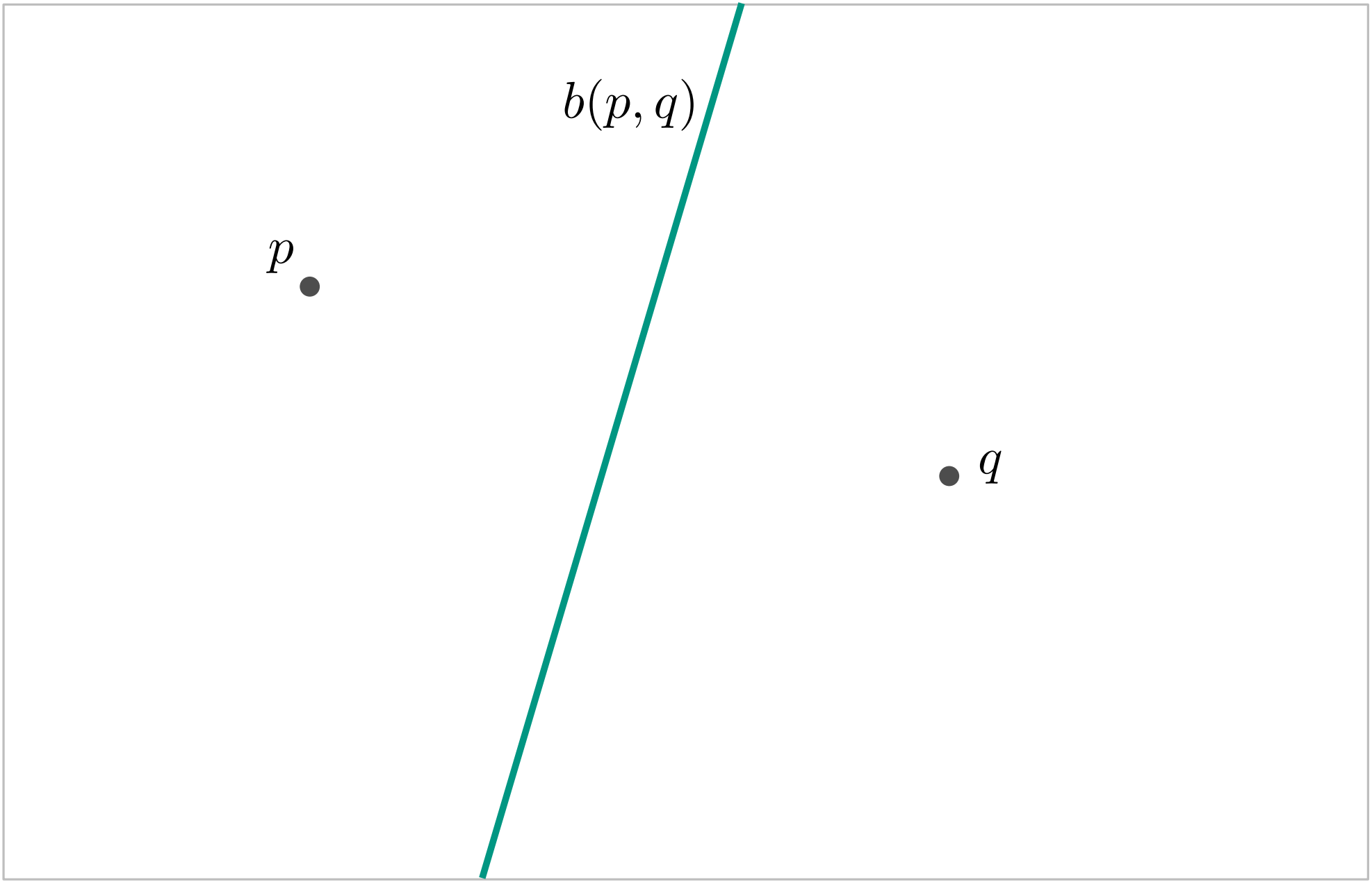

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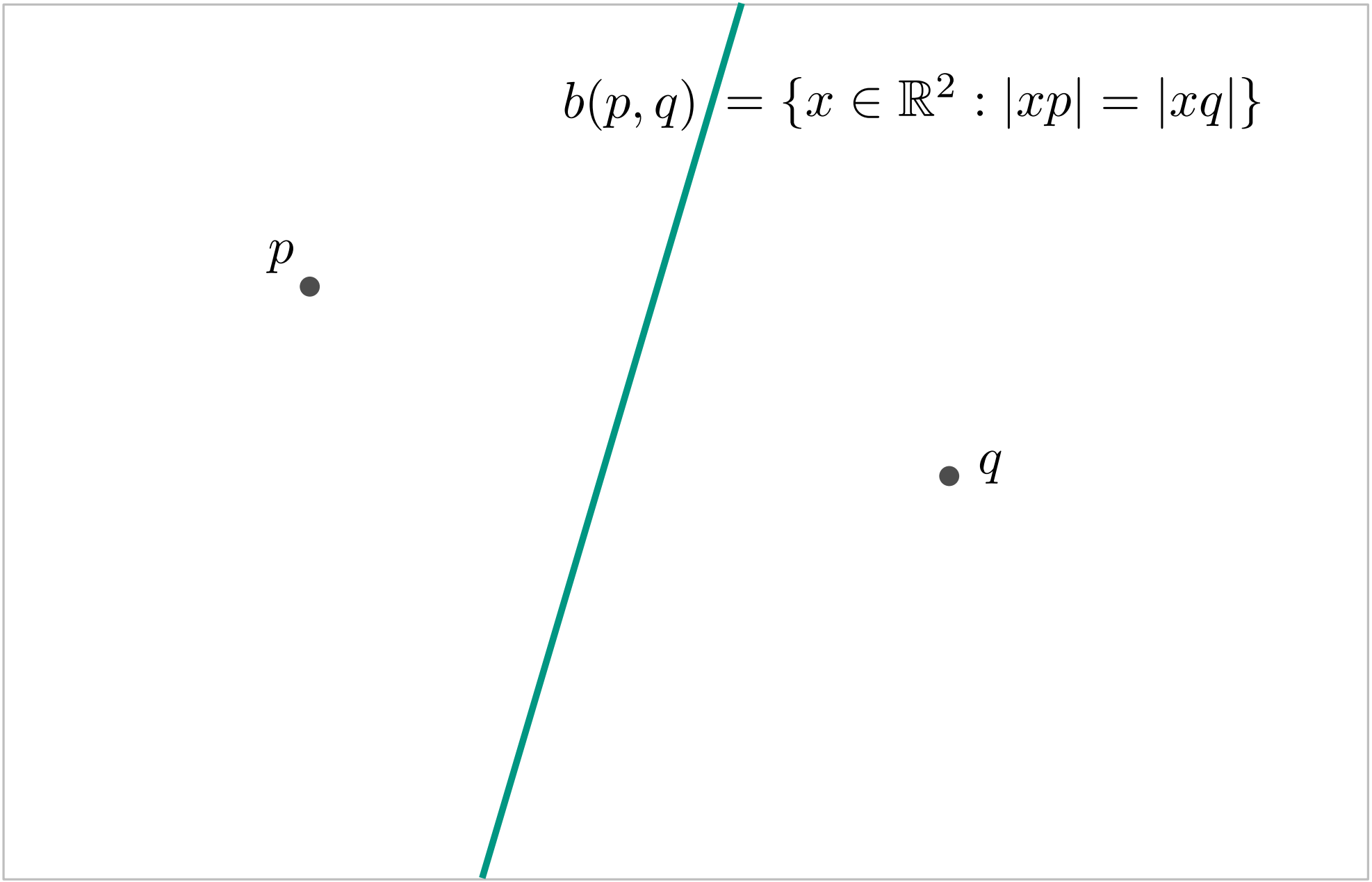
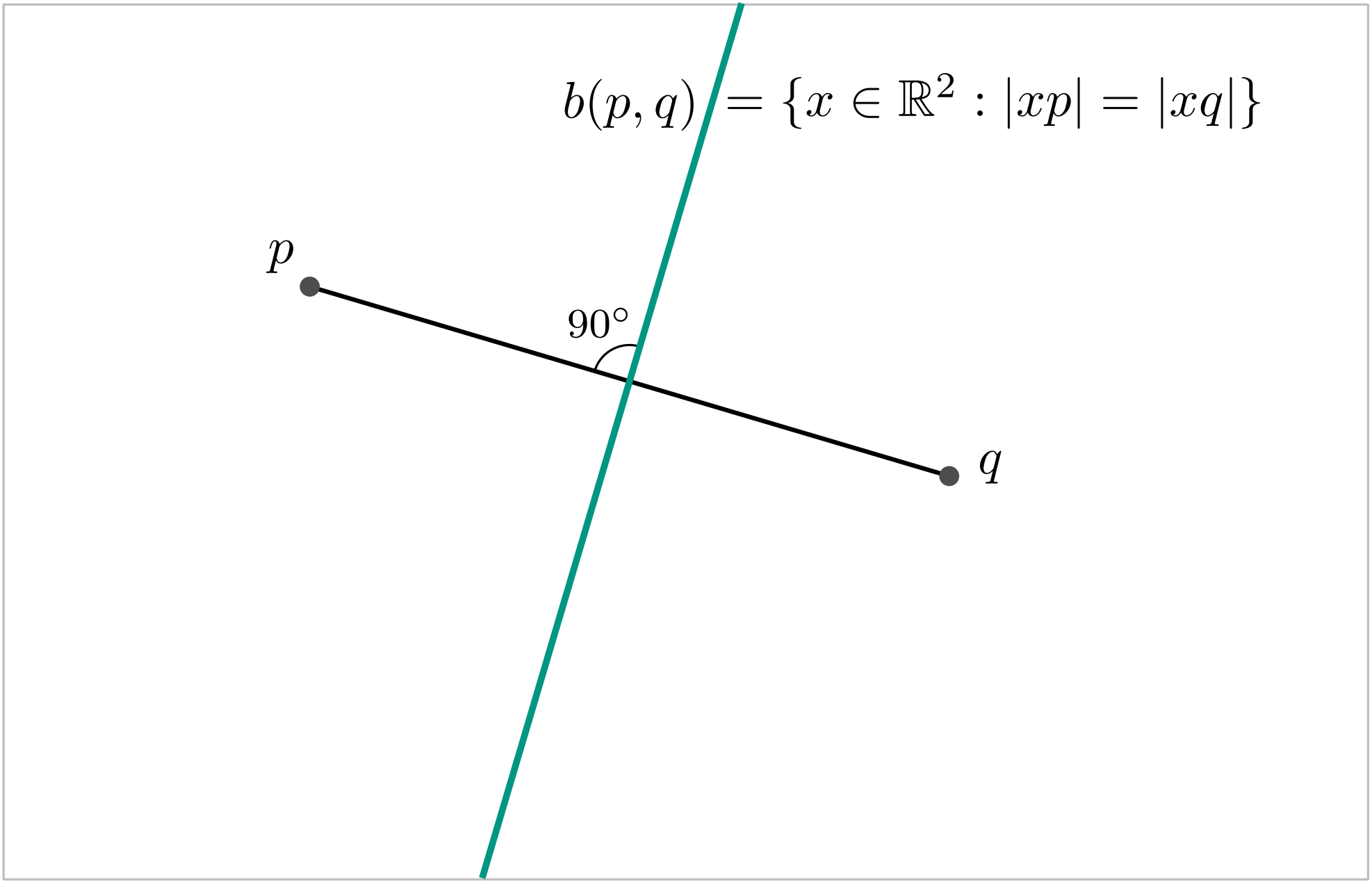


Bisector and Half-Plane



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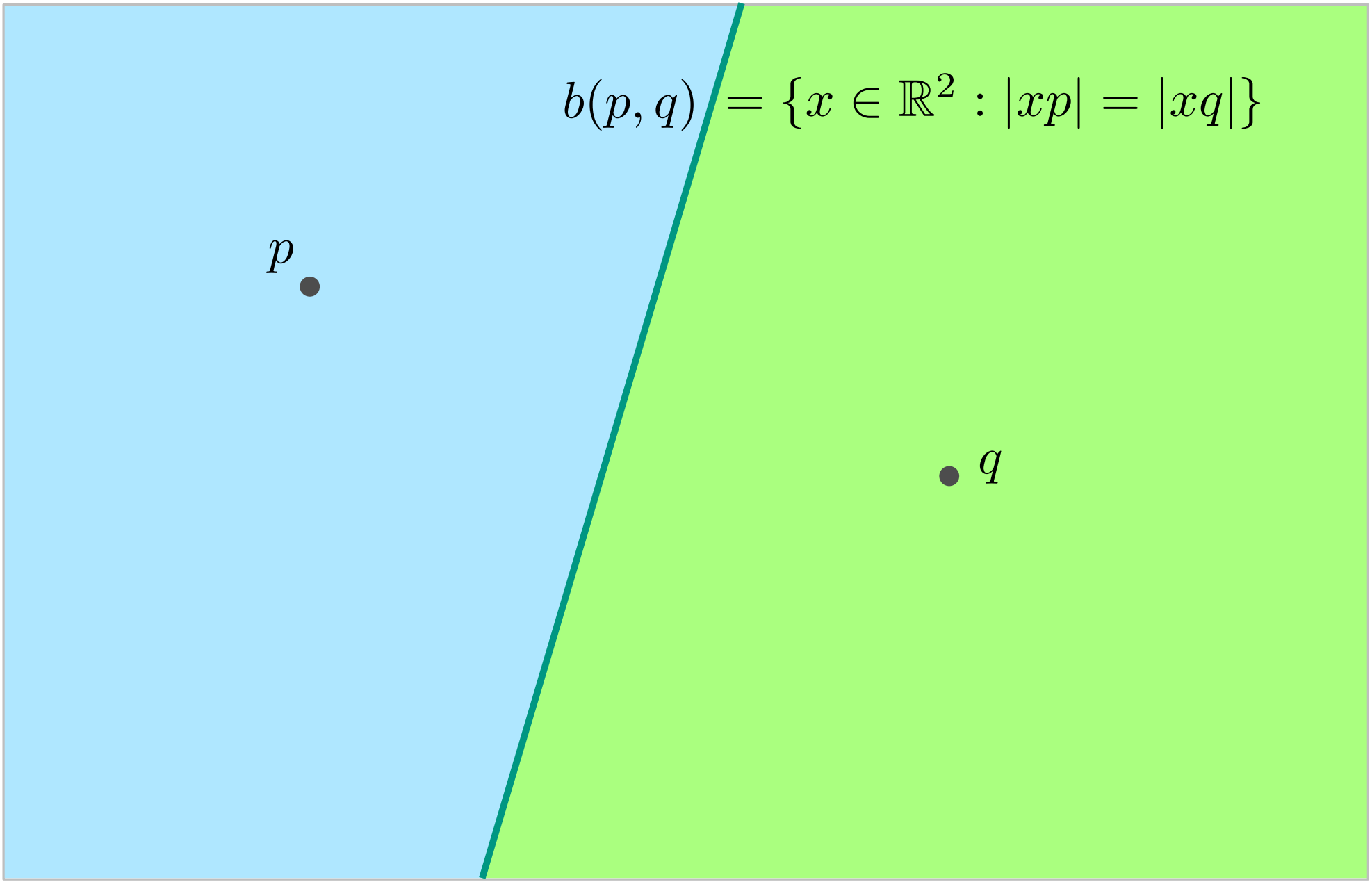
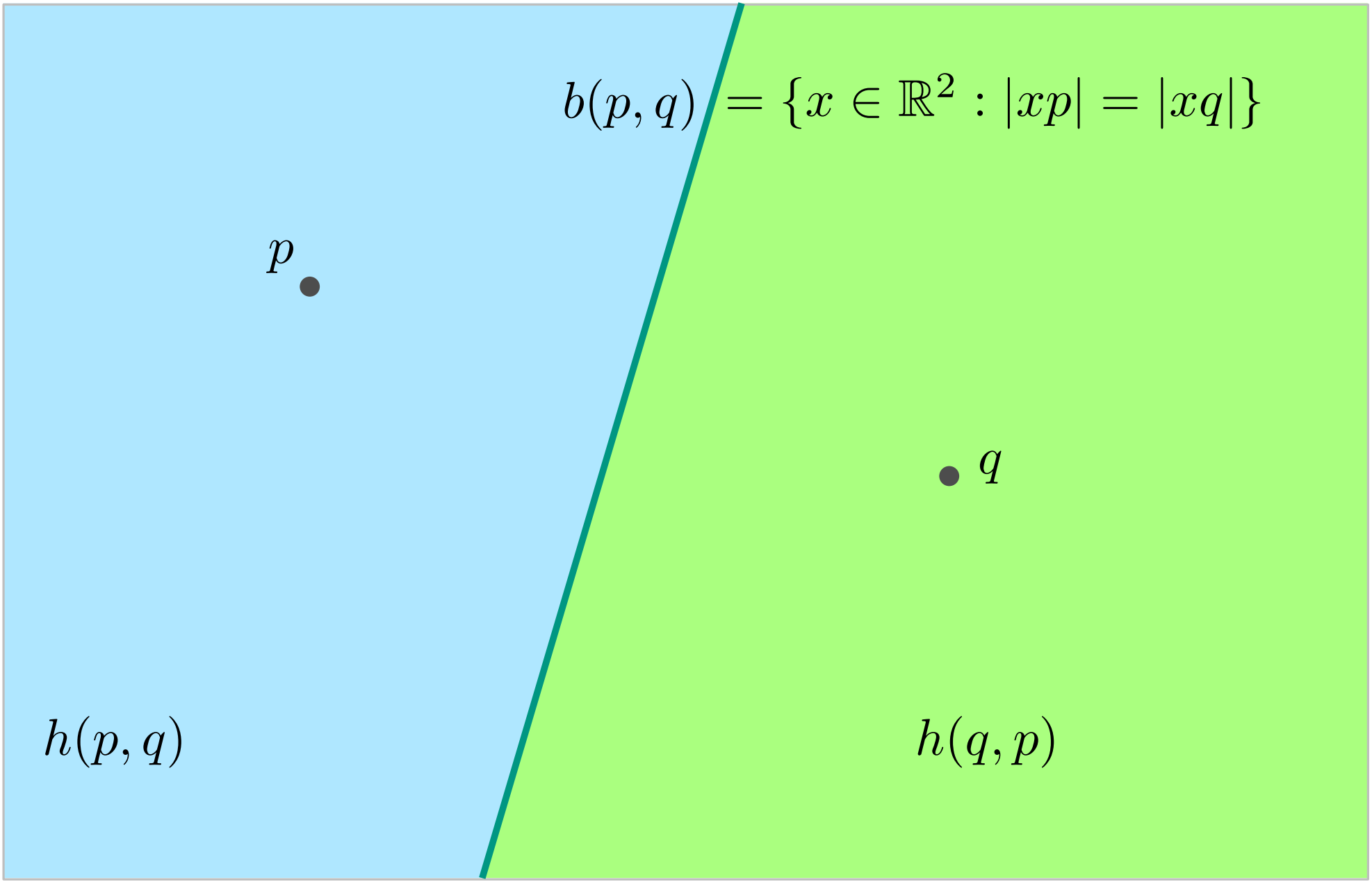
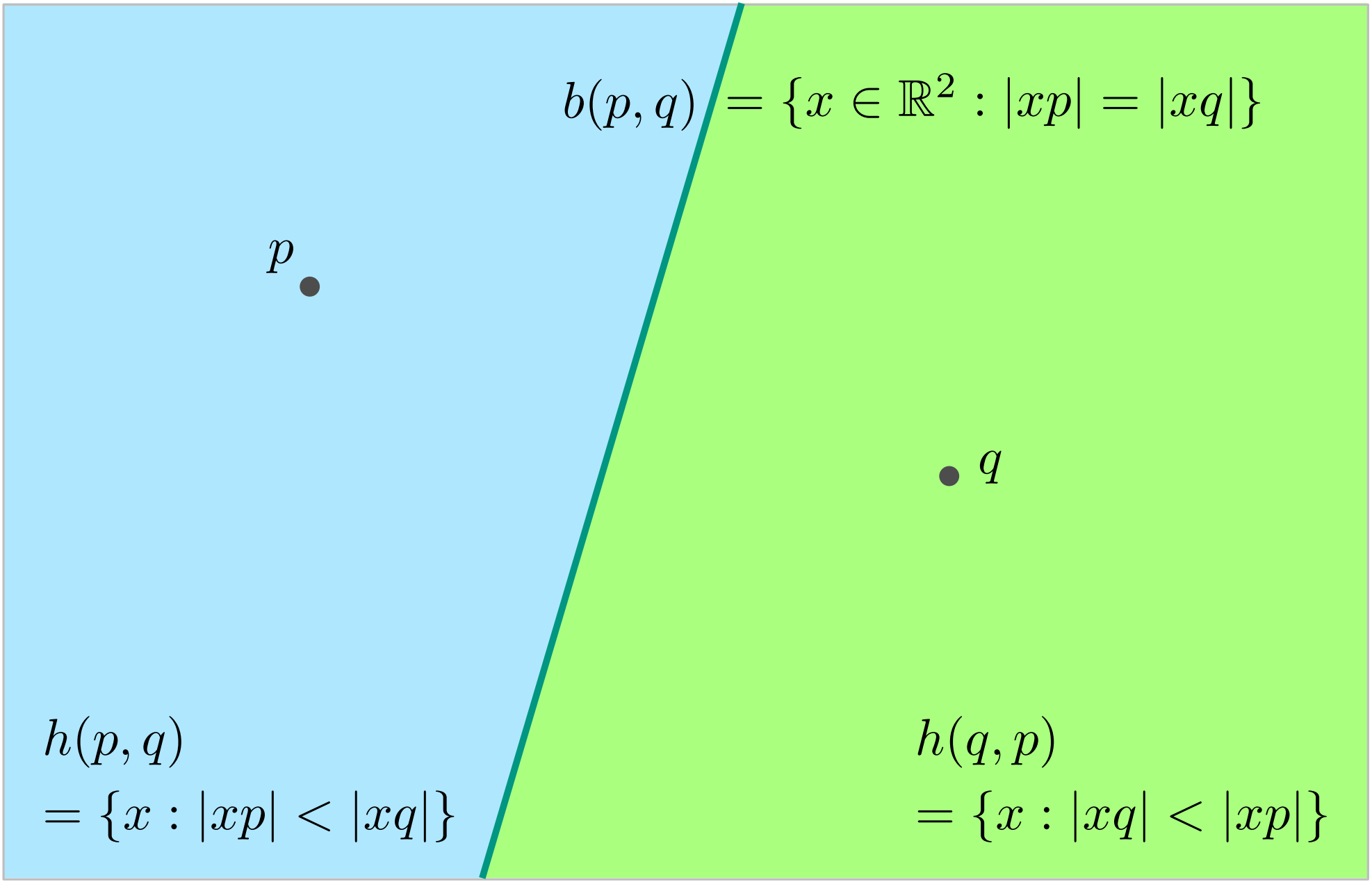


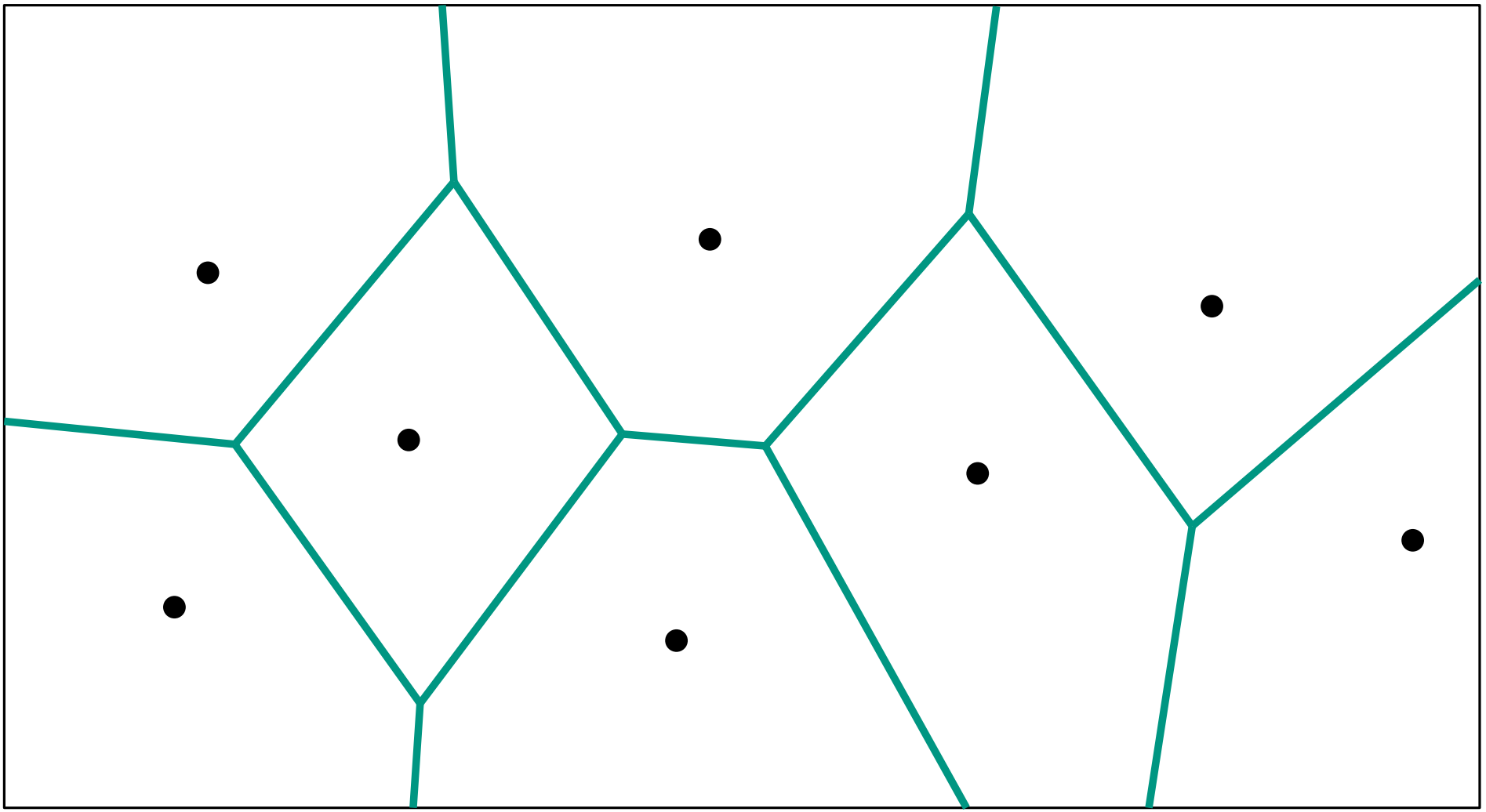
Bisector and Half-Plane



Bisector and Half-Plane



Voronoi Cell



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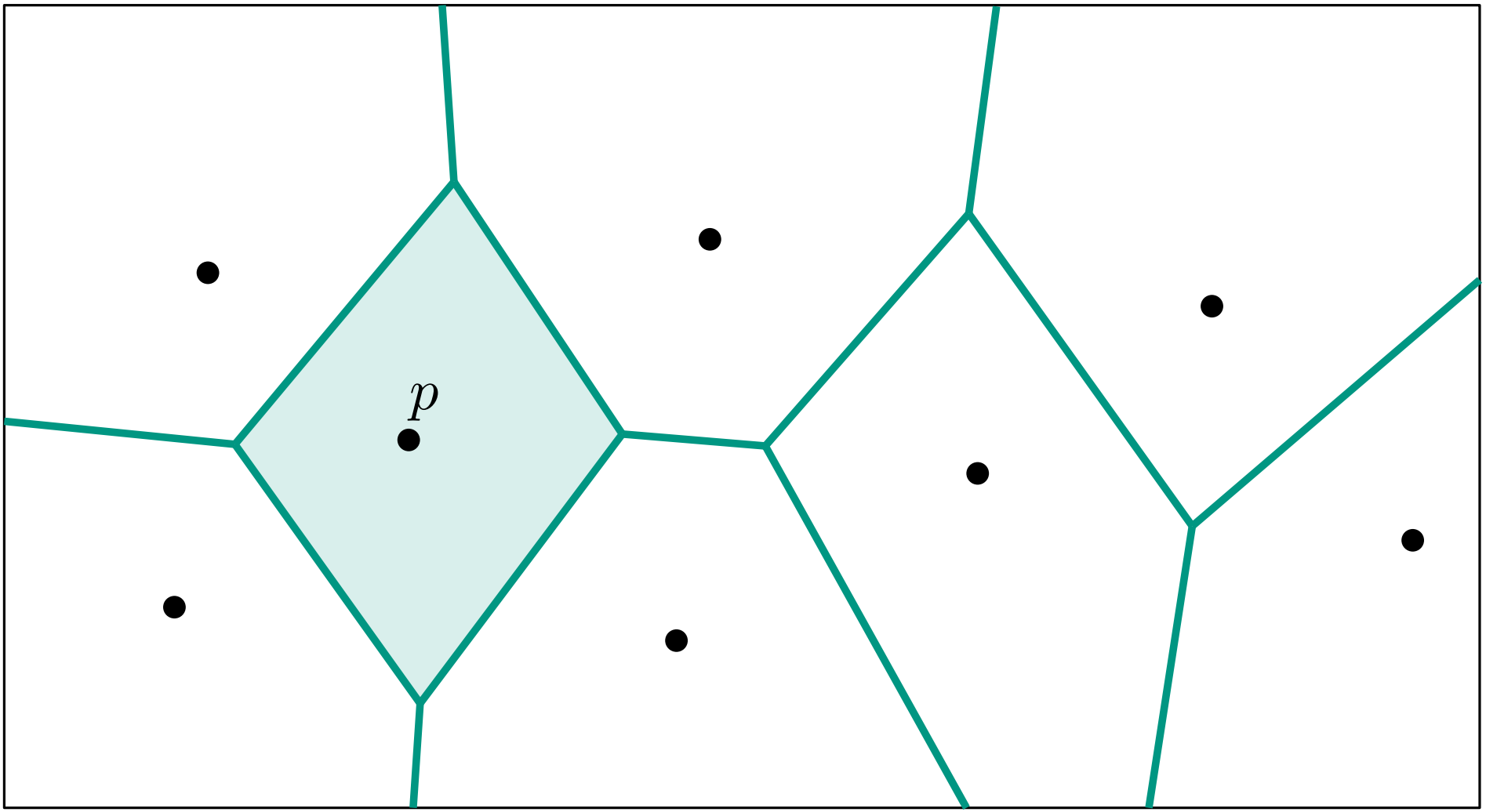
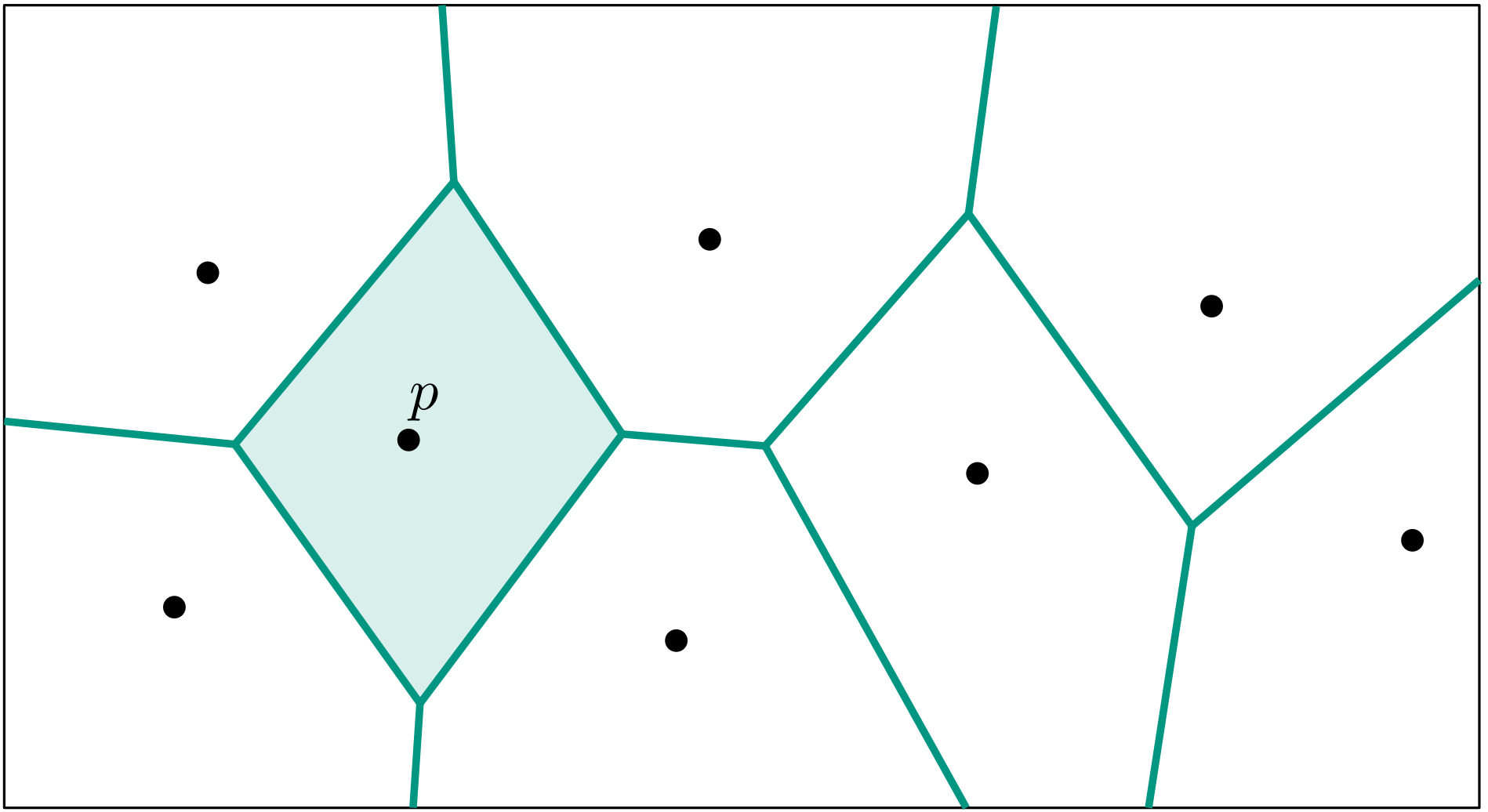


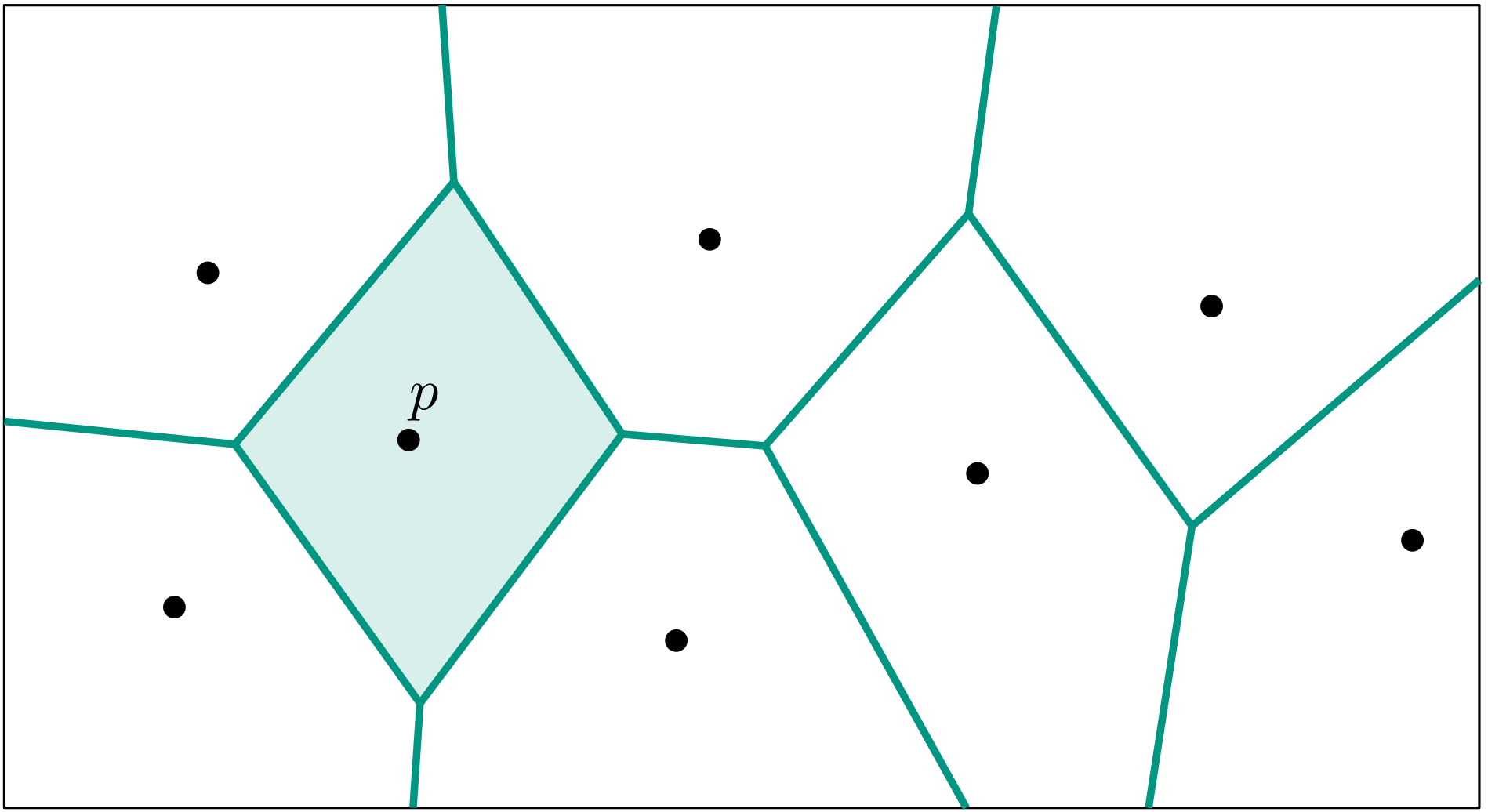
Voronoi Cell

$\mathcal{V}(p)$



Voronoi Cell

$$\mathcal{V}(p) = \{x \in \mathbb{R}^2 : |xp| < |xq|, \forall q \in P \setminus \{p\}\}$$



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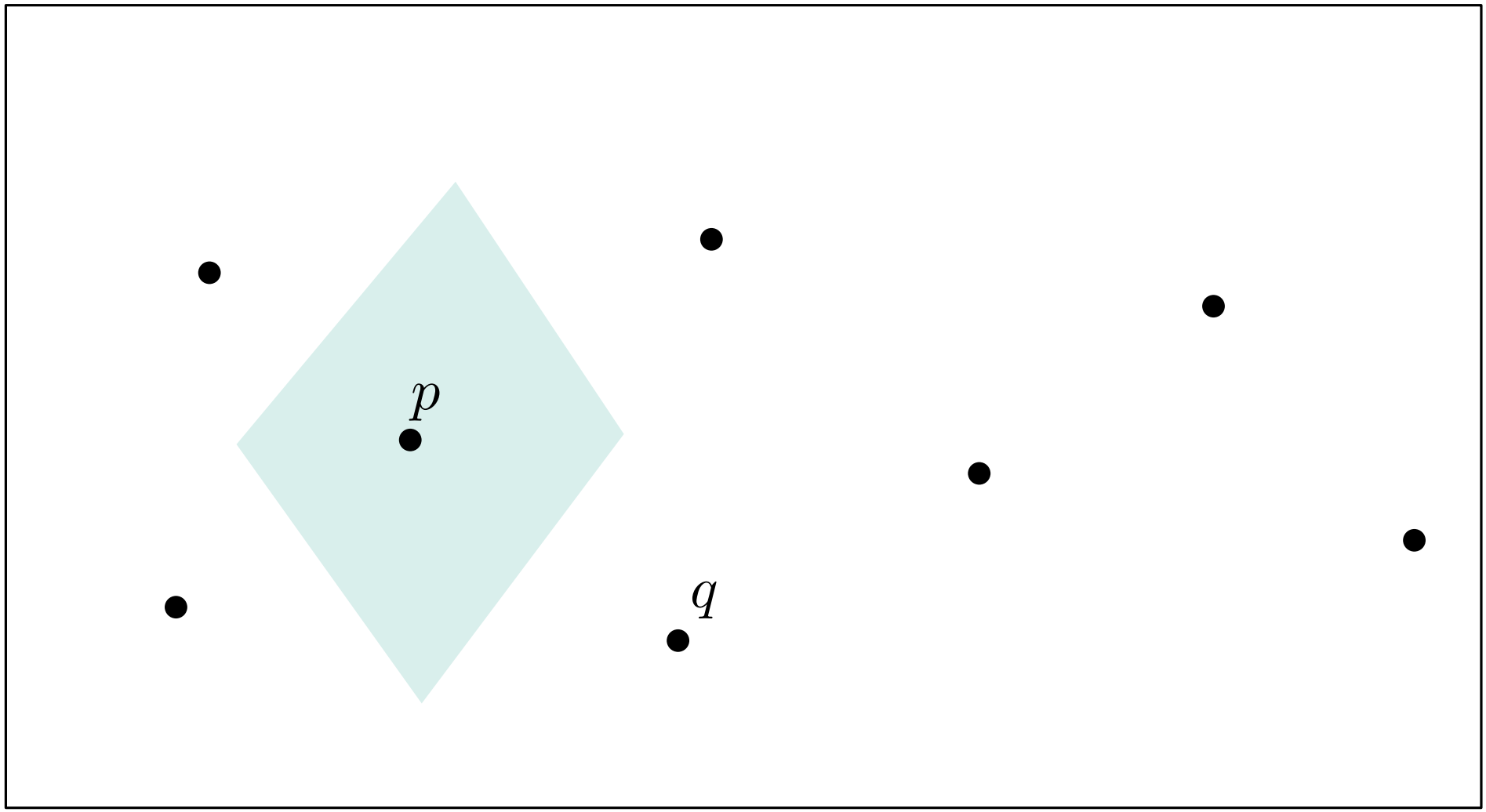

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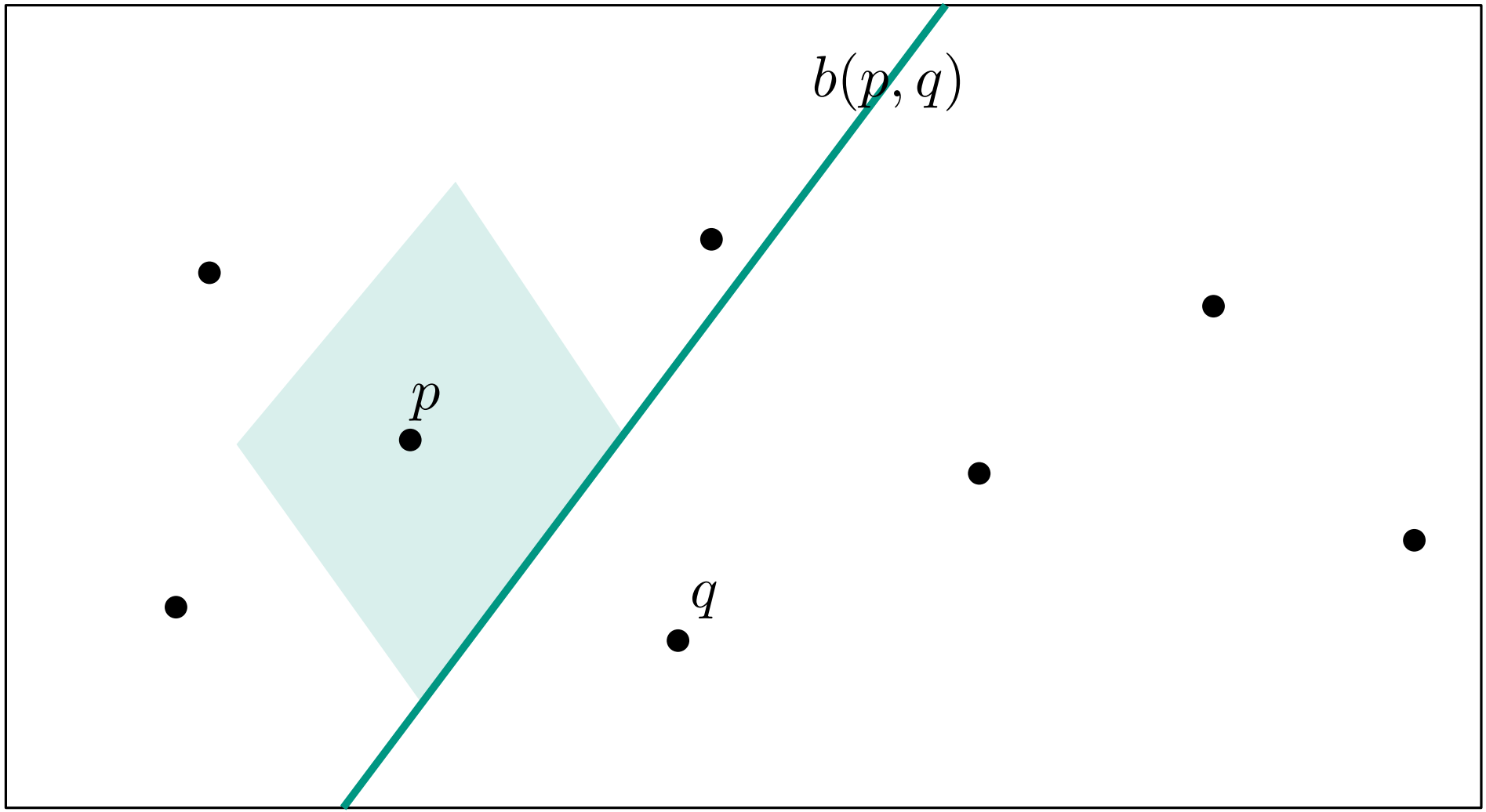

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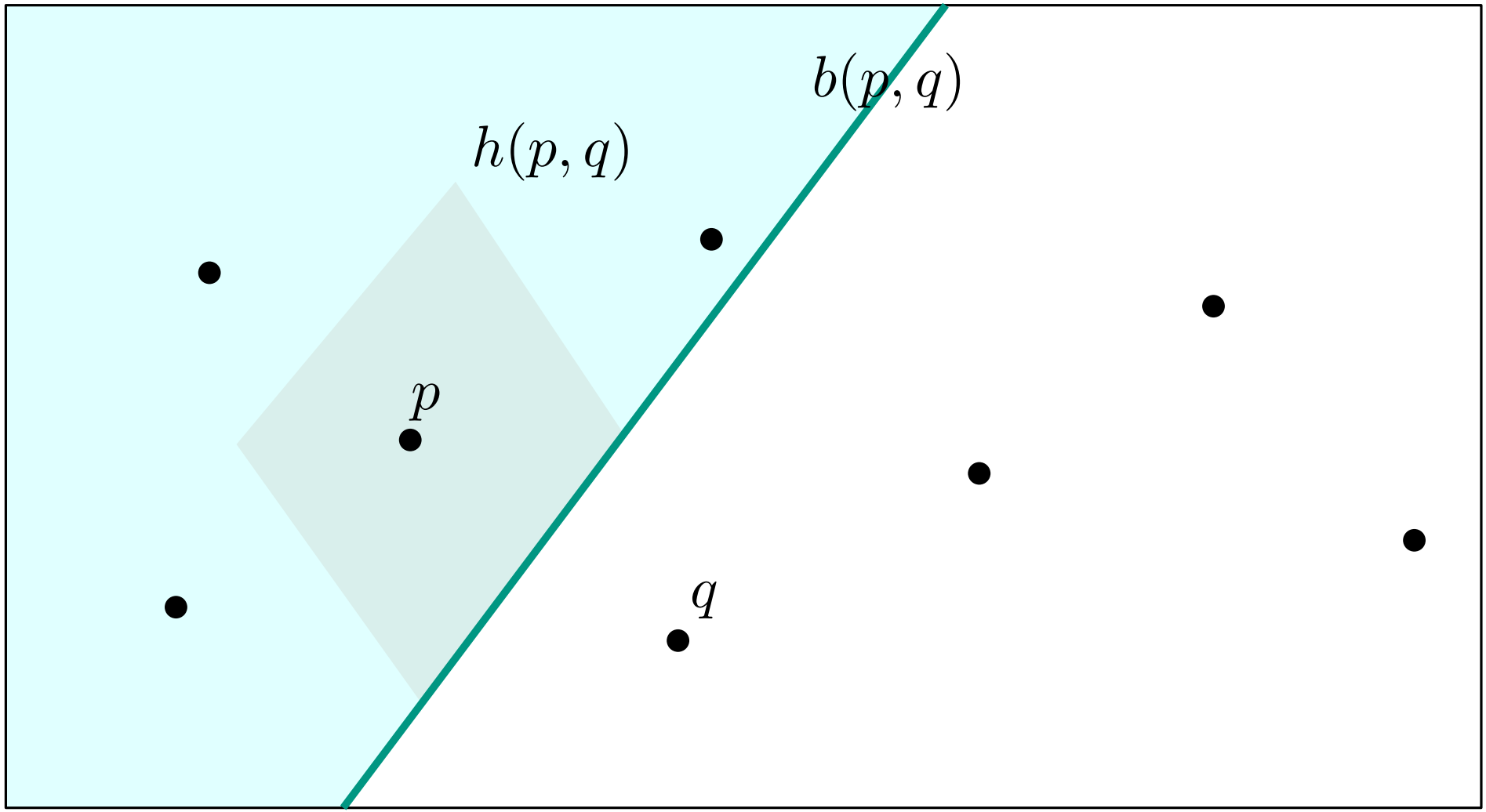

