Computational Geometry Lecture
Point Location

INSTITUT FÜR THEORETISCHE INFORMATIK · FAKULTÄT FÜR INFORMATIK

Chih-Hung Liu · Tamara Mchledidze

20.06.2018

Motivation

Given a position $p = (p_x, p_y)$ in a map, determine in which country p lies.

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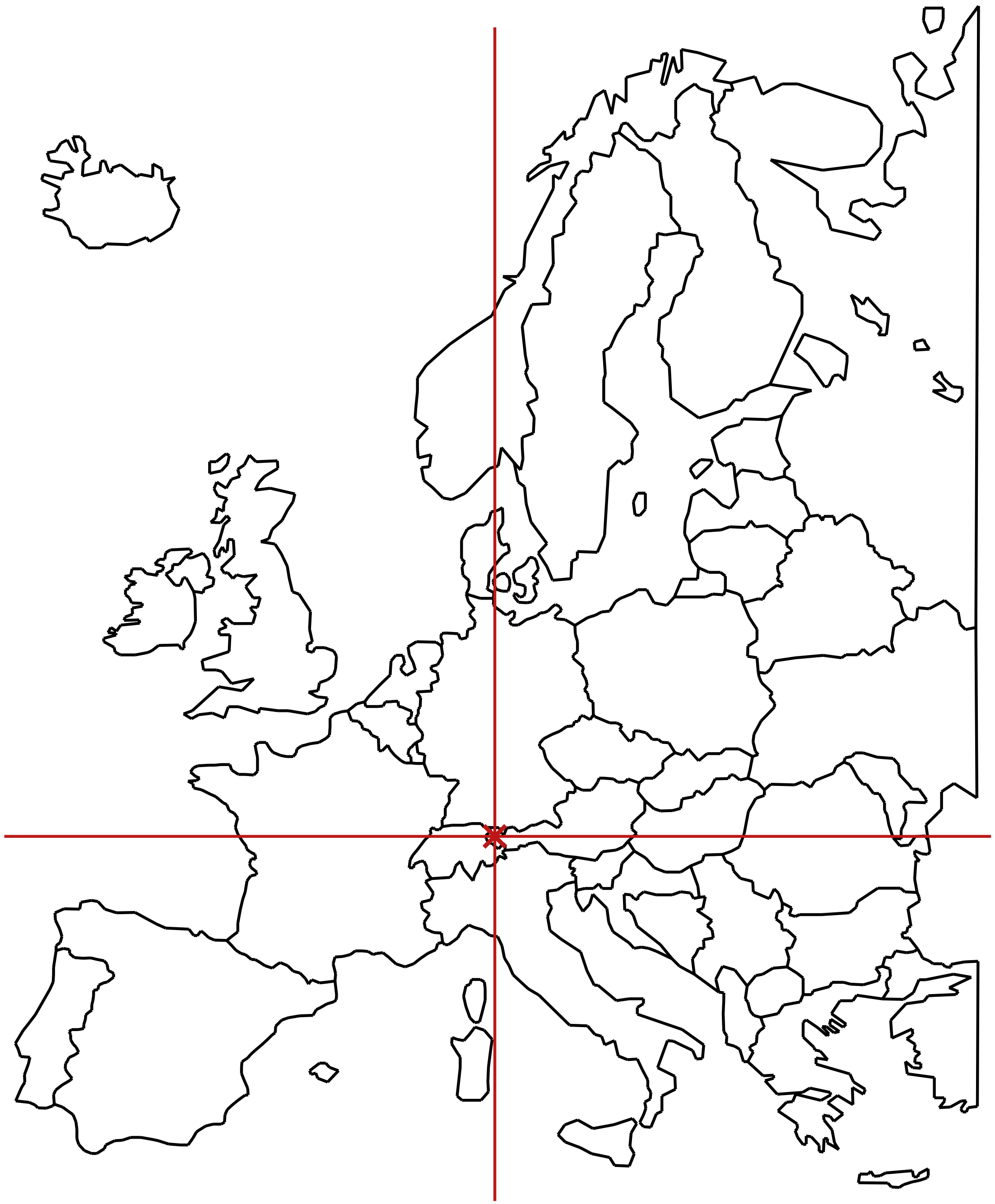

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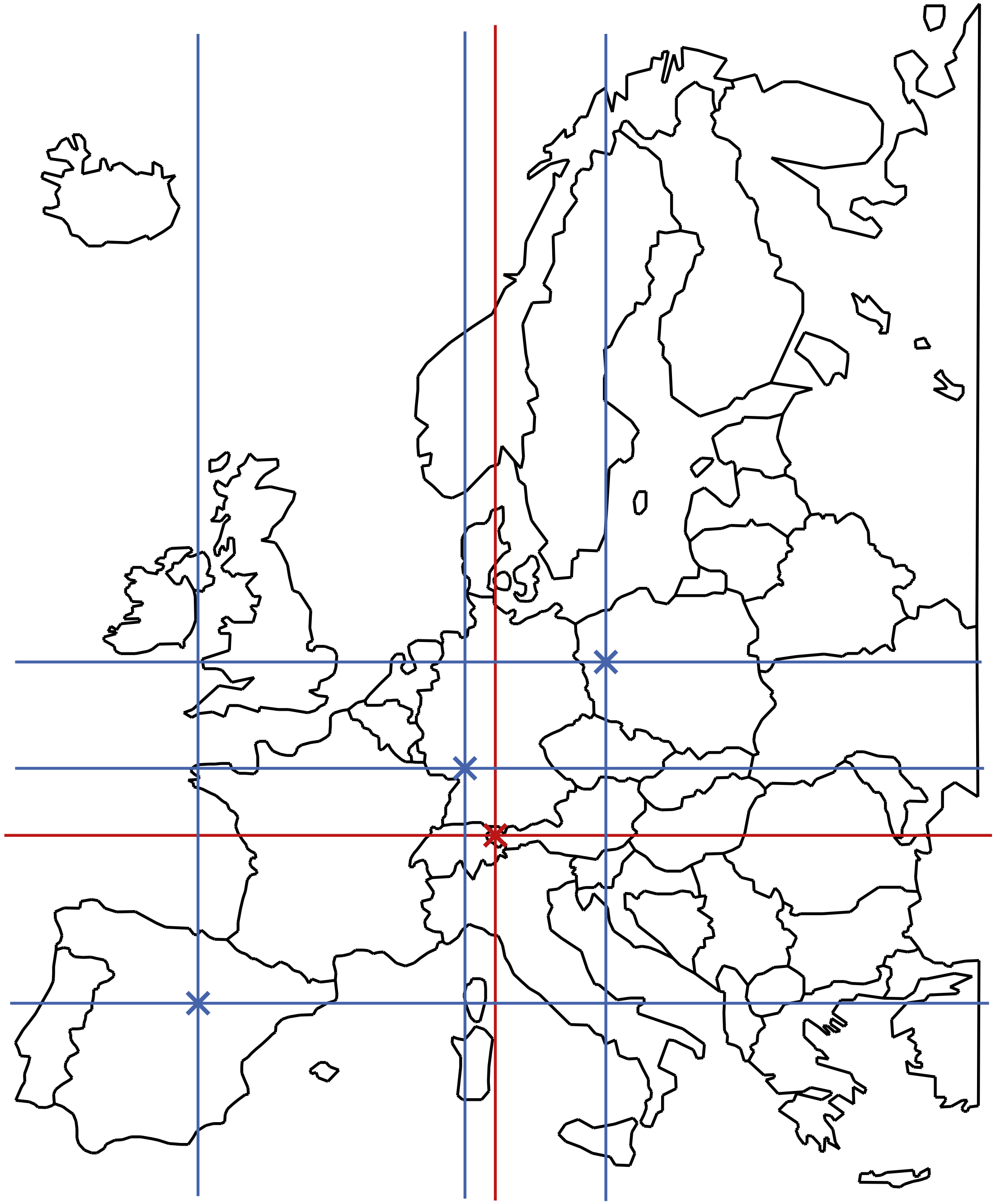

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more precisely:
Find a data structure for efficiently answering such point location queries.

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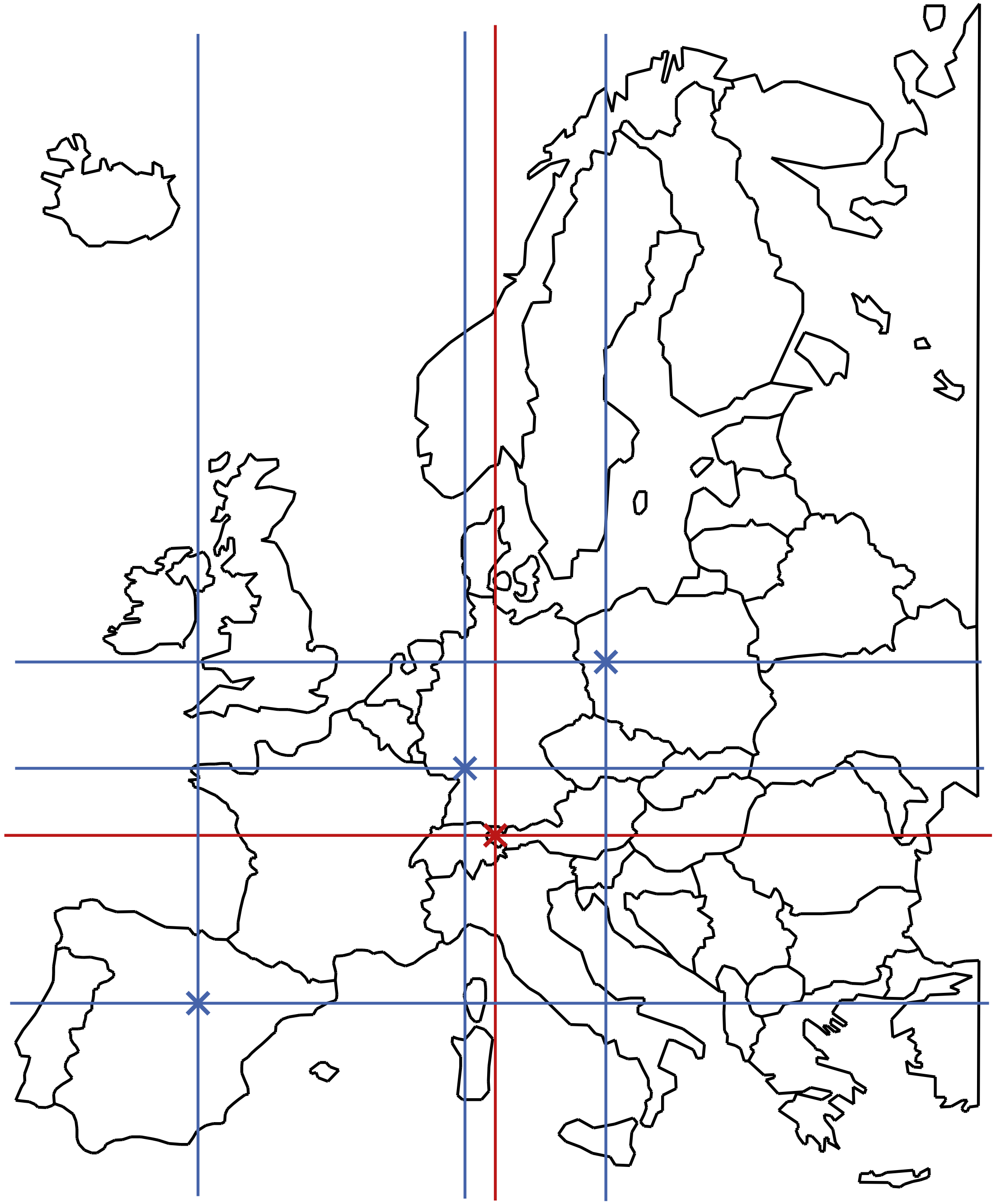
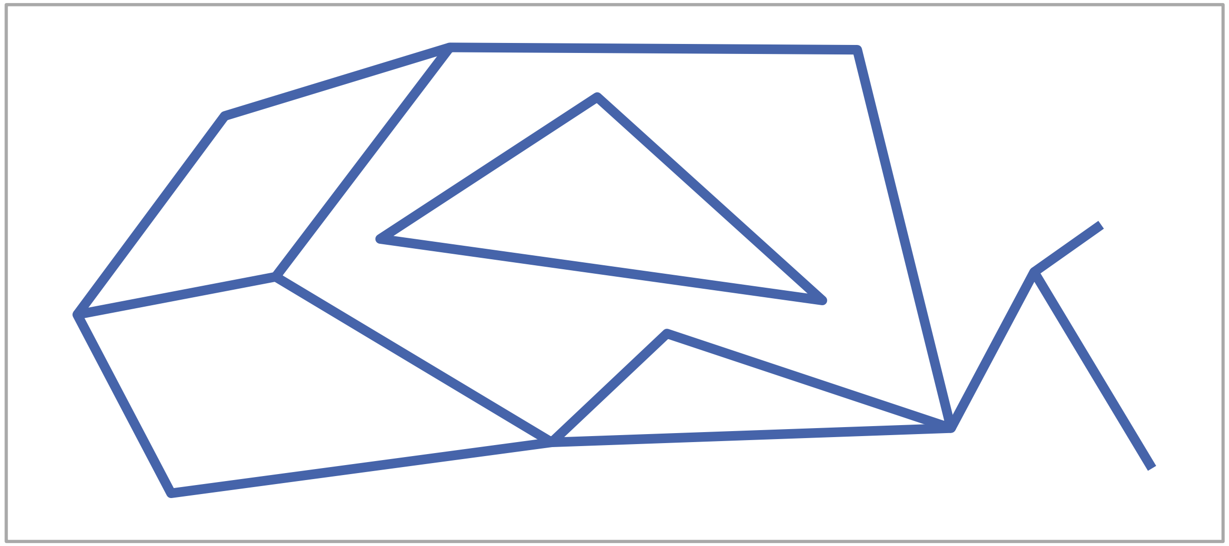


Given a position $p = (p_x, p_y)$ in a map, determine in which country p lies.

more precisely:
Find a data structure for efficiently answering such point location queries.

The map is modeled as a subdivision of the plane into disjoint polygons.

Problem Setting



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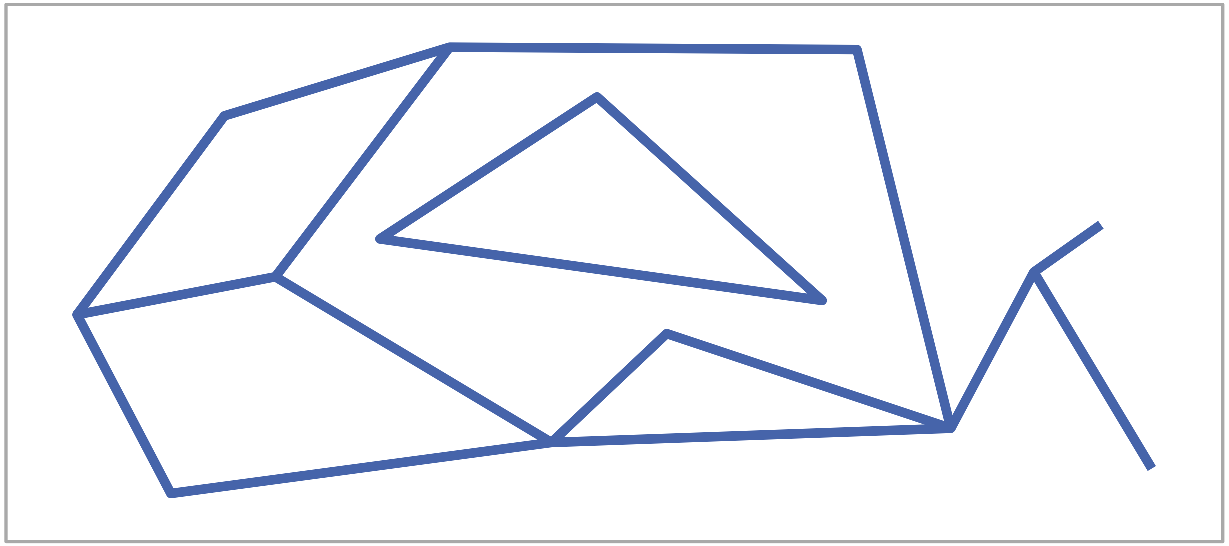

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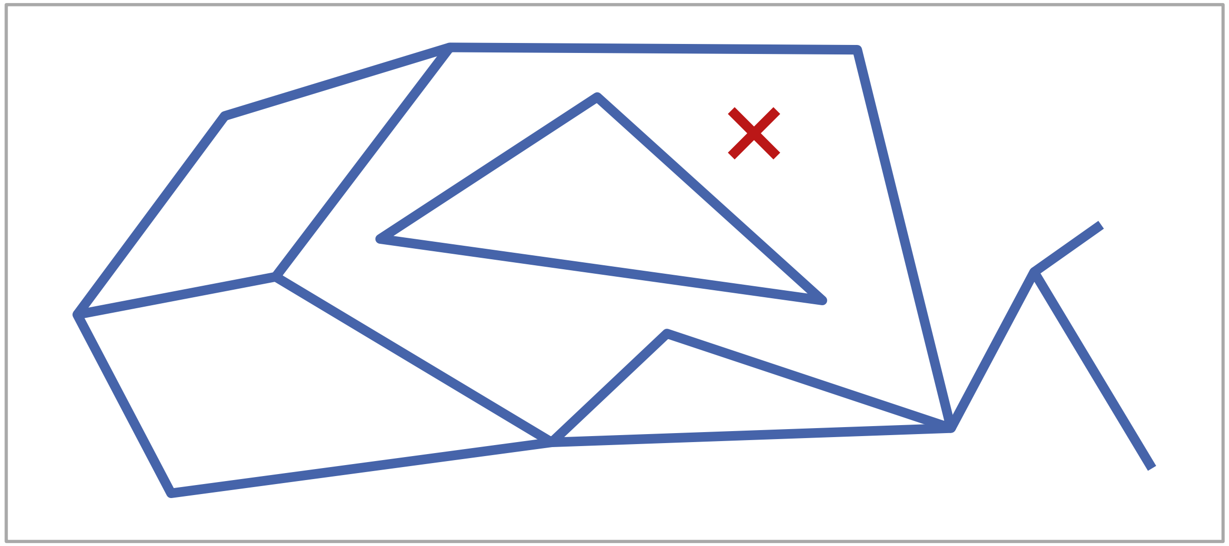

