Computational Geometry Lecture
Range Searching

INSTITUT FÜR THEORETISCHE INFORMATIK · FAKULTÄT FÜR INFORMATIK

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20.06.2018

Geometry in Databases

In a personnel database, the employees of a company are anonymized and their monthly income and birth year are saved. We now want to perform a search: which employees have an income between 2,000 and 3,000 Euro and were born between 1960 and 1980?

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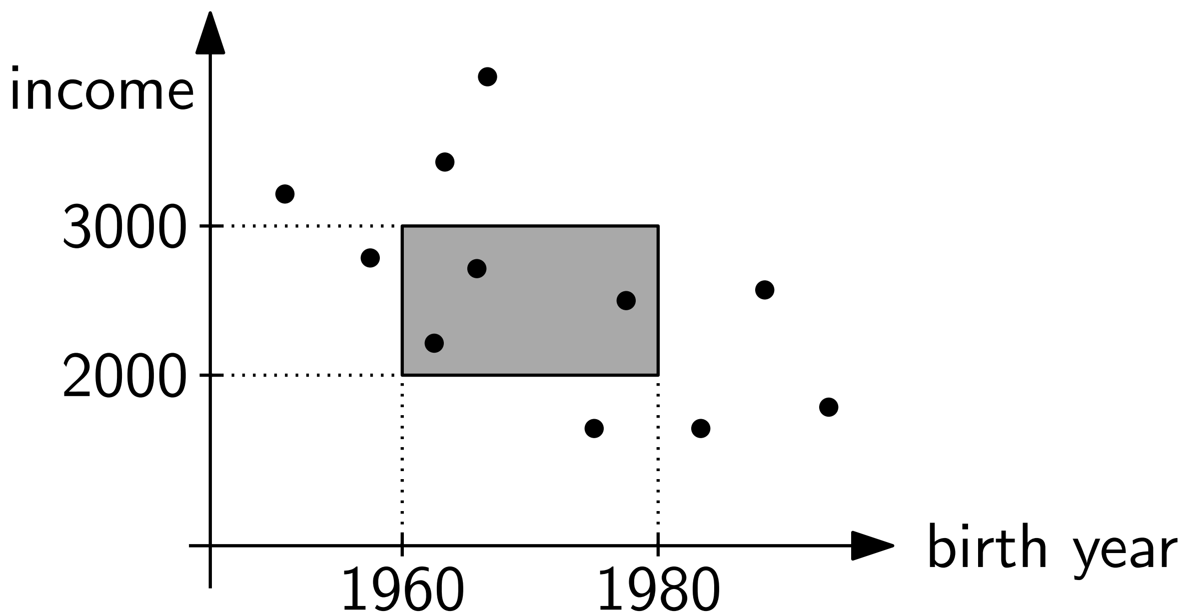

Geometric Interpretation:
Entries are points: (birth year, income level) and the query is an axis-parallel rectangle

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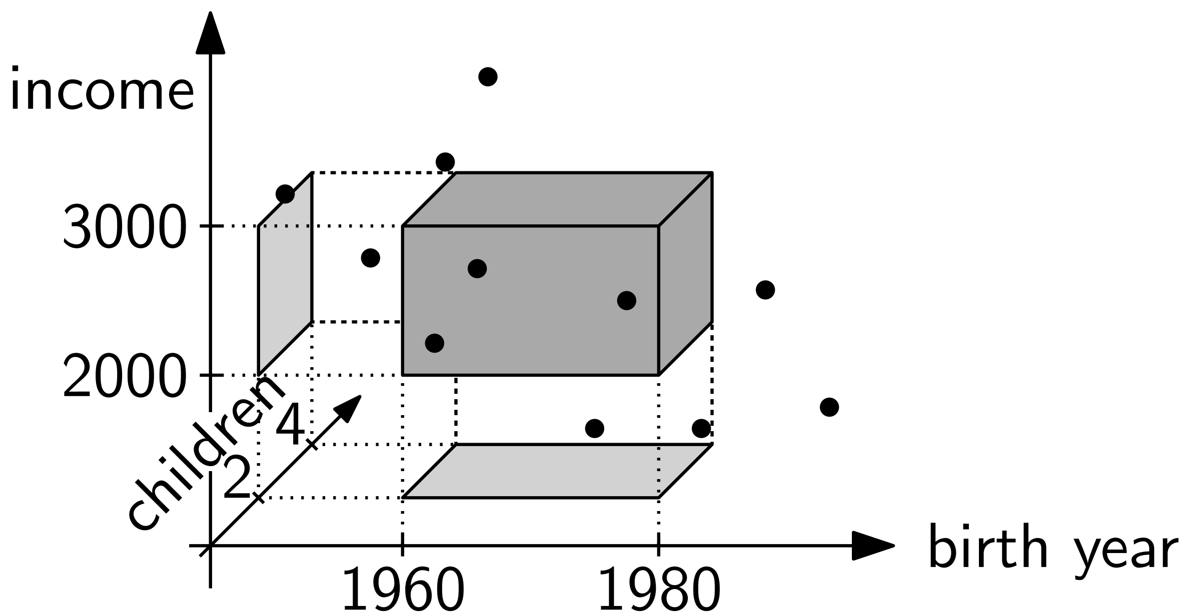

This problem can easily be generalized to d dimensions.

Orthogonal Range Queries

Given: n points in $\mathbb{R}^d$

Output: A data structure that efficiently answers queries of the form $[a_1, b_1] \times \cdots \times [a_d, b_d]$

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- Stores points in the leaves
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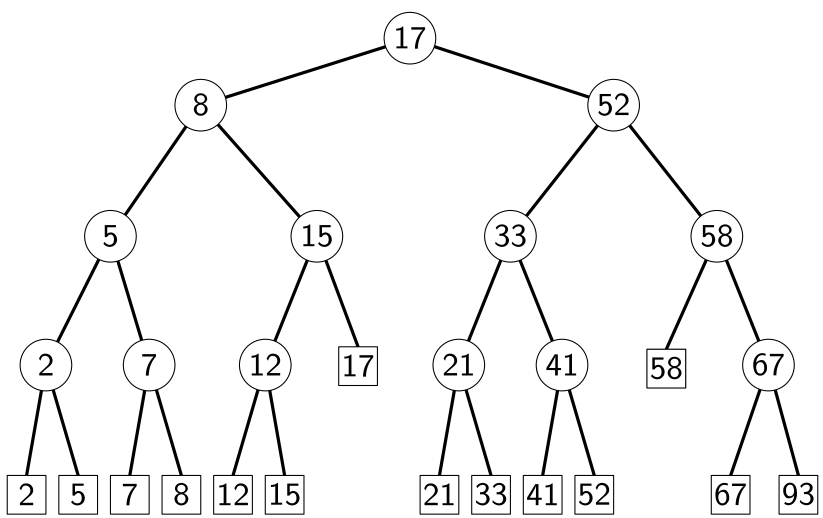

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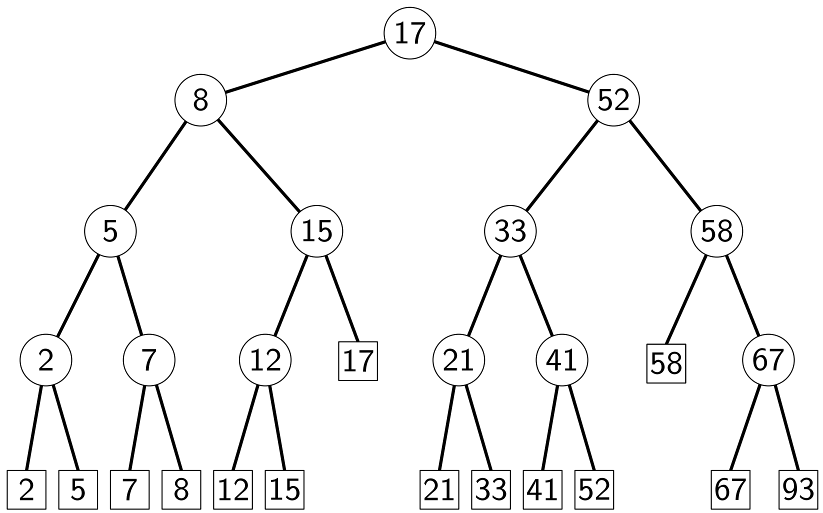

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Search for all points in [6,50]

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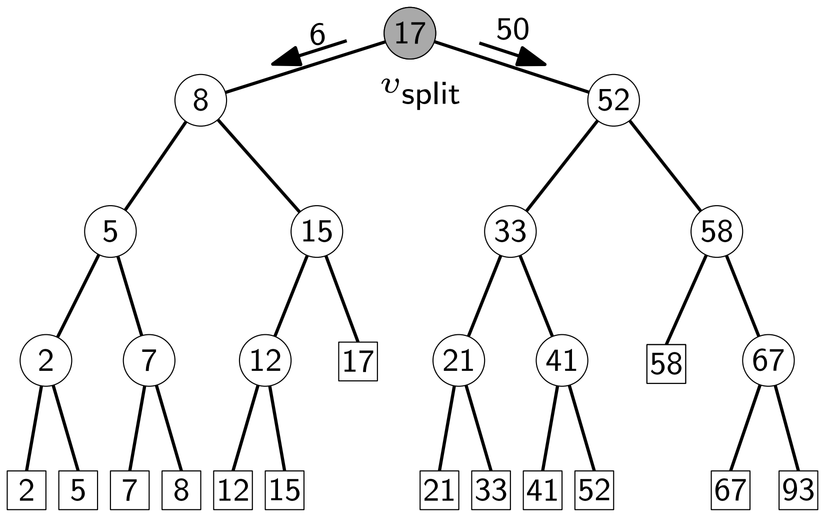

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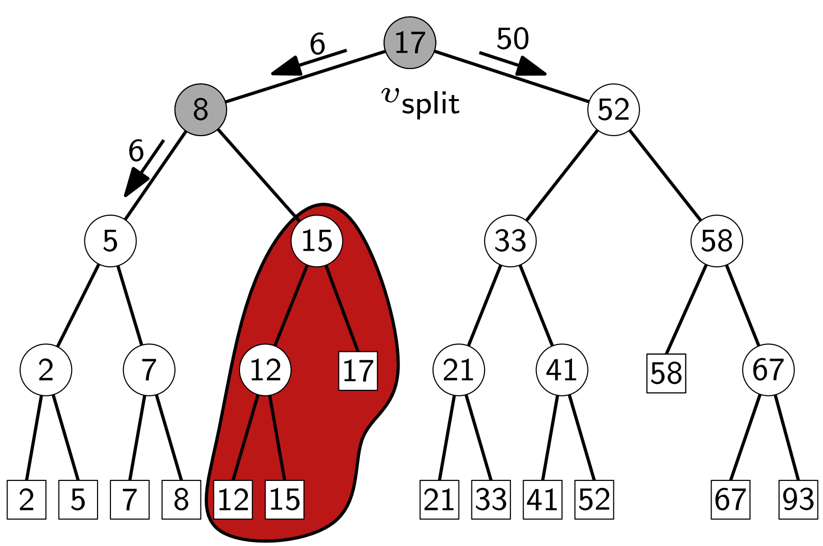

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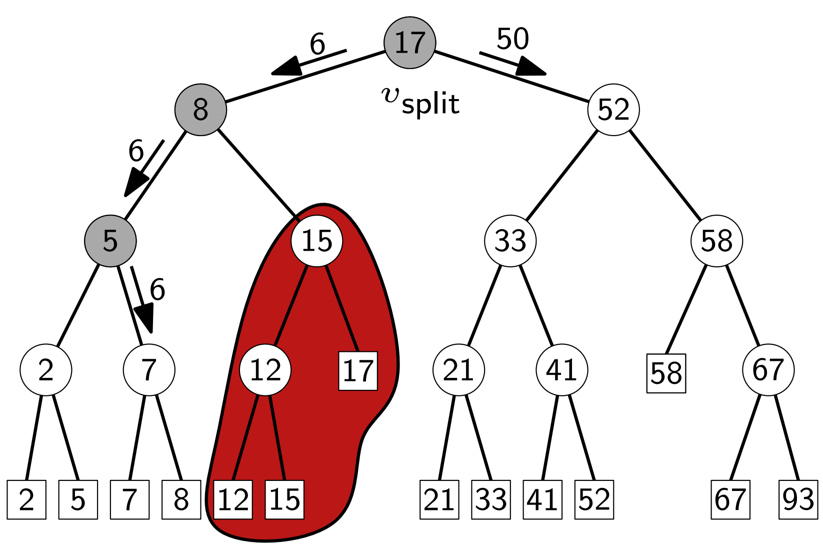

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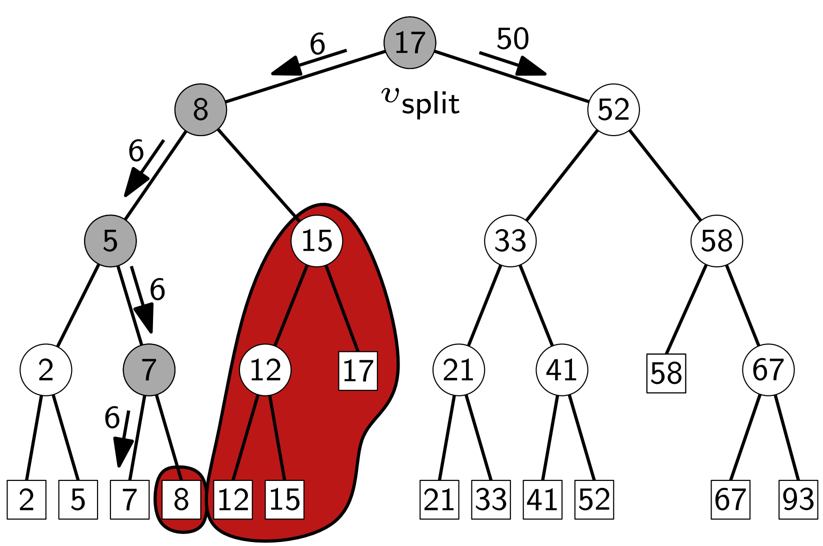

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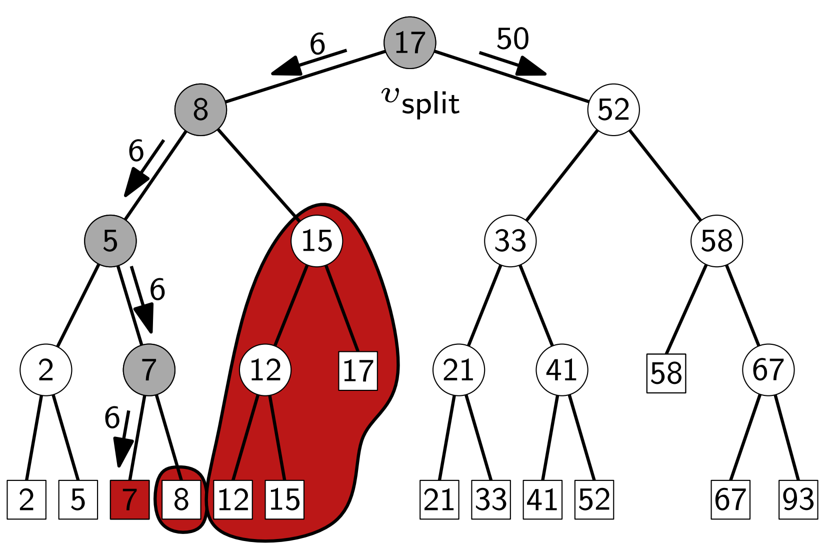

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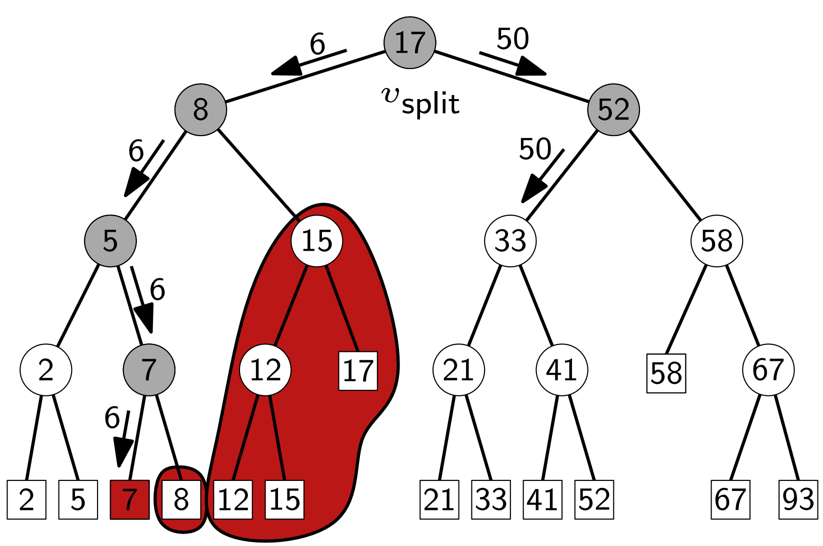

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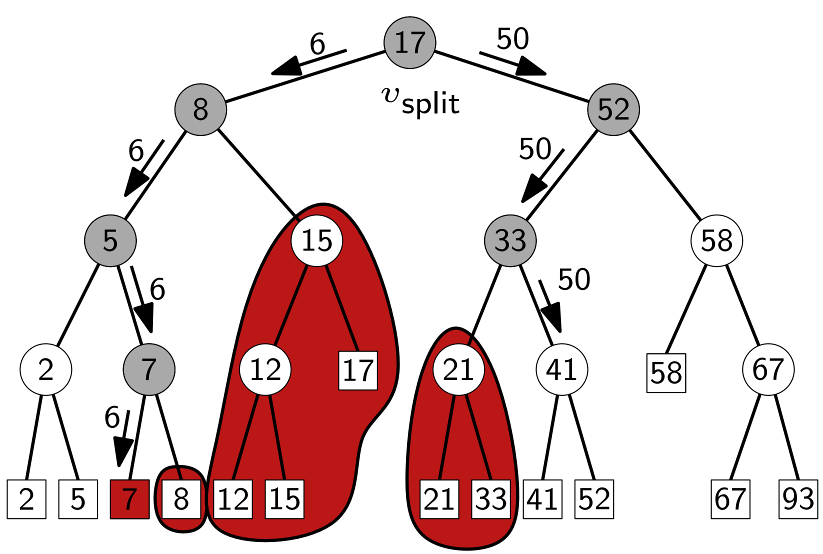

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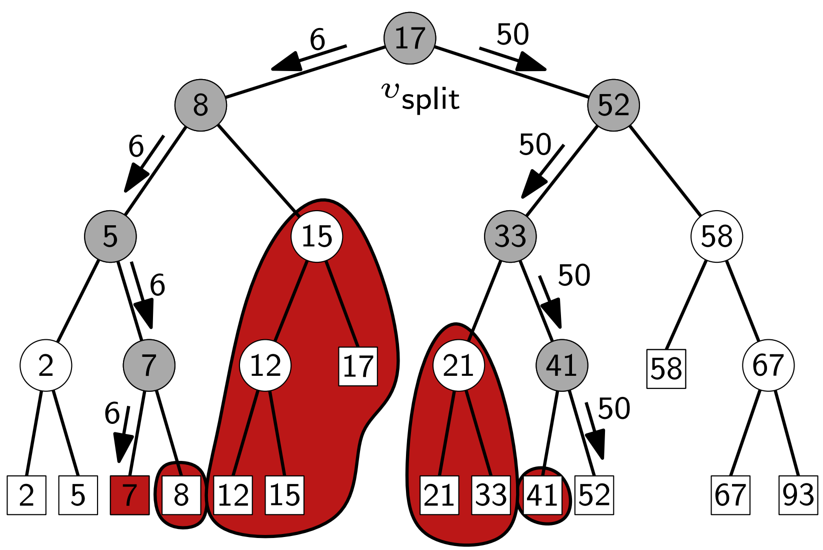

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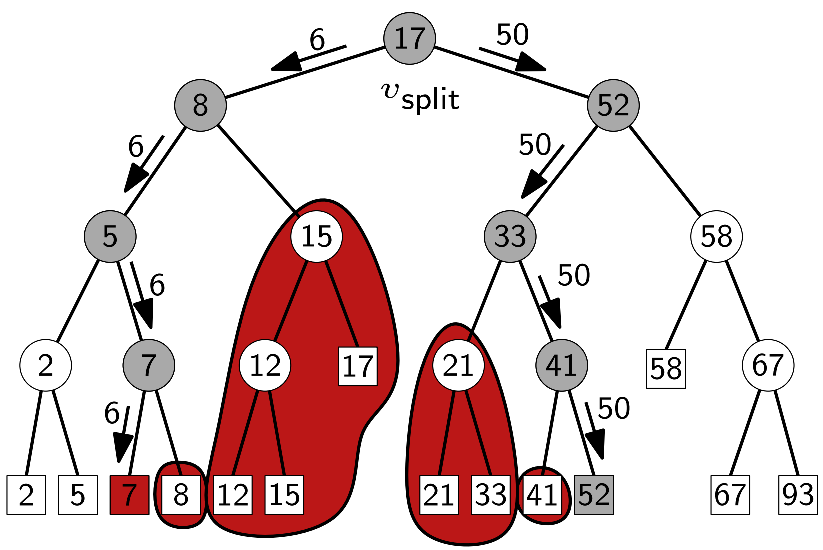

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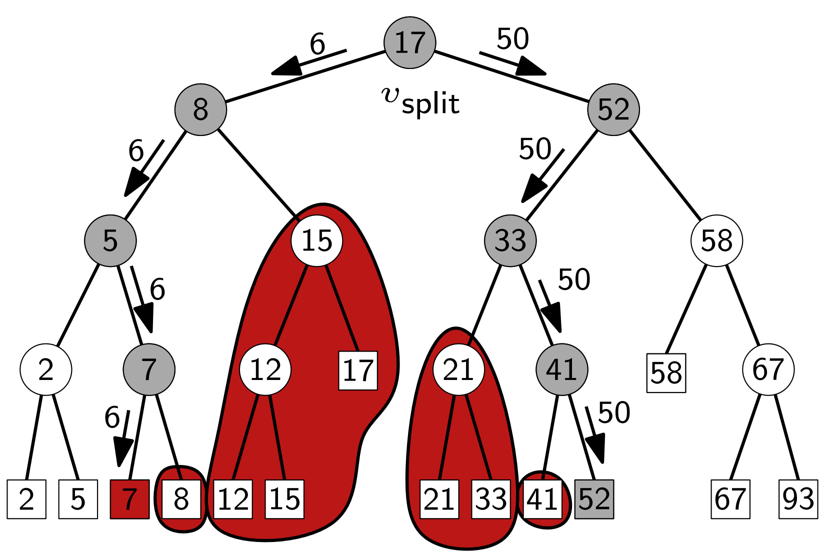

Example:

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Answer:

Points in the leaves between the search paths, (i.e., {7,8,12,15,17,21,33,41})

1dRangeQuery

FindSplitNode(T, x, x')

$v \leftarrow$ root(T)
while v not a leaf and $(x' \leq x_v$ or $x > x_v)$ **do**
  if $x' \leq x_v$ **then** $v \leftarrow$ lc(v) **else** $v \leftarrow$ rc(v)
return v

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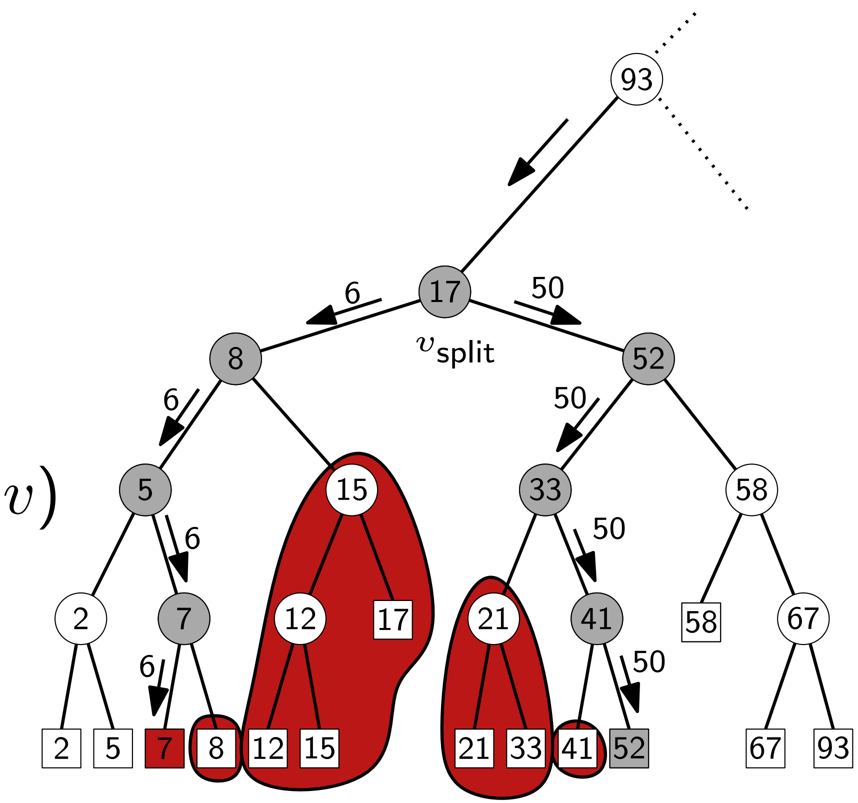

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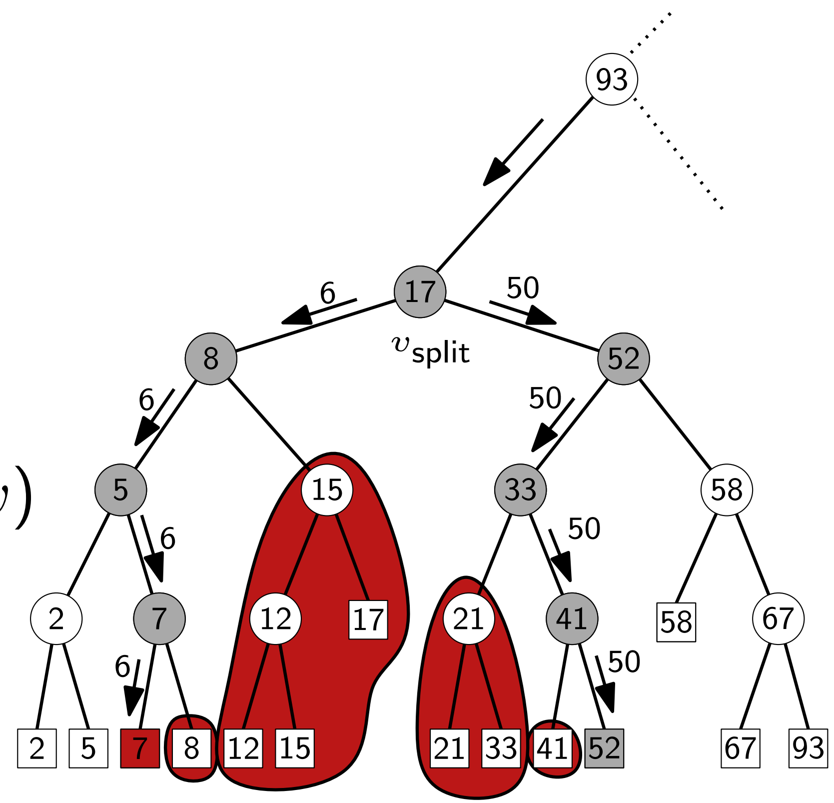
Can find *canonical subset* in linear time



Analysis of 1dRangeQuery

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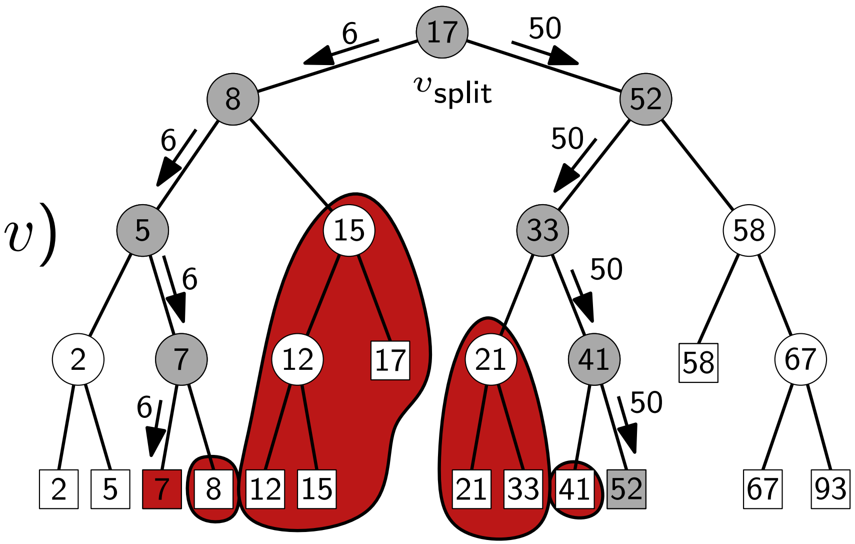

Theorem 1: A set of n points in $\mathbb{R}$ can preprocessed in $O(n \log n)$ time and stored in $O(n)$ space so that we can answer range queries in $O(k + \log n)$ time, where k is the number of reported points.