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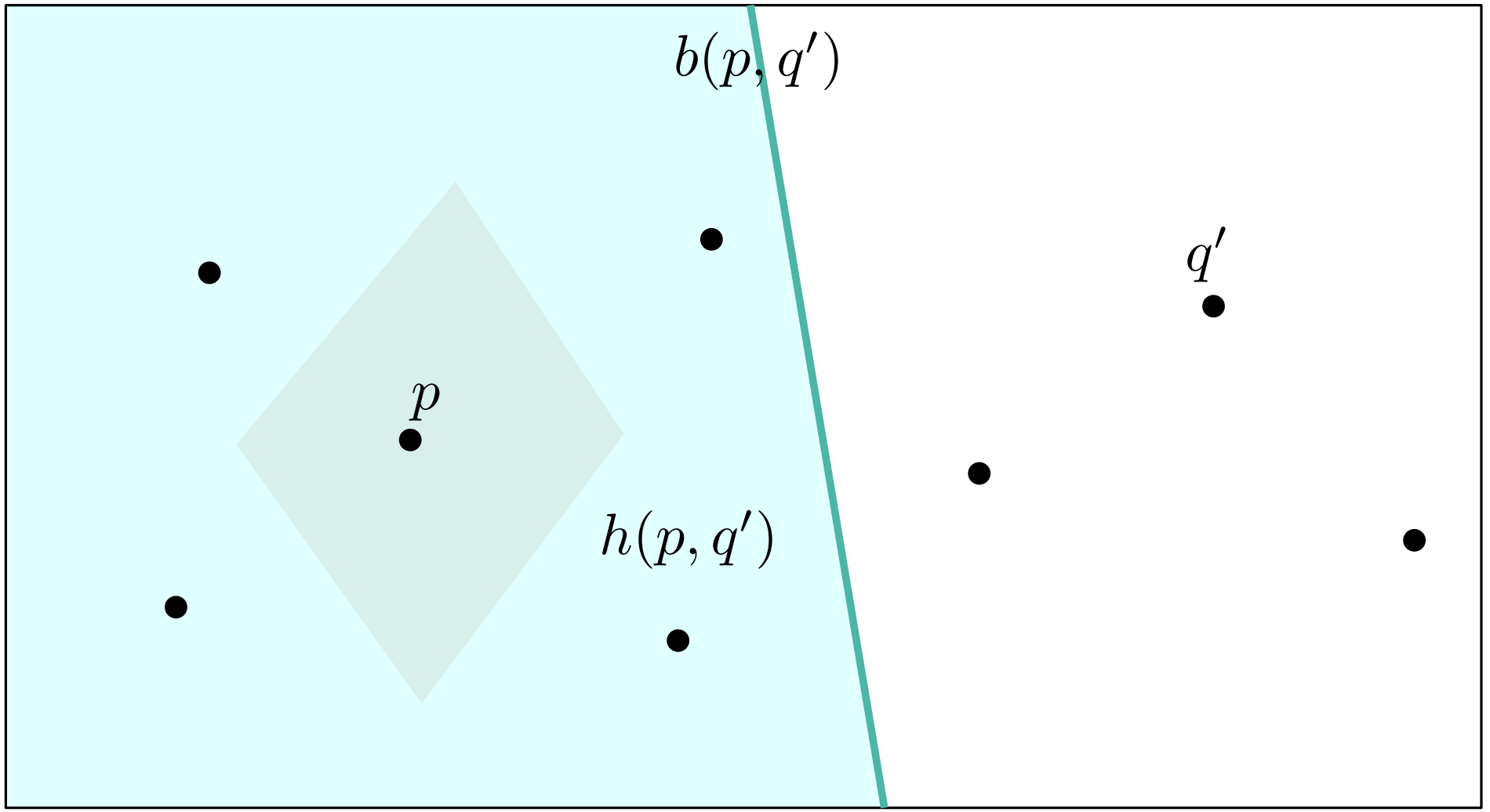
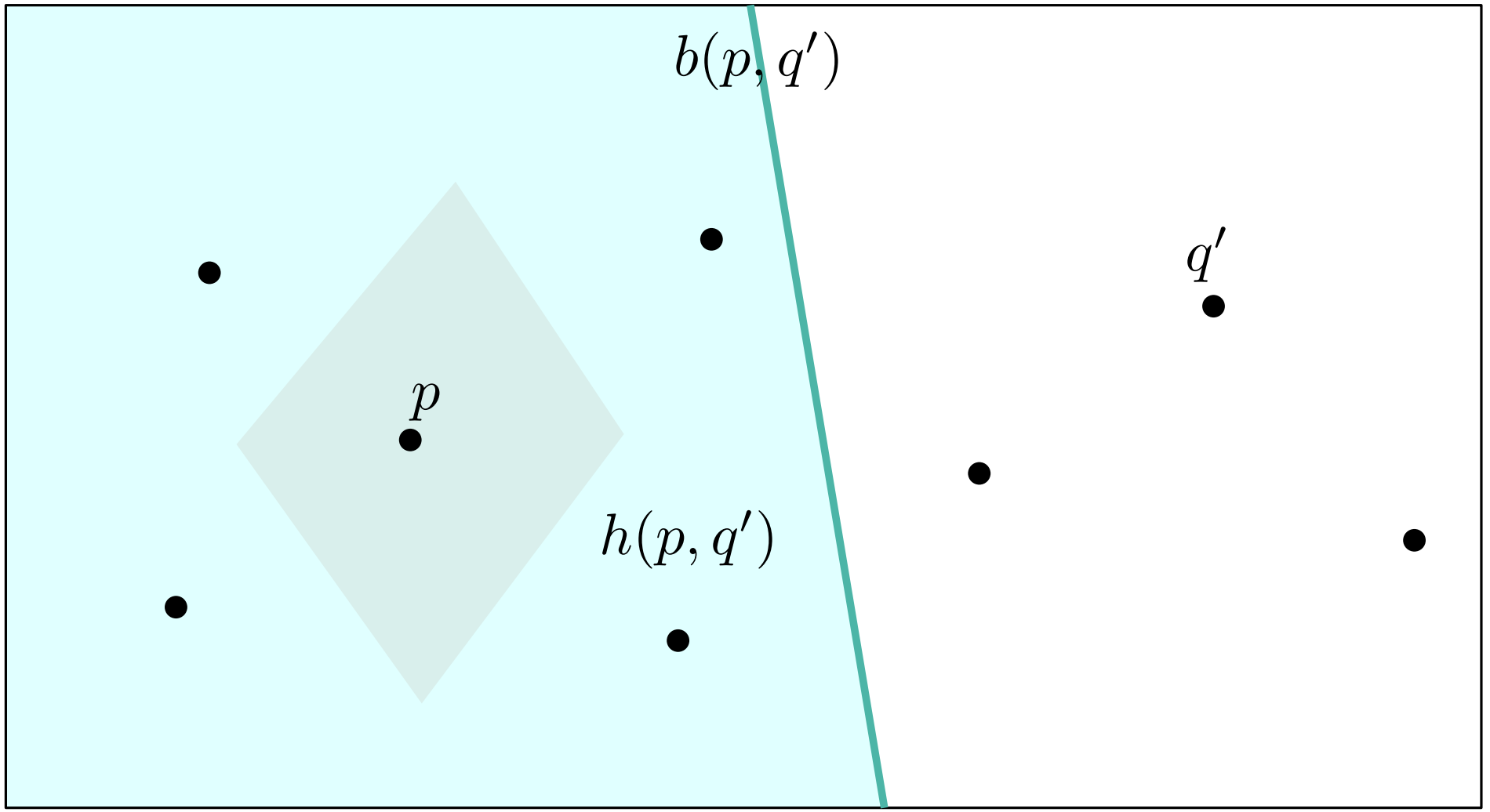
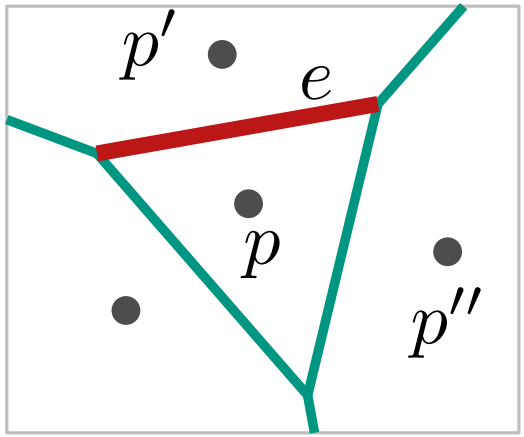


Voronoi Cell

$$\mathcal{V}(p) = \{x \in \mathbb{R}^2 : |xp| < |xq|, \forall q \in P \setminus \{p\}\}$$
$$= \bigcap_{q \in P \setminus \{p\}} h(p, q)$$

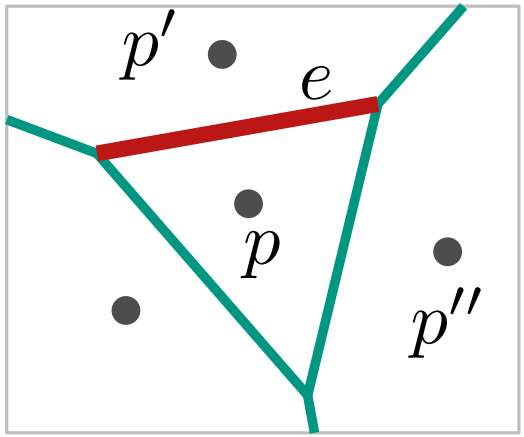


Voronoi Edges and Vertices



- Voronoi edge e between $\mathcal{V}(p)$ and $\mathcal{V}(p')$

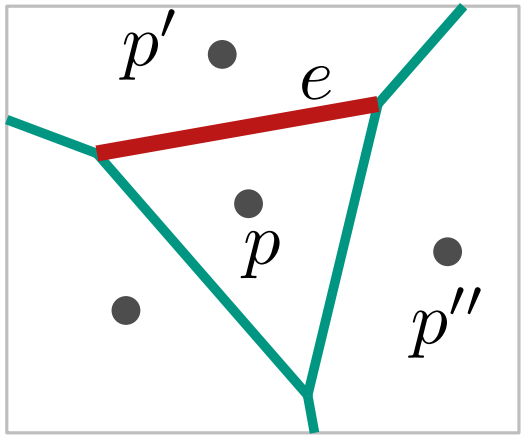
Voronoi Edges and Vertices



- Voronoi edge e between $\mathcal{V}(p)$ and $\mathcal{V}(p')$

$$e = \partial\mathcal{V}(p) \cap \partial\mathcal{V}(p')$$

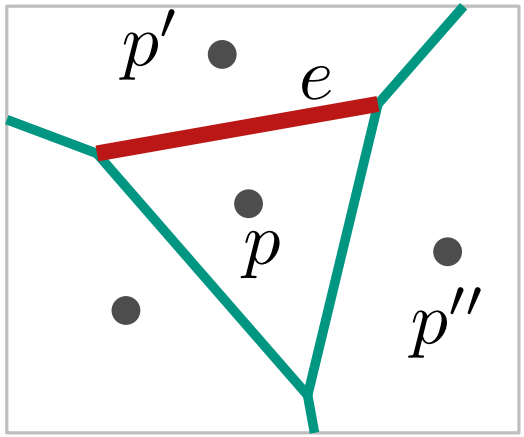
Voronoi Edges and Vertices



- Voronoi edge e between $\mathcal{V}(p)$ and $\mathcal{V}(p')$

$$e = \text{int}\Big(\partial\mathcal{V}(p) \cap \partial\mathcal{V}(p') \Big)$$ d.h. without endpoints

Voronoi Edges and Vertices

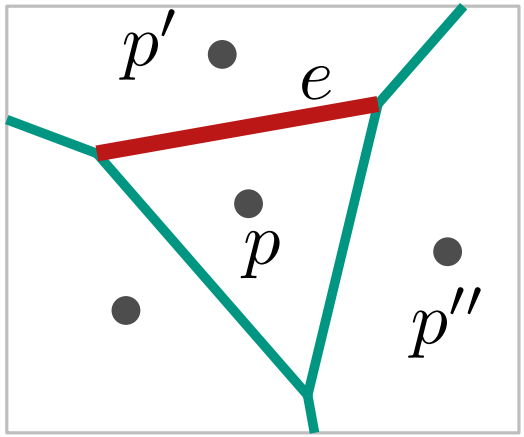


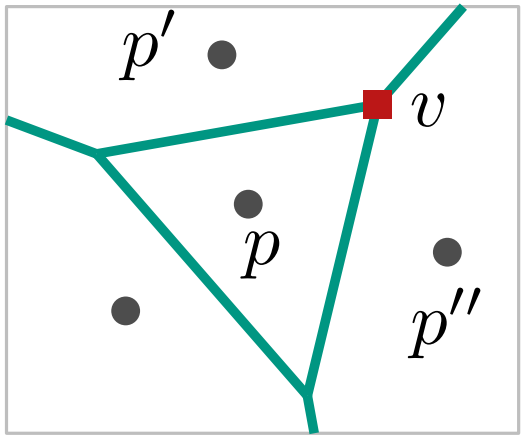
- Voronoi edge e between $\mathcal{V}(p)$ and $\mathcal{V}(p')$

$$e = \text{int}\big(\ \partial\mathcal{V}(p) \cap \partial\mathcal{V}(p')\ \big)$$ d.h. without endpoints

$$= \{x \in \mathbb{R}^2 : |xp| = |xp'| \text{ and } |xp| < |xq| \ \ \forall q \neq p, p'\}$$

Voronoi Edges and Vertices



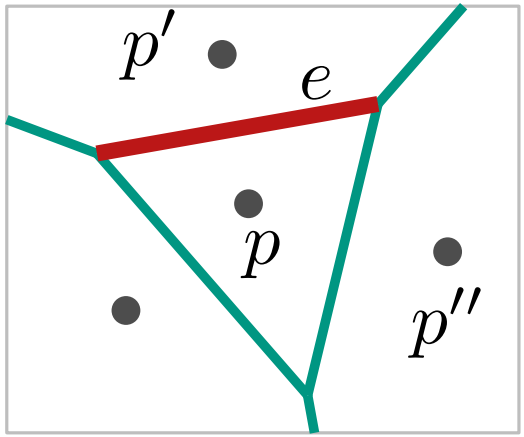


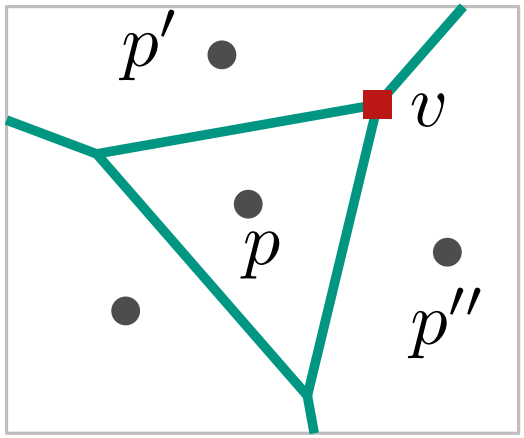
- Voronoi edge e between $\mathcal{V}(p)$ and $\mathcal{V}(p')$

$$e = \text{int}\big(\ \partial\mathcal{V}(p) \cap \partial\mathcal{V}(p')\ \big)$$ d.h. without endpoints

$$= \{x \in \mathbb{R}^2 : |xp| = |xp'| \text{ and } |xp| < |xq| \ \ \forall q \neq p, p'\}$$

Voronoi Edges and Vertices





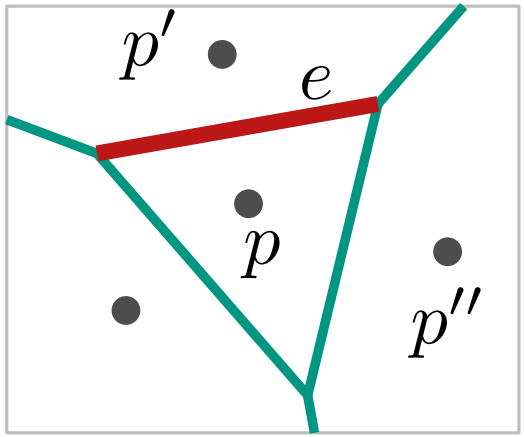
- Voronoi edge e between $\mathcal{V}(p)$ and $\mathcal{V}(p')$

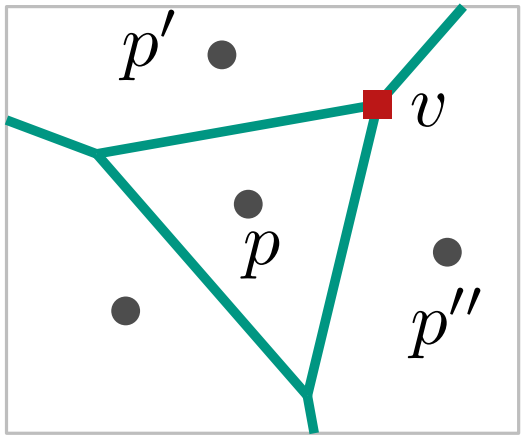
$$e = \text{int}\big(\ \partial\mathcal{V}(p) \cap \partial\mathcal{V}(p')\ \big)$$ d.h. without endpoints

$$= \{x \in \mathbb{R}^2 : |xp| = |xp'| \text{ and } |xp| < |xq| \ \ \forall q \neq p, p'\}$$

- Voronoi vertex v among $\mathcal{V}(p)$, $\mathcal{V}(p')$, and $\mathcal{V}(p'')$

Voronoi Edges and Vertices





- Voronoi edge e between $\mathcal{V}(p)$ and $\mathcal{V}(p')$

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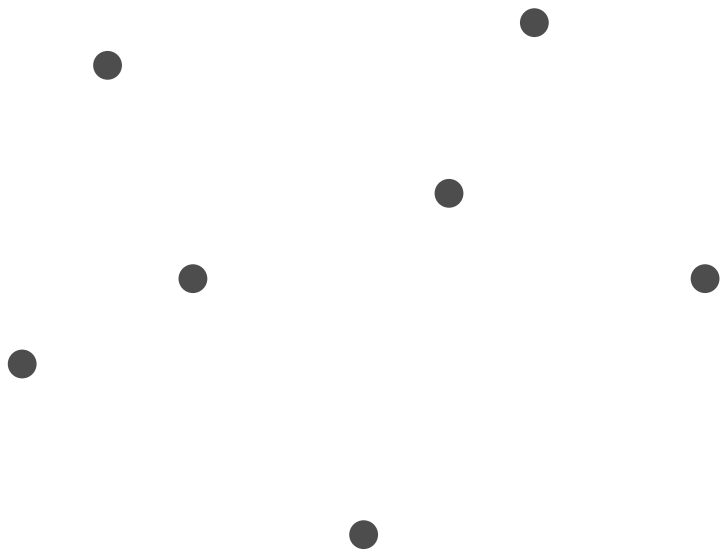
$$= \{x \in \mathbb{R}^2 : |xp| = |xp'| \text{ and } |xp| < |xq| \ \ \forall q \neq p, p'\}$$

- Voronoi vertex v among $\mathcal{V}(p)$, $\mathcal{V}(p')$, and $\mathcal{V}(p'')$

$$v = \partial\mathcal{V}(p) \cap \partial\mathcal{V}(p') \cap \partial\mathcal{V}(p'')$$

Growing Circle

- Growing a circle from a point $x \in \mathbb{R}^2$ first hits



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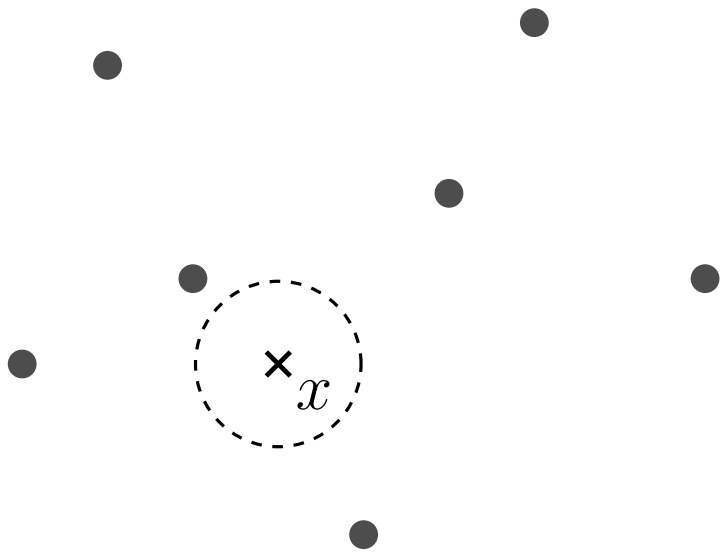
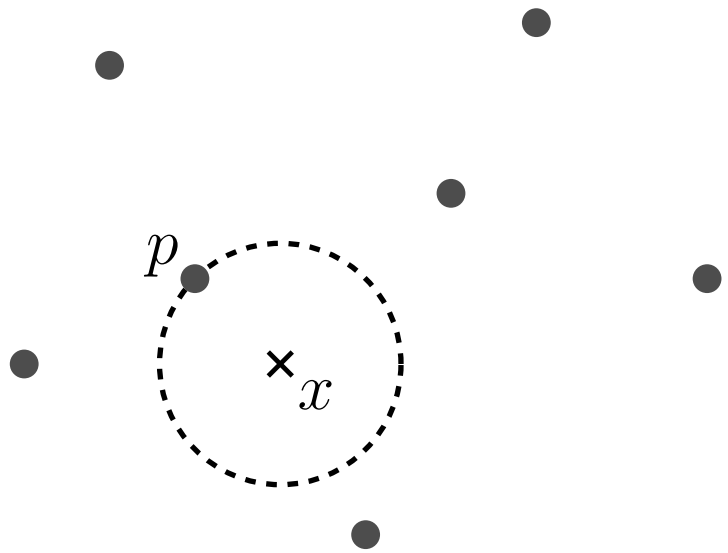


Growing Circle

- Growing a circle from a point $x \in \mathbb{R}^2$ first hits
  - one site $p \rightarrow x \in \mathcal{V}(p)$



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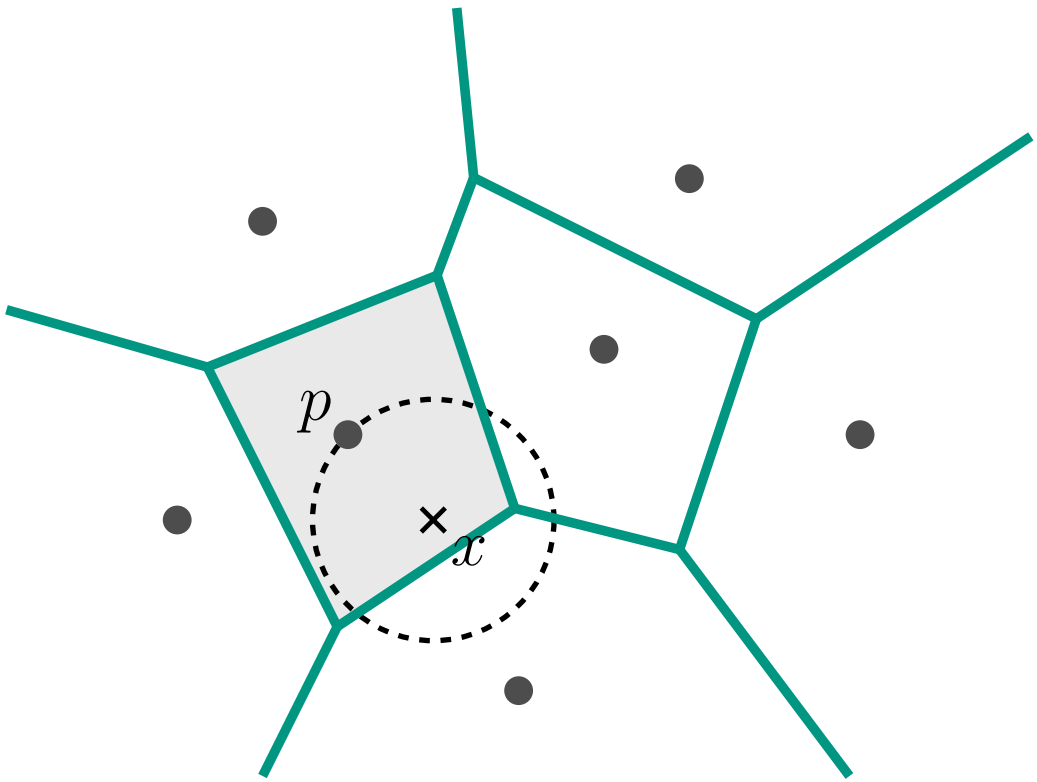
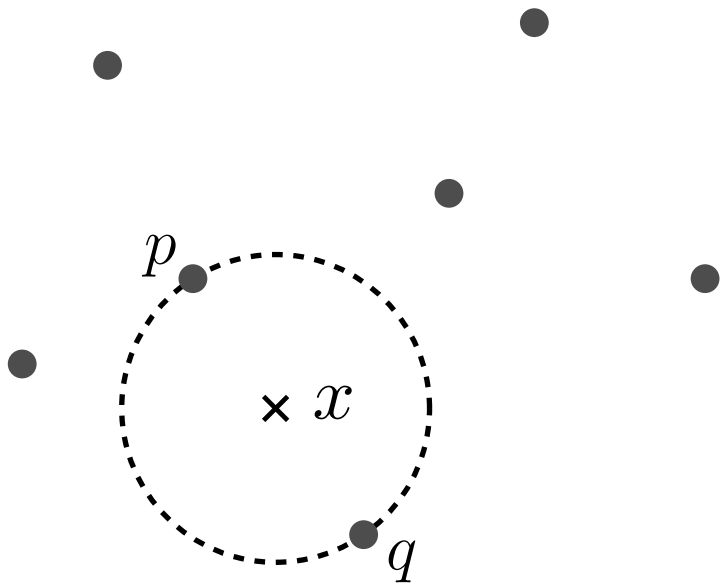


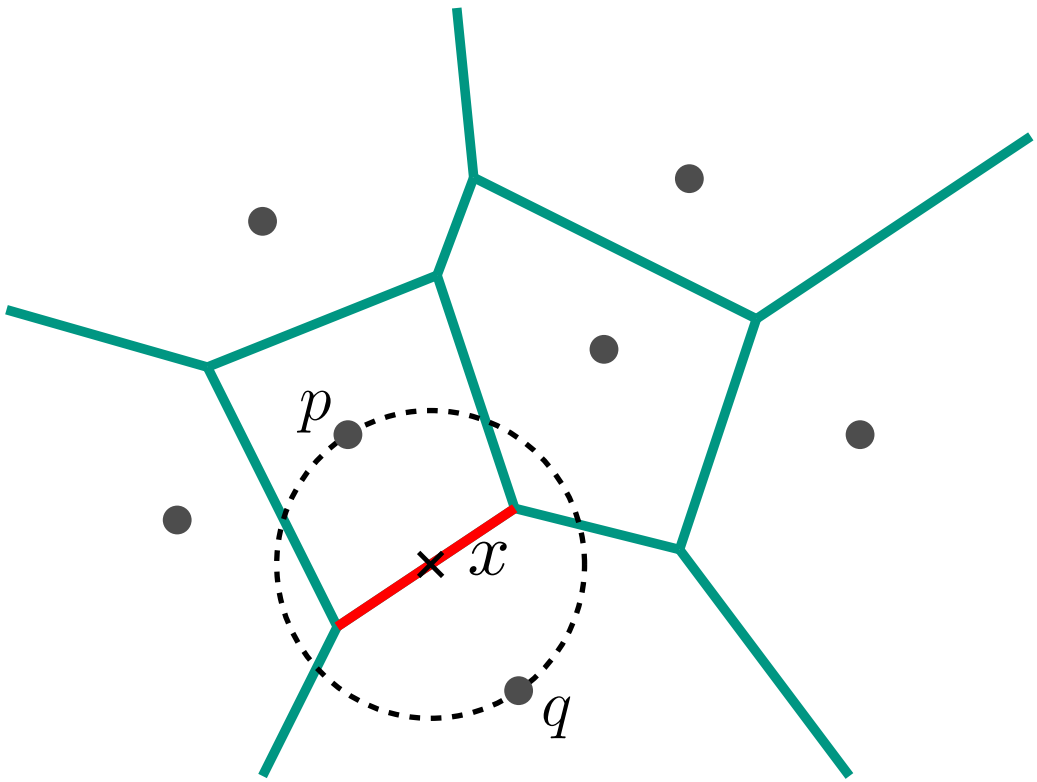
Growing Circle

- Growing a circle from a point $x \in \mathbb{R}^2$ first hits
  - one site $p \rightarrow x \in \mathcal{V}(p)$
  - two sites p, $q \rightarrow x \in$ edge $= \partial\mathcal{V}(p) \cap \partial\mathcal{V}(q)$



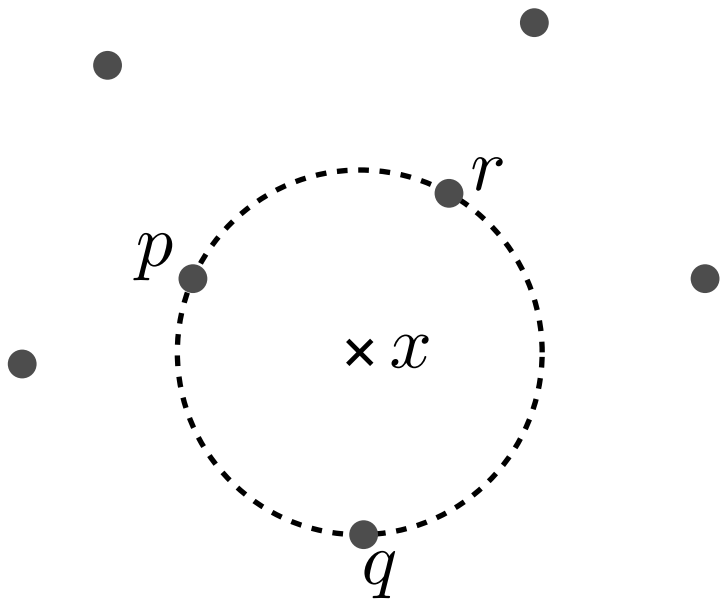
Growing Circle

- Growing a circle from a point $x \in \mathbb{R}^2$ first hits
  - one site $p \rightarrow x \in \mathcal{V}(p)$
  - two sites p, $q \rightarrow x \in$ edge $= \partial\mathcal{V}(p) \cap \partial\mathcal{V}(q)$



Growing Circle

- Growing a circle from a point $x \in \mathbb{R}^2$ first hits
  - one site $p \rightarrow x \in \mathcal{V}(p)$
  - two sites p, $q \rightarrow x \in$ edge $= \partial\mathcal{V}(p) \cap \partial\mathcal{V}(q)$
  - $>$ two sites p, q , r, ...
    $\rightarrow x =$ vertex $= \partial\mathcal{V}(p) \cap \partial\mathcal{V}(q) \cap \partial\mathcal{V}(r)$



Growing Circle

- Growing a circle from a point $x \in \mathbb{R}^2$ first hits
  - one site $p \rightarrow x \in \mathcal{V}(p)$
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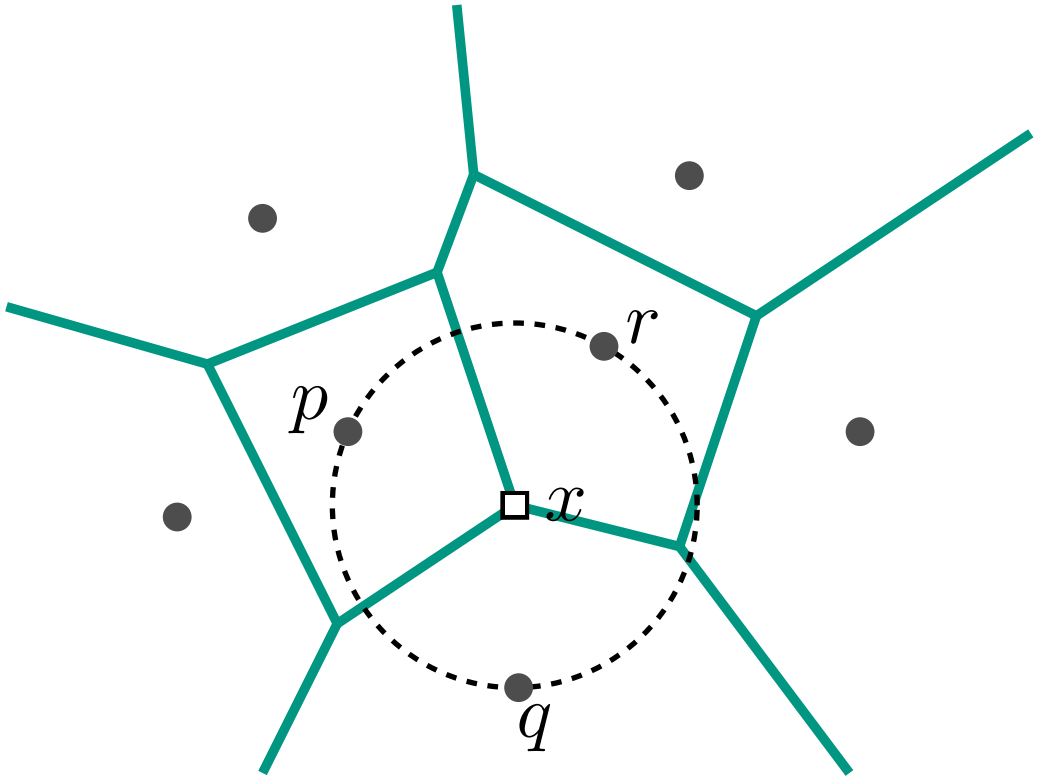
  - $>$ two sites p, q , r, ...
    $\rightarrow x =$ vertex $= \partial\mathcal{V}(p) \cap \partial\mathcal{V}(q) \cap \partial\mathcal{V}(r)$



Wavefront Model (Growth Model)

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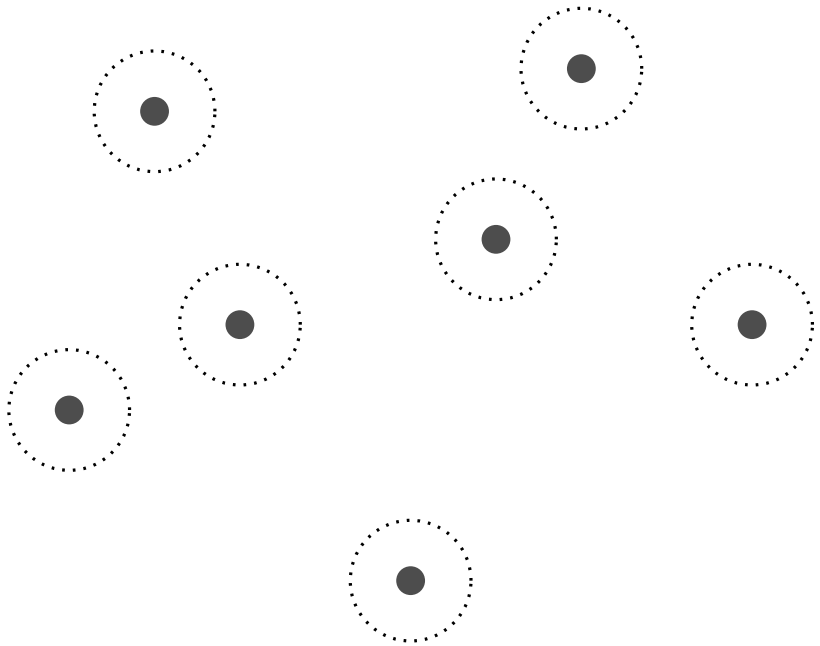

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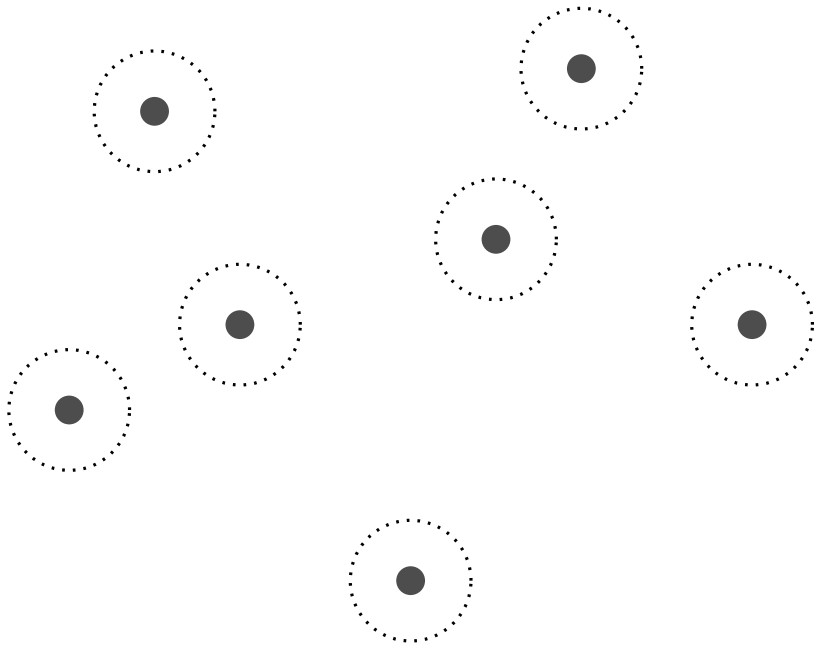

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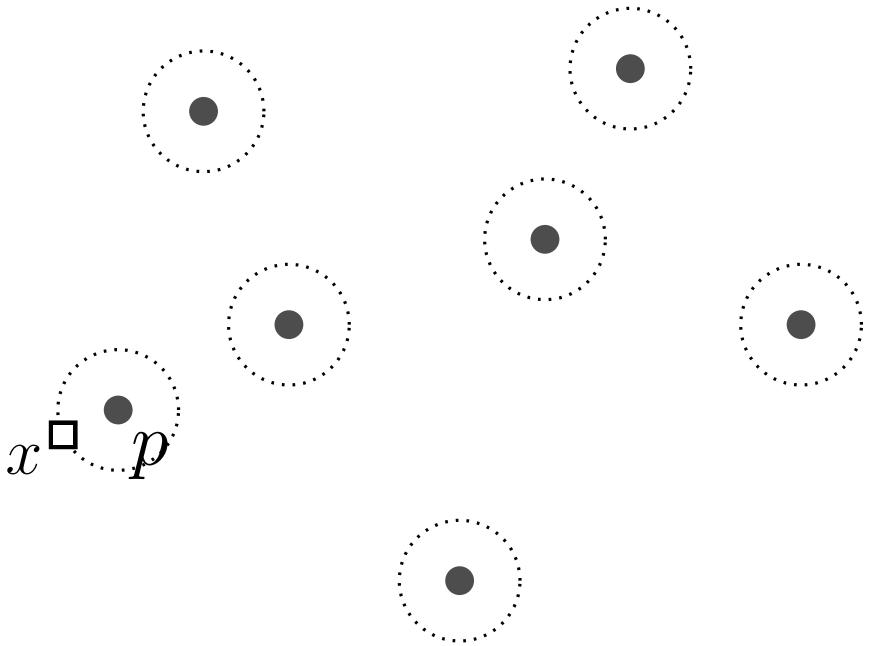

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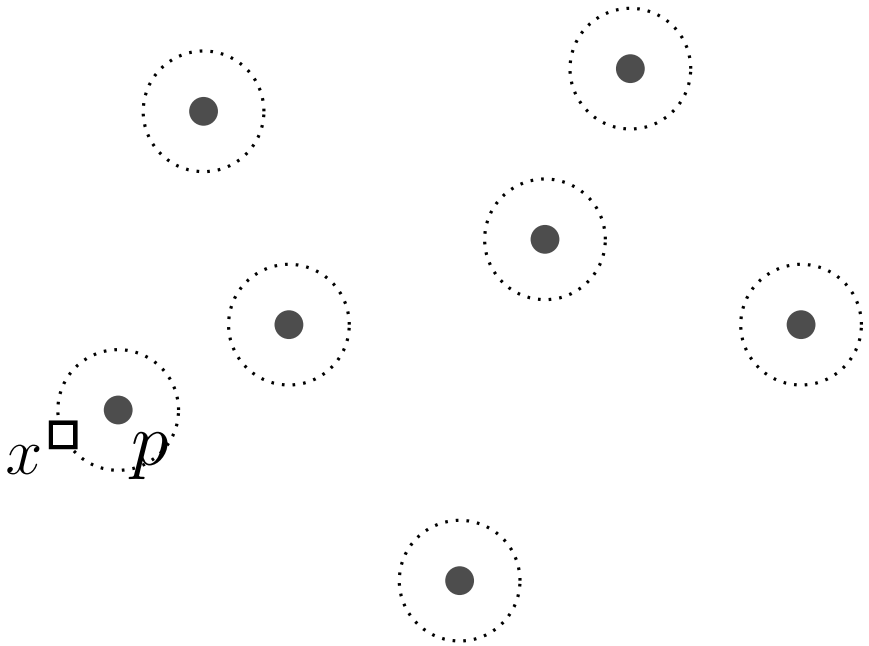

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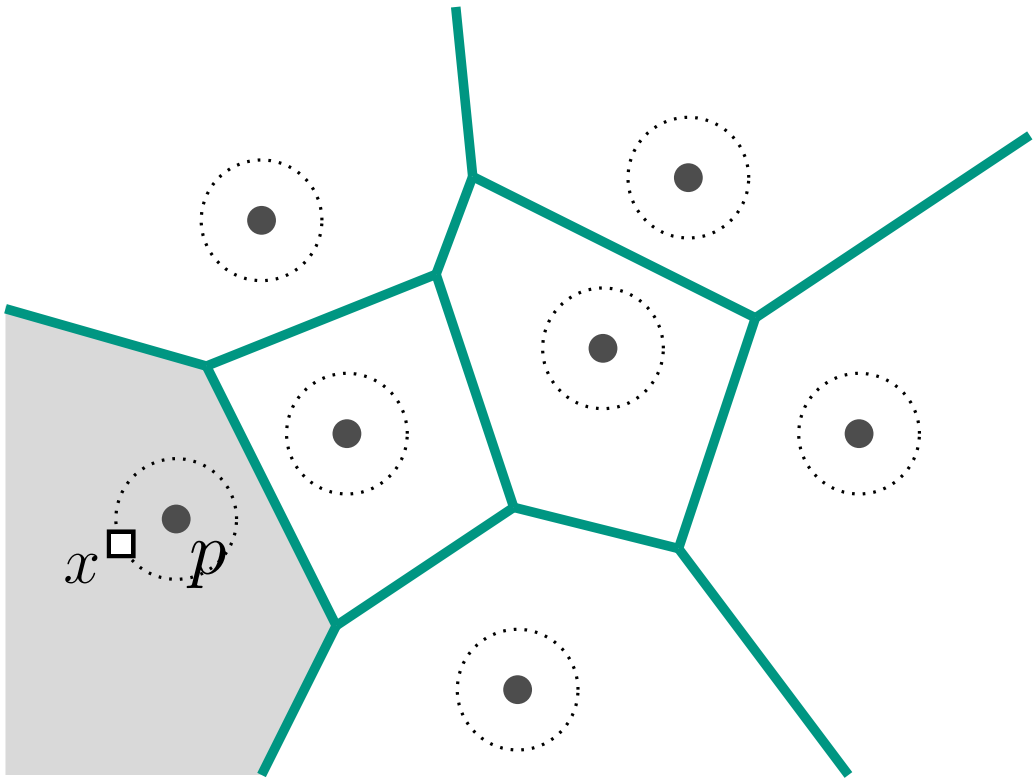

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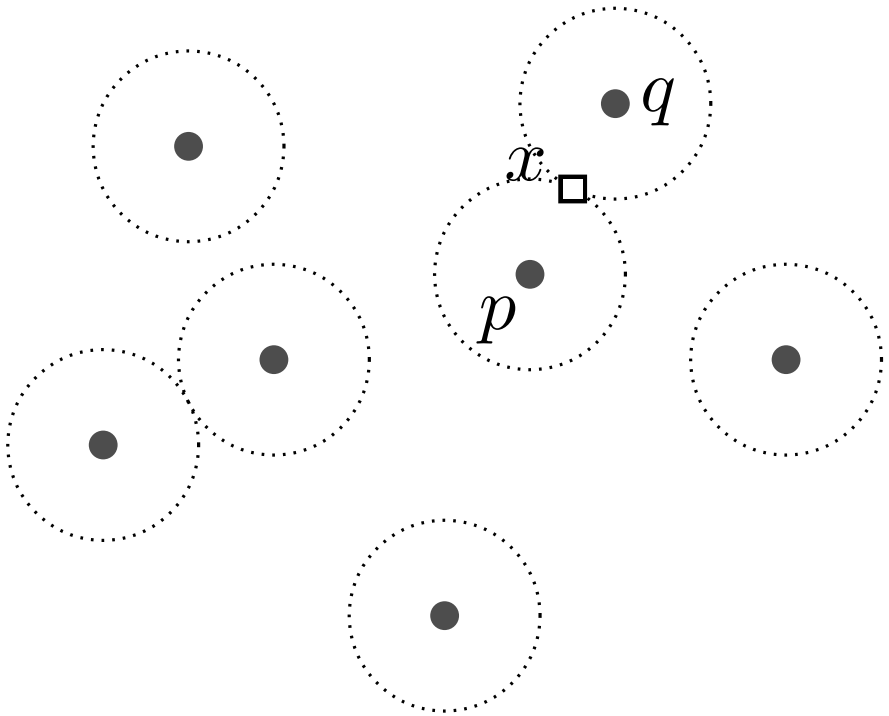

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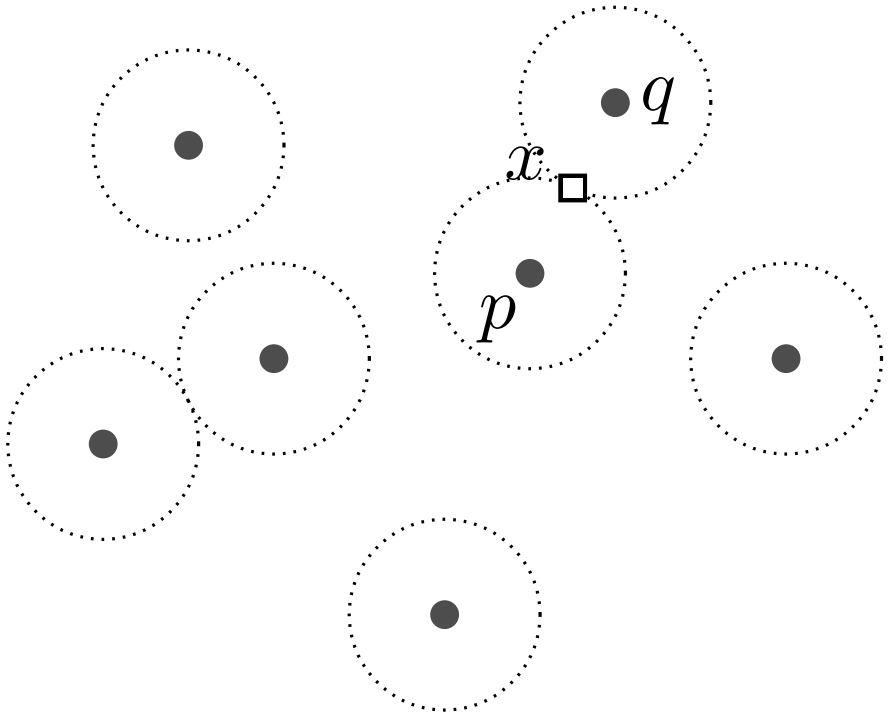

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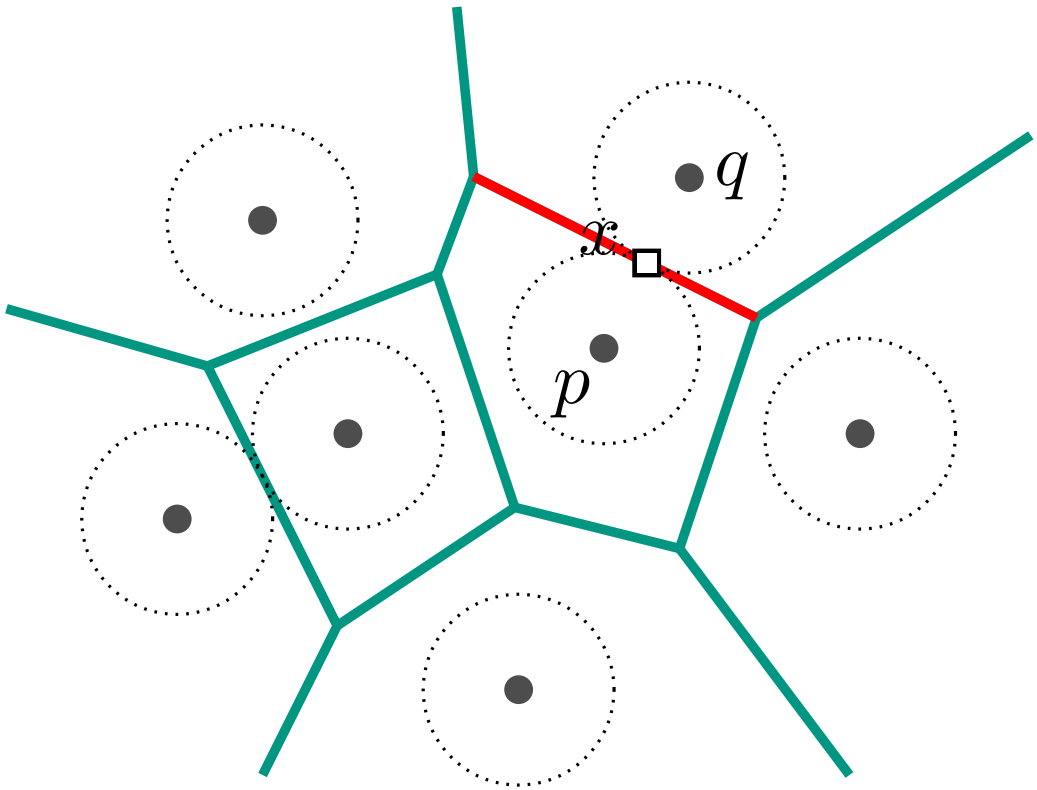

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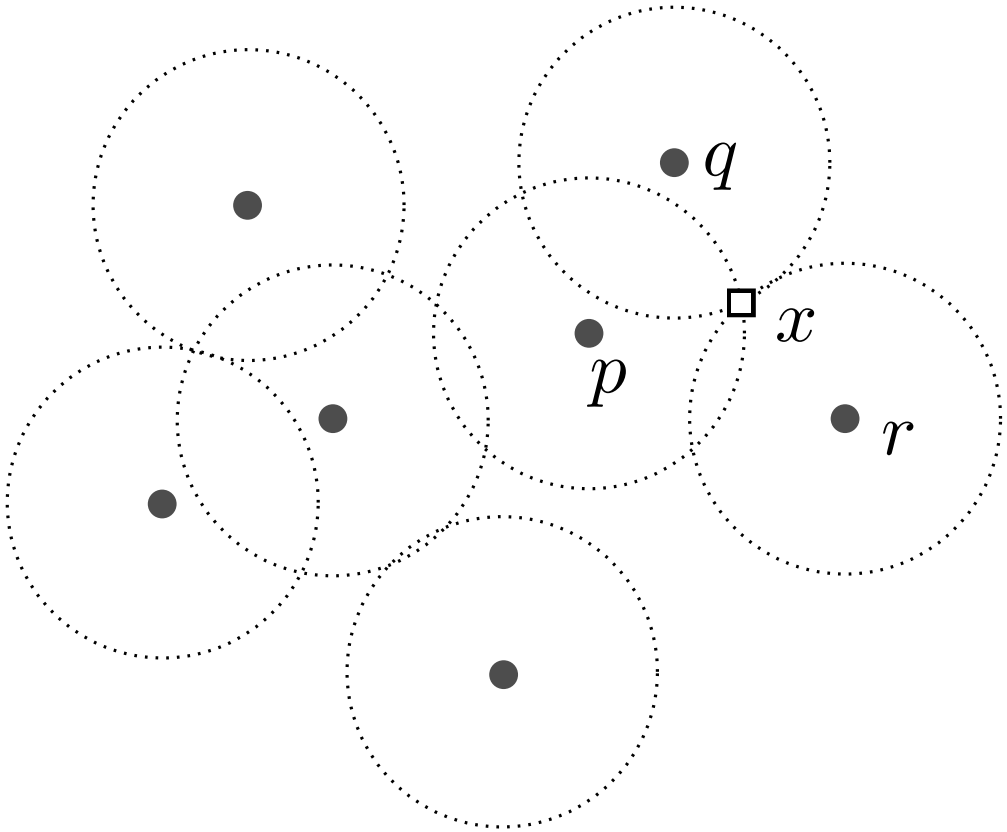

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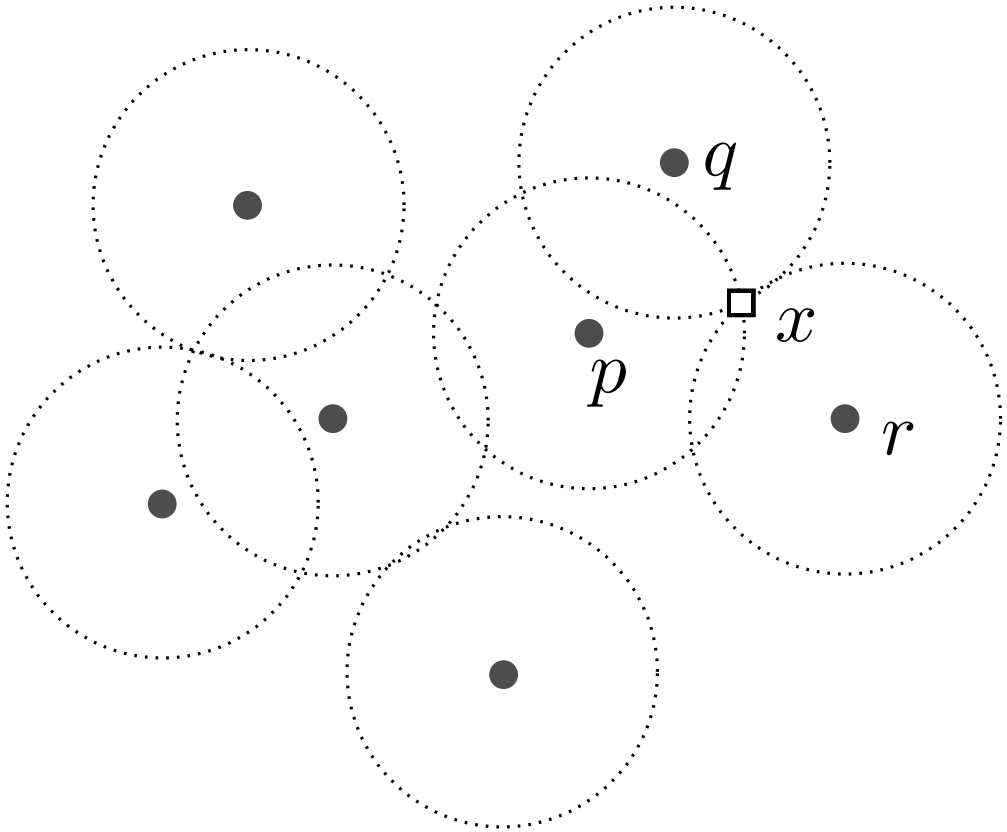

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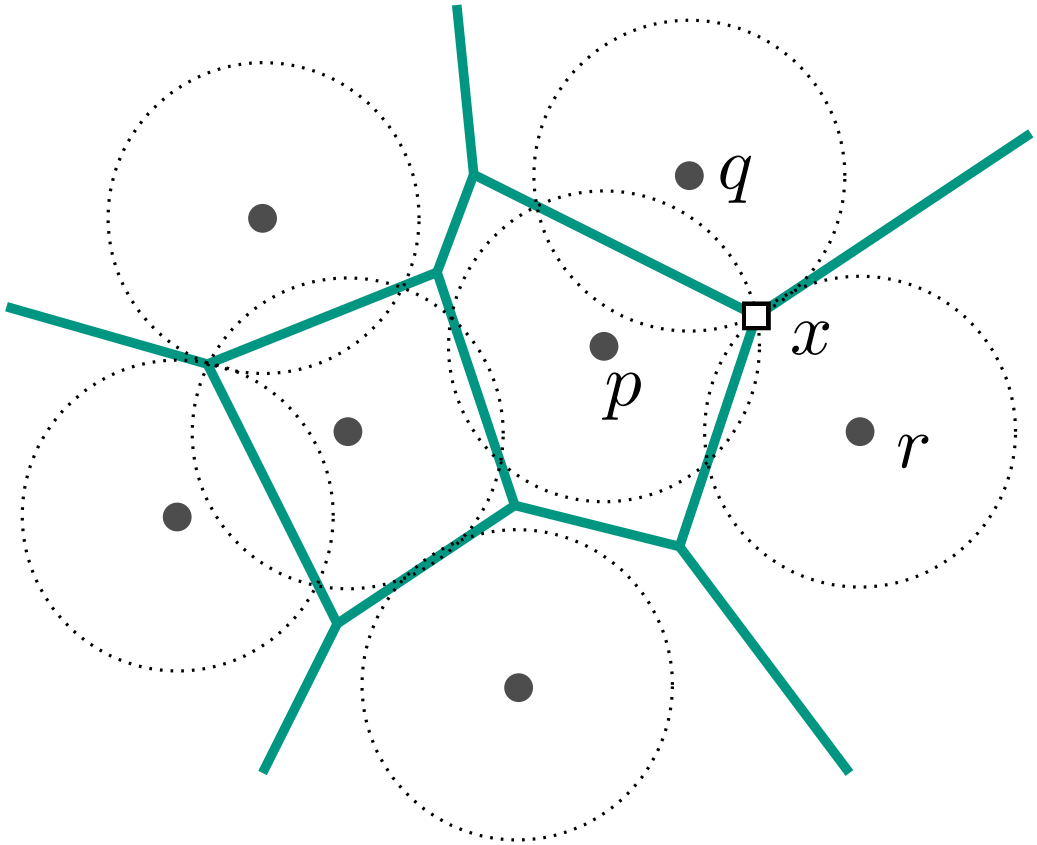