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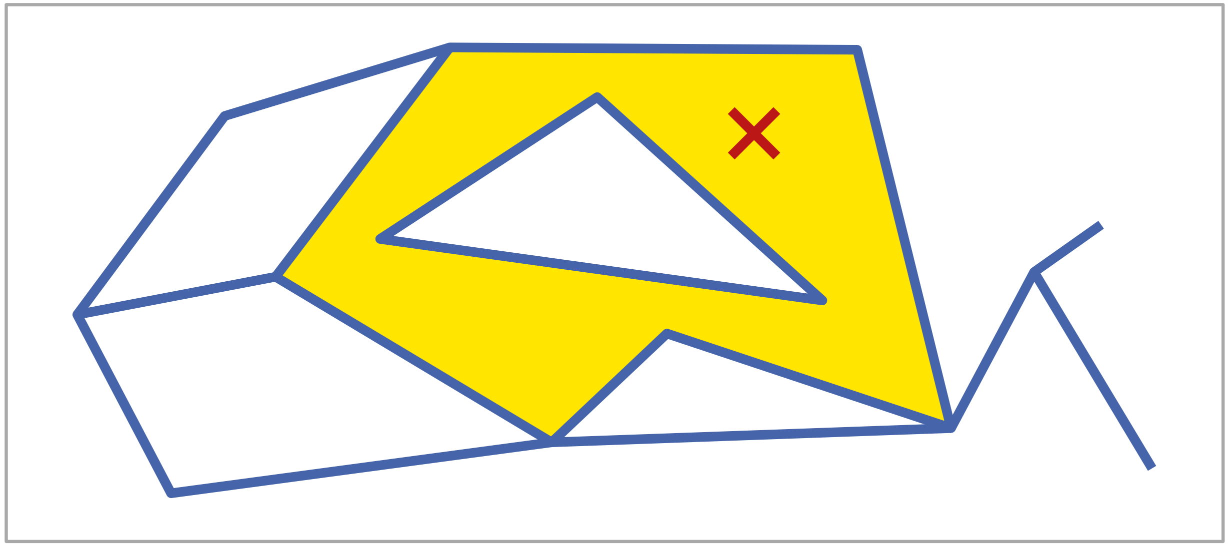

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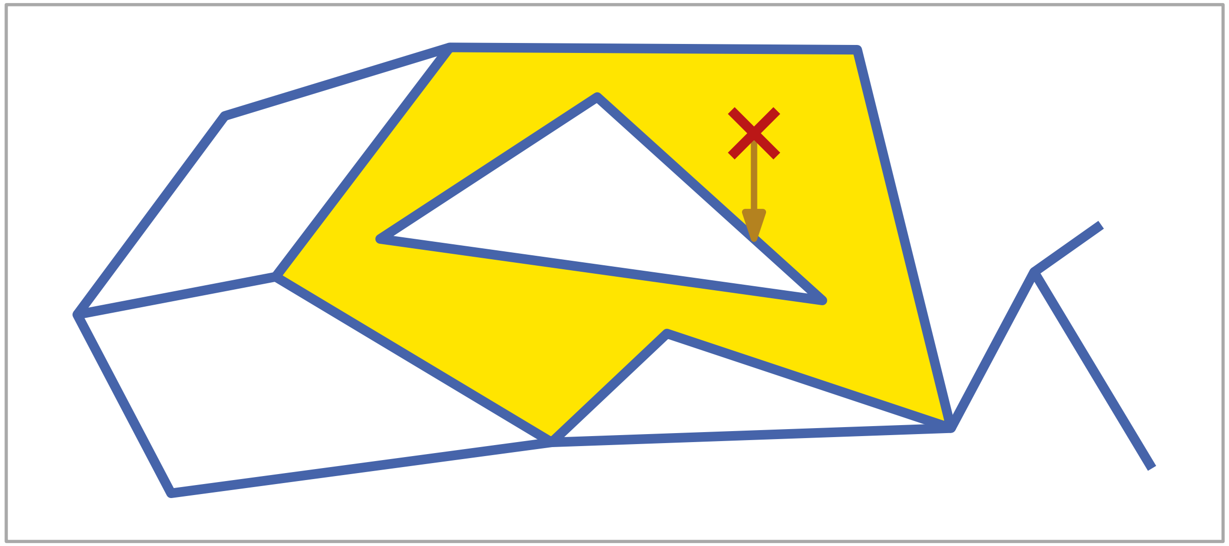

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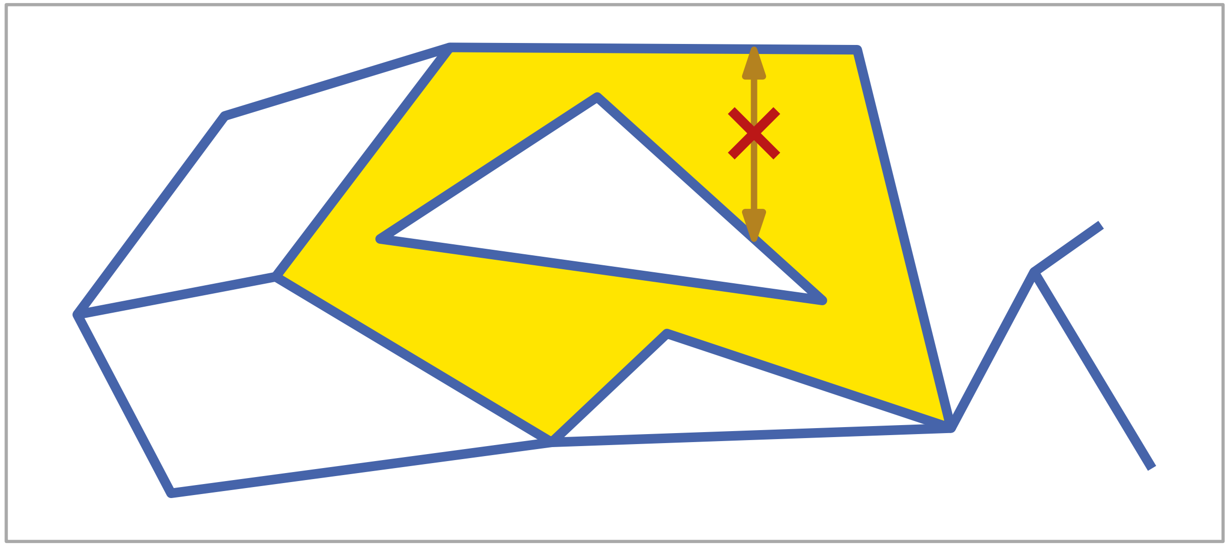

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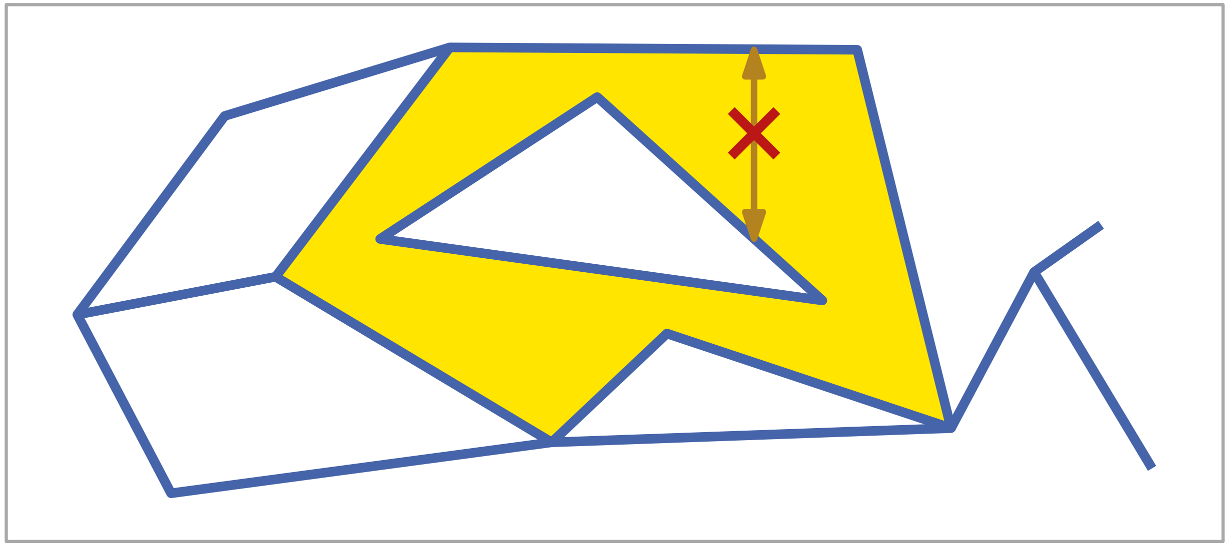

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Think for 2 minutes!

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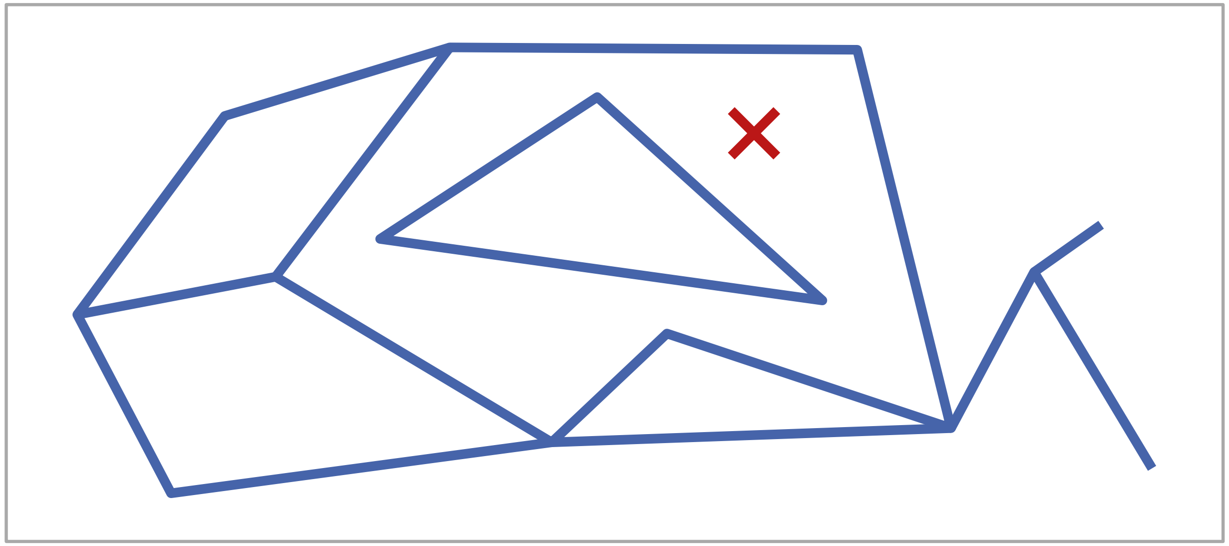

Goal: Given subdivision $\mathcal{S}$ of the plane with n segments, construct data structure for fast point location queries.

Solution: Partition $\mathcal{S}$ at points into vertical slabs.

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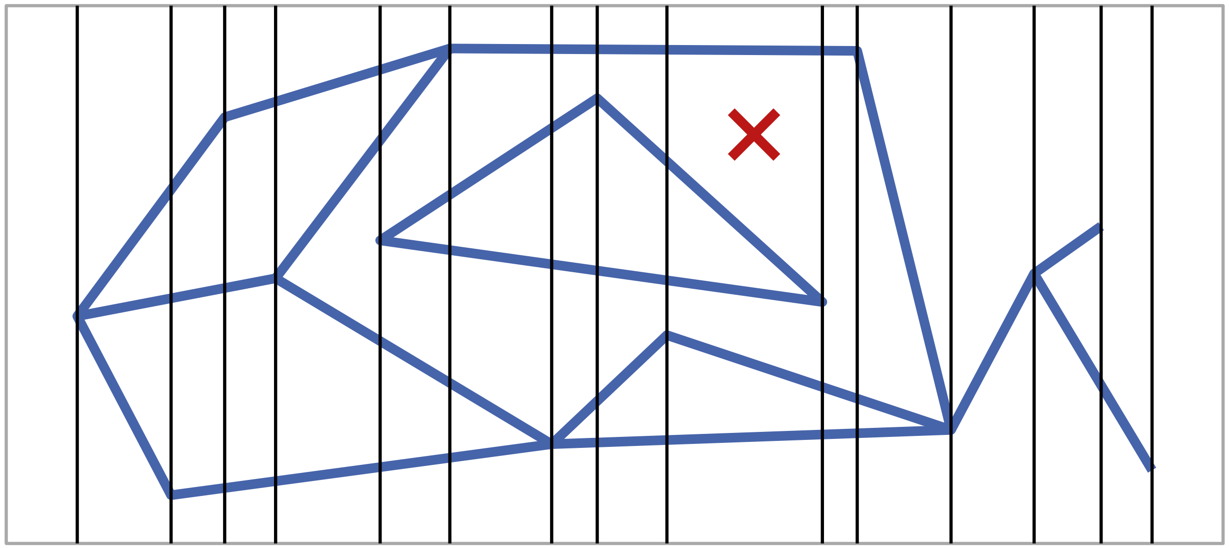

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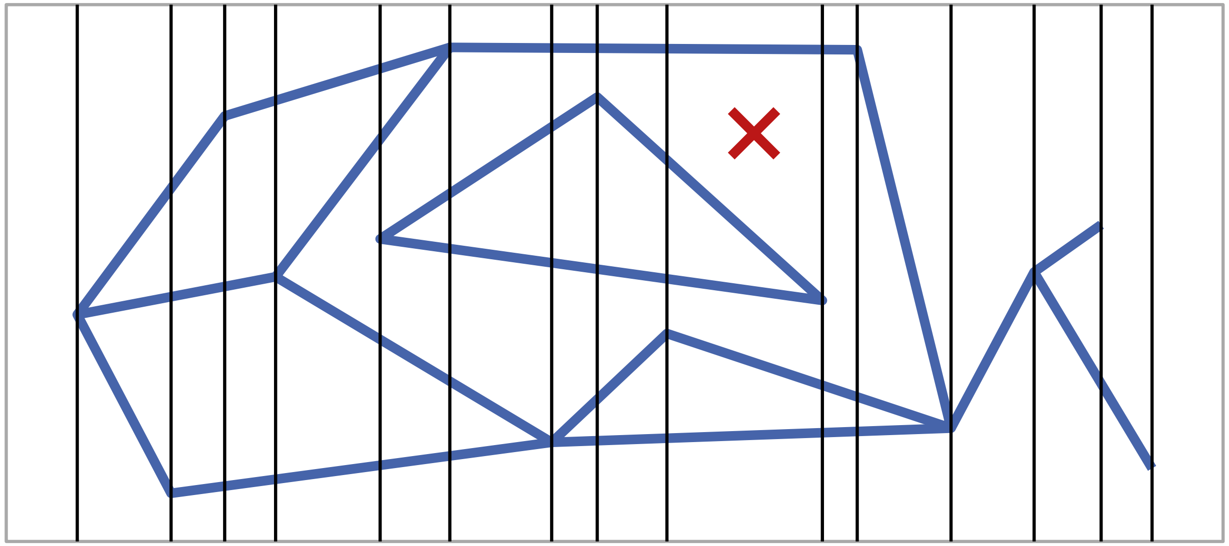

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Query:

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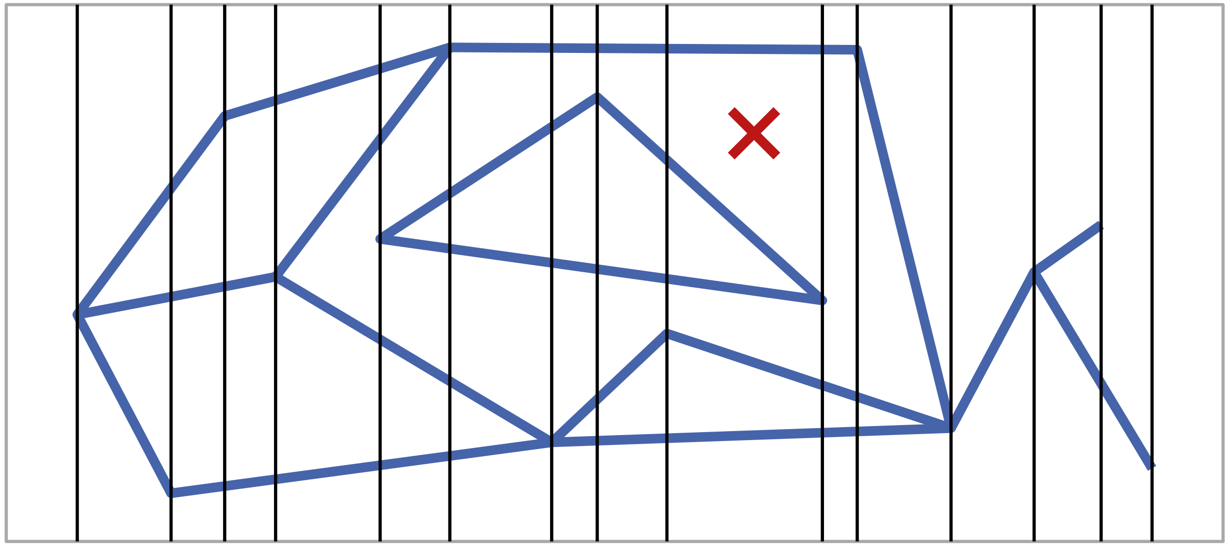

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- find correct slab

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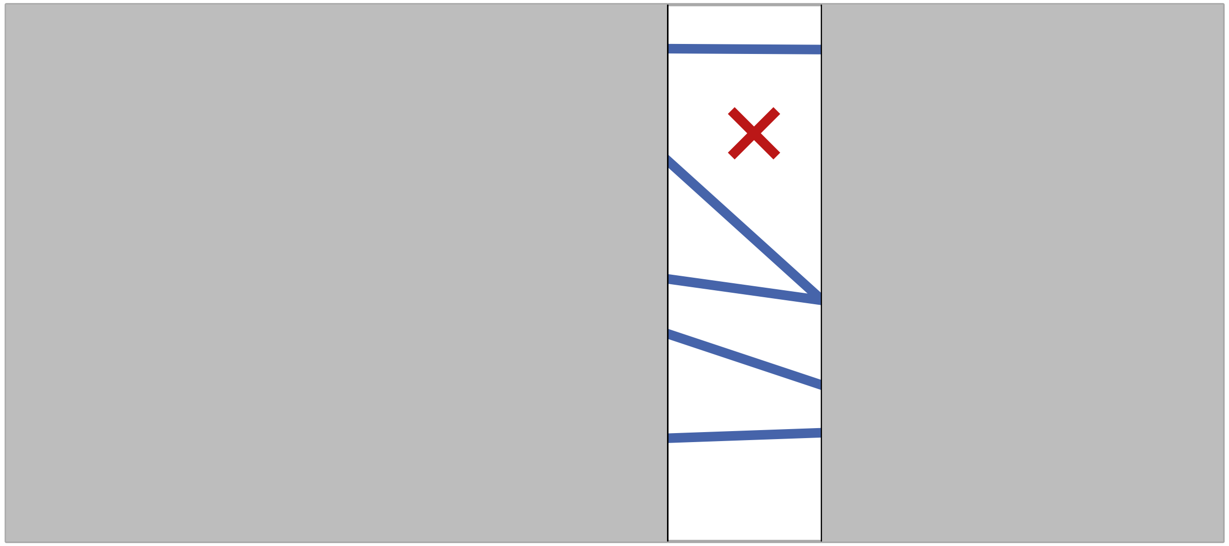