Orthogonal Range Queries for $d = 2$

Given: Set P of n points in $\mathbb{R}^2$

Goal: A data structure to efficiently answer range queries of the form $R = [x, x'] \times [y, y']$

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Solutions:

- *one* search tree, alternate search for x and y coordinates
  $\rightarrow$ kd-**Tree**
- *primary* search tree on x-coordinates,
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Temporary assumption: general position, that is no two points have the same x- or y-coordinates

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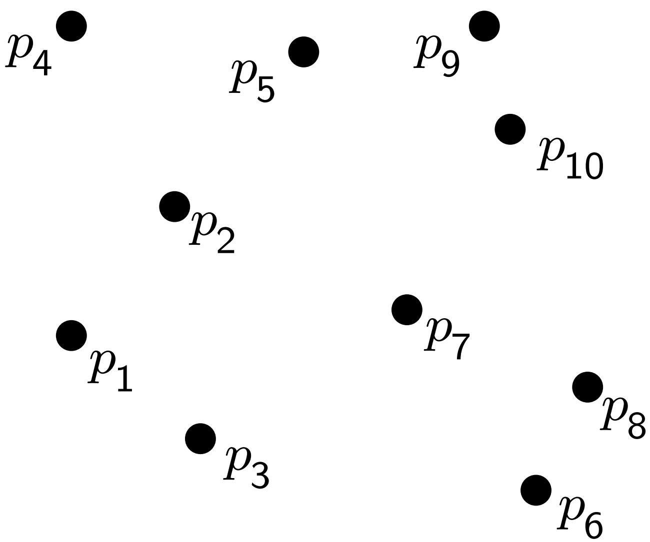

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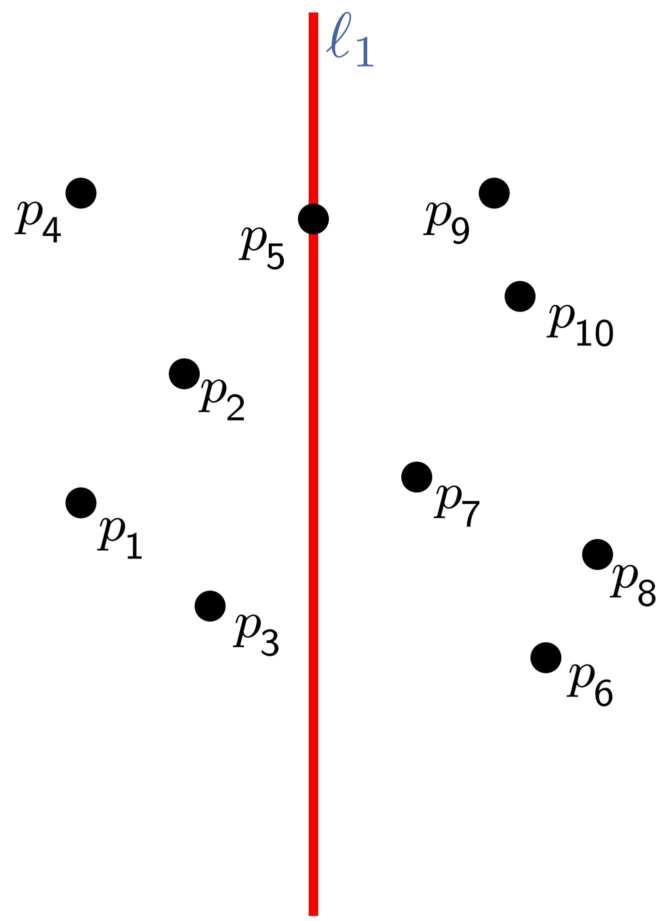

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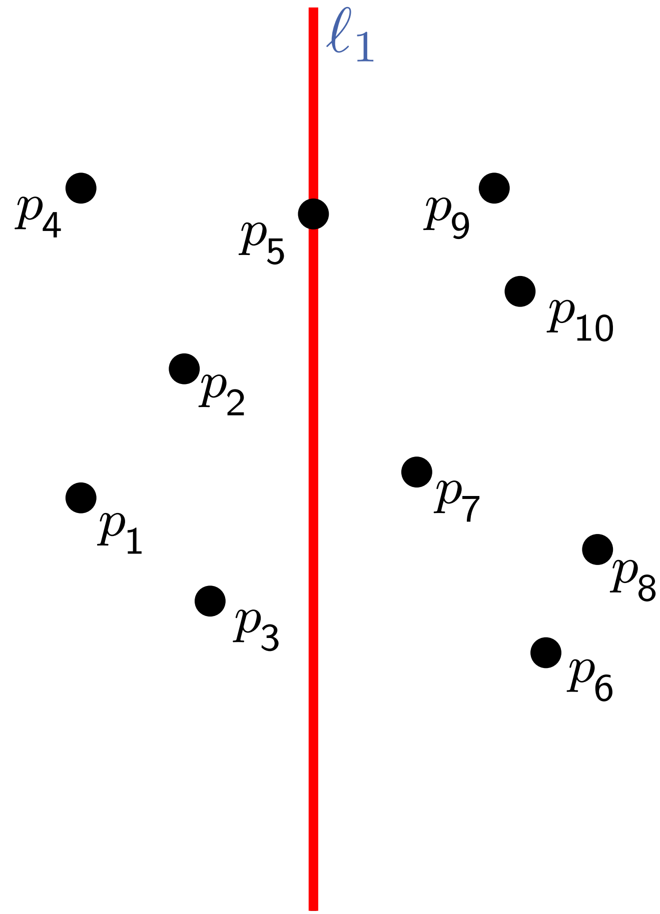

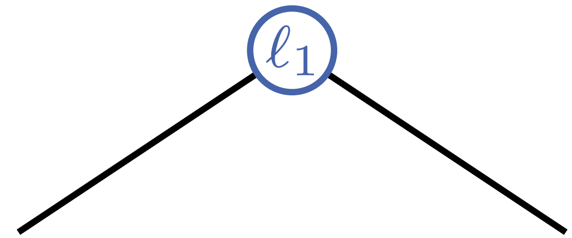

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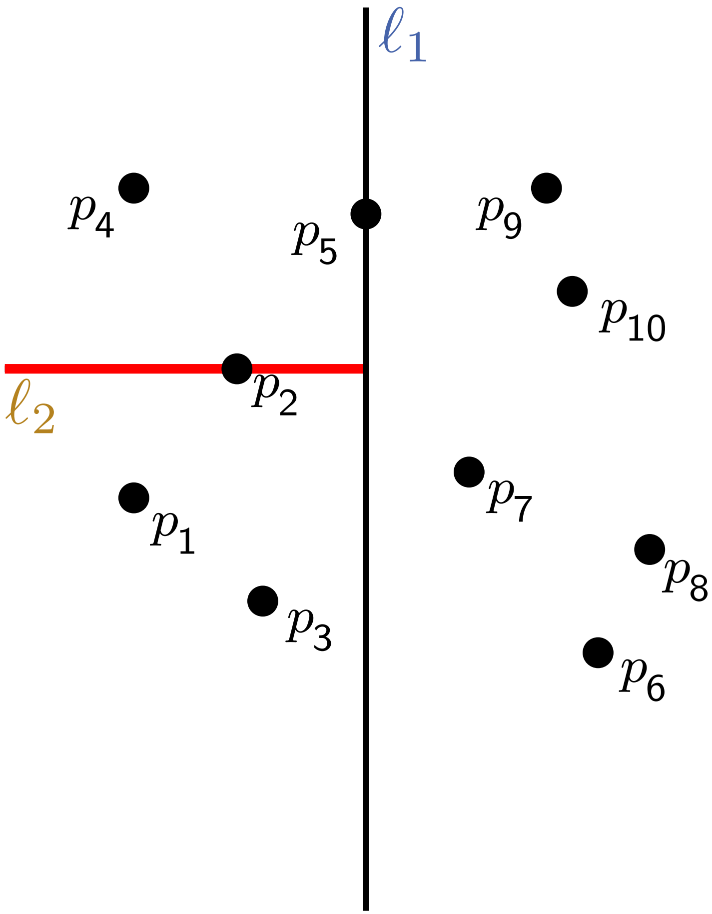

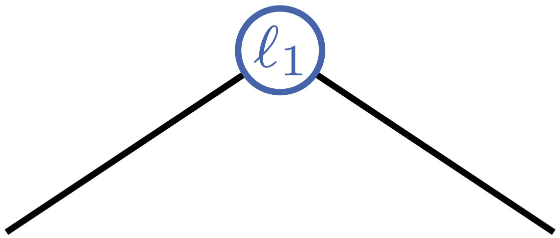

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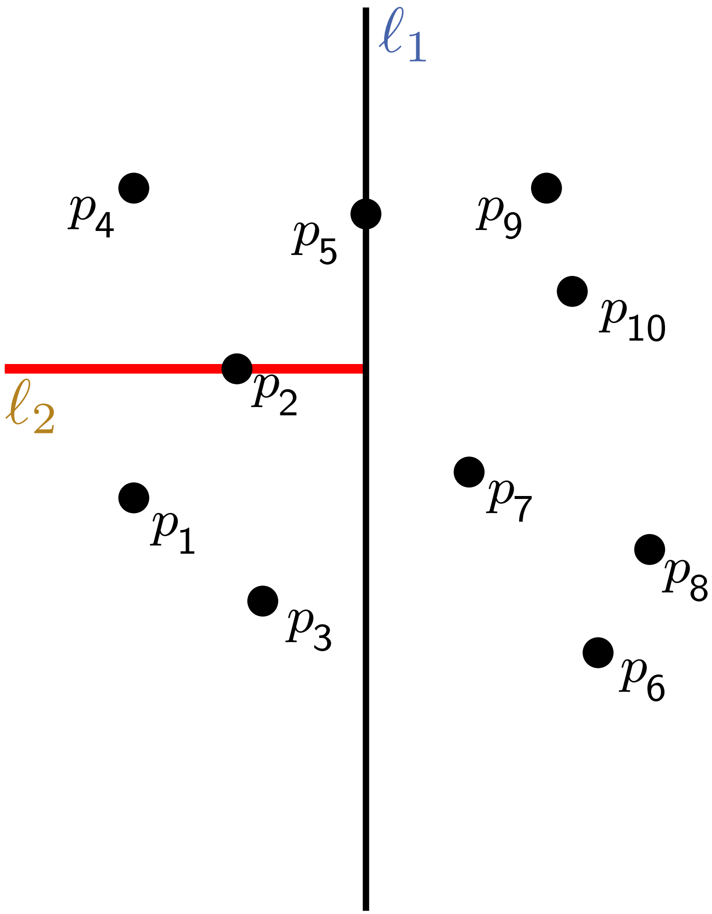

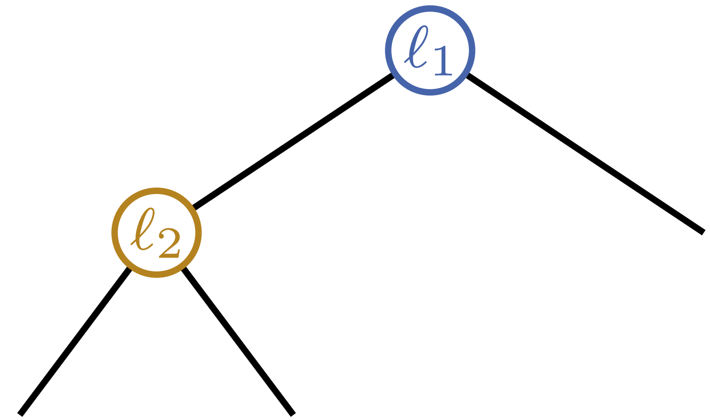

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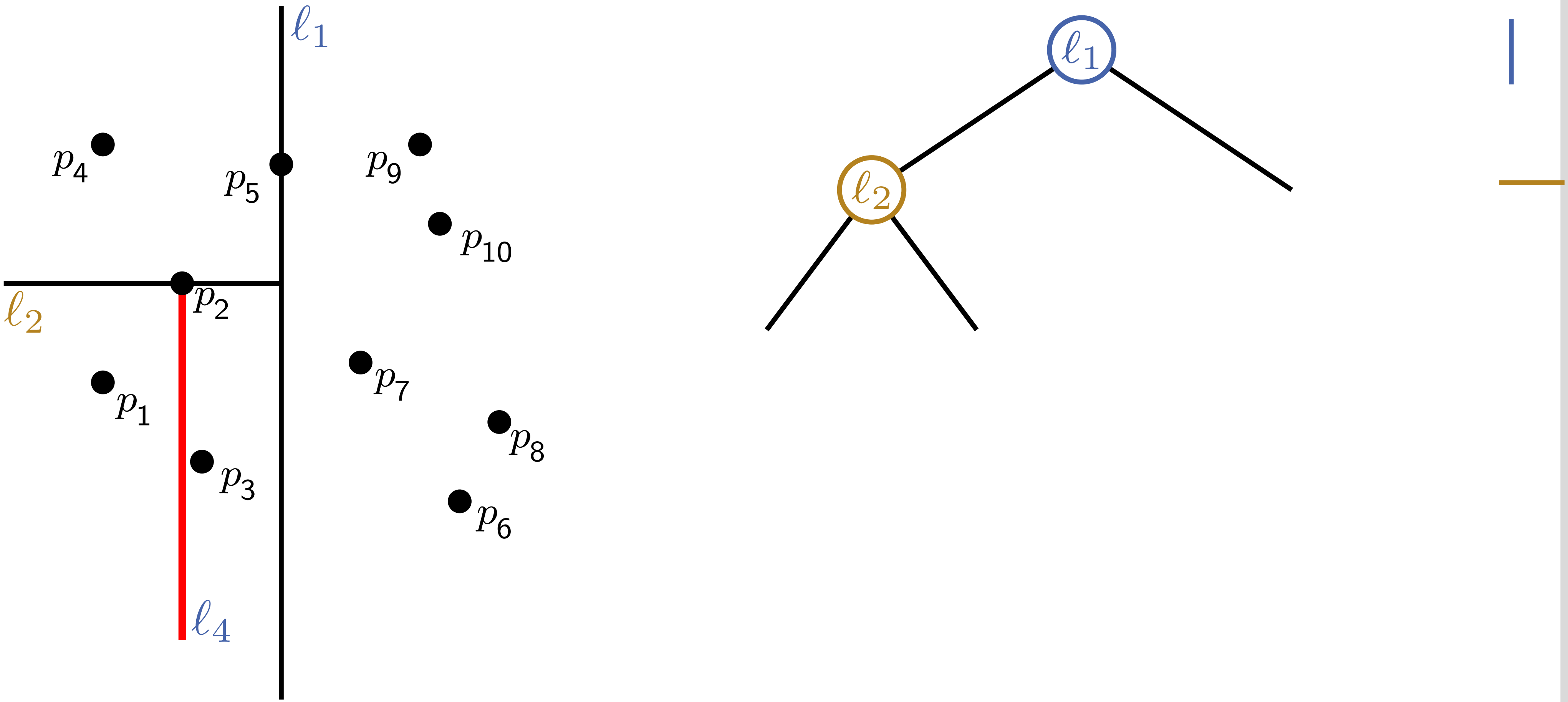

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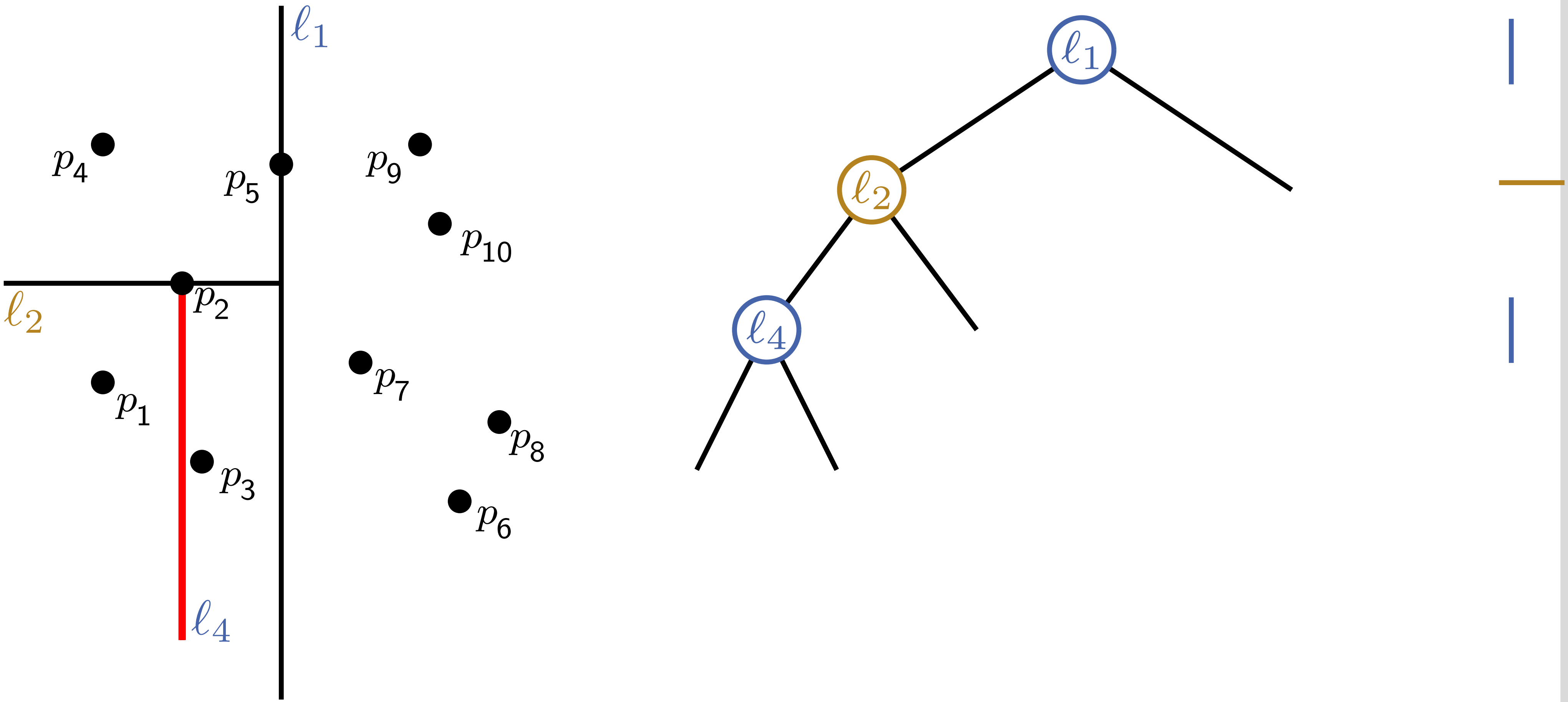

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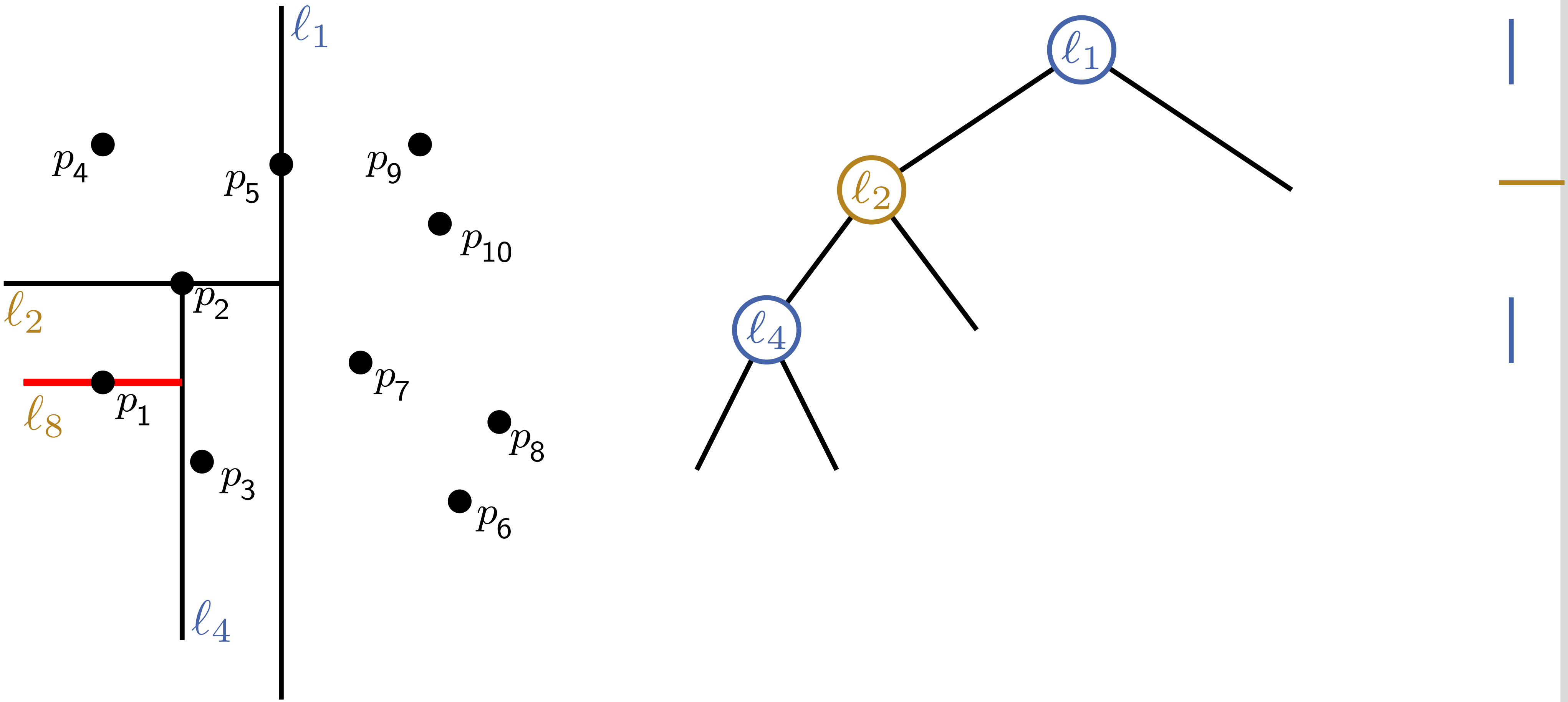

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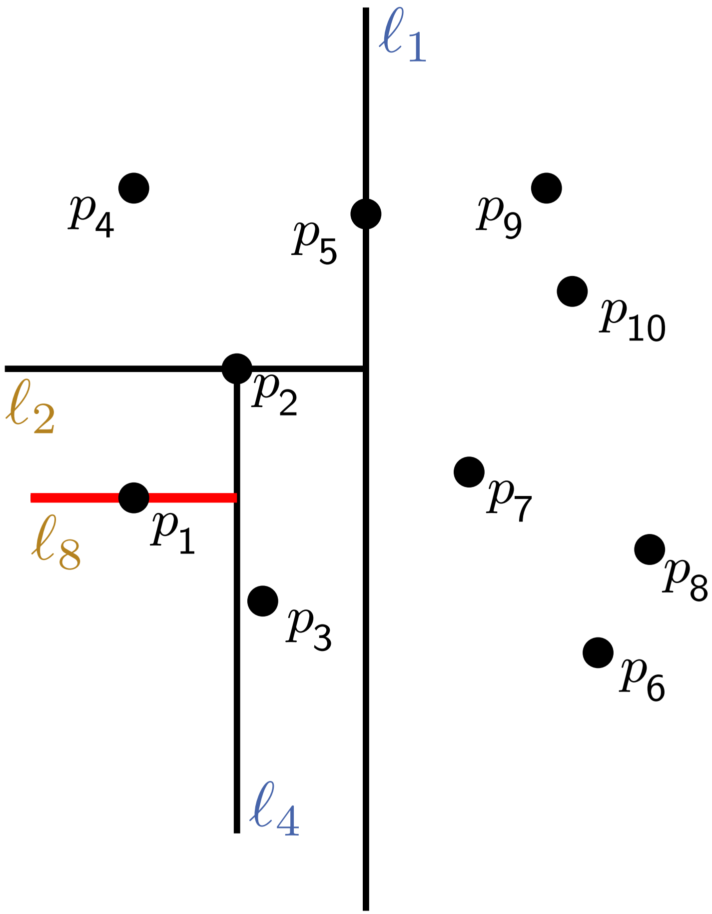

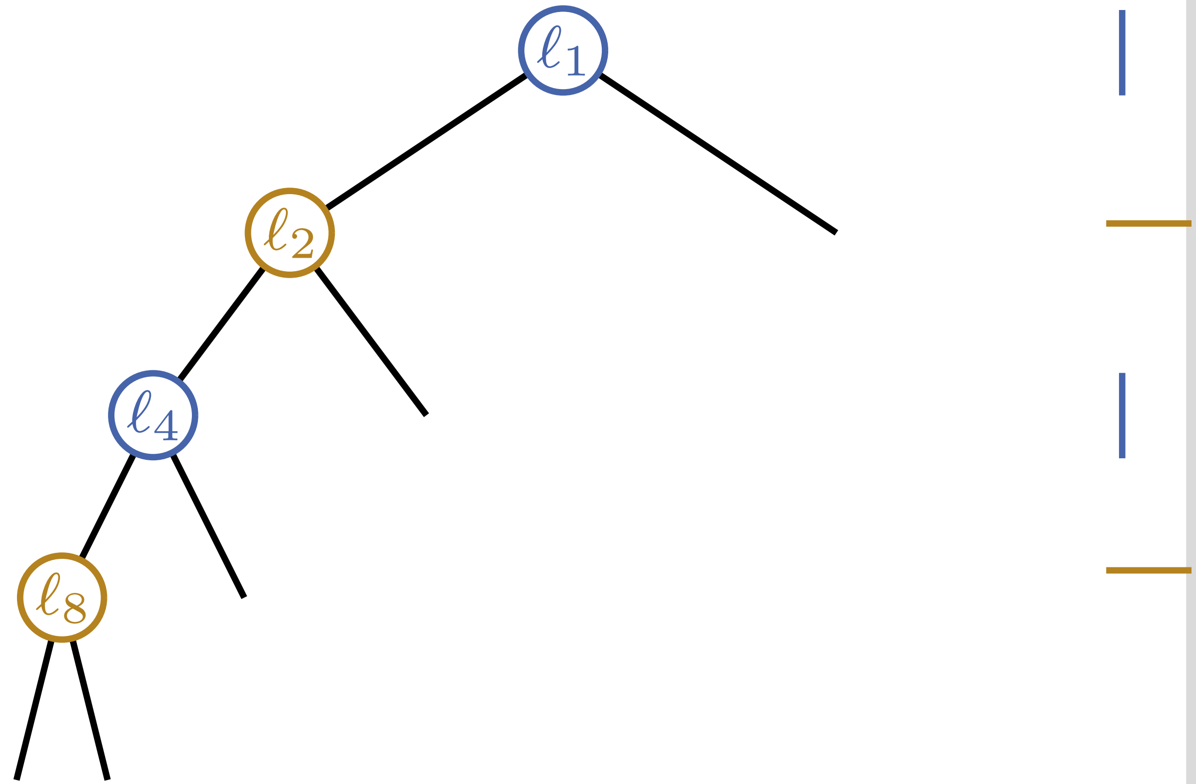

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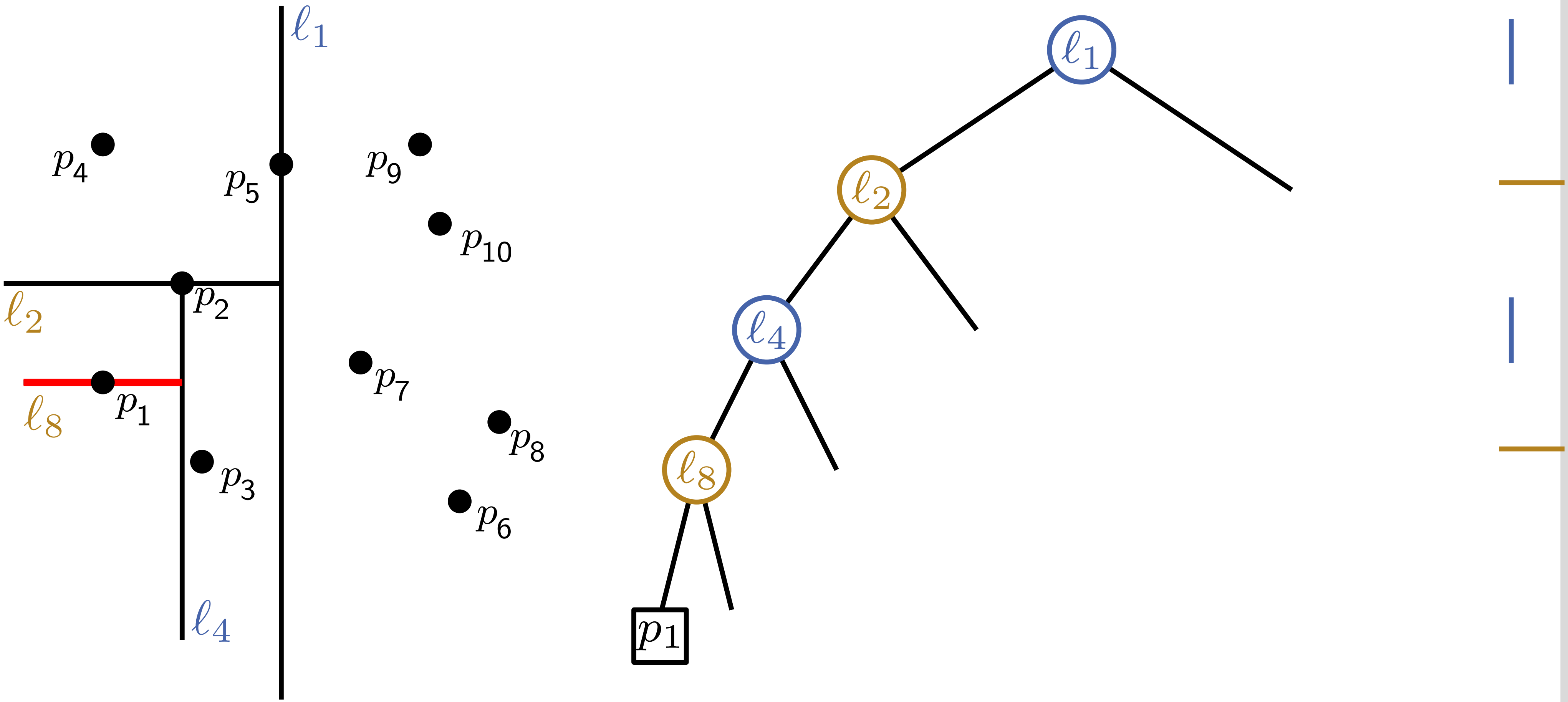