Unbounded Cell

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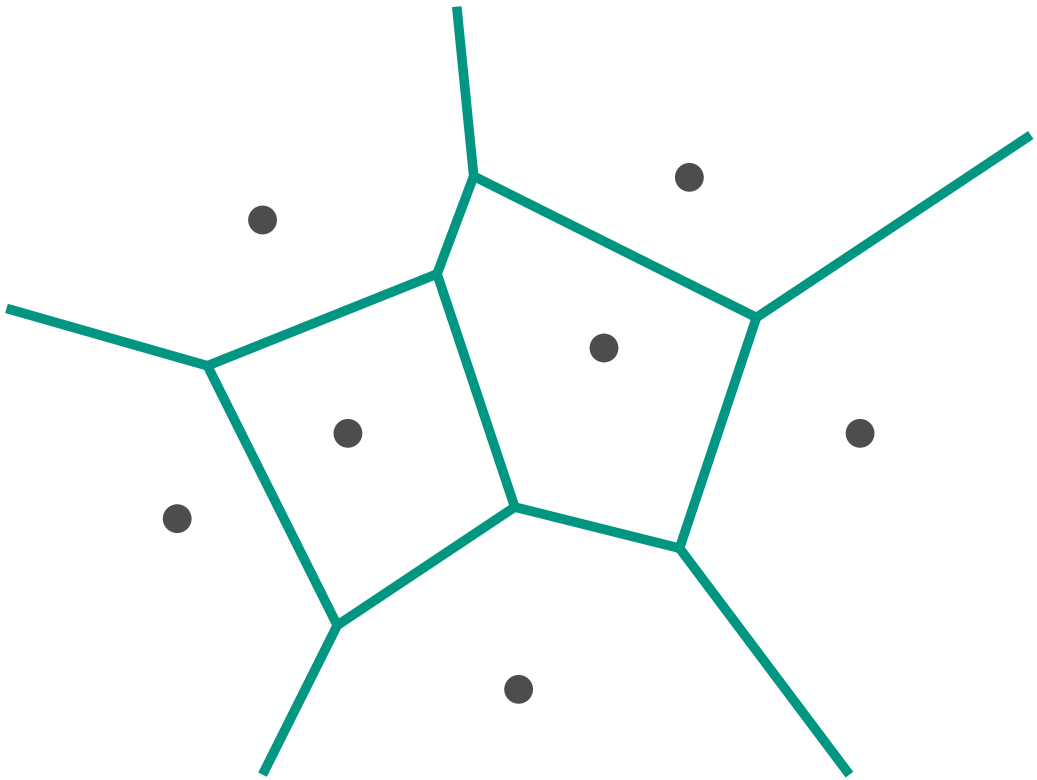

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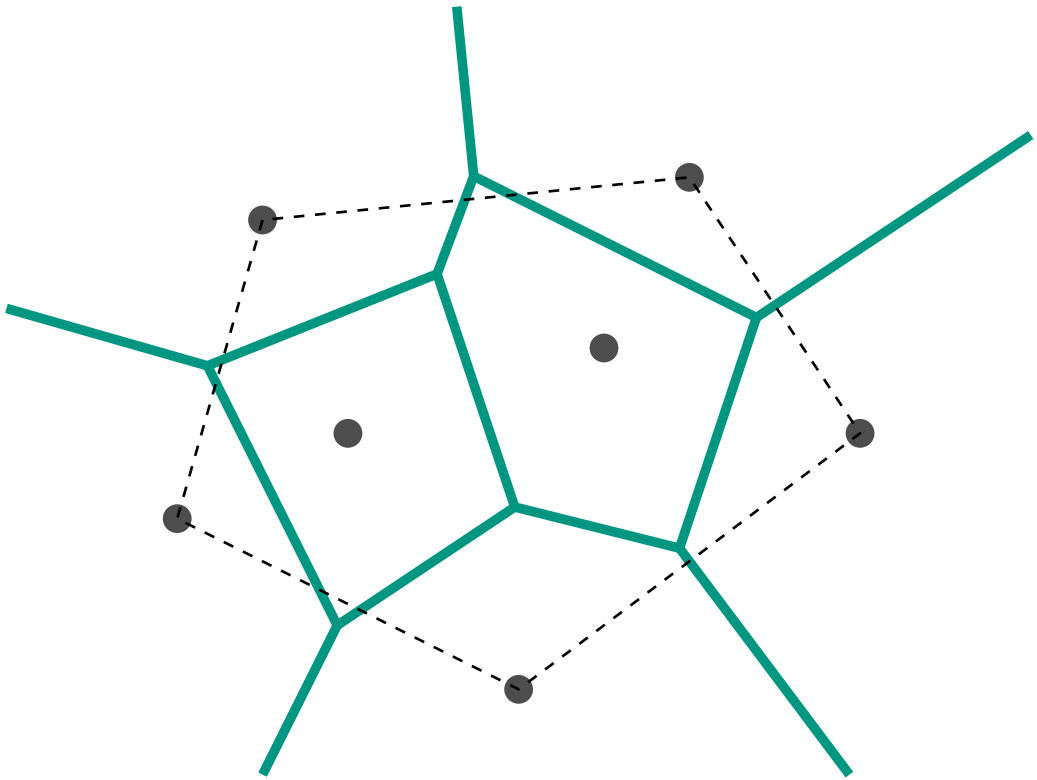

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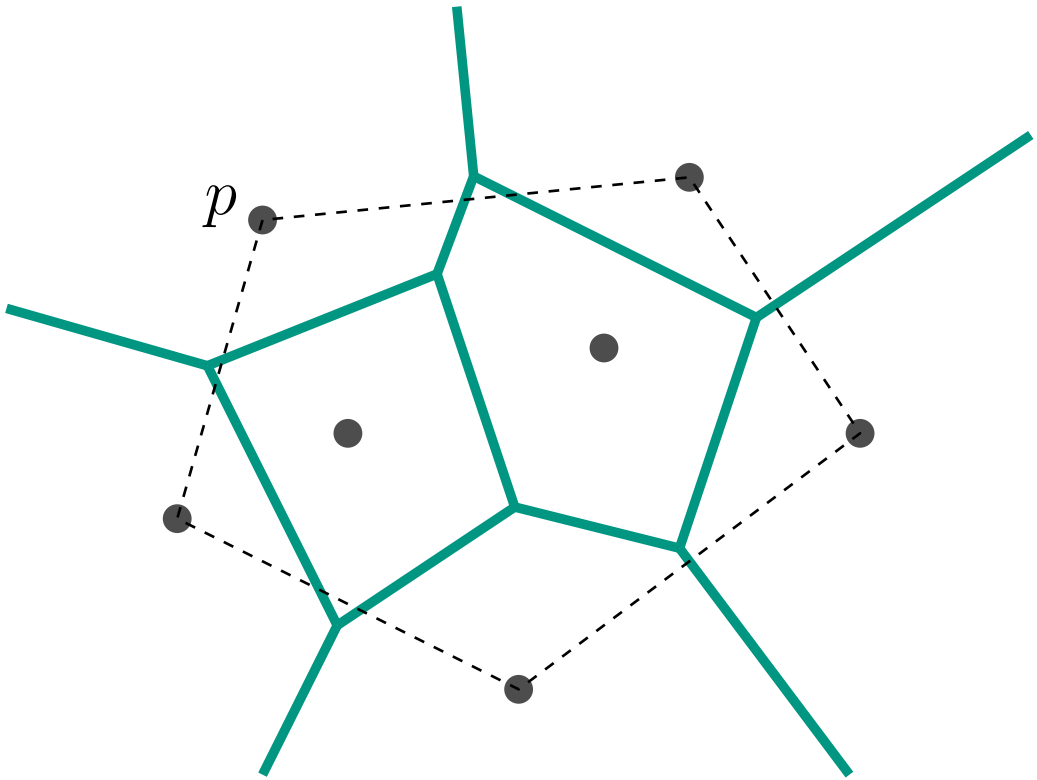

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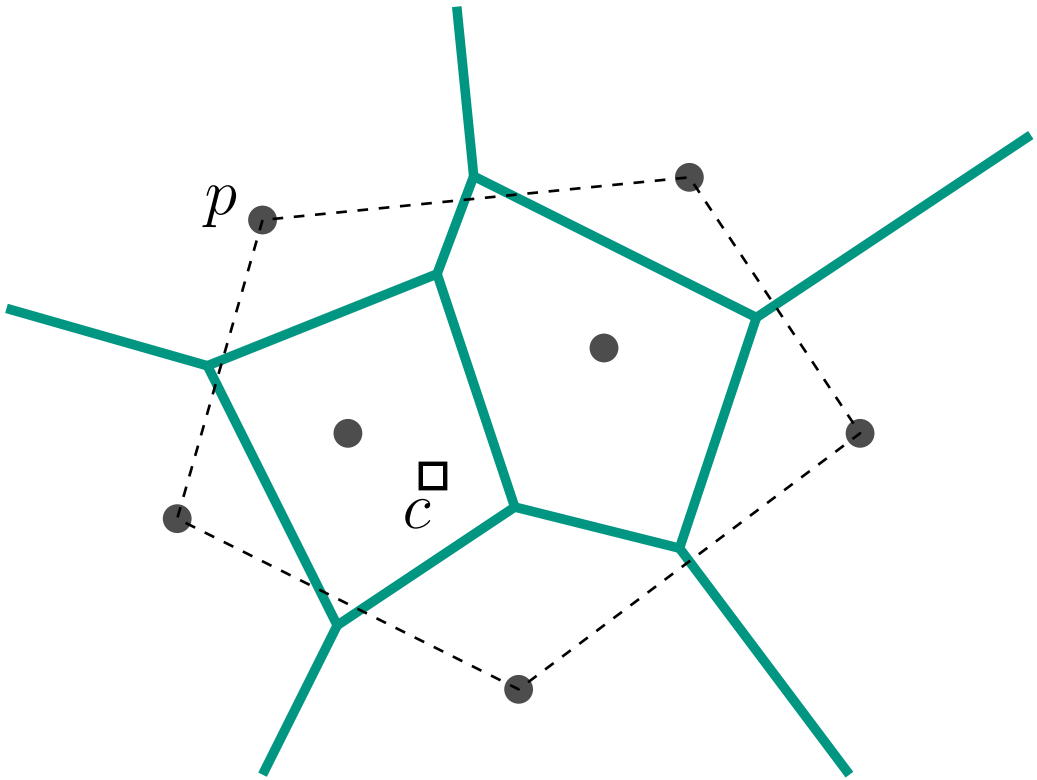

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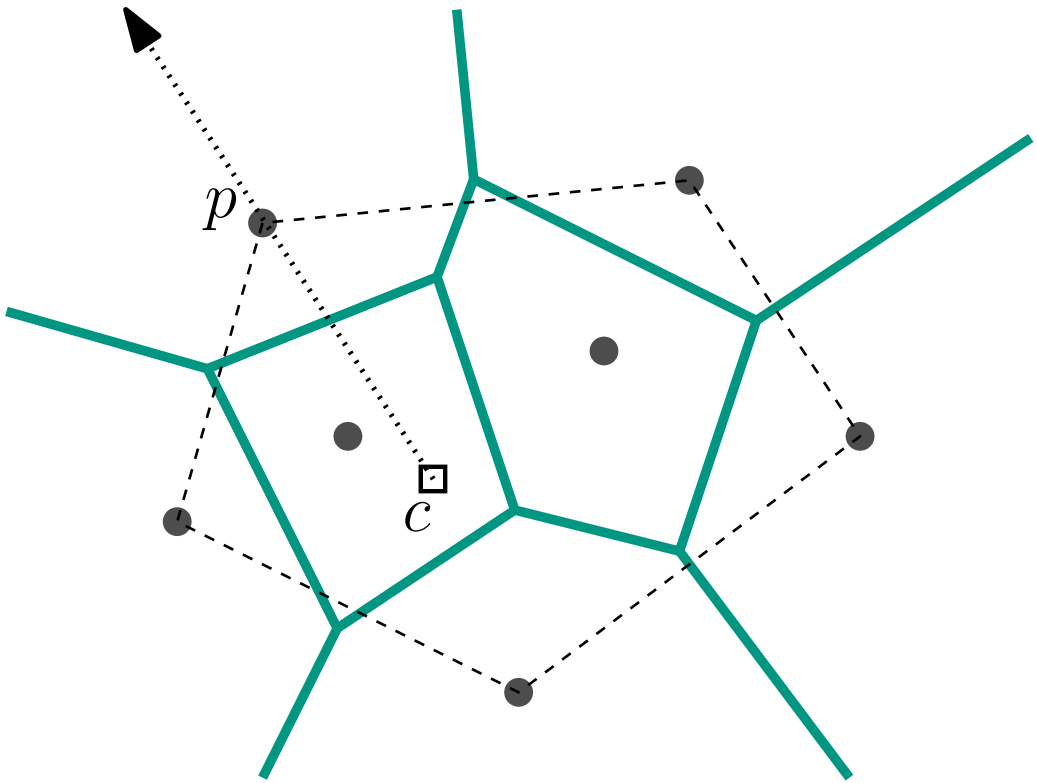

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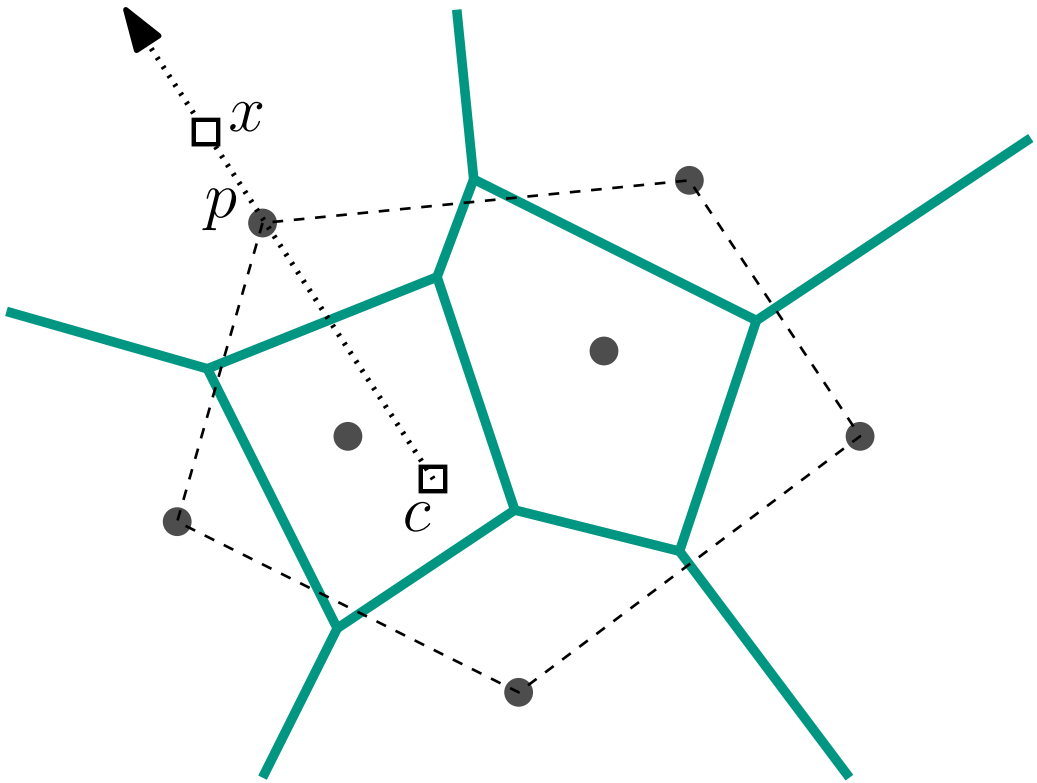

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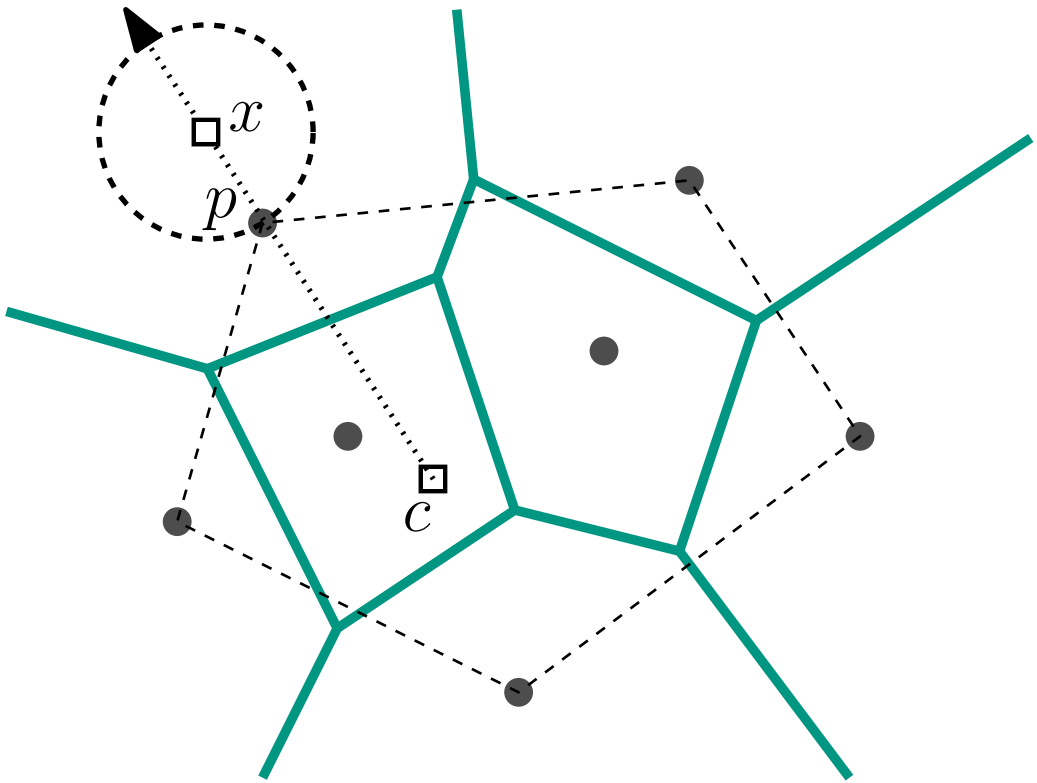

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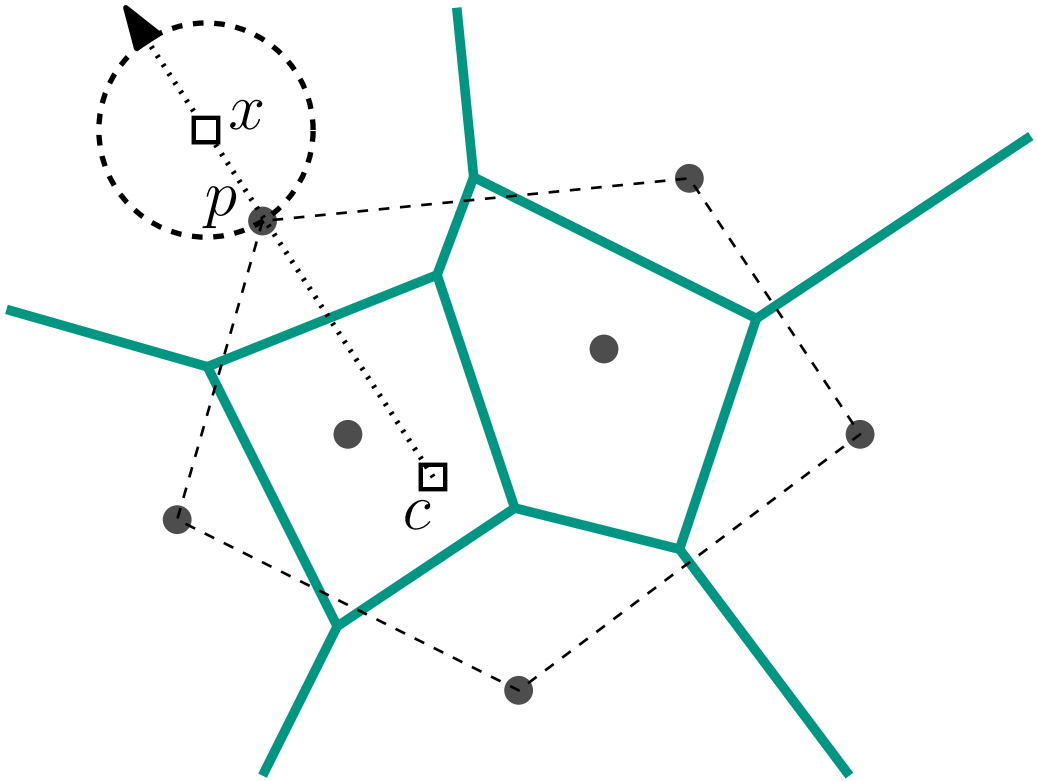

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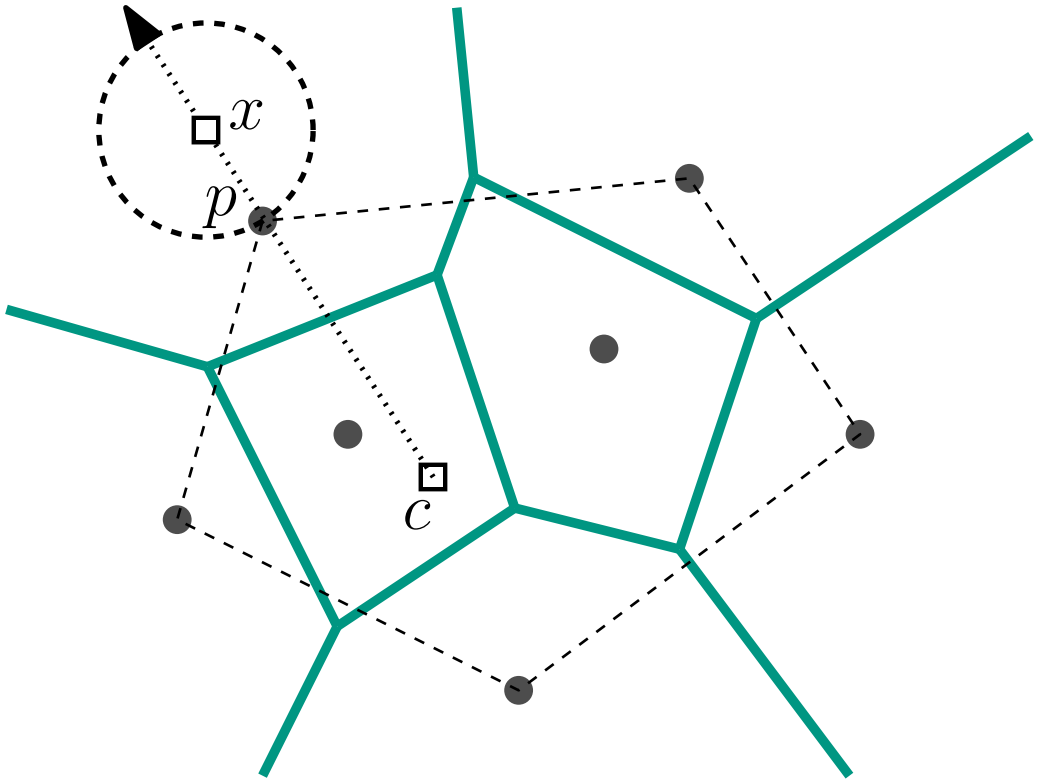
- If P is in convex position, $\mathrm{Vor}(P)$ is a tree.



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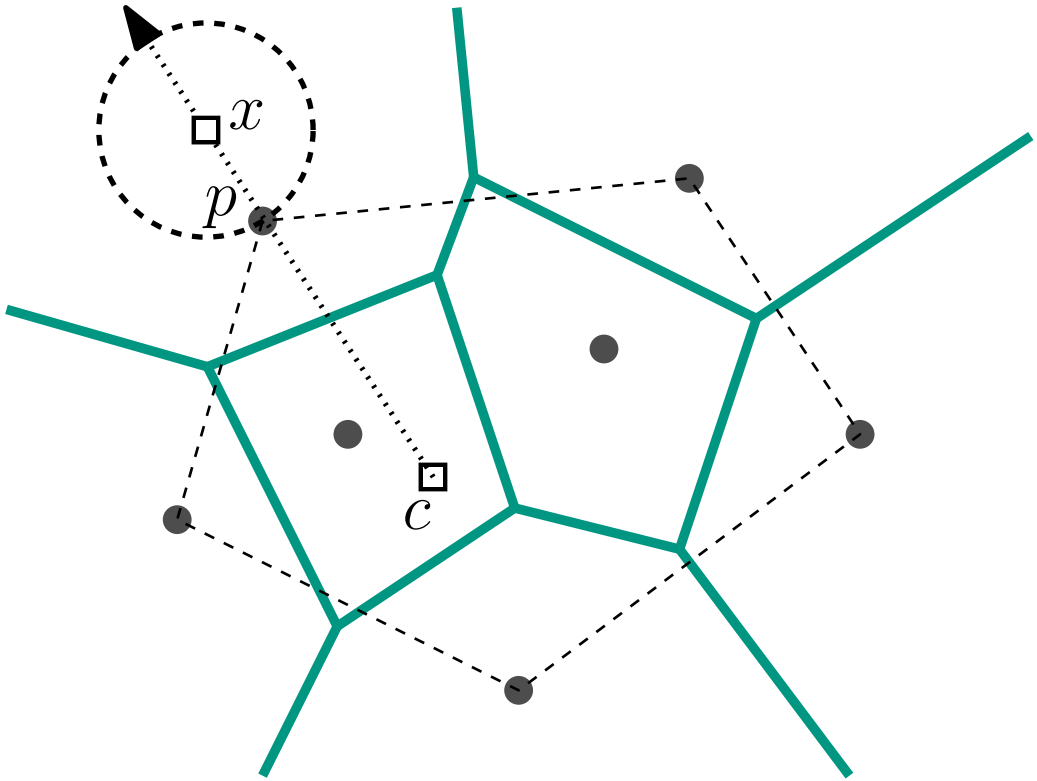

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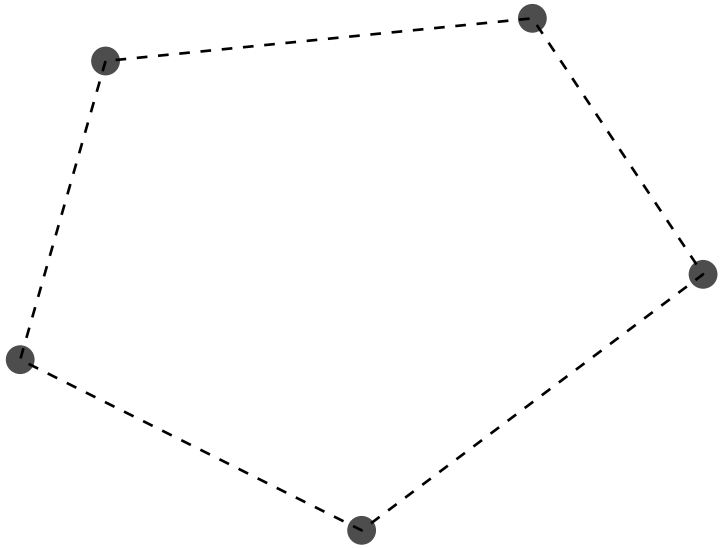

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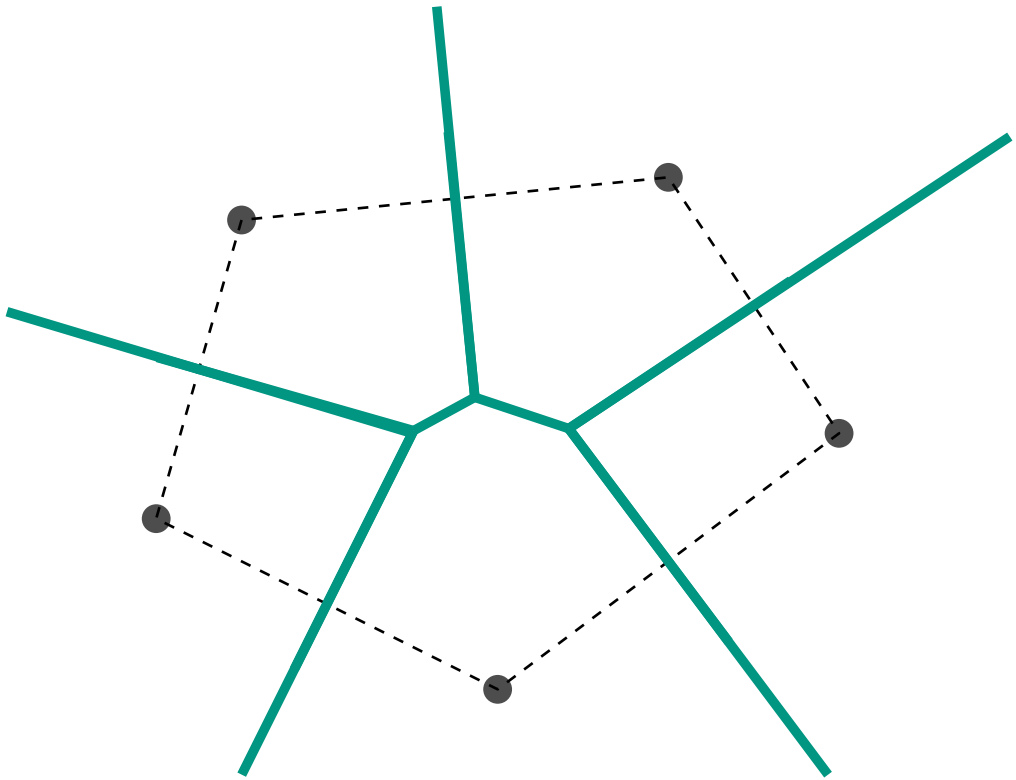

Properties

Theorem 1: Let $P \subset \mathbb{R}^2$ be a set of n points. If all points are collinear, then $\mathrm{Vor}(P)$ consists of $n-1$ parallel lines. Otherwise $\mathrm{Vor}(P)$ is connected and its edges are either segments or half lines.

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Exercise!

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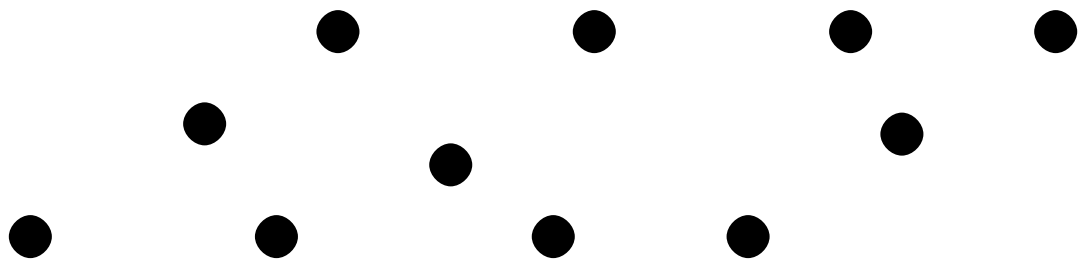

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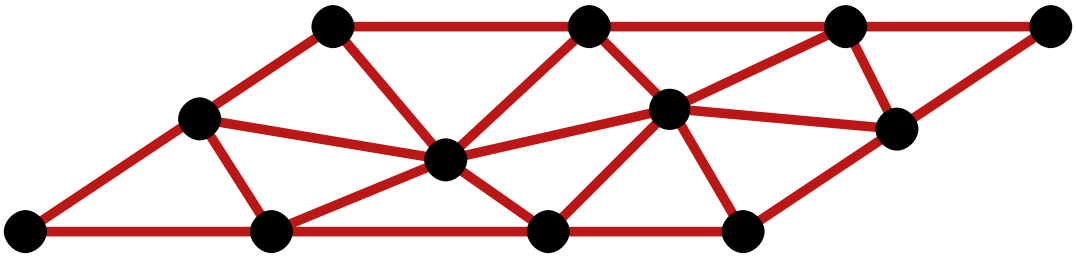

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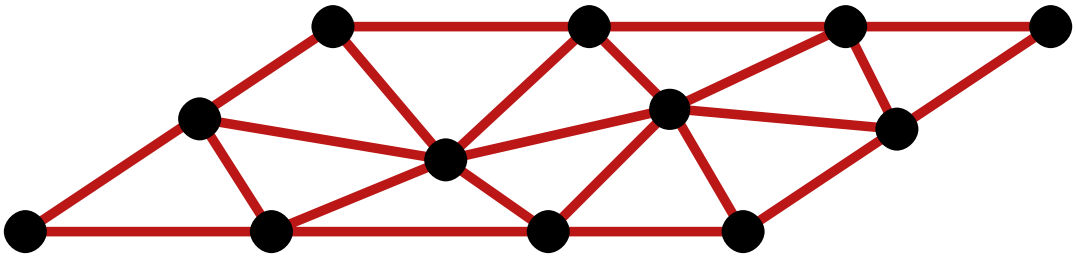

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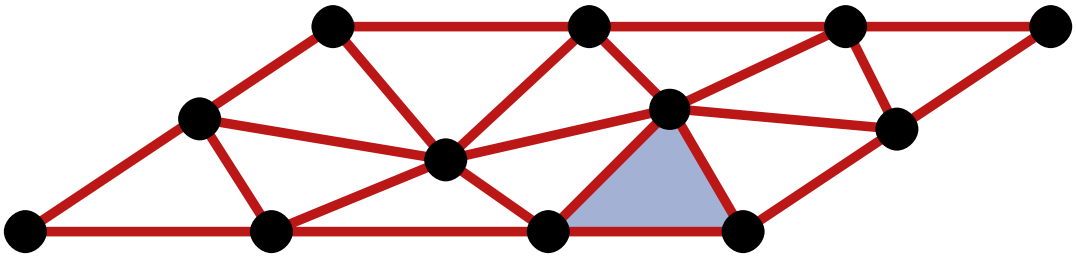

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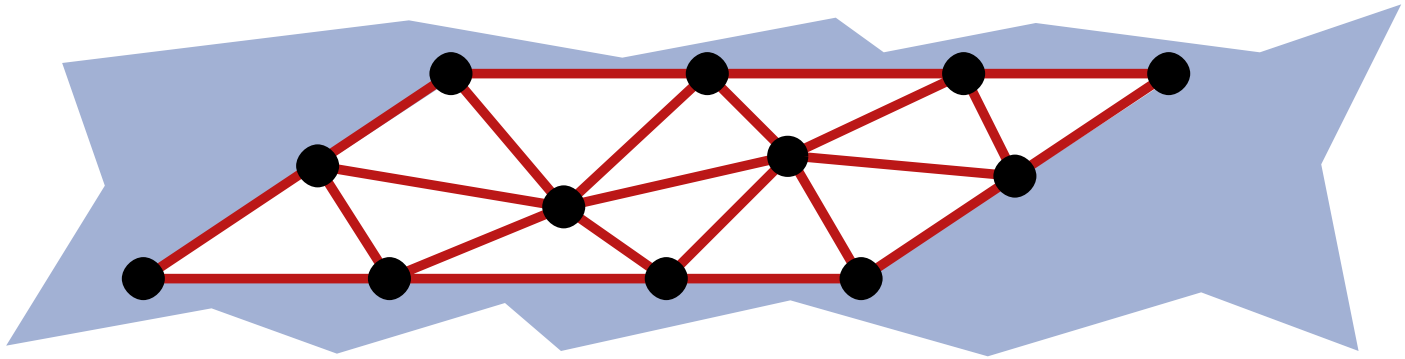

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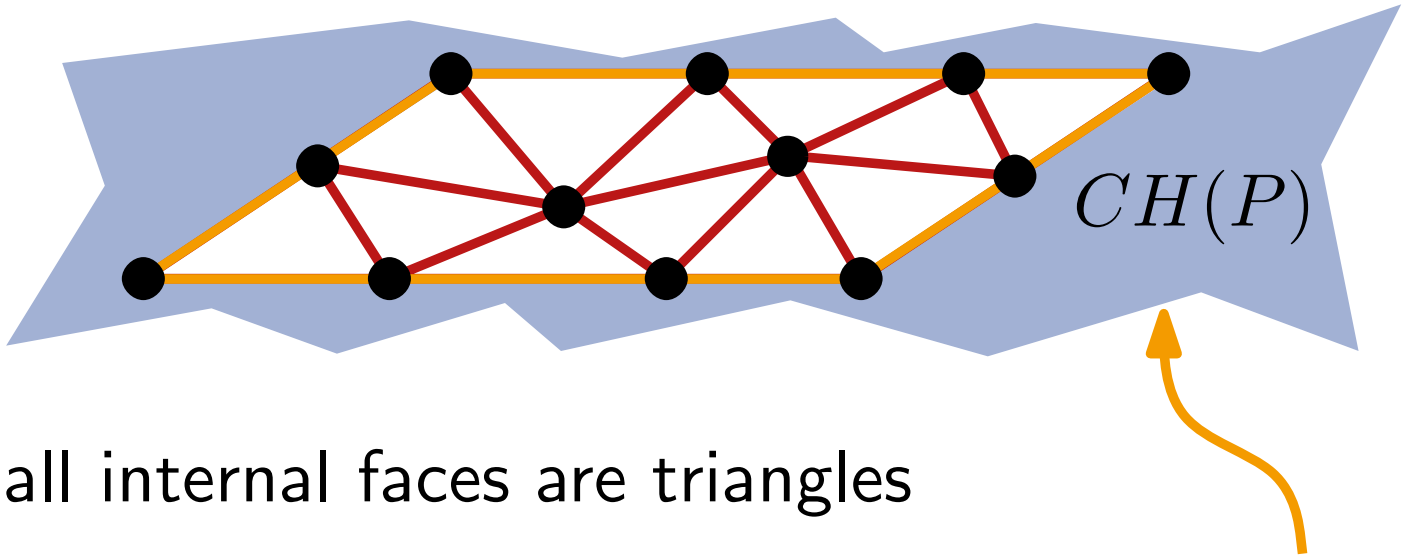

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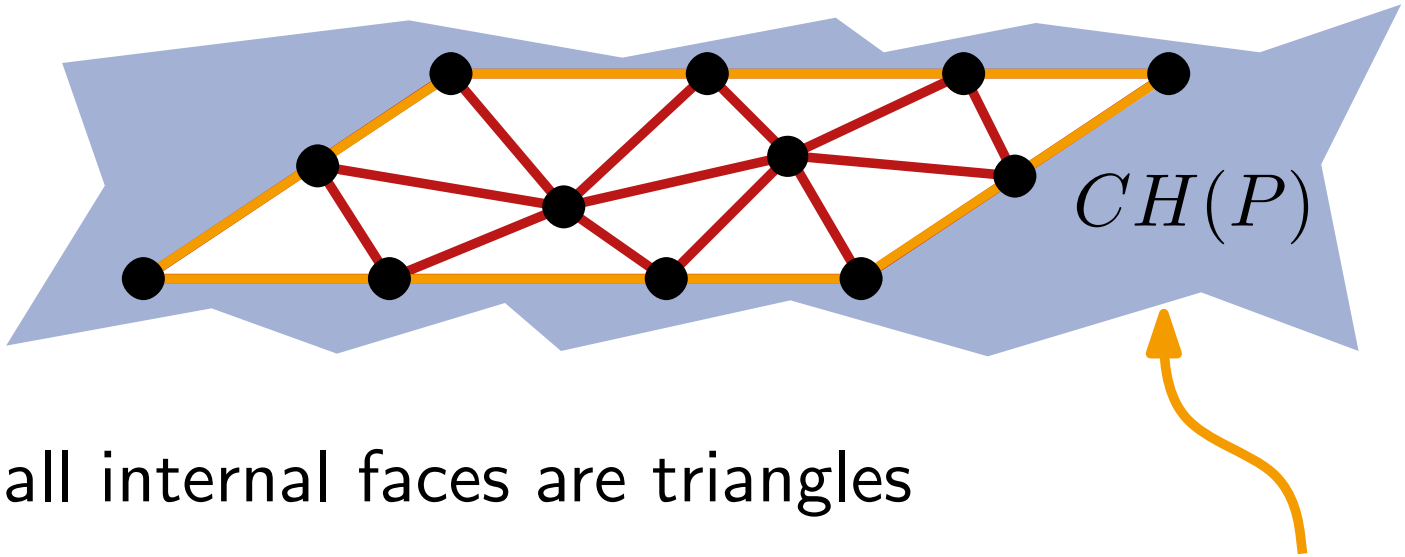

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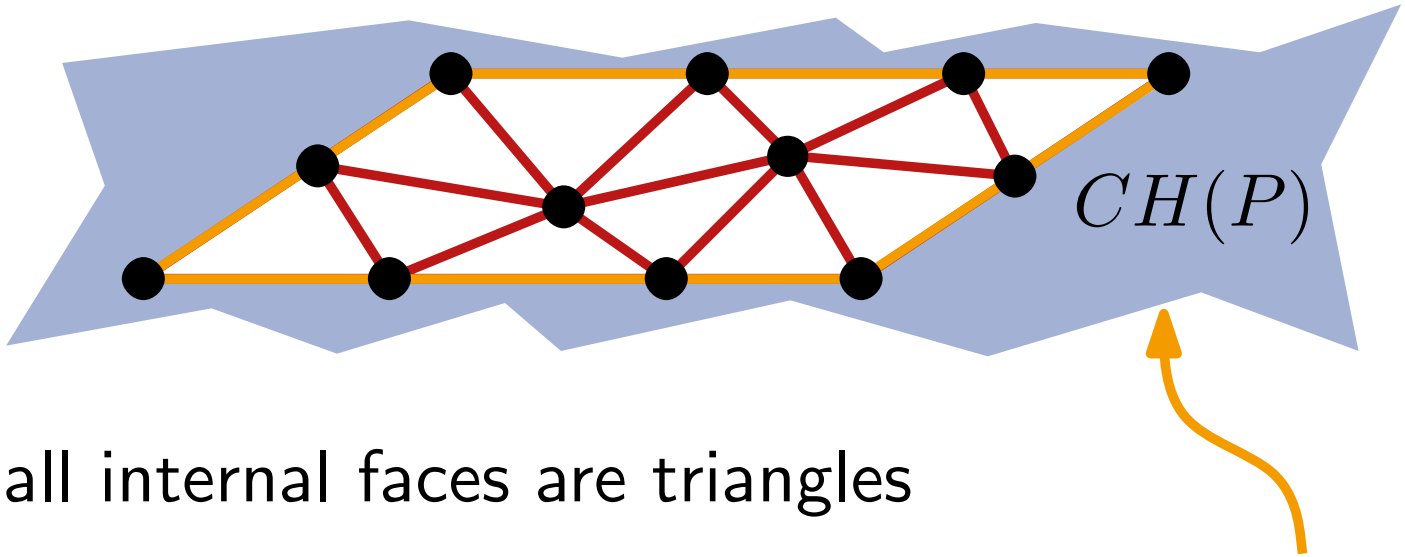

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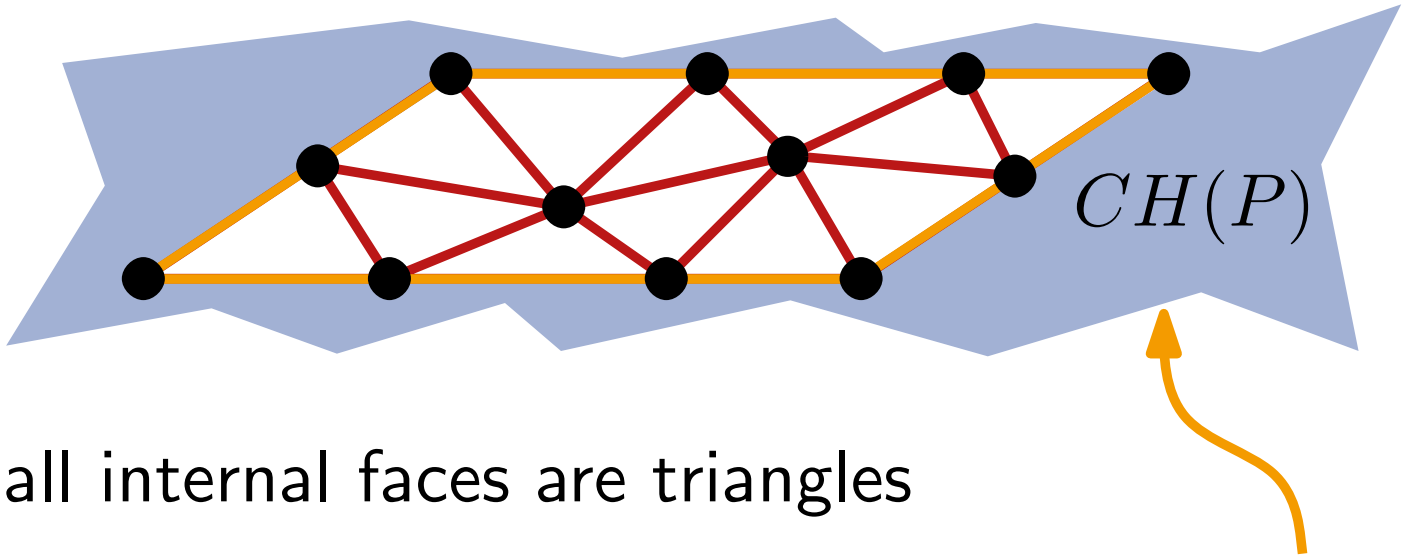

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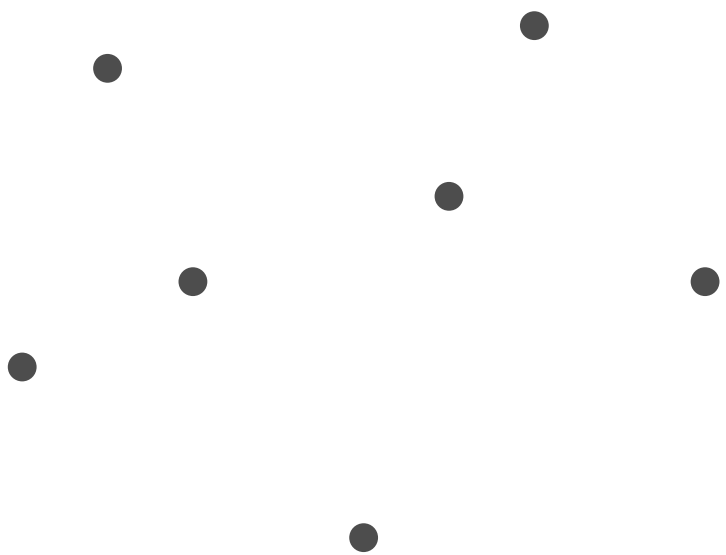
An edge $\overline{pq}$ is called **Delaunay** if there exists a circle passing through p and q and containing no other point in its interior.