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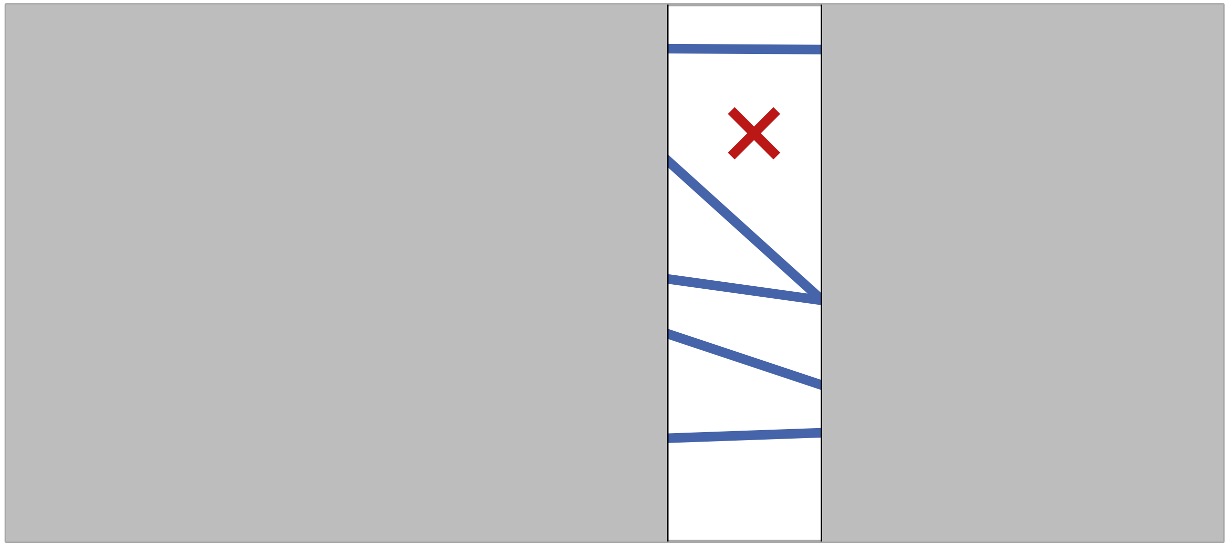

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- search this slab

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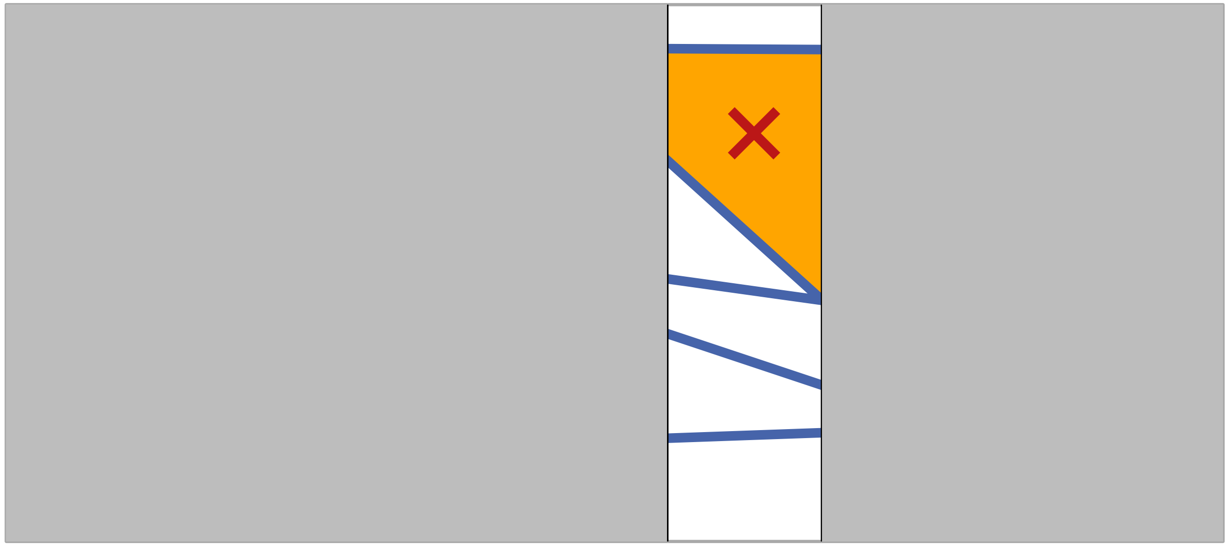

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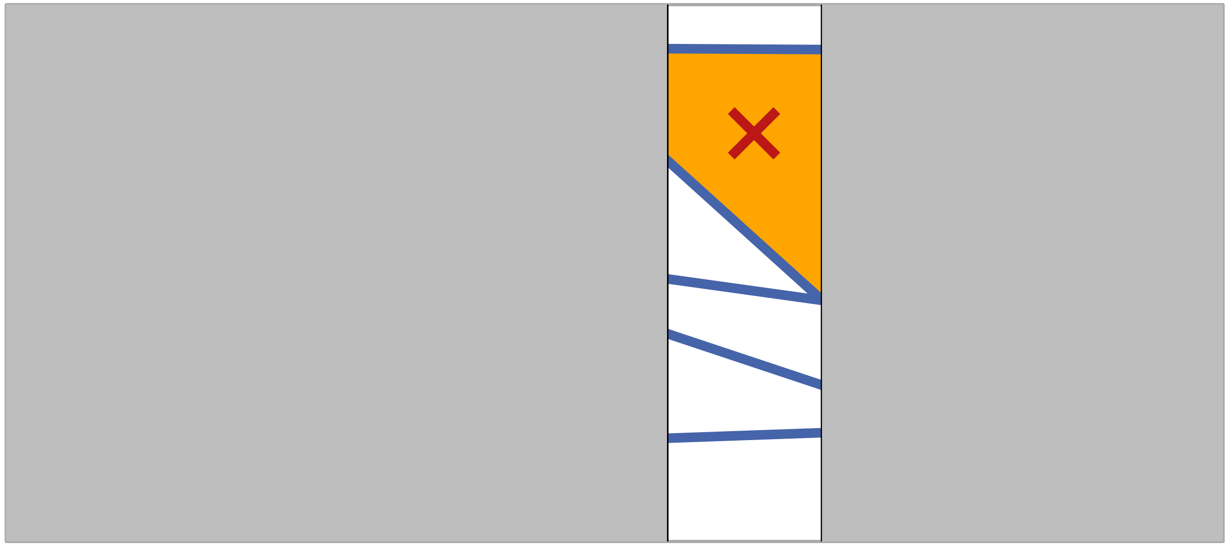

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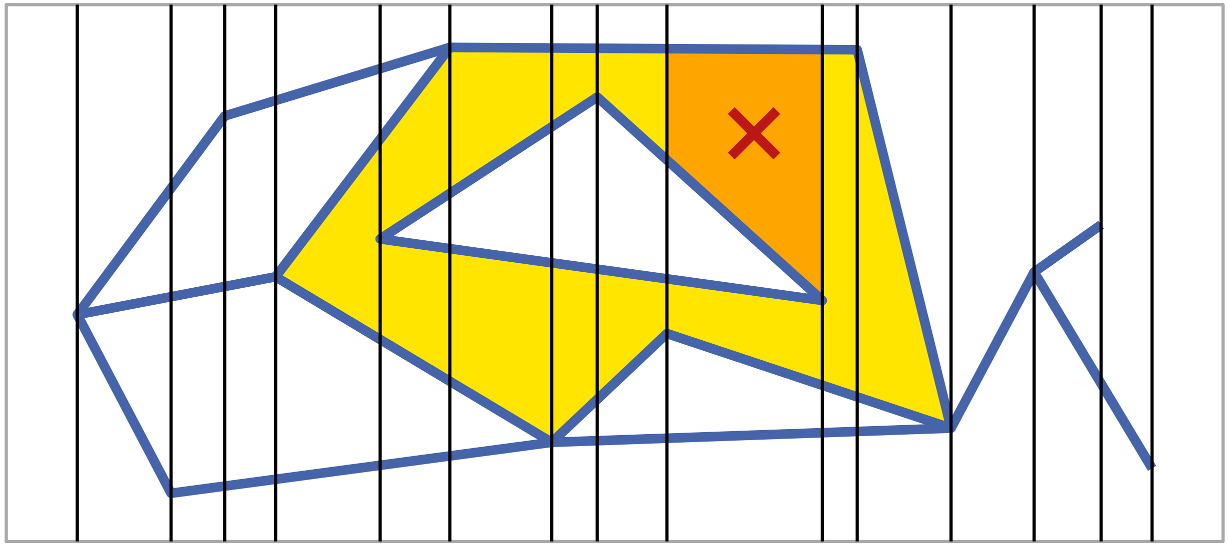

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$O(\log n)$ time

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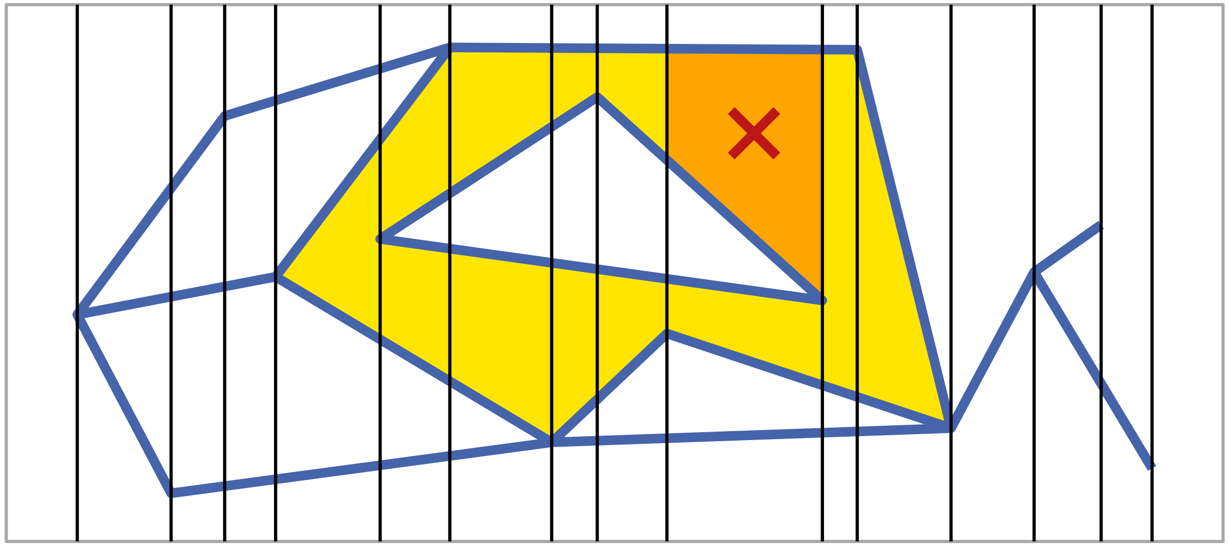

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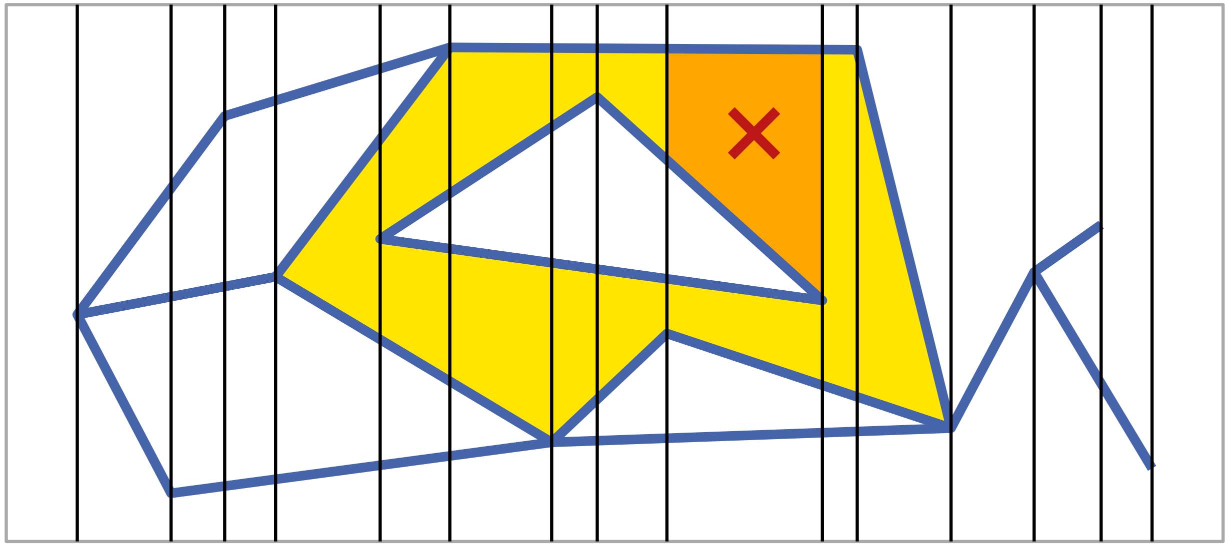

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But: Space?

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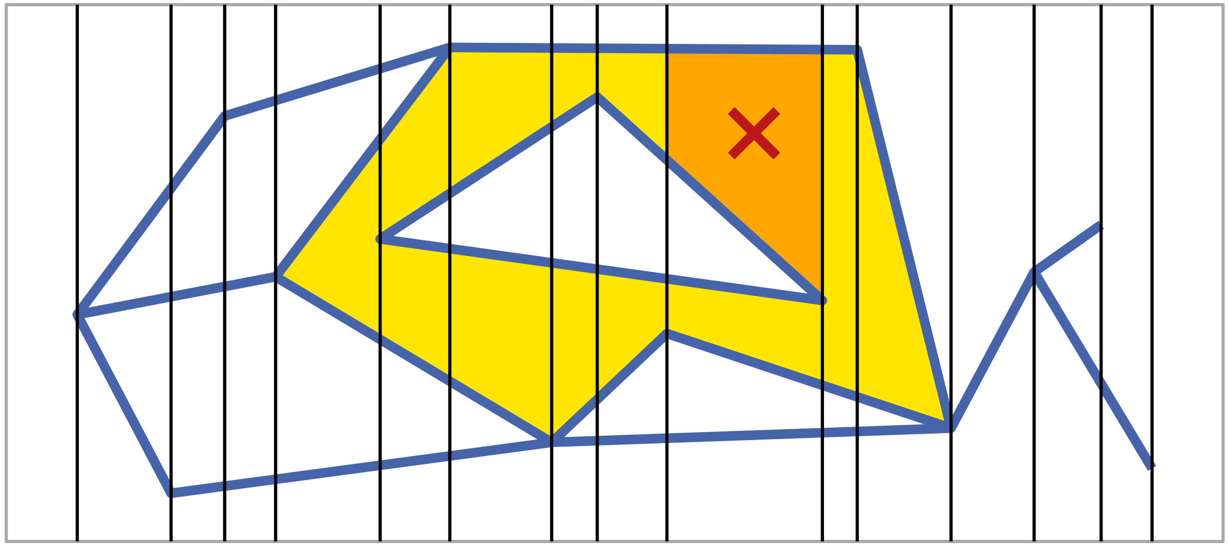

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But: Space? $\Theta(n^2)$

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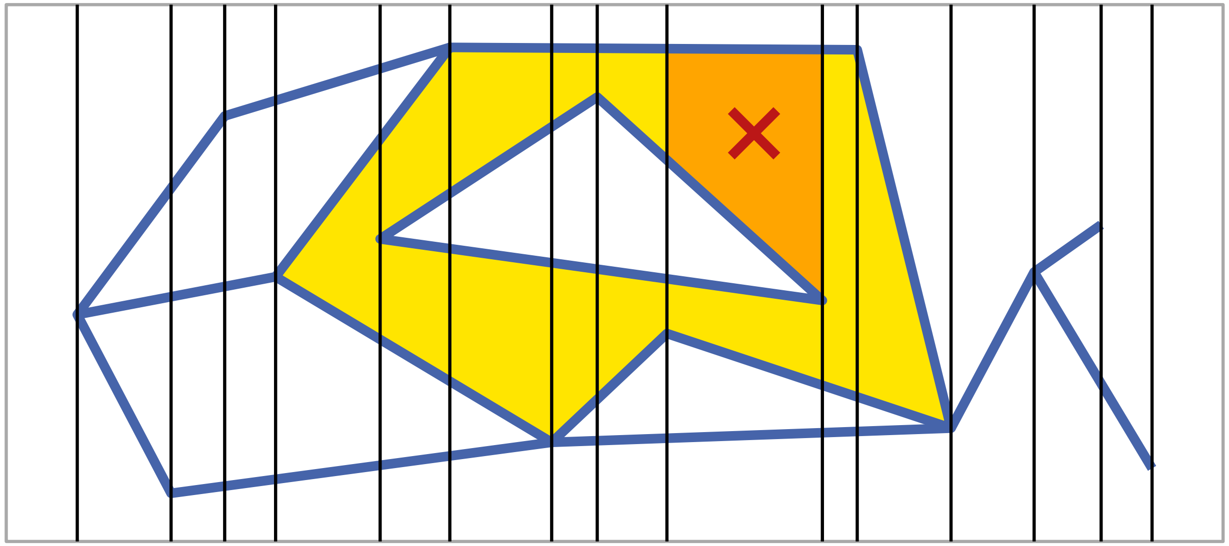

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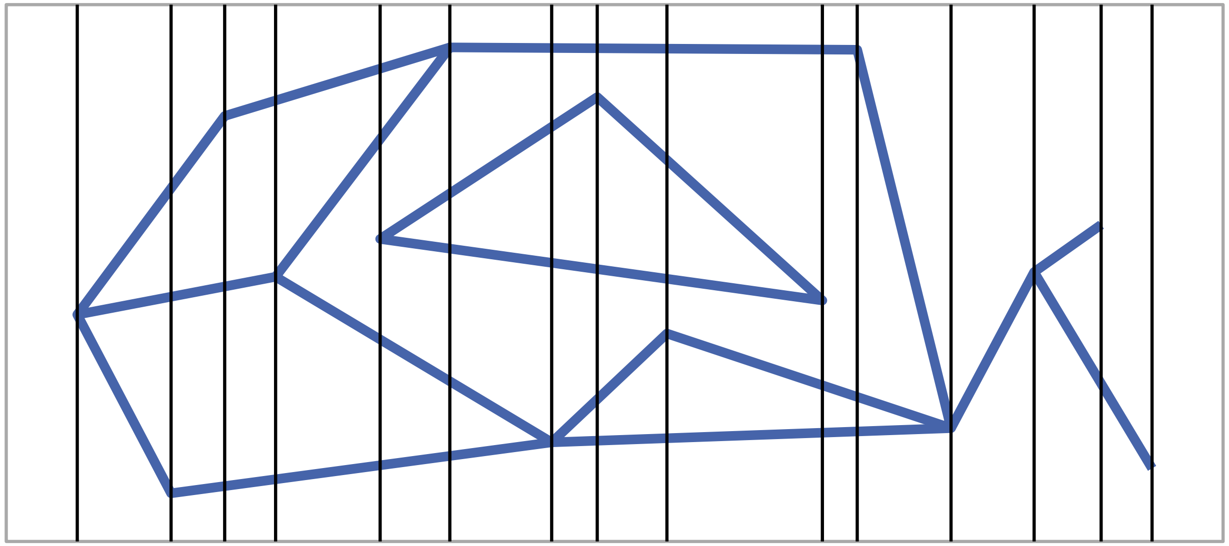
$\left.\right\}$ 2 binary searches — $O(\log n)$ time

But: Space? $\Theta(n^2)$ **Question:** lower bound example?