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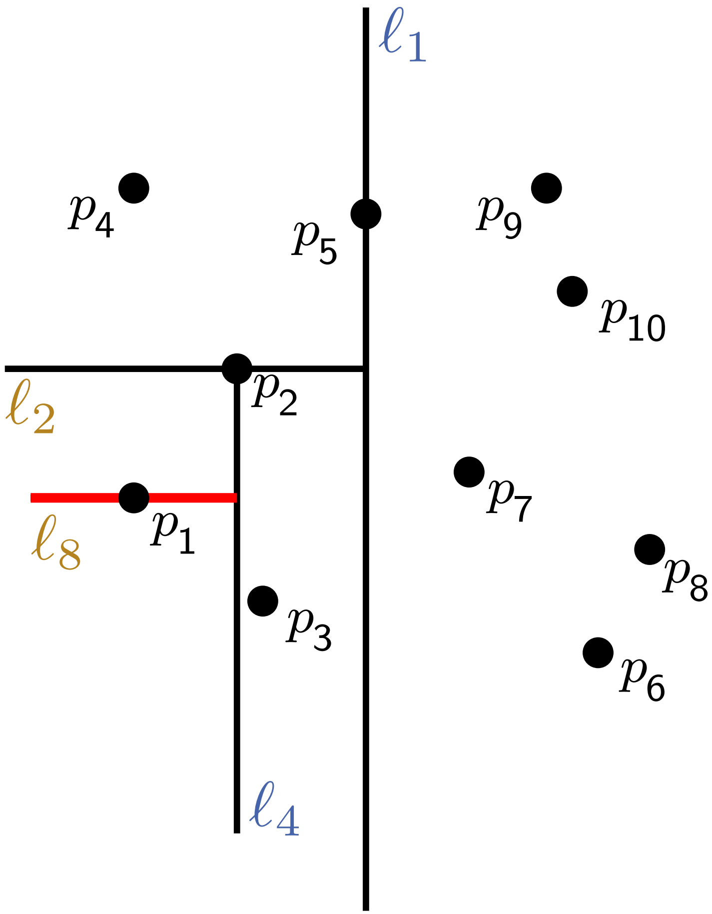

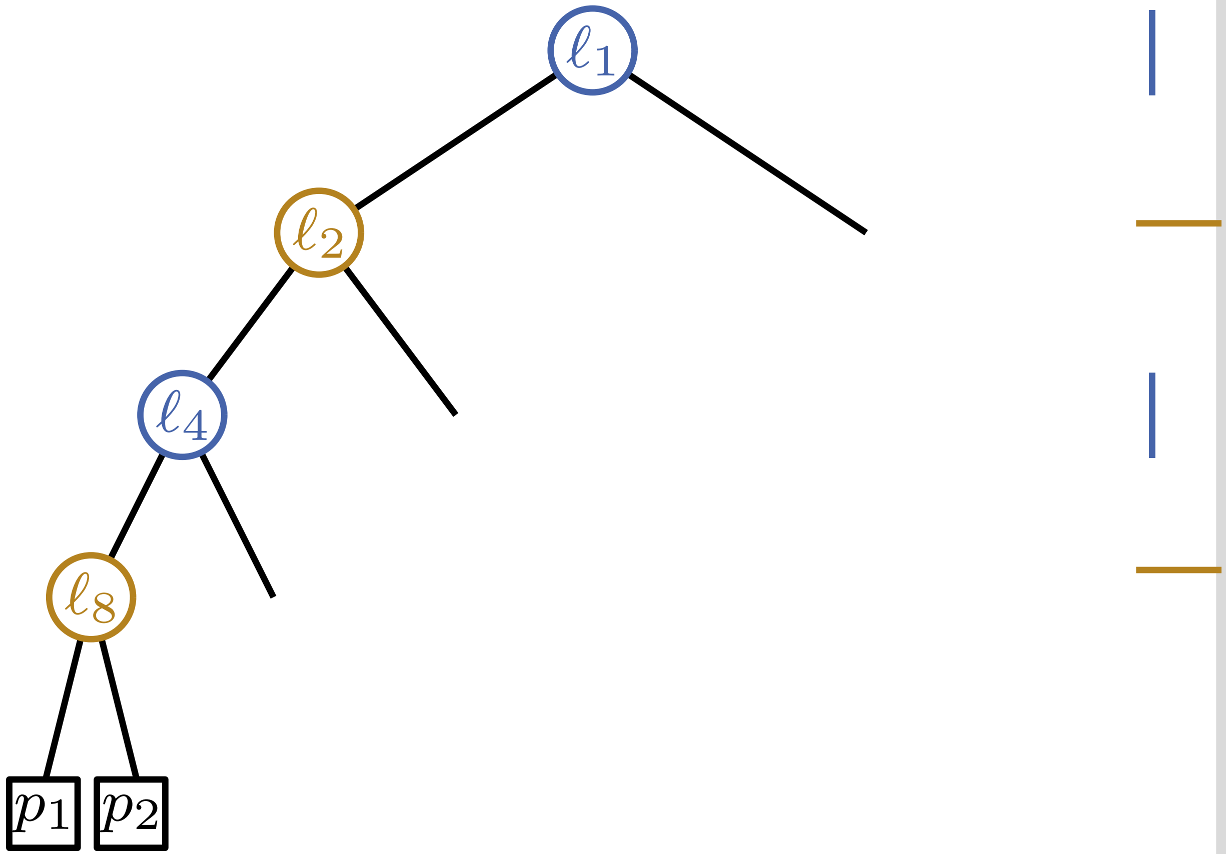

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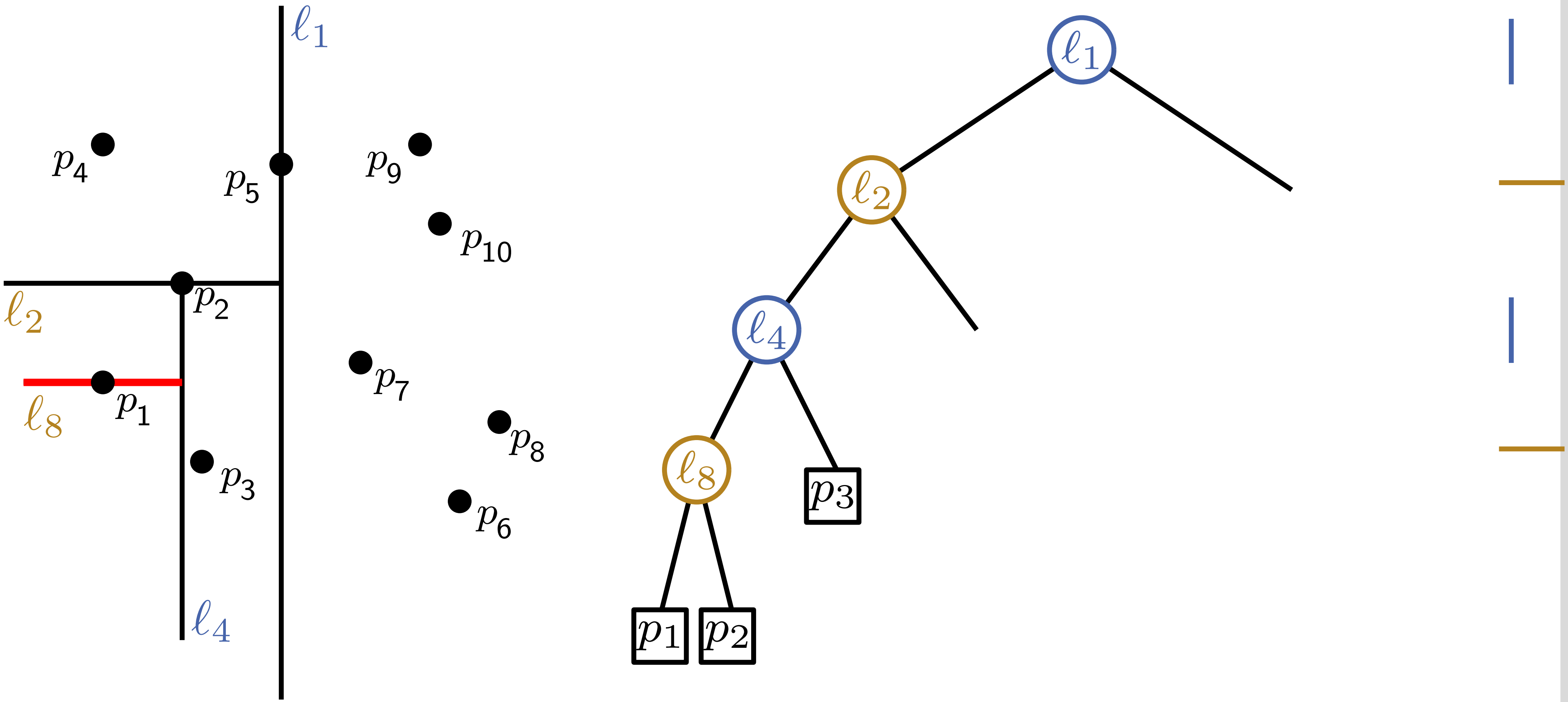

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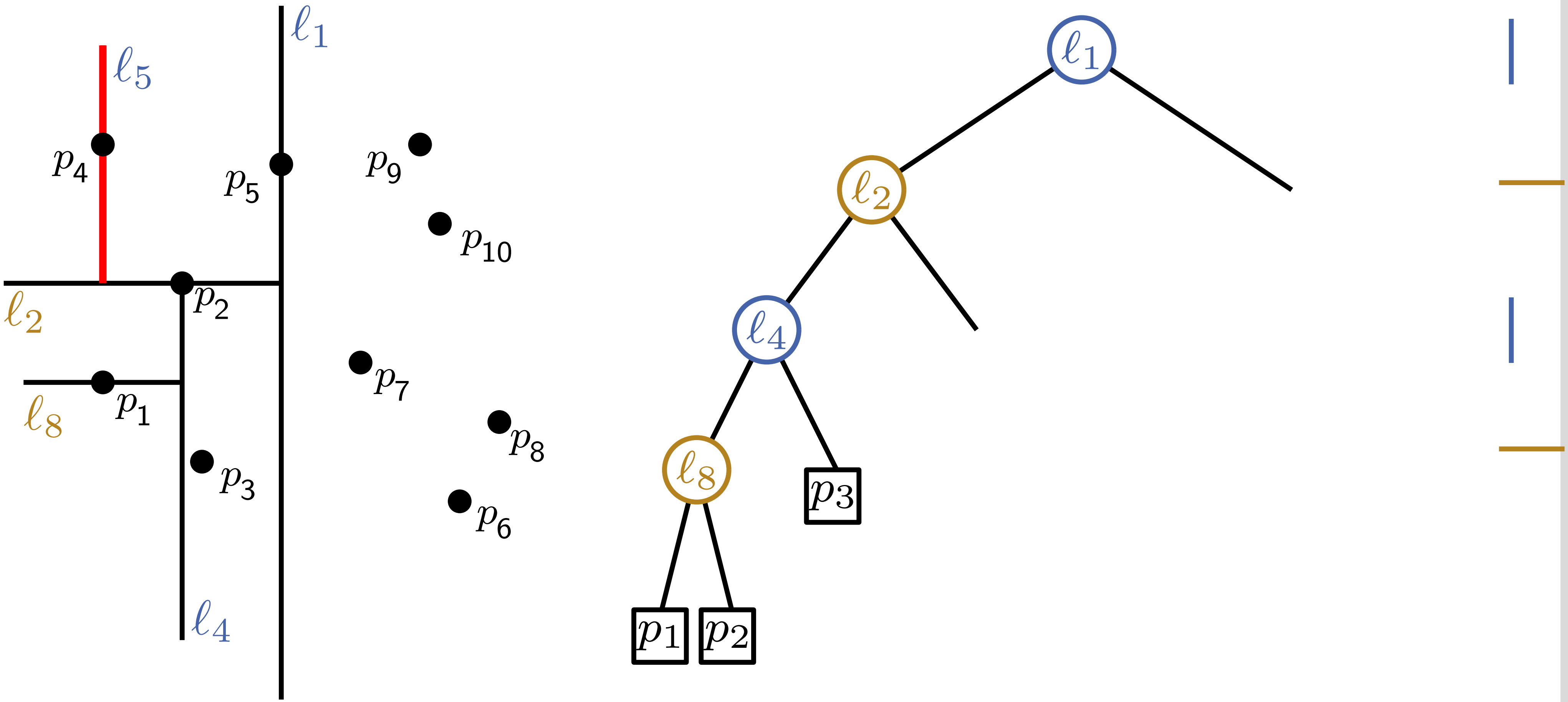

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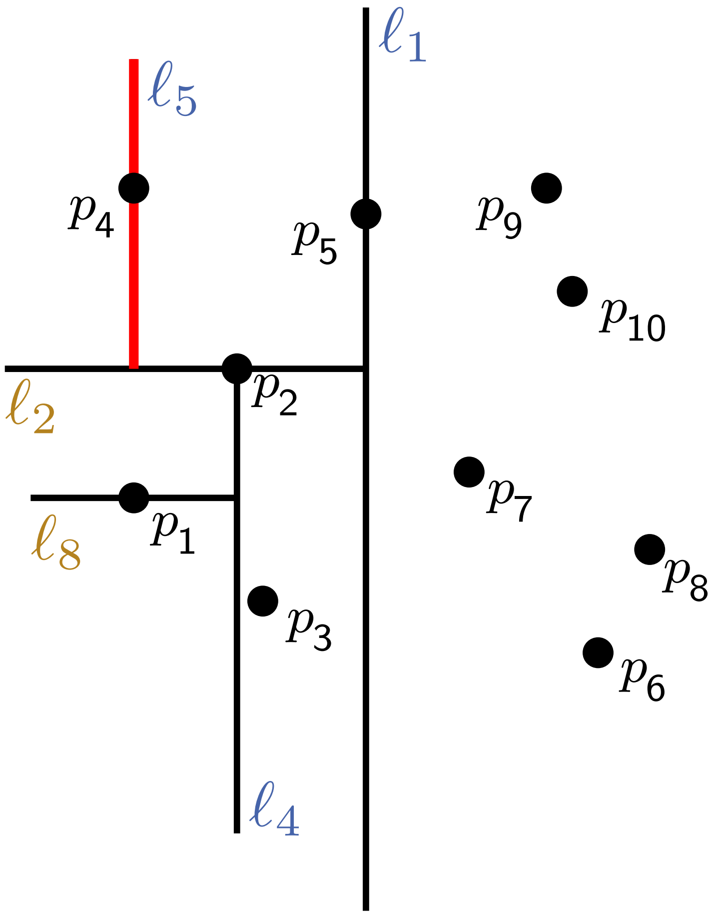

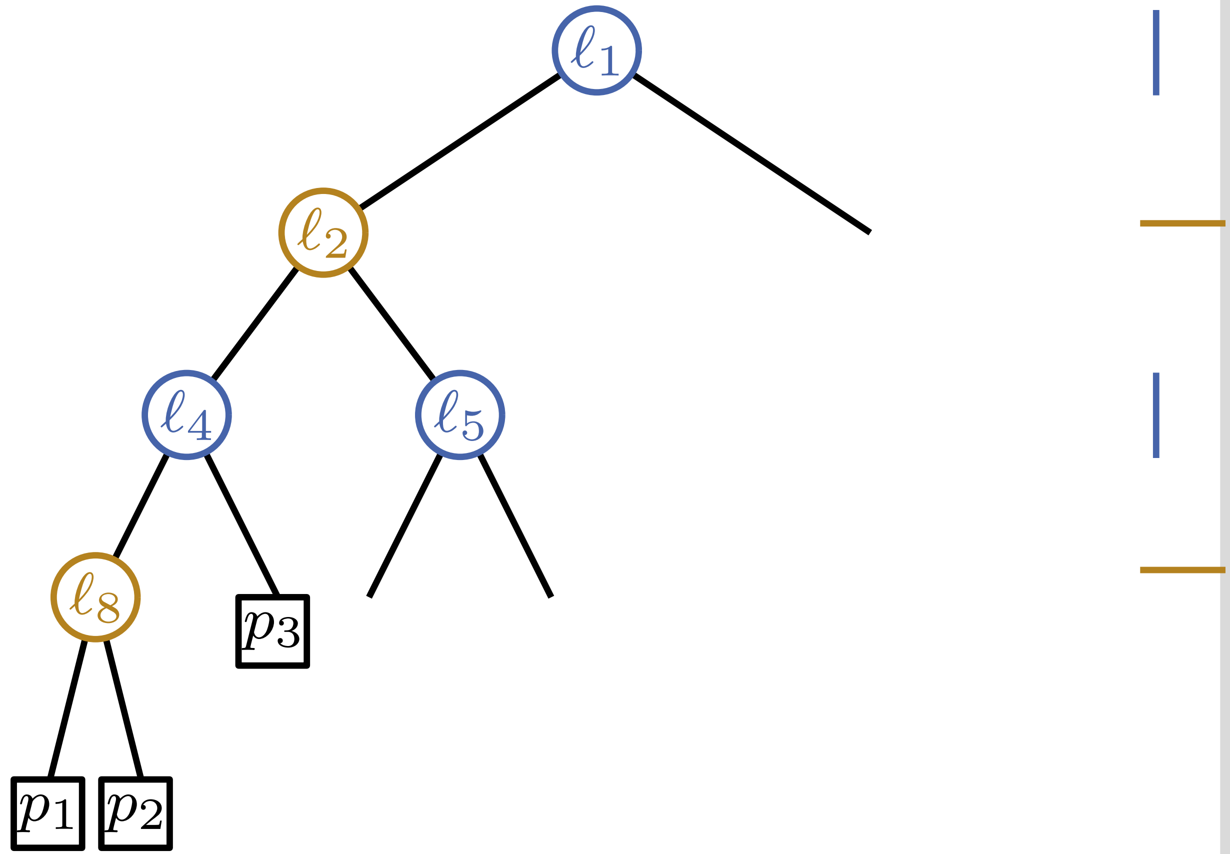

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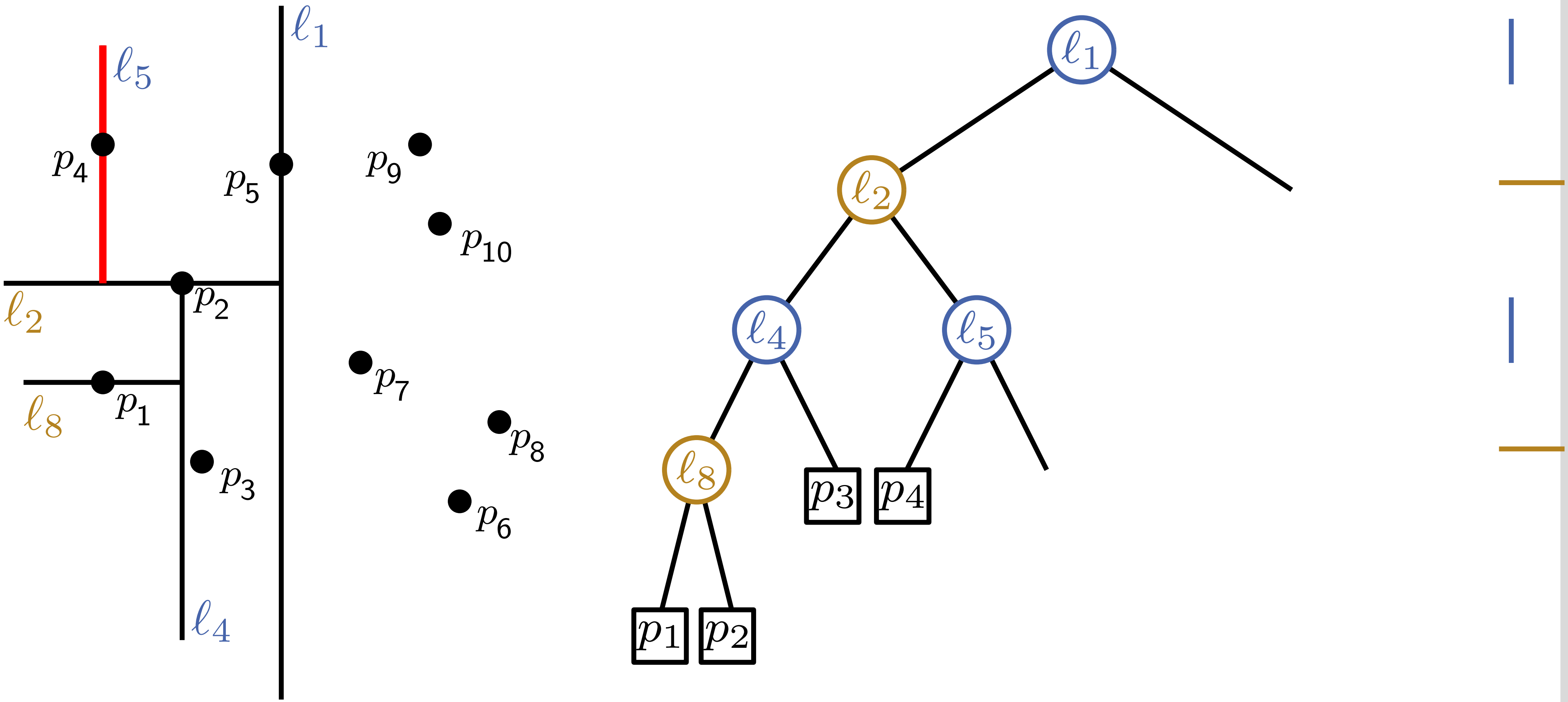

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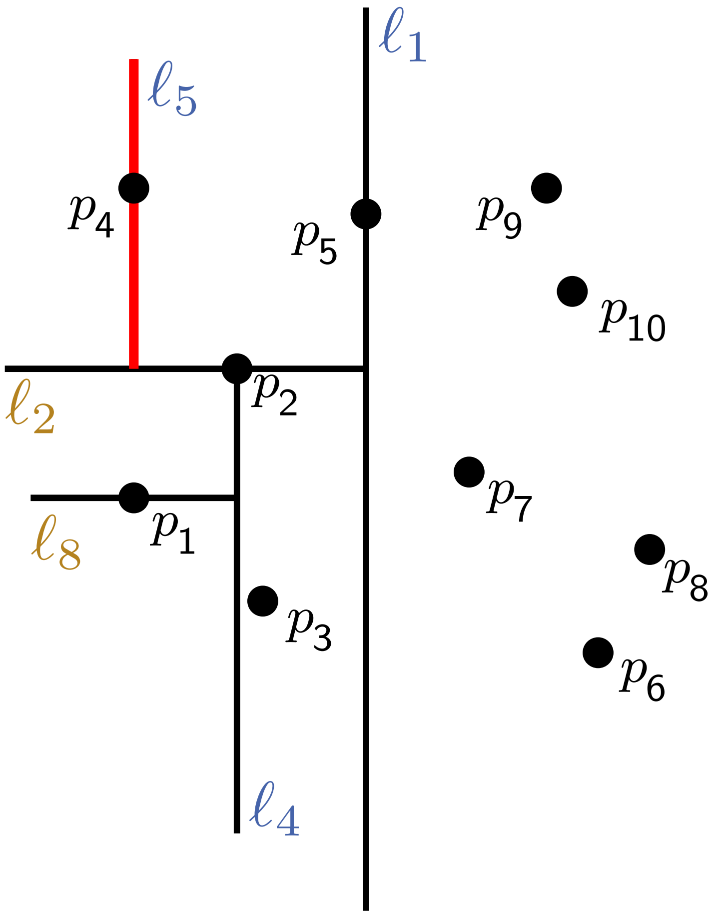