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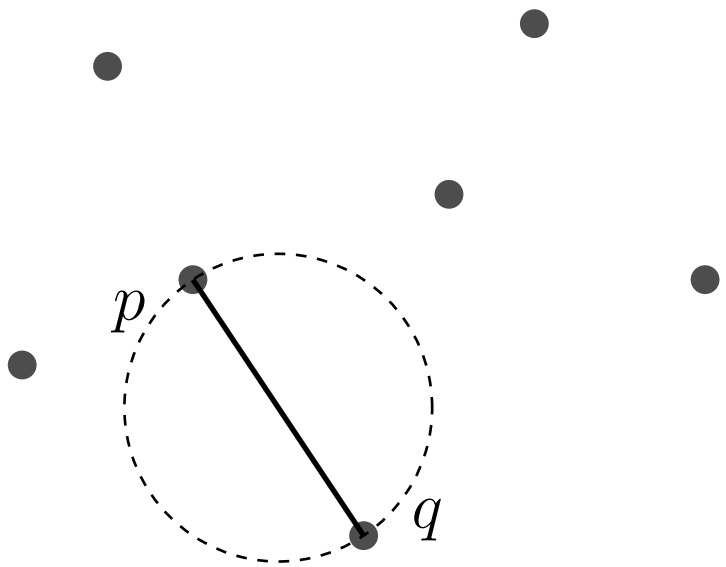

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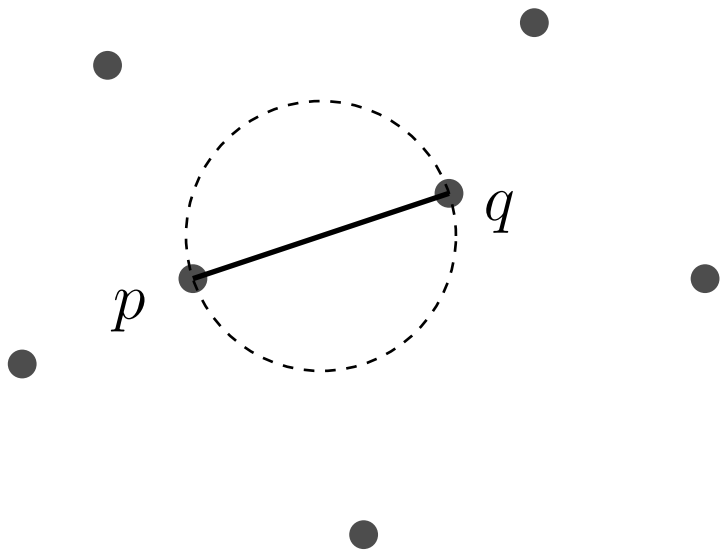

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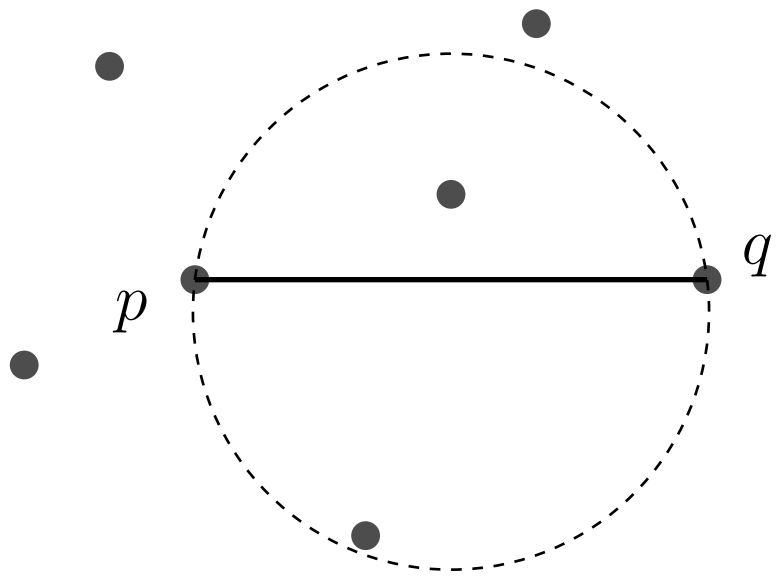

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Delaunay Triangulation

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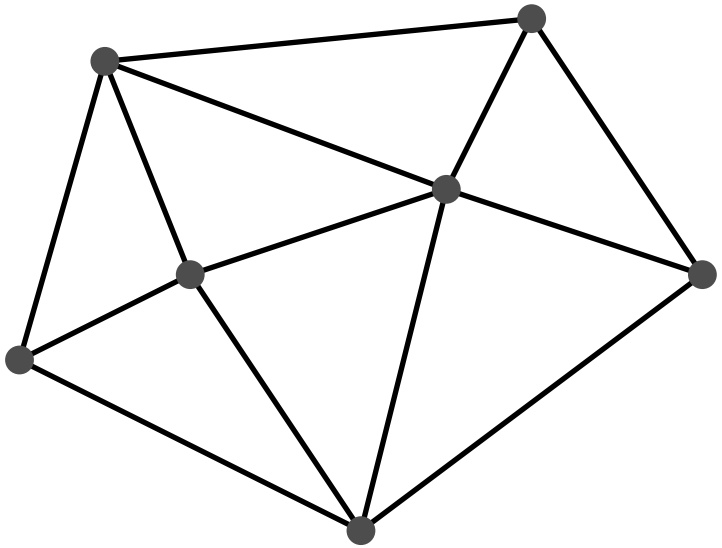

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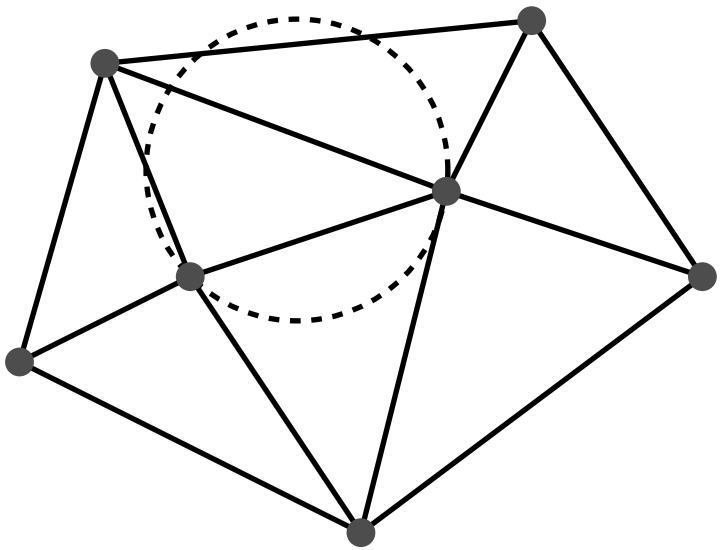

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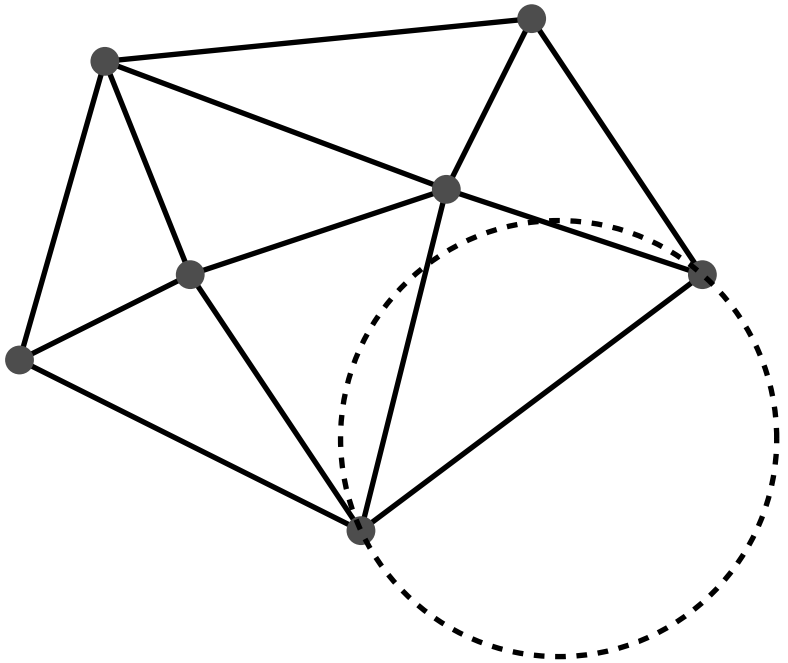

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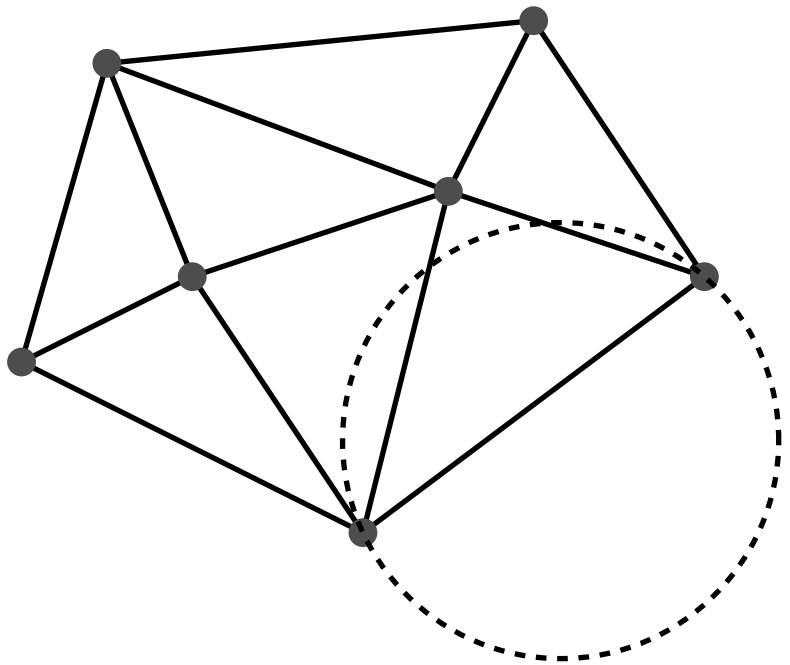

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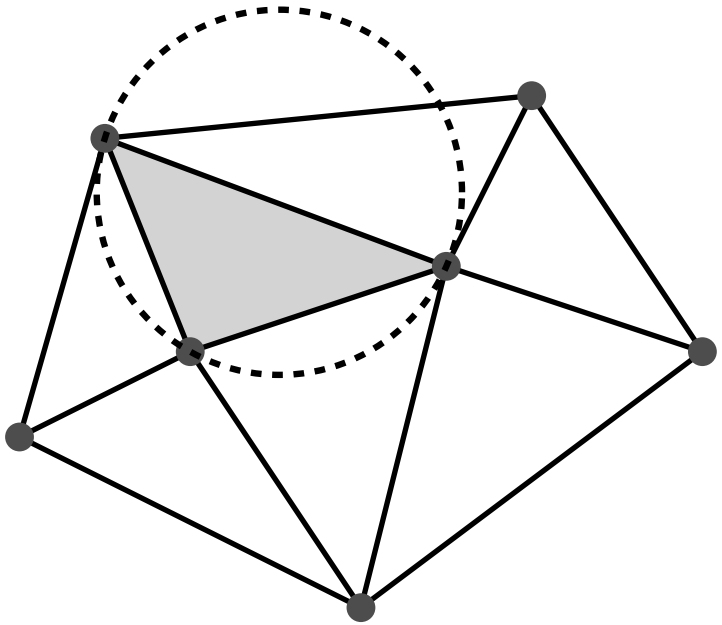

Angle-optimal Triangulations

Def.: Let $P \subset \mathbb{R}^2$ be a set of points, $\mathcal{T}$ be a triangulation of P and m be the number of the triangles.
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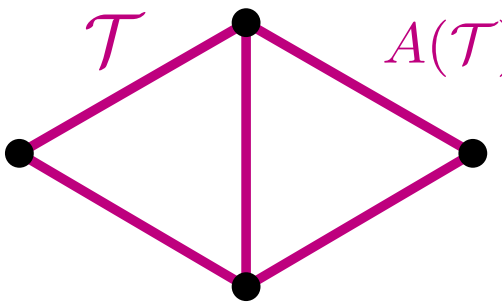

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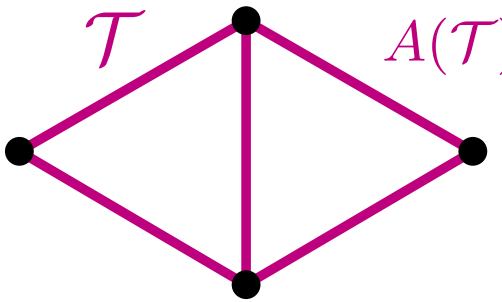

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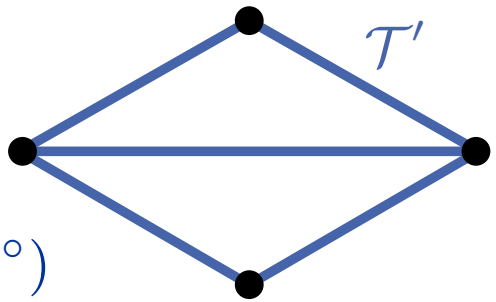

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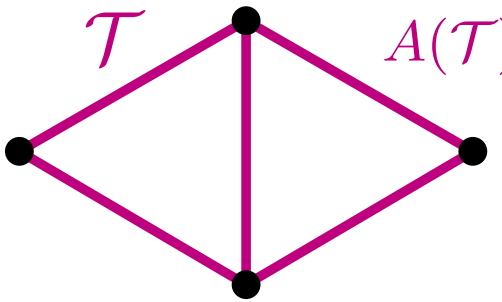

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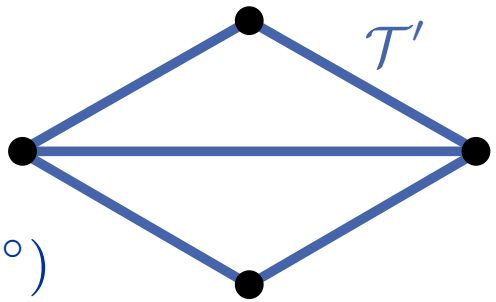

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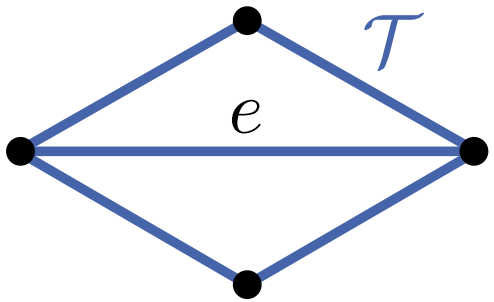

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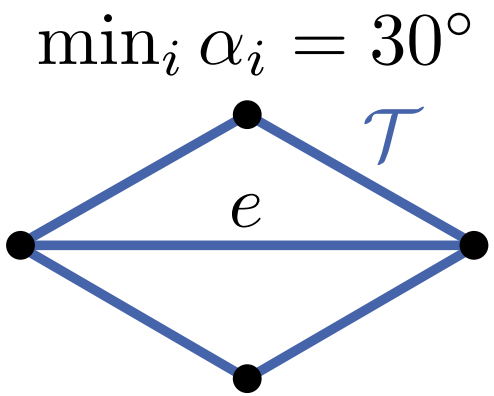

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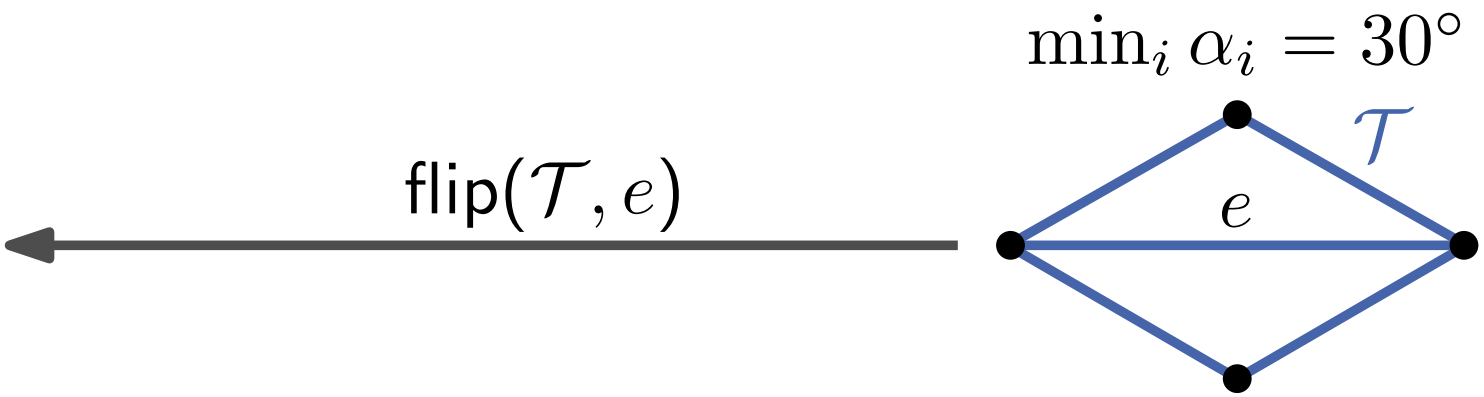

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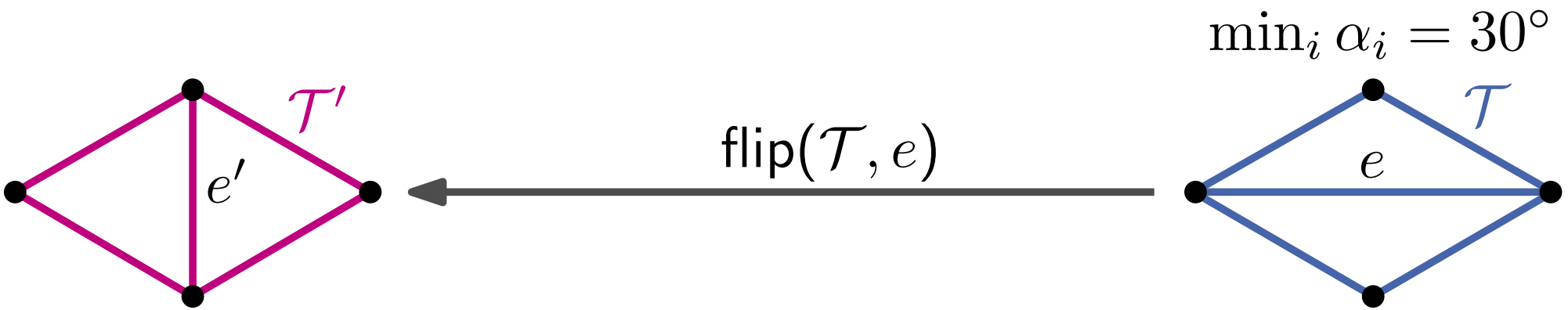

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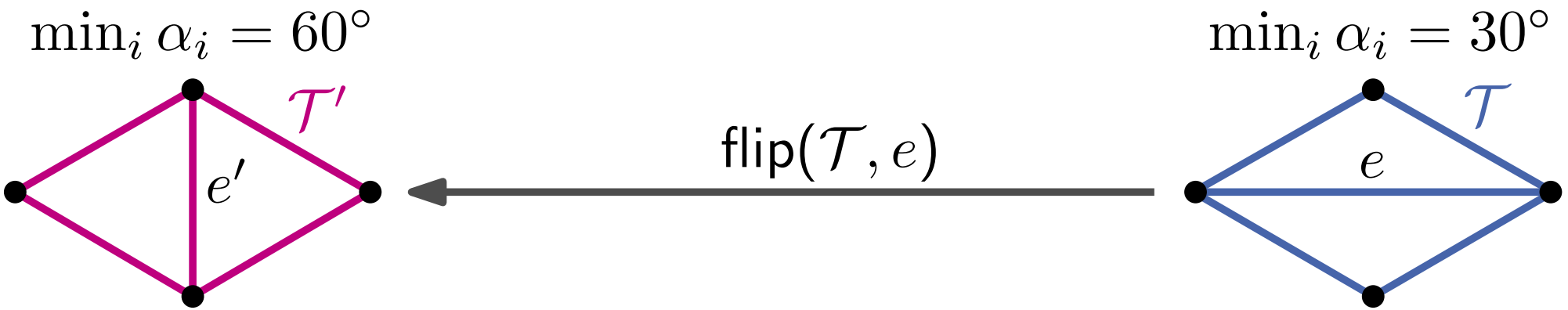

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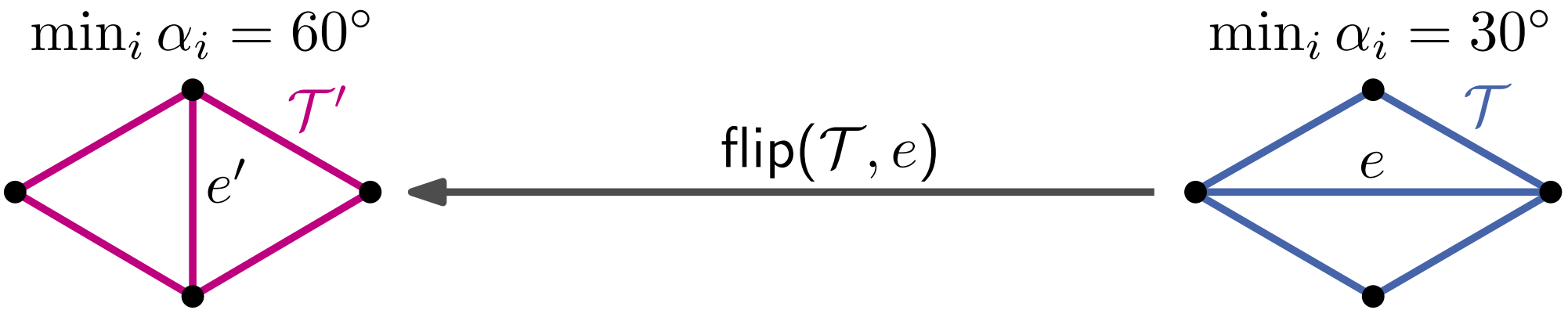
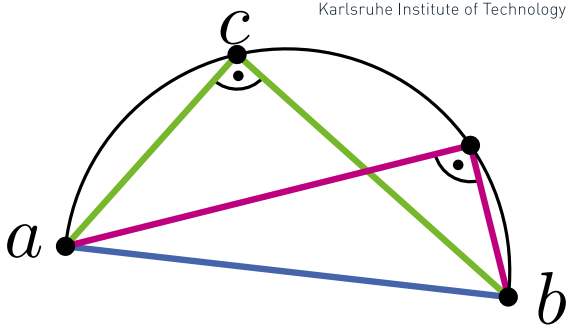
Obs.: Let e be an illegal edge of $\mathcal{T}$ and $\mathcal{T}' = \text{flip}(\mathcal{T}, e)$. Then $A(\mathcal{T}') > A(\mathcal{T})$.



Thales's Theorem

Theorem 4: If a, b and c are points on a circle where the segment ab is a diameter of the circle, then the angle $\angle bca$ is a right angle.



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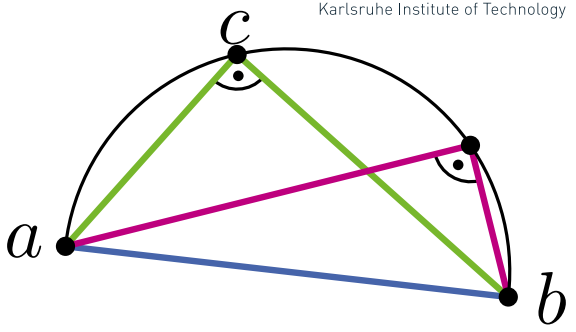
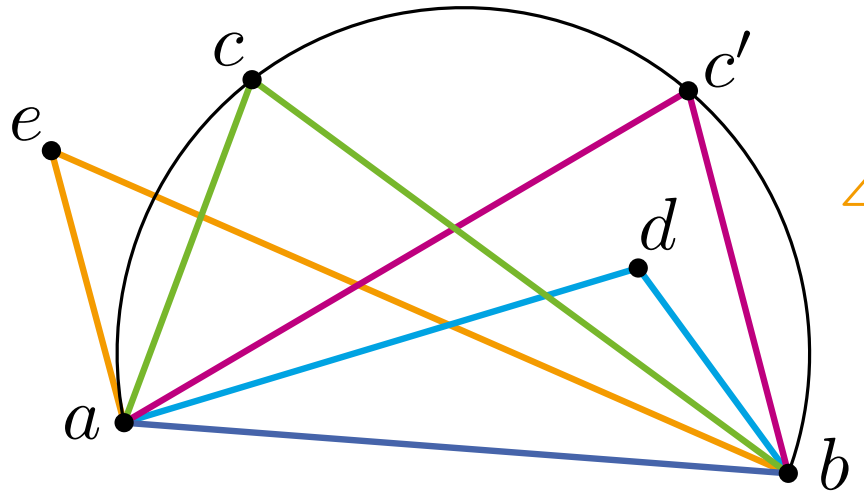


Theorem 4′: Consider a circle C through a, b, c. For any point c' on C on the same side of ab as c , holds that $\angle acb = \angle ac'b$. For any point d inside C holds that $\angle adb > \angle acb$, and for point e outside C, holds that $\angle aeb < \angle acb$.



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Lemma 5: Let Δpqr and Δpqs be two triangles in $\mathcal{T}$ and let C be the circle through Δpqr. Then $\overline{pq}$ is illegal iff $s \in \text{int}(C)$. If p, q, r, s form a convex quadrilateral and do not lie on a common circle ($s \notin \partial C$) then exactly one of $\overline{pq}$ and $\overline{rs}$ is an illegal edge.

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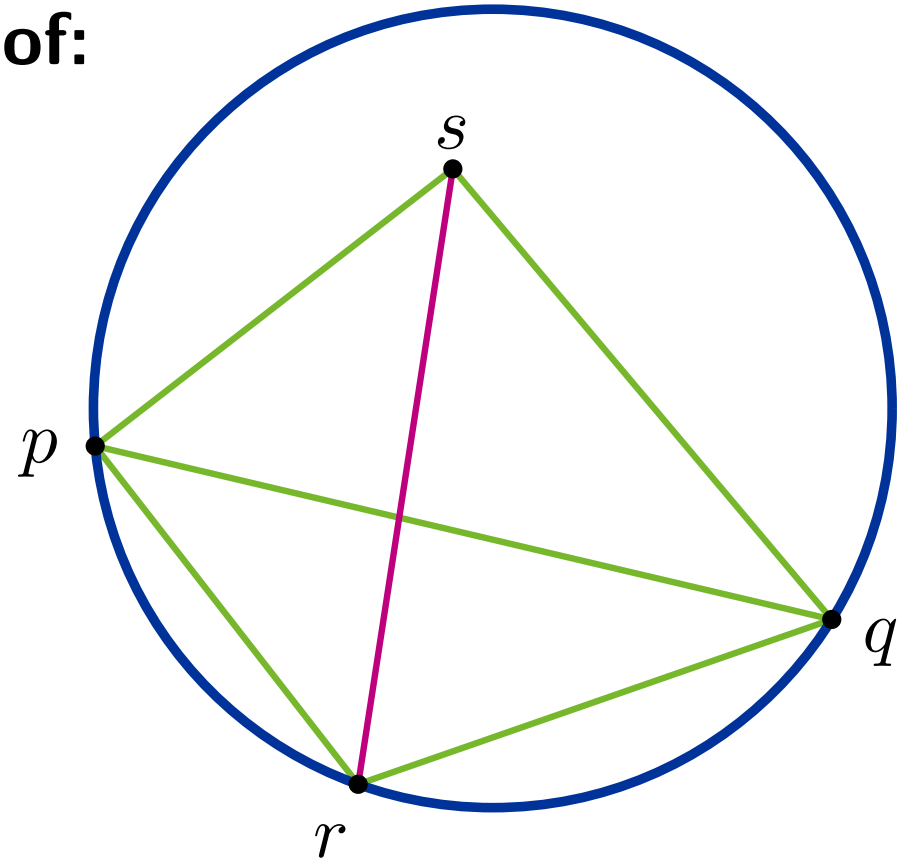

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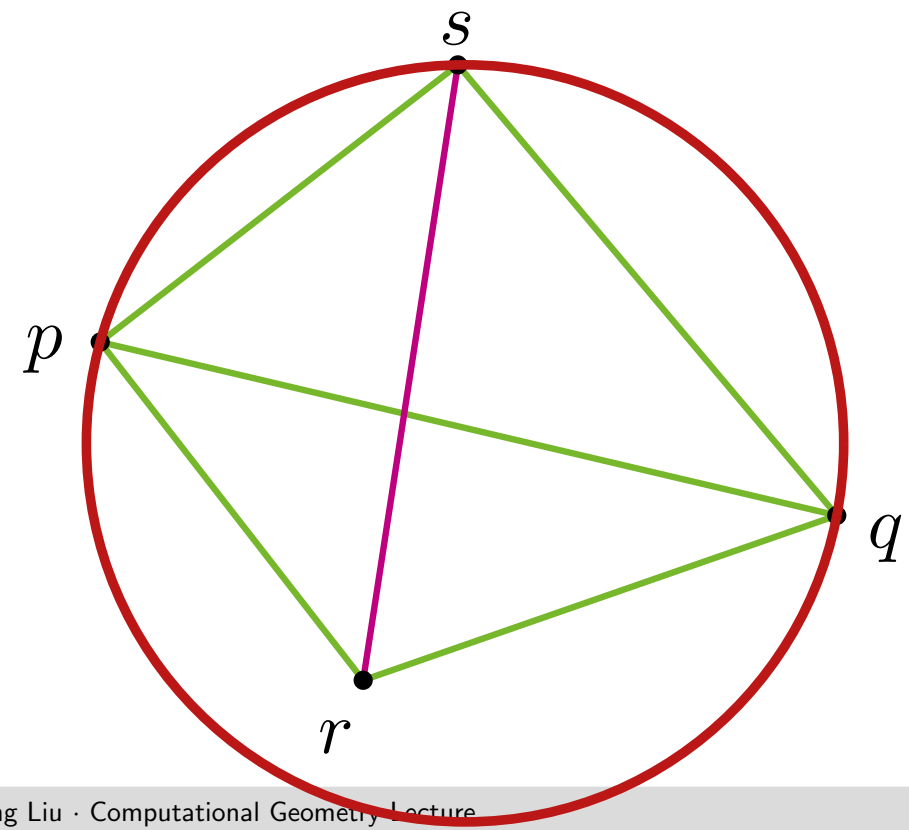

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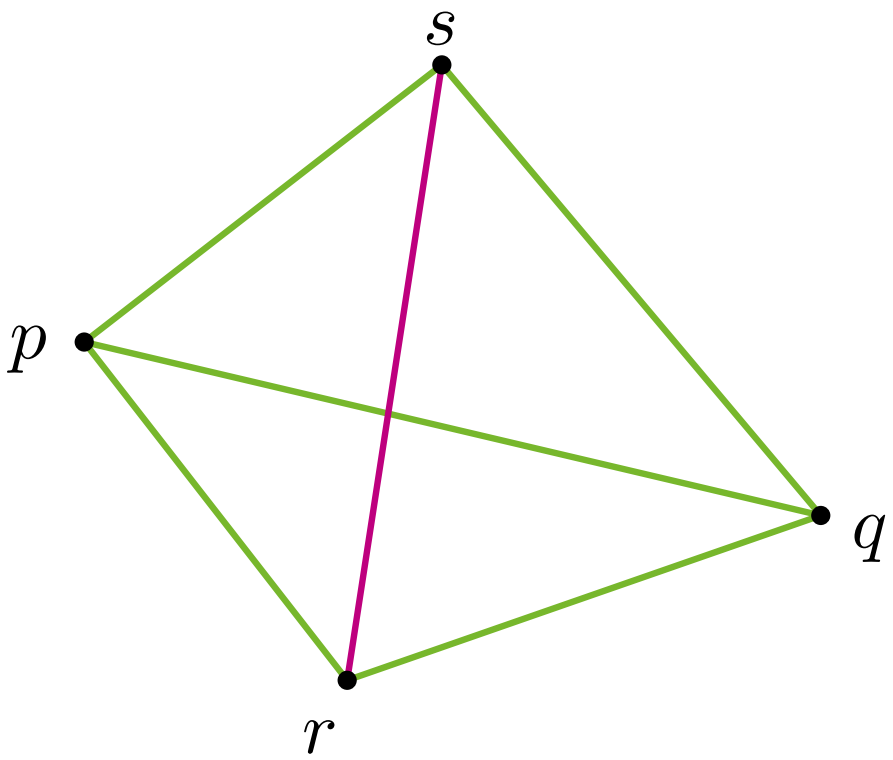

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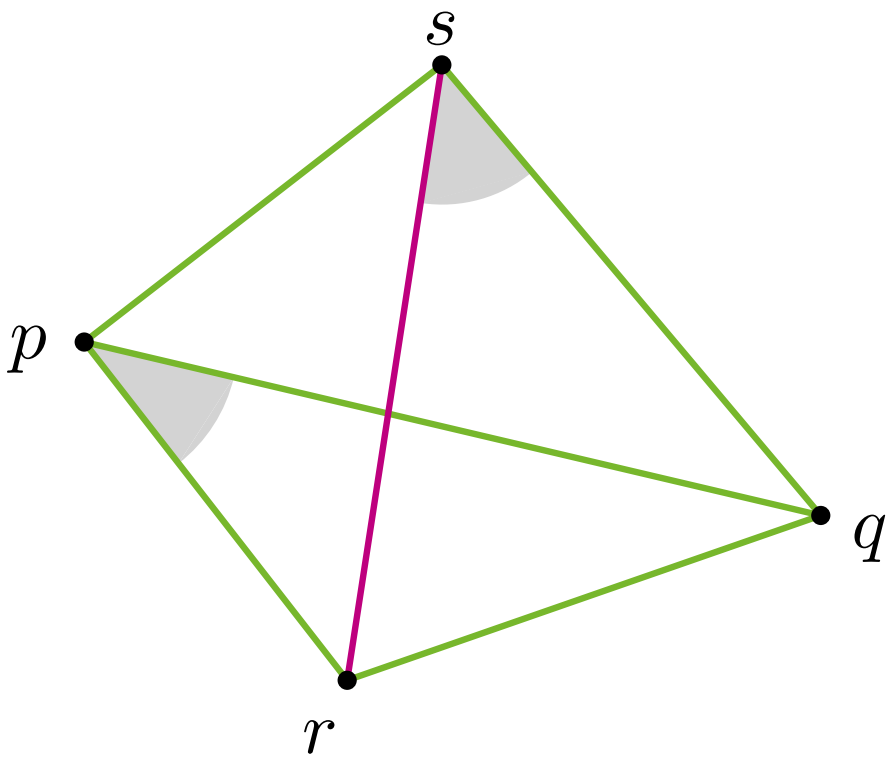

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Is there always a legal triangulation?

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terminates, since $A(\mathcal{T})$ increases and #Triangulations is finite ($< 30^n$, [Sharir, Sheffer 2011])

Reverse statement?

It holds that: each angle-optimal triangulation is legal.

But is each legal triangulation also angle-optimal?

Legality and Delaunay-Triangulation

Karlsruhe Institute of Technology

Theorem 6: Let P be a set of points and $\mathcal{T}$ a triangulation of P. $\mathcal{T}$ is legal $\Leftrightarrow$ $\mathcal{T}$ is Delaunay-Triangulation.

Sketch of proof:

„$\Leftarrow$“ clear; use

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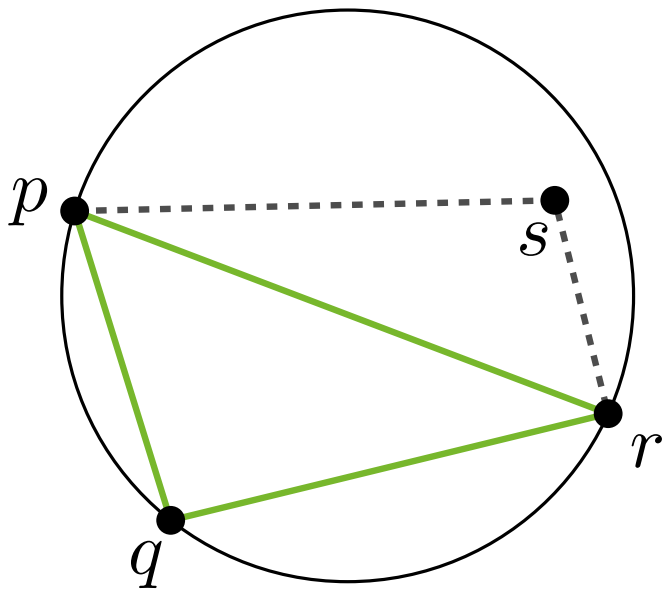

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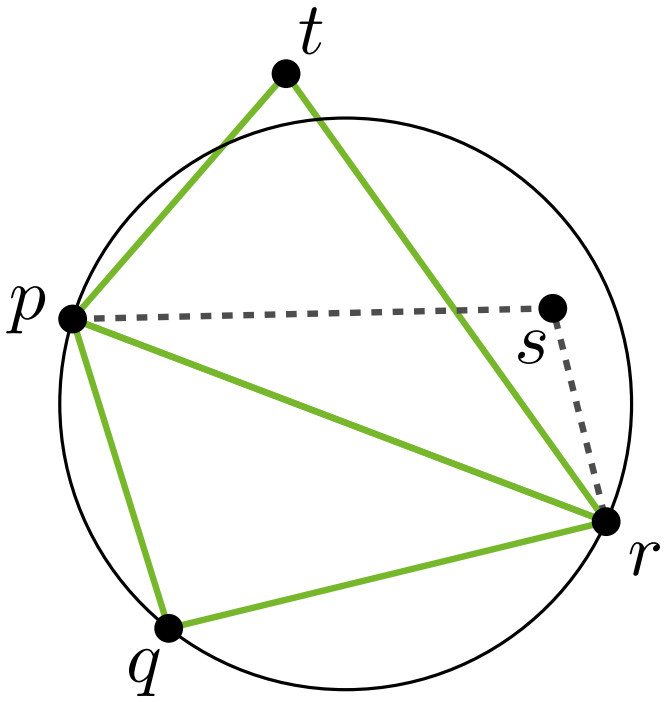

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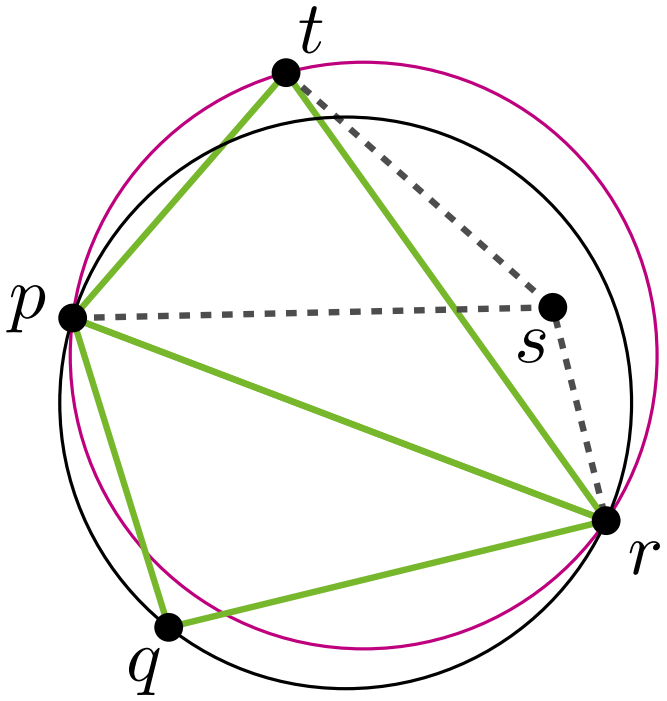

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- Delaunay Triangulation is unique
  - Angle-Optimal!!

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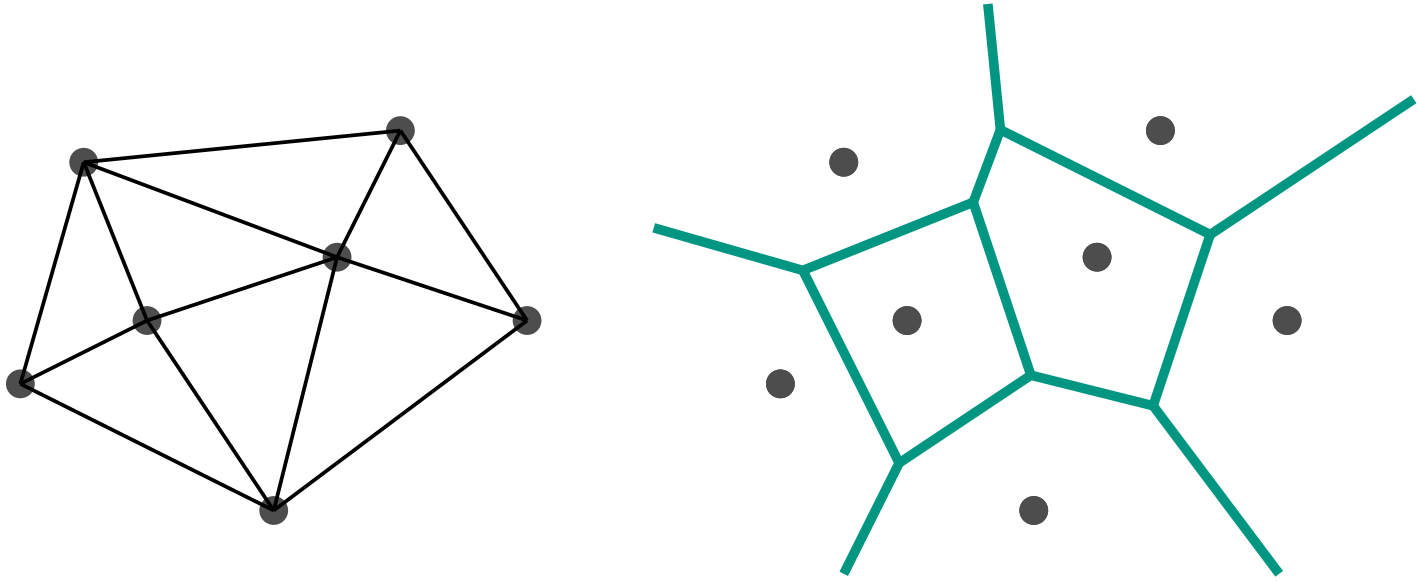

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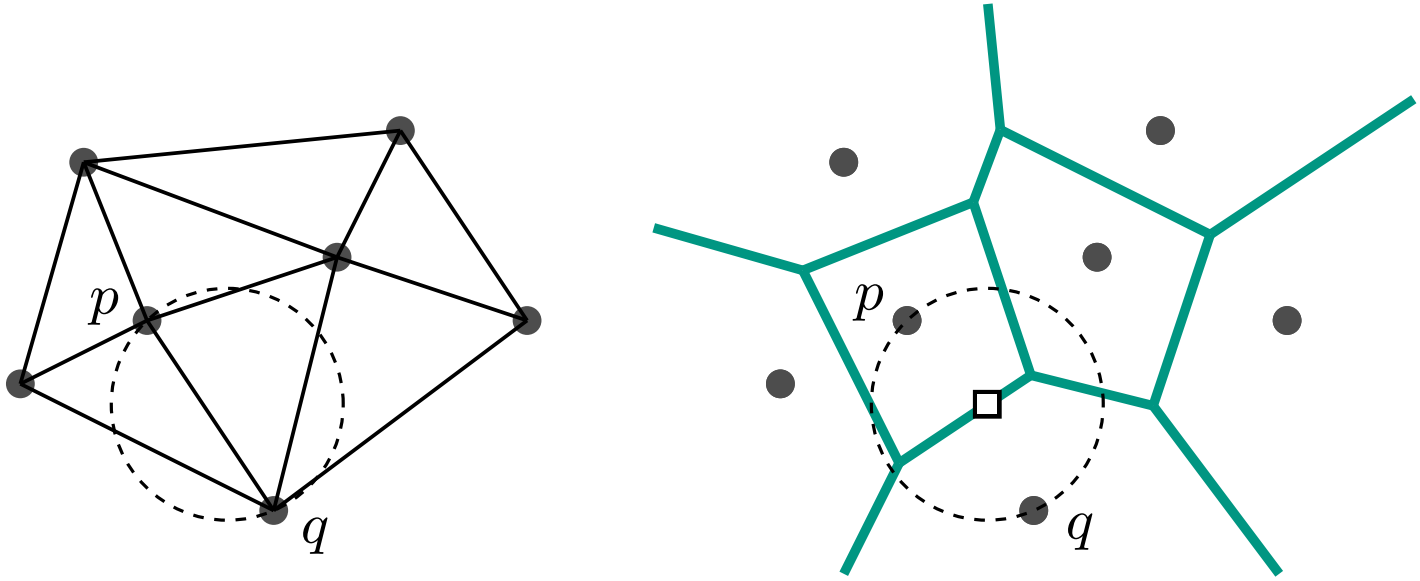

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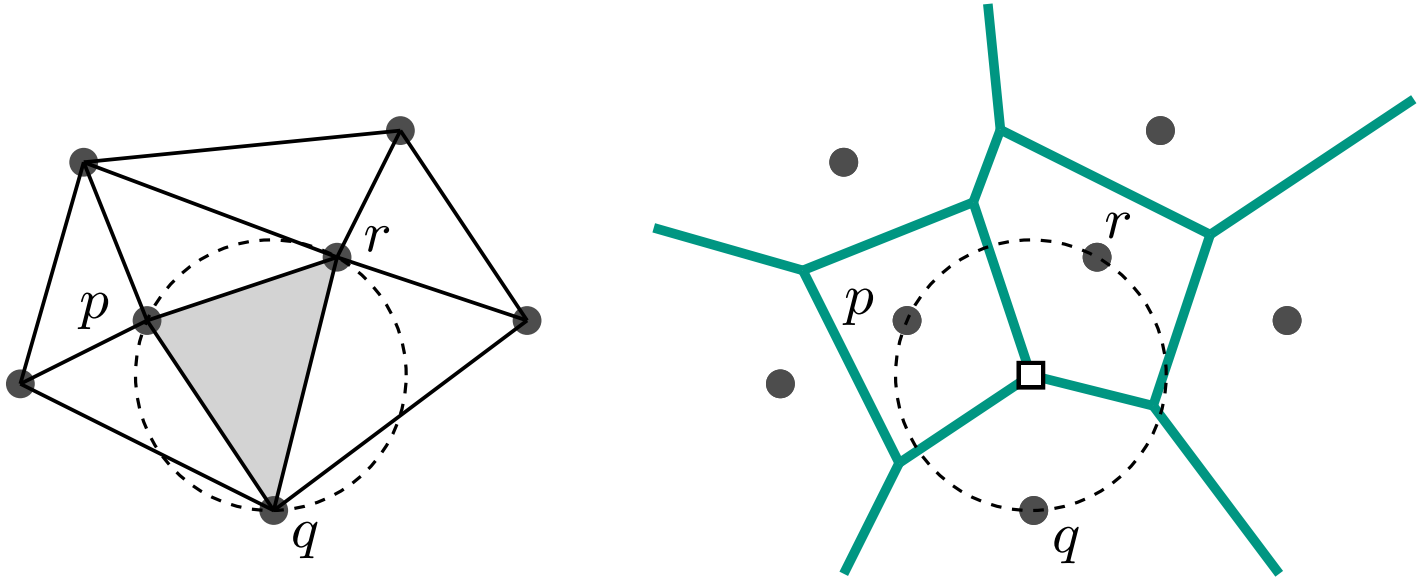

Computing Vor(P) (Upper Bound)

Karlsruhe Institute of Technology

How can we calculate Vor(P) with methods we already know?

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Is $O(n^2 \log n)$ running time for a linear-size object necessary?

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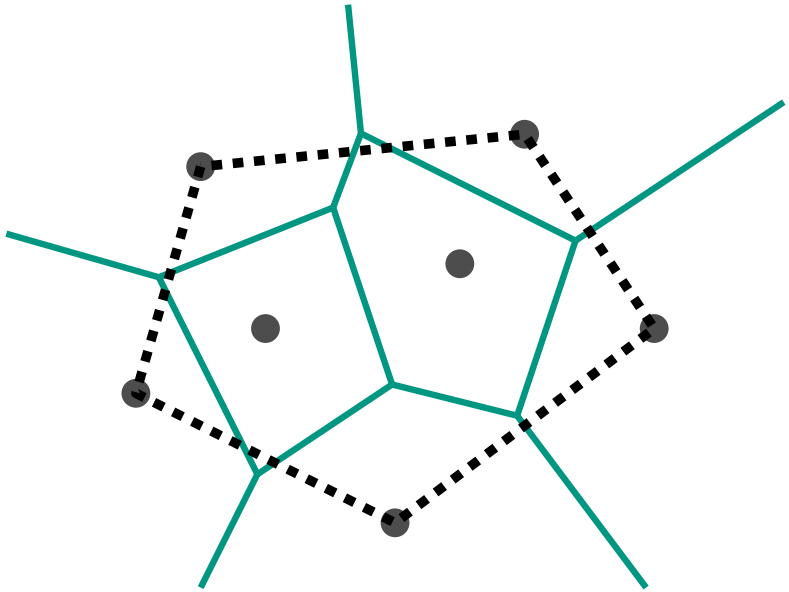

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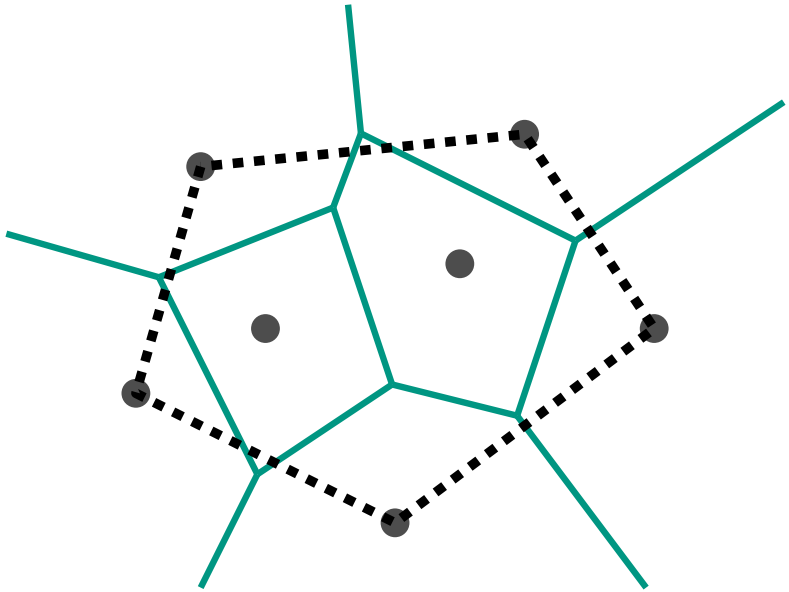

- $O(n \log n)$-Time Algorithms
  - Plane Sweep
  - Divide and Conquer

Plane-Sweep Paradigm

- Move a line H to sweep the plane (from left to right)

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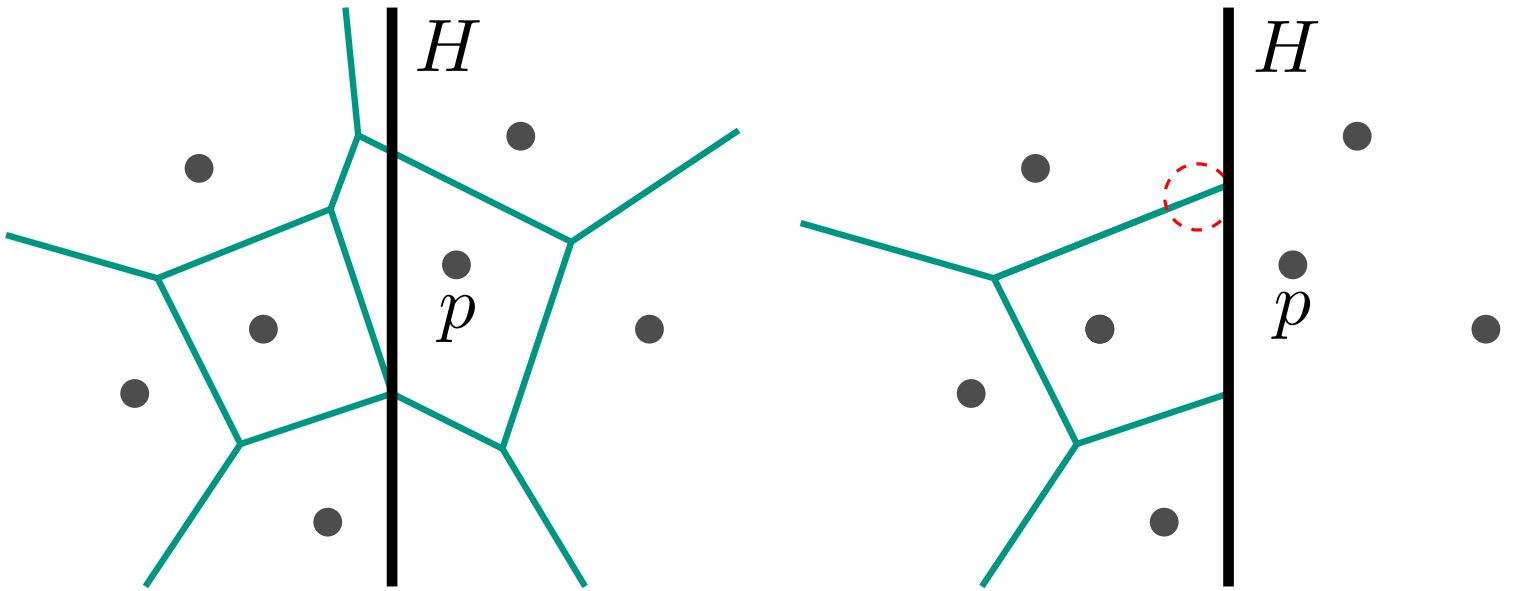

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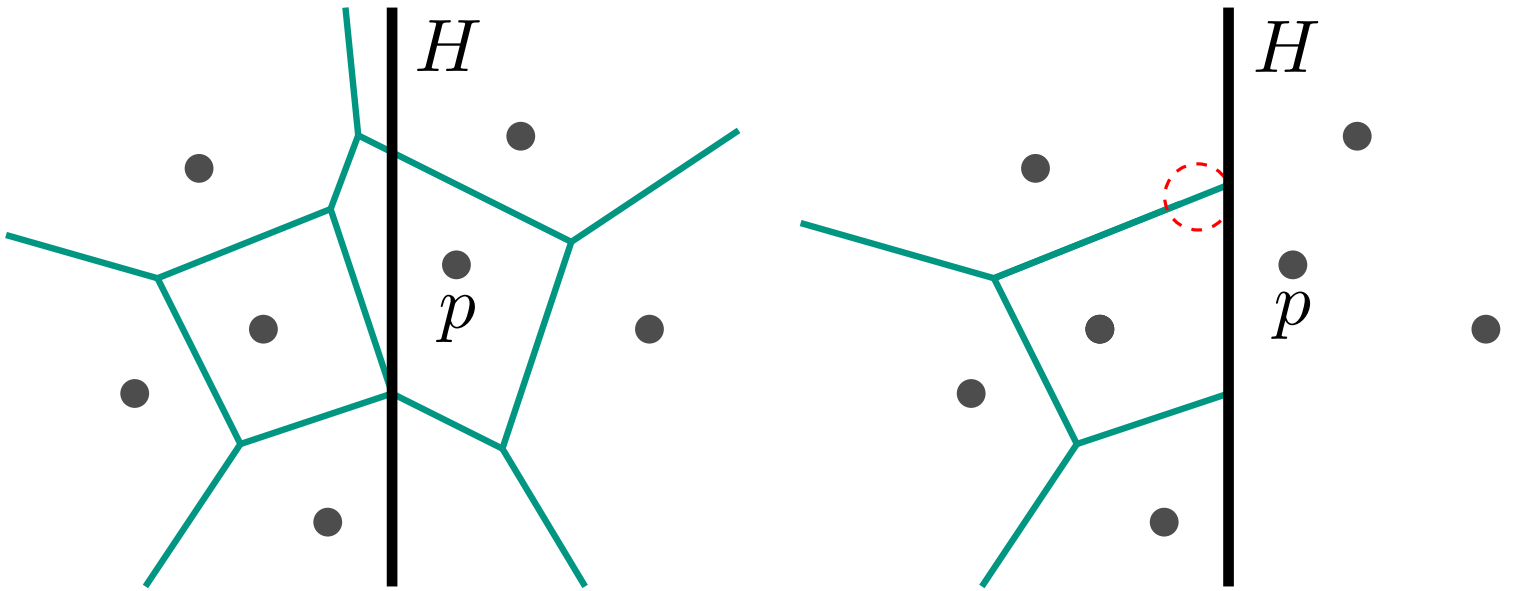
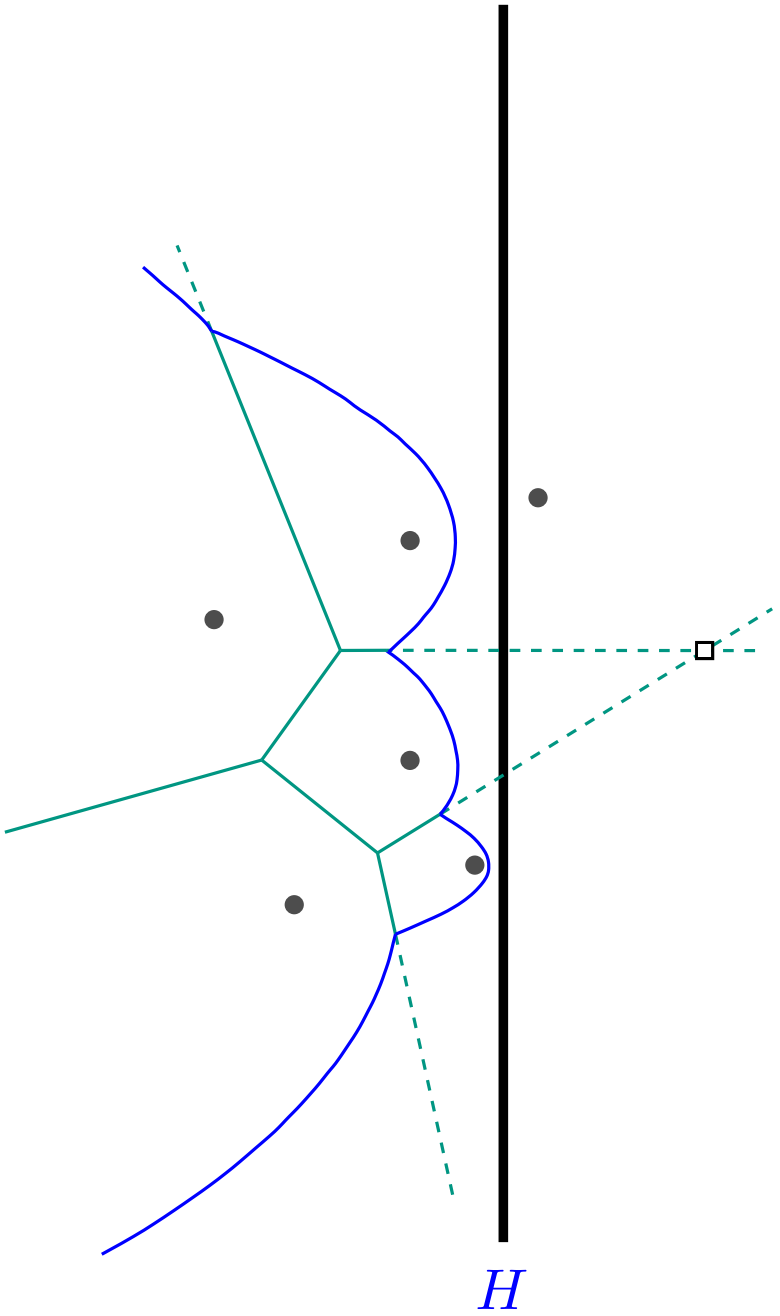
  - only consider sites left to $H \rightarrow$ wrong substructure