Reducing the Complexity

Observation: Slab partition is a refinement $\mathcal{S}'$ of $\mathcal{S}$ into (possibly degenerate) trapezoids.



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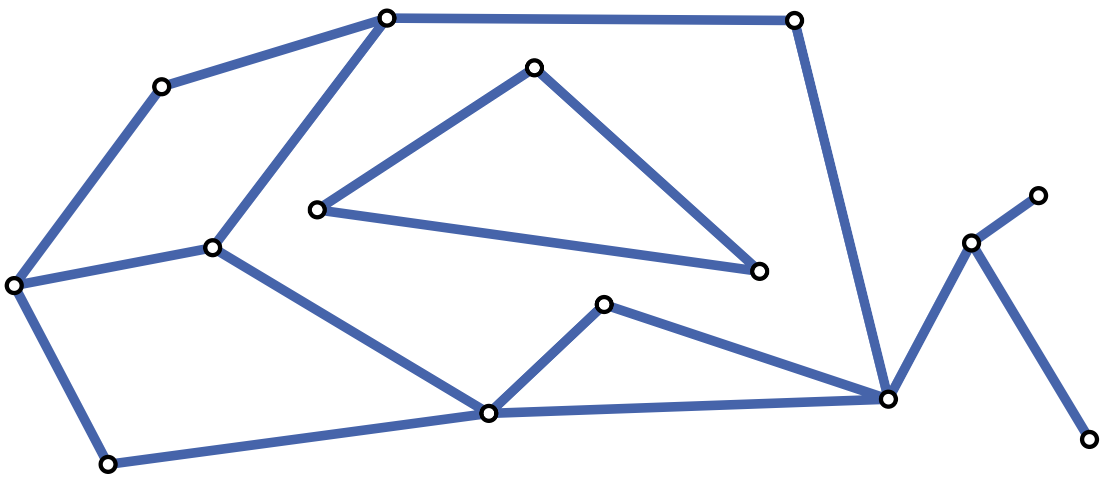

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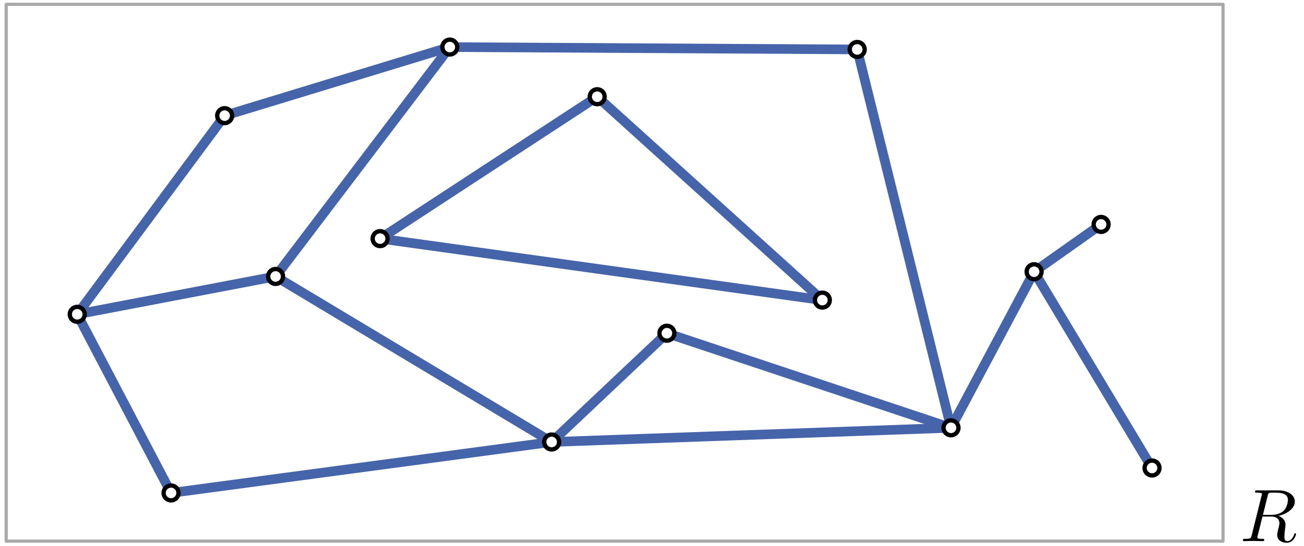

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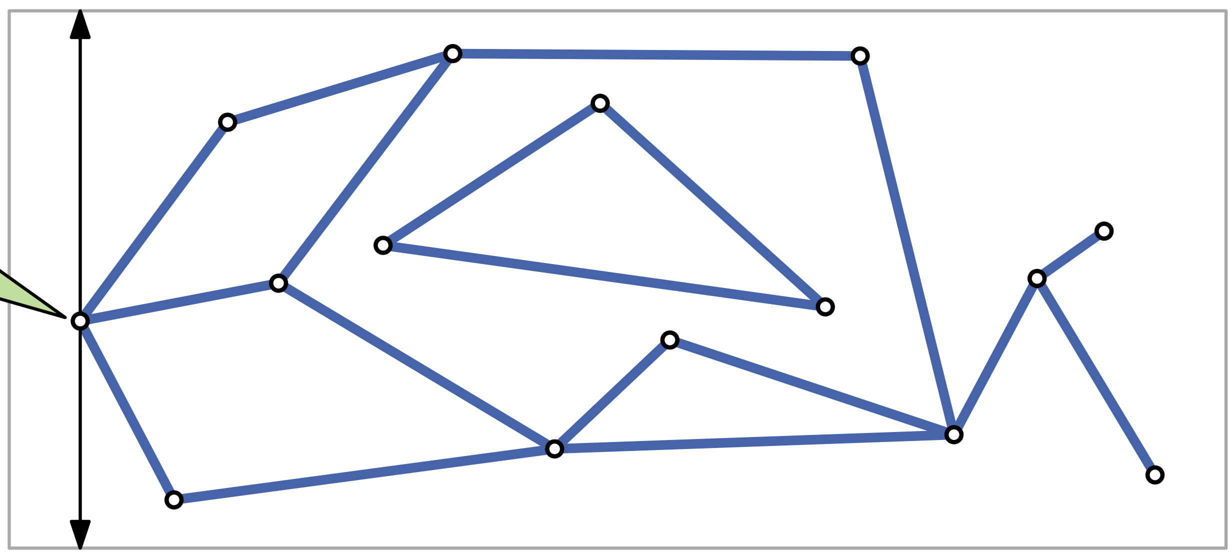
edge from each endpoint vertically up and down to neighboring segment



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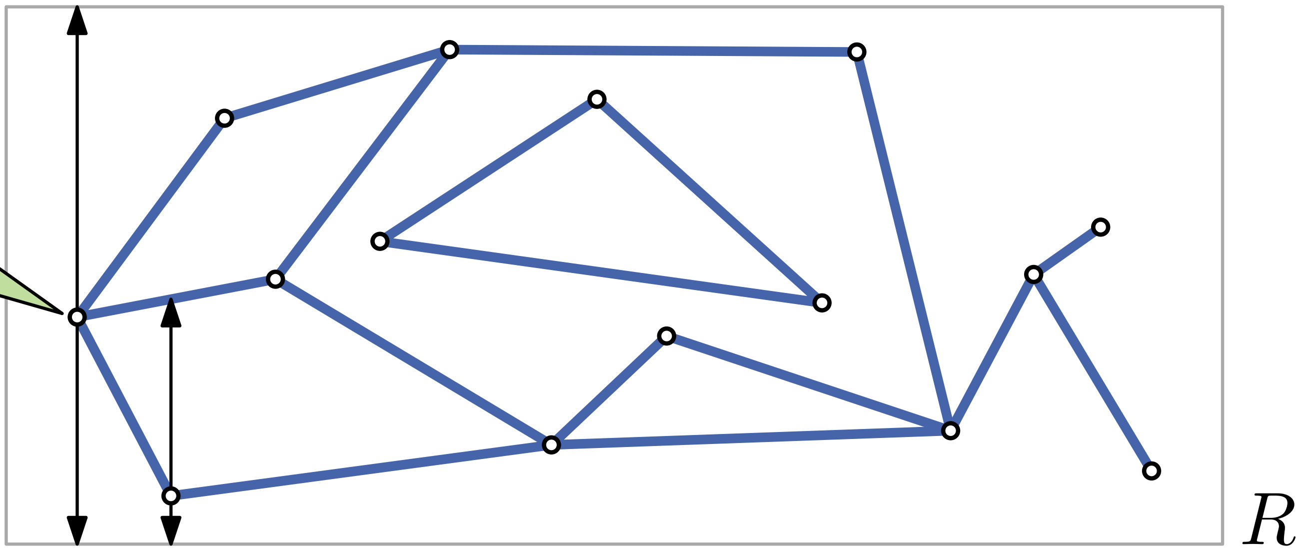

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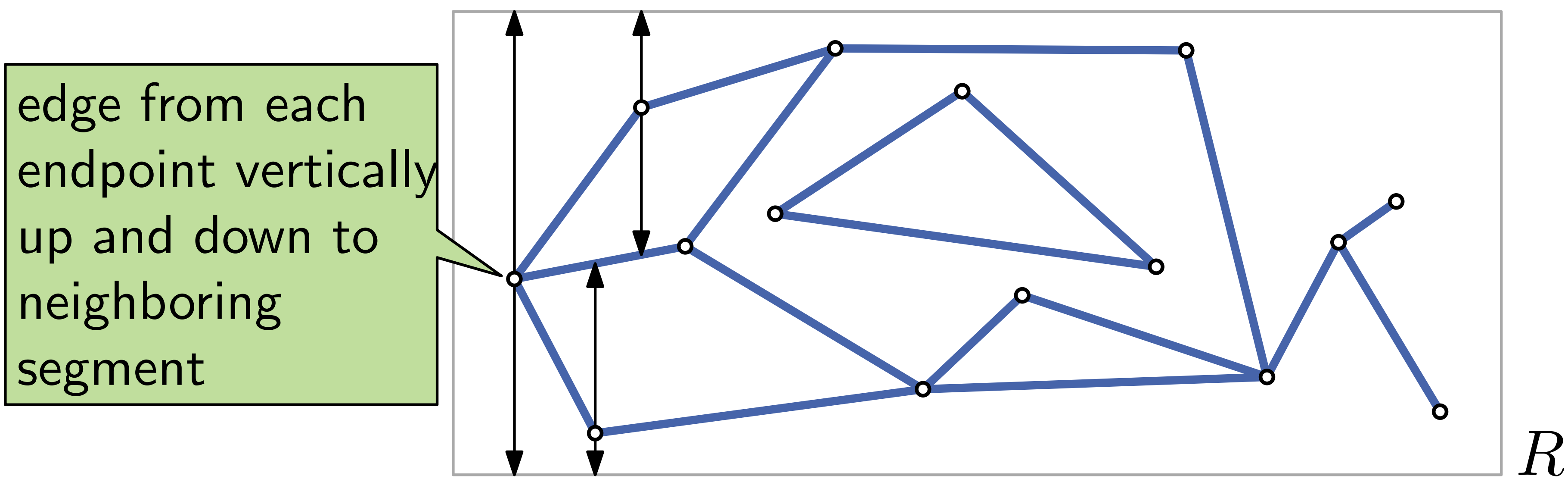

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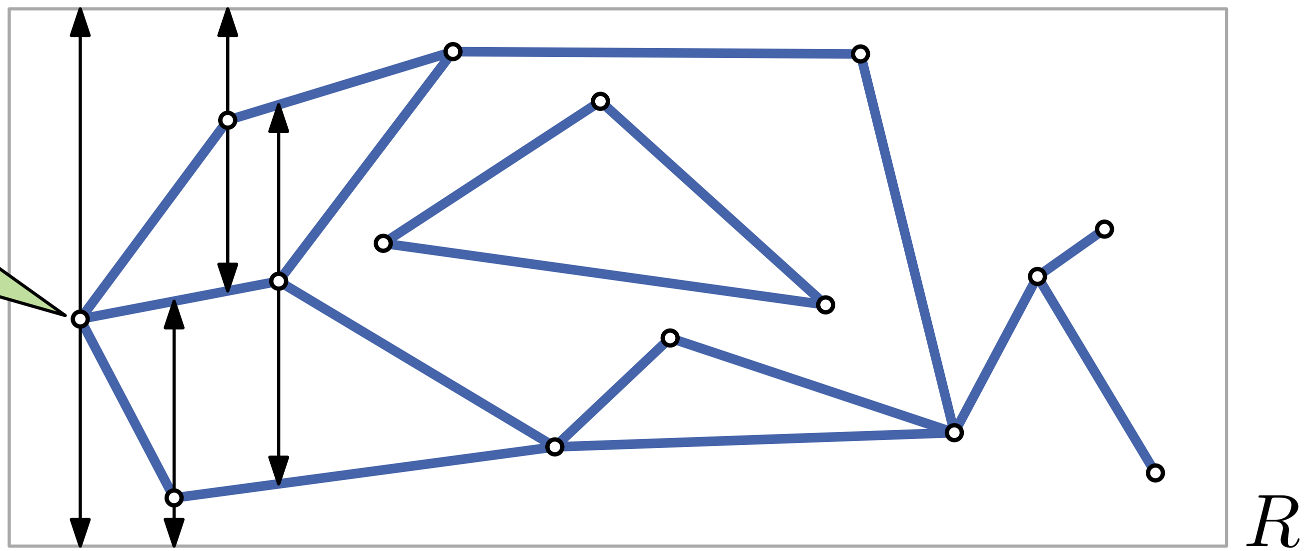

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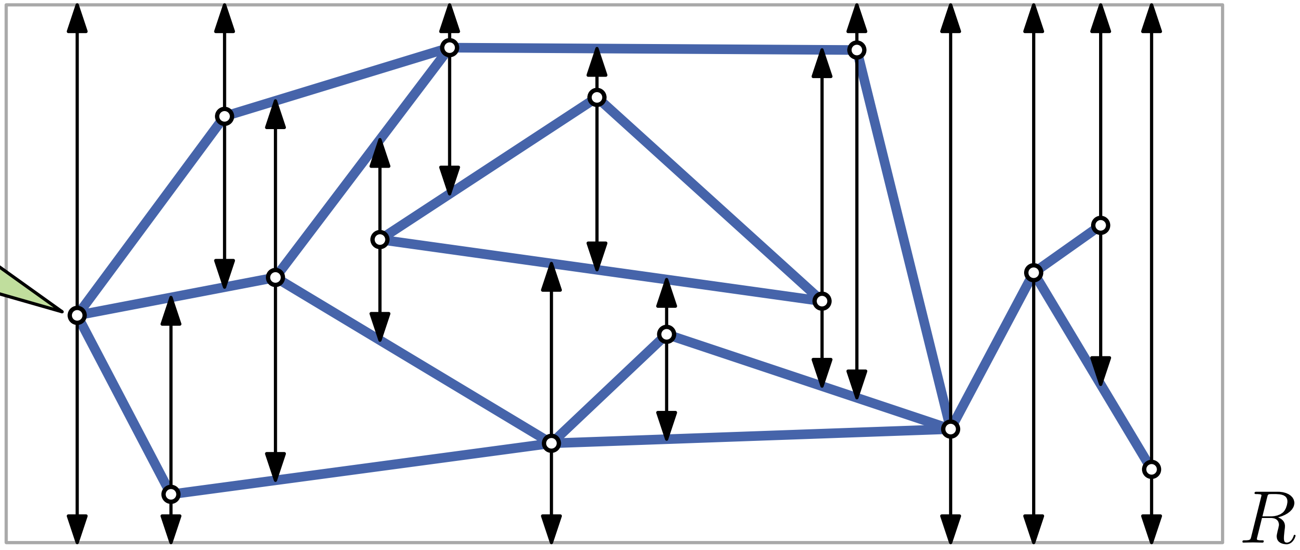

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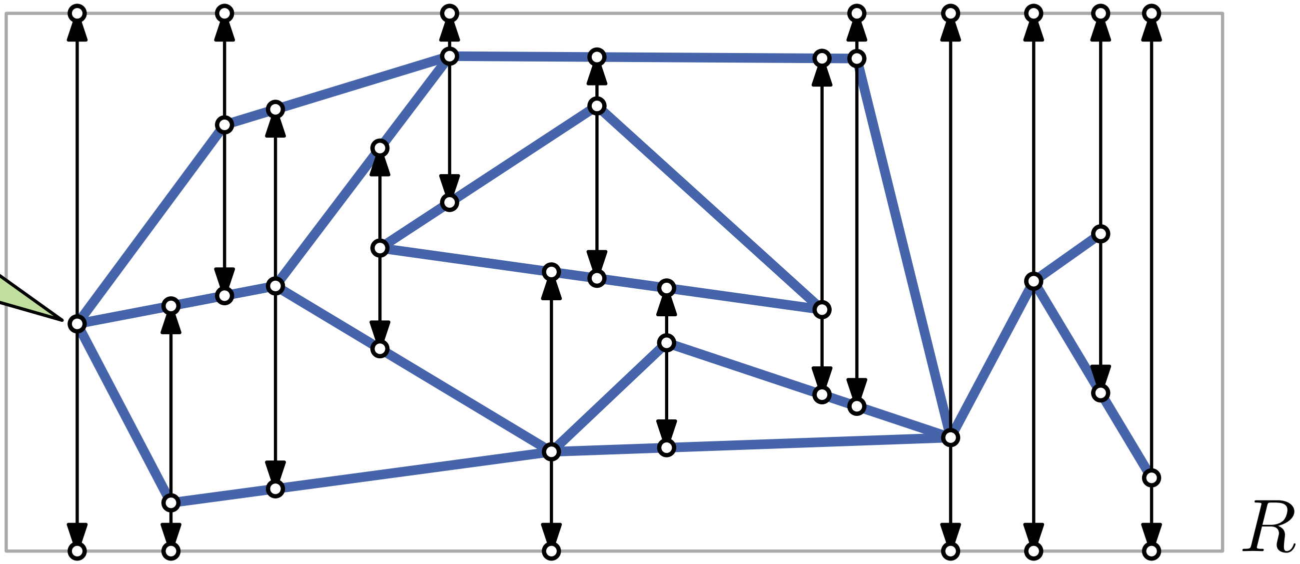