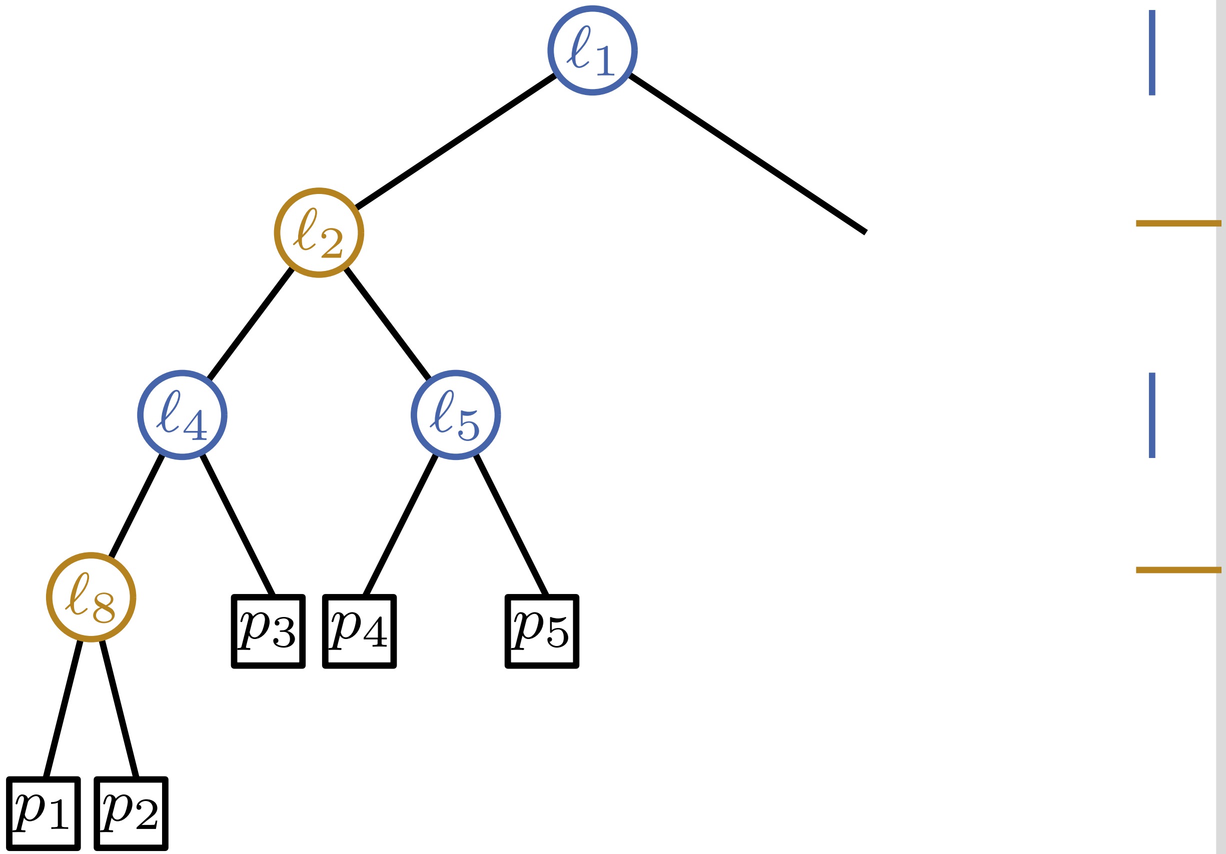

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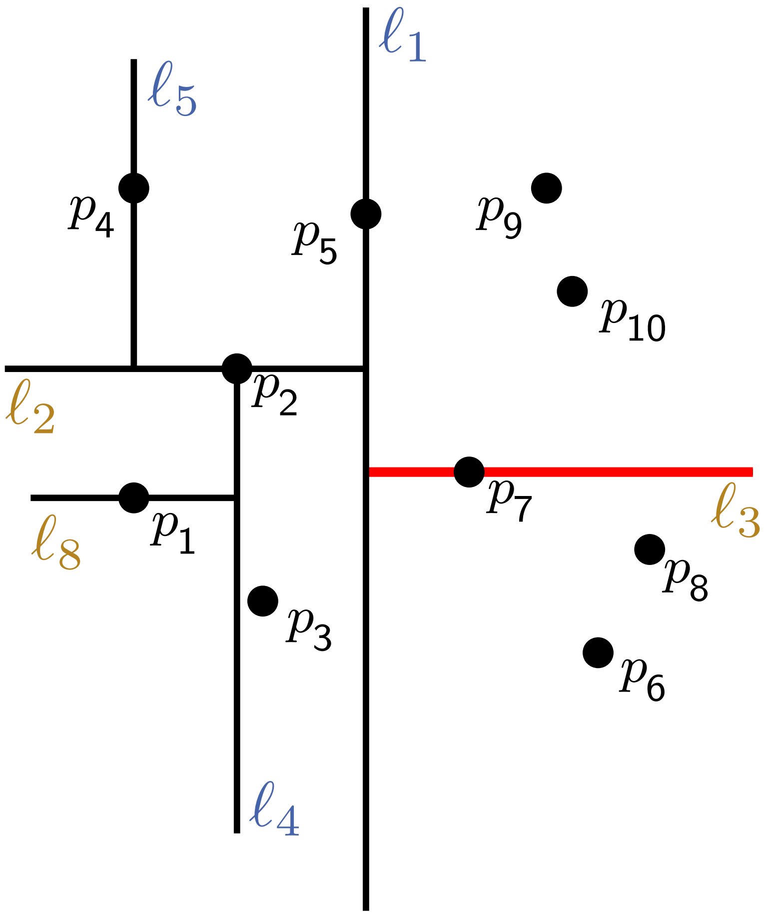

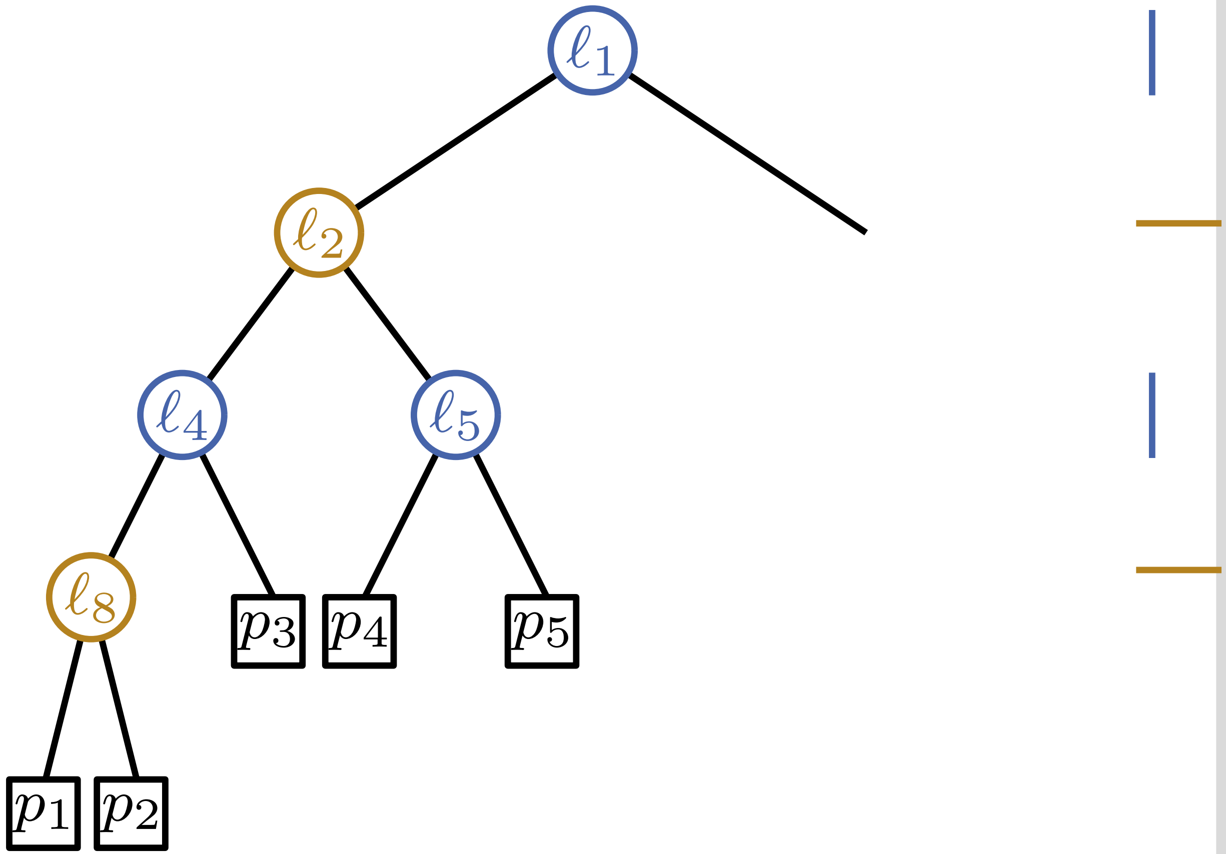

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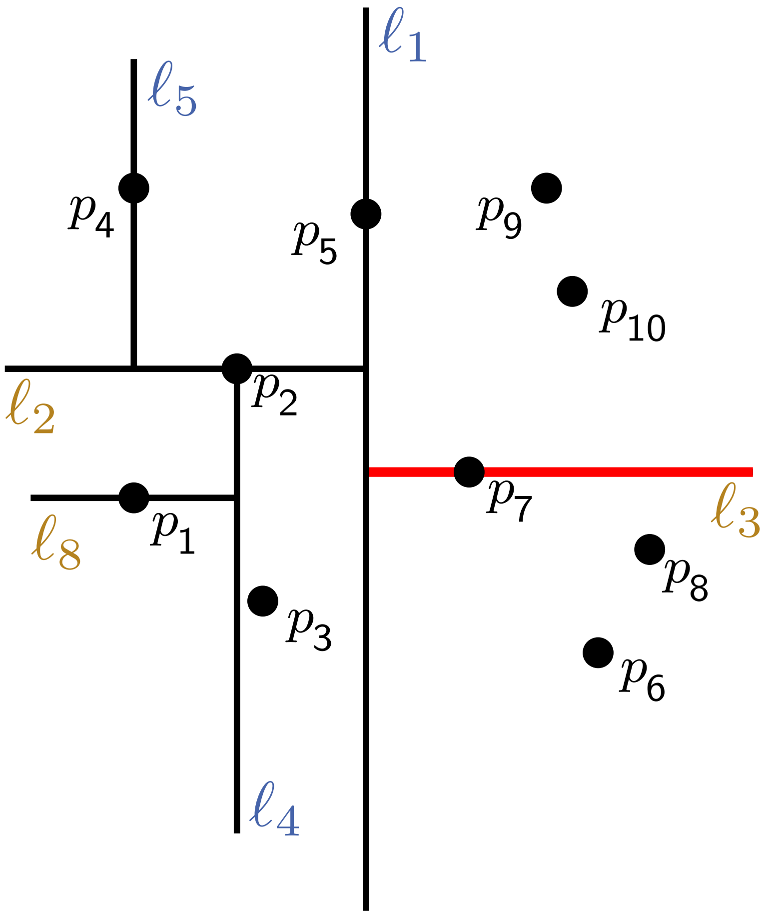

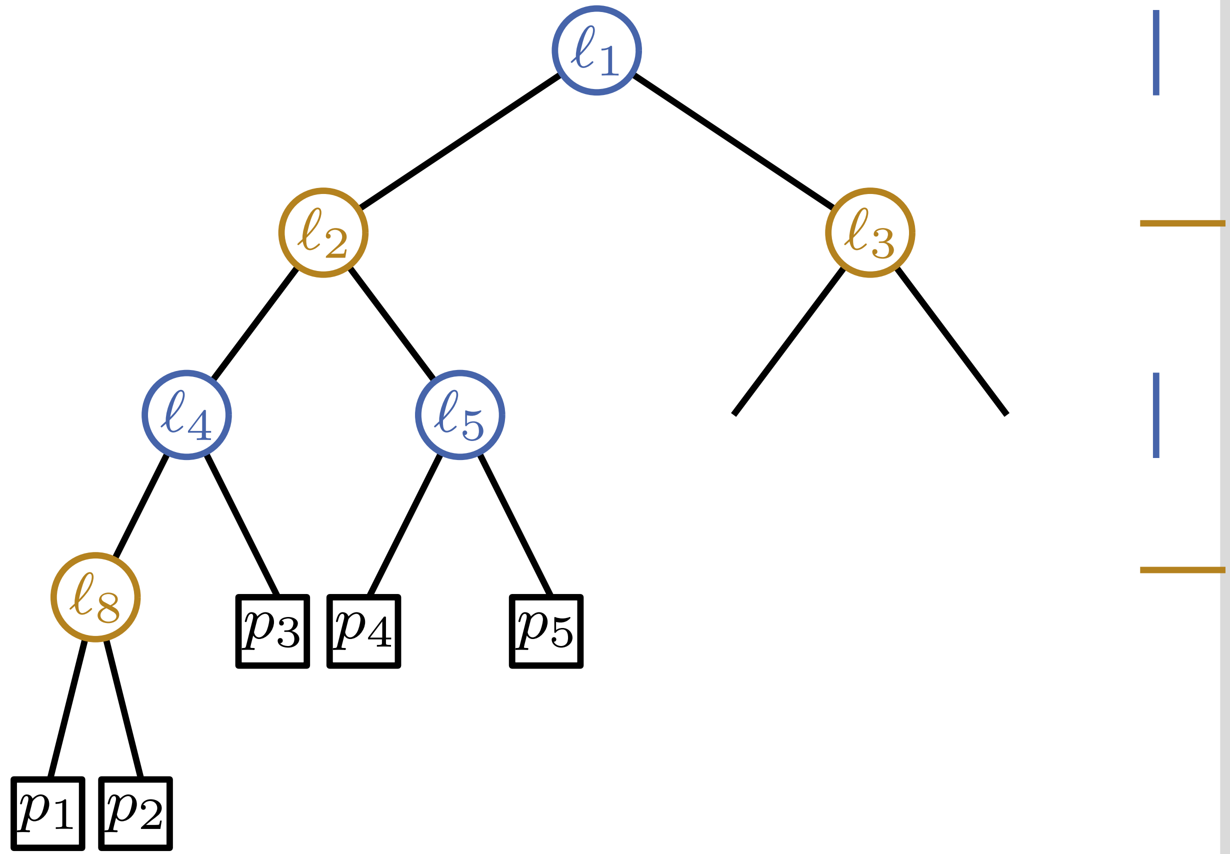

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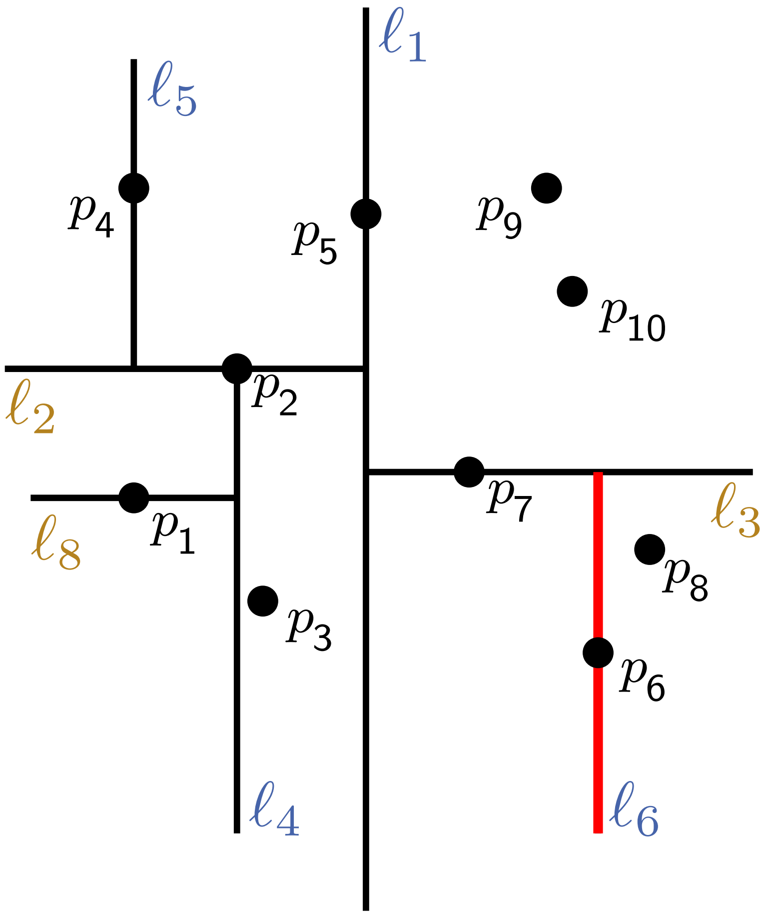

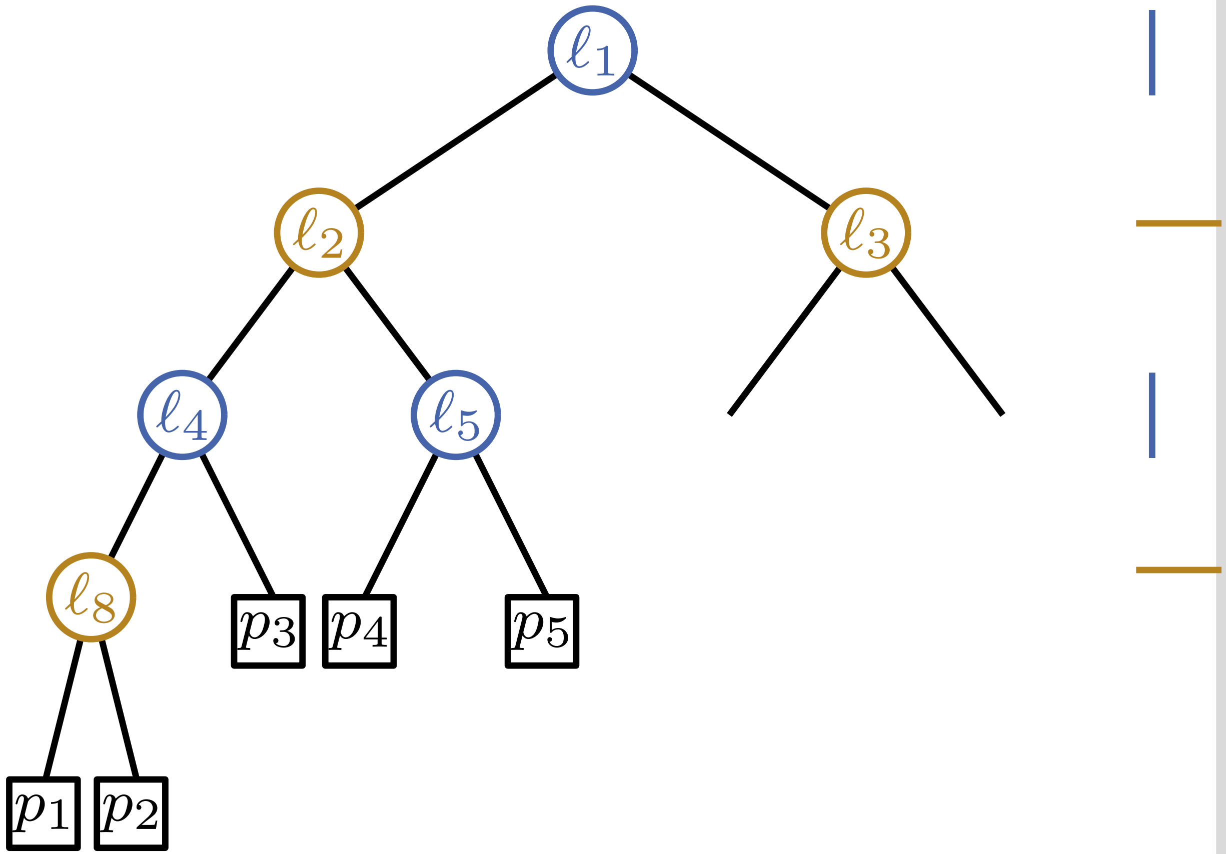

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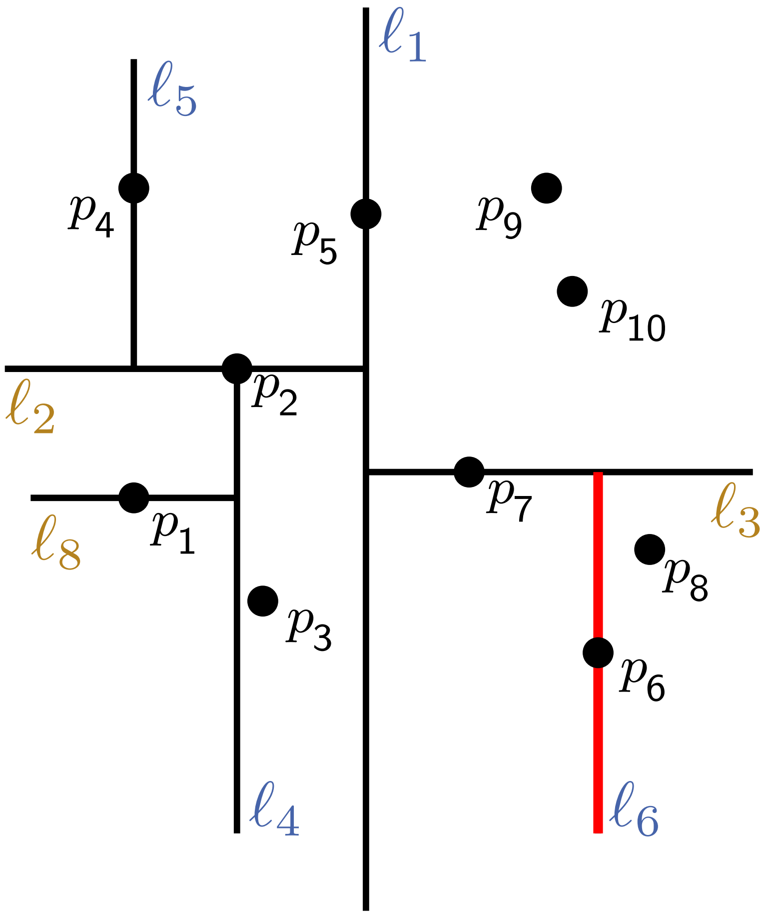

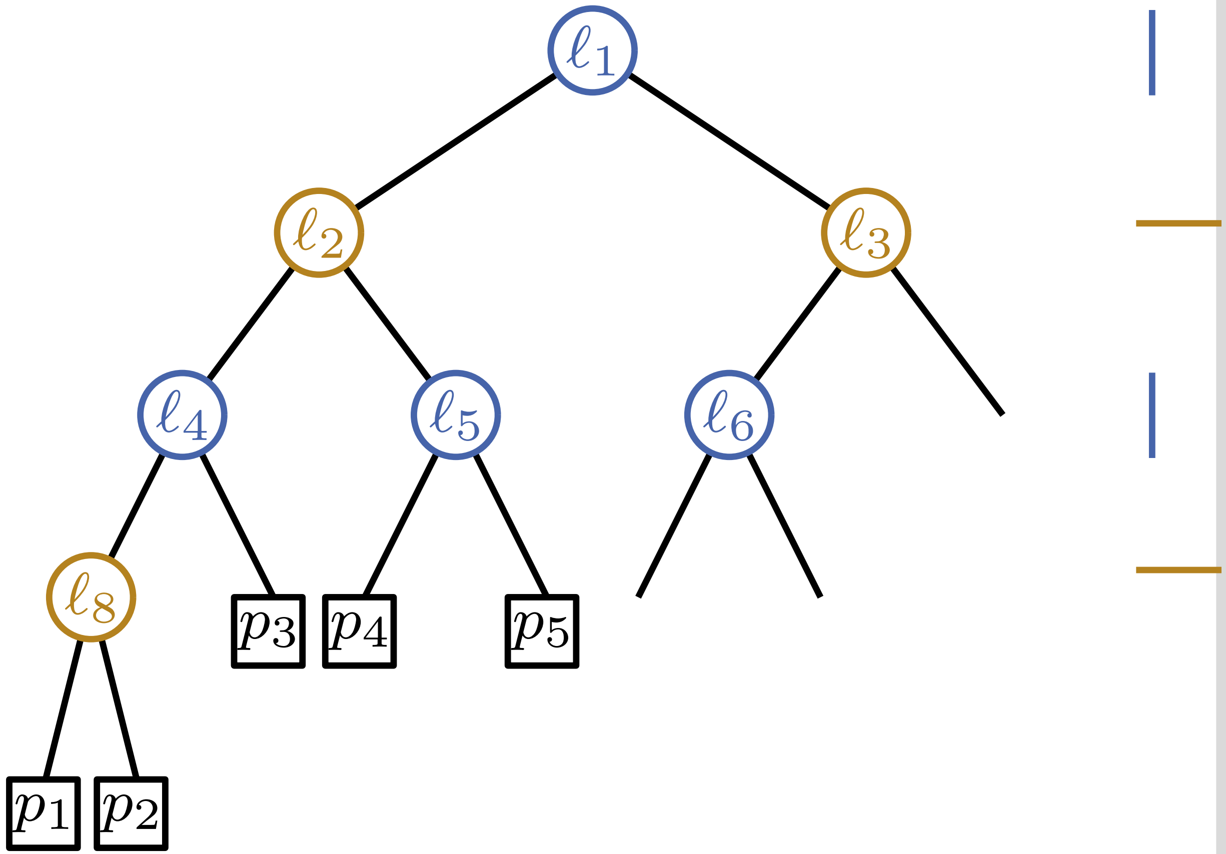

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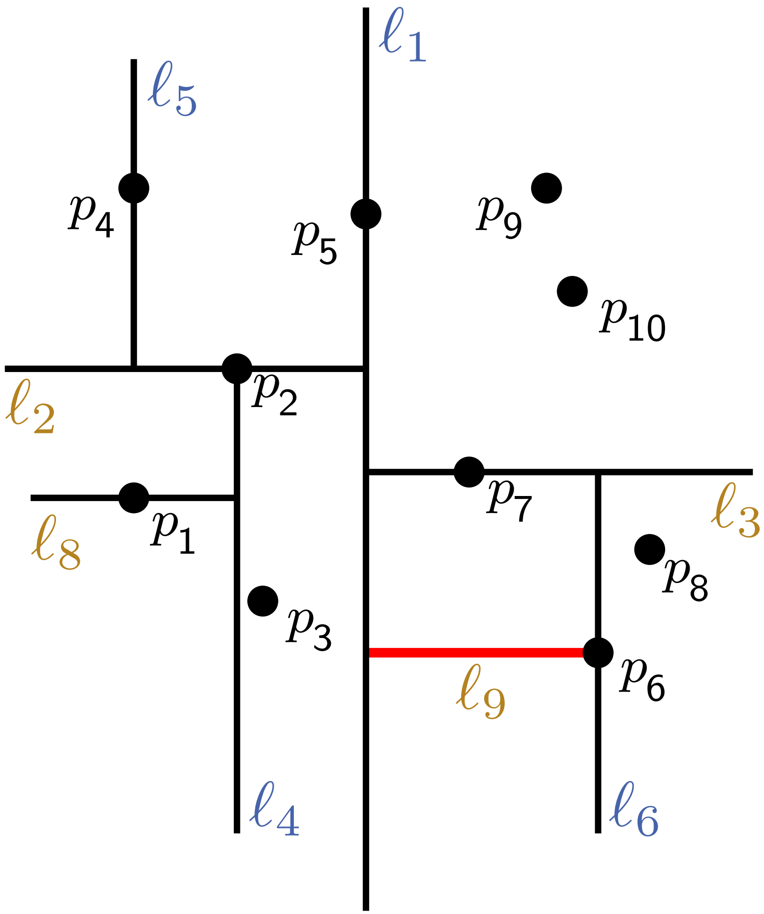