Voronoi Diagram with Sweep Line



- View H as a site

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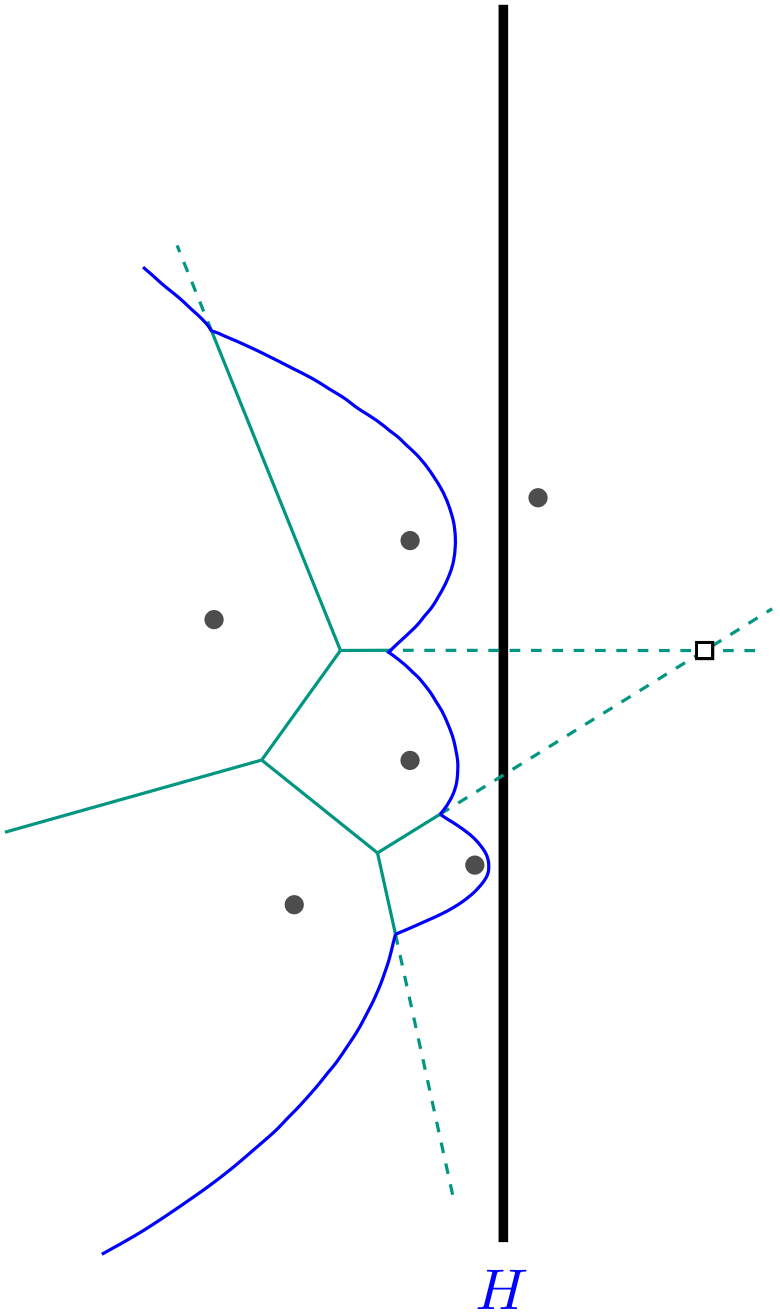

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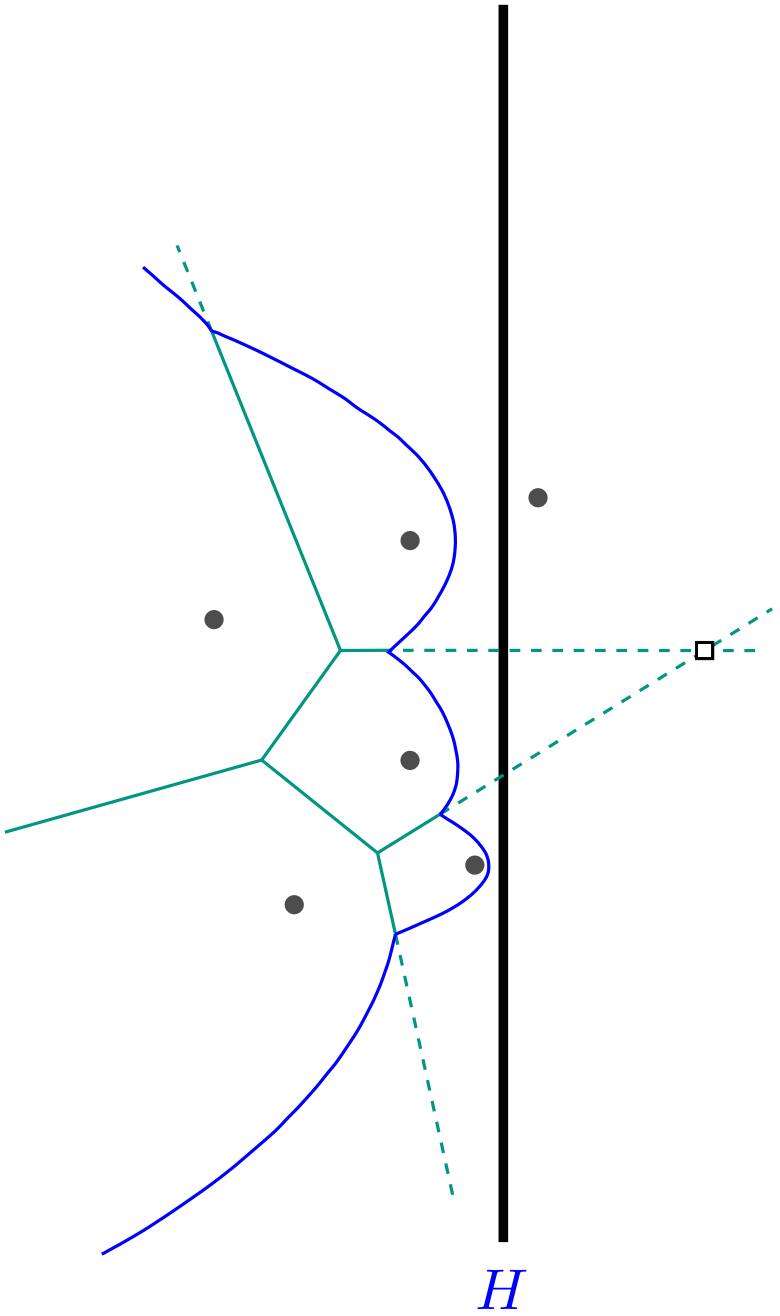

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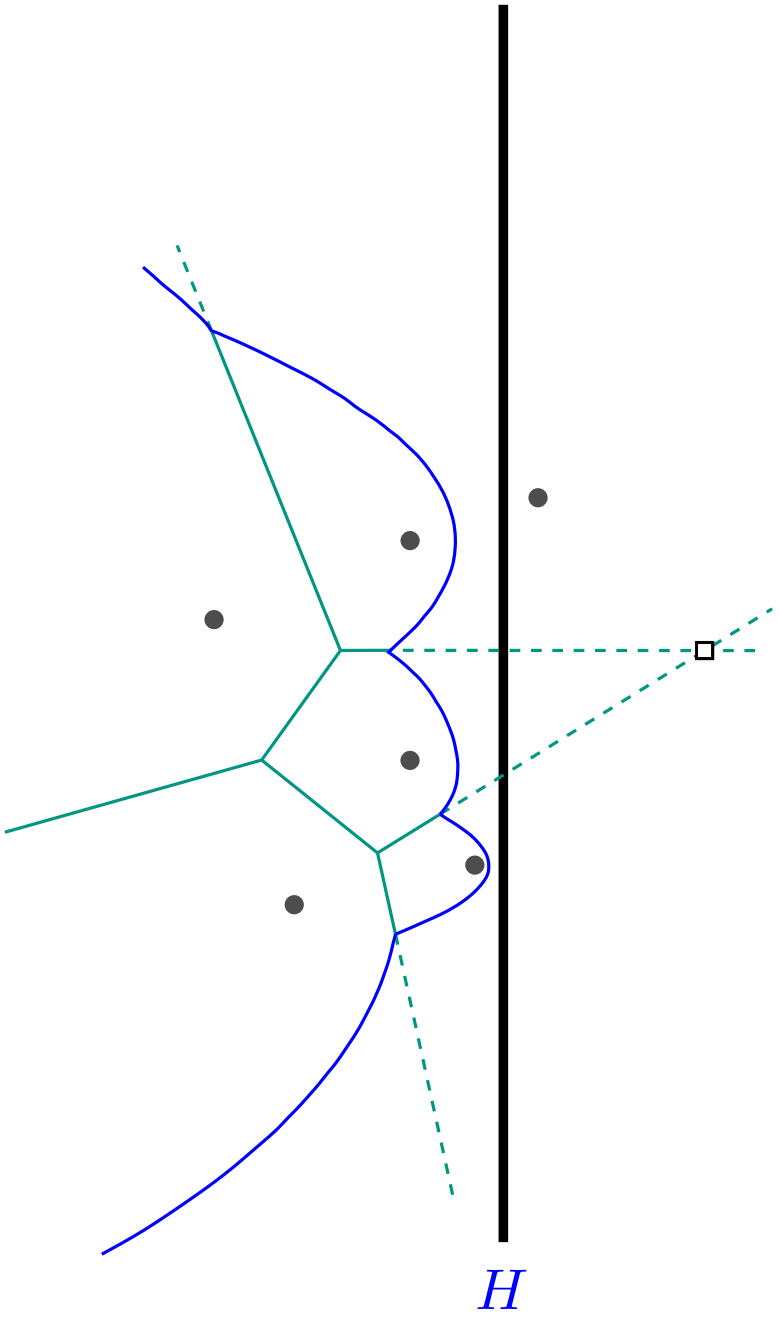

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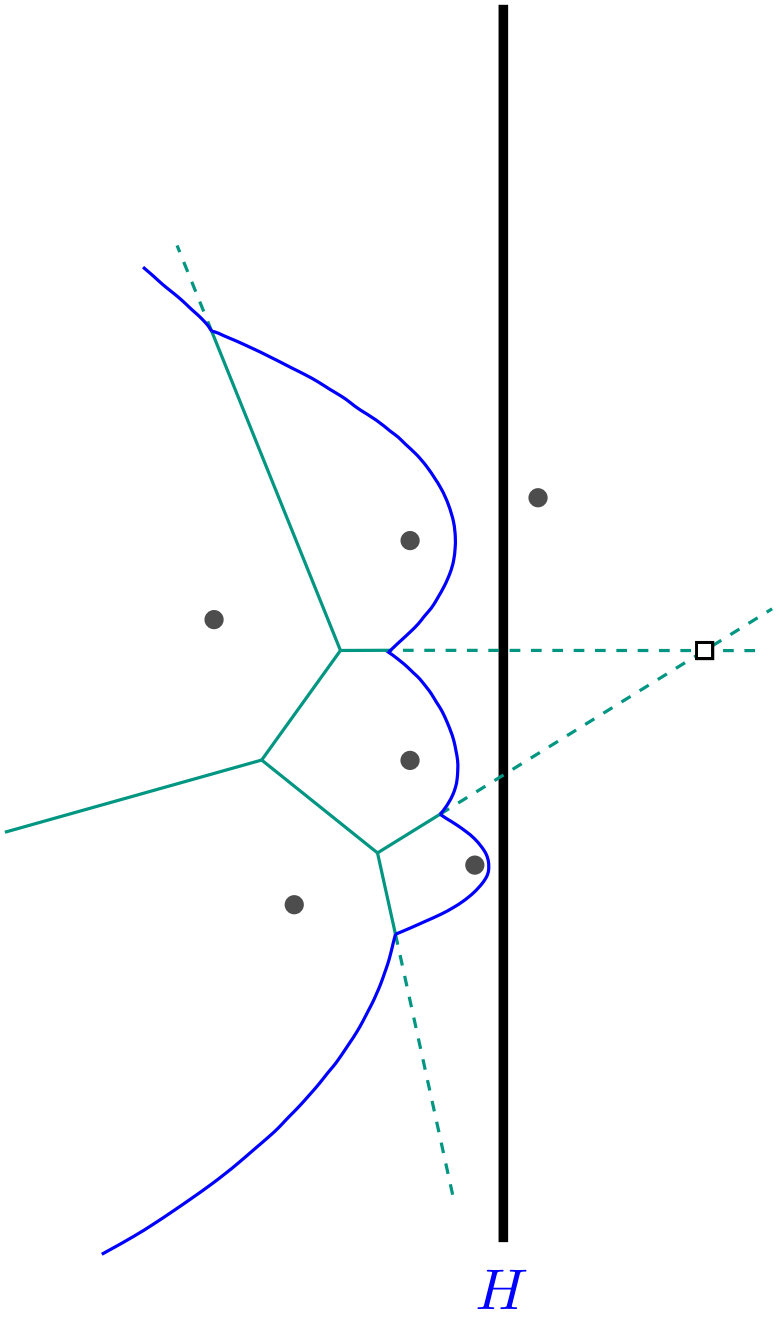

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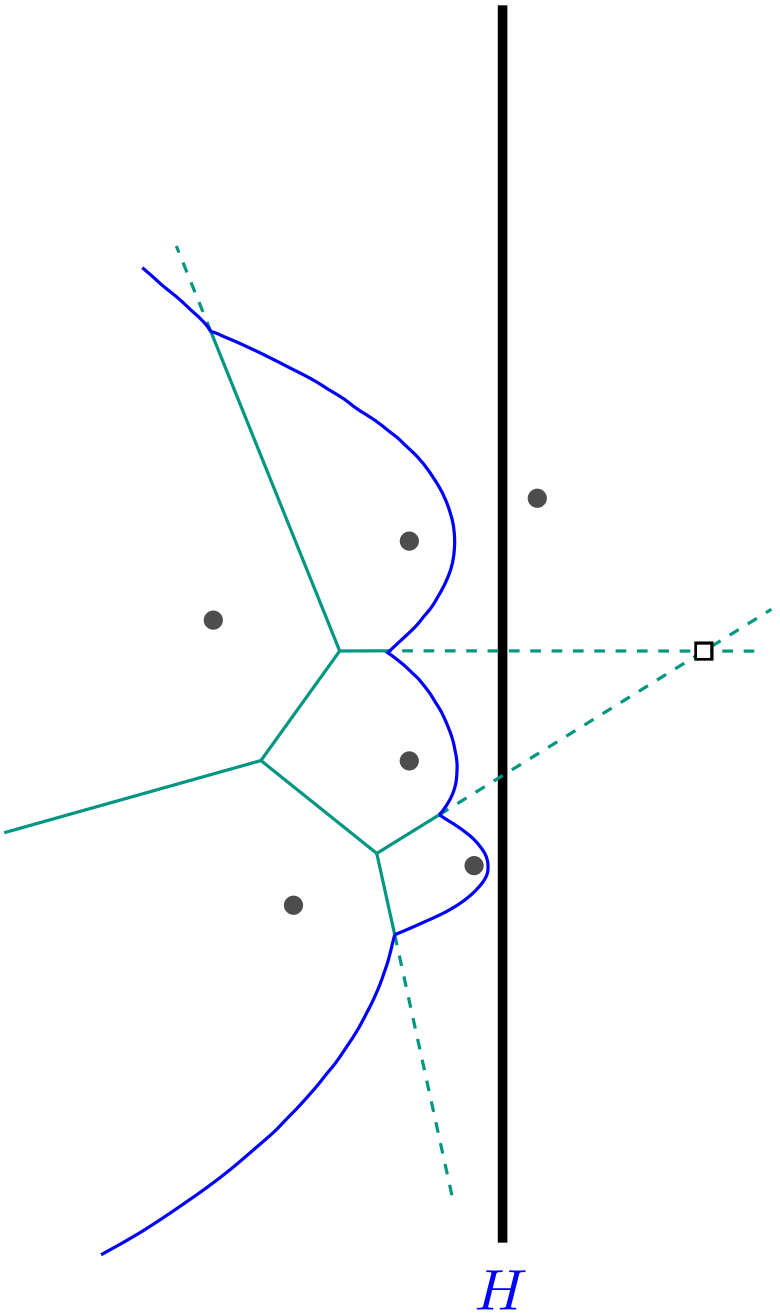
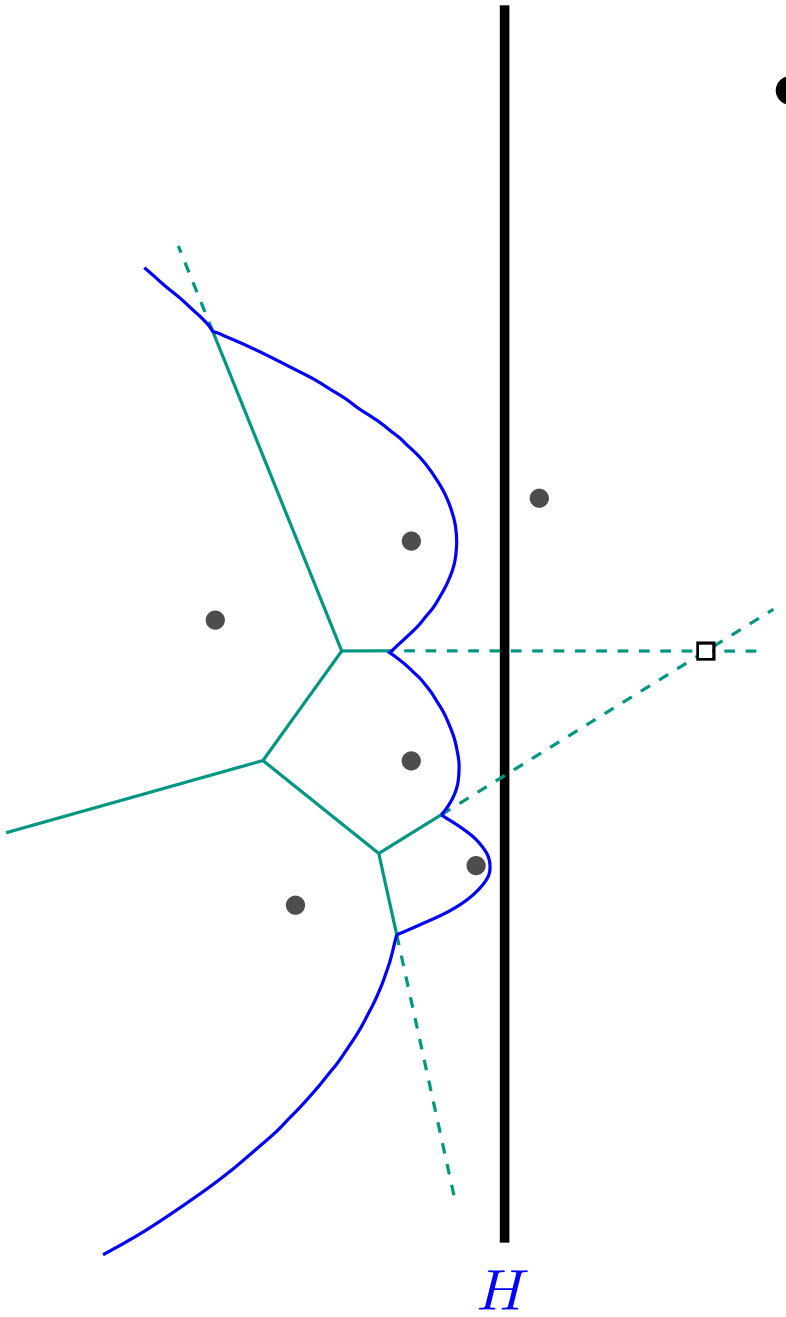
$$b(p, H) = \{x \in R^2 : d(x, p) = d(x, H)\}$$

Wavefront and Spikes



Wavefront and Spikes



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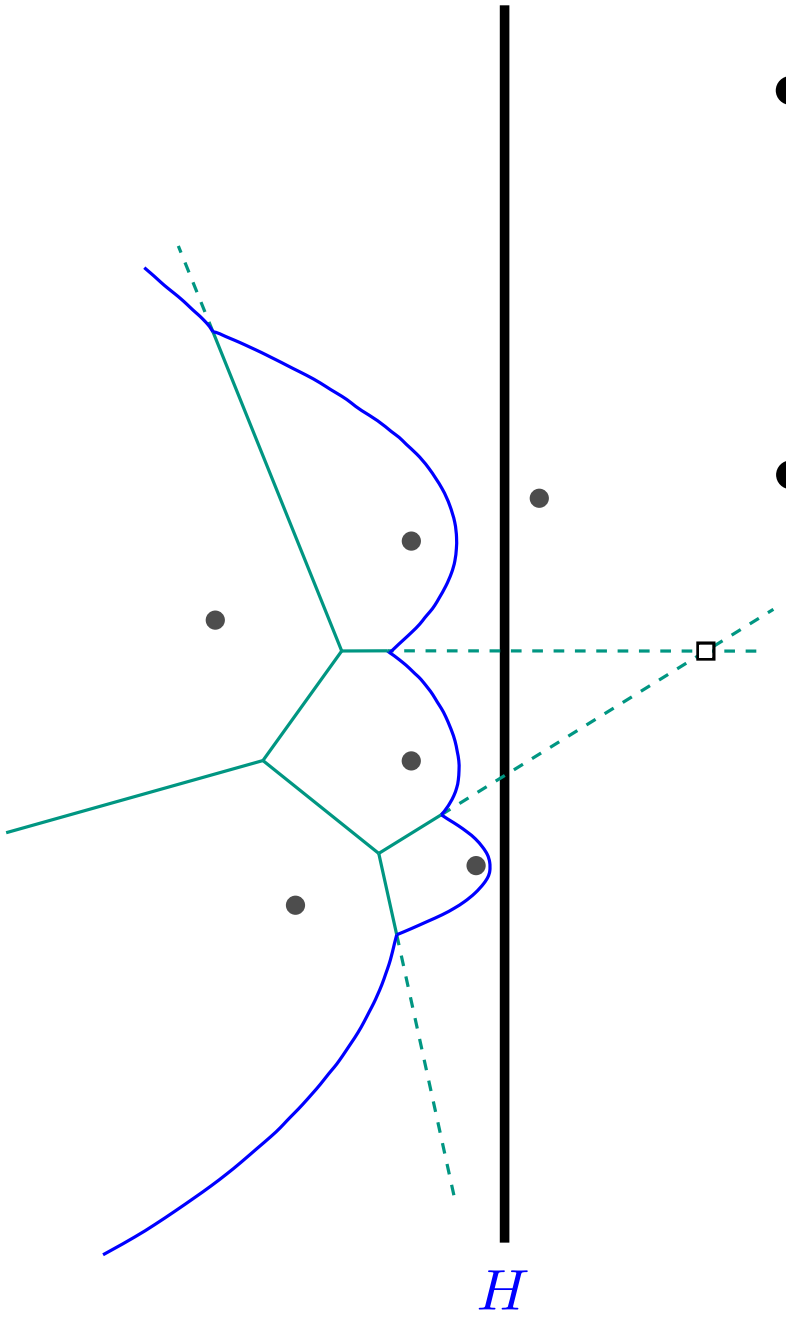
  - A y-monotone curve
  - A wavelet: an edge of $\mathcal{V}(H)$
  - A point site may have > 1 wavelets

Wavefront and Spikes



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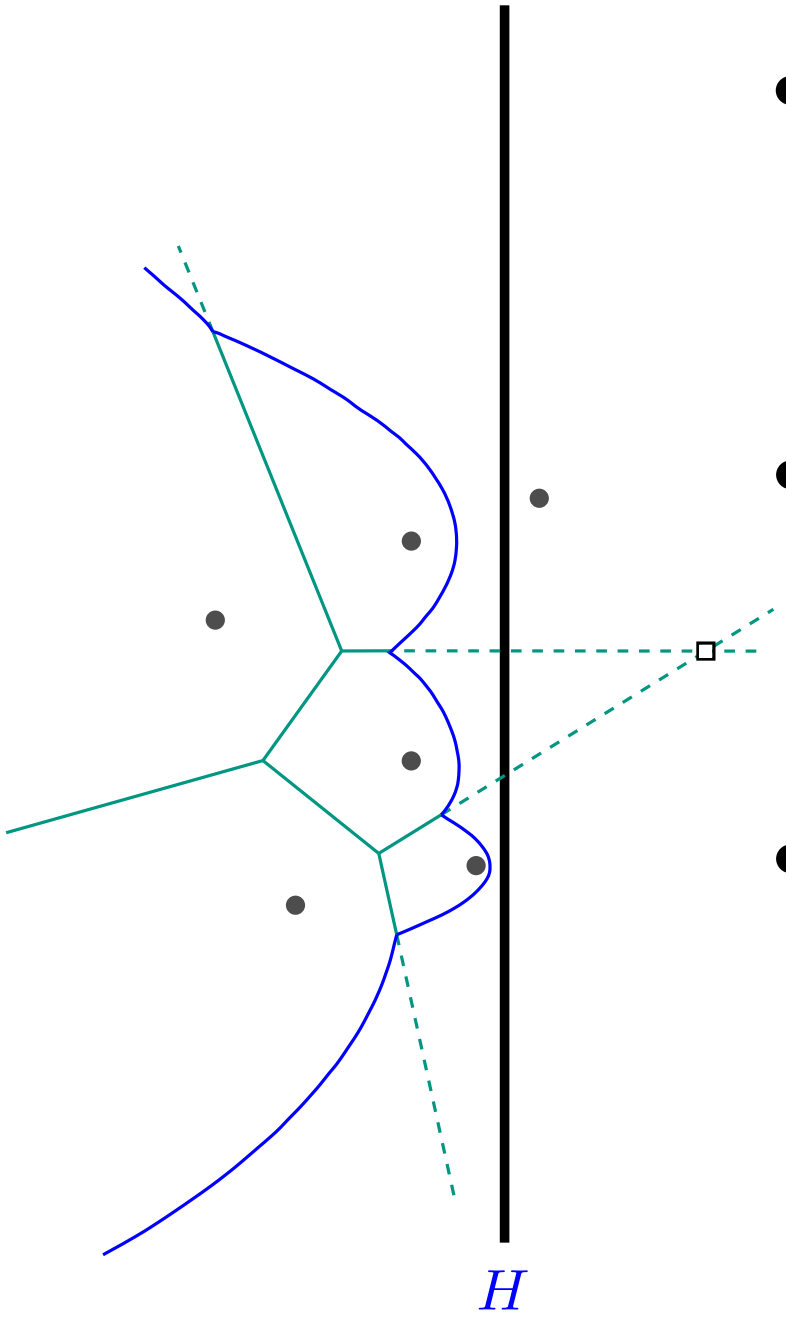

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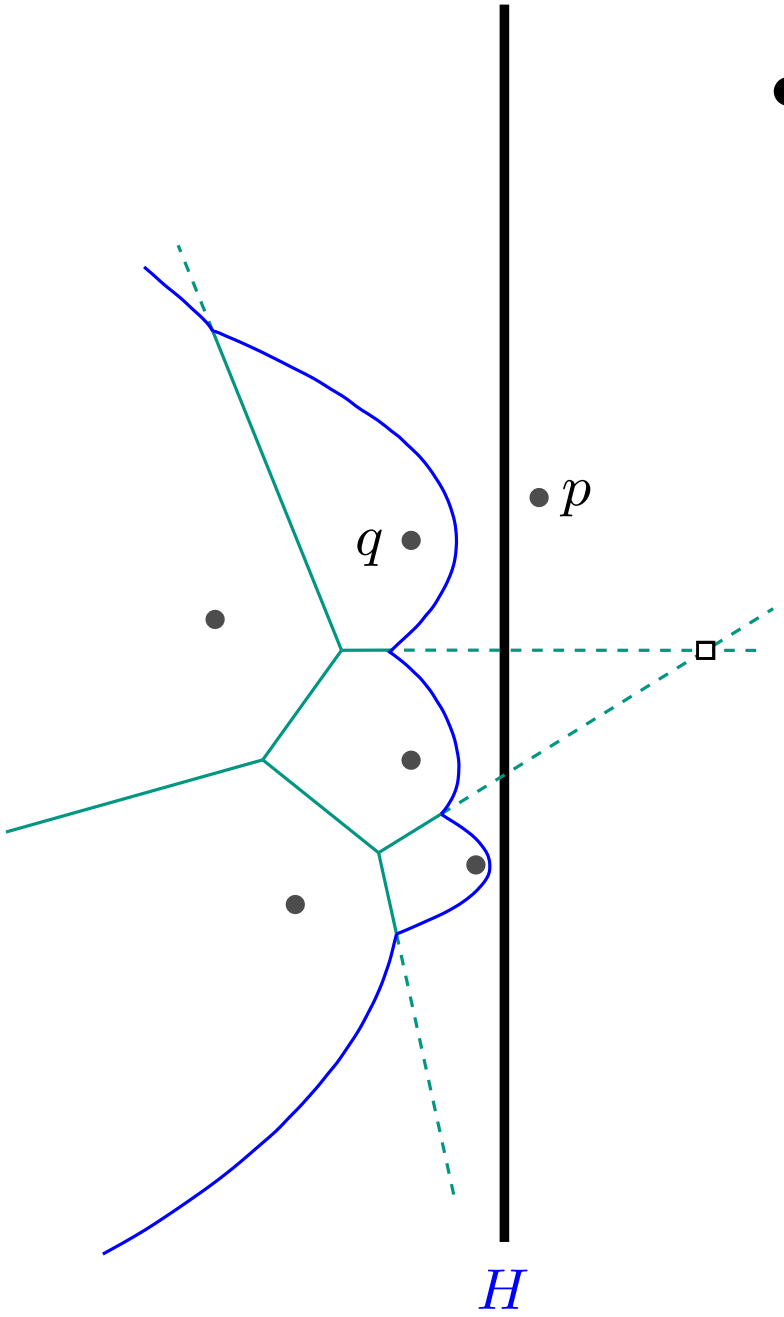
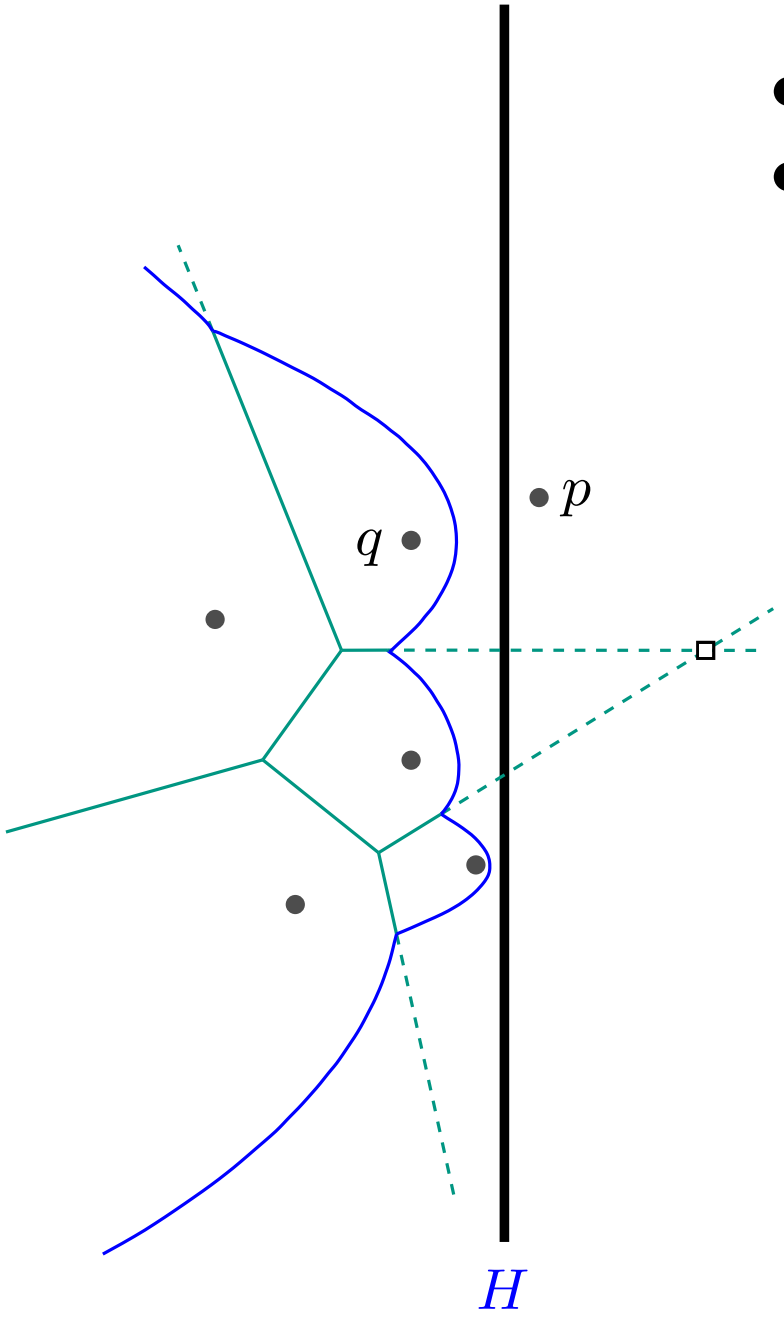
- H moves right
  - W moves right
  - Wavelets move along their spikes

Site Event



- H hits a new site p

Site Event



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  - Where ?

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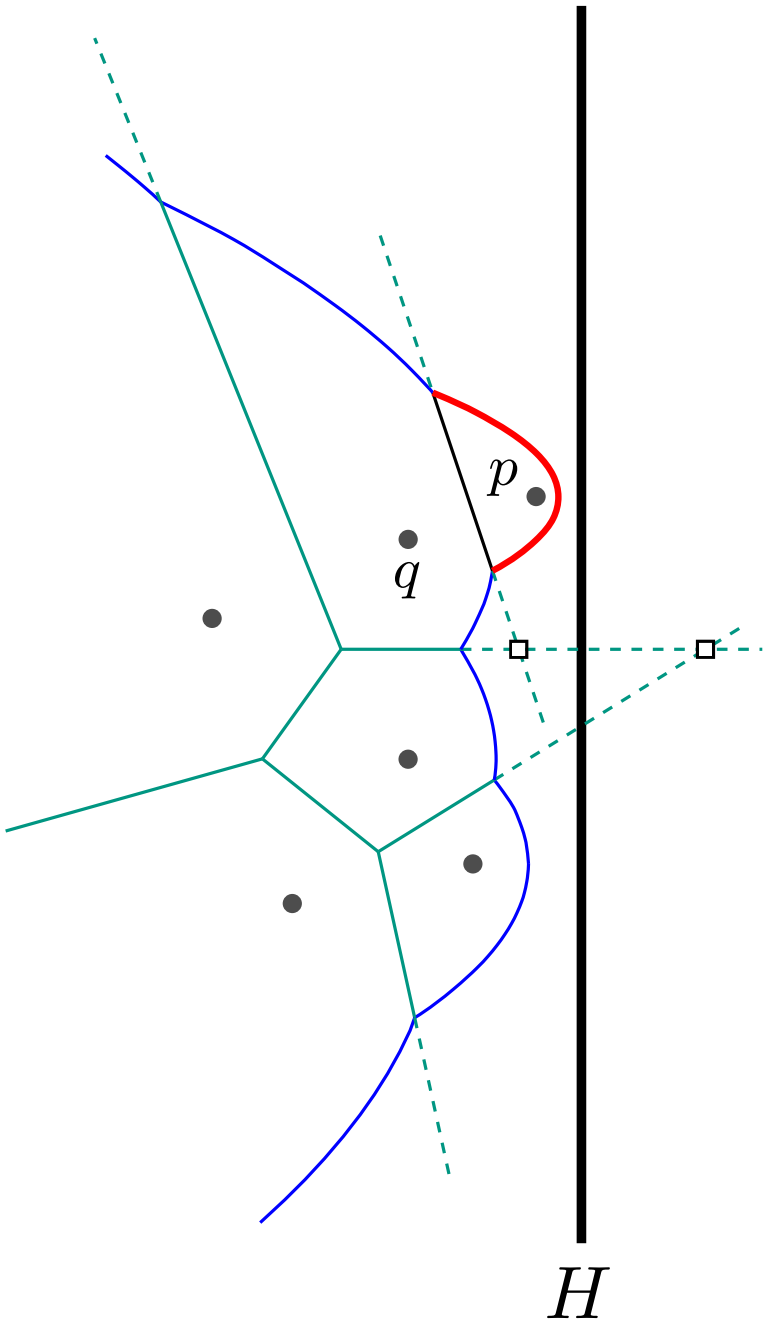

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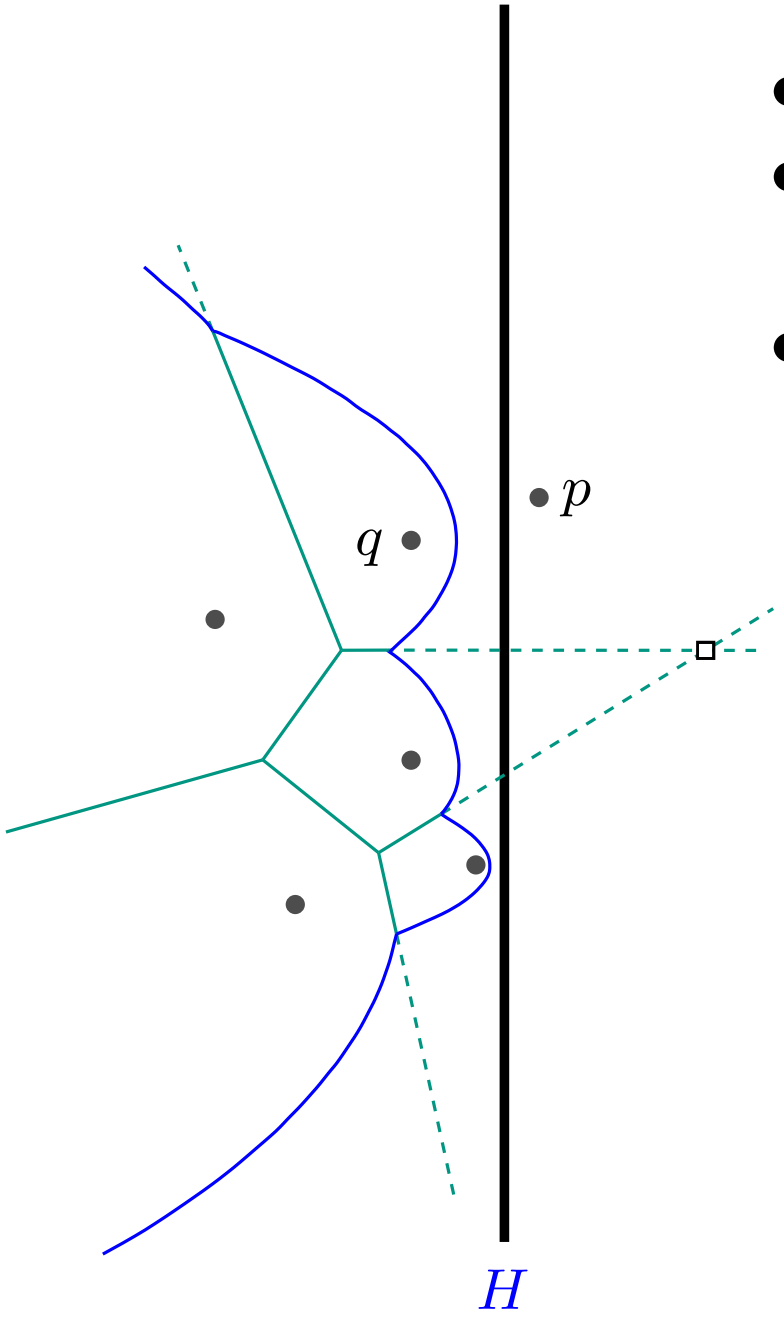

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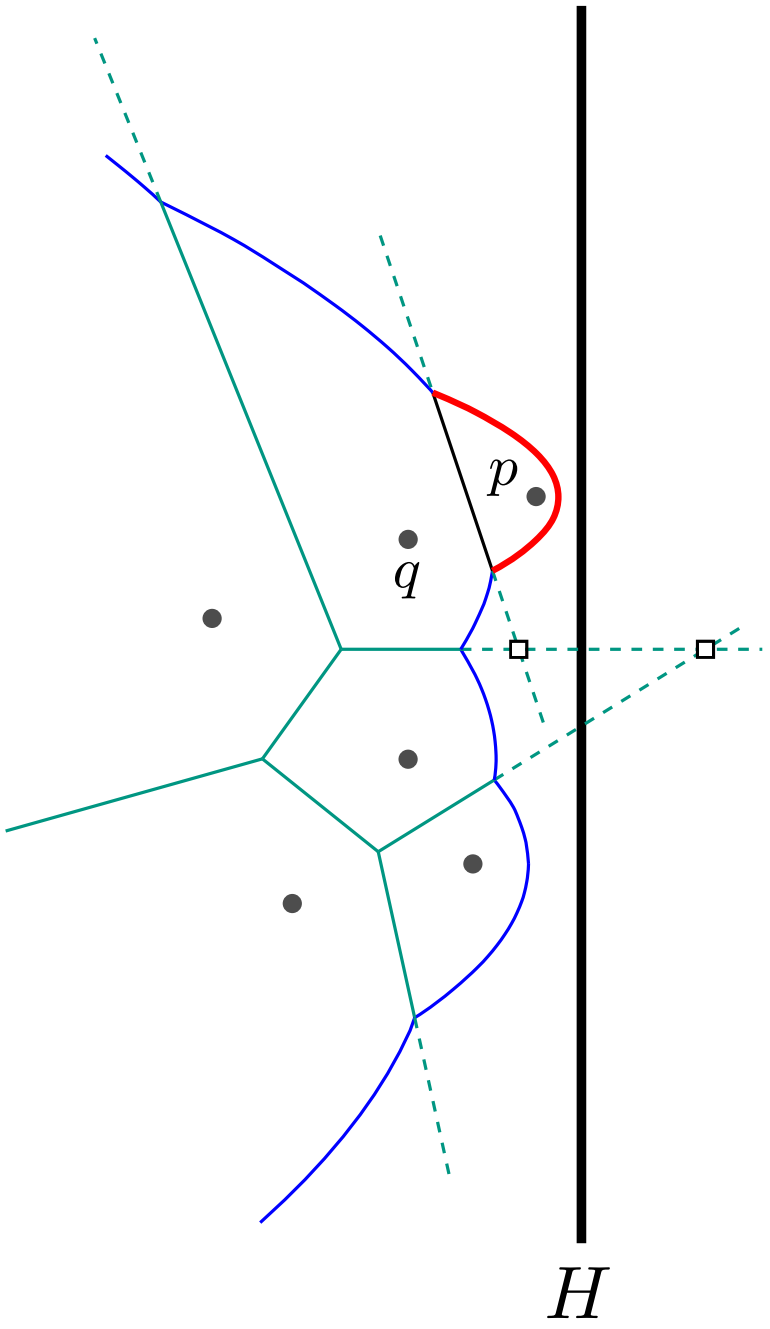

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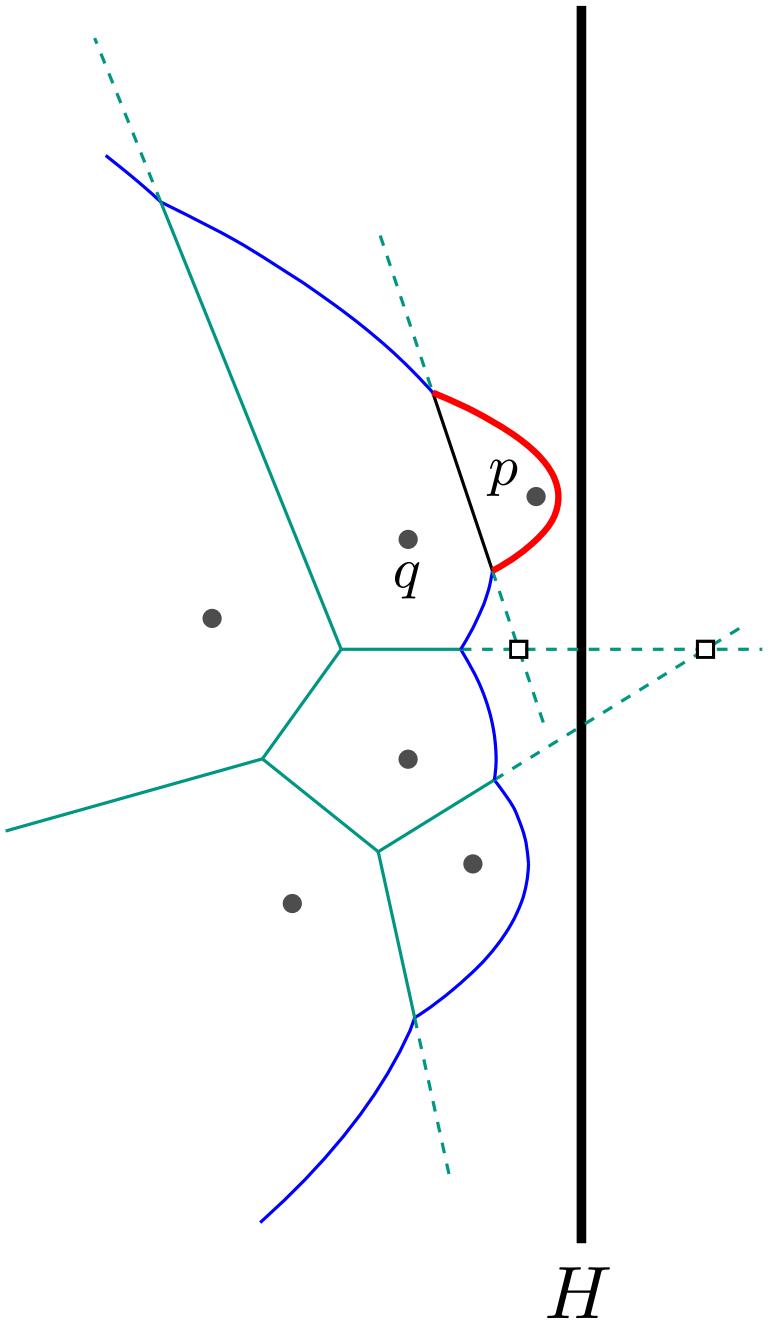

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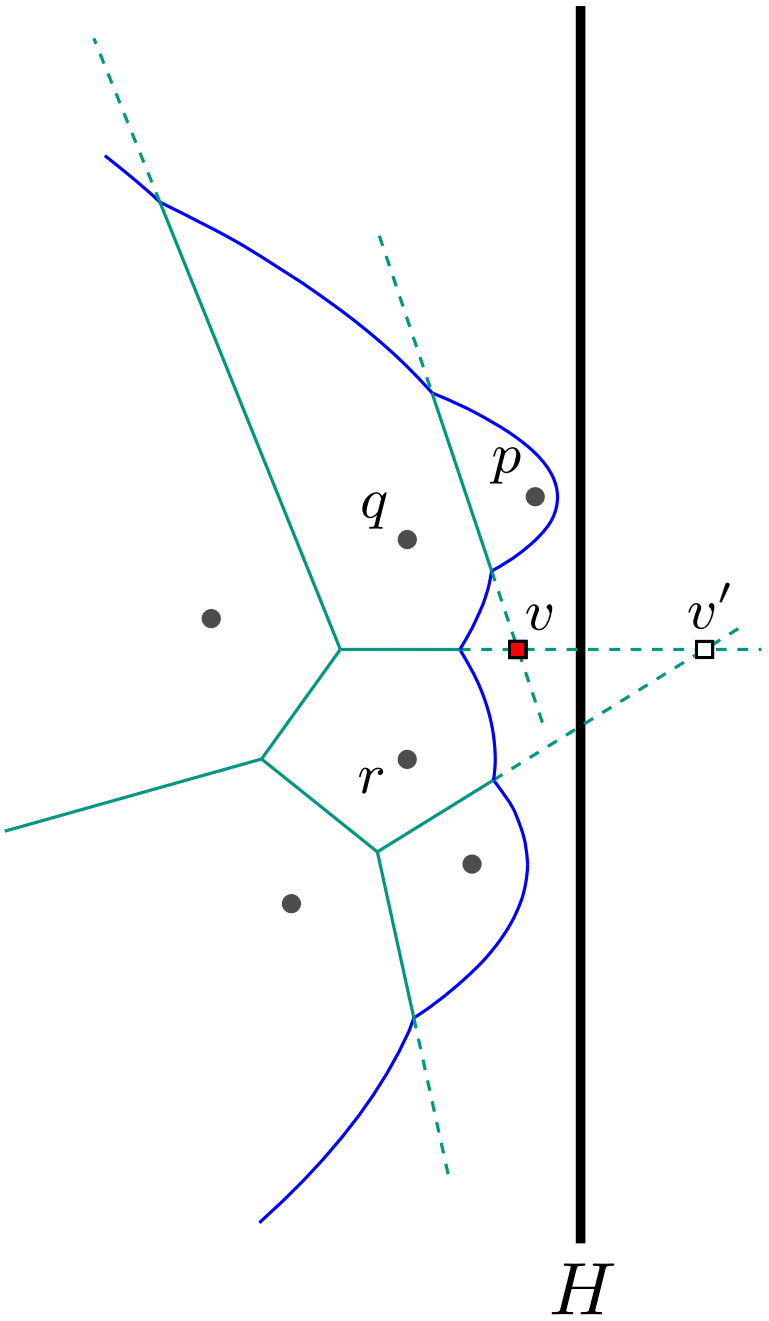
1. Add one new wavelet
2. Separate one old wavelet into two
3. Add one new Voronoi edge between p's and q's regions
4. Add two new spikes
   - Add at most two intersections with two adjacent spikes

Spike Event



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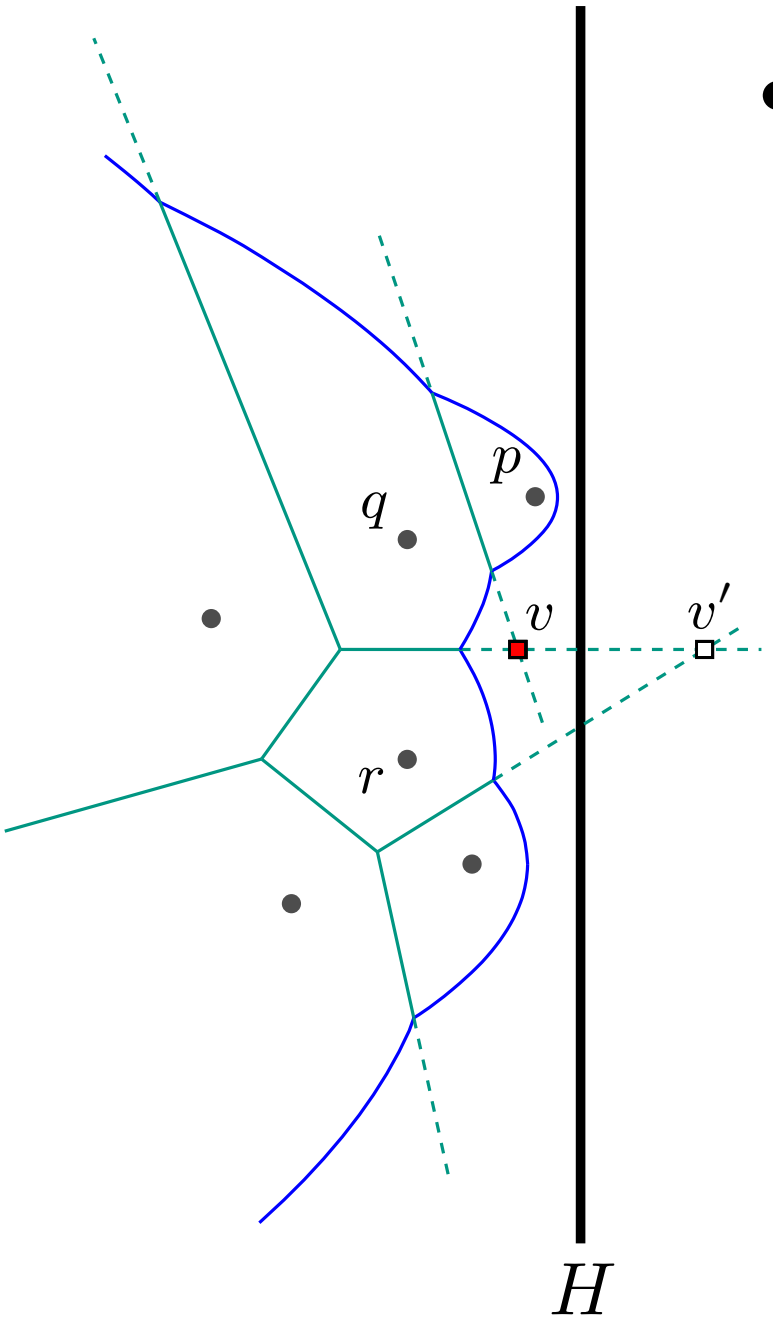

- W hits the intersection between two spikes

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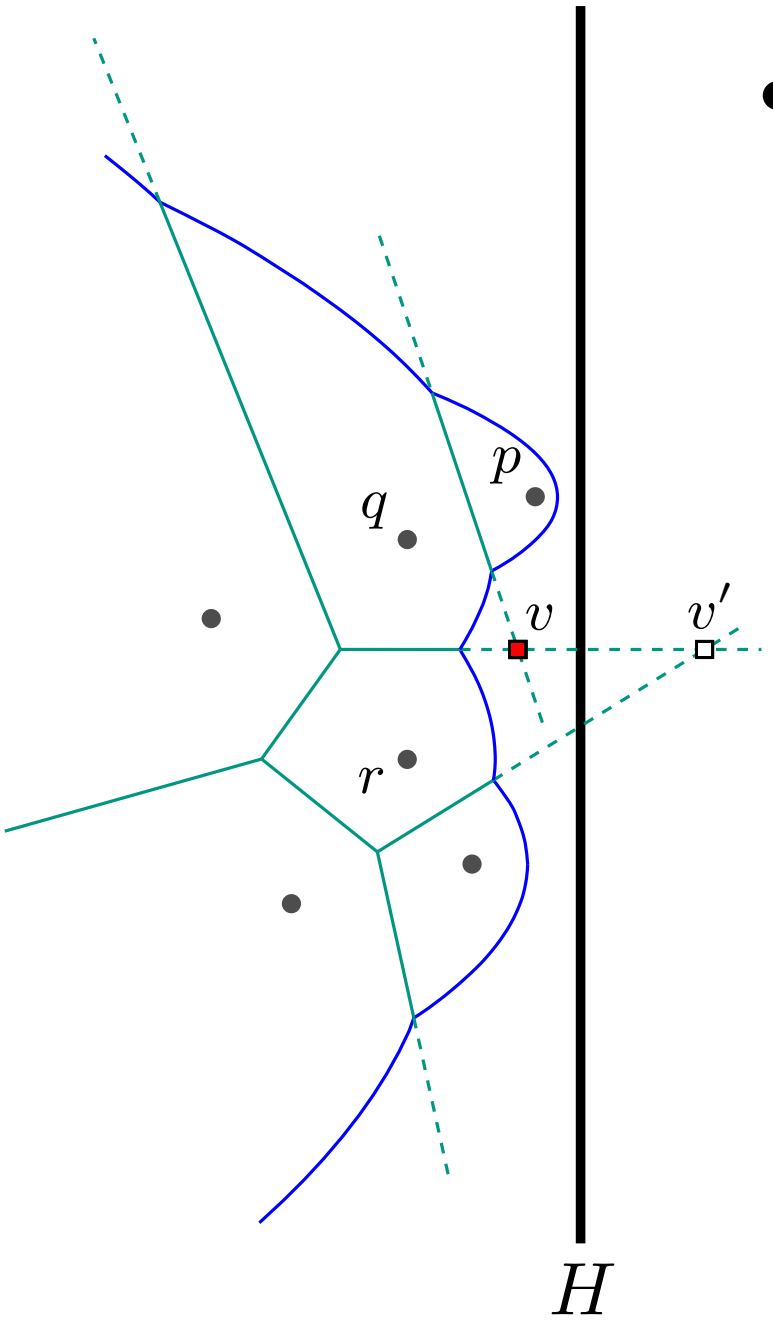

- W hits the intersection between two spikes
  - H leaves the circumcircle.