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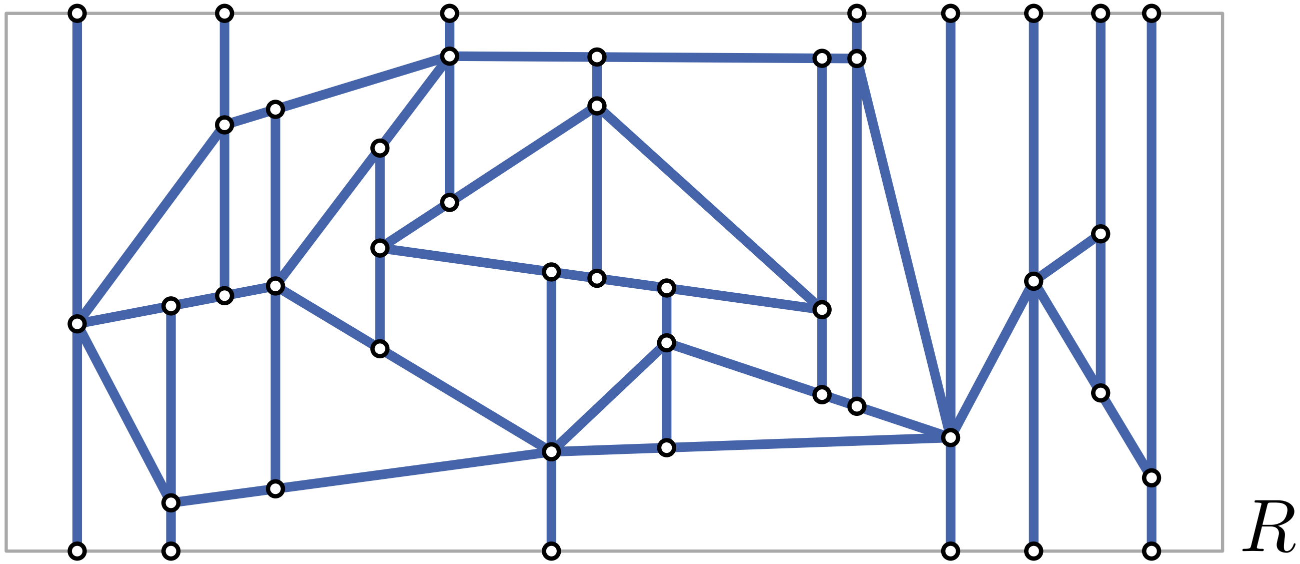

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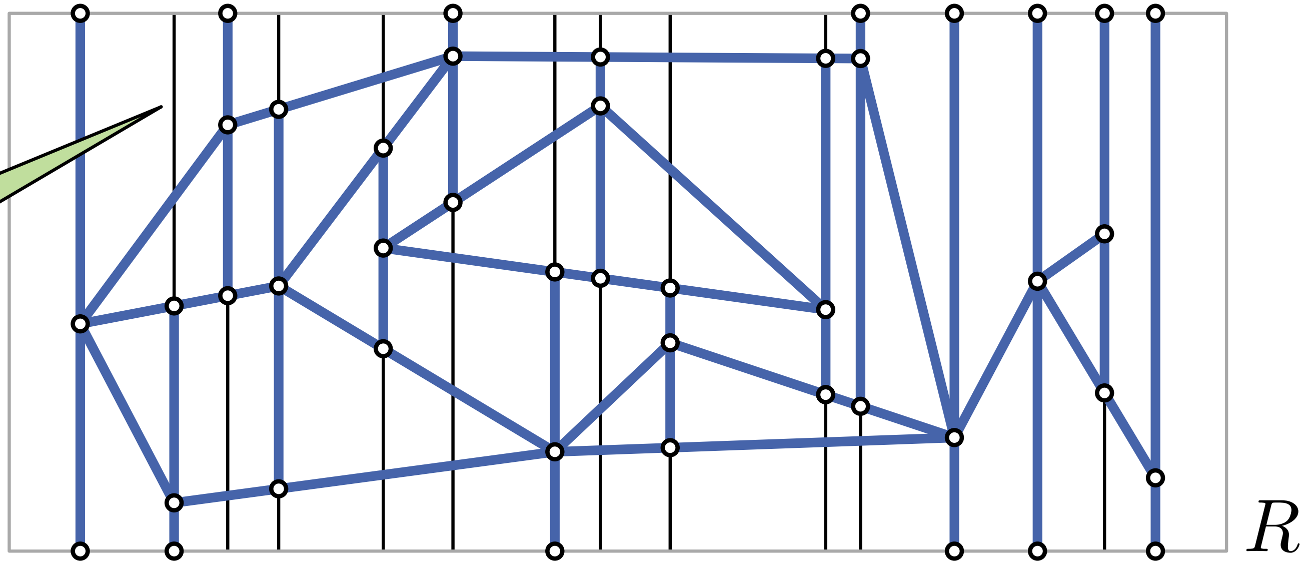
Compare to slab partition



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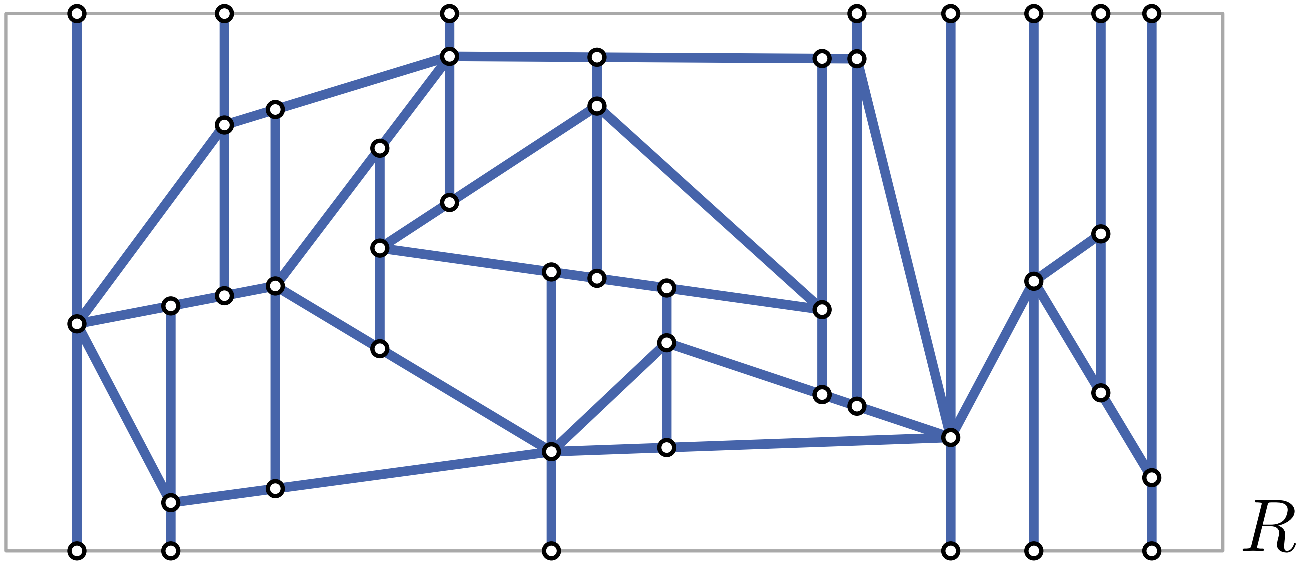
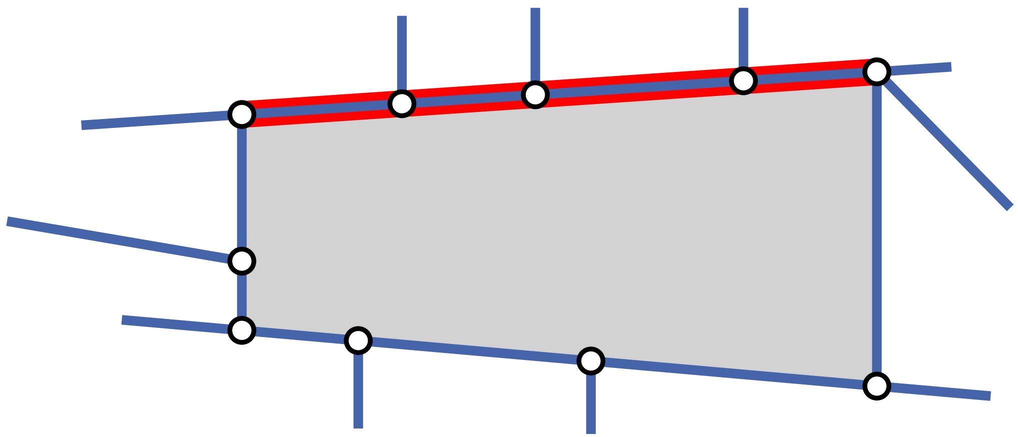


Assumption: $\mathcal{S}$ is in *general position*, i.e., no two segment endpoints have the same x-coordinate

Notation

Definition: A *side* of a face of $\mathcal{T}(\mathcal{S})$ is a segment of maximal length contained in face boundary.



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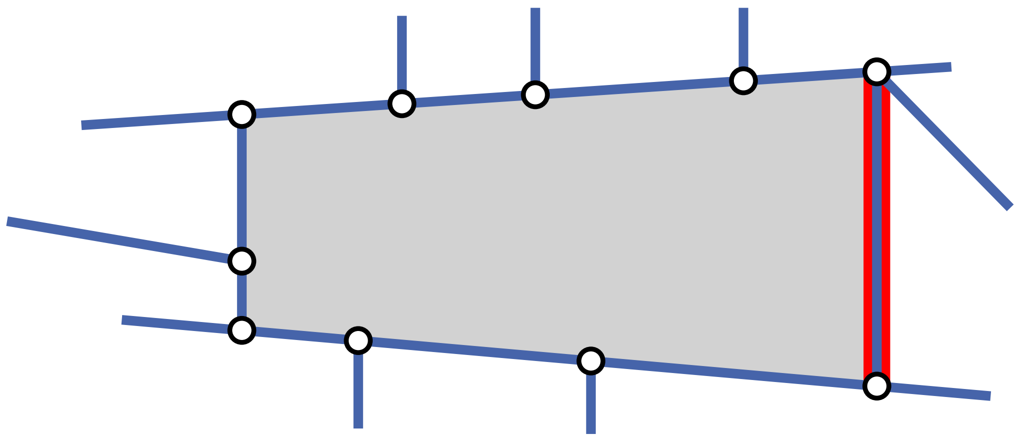

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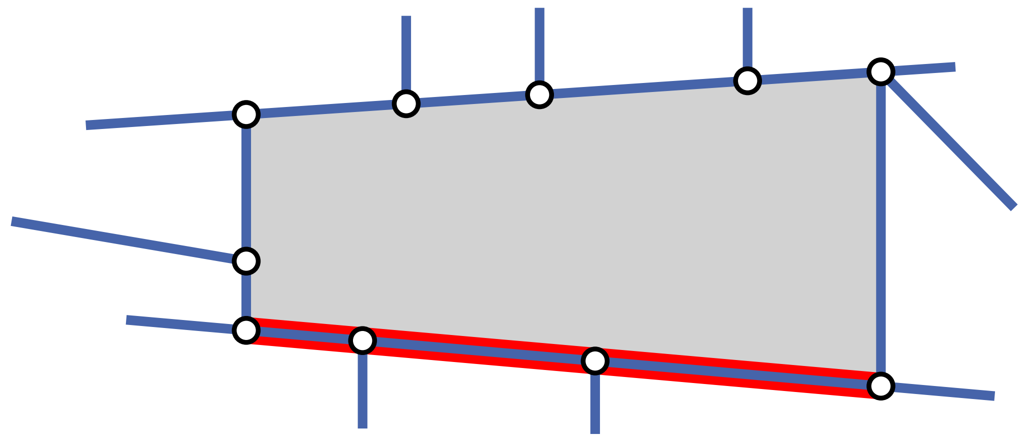

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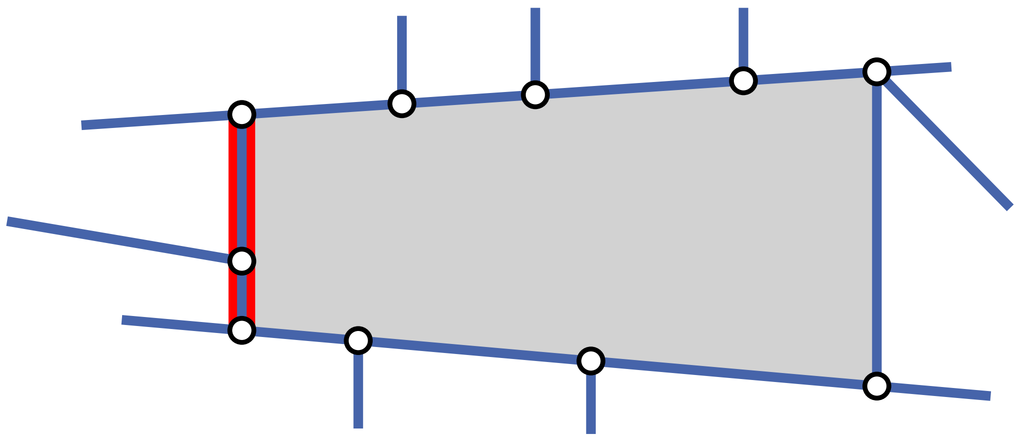

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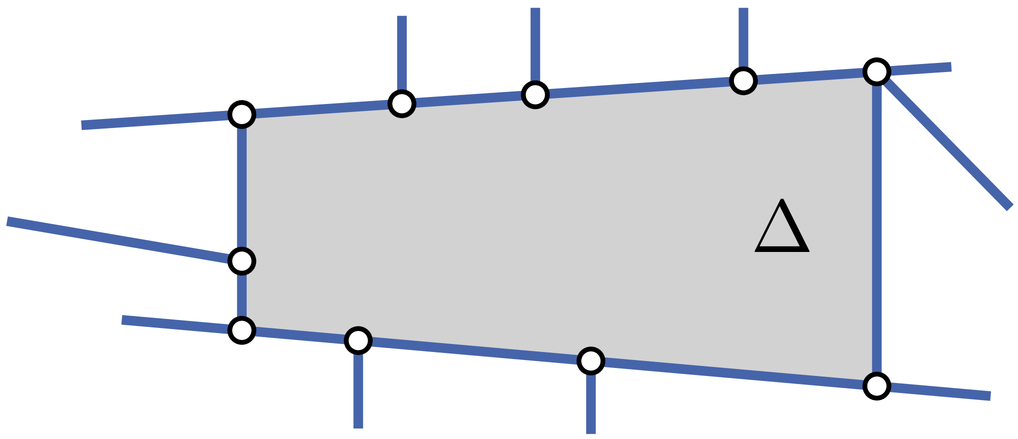

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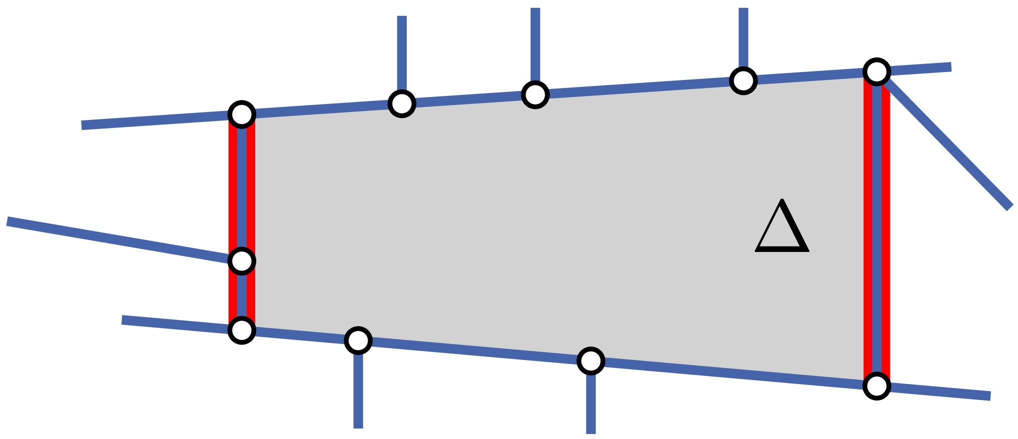

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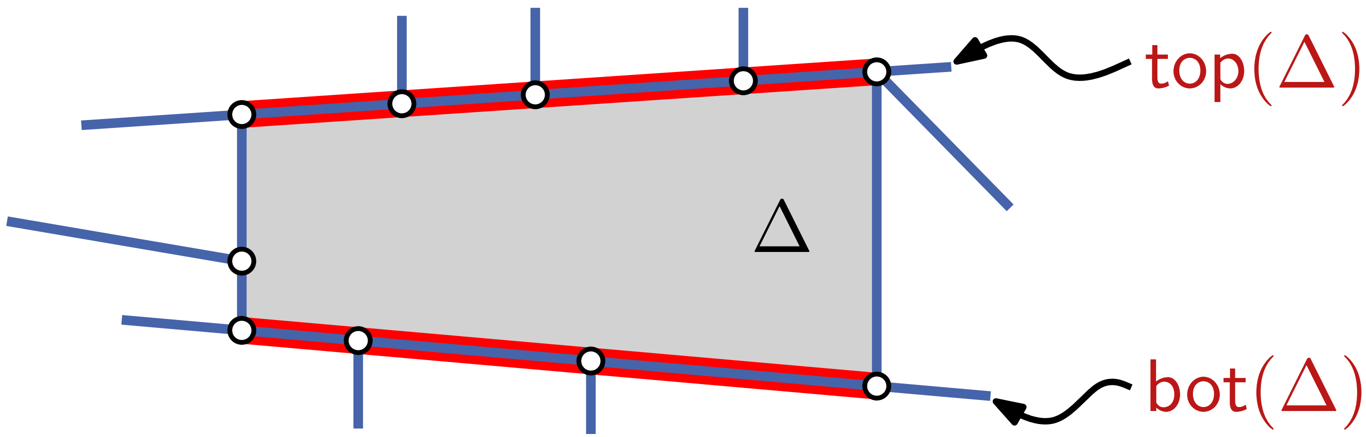