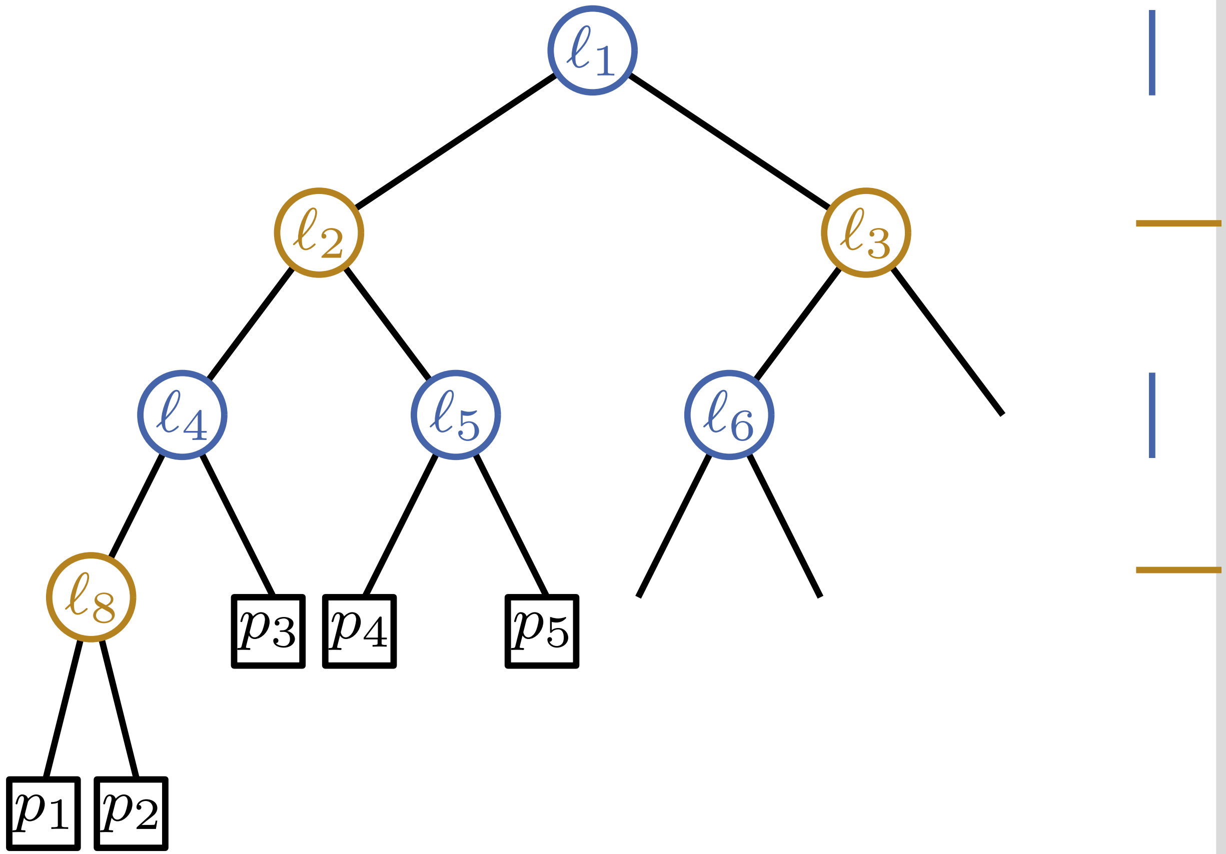

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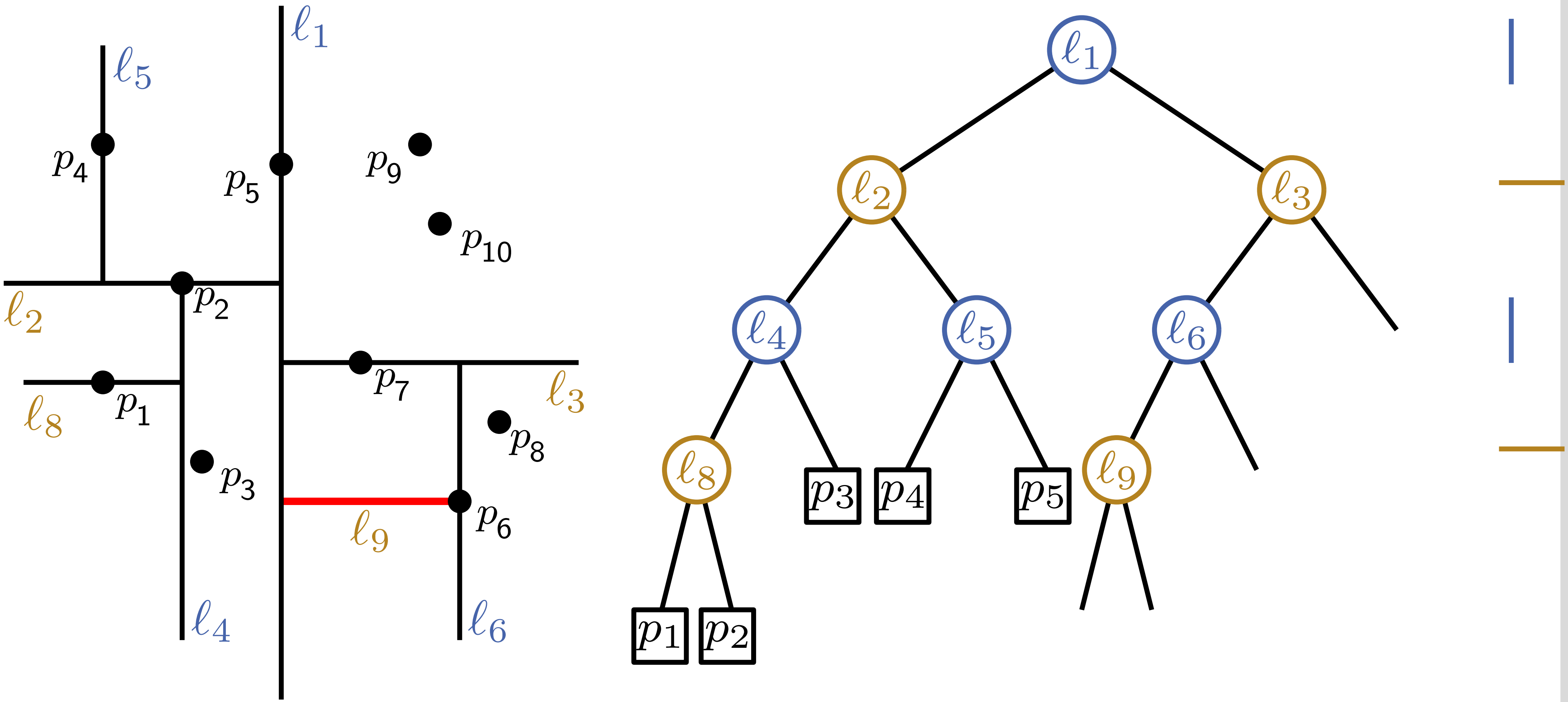

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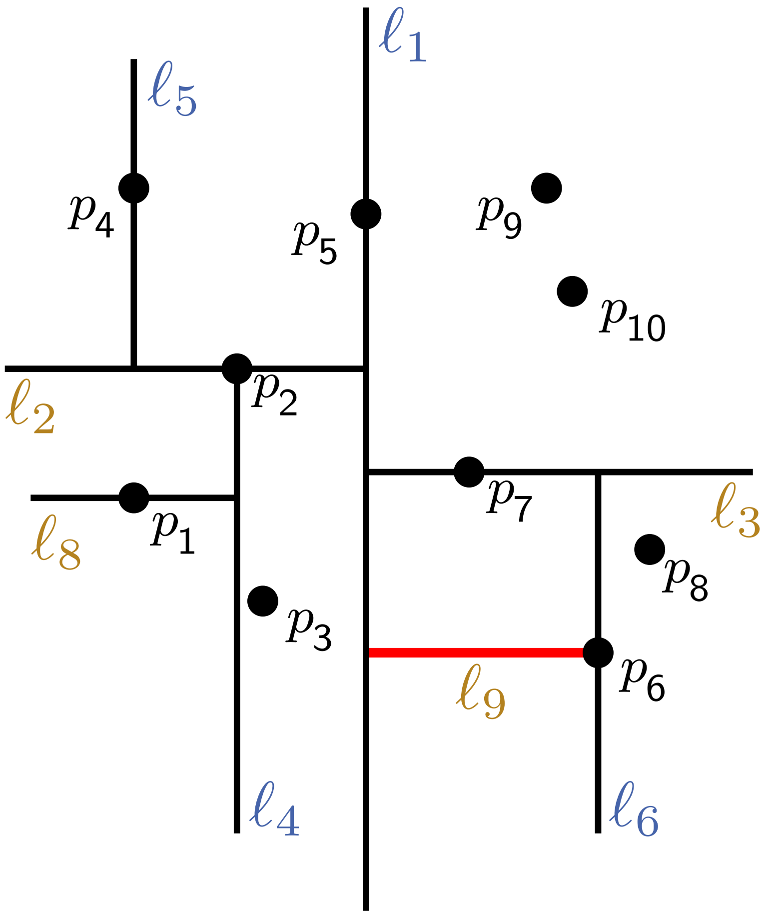

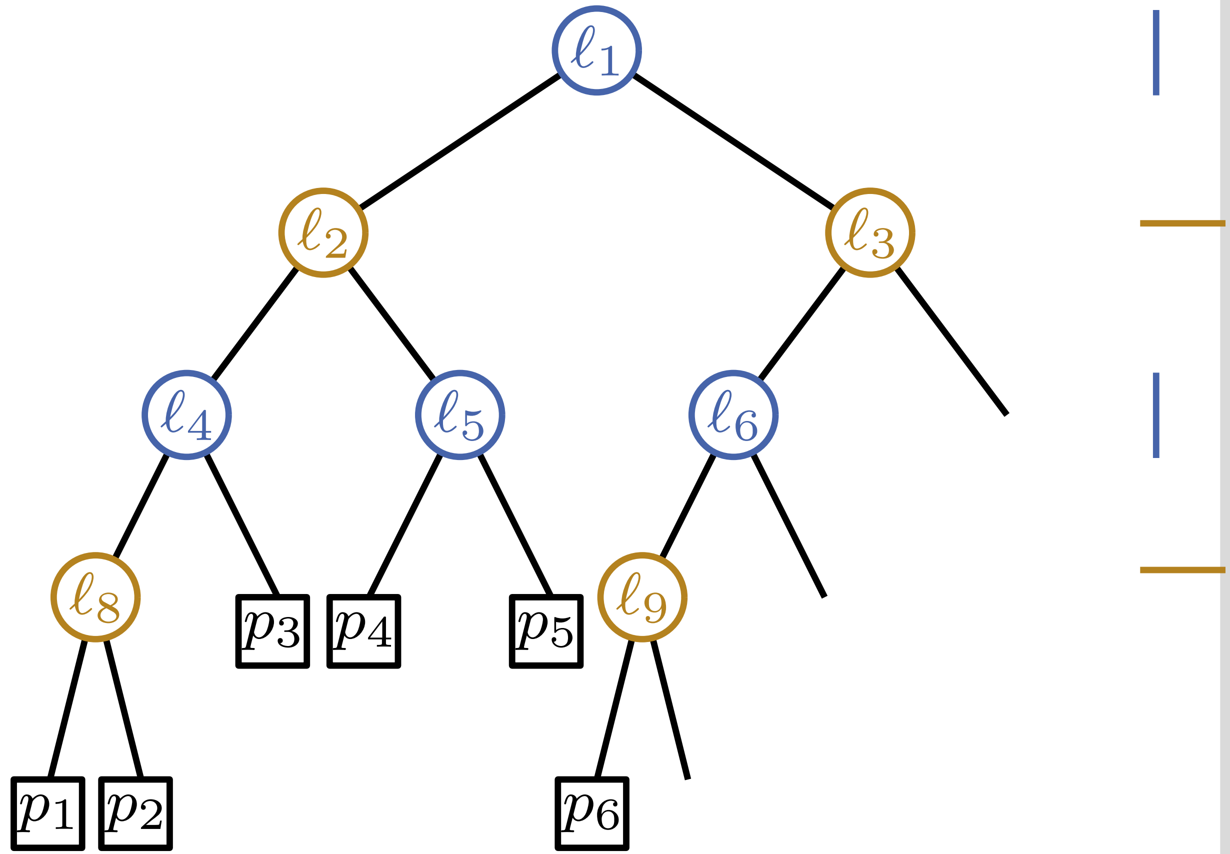

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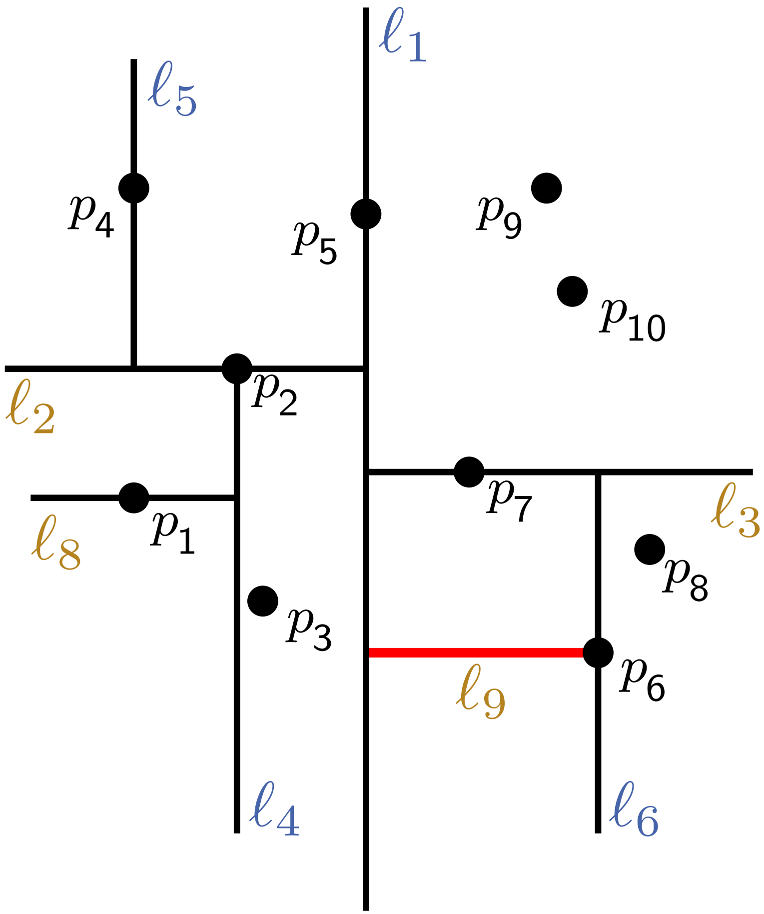

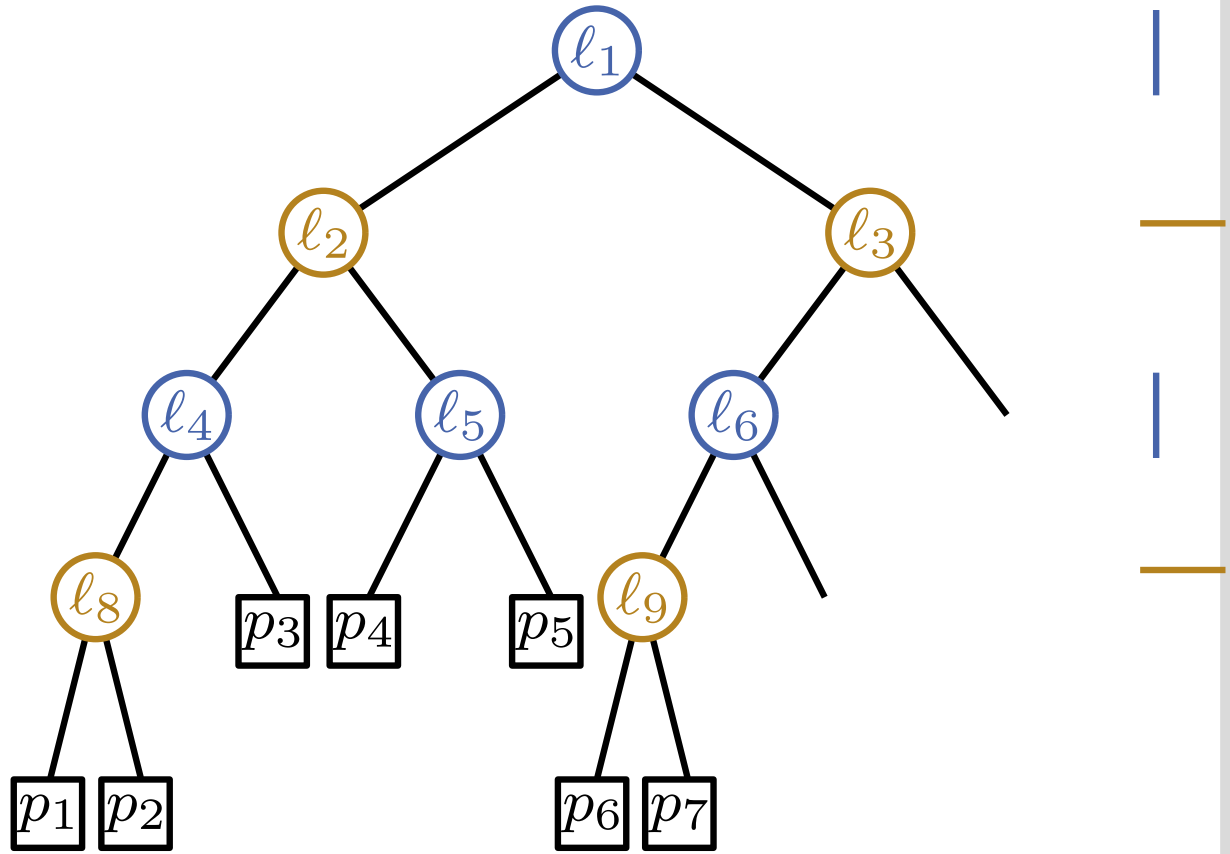

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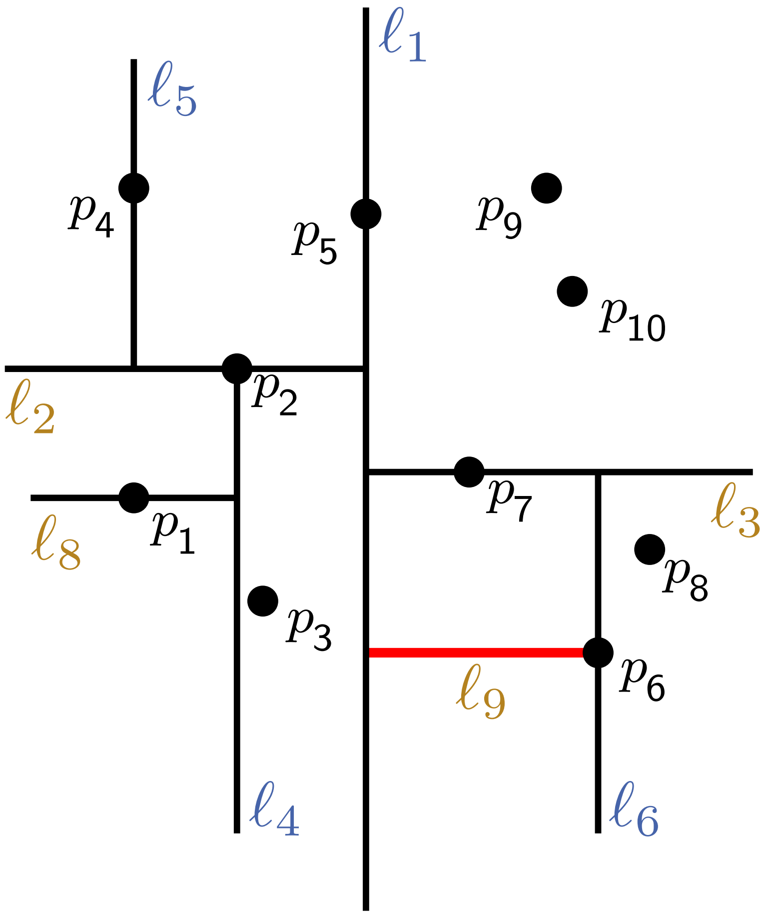

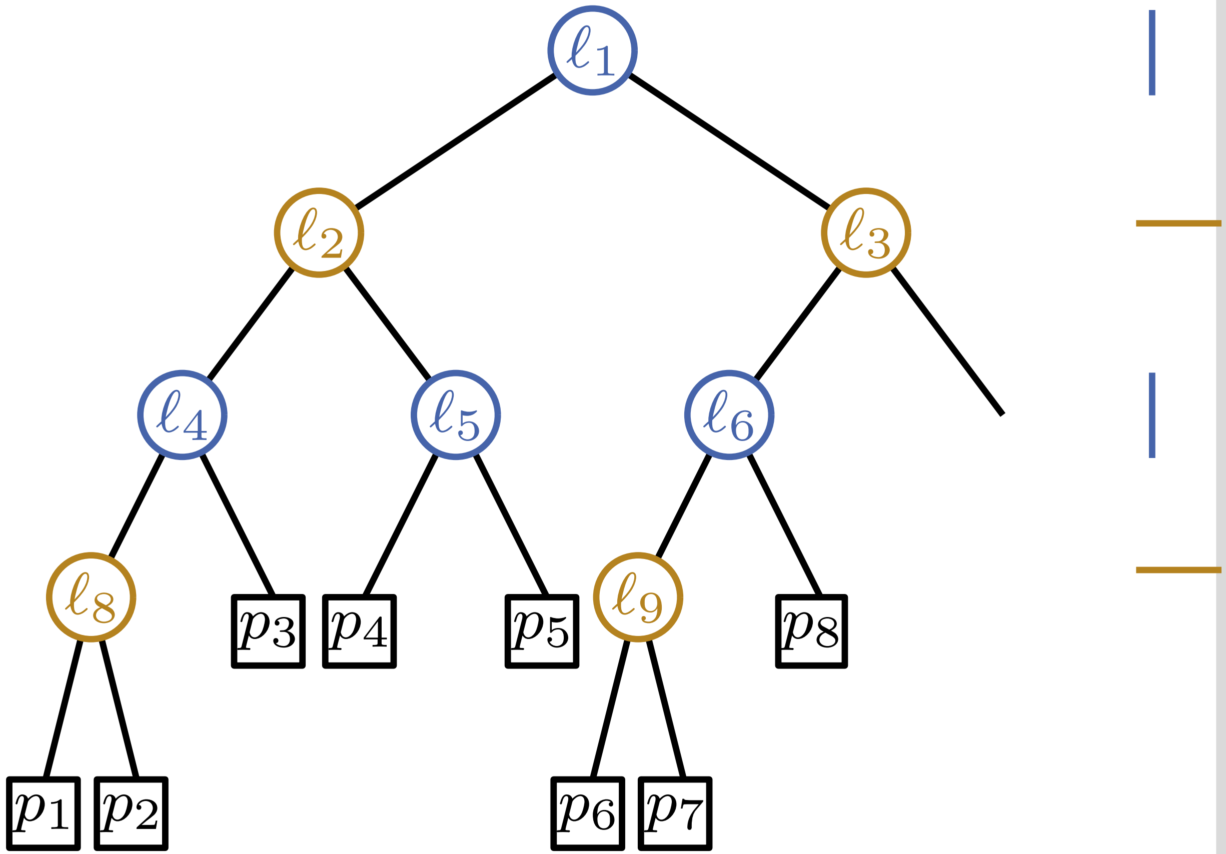

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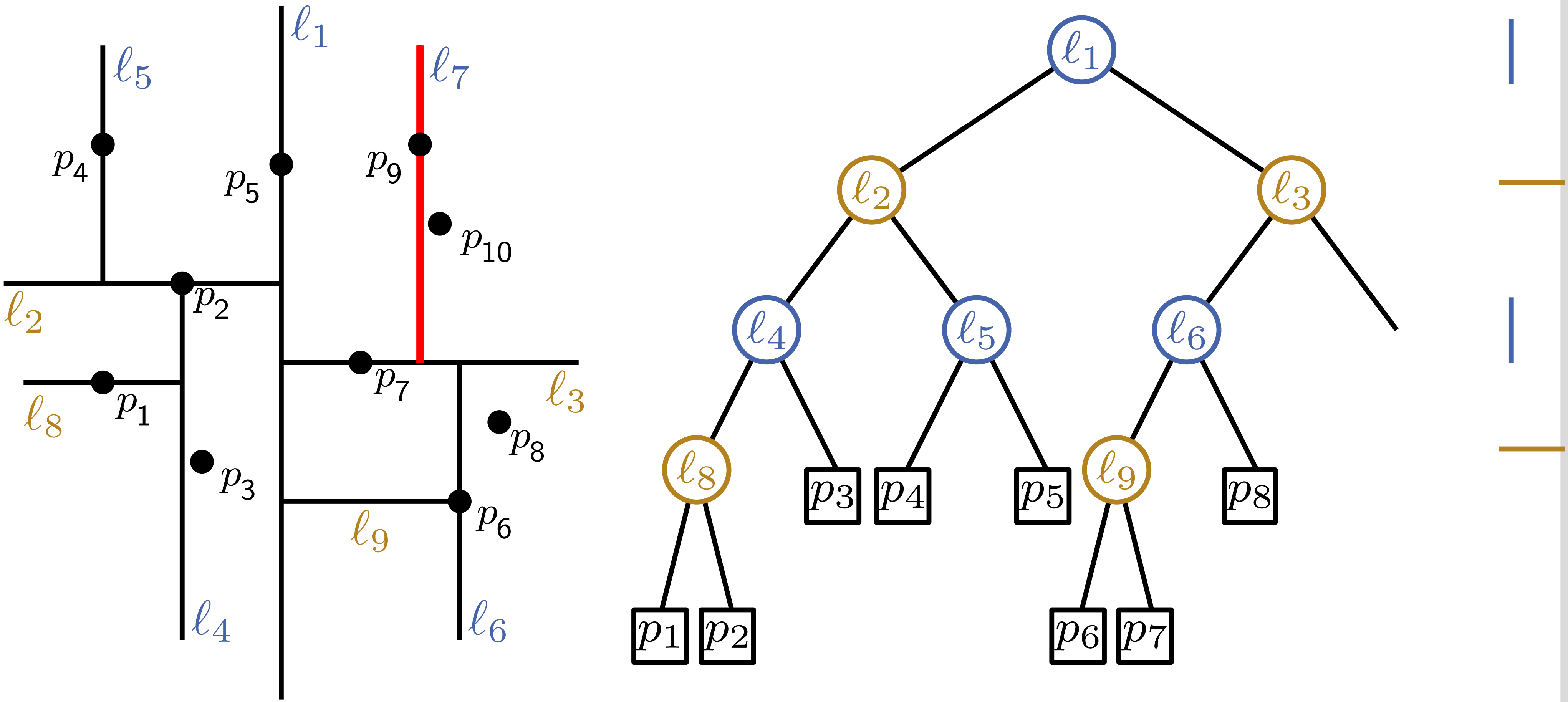

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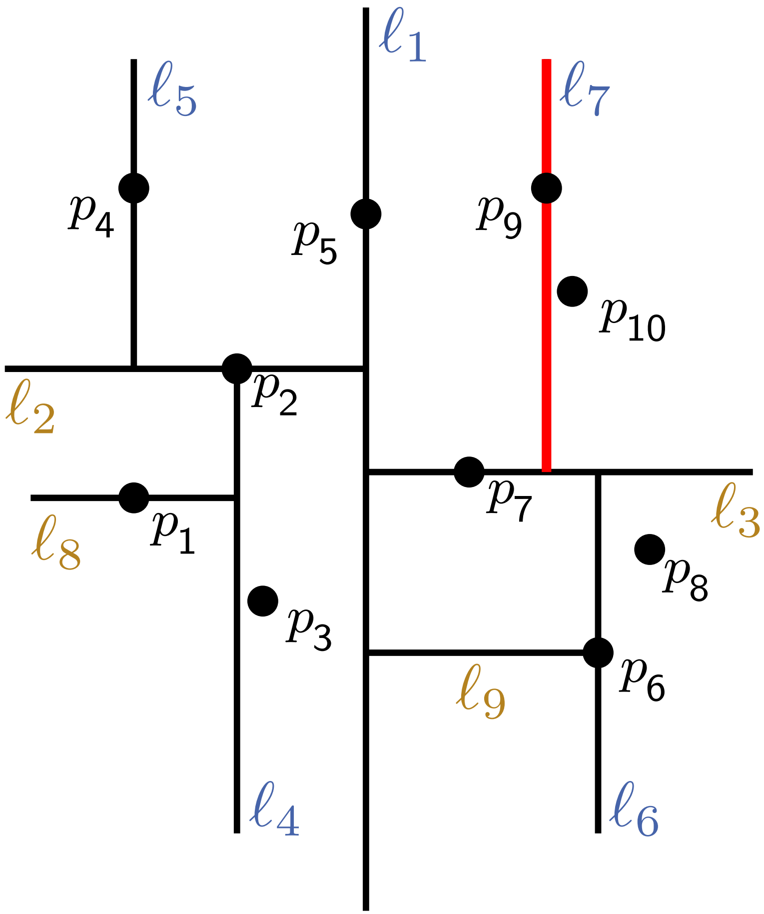