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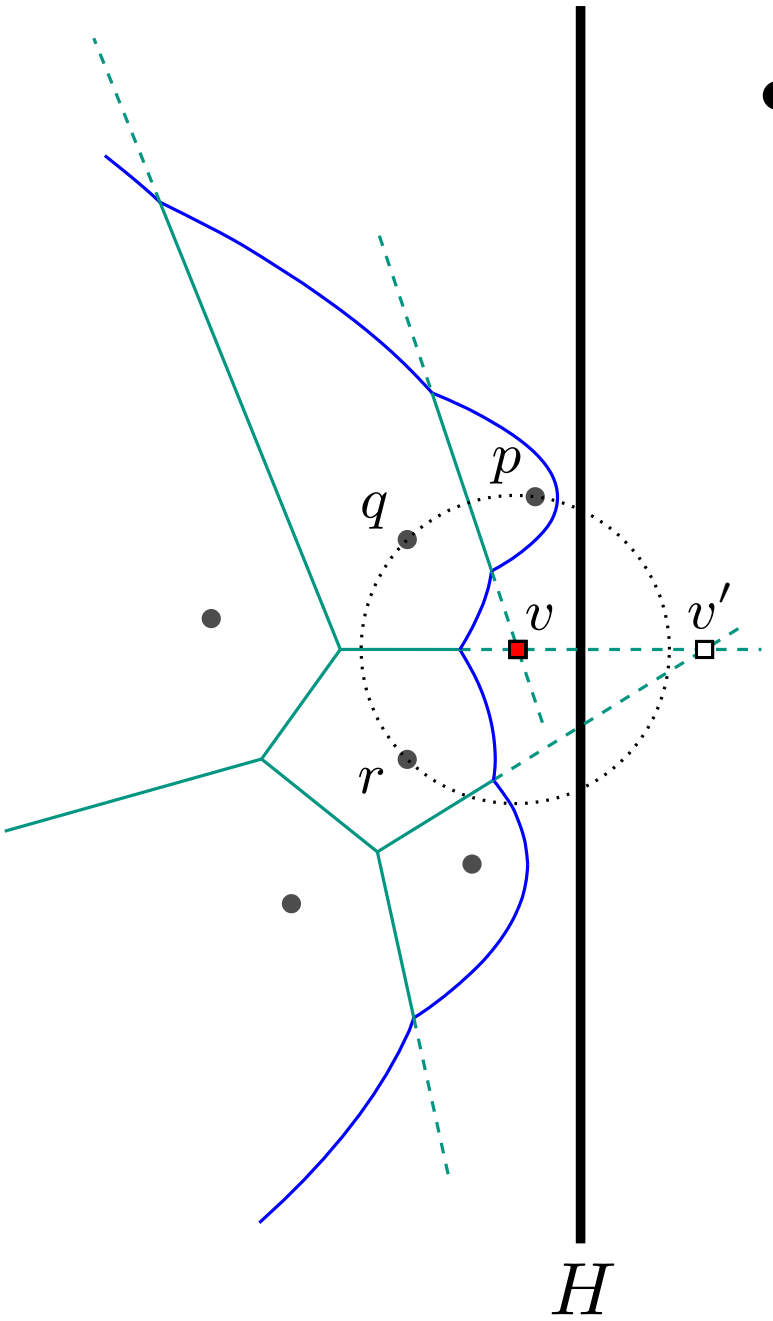

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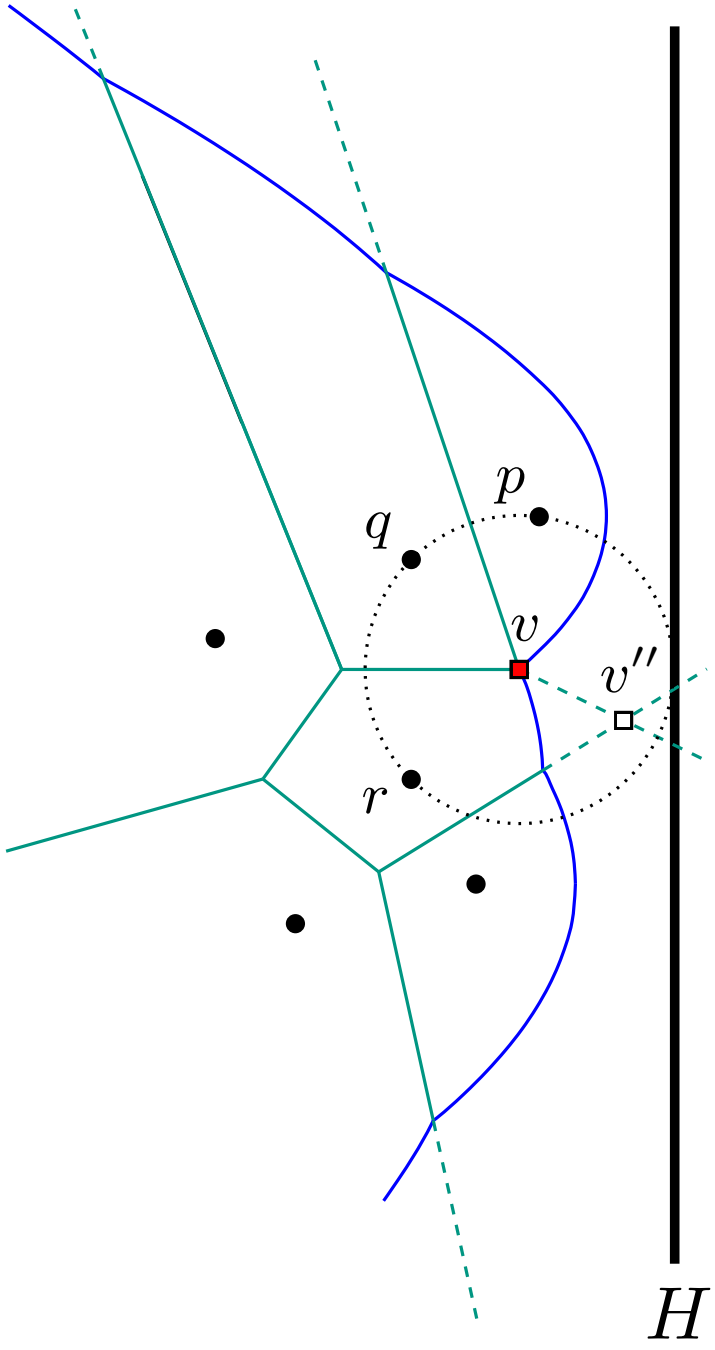

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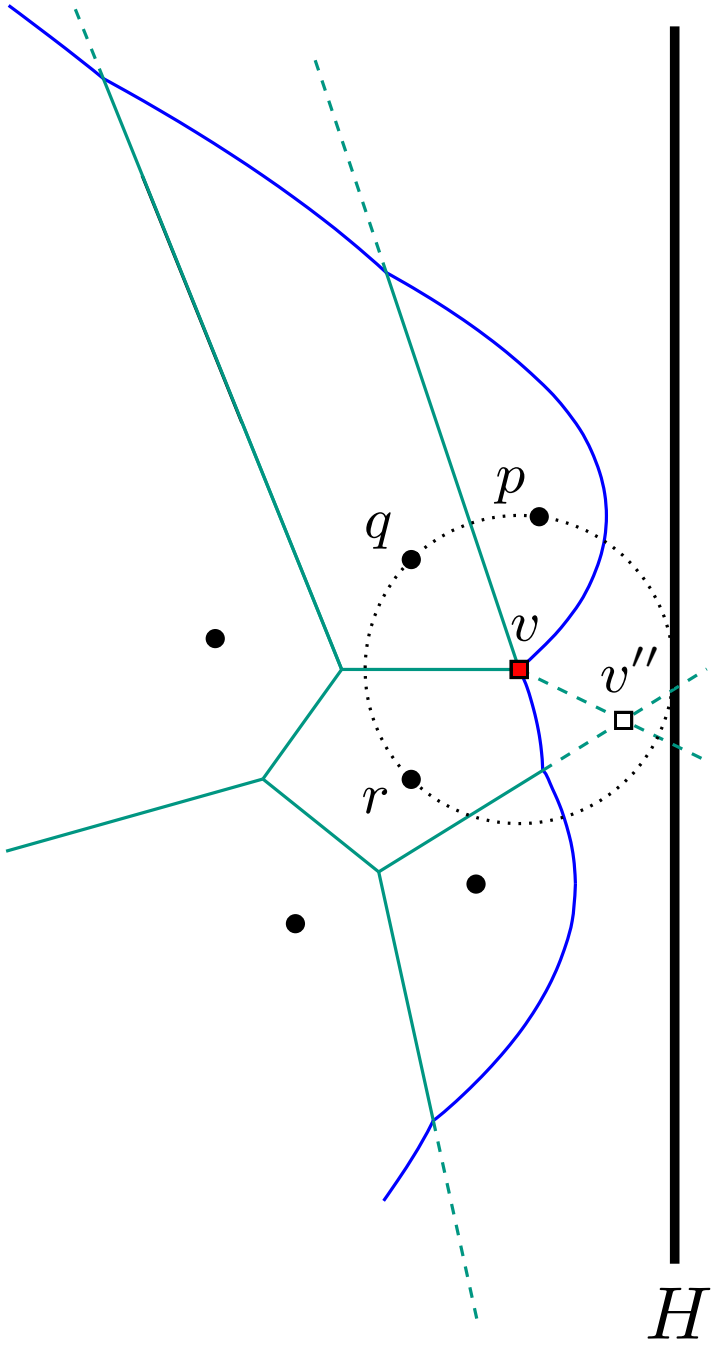

- W hits the intersection between two spikes
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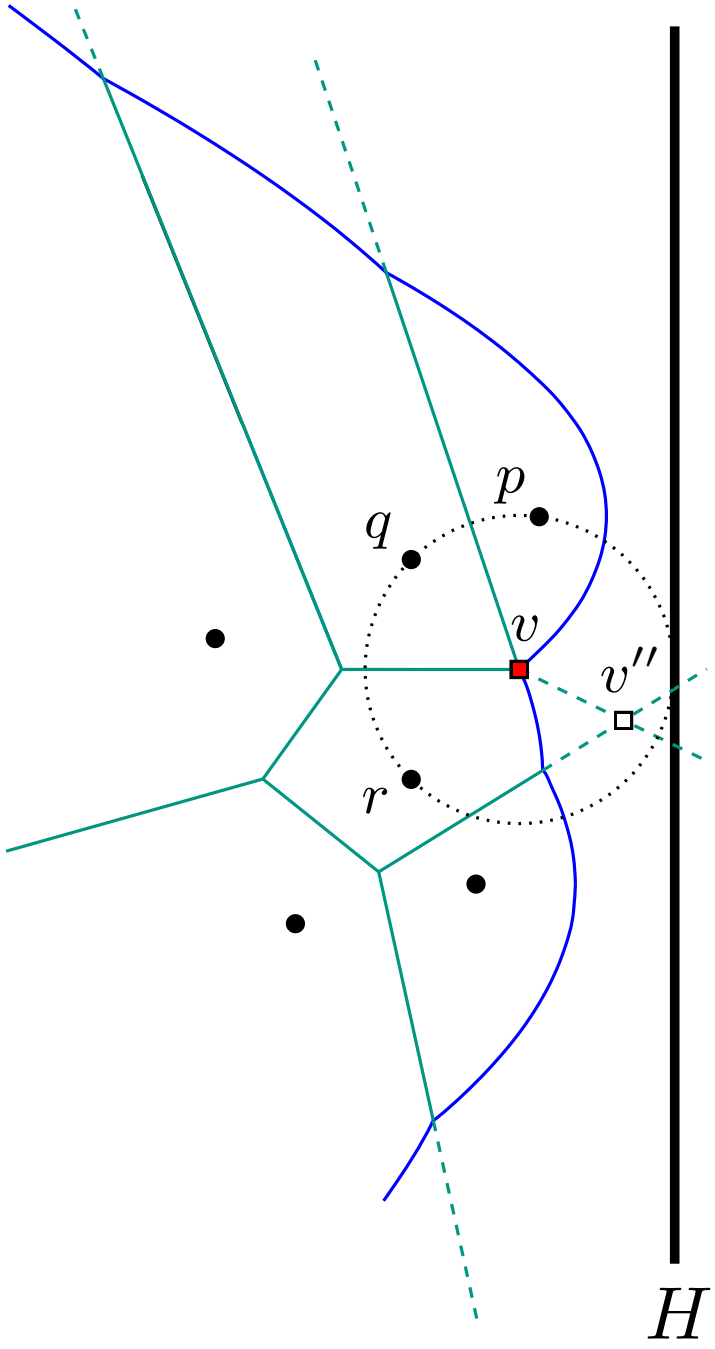

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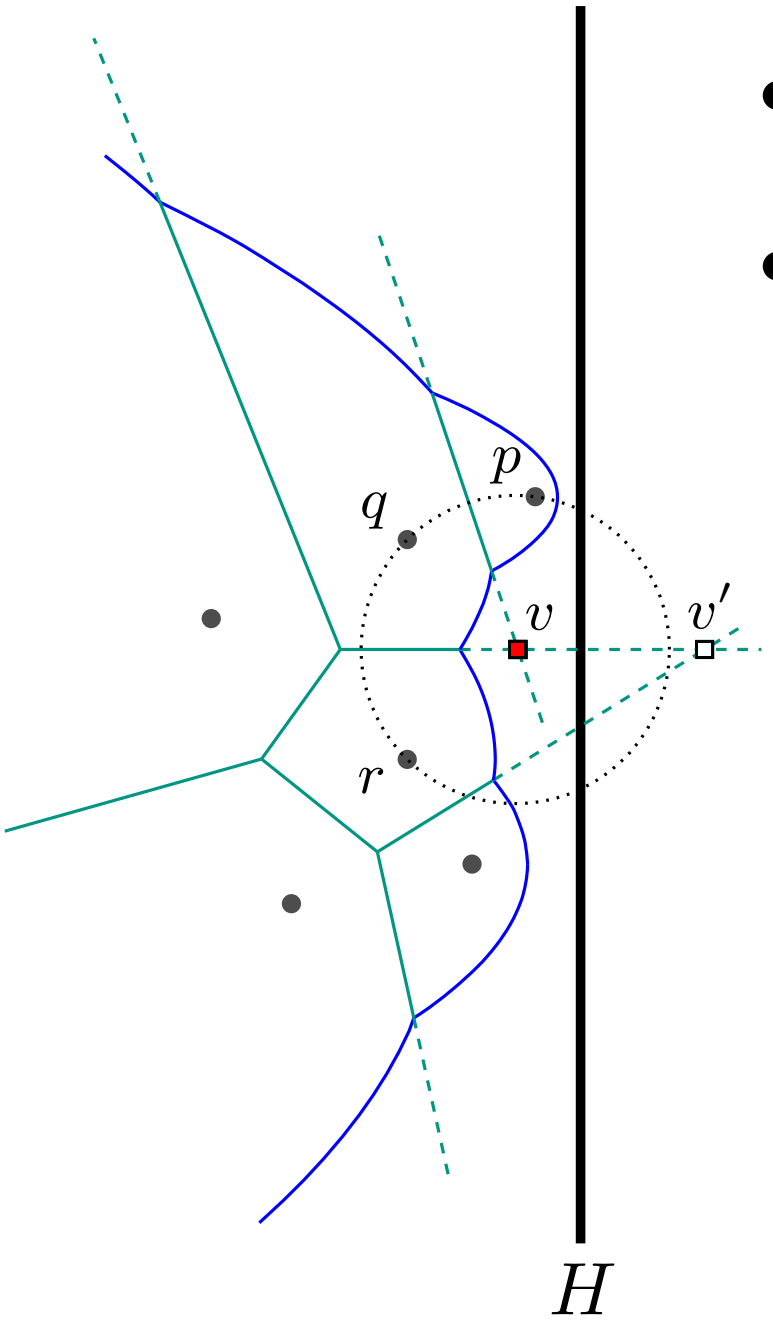
Spike Event



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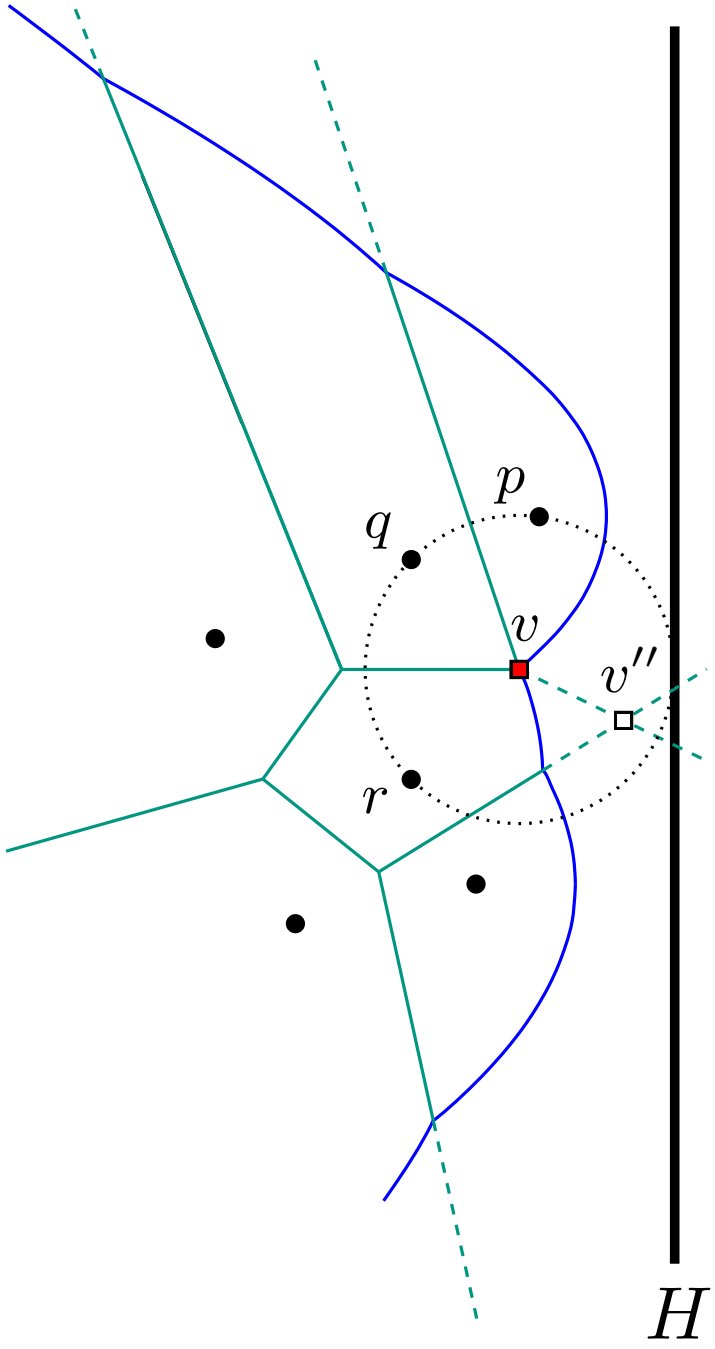

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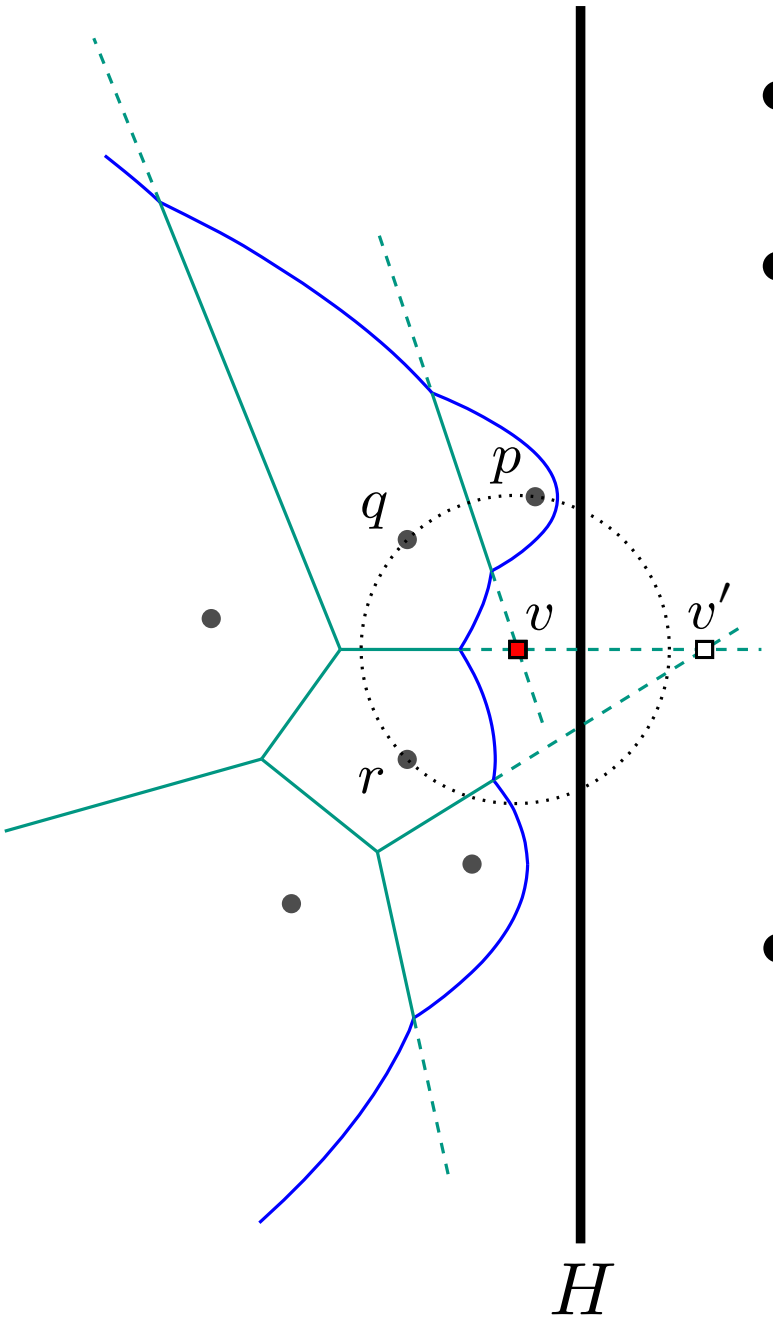

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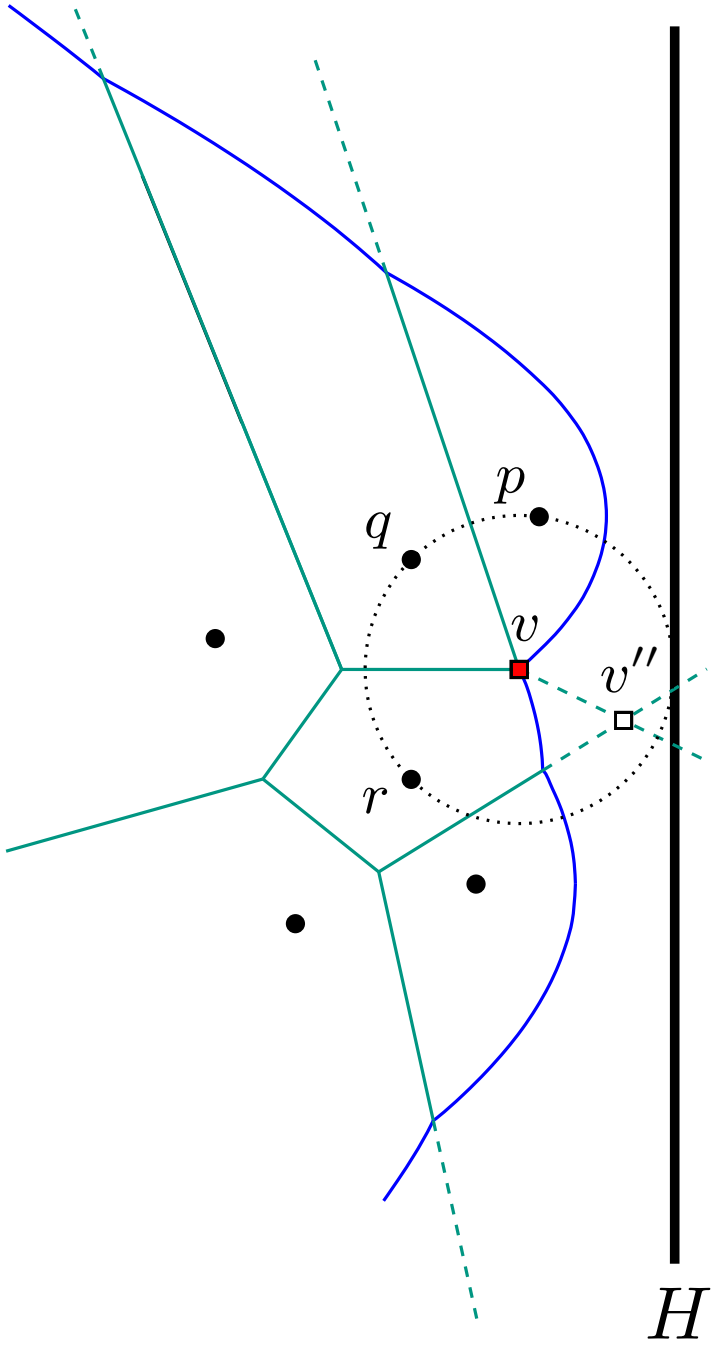

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Data Structures

Wavefront

A balanced binary tree

- wavelets
- y-coordinate

Events

A priority queue with respect to x-coordinate

- Site event: x-coordinate of a site
- Spike event: x-coordinate of H when W hits an intersection between two spikes

Construction Time

- A site event takes $O(\log n)$ time
  - $O(1)$ operations on the wavefront
    - Add one wavelet and Separate one old wavelet into two.
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Correctness

Theorem 8:

For a point $x \in \mathbb{R}^2$, if $x \in \mathcal{V}(p, S_L \cup \{H\})$, $x \in \mathcal{V}(p, P)$

Proof:

Correctness

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- $d(x,H) \leq d(x,q')$, $\forall q' \in P_R$

Correctness

Theorem 8:

For a point $x \in \mathbb{R}^2$, if $x \in \mathcal{V}(p, S_L \cup \{H\})$, $x \in \mathcal{V}(p, P)$

Proof:

- Let P_R be $P \setminus P_L$
- $d(x,p) < d(x,q)$, $\forall q \in P_L \setminus \{p\}$ and $d(x,p) < d(x,H)$
- $d(x,H) \leq d(x,q')$, $\forall q' \in P_R$
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Divide and Conquer

- Basic Steps
  1. Divide an instance into c equal-size sub-instances
     - If the instance is extremely small, solve it directly
  2. Recursively compute the sub-solution for each sub-instance
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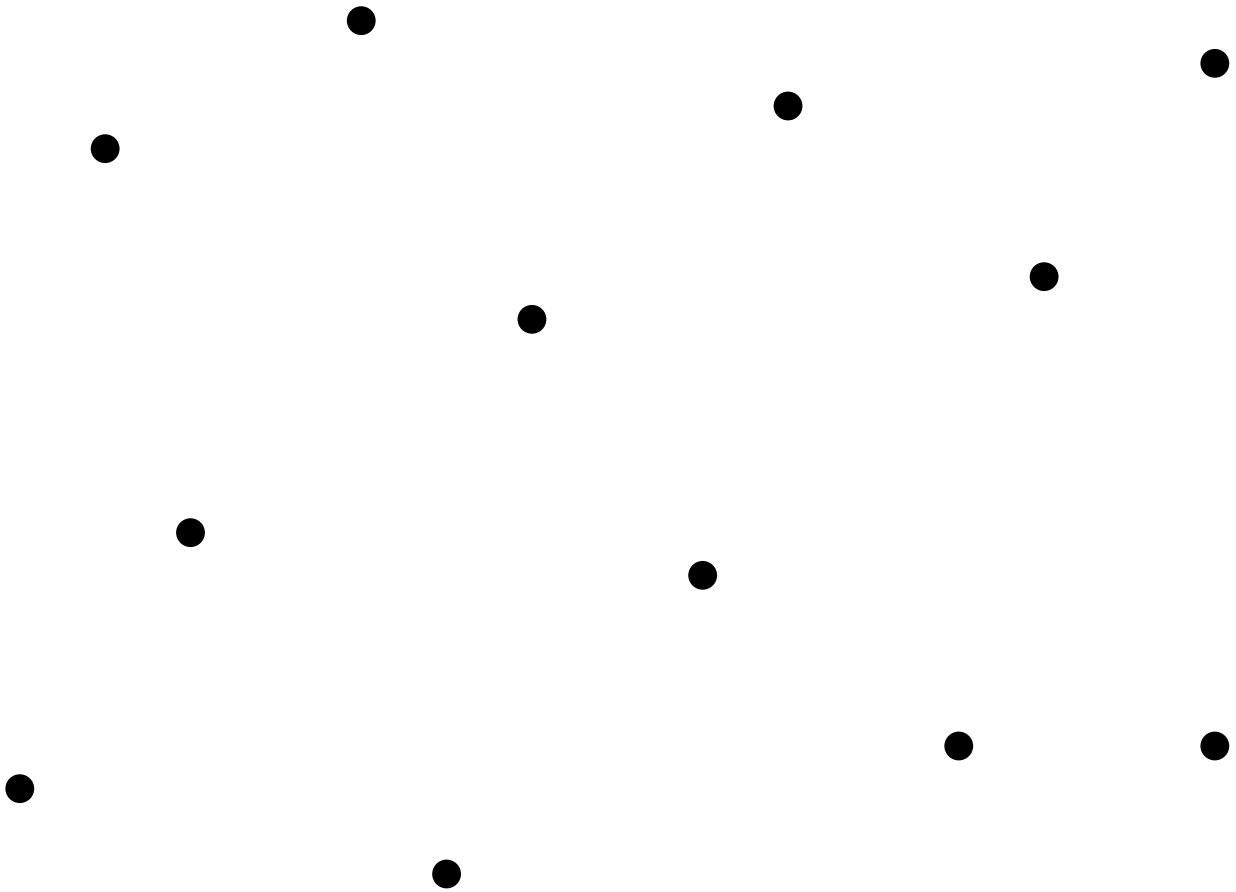
- If both Divide and Merge take linear time,

$$T(n) = c \cdot T(n/c) + O(n) => O(n \log n)$$

  - The i^{th} level has c^i parts and each part has n/c^i elements
  - A level takes $c^i \times O(n/c^i) = O(n)$ time
  - $O(\log n)$ levels

Example: Convex Hull

- $n = 12$ and $c = 2$



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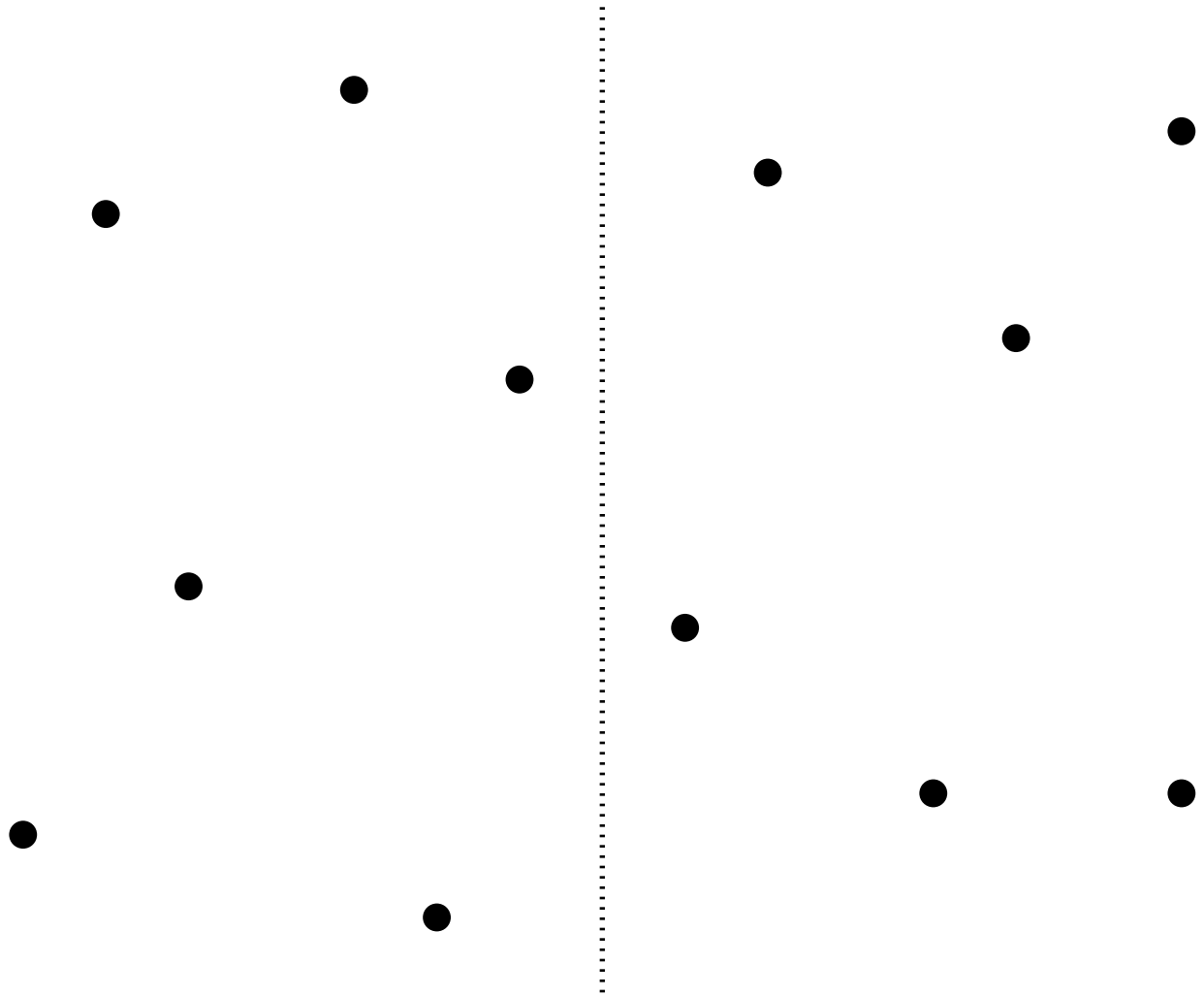

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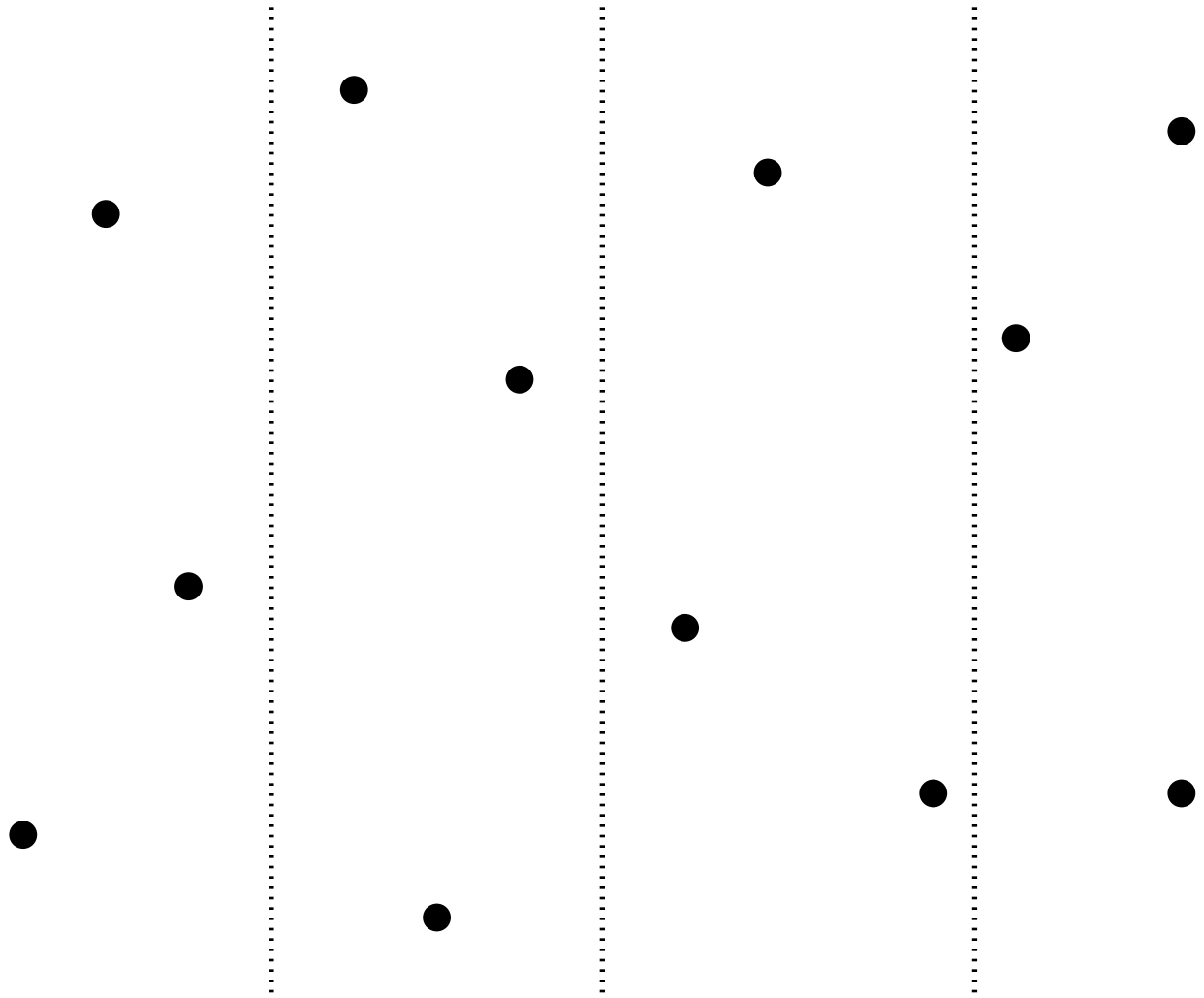

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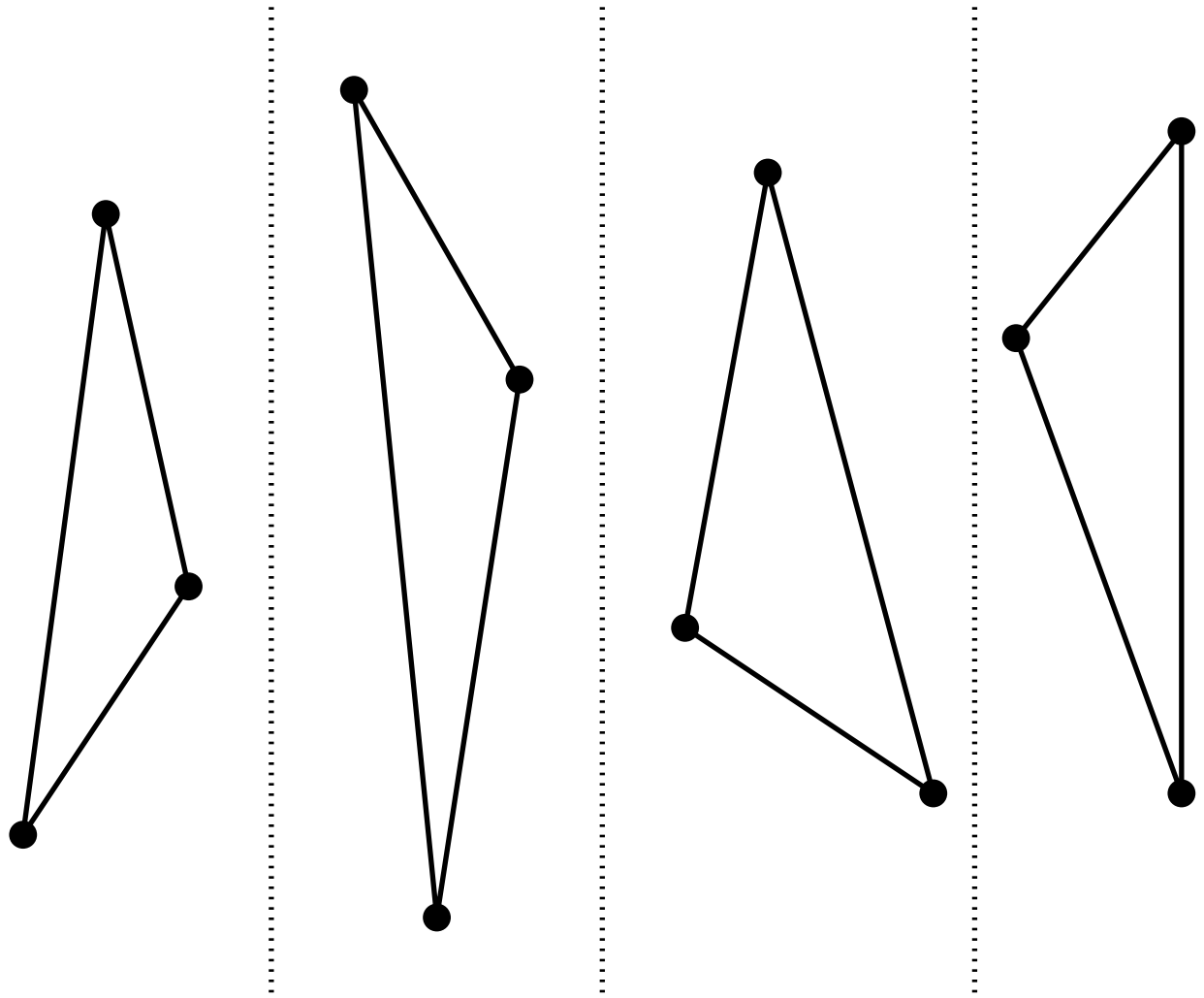

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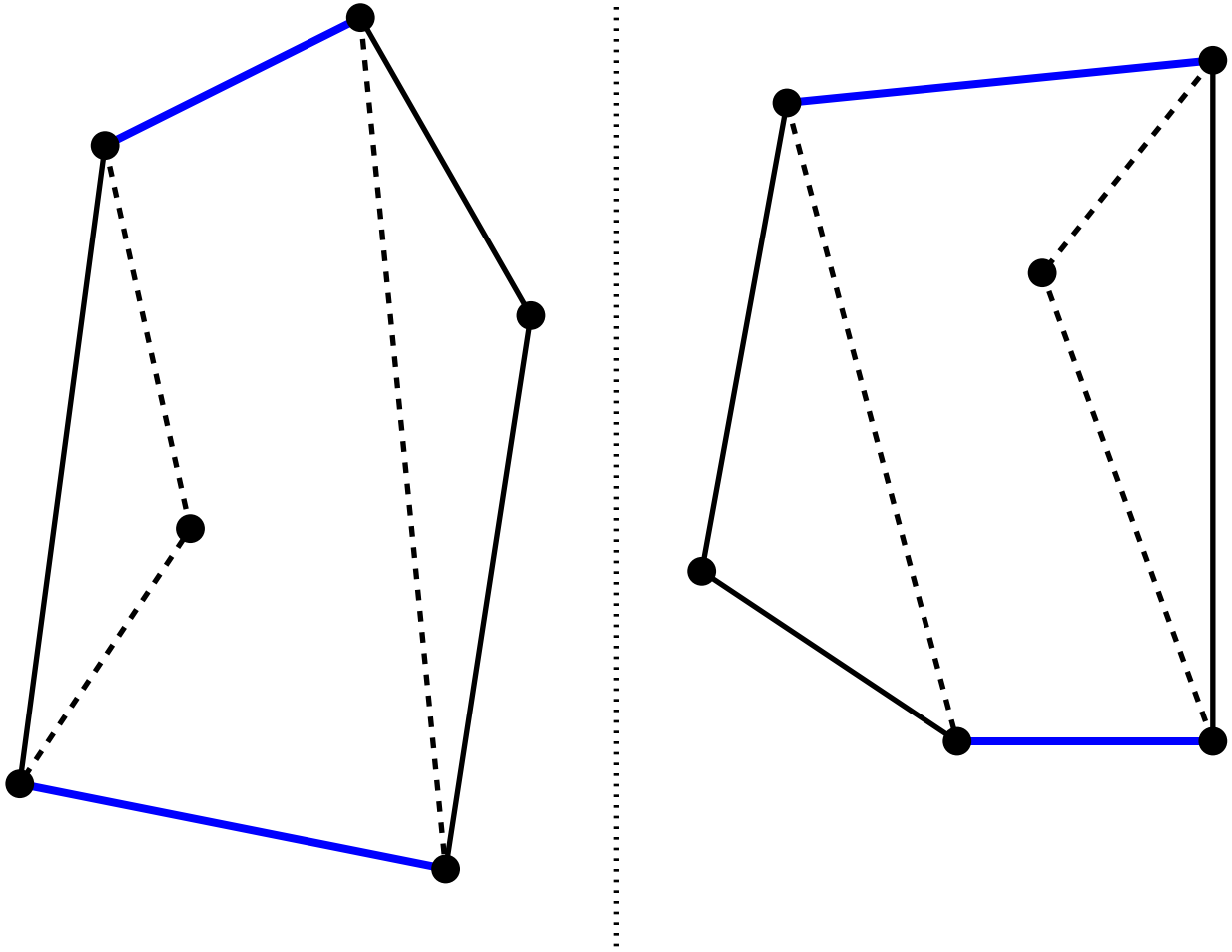

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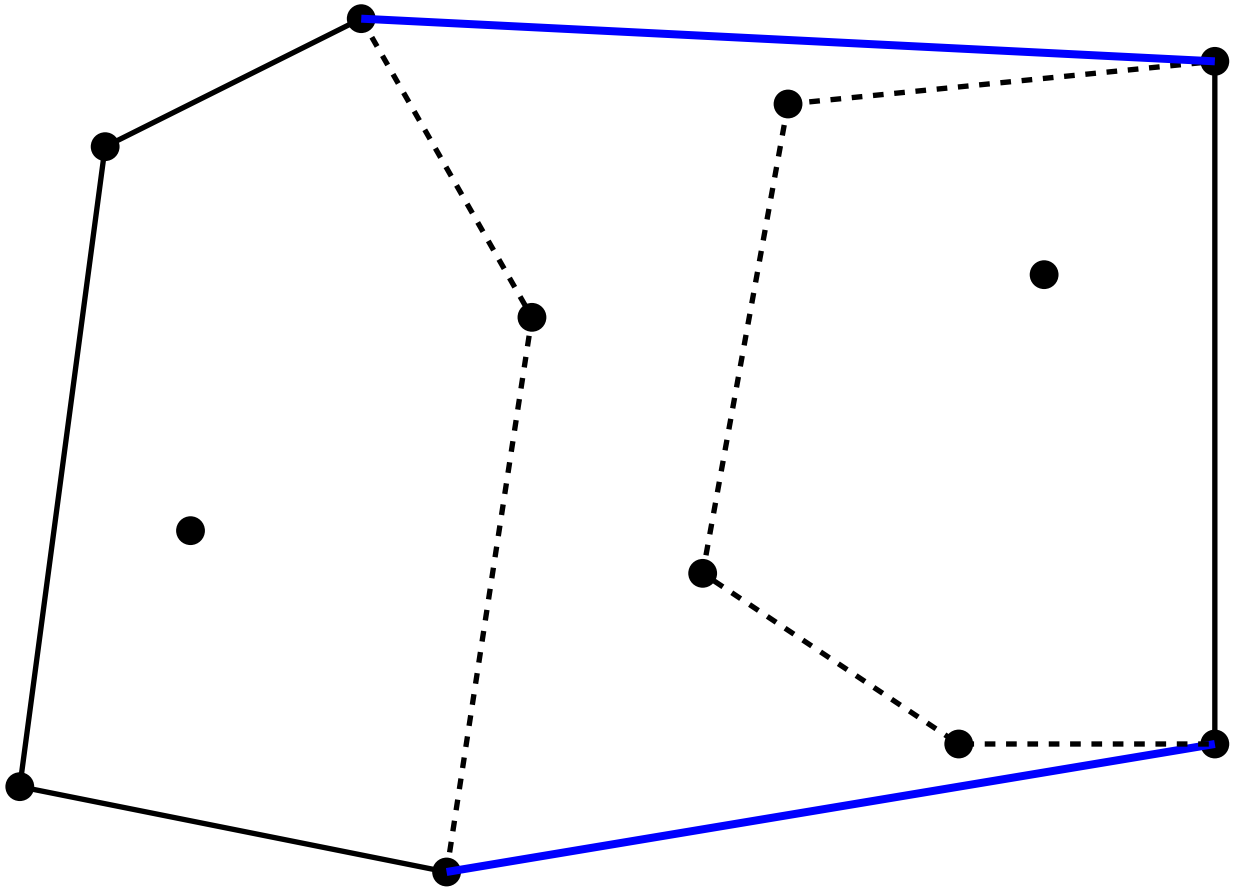