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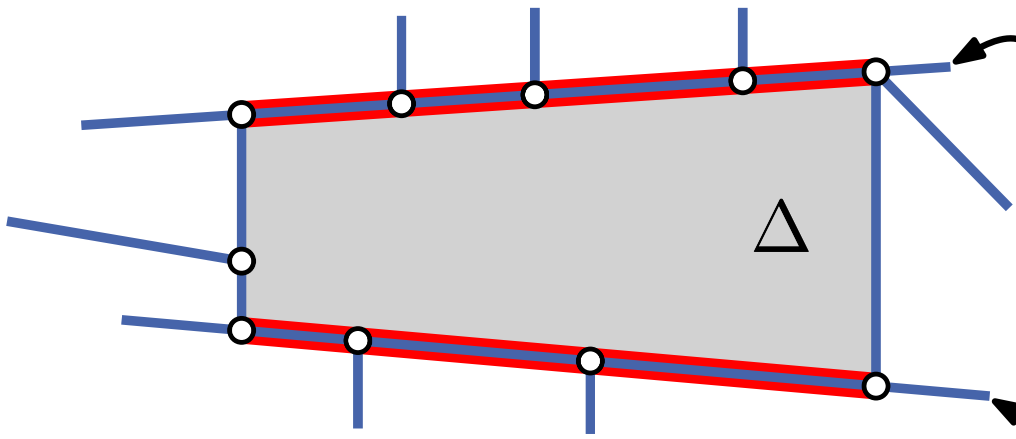

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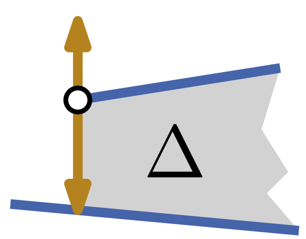

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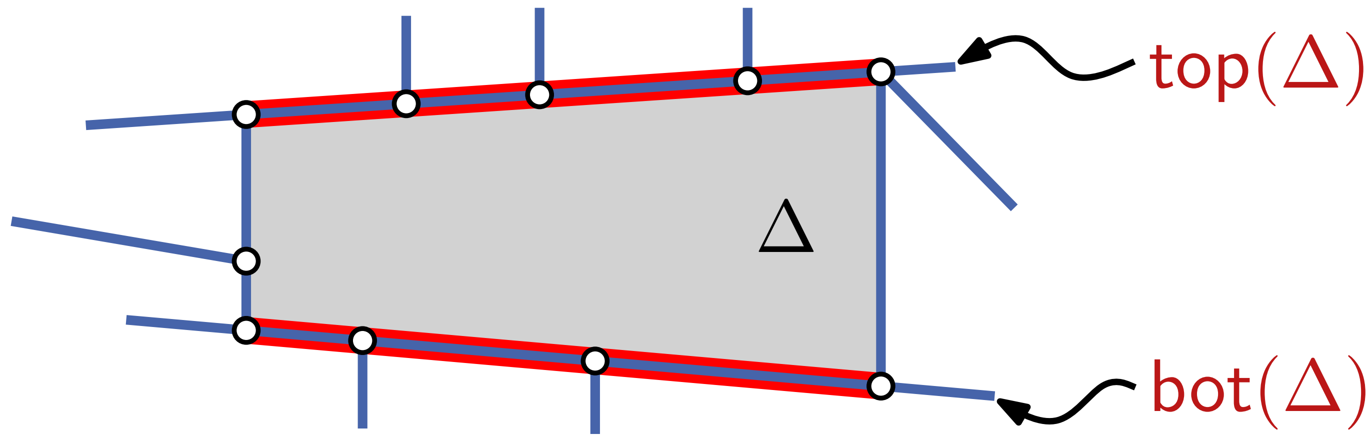

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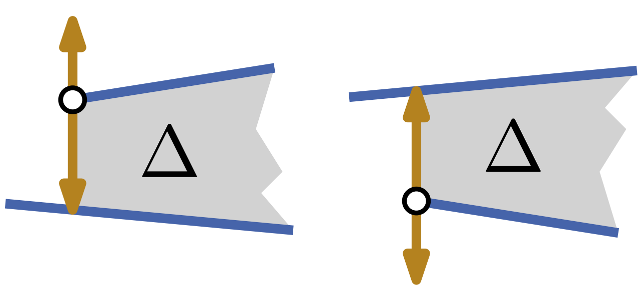

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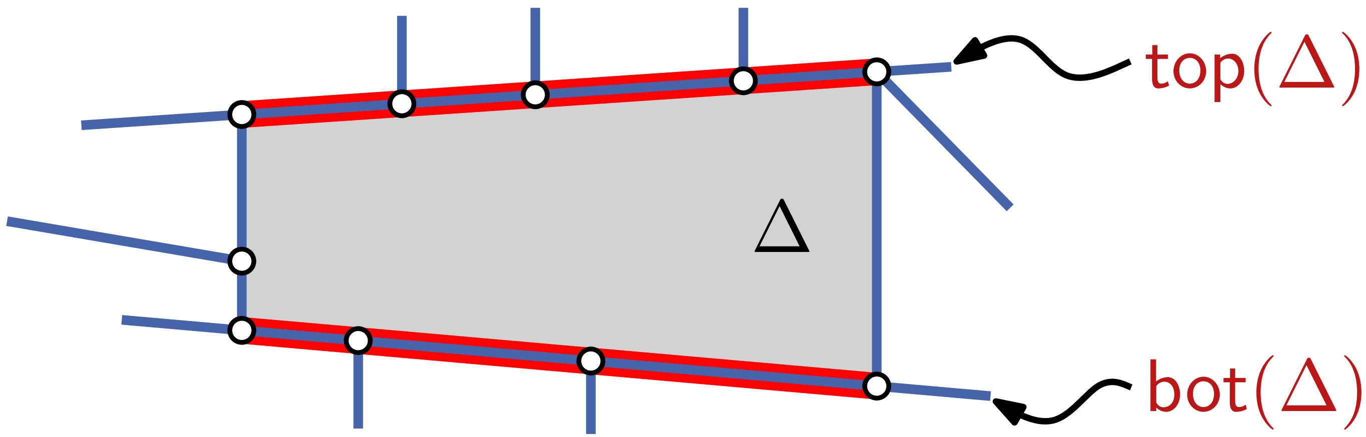

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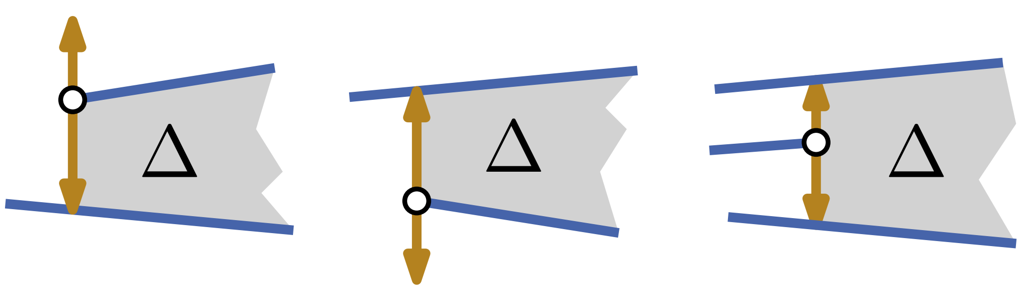

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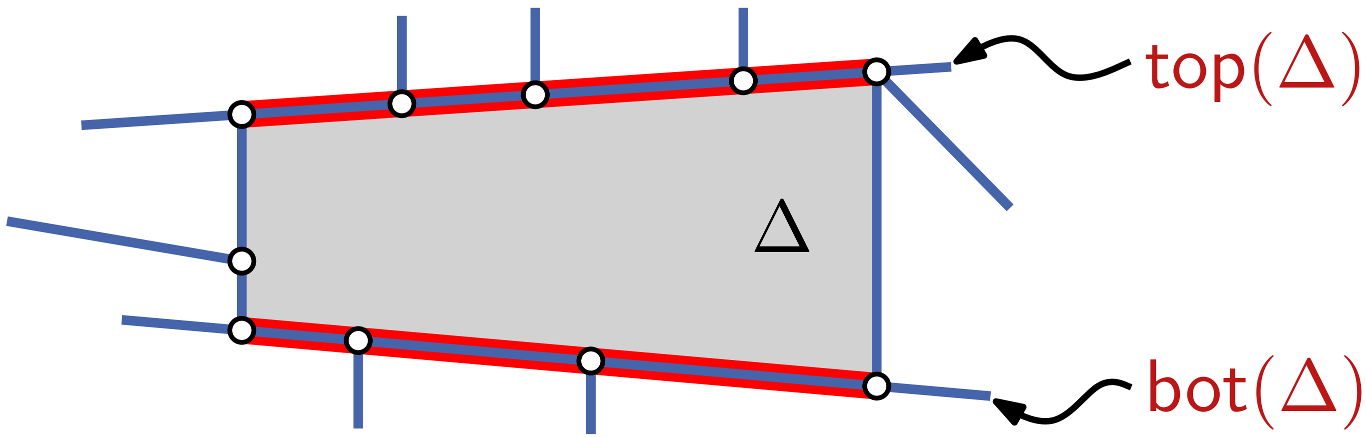

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- one or two vertical sides
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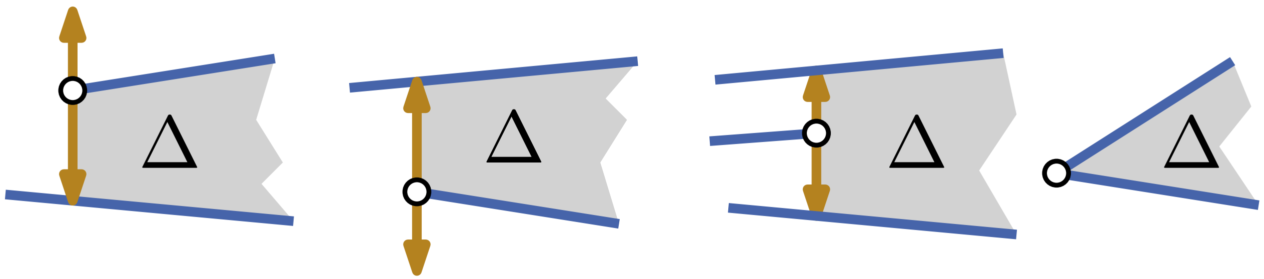

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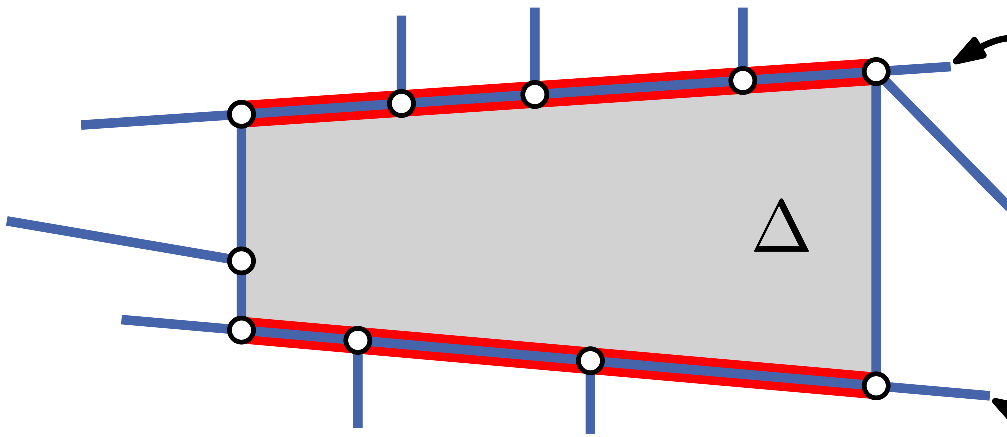

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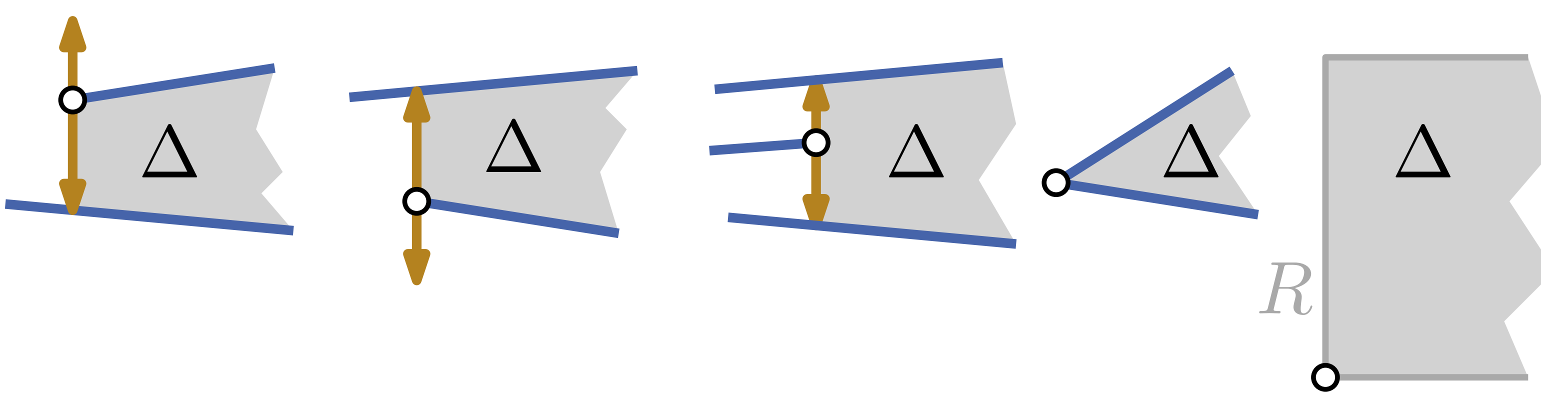

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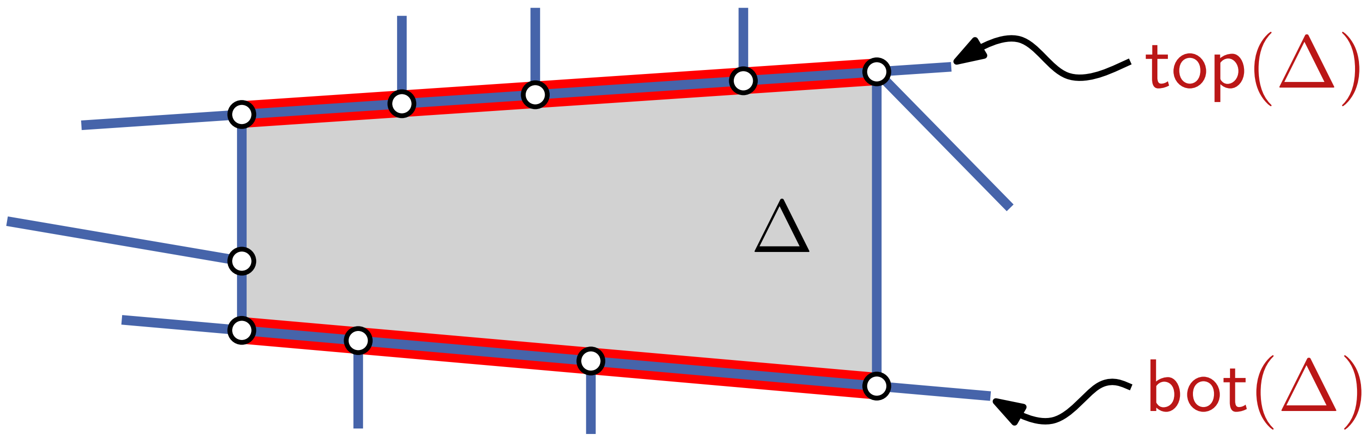

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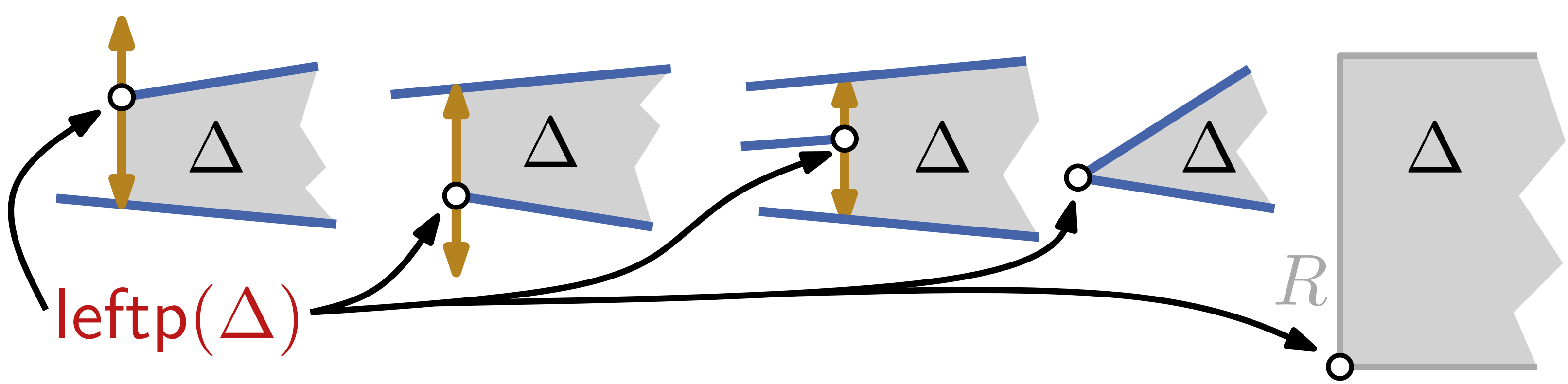

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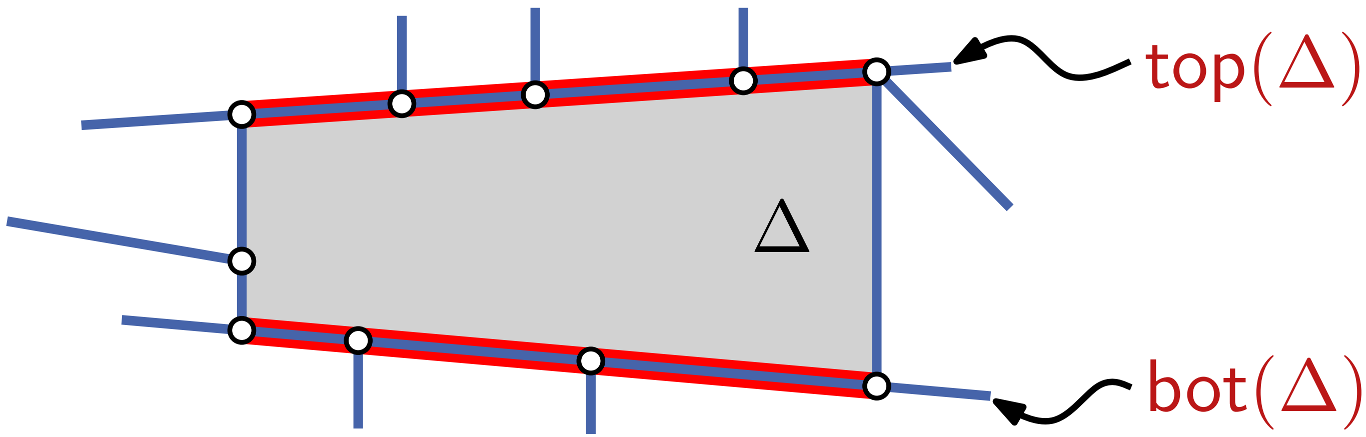

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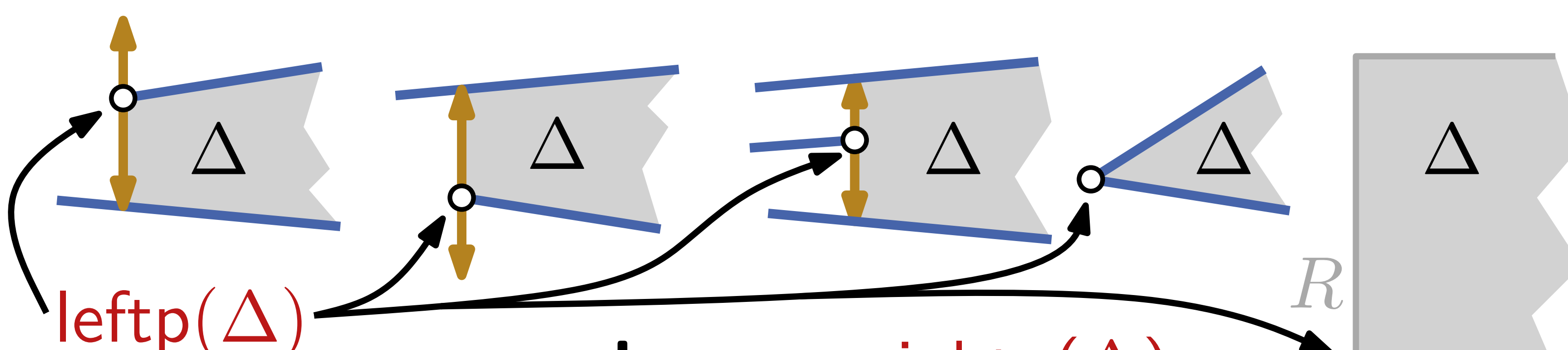
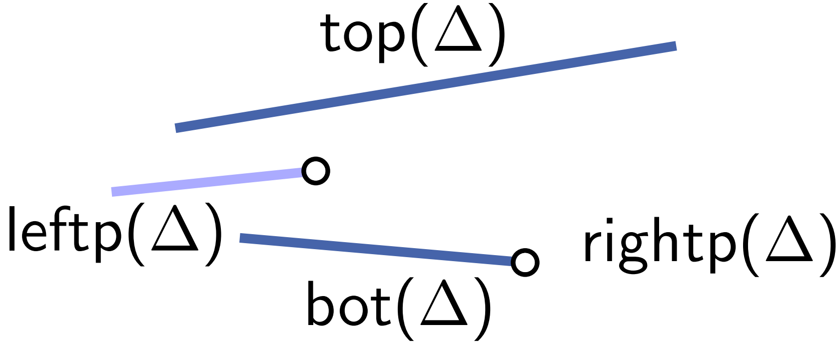


analogous: rightp(Δ)

Complexity of the Trapezoidal Map

Obs.: A trapezoid Δ is uniquely defined by bot(Δ), top(Δ), leftp(Δ) and rightp(Δ).