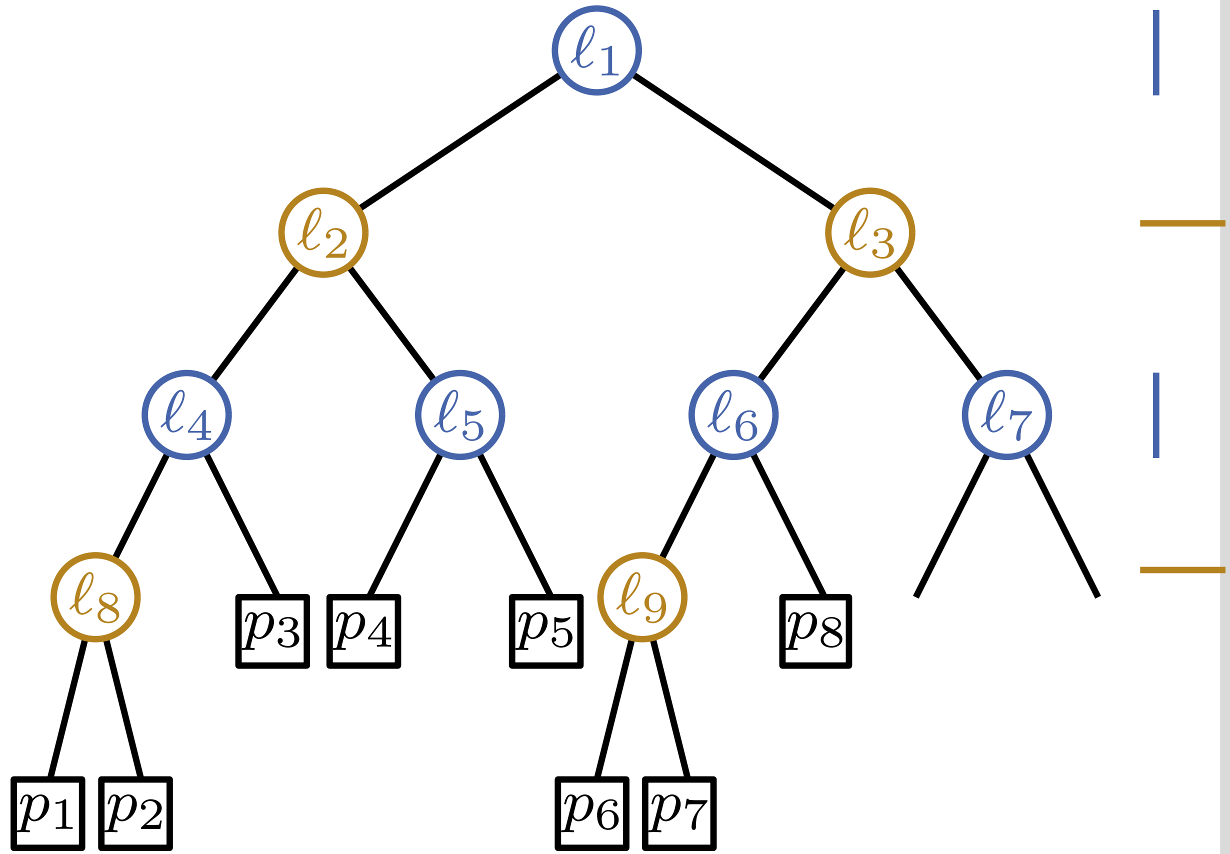

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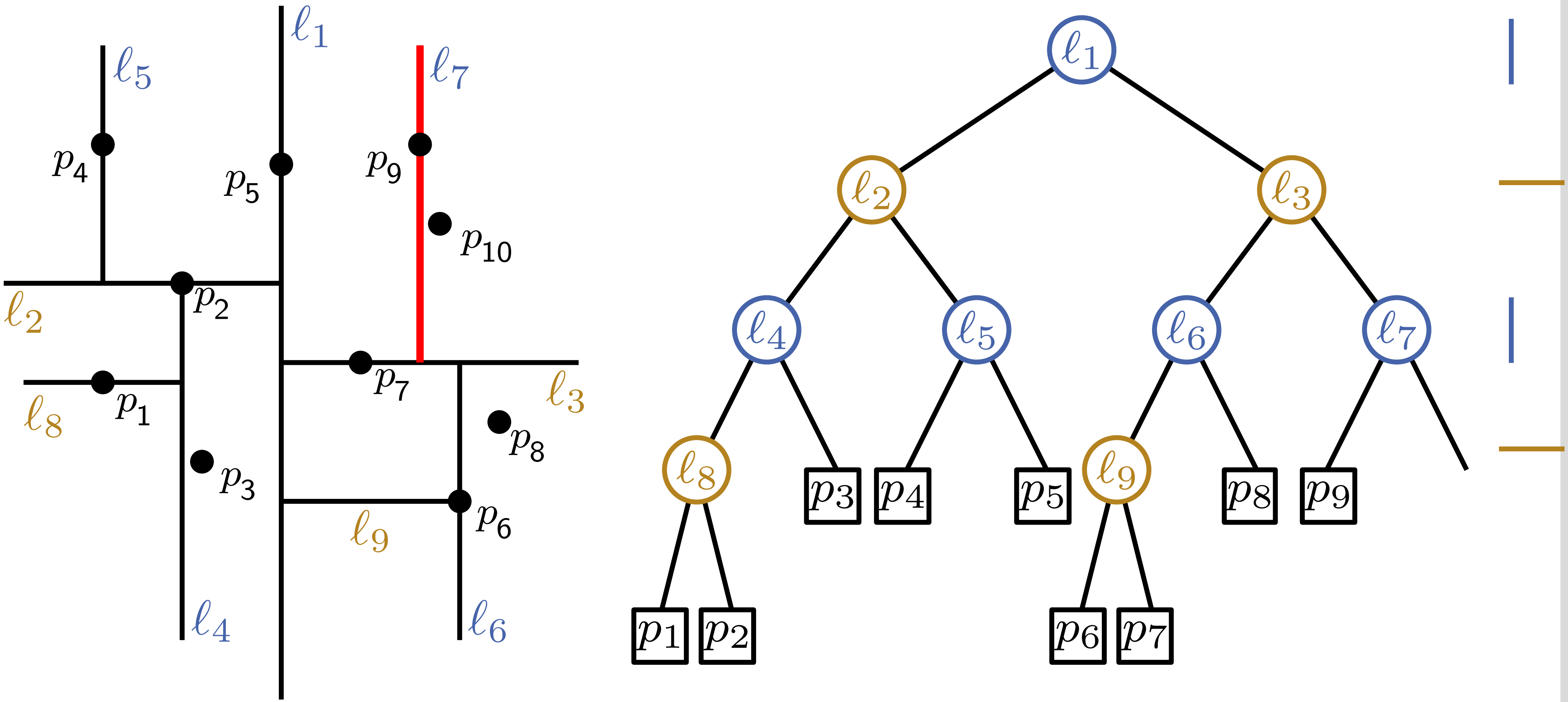

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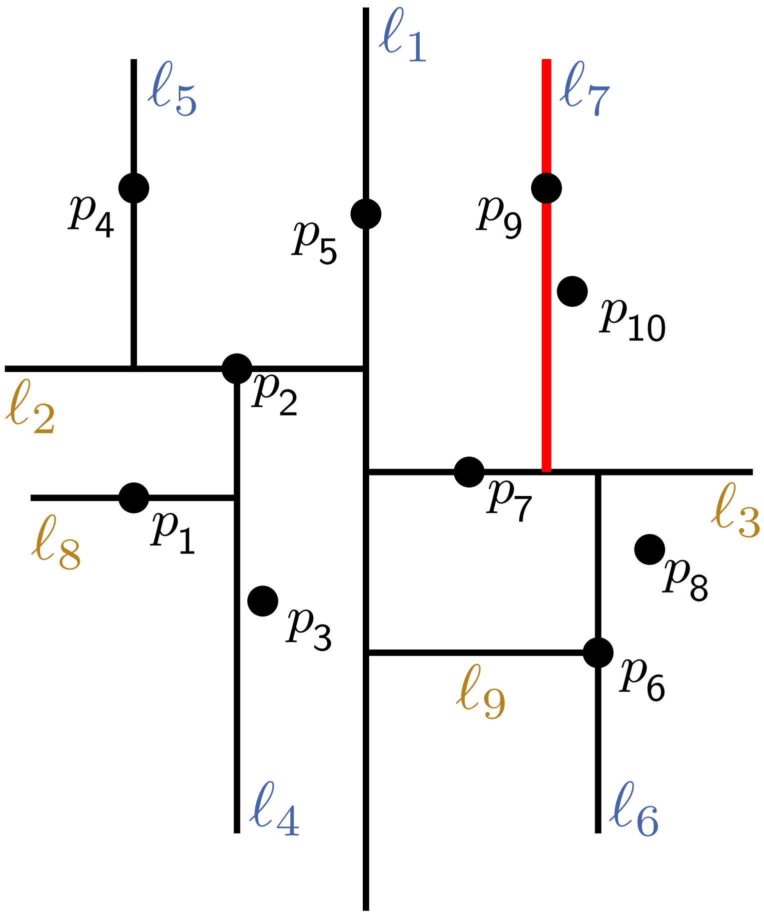
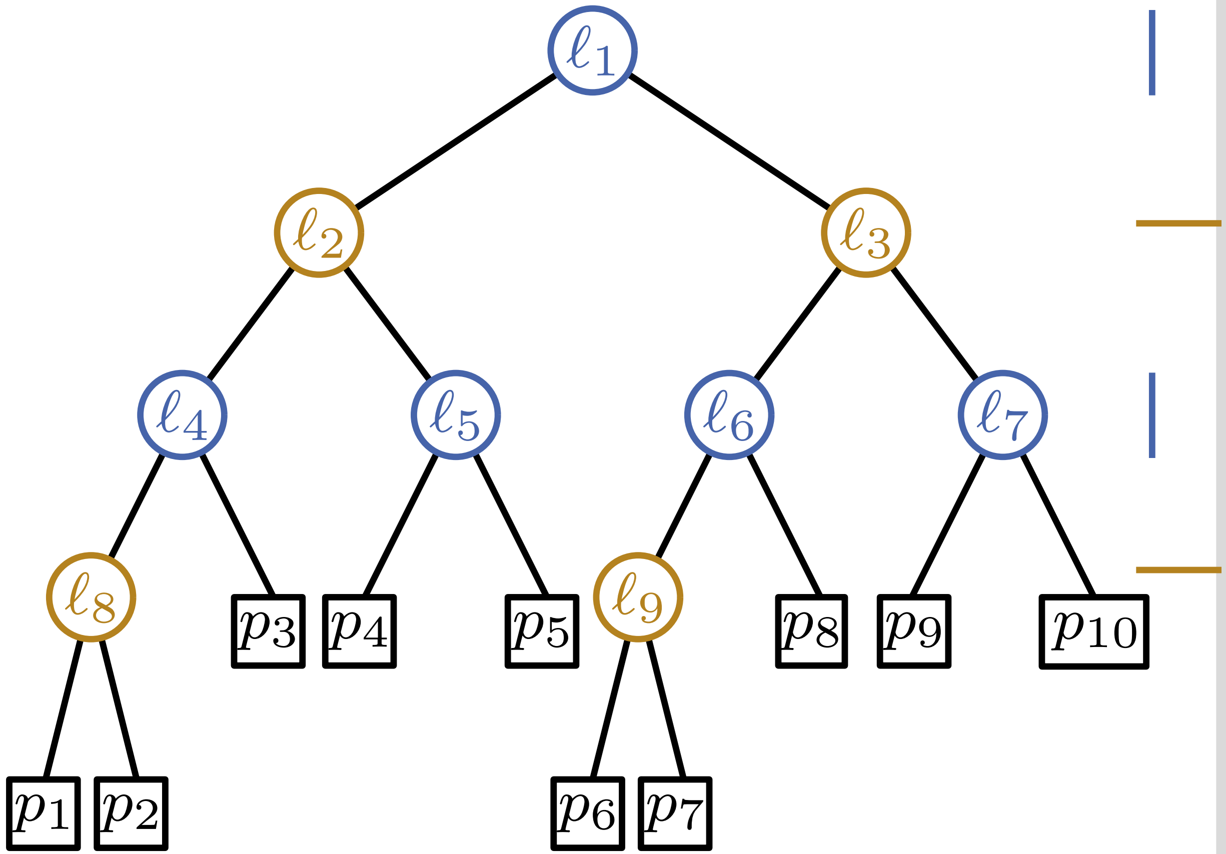
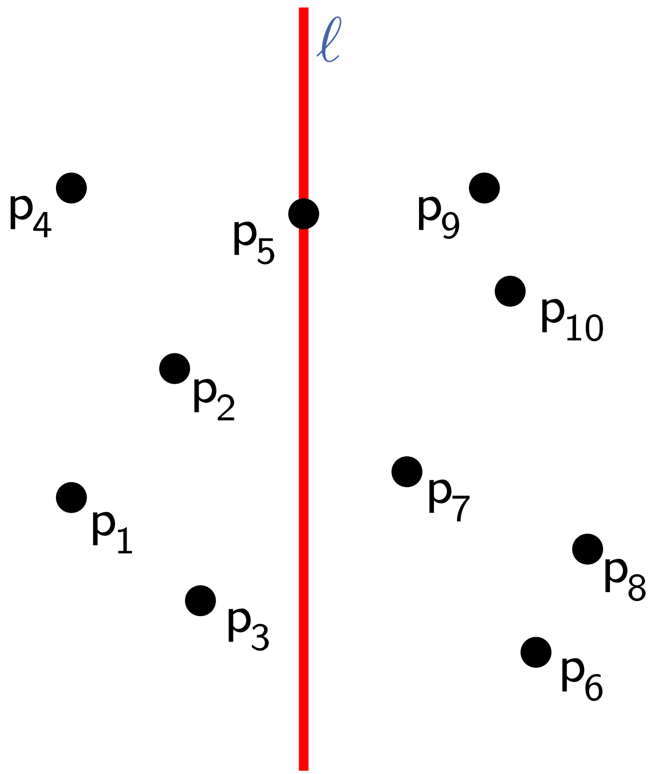


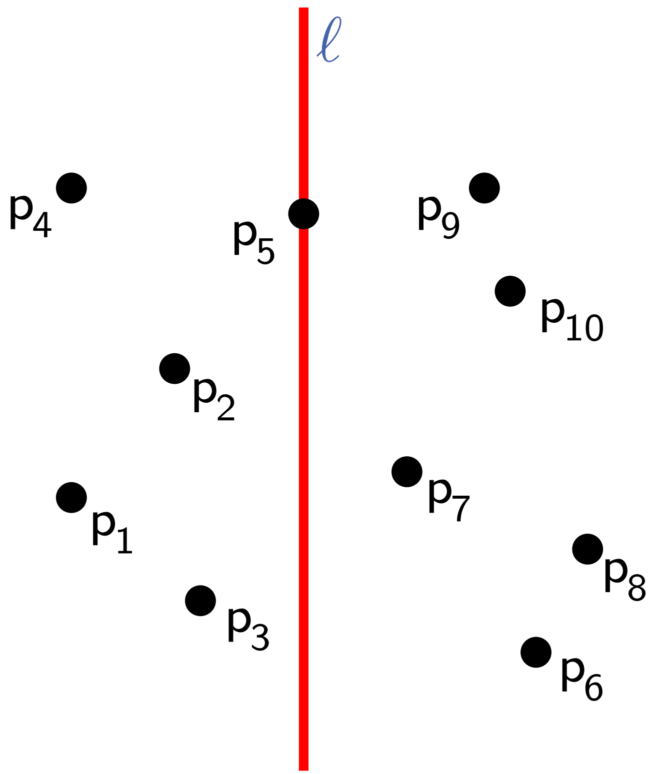


kd-Trees: Construction



BuildKdTree(P, $depth$)

kd-Trees: Construction



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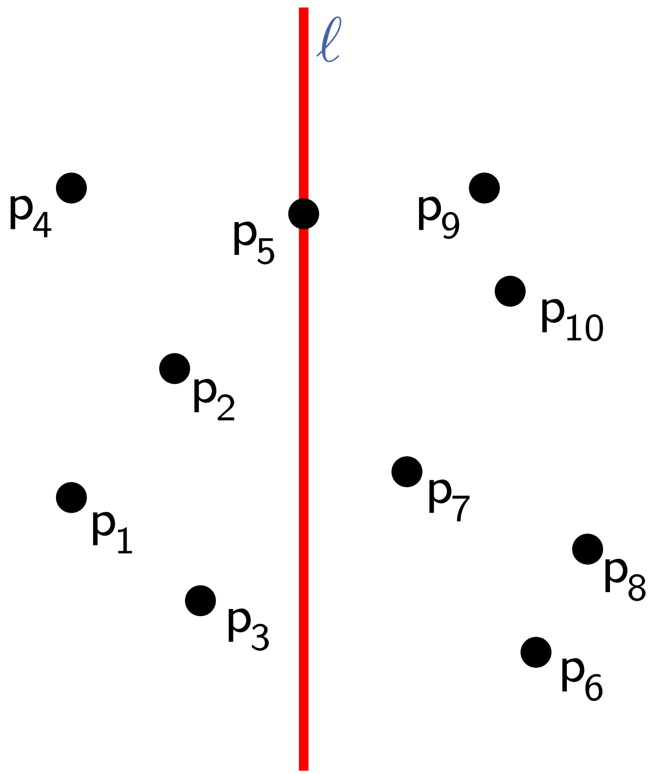

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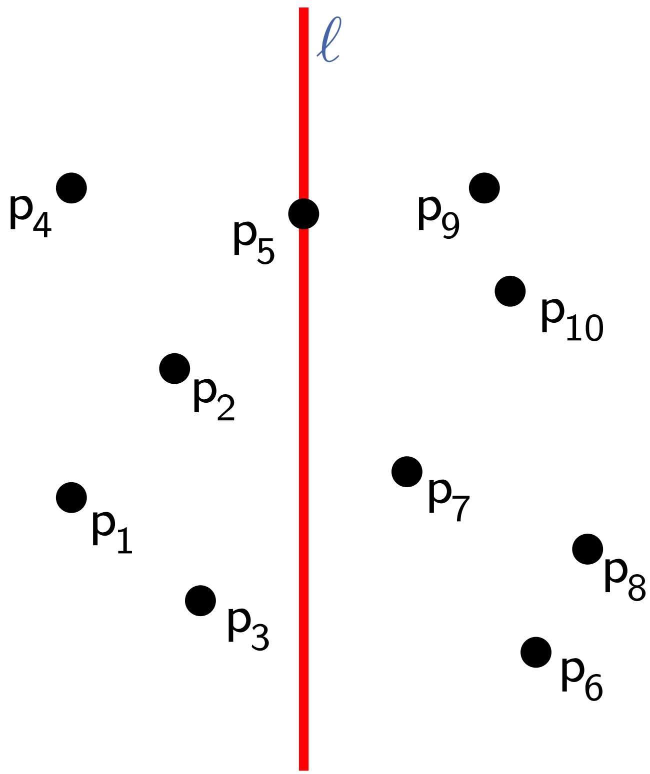

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Point $\lceil |P|/2 \rceil$

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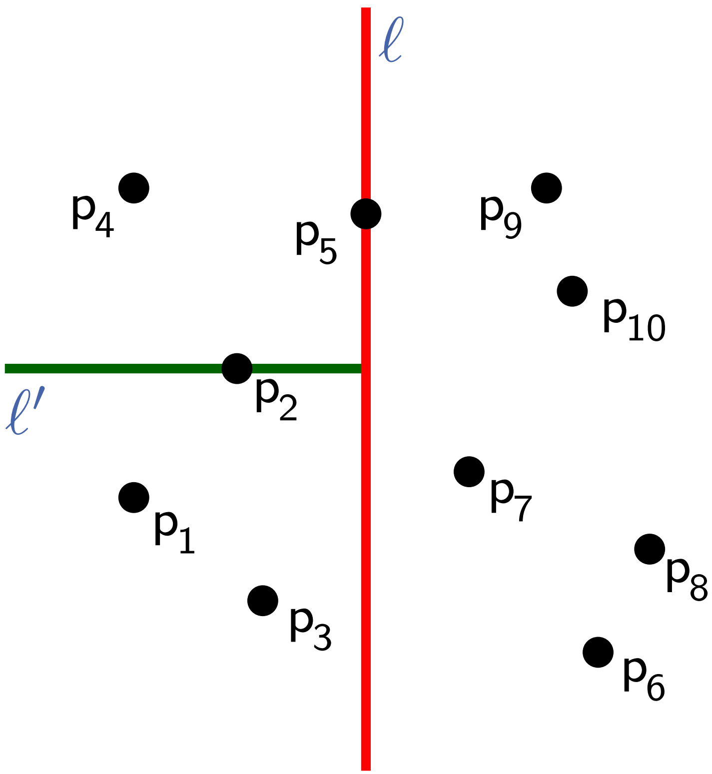

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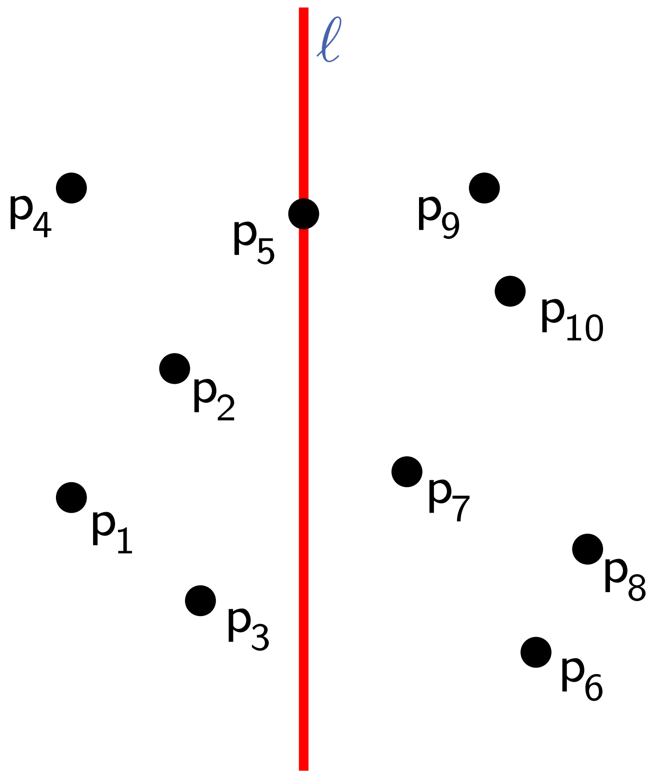

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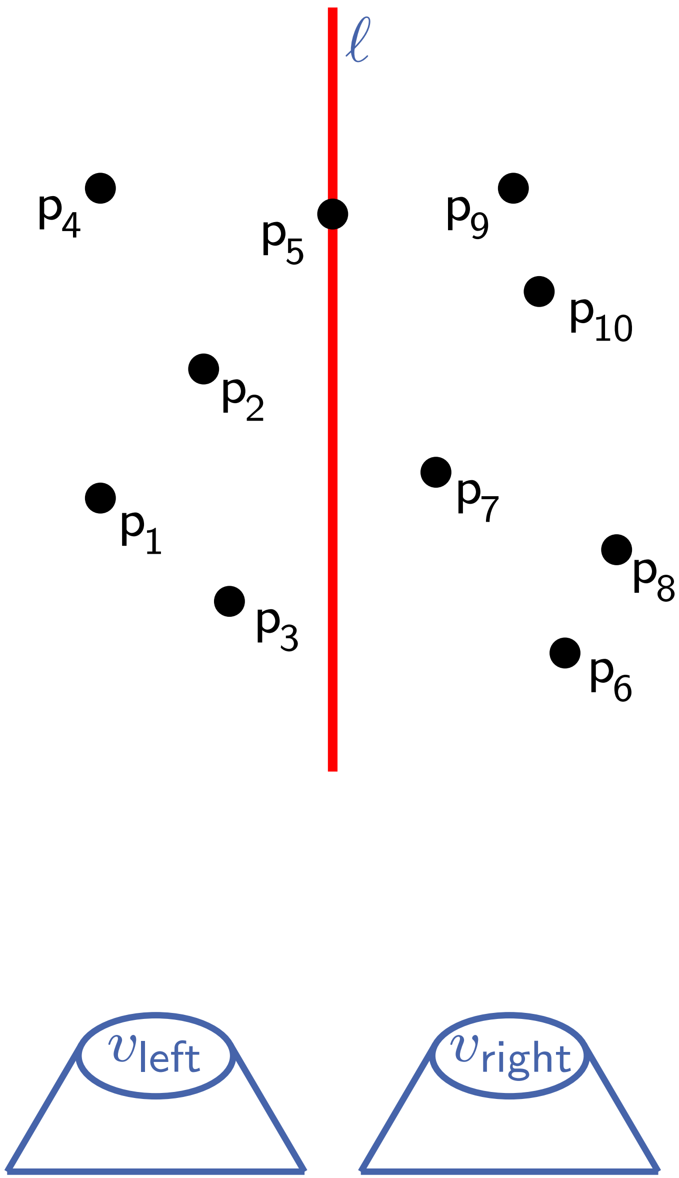

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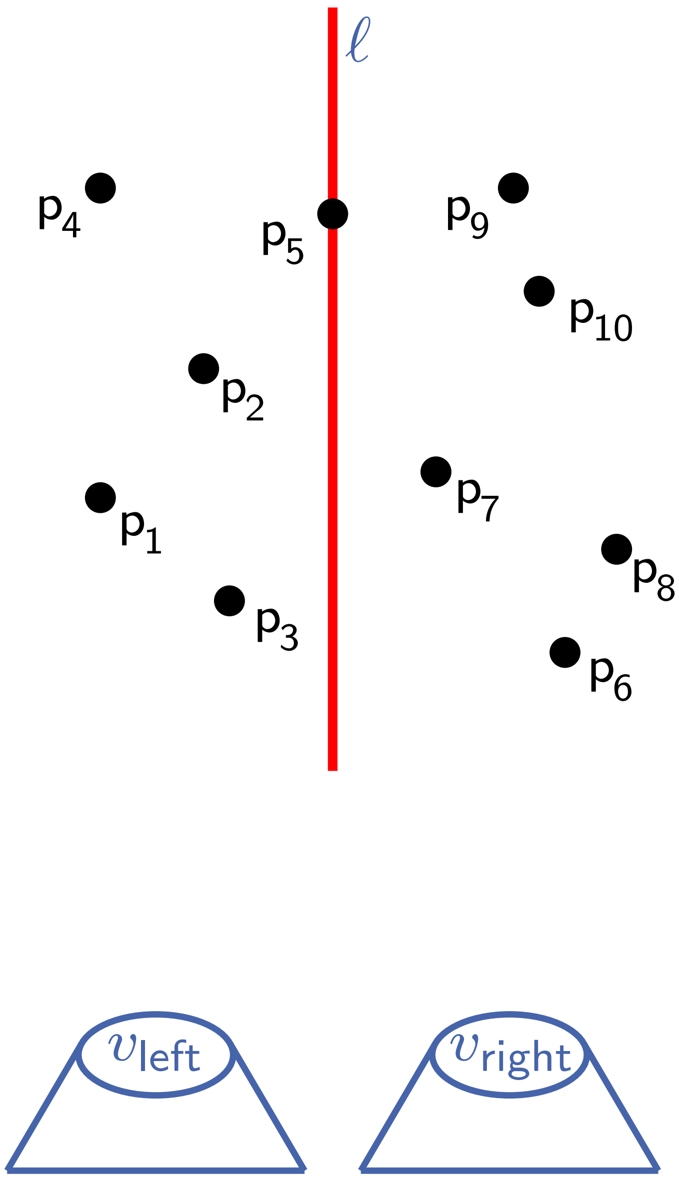

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```
if |P| = 1 then
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else
    if depth is even then
        divide P vertically at
         ℓ : x = x_median(P) in
        P1 (Points left to or on ℓ) and
         P2 = P \ P1
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    v_left ← BuildKdTree(P1, depth + 1)
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    Create node v, which stores ℓ
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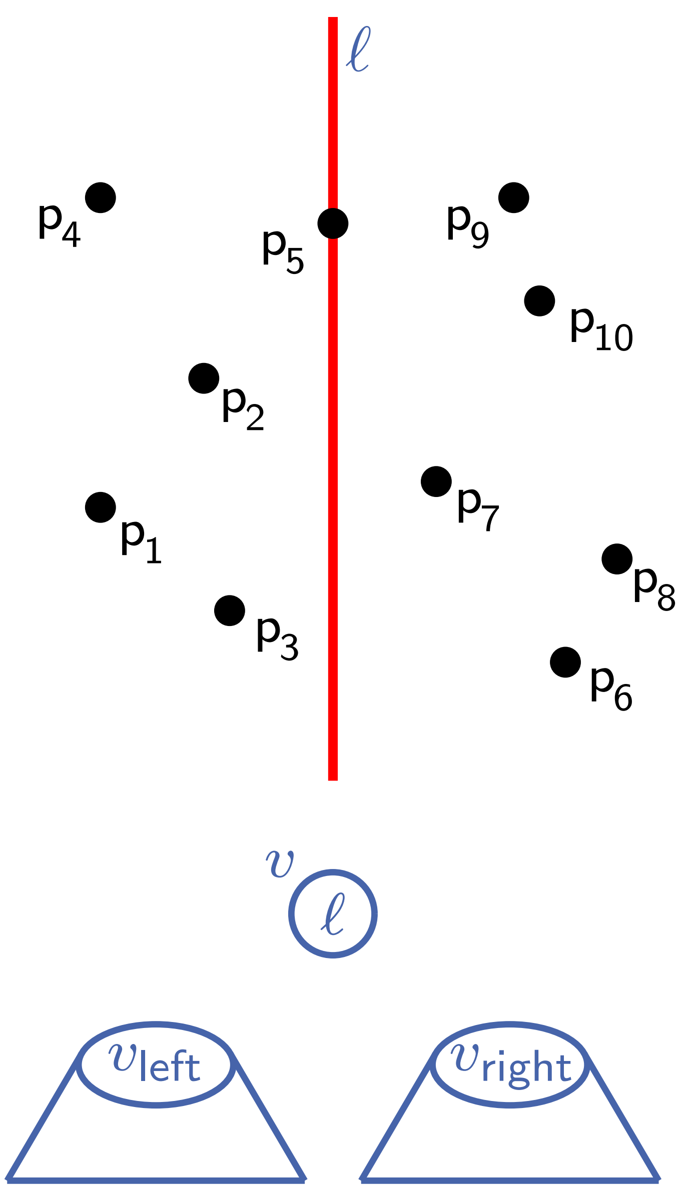
kd-Trees: Construction