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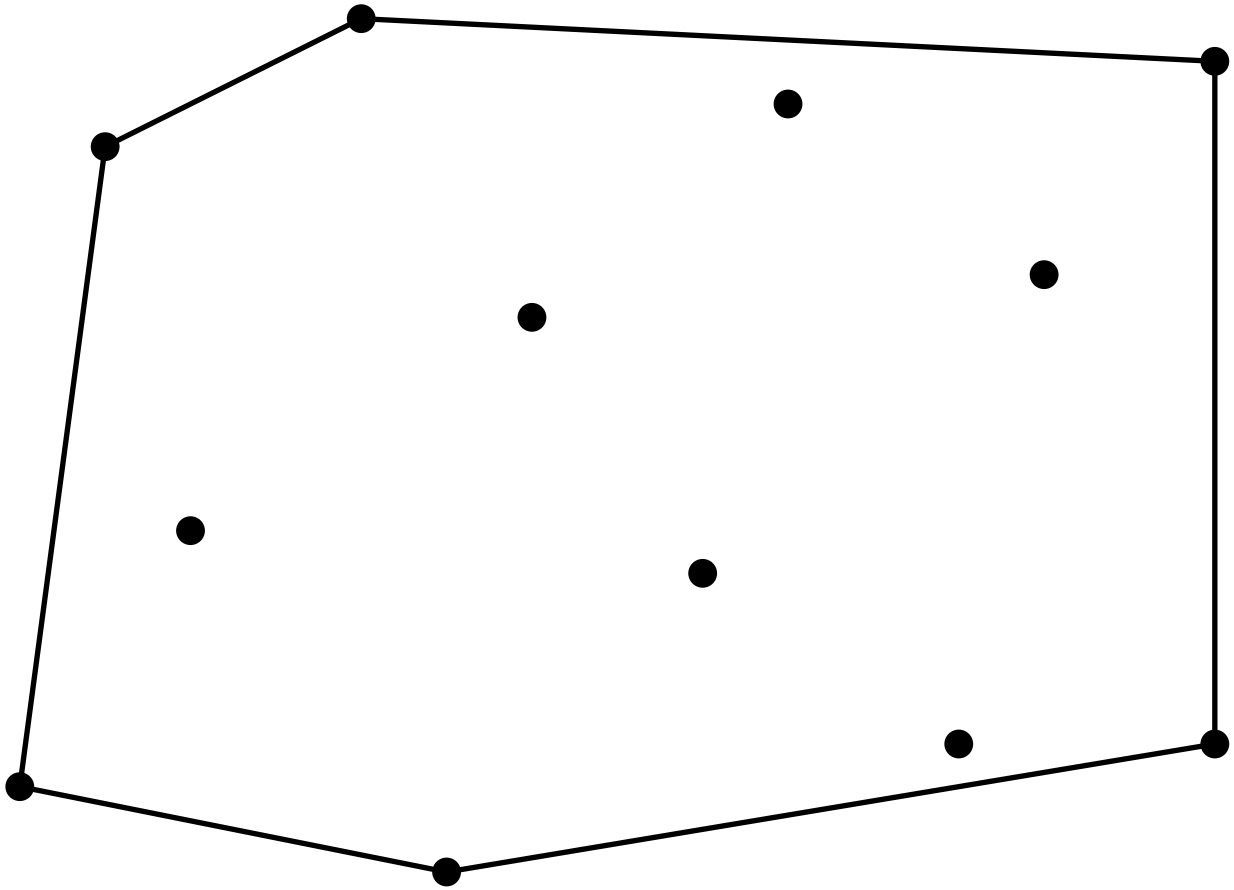

D & C for Voronoi Diagram

Divide and Conquer to Compute $\text{Vor}(P)$

1. Use a vertical line to partition P into L and R with $|L| \sim |R|$
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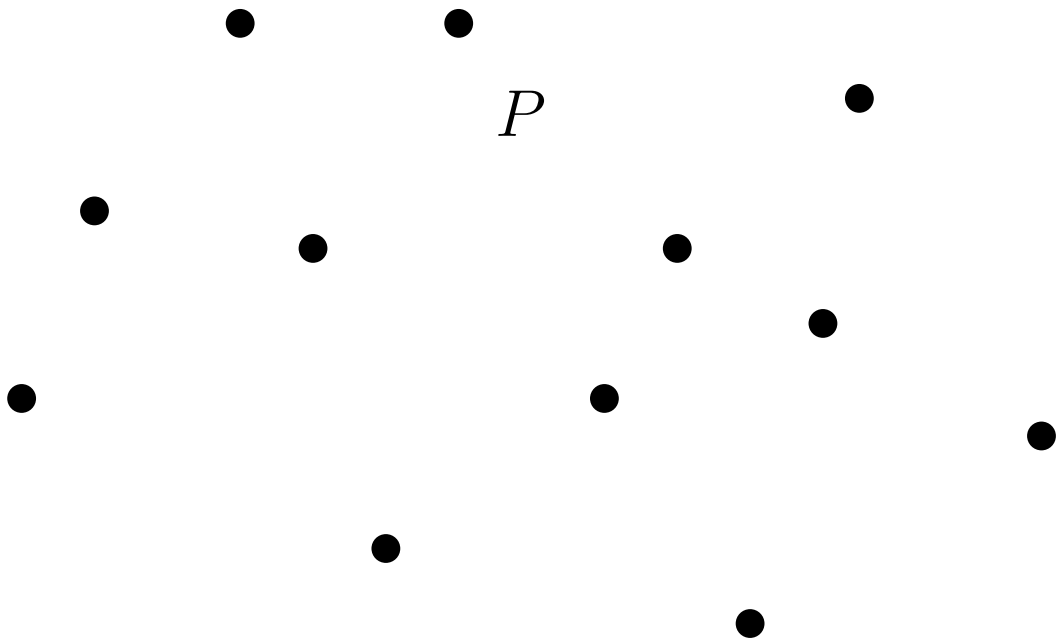

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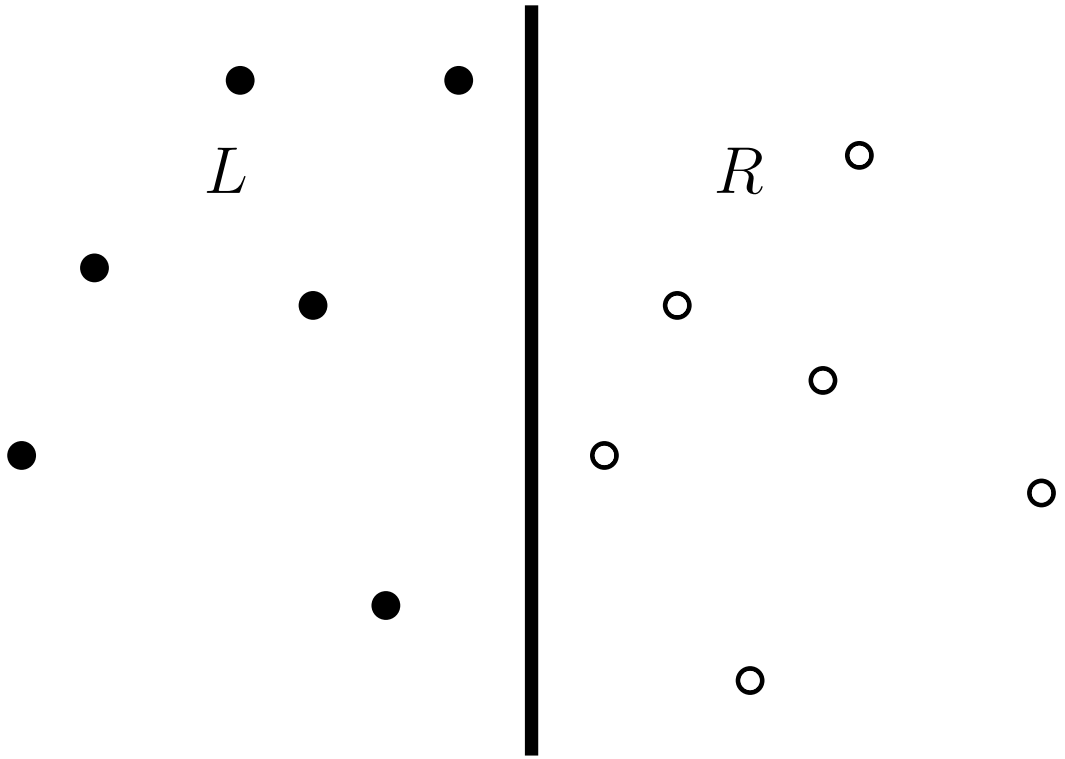

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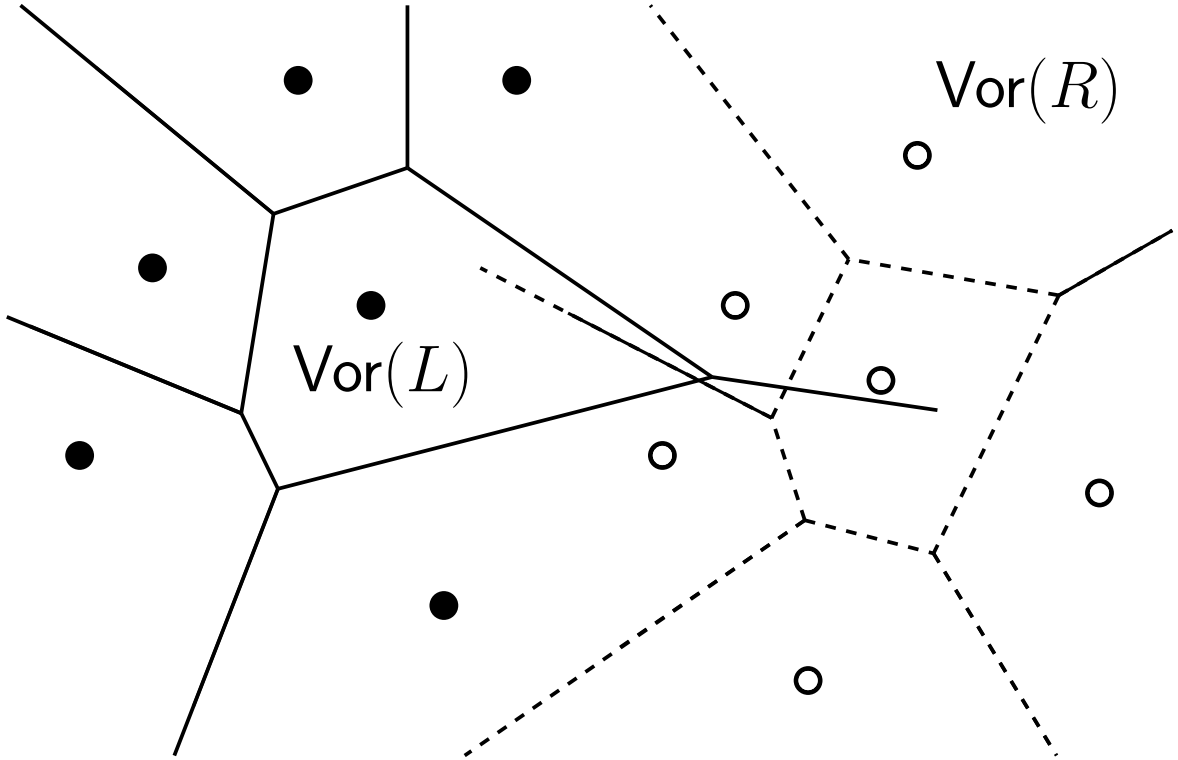

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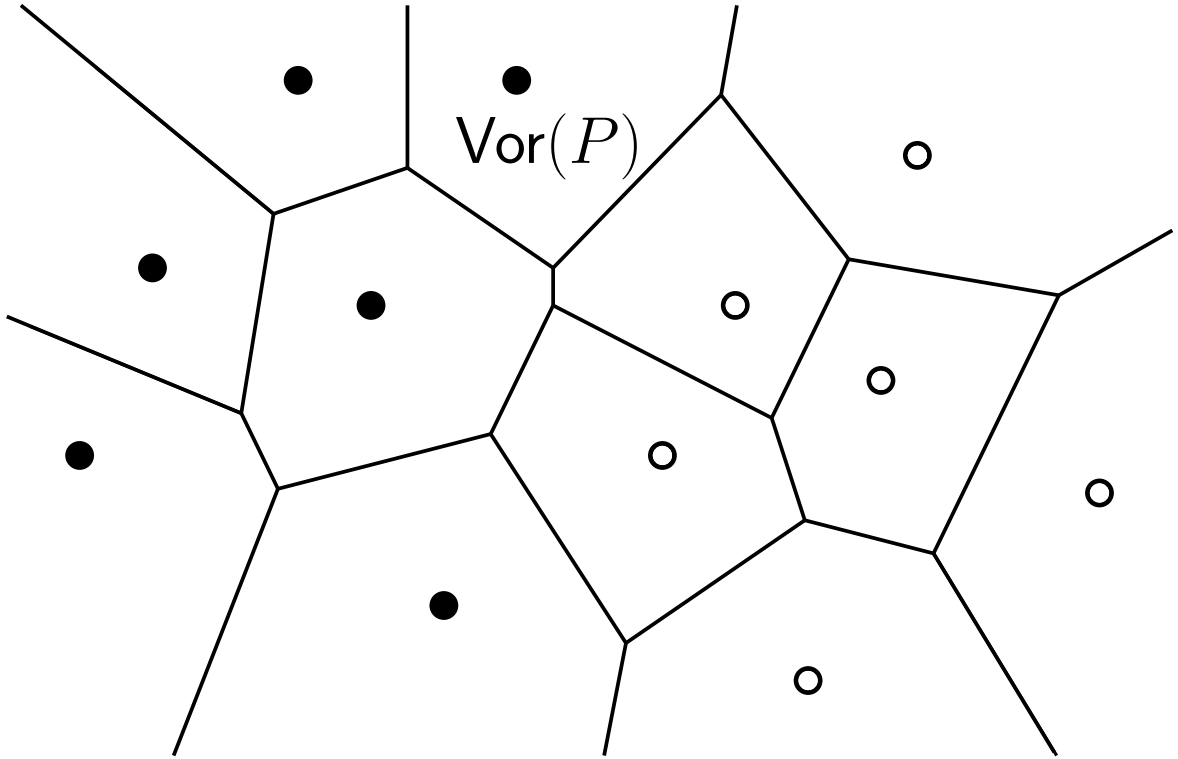
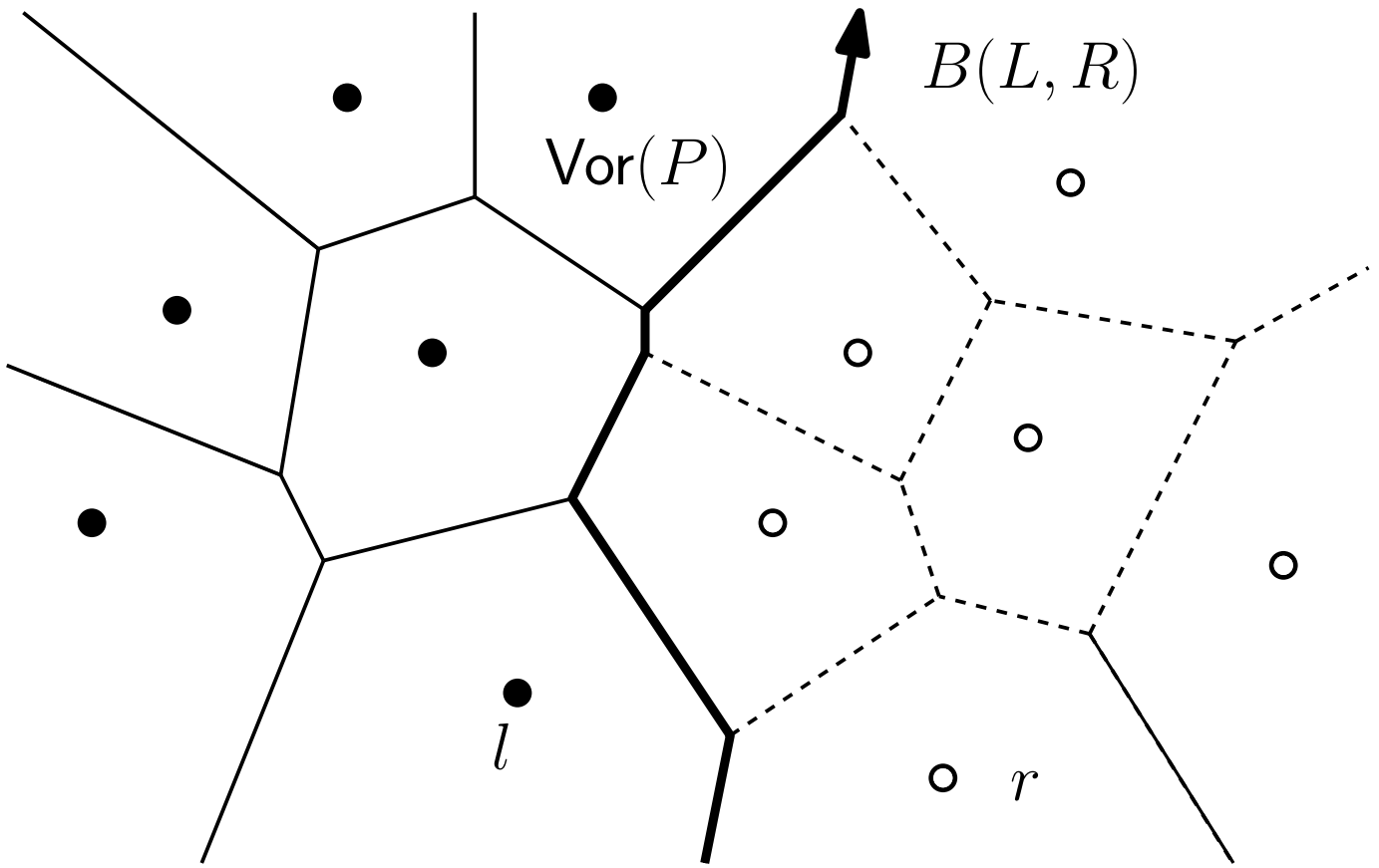


Merge Chain

- **Merge Chain** $B(L, R)$ consists of Voronoi edges between $\mathcal{V}(l)$ and $\mathcal{V}(r)$ where $l \in L$ and $r \in R$.



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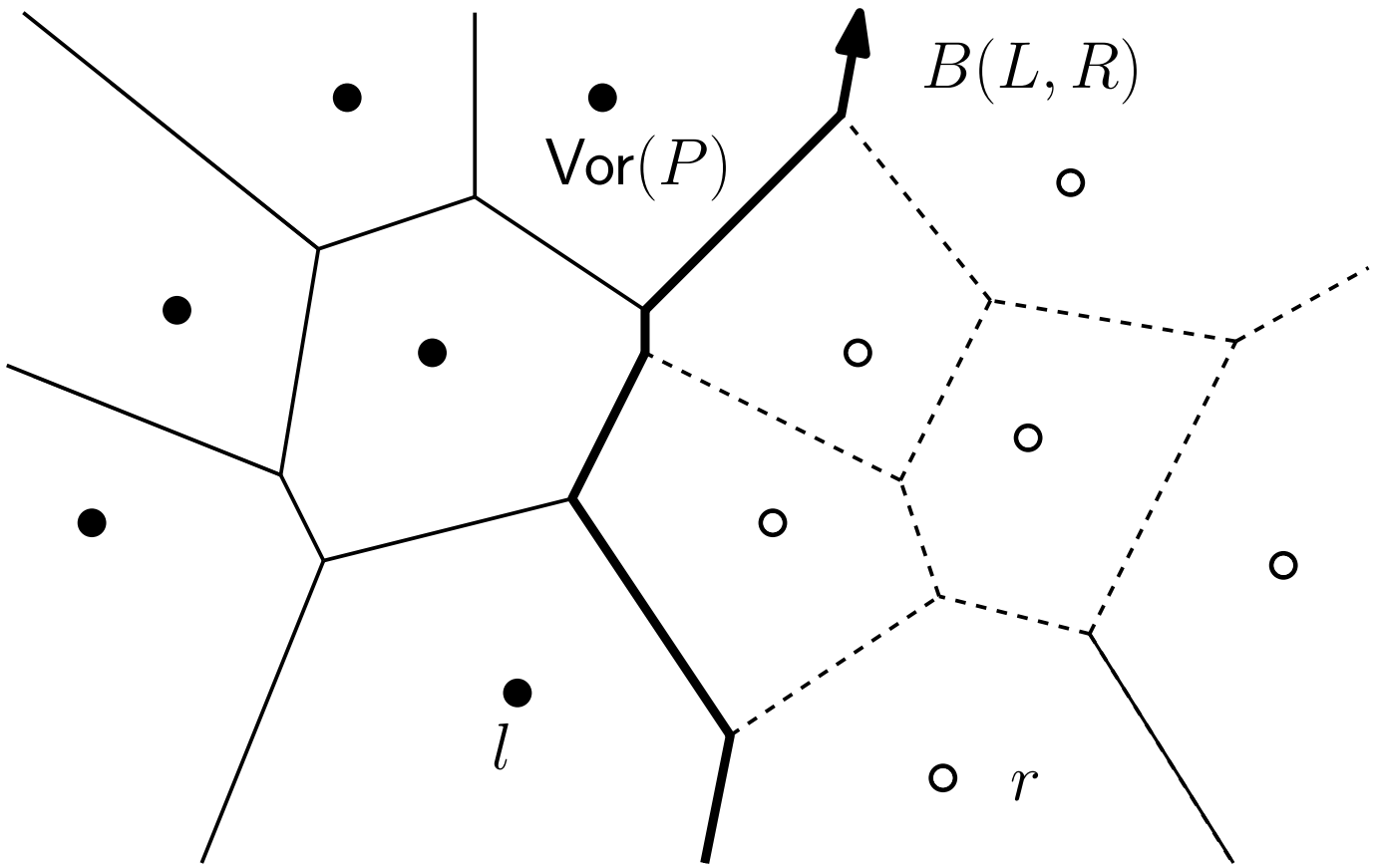

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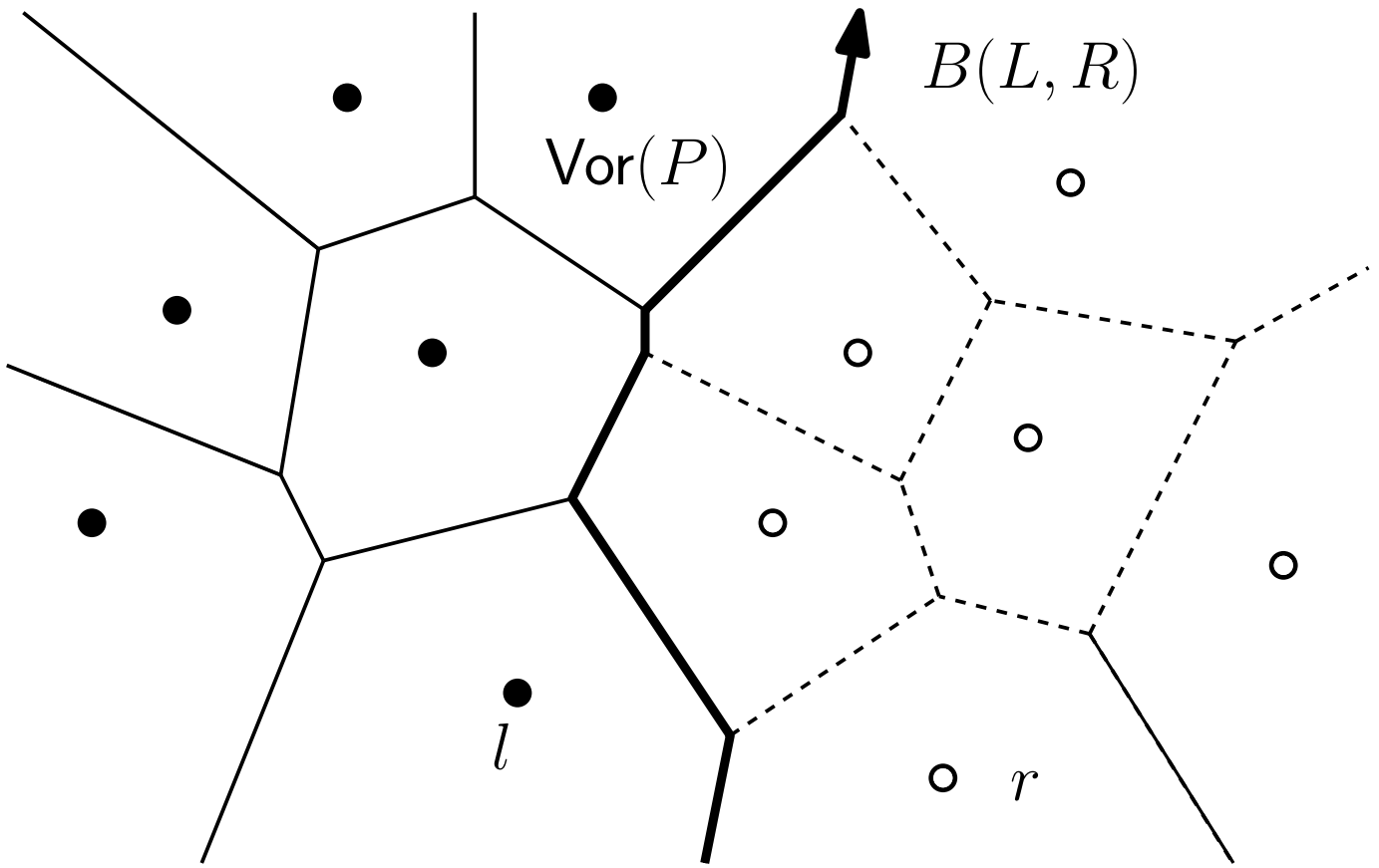

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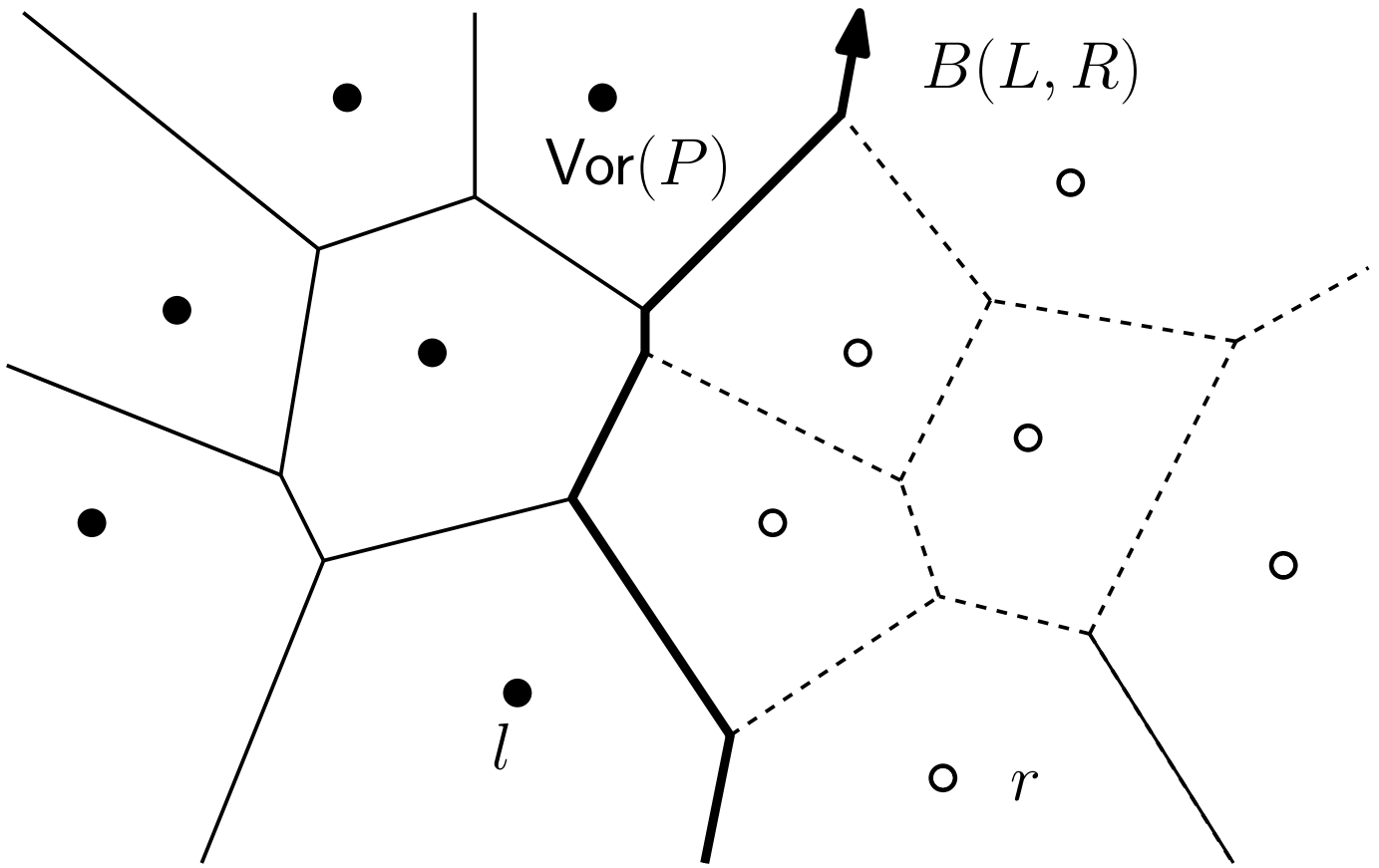

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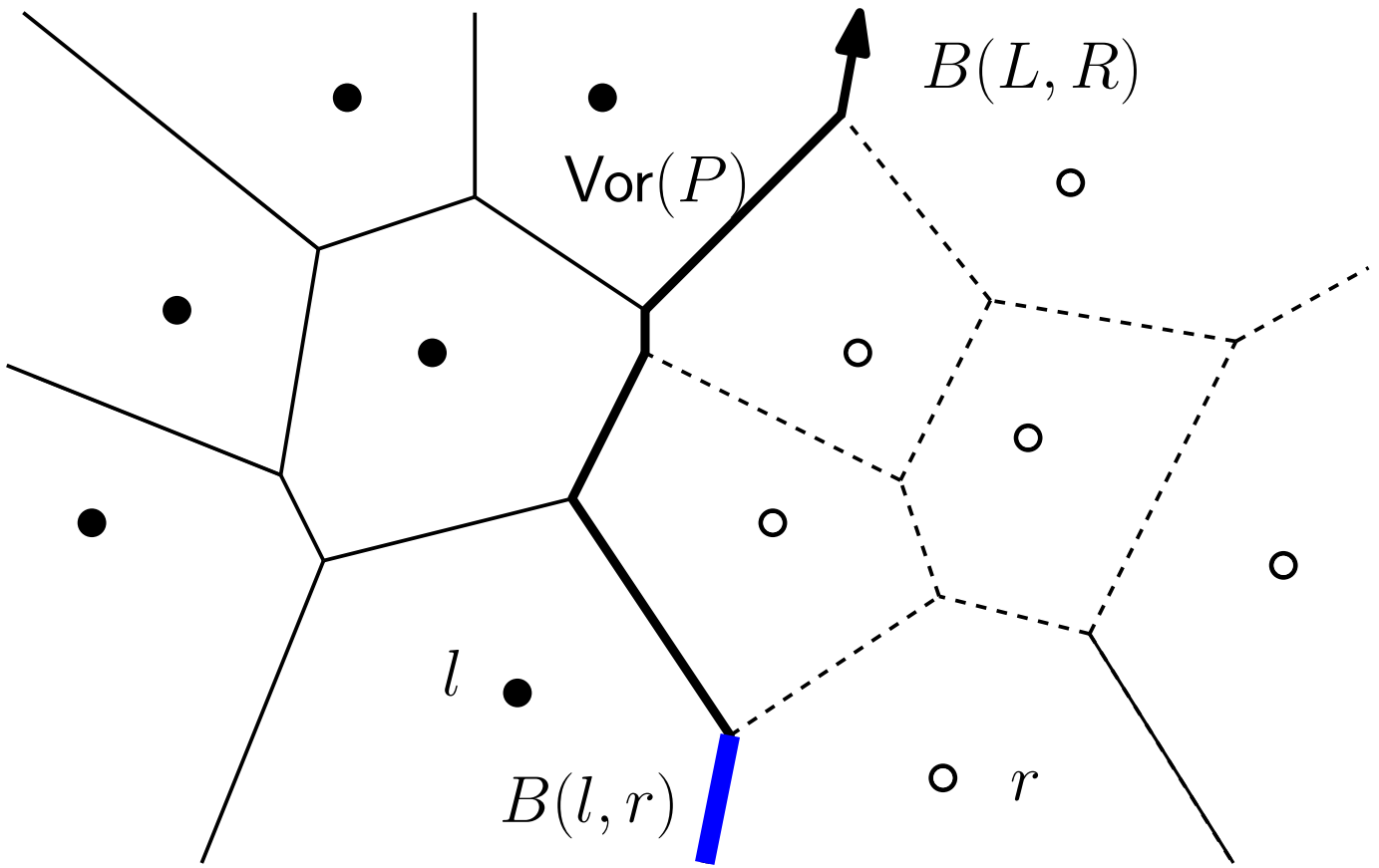
    - $\forall e \in B(L, R)$ belongs to $B(l, r)$, $l \in L$ and $r \in R$



If We Have the Merge Chain...

- Compute Vor(P)
  - Remove the part of Vor(L) right to $B(L, R)$
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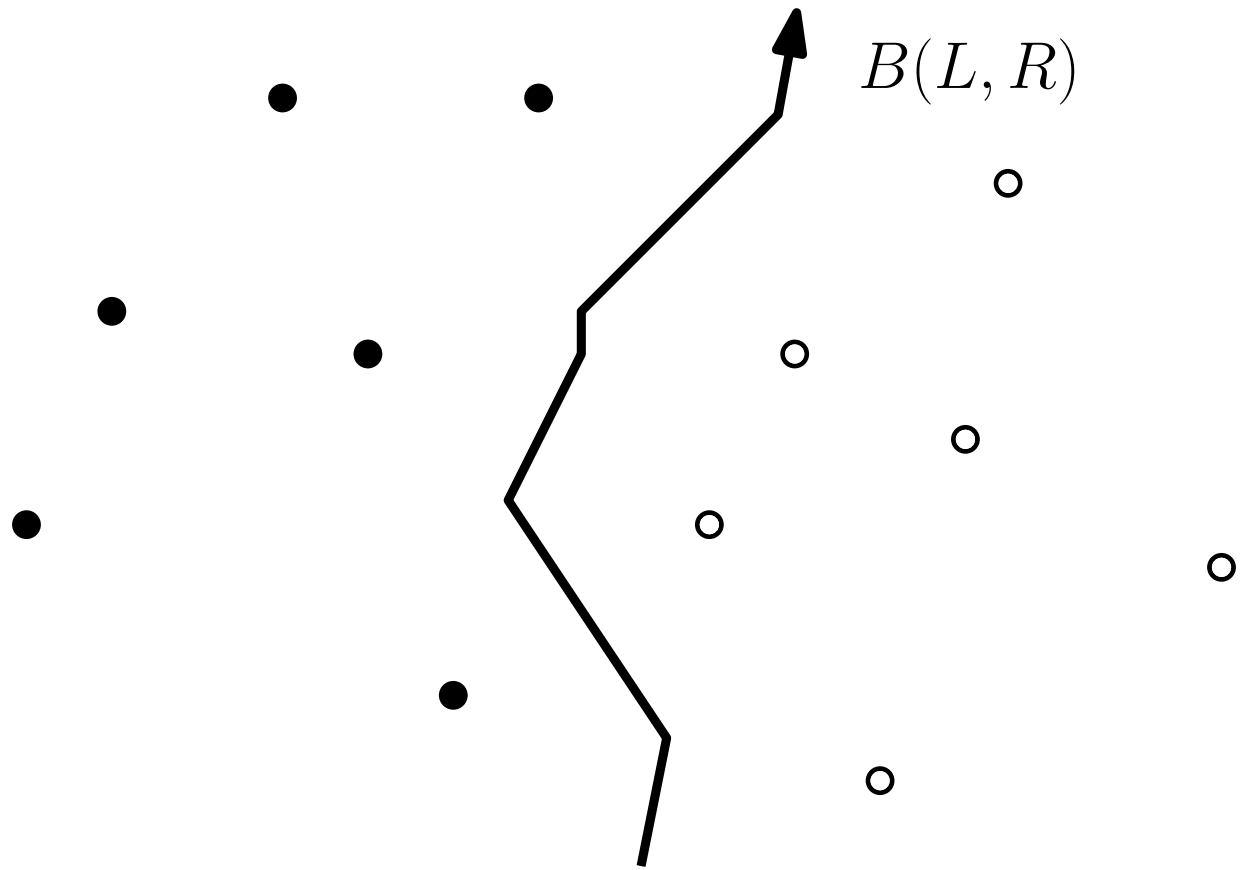
  - Glue the two remaining parts



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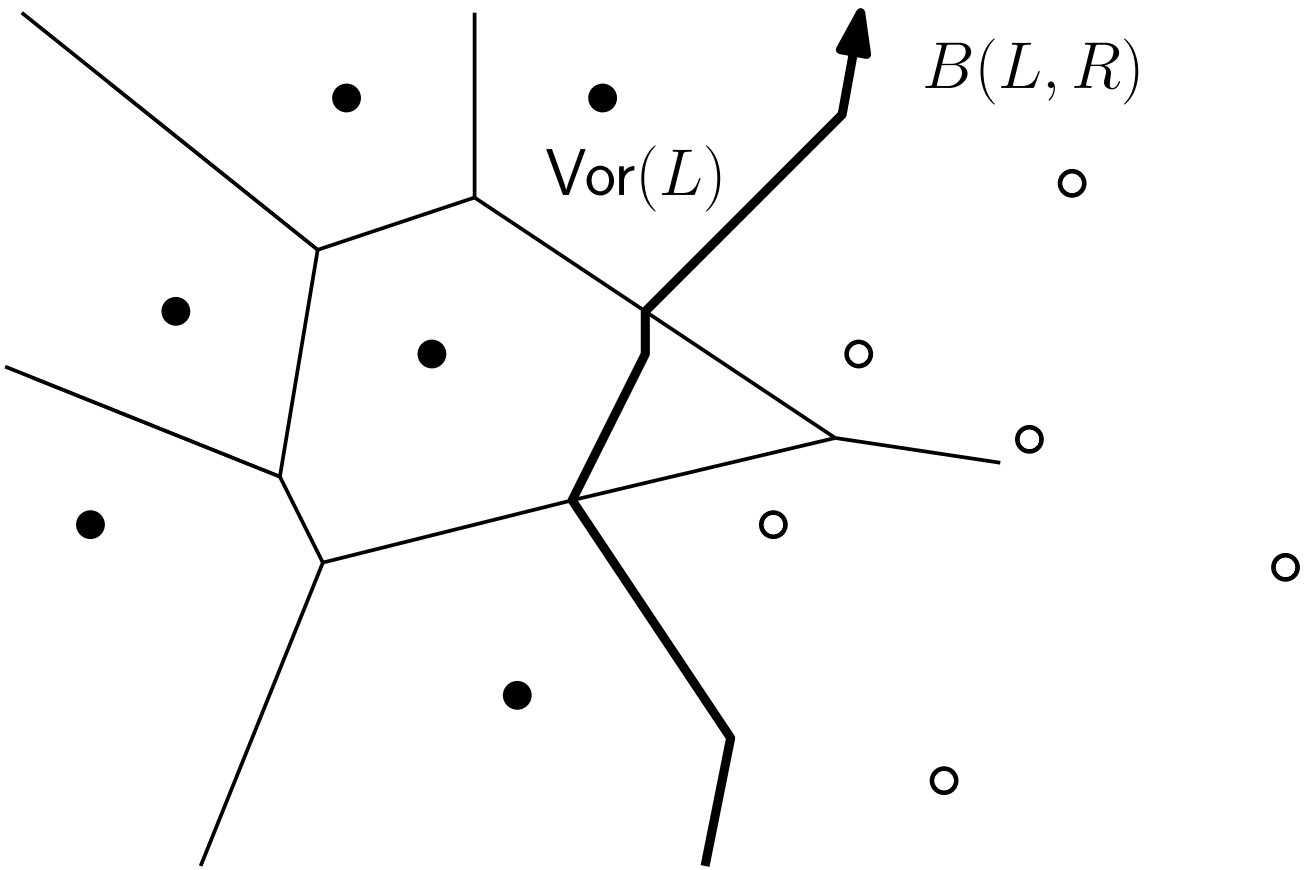

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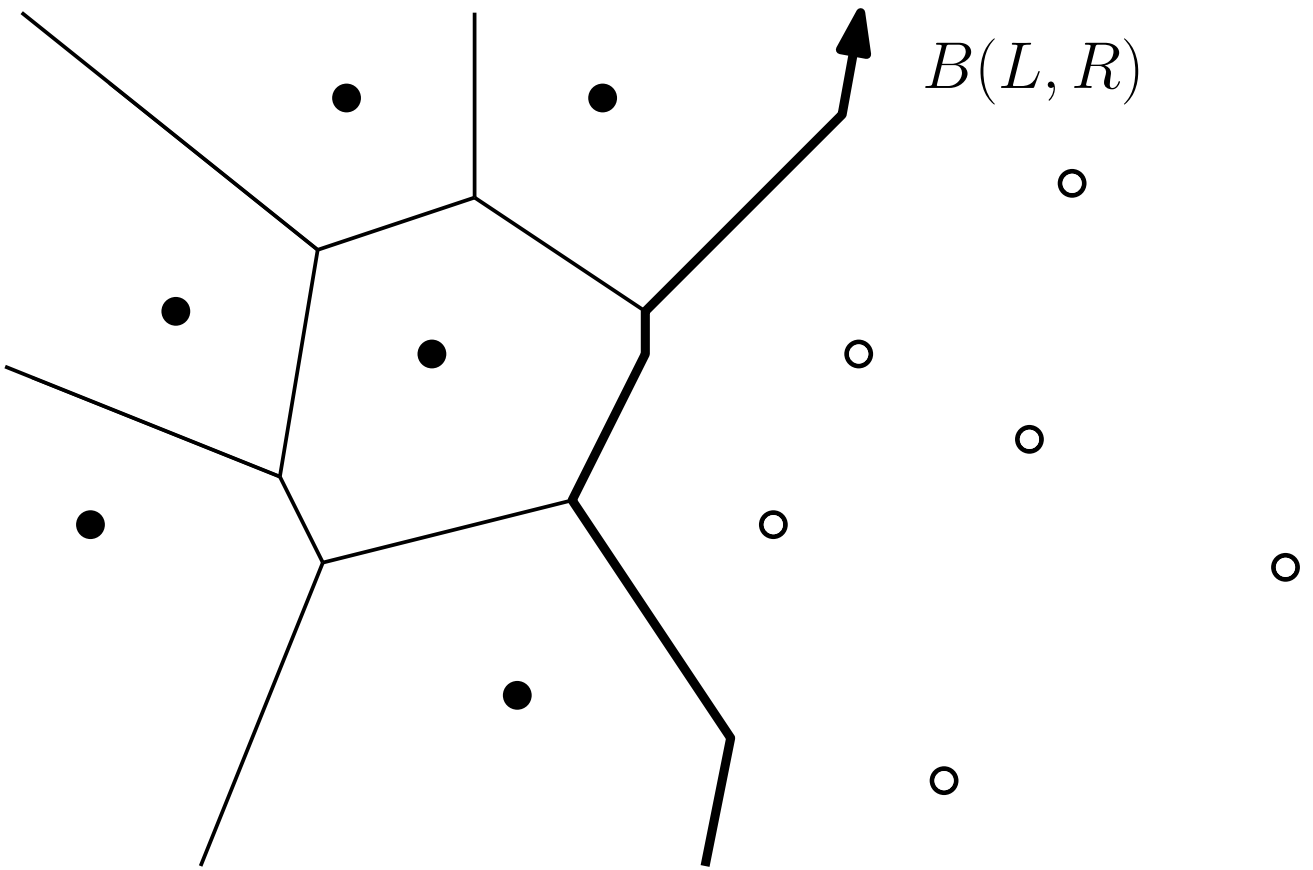

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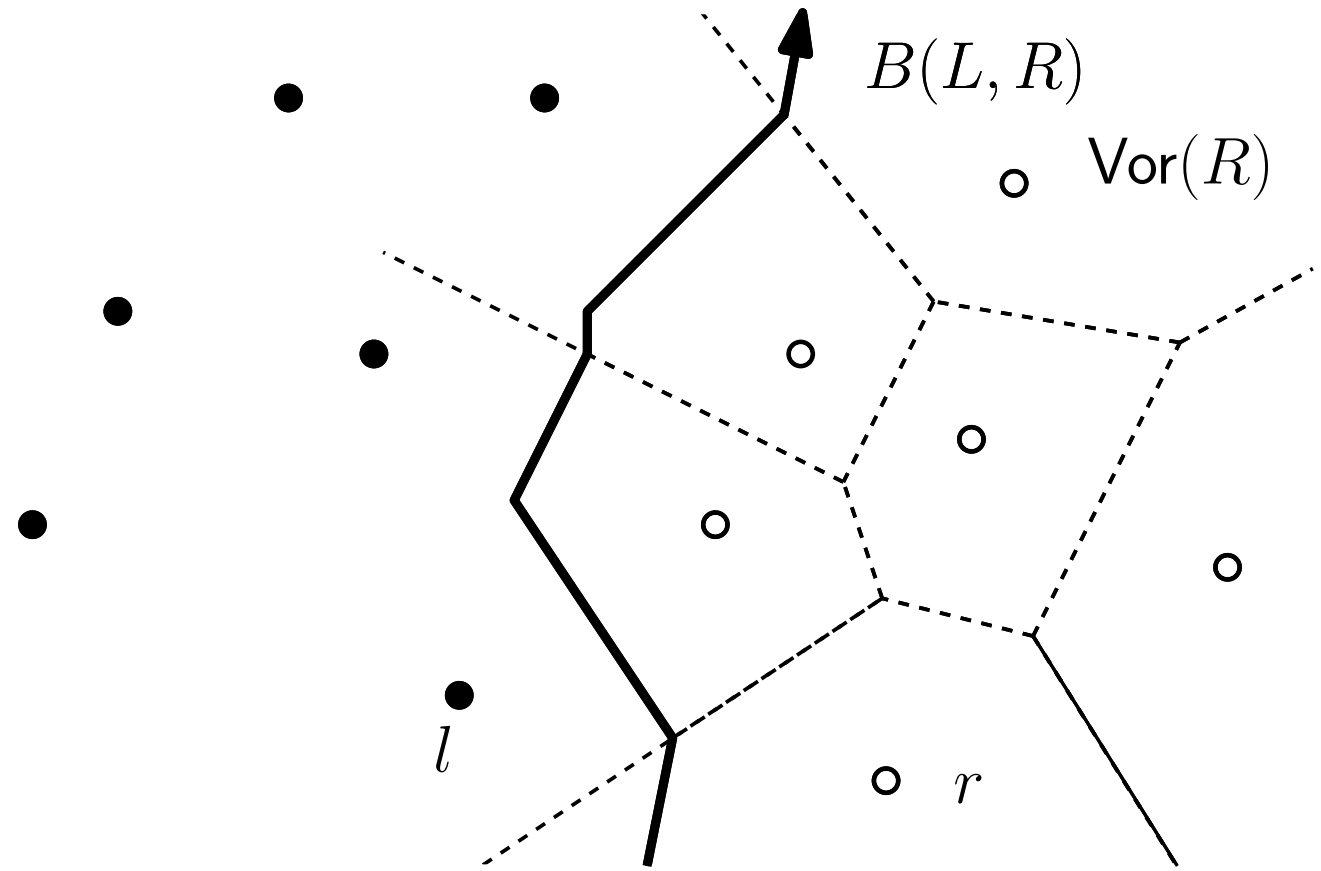

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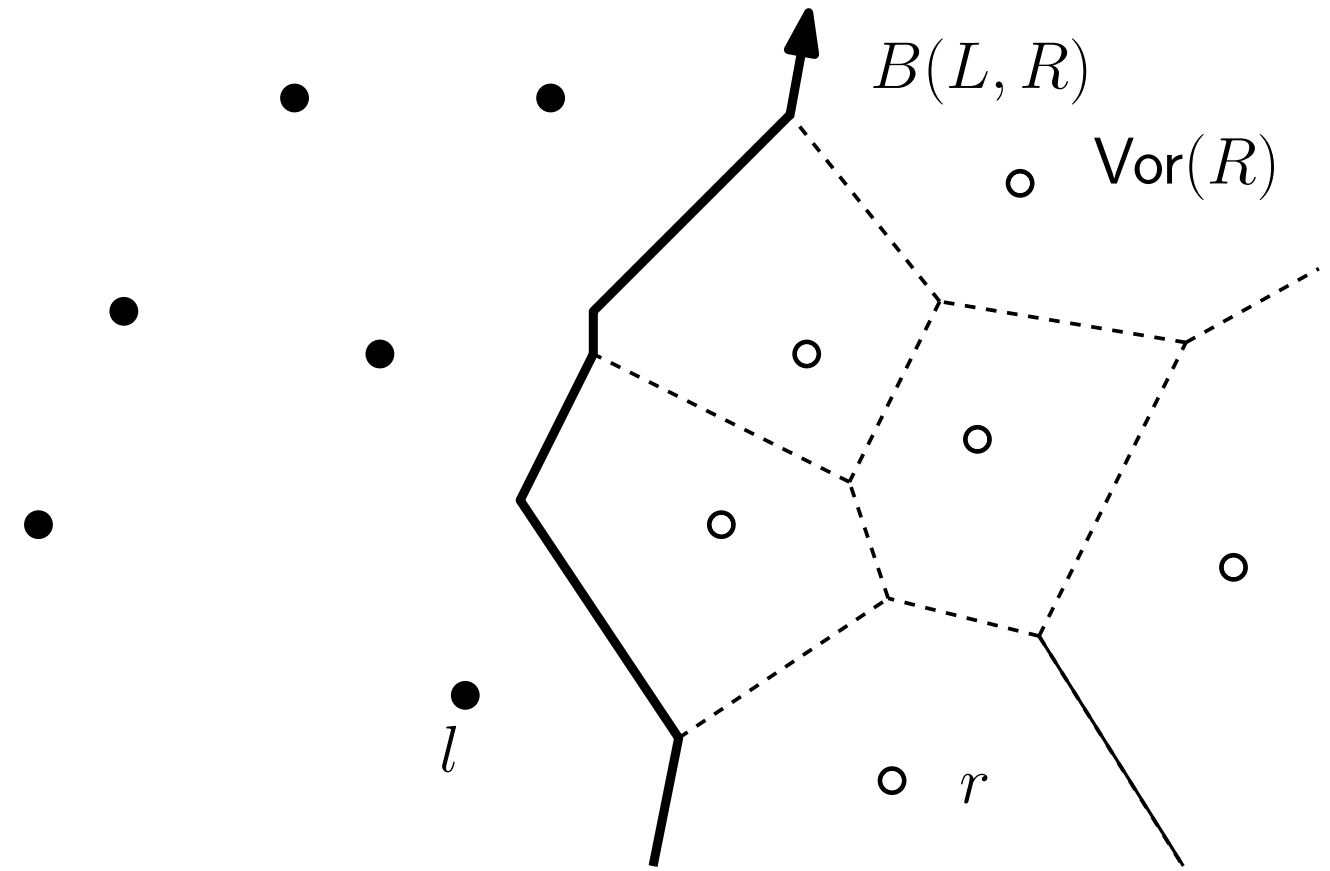

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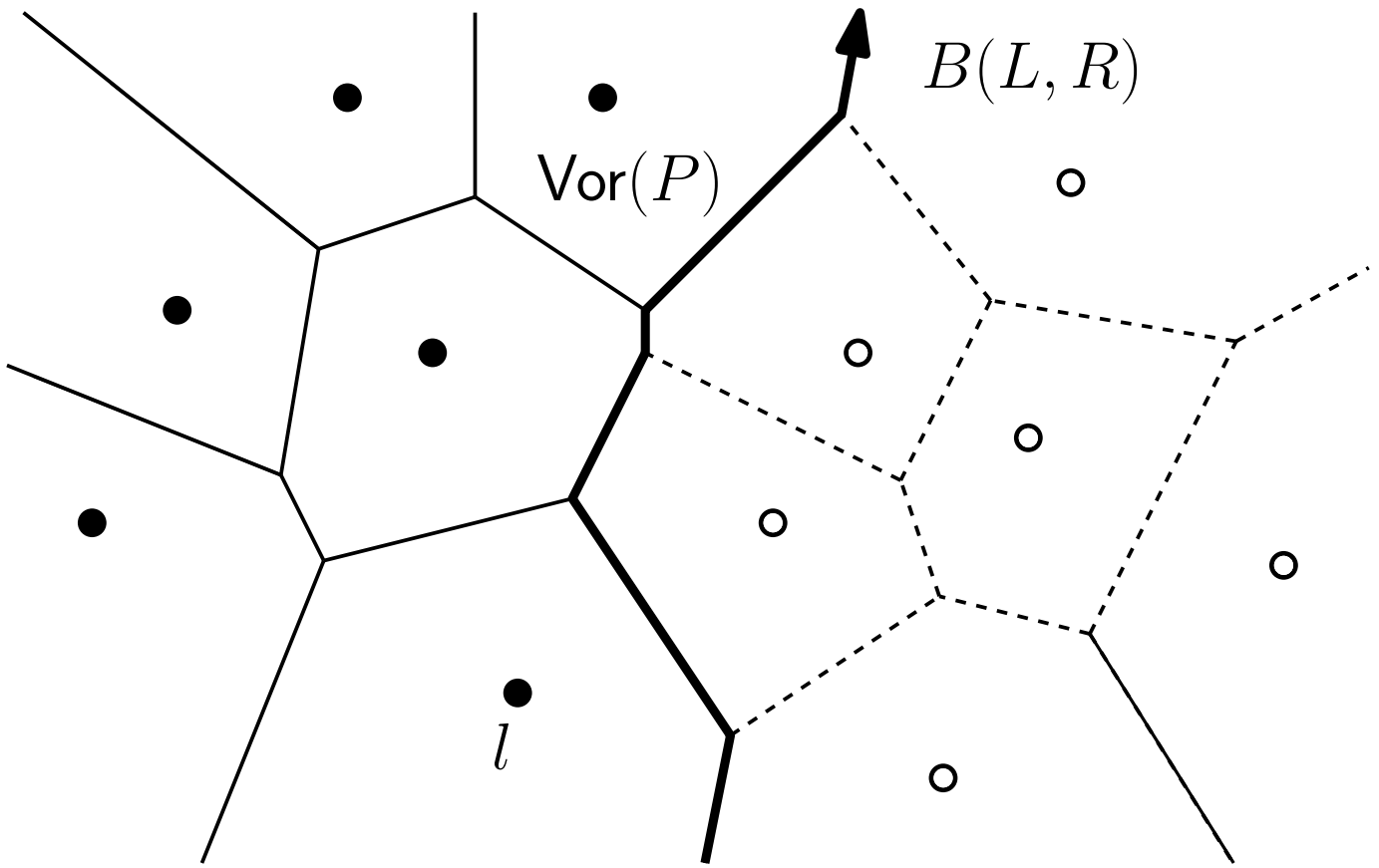