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Karlsruhe Institute of Technology

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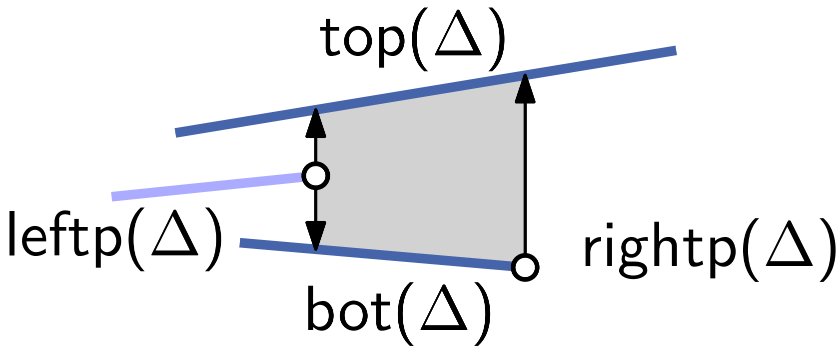

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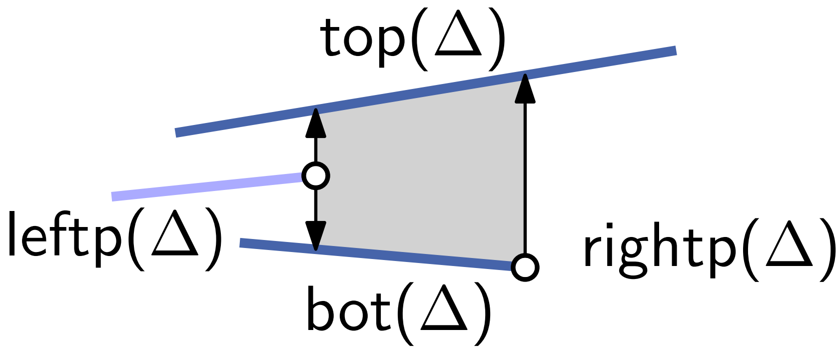

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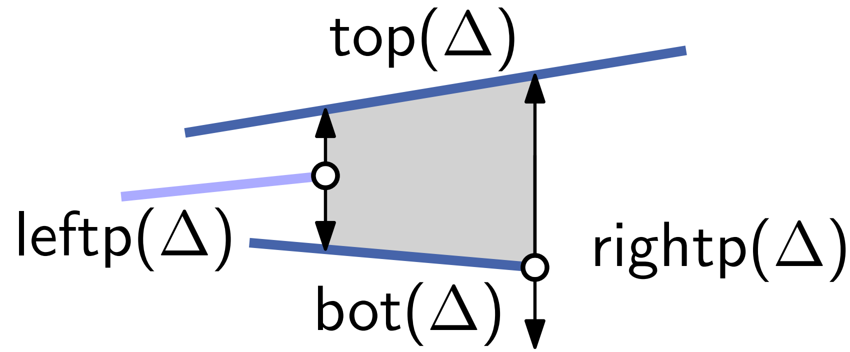
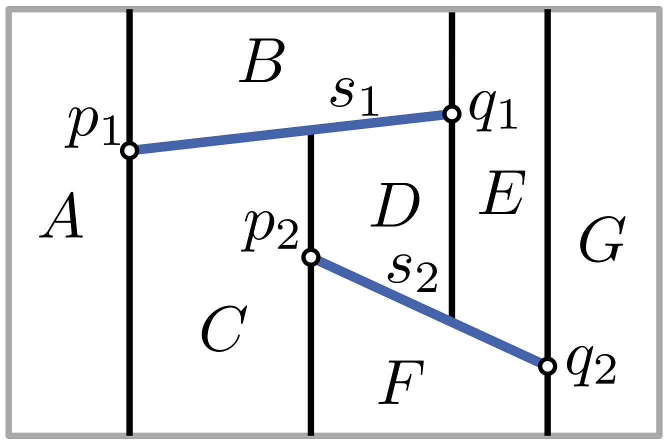


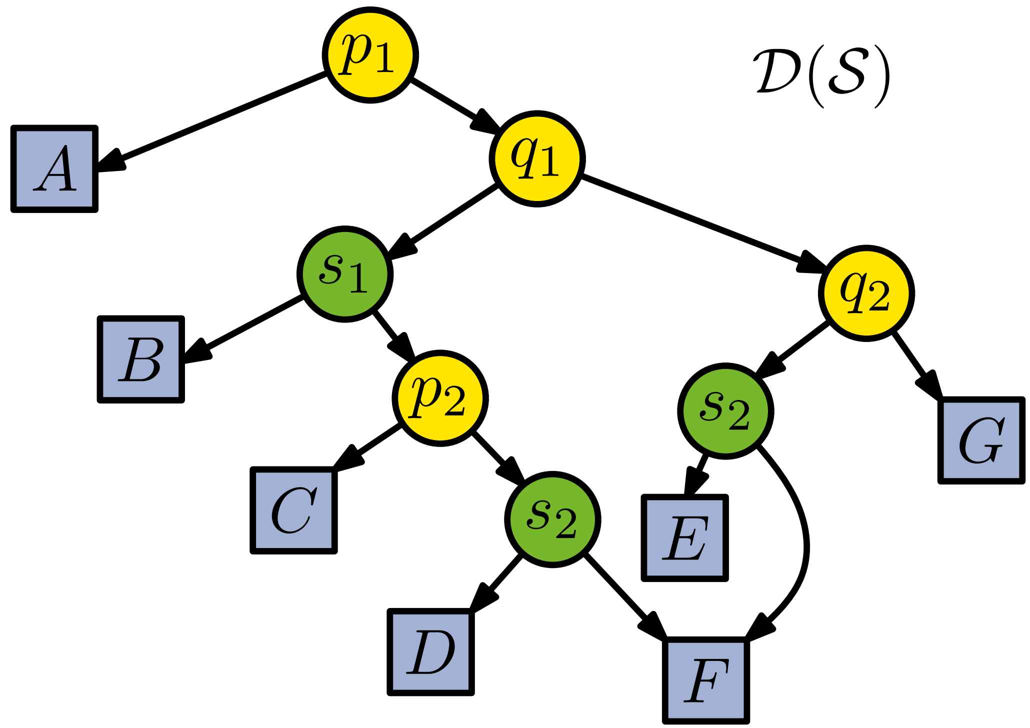
Lemma 1: The trapezoidal map $\mathcal{T}(\mathcal{S})$ of a set $\mathcal{S}$ of n segments in general position contains at most $6n + 4$ vertices and at most $3n + 1$ trapezoids.

Search Structure

Goal: Compute the trapzoidal map $\mathcal{T}(\mathcal{S})$ and simultaneously a data structure $\mathcal{D}(\mathcal{S})$ for point location in $\mathcal{T}(\mathcal{S})$.



$\mathcal{T}(\mathcal{S})$



$\mathcal{D}(\mathcal{S})$ is a DAG with:

- p — x-node for point p tests left/right of p
- s — y-node for segment s tests above/below s
- Δ — leaf node for trapezoid Δ

Incremental Algorithm

TrapezoidalMap($\mathcal{S}$)

Input: set $\mathcal{S} = \{s_1, \dots, s_n\}$ of crossing-free segments
Output: trapezoidal map $\mathcal{T}(\mathcal{S})$ and search structure $\mathcal{D}(\mathcal{S})$
initialize $\mathcal{T}$ and $\mathcal{D}$ for $R =$ BBox($\mathcal{S}$)

for $i \leftarrow 1$ **to** n **do**
  $H \leftarrow \{\Delta \in \mathcal{T} \mid \Delta \cap s_i \neq \emptyset\}$
  $\mathcal{T} \leftarrow \mathcal{T} \setminus H$
  $\mathcal{T} \leftarrow \mathcal{T} \cup$ newly created trapezoids of s_i
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Solution: Randomization!

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Invariant: $\mathcal{T}$ is trapezoidal map for $\mathcal{S}_i = \{s_1, \ldots, s_i\}$ and $\mathcal{D}$ is corresponding search structure

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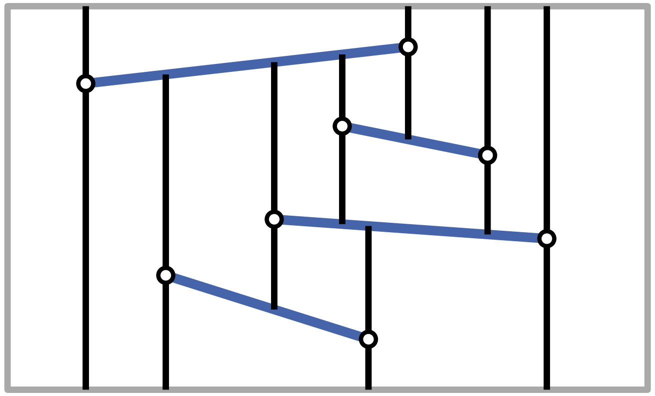

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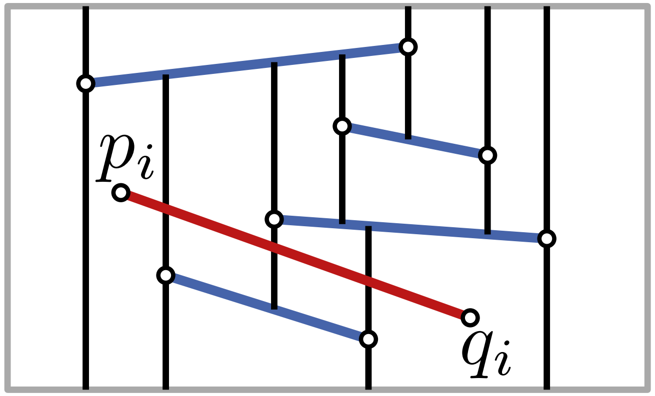

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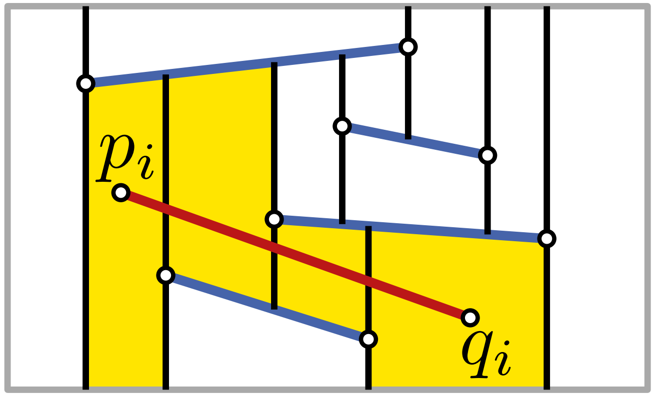

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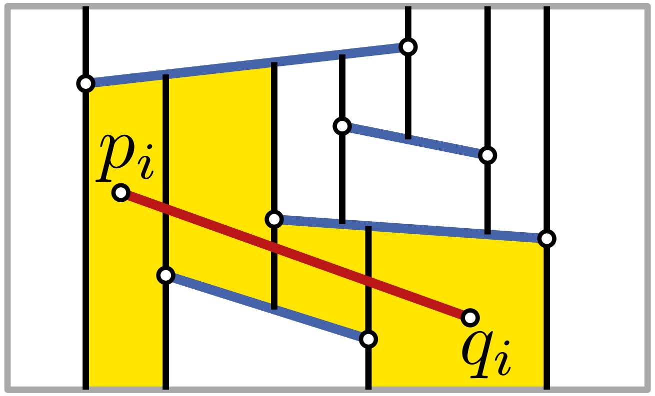

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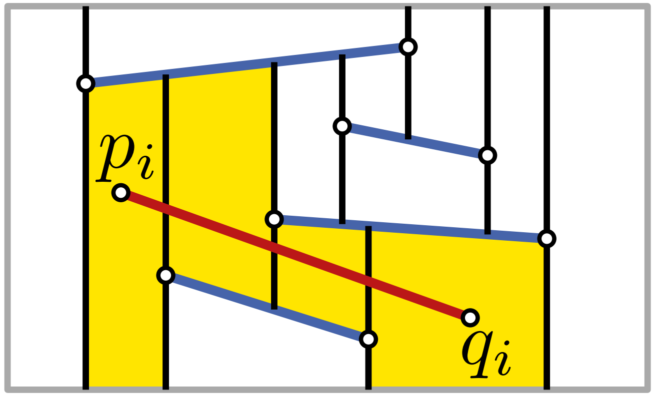

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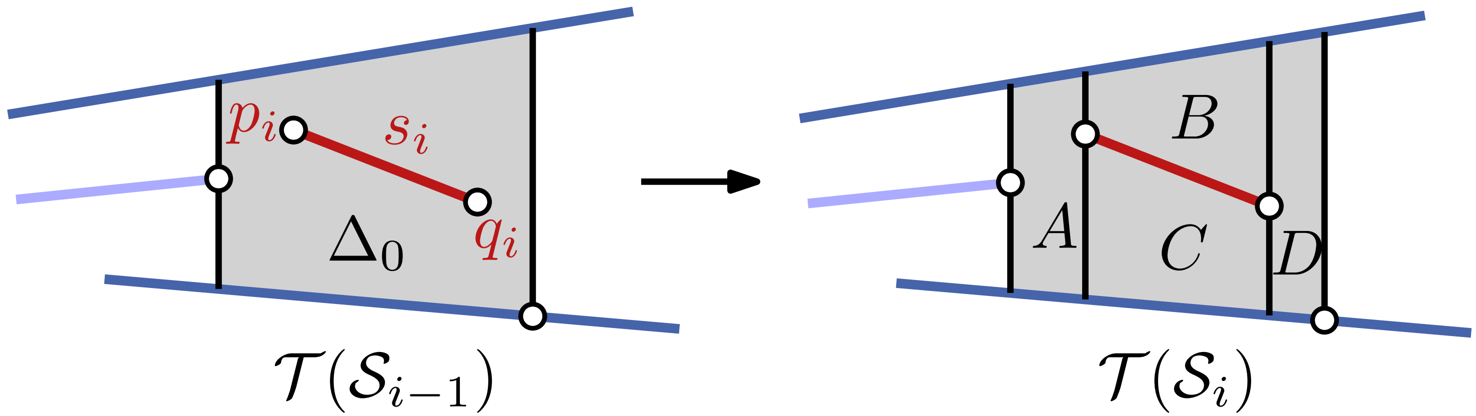
$\Delta_0 \leftarrow$ FindTrapezoid$(p_i, \mathcal{D})$; $j \leftarrow 0$
while right endpoint q_i right of rightp(Δ_j) **do**
    if rightp(Δ_j) above s_i **then**
        $\Delta_{j+1} \leftarrow$ lower right neighbor of Δ_j
    else
        $\Delta_{j+1} \leftarrow$ upper right neighbor of Δ_j
    $j \leftarrow j + 1$
return $\Delta_0, \ldots, \Delta_j$