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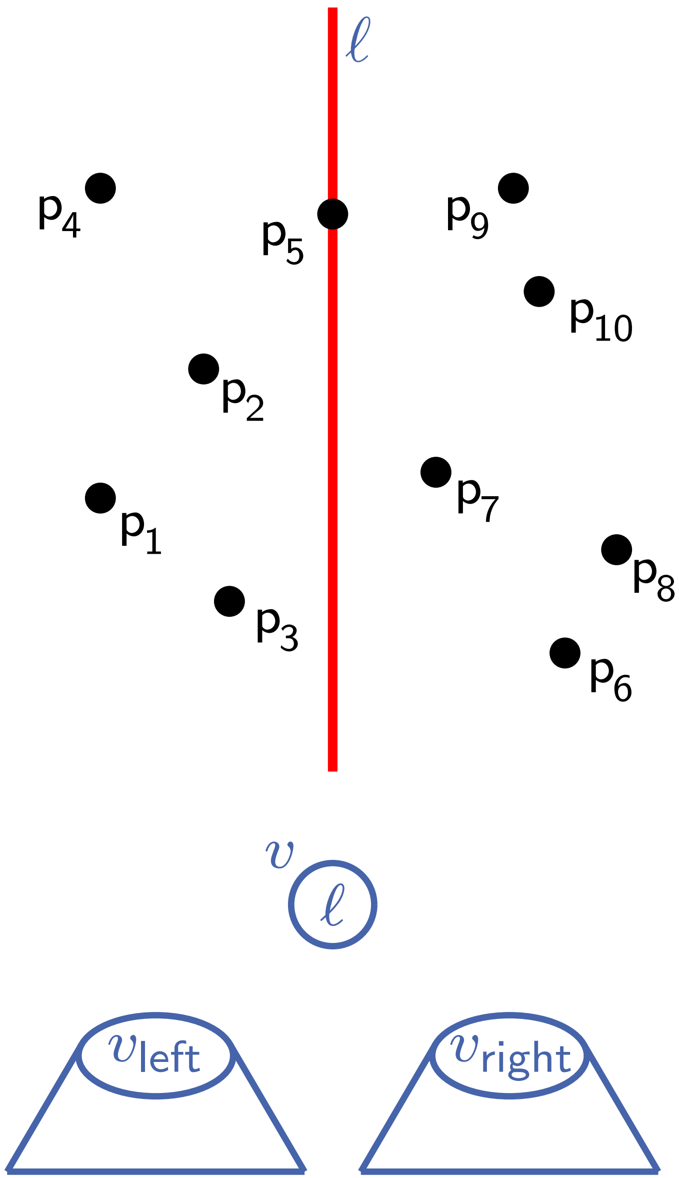

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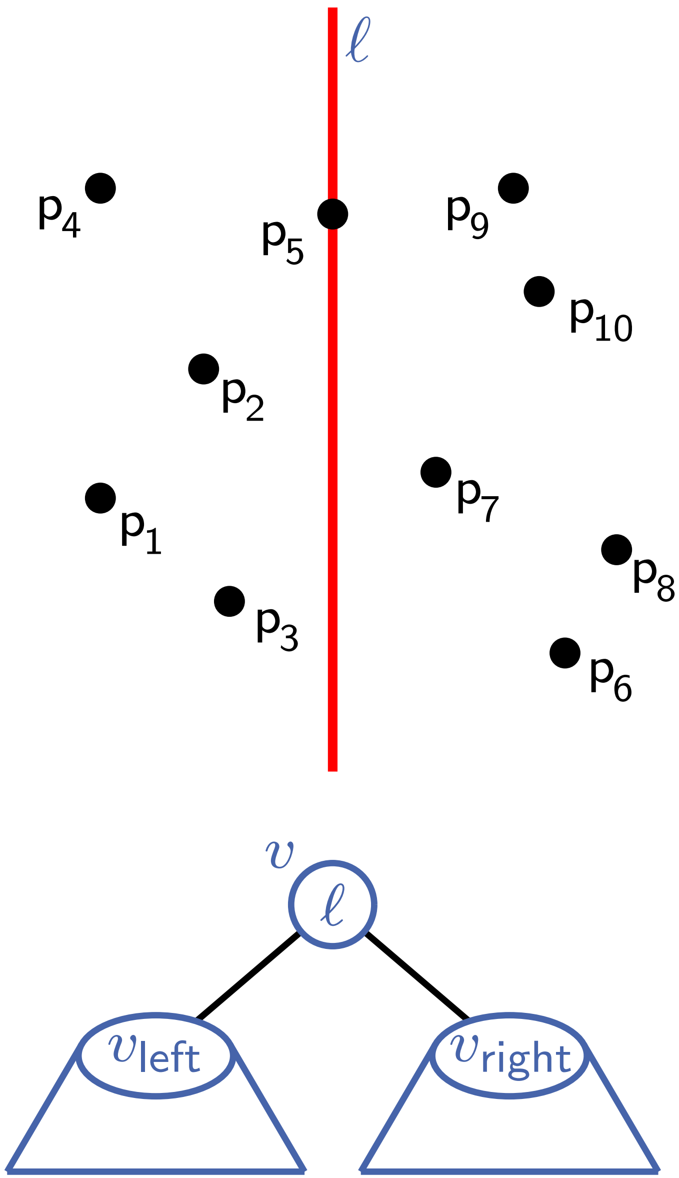

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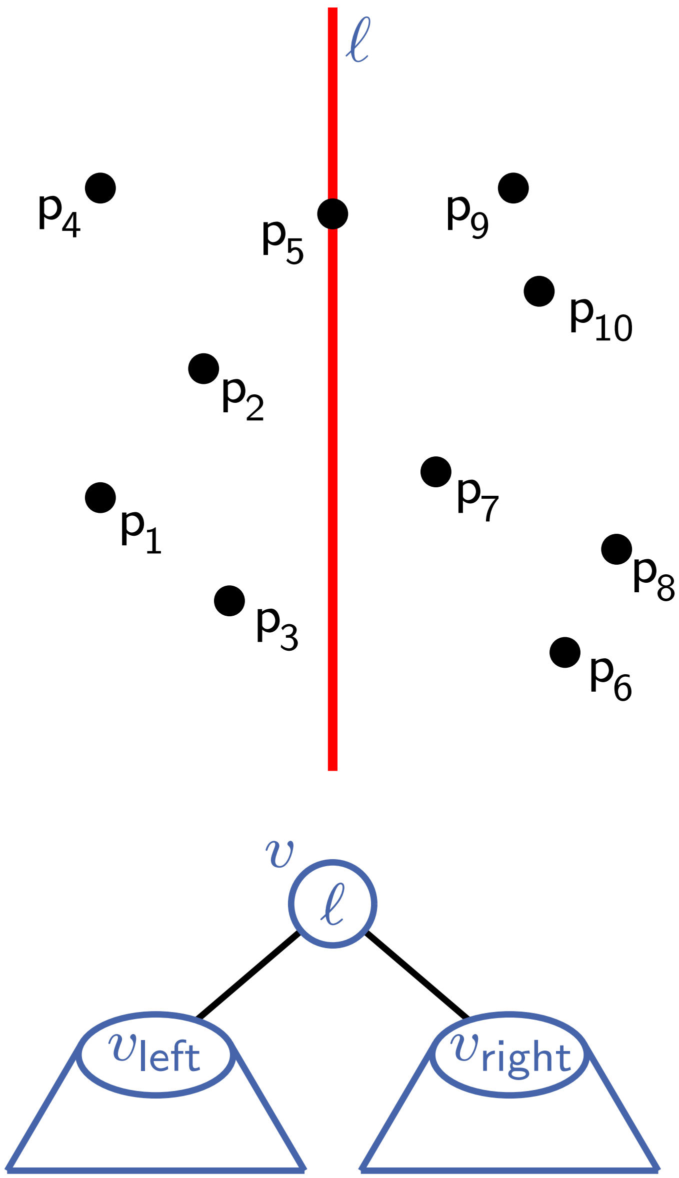

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Analysis of kd-Tree Construction

Lemma 1: A kd-tree for n points in $\mathbb{R}^2$ can be constructed in $O(n \log n)$ time, using $O(n)$ space.

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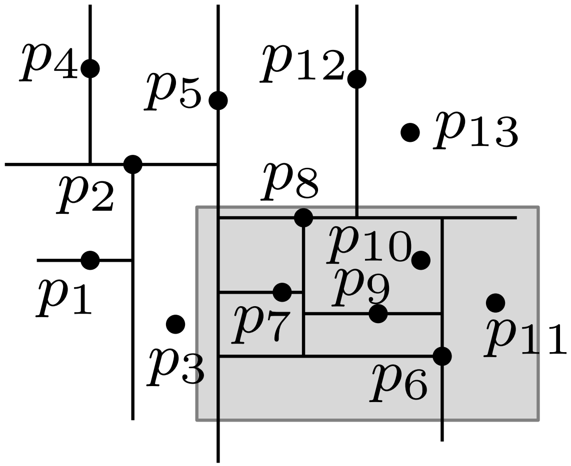
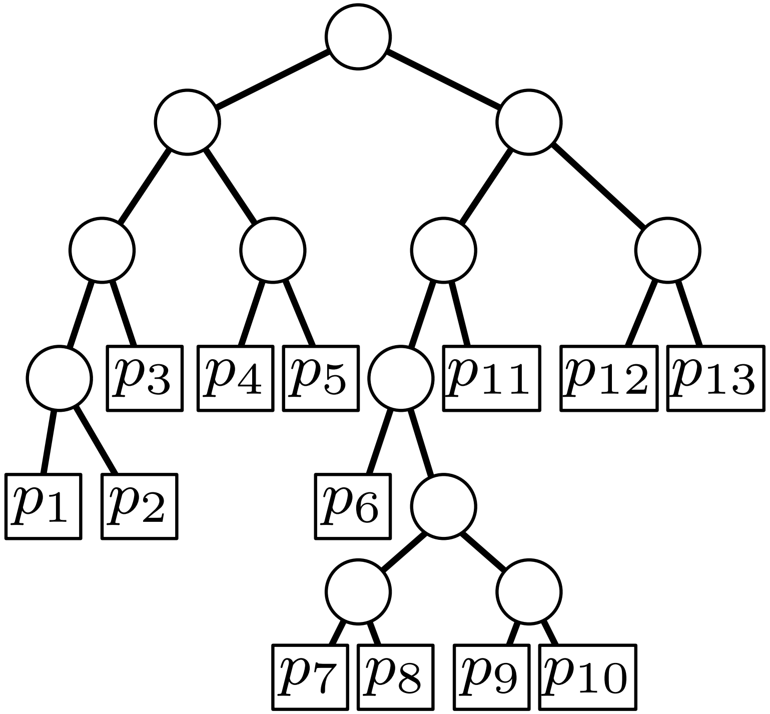
- Linear space, since we are using a binary tree with n leaves.

Range Queries in a kd-Tree





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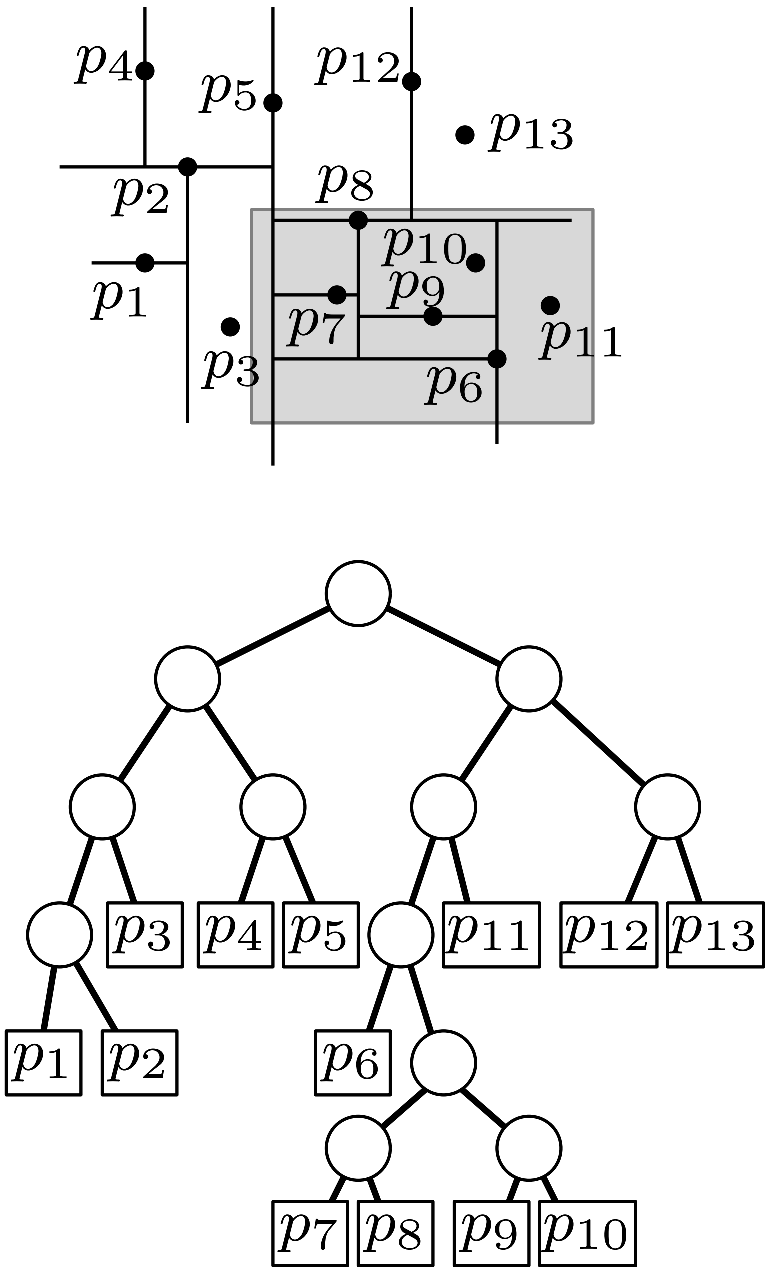

SearchKdTree(v, R)

if v leaf **then**
- report point p in v when $p \in R$

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  - ReportSubtree(lc(v))
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  - **if** region(lc(v))$\cap R \neq \emptyset$ **then**
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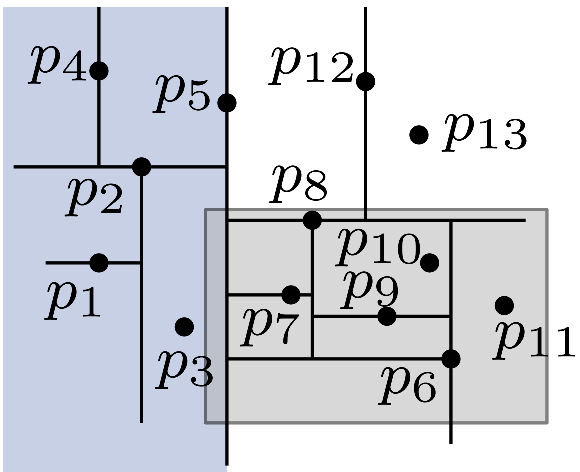

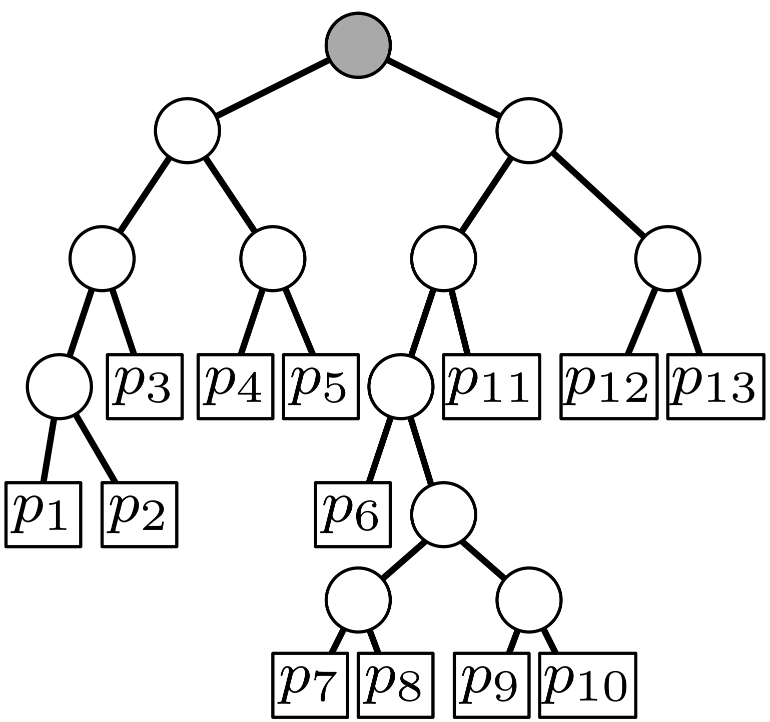

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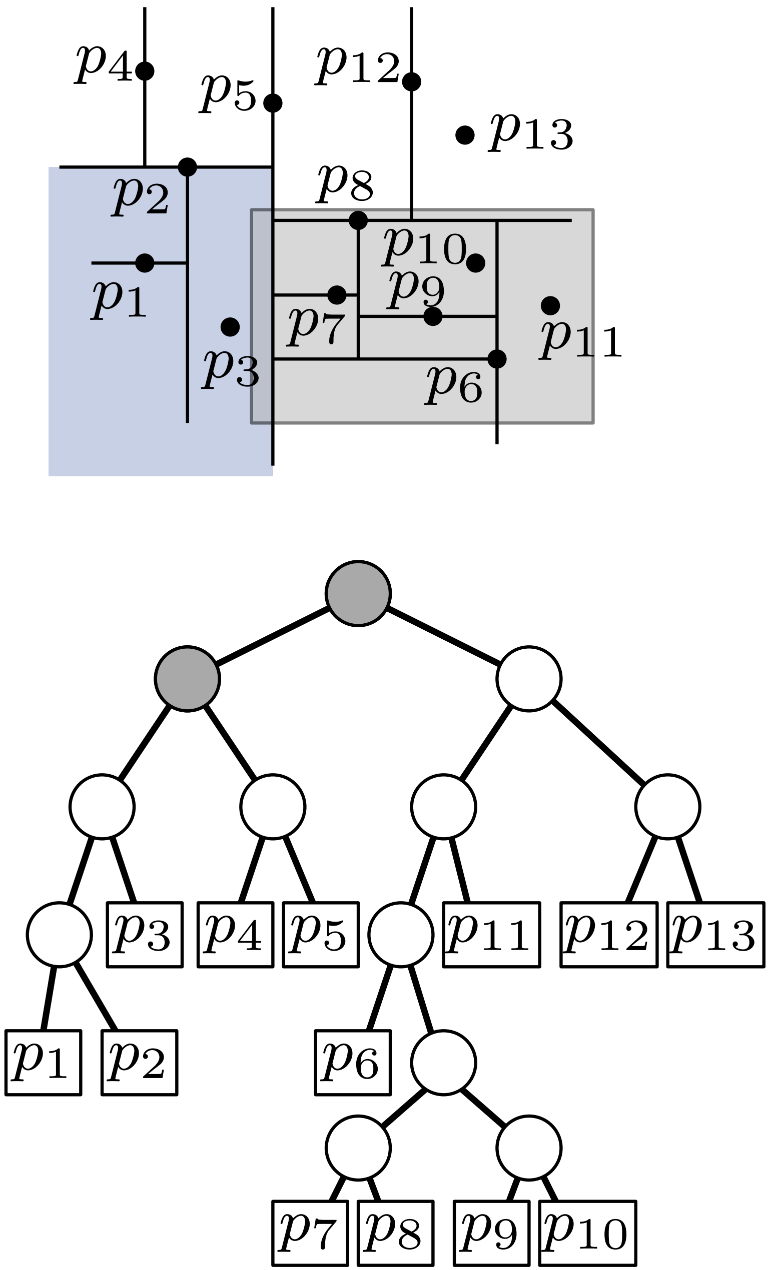

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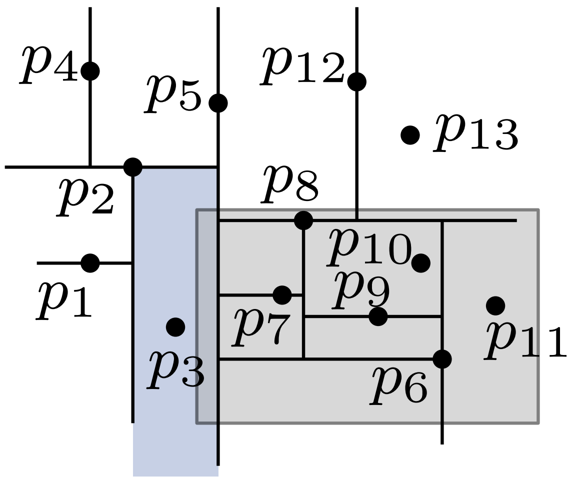

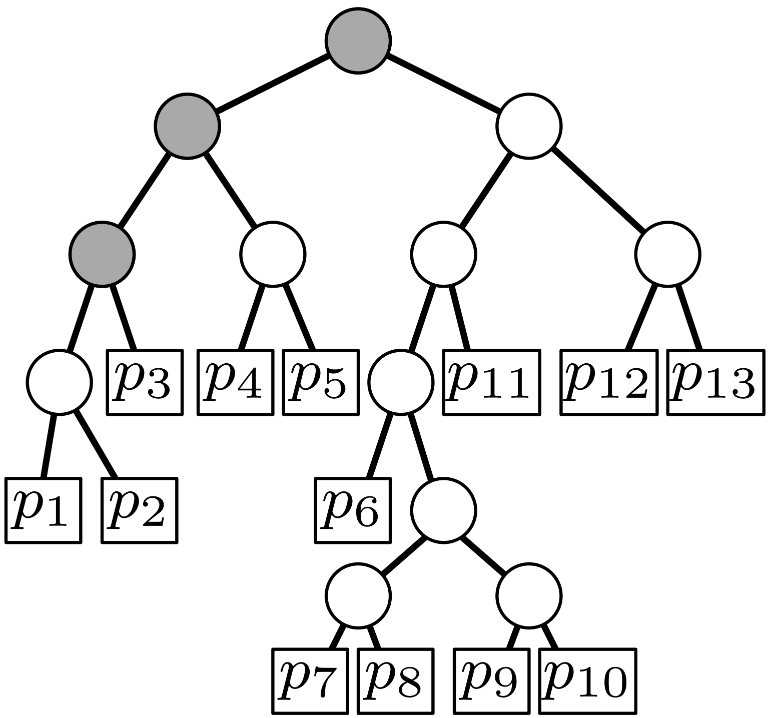

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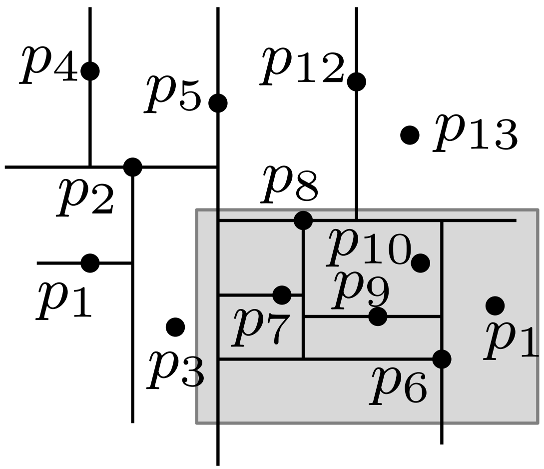

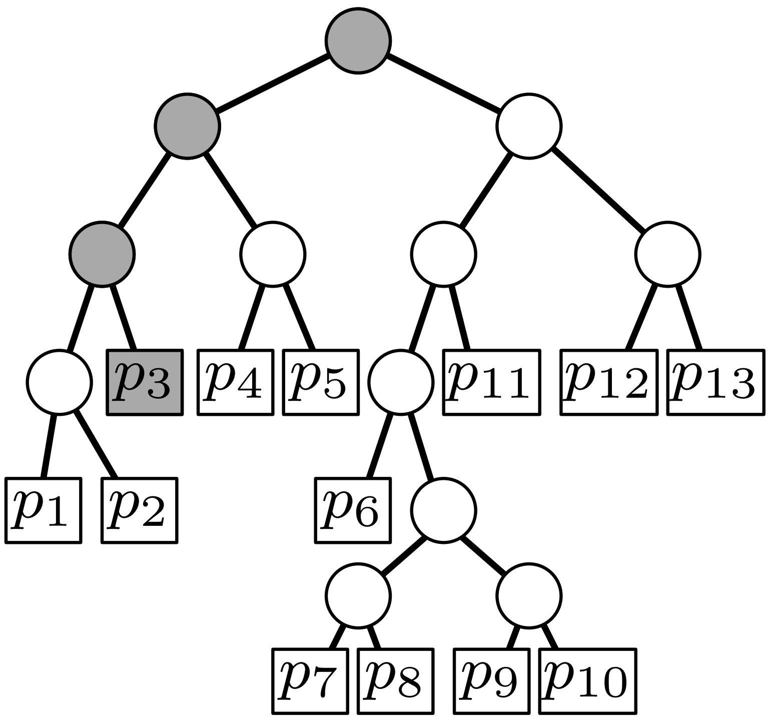

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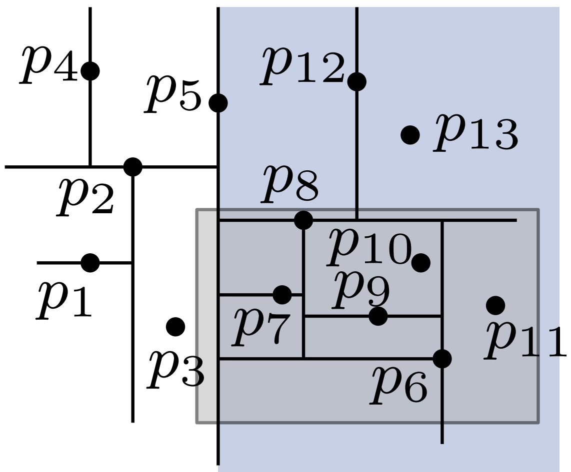

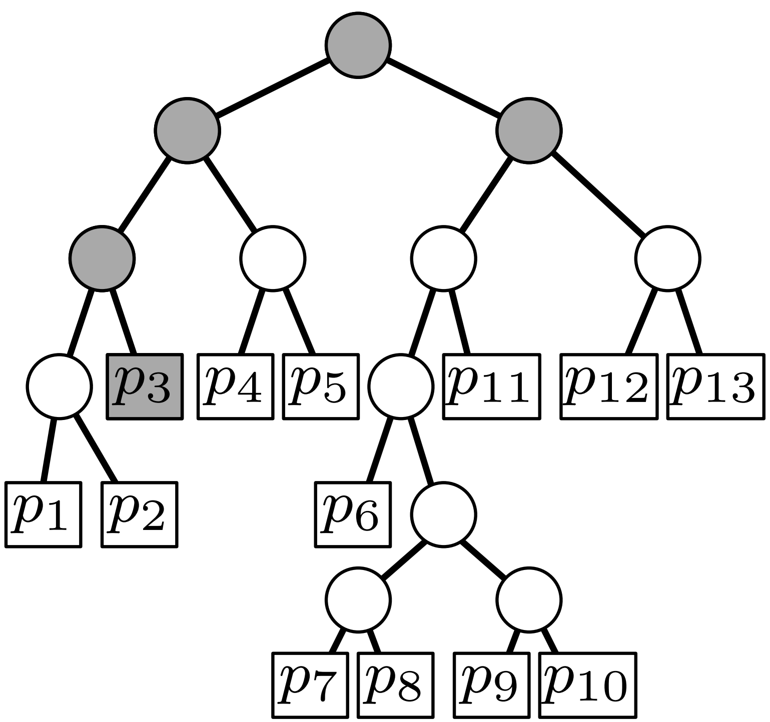

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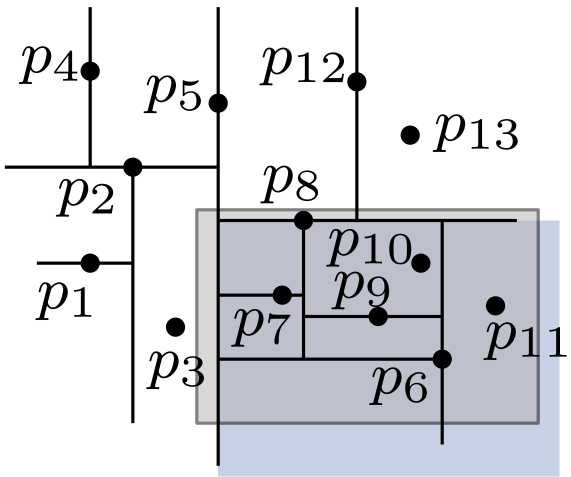
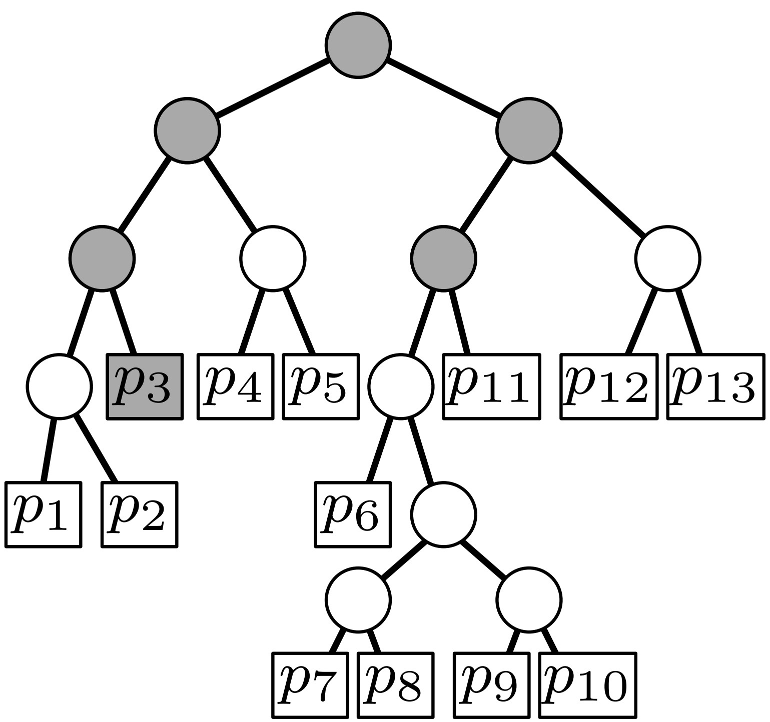
Range Queries in a kd-Tree