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  - Remove the part of Vor(L) right to $B(L, R)$
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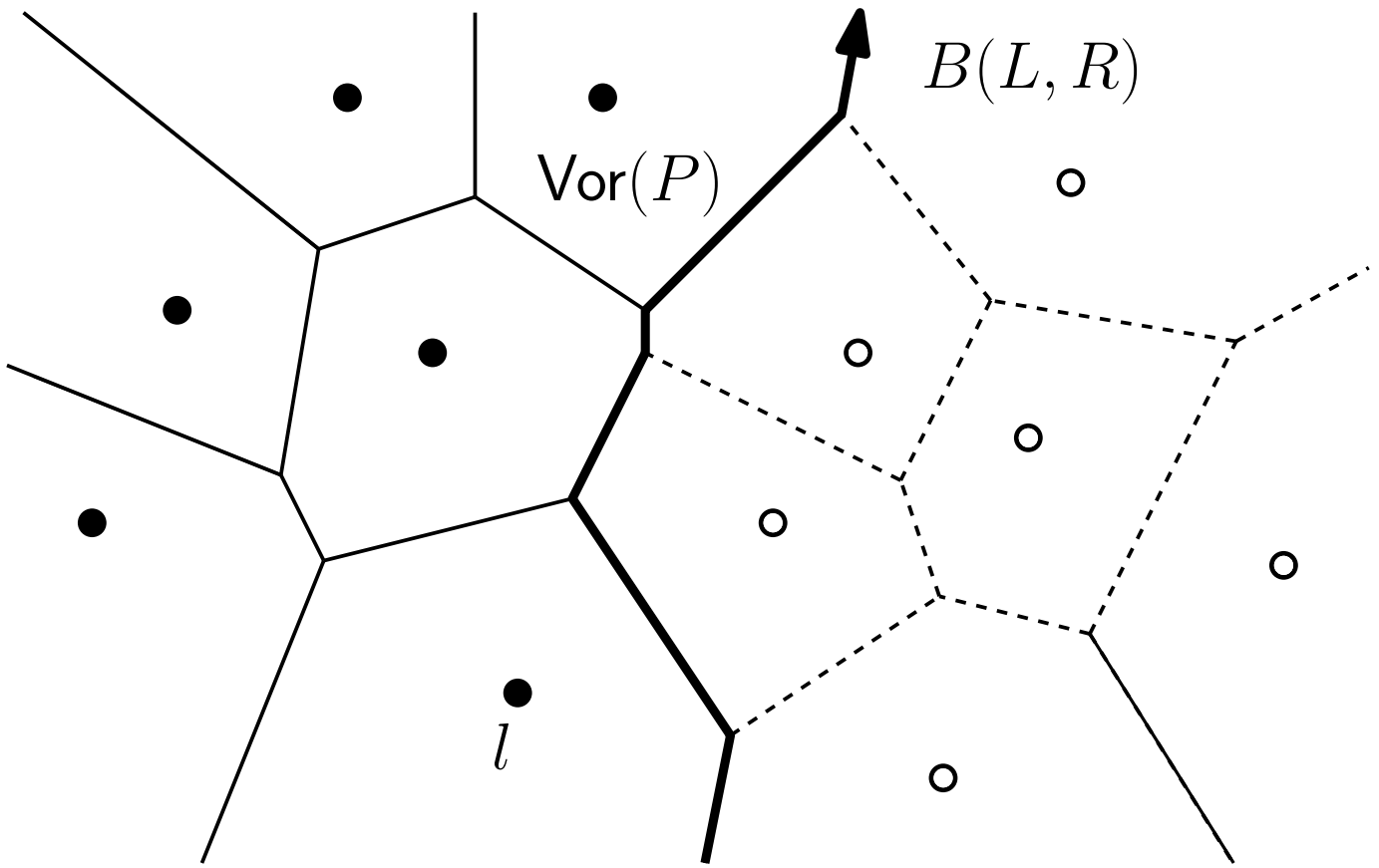
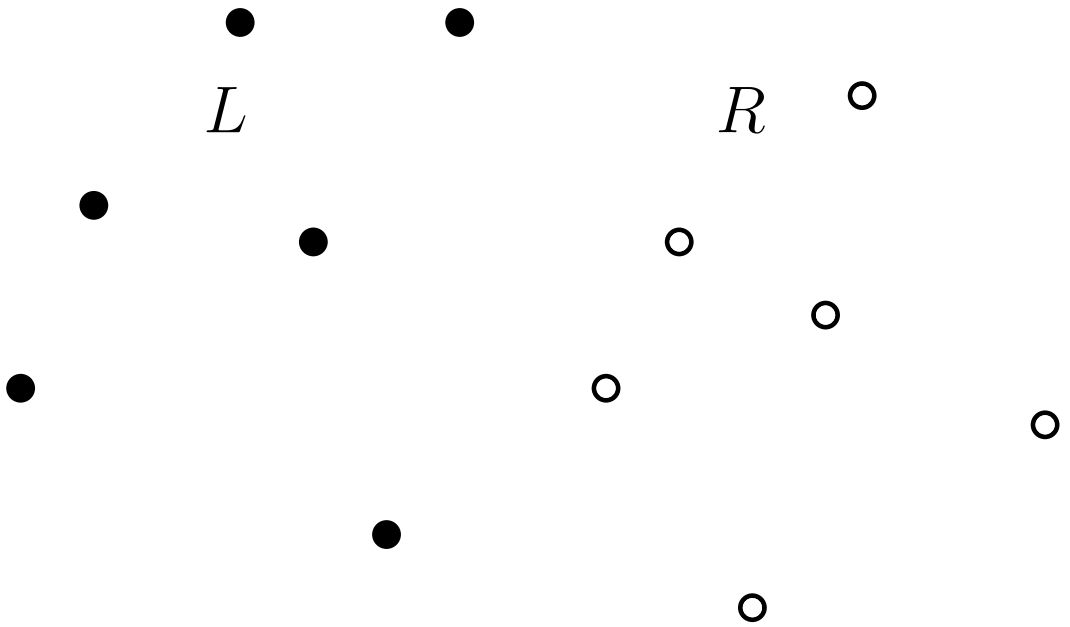
- $B(L, R)$ is a part of Vor(P)



Finding a starting edge of $B(L, R)$

- An unbounded Voronoi edge $\leftrightarrow$ an edge of the convex hull $conv(P)$



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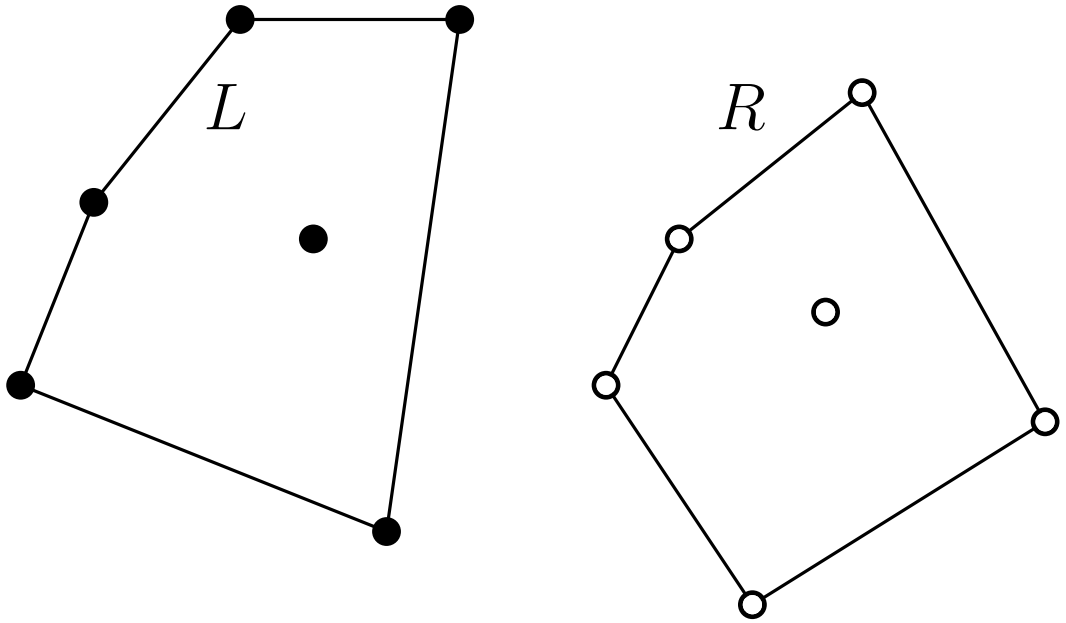

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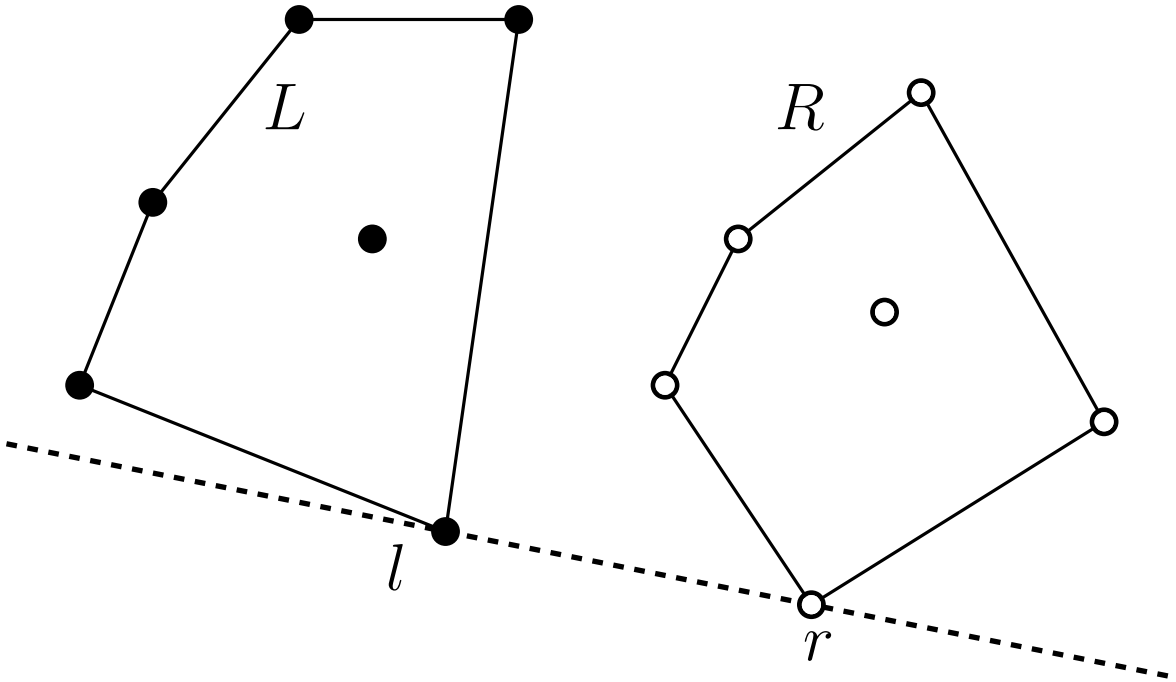

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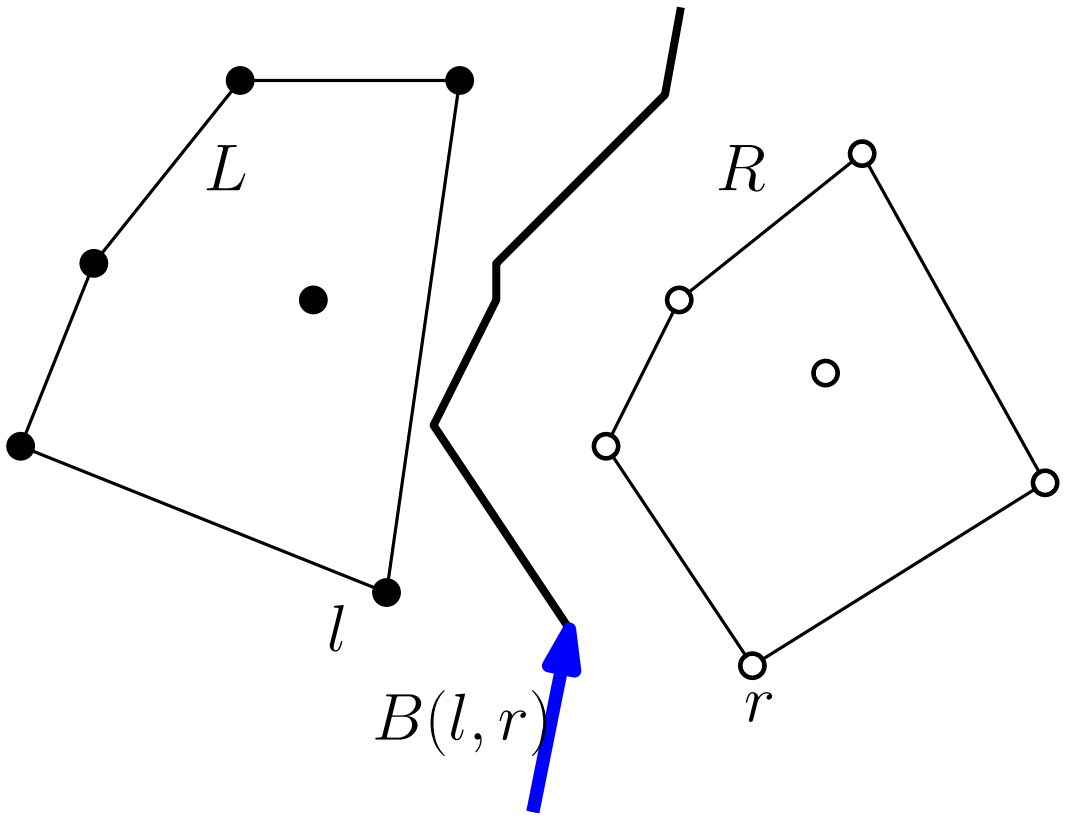
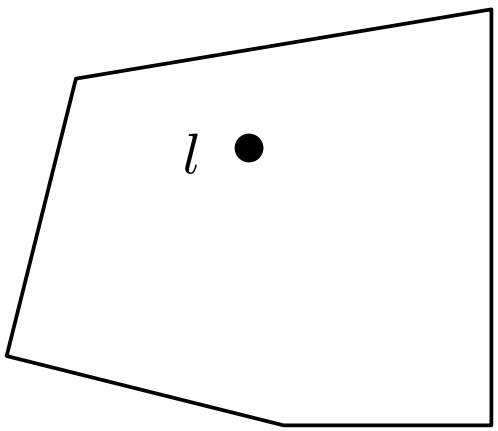


Trace $B(L, R)$ in $\mathcal{V}(l, L)$ (1)

$B(L, R) \cap \mathcal{V}(l, L)$



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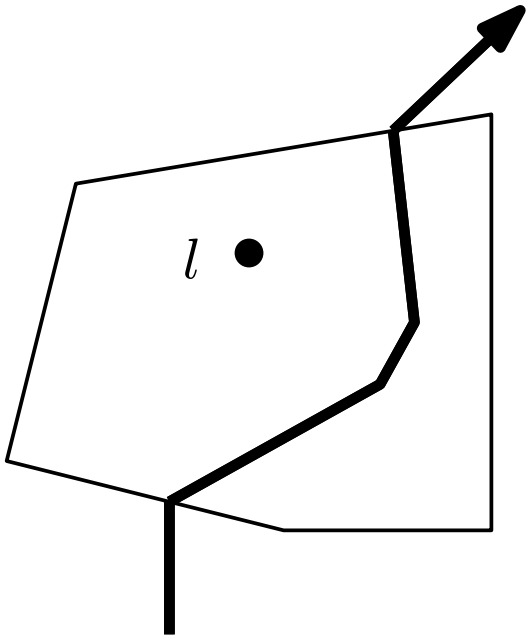

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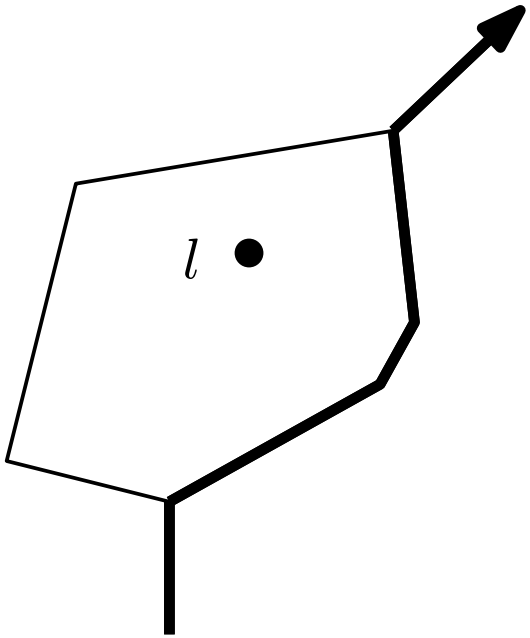

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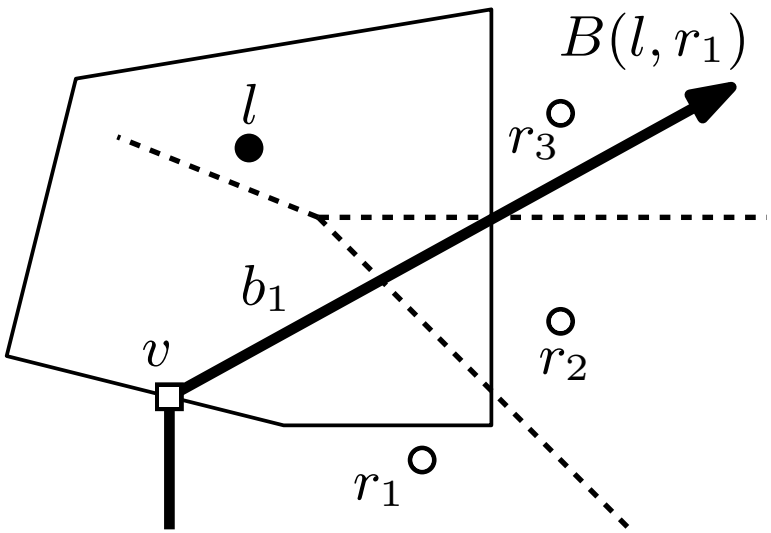

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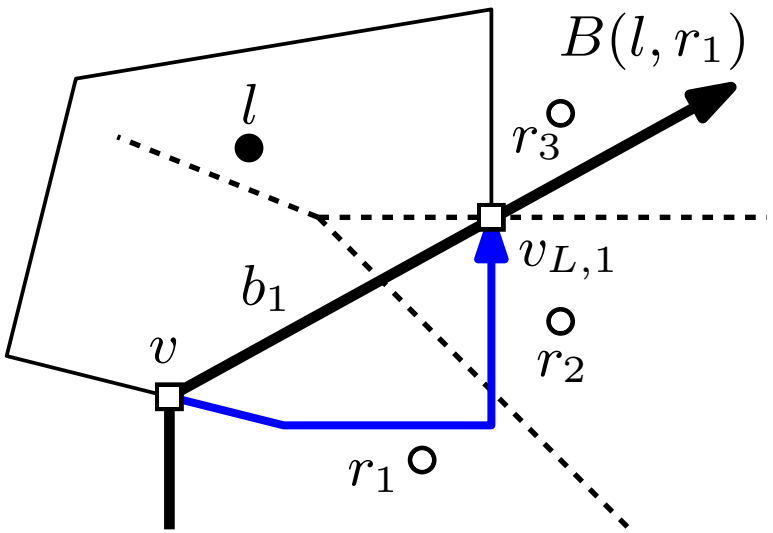
  - After $v_{L,1}$, $\mathcal{V}(l, L) \to \mathcal{V}(l', L)$, and $B(l, r_1) \to B(l', r_1)$



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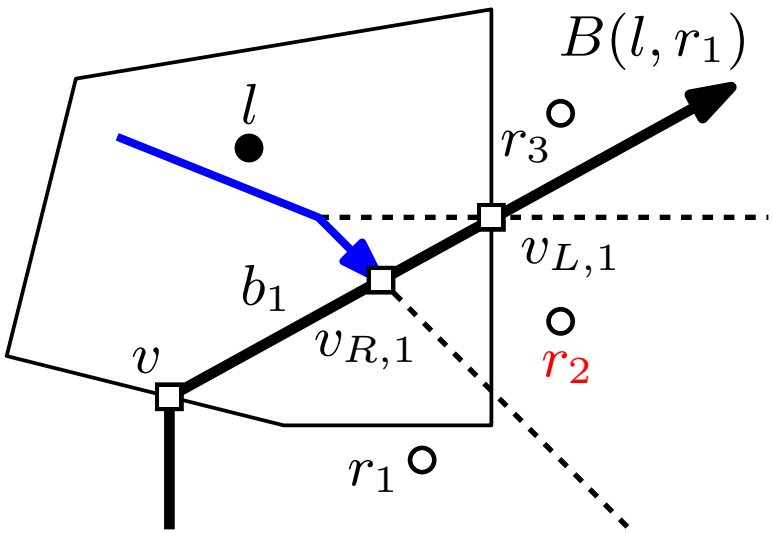
- Clockwise along $\partial\mathcal{V}(r_1, R)$ to find $v_{R,1} \in B(l, r_1)$
  - After $v_{R,1}$, $\mathcal{V}(r_1, R) \to \mathcal{V}(r_2, R)$, and $B(l, r_1) \to B(l, r_2)$



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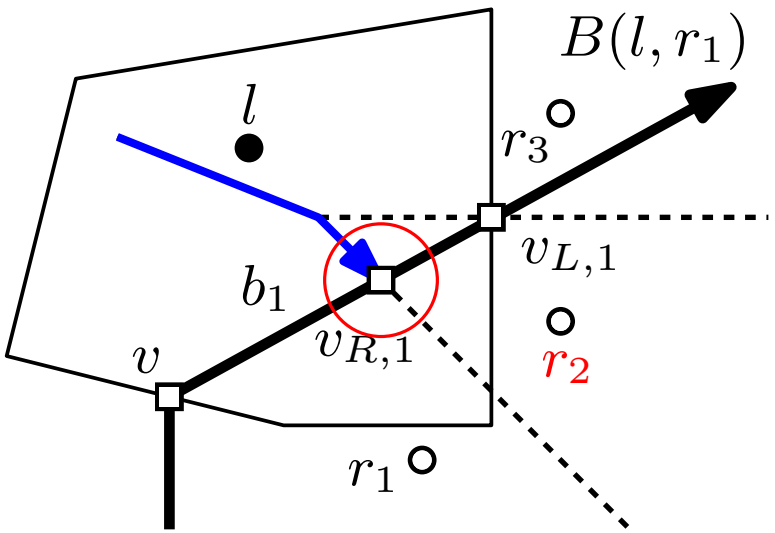
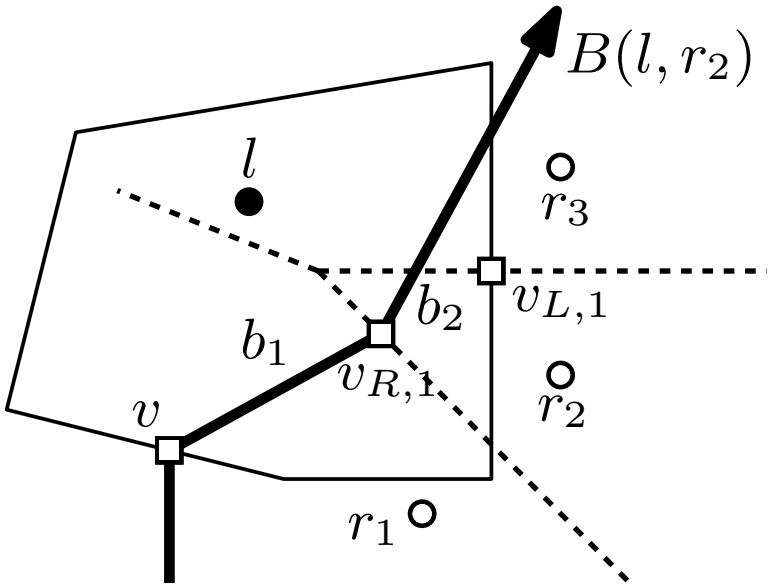
  - After $v_{R,1}$, $\mathcal{V}(r_1, R) \to \mathcal{V}(r_2, R)$, and $B(l, r_1) \to B(l, r_2)$
- $v_{R,1}$ earlier than $v_{L,1}$, and $B(l, r_1) \to B(l, r_2)$ at $v_{R,1}$



Trace $B(L, R)$ in $\mathcal{V}(l, L)$ (2)

- $B(L, R) \cap V(l, L)$
  - $B(L, R)$ from $v_{R,1}$ along $B(l, r_2)$



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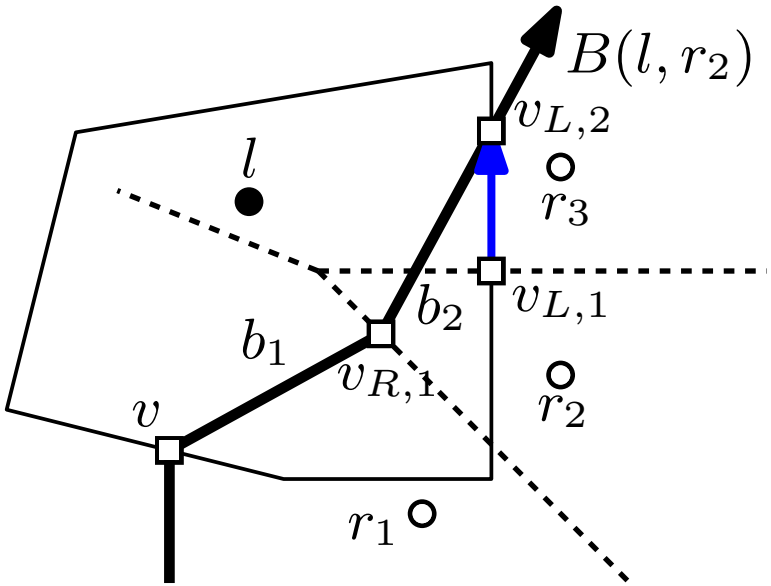
    - After $v_{L,2}$, $\mathcal{V}(l, L) \to \mathcal{V}(l', L)$, and $B(l, r_2) \to B(l', r_2)$



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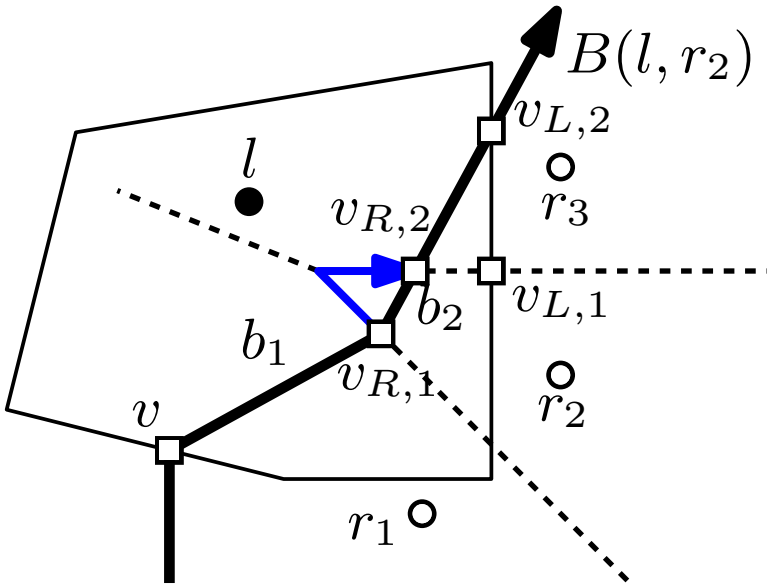

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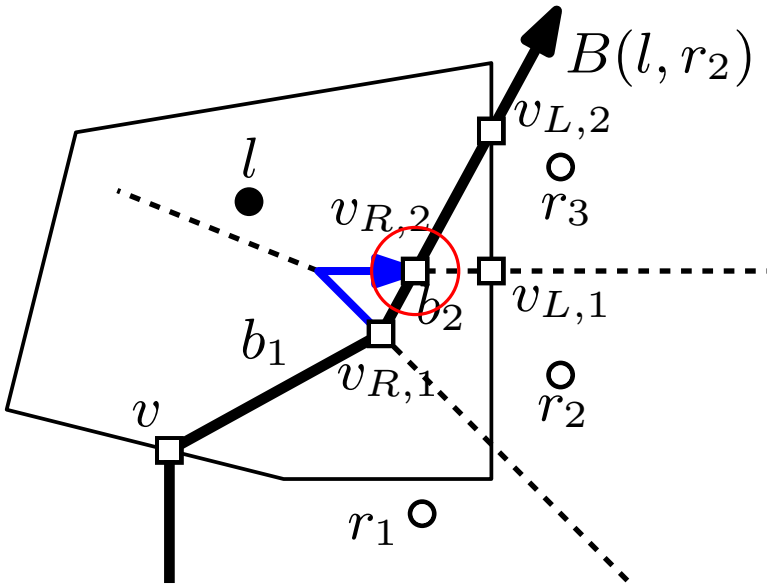
    - After $v_{R,2}$, $\mathcal{V}(r_2, R) \to \mathcal{V}(r_3, R)$, and $B(l, r_2) \to B(l, r_3)$
  - $v_{R,2}$ earlier than $v_{L,2}$, and $B(l, r_2) \to B(l, r_3)$ at $v_{R,2}$



Trace $B(L, R)$ in $\mathcal{V}(l, L)$ (3)

- $B(L, R) \cap \mathcal{V}(l, L)$
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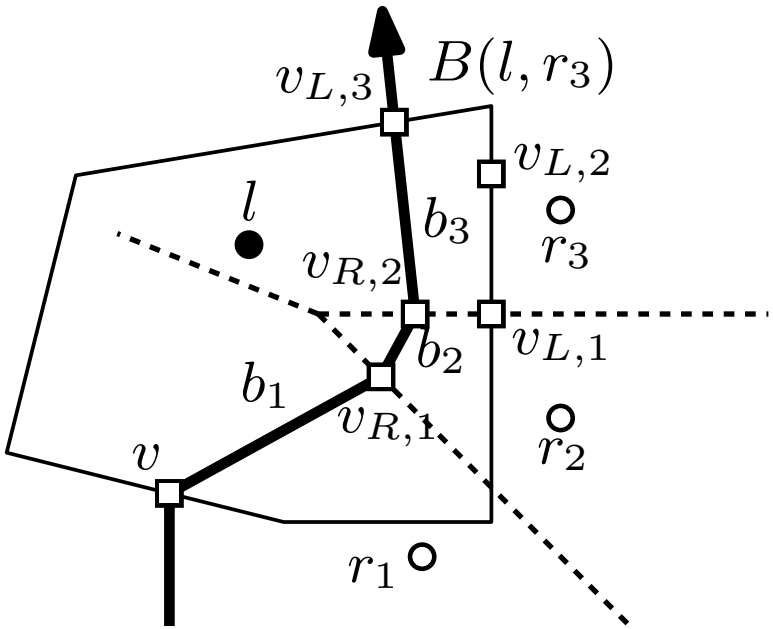

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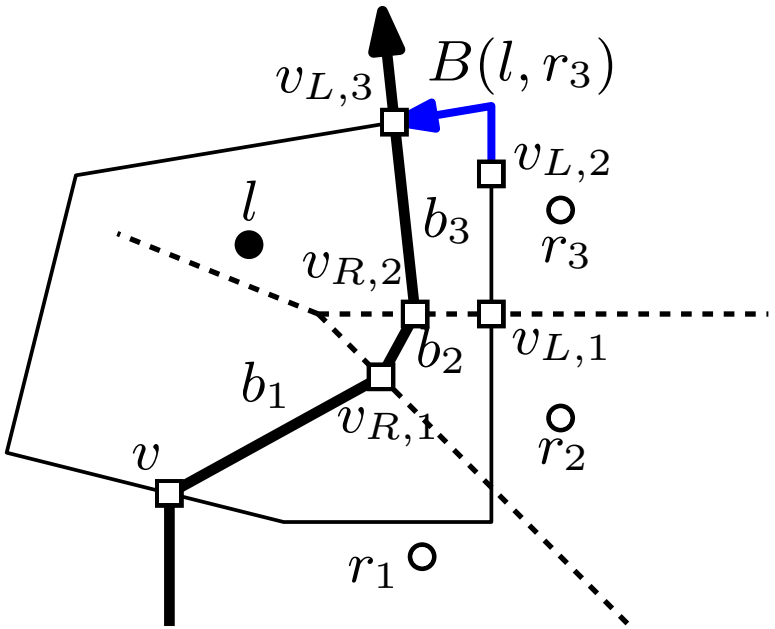

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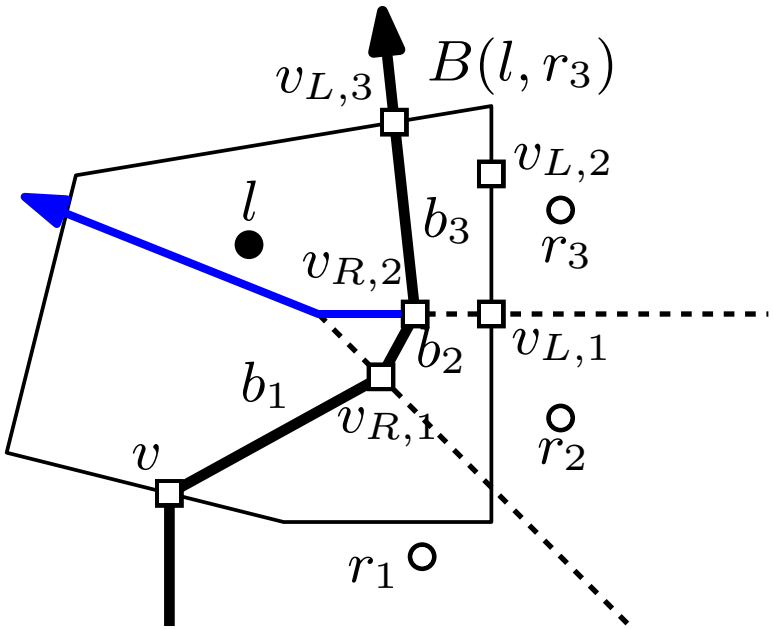
  - Clockwise along $\partial\mathcal{V}(r_3, R)$ from $v_{R,2}$ to find $v_{R,3} \in B(l, r_3)$
    - No $v_{R,3}$



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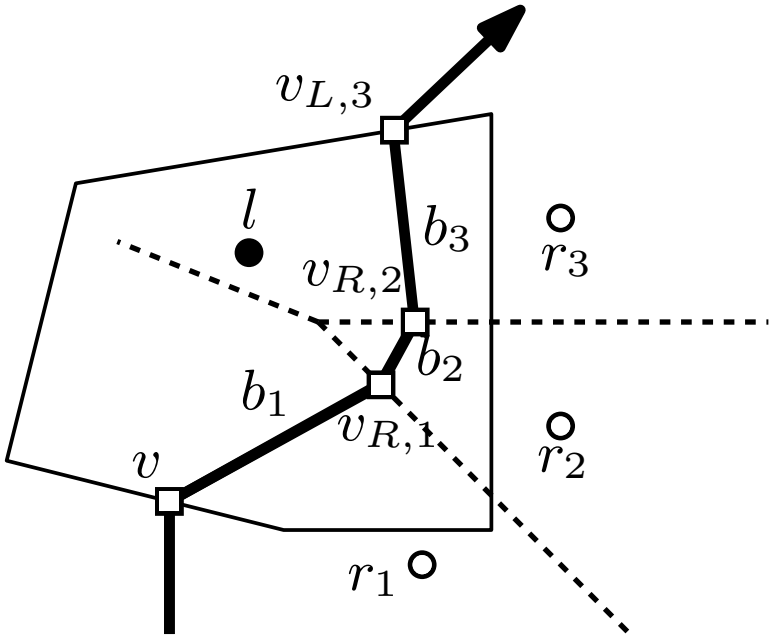
  - Clockwise along $\partial\mathcal{V}(r_3, R)$ from $v_{R,2}$ to find $v_{R,3} \in B(l, r_3)$
    - No $v_{R,3}$
  - $B(l, r_3) \to B(l'', r_3)$ at $v_{L,3}$



Time to Compute $B(L, R)$

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Lemma 9:

Computing $B(L, R)$ takes $O(|L| + |R|) = O(|P|)$ time

proof:

- Finding the starting edge takes $O(|L| + |R|)$ time
- Traversing $\text{Vor}(L)$ takes $O(|\text{Vor}(L)|)$ time ($\text{Vor}(R) \rightarrow O(|\text{Vor}(R)|)$)
  - Enter each $\mathcal{V}(l, L)$ and its boundary at most once
- There are $O(|B(L, R)|)$ intersections
  - Each edge of $B(L, R)$ makes two intersections.
- $O(|L| + |R|) + O(|\text{Vor}(L)|) + O(|\text{Vor}(R)|) + O(|B(L, R)|)$
  $= O(|L| + |R| + |L| + |R| + |P|) = O(|P|)$
  - $O(|\text{Vor}(L)|) = O(|L|)$, $O(|\text{Vor}(R)|) = O(|R|)$, and $O(|B(L, R)|) = O(|P|)$

Construction Time for Divide and Conquer

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Theorem 10:

The divide-and-conquer algorithm computes $\mathrm{Vor}(P)$ in $O(n \log n)$ time

proof:

- Sorting takes $O(n \log n)$ time (only do once)
- A sub-instance $P' \subset P$ takes $O(|P'|)$ time
  - Partitioning P' into L' and R' takes $O(|P'|)$ time
  - Computing $B(L', R')$ takes $O(|P'|)$ time
- The i^{th}-level takes $O(2^i) \times O(n/2^i) = O(n)$ time
  - $O(2^i)$ sub-instances each with $O(n/2^i)$ sites
- $O(\log n)$ levels
- $T(n) = O(n) + 2 \cdot T(n/2) => T(n) = O(n \log n)$

Comparison between two Algorithms

Plane Sweep

- $O(n)$ Voronoi vertices
  - A vertex must belong to $\mathrm{Vor}(P)$

Divide and Conquer

- $\Theta(n \log n)$ Voronoi vertices
  - Each level computes $\Theta(n)$ vertices.
  - A vertex does not necessarily belong to $\mathrm{Vor}(P)$

Comparison between two Algorithms

Plane Sweep

- $O(n)$ Voronoi vertices
  - A vertex must belong to $\mathrm{Vor}(P)$
- Better Space
  - Only keep the wavefront W

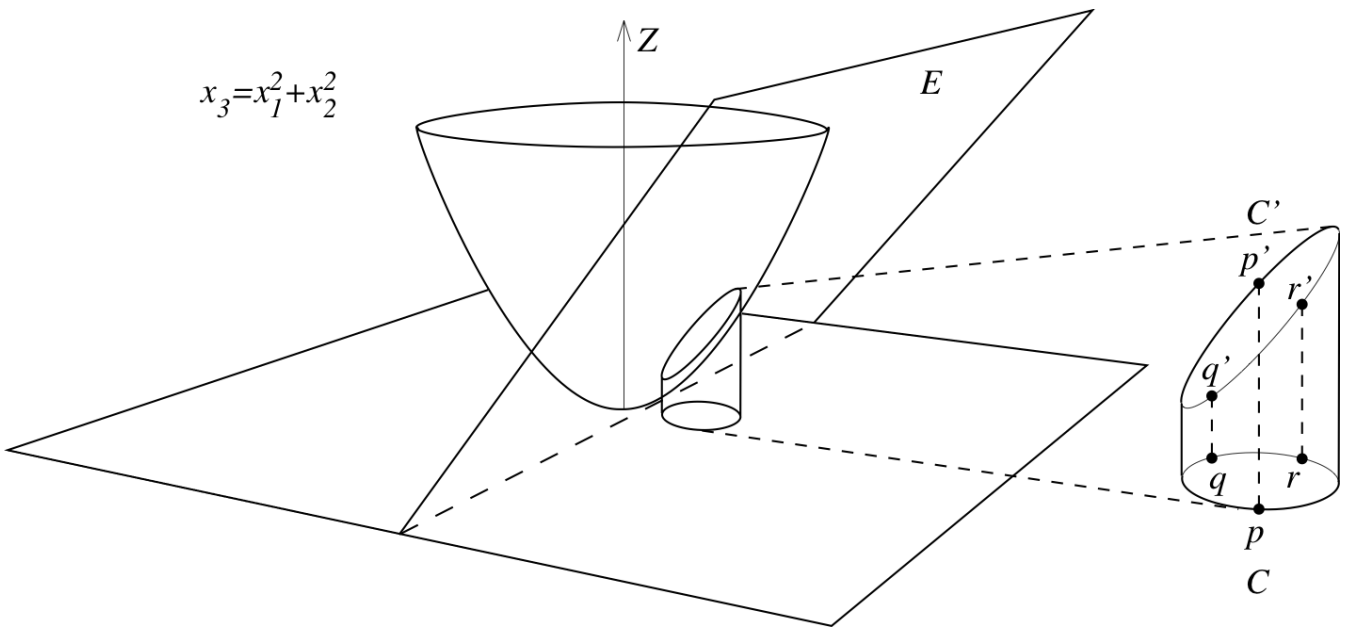
Divide and Conquer

- $\Theta(n \log n)$ Voronoi vertices
  - Each level computes $\Theta(n)$ vertices.
  - A vertex does not necessarily belong to $\mathrm{Vor}(P)$
- Parallelism (n machines and P is sorted)

$$\sum_{i=0}^{O(\log n)} O(n/2^i) = O(n)$$

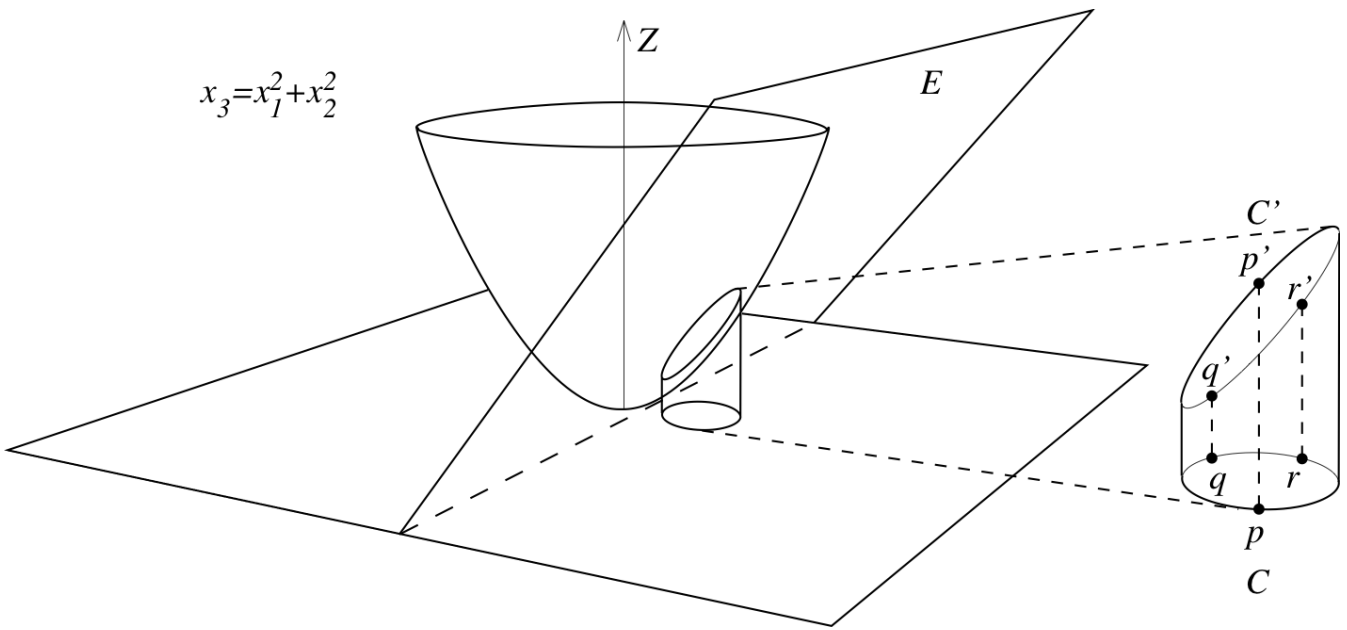
Geometric Transformation from 2D to 3D

- A paraboloid $\mathcal{P} = \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 = x_3\}$ in 3D



Geometric Transformation from 2D to 3D

- A paraboloid $\mathcal{P} = \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 = x_3\}$ in 3D
- For a point $x = (x_1, x_2)$ in 2D, $x' = (x_1, x_2, x_1^2 + x_2^2)$ is its lifted image in 3D
  - $x' \leftarrow$ vertical projection from x to $\mathcal{P}$



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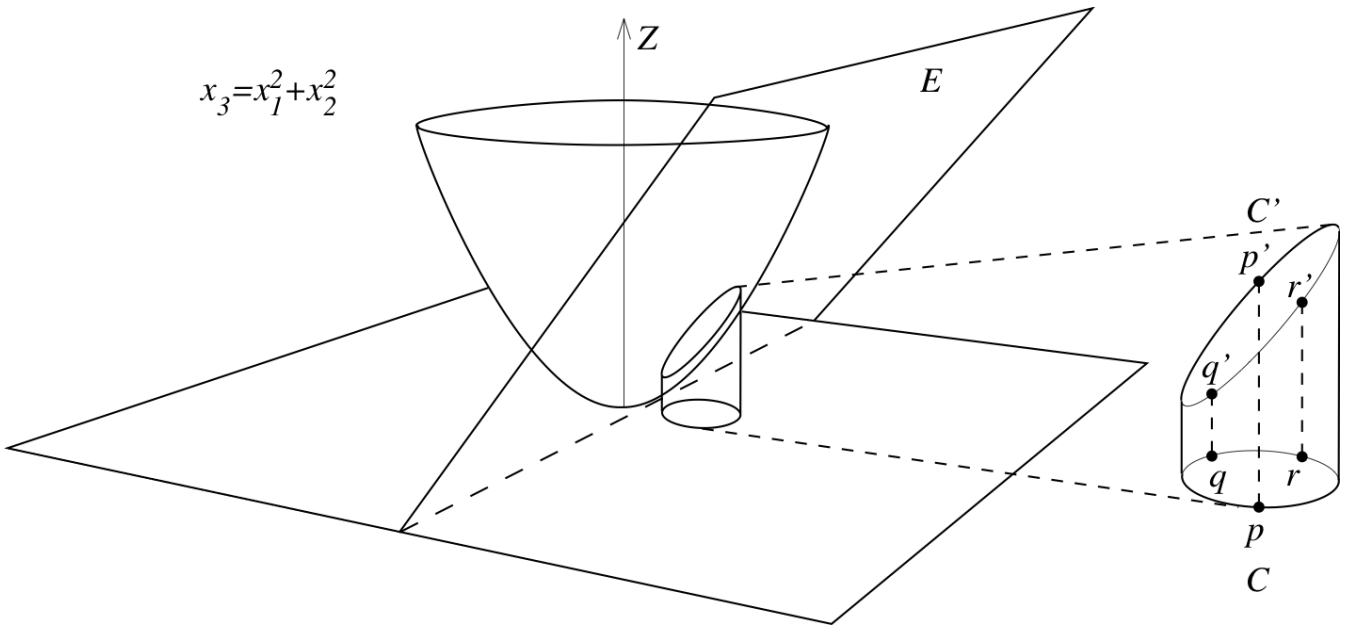
  - $x' \leftarrow$ vertical projection from x to $\mathcal{P}$
- For a set A of points in 2D, its lifted image $A' = \{x' = (x_1, x_2, x_1^2 + x_2^2) \mid x = (x_1, x_2) \in A\}$



Circle in 2D $\leftrightarrow$ Planar Curve in $\mathcal{P}$

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Lemma 11:

Let C be a circle in the plane. Then C' is a planar curve on the paraboloid $\mathcal{P}$

proof:

- C is given by $r^2 = (x_1 - c_1)^2 + (x_2 - c_2)^2$
  - $r^2 = x_1^2 + x_2^2 - 2x_1c_1 - 2x_2c_2 + c_1^2 + c_2^2$
- C' satisfies $x_1^2 + x_2^2 = x_3$
- Substituting $x_1^2 + x_2^2$ by x_3, we obtain a plane E

$$x_3 - 2x_1c_1 - 2x_2c_2 + c_1^2 + c_2^2 - r^2 = 0$$

- $C' = \mathcal{P} \cap E$
- Intersection between E and $\mathcal{P}$ is a planar curve

Lower Convex Hull

- $P' =$ lifted image of P on $\mathcal{P} \rightarrow P'$ in convex position
- Each point of P' is a vertex of $conv(P')$
- Lower convex hull $lconv(P')$ of P' is the part of $conv(P')$ visible from $x_3 = -\infty$

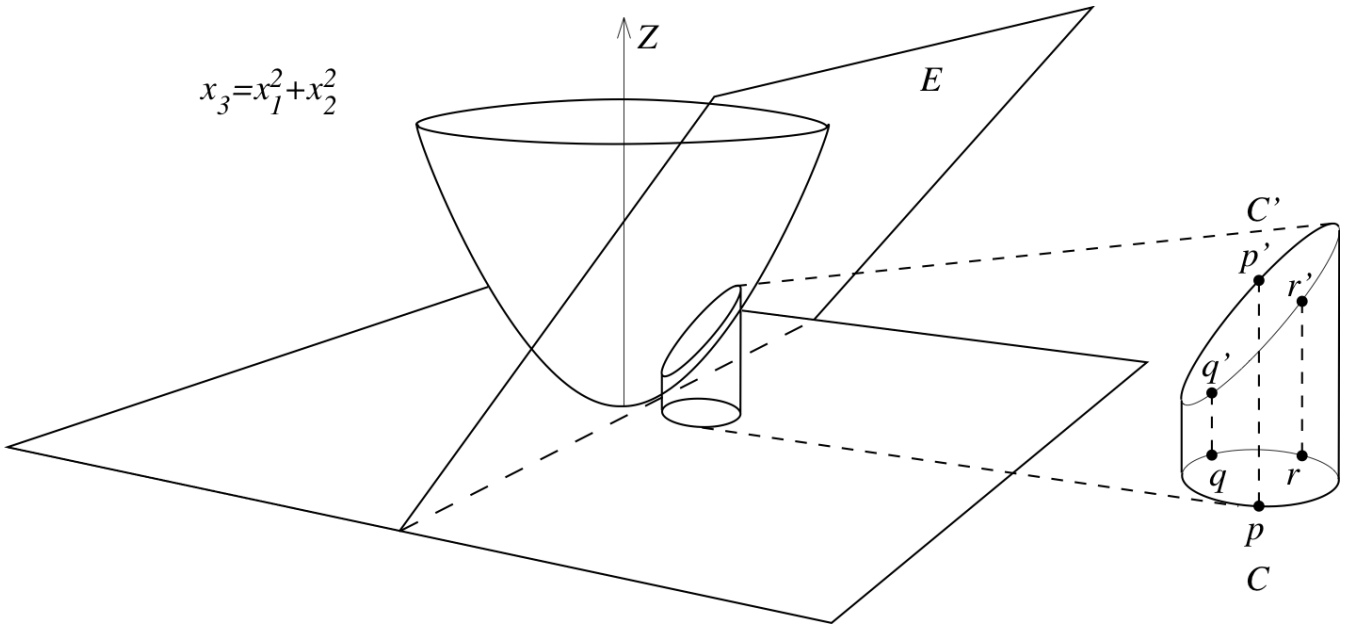
Duality between $\mathrm{DT}(P)$ and $lconv(P')$ (1)

Theorem 12:

Delaunay triangulation $\mathrm{DT}(P)$ equals to the vertical projection onto the x_1x_2-plane of lower convex hull of P'

proof:

- $p, q, r \in P$. C: circumcircle of p, q, r
- C' lies on a plane E defined by p', q', r'
- a point x inside $C \leftrightarrow$ lifted image x' below E



Duality between $\mathrm{DT}(P)$ and $lconv(P')$ (2)

Theorem 12:

Delaunay triangulation $\mathrm{DT}(P)$ equals to the vertical projection onto the x_1x_2-plane of lower convex hull of P'

proof:

- p, q, r defines a triangle of $\mathrm{DT}(P)$
  $\leftrightarrow$ no point of P lies in C defined by p, q, r
  $\leftrightarrow$ no point of P' lies below E defined by p', q', r'
  $\leftrightarrow$ p', q', r' defines a facet of $lconv(P')$
- Computing a convex hull in 3D takes $O(n \log n)$ time
  - Vor(P) in $O(n \log n)$ time