Updating $\mathcal{T}(\mathcal{S})$ and $\mathcal{D}(\mathcal{S})$

Step 2: Update $\mathcal{T}$ and $\mathcal{D}$

- Case 1: $s_i \subset \Delta_0$



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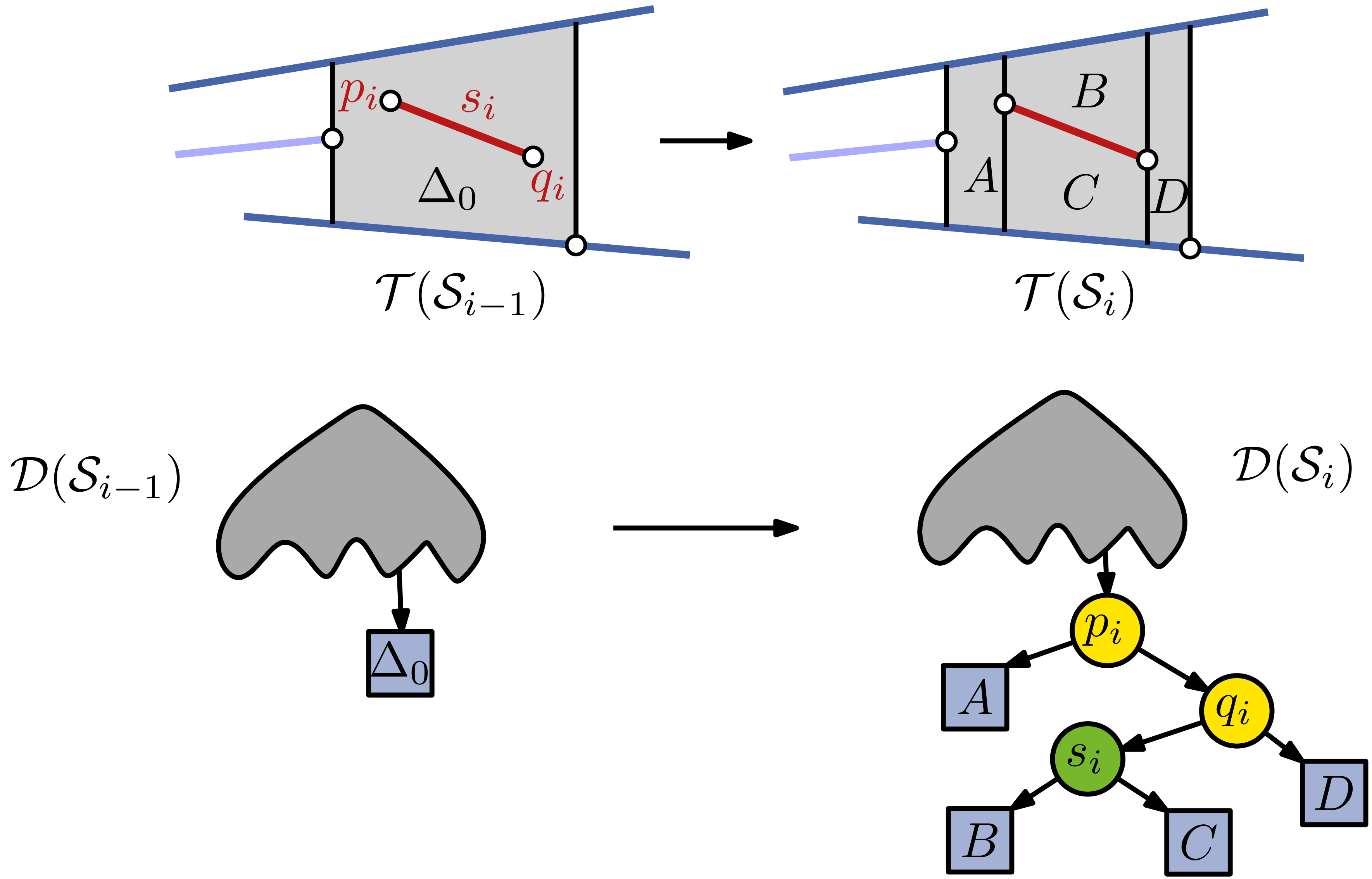

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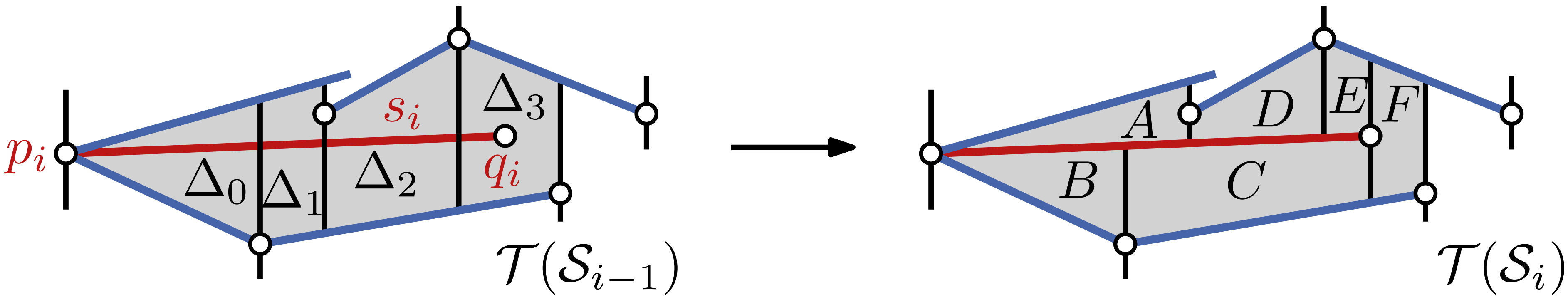

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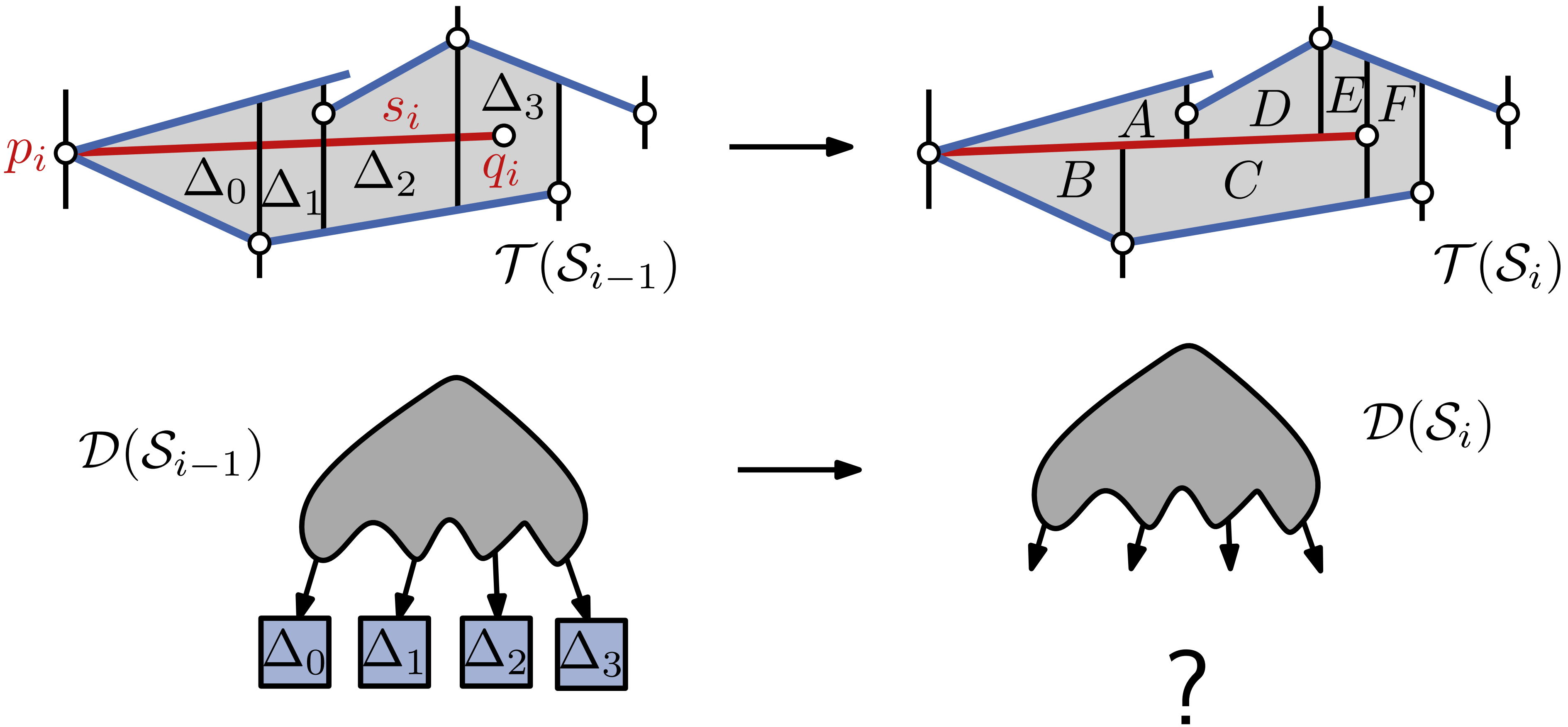

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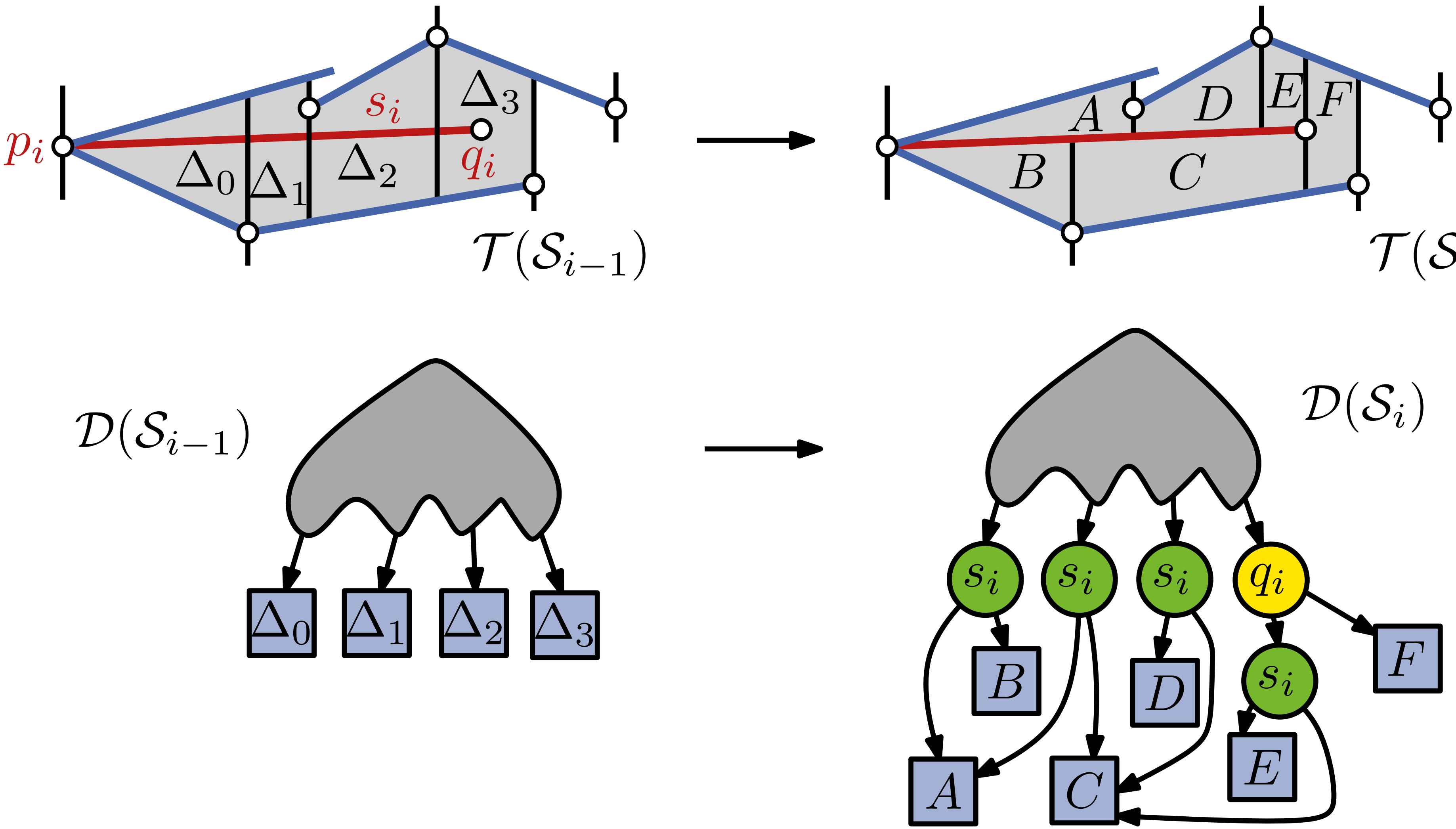

Analysis

Thm 1: The algorithm computes the trapezoidal map $\mathcal{T}(\mathcal{S})$ and the search structure $\mathcal{D}$ for a set $\mathcal{S}$ of n segments in *expected* $O(n \log n)$ time. The *expected* size of $\mathcal{D}$ is $O(n)$ and the *expected* query time is $O(\log n)$.

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- worst case: size of $\mathcal{D}$ is quadratic and query time is linear
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Proof:

- define random variables and consider their expected values
- perform *backward analysis*

$\rightarrow$ details on blackboard

Worst-Case Consideration

So far: expected query time for arbitrary point is $O(\log n)$

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Pr[search path for q longer than $3\lambda \ln(n+1)$]
$\leq 1/(n+1)^{\lambda \ln 1.25 - 1}$.

No proof. (or see Chapter 6.4)

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Thm 2: Let $\mathcal{S}$ be a subdivision of the plane with n edges. There is a search structure for point location within $\mathcal{S}$ that has $O(n)$ space and $O(\log n)$ query time.

Degenerate Inputs

Two assumptions:

- no two segment endpoints have the same x-coordinates
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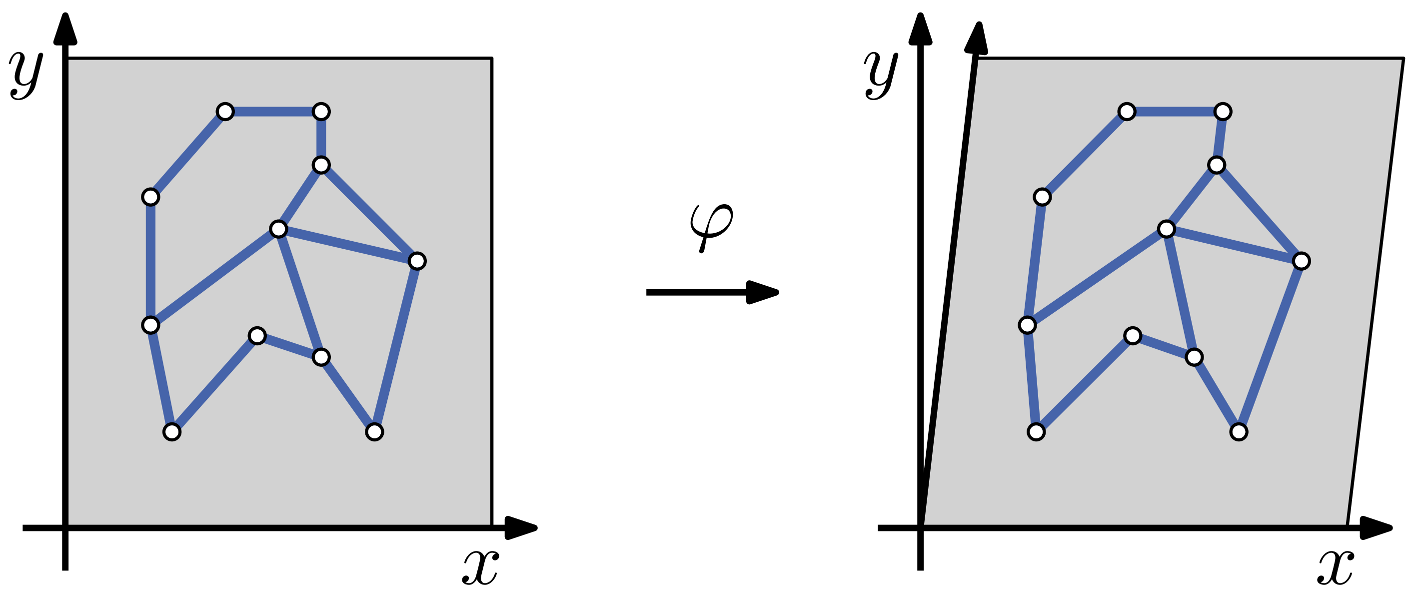

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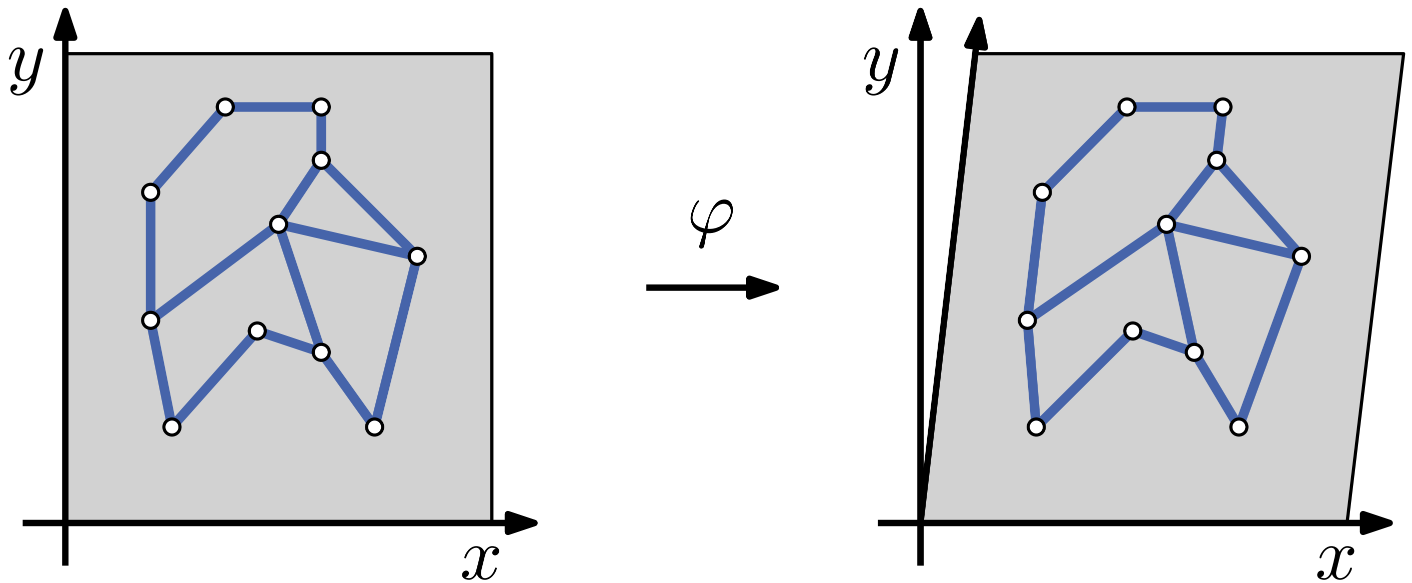

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- Two basic operations for constructing $\mathcal{T}$ and $\mathcal{D}$:
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- Locating a point q in $\mathcal{T}(\mathcal{S})$ works by locating φq in $\mathcal{T}(\varphi\mathcal{S})$.

$\rightarrow$ see Chapter 6.3 in [De Berg et al. 2008]

Discussion

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The currently best three-dimensional data structure uses $O(n \log n)$ space and $O(\log^2 n)$ query time [Snoeyink '04]. Whether linear space and $O(\log n)$ query time is possible is an open question. In even higher dimensions efficient methods are known only for special hyper plane subdivisions.

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Are there dynamic data structures that allow insertions and deletions?

Dynamic data structures for point location are well known, see the survey by [Chiang, Tamassia '92]. A more recent example by [Arge et al. '06] needs $O(n)$ space, $O(\log n)$ query time and $O(\log n)$ update time (insertions).