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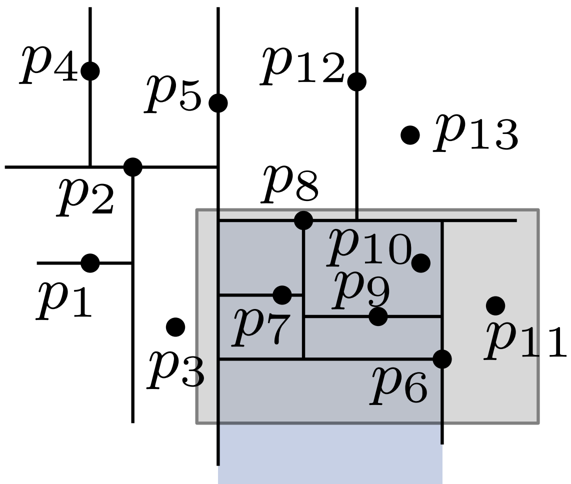

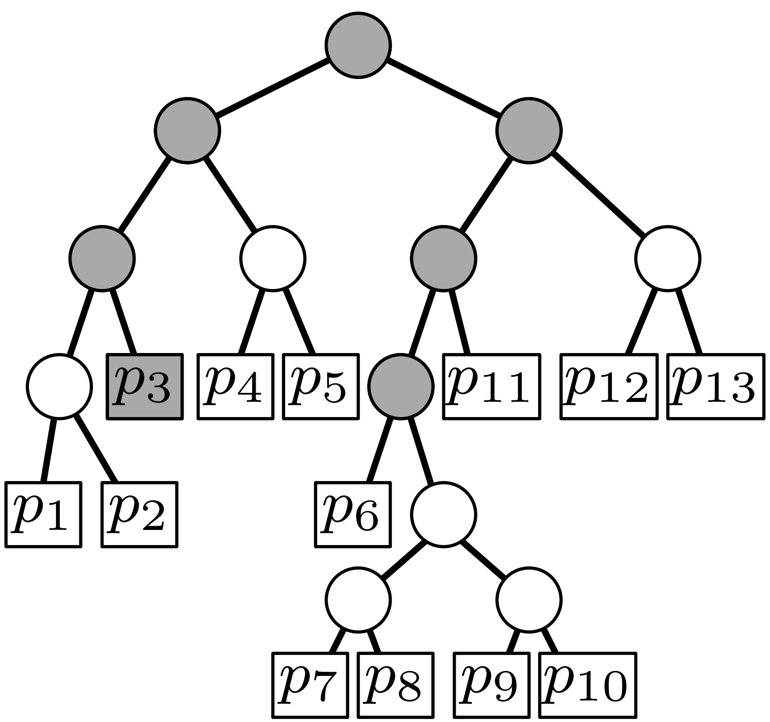

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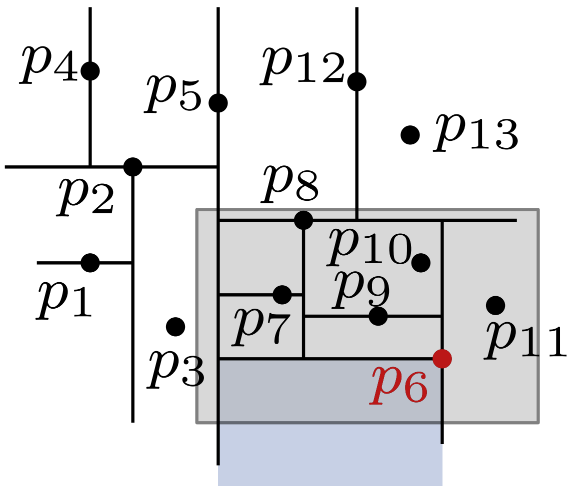

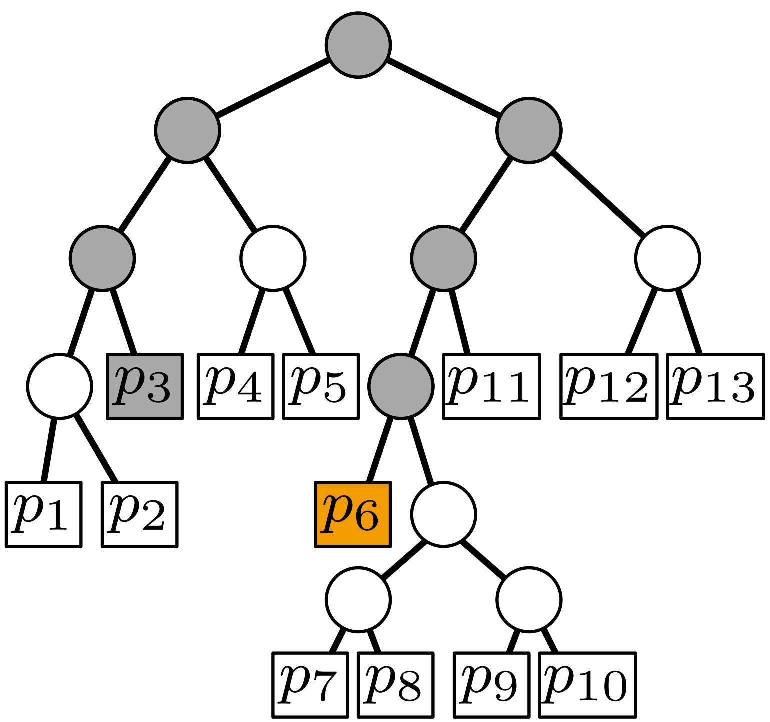

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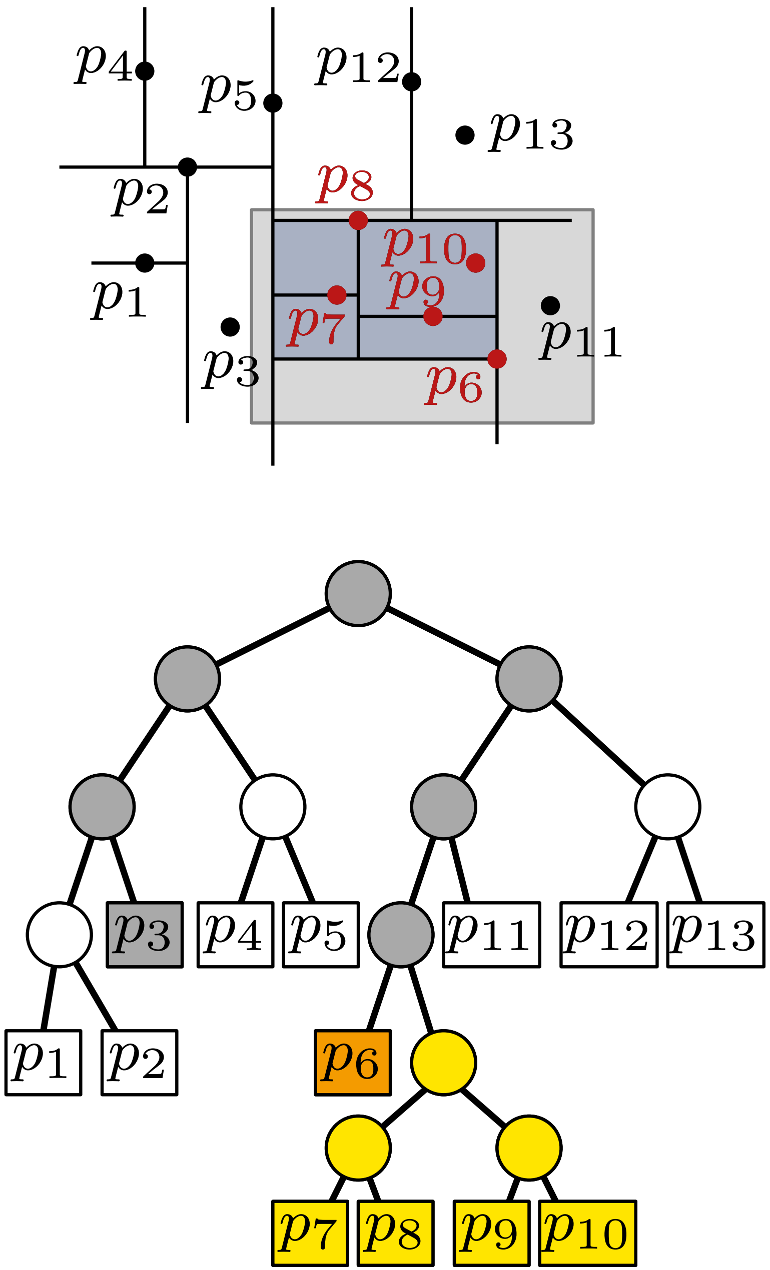

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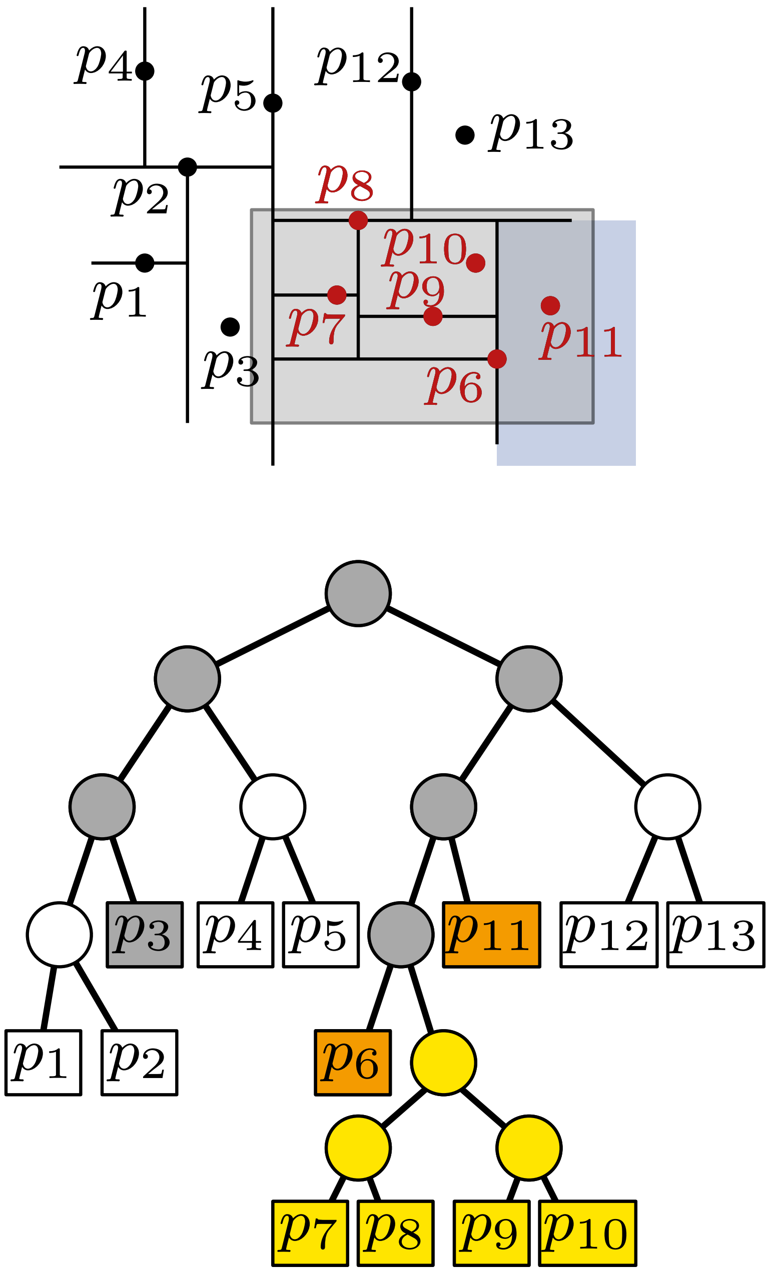

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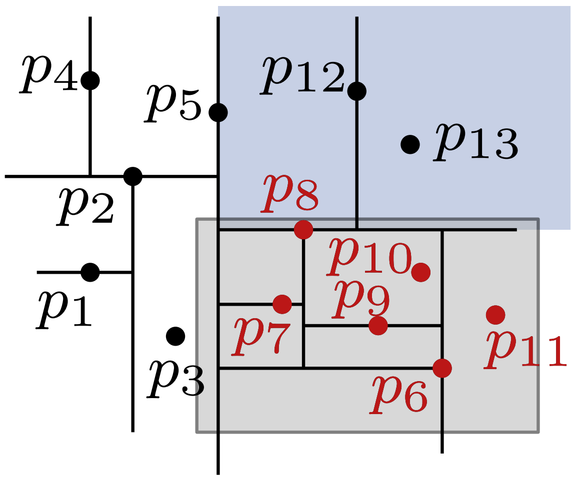

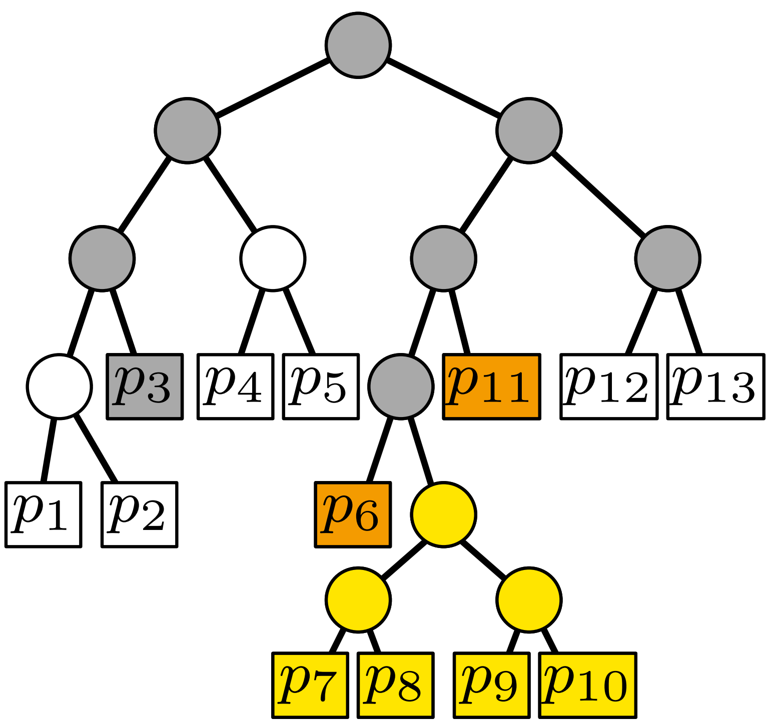

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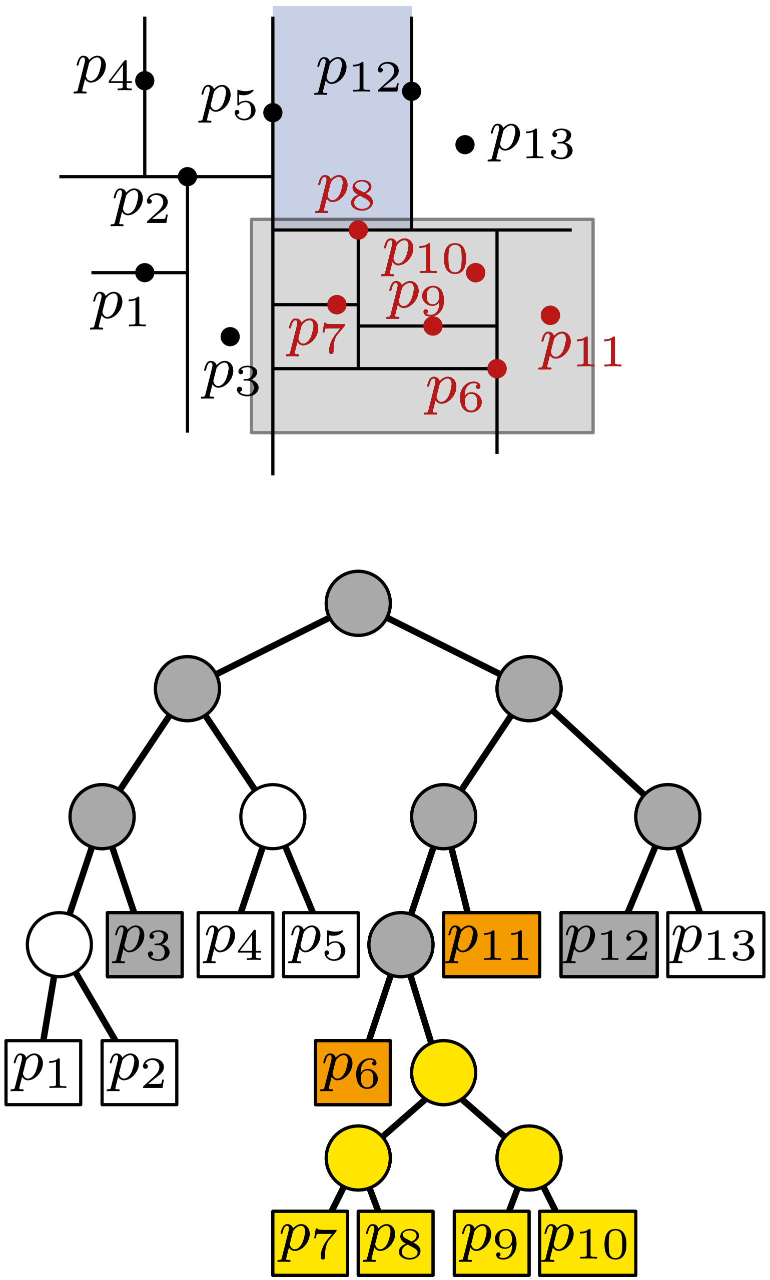

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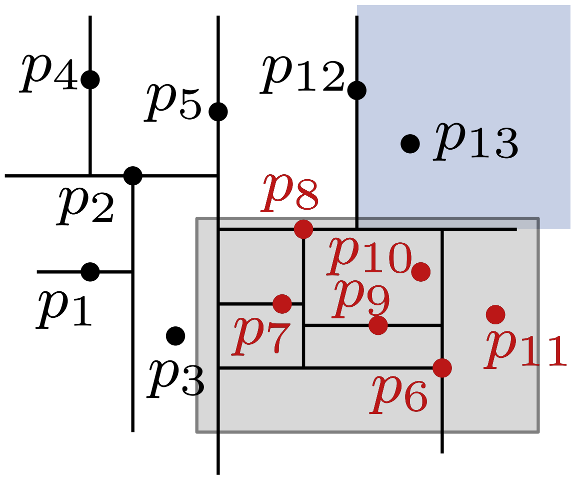

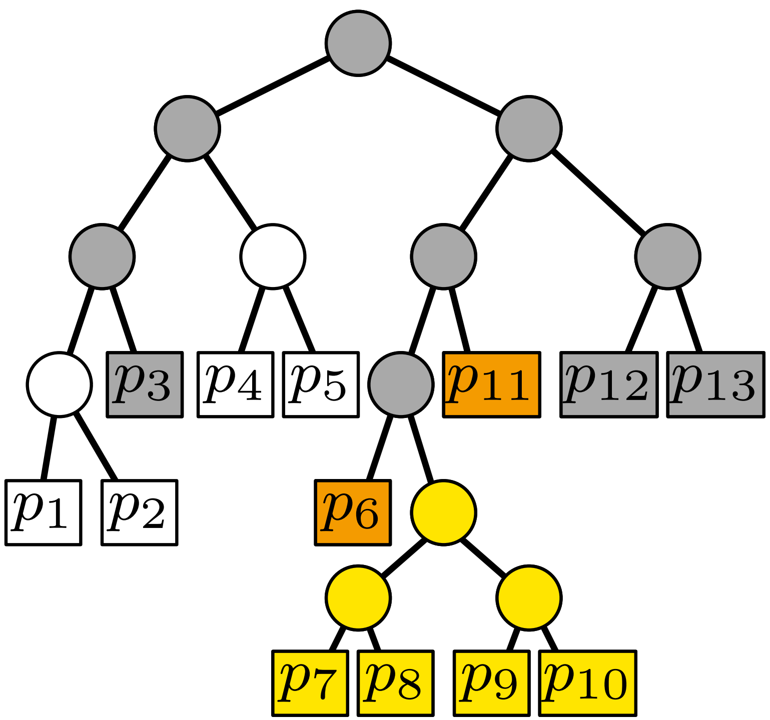

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Analysis of Queries in kd-Trees

Lemma 2: A range query with an axis-aligned rectangle R in a kd-tree on n points may use $O(\sqrt{n} + k)$ time, where k is the number of reported points.

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Proof sketch:

- Calls to ReportSubtree take $O(k)$ time in total
- Still missing:
  Number of remaining nodes visited
  $\rightarrow$ Exercise

Orthogonal Range Queries for $d = 2$

Given: Set P of n points in $\mathbb{R}^2$

Goal: A data structure to efficiently answer range queries of the form $R = [x, x'] \times [y, y']$

Ideas for generalizing the 1d case?

Solutions:

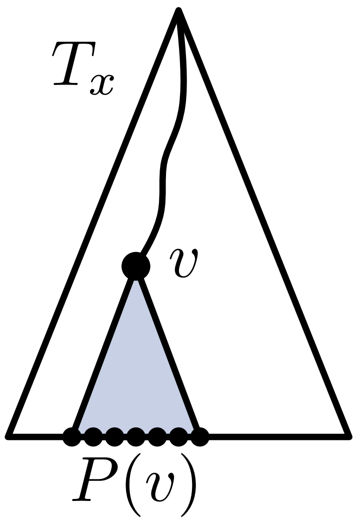
- *one* search tree, alternate search for x and y coordinates
  $\rightarrow$ kd-**Tree** ✓
- *primary* search tree on x-coordinates,
  several *secondary* search trees on y-coordinates
  $\rightarrow$ **Range Tree**

Temporary assumption: general position, that is no two points have the same x- or y-coordinates

Range Trees

Idea: Use 1-dimensional search trees on two levels:

- a 1d search tree T_x on x-coordinates



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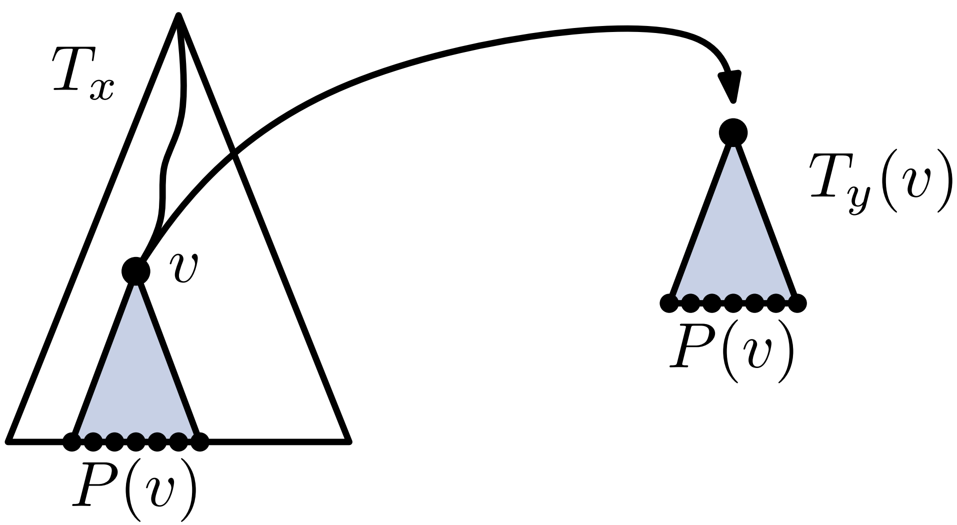

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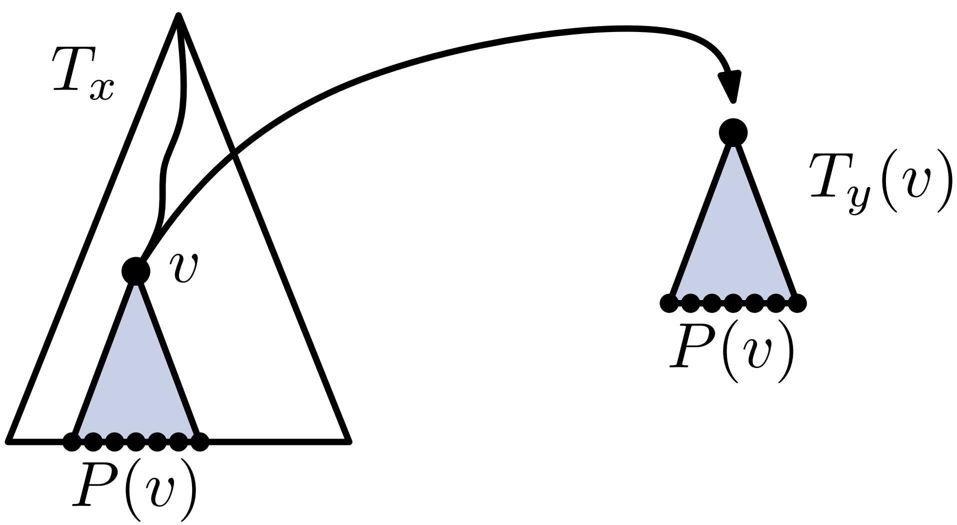
- in each node v of T_x a 1d search tree $T_y(v)$ stores the canonical subset $P(v)$ on y-coordinates
- compute the points by x-query in T_x and subsequent y-queries in the auxiliary structures T_y for the subtrees in T_x



Range Trees: Construction

BuildRangeTree(P)

if $|P| = 1$ **then**
    Create leaf v for the point in P
else
    Split P at x_{median} into $P_1 = \{p \in P \mid p_x \leq x_{\text{median}}\}$, $P_2 = P \setminus P_1$
    $v_{\text{left}} \leftarrow$ BuildRangeTree(P_1)
    $v_{\text{right}} \leftarrow$ BuildRangeTree(P_2)
    Create node v with pivot x_{median} and children v_{left} and v_{right}
$T_y(v) \leftarrow$ binary search tree for P w.r.t y-coordinates
return v

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Lemma 3: A Range Tree for n points in $\mathbb{R}^2$ uses $O(n \log n)$ space and can be constructed in $O(n \log n)$ time.

Range Queries in a Range Tree

Reminder:

1dRangeQuery(T, x, x')

```
v_split ← FindSplitNode(T, x, x')
if v_split is leaf then report v_split
else
    v ← lc(v_split)
    while v not leaf do
        if x ≤ x_v then
            ReportSubtree(rc(v))
            v ← lc(v)
        else  v ← rc(v)
    report v
    // analogous for x' and rc(v_split)
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Range Queries in a Range Tree

Reminder:

~~**1dRangeQuery**(T, x, x')~~ **2dRangeQuery**$(T, [x, x'] \times [y, y'])$

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Lemma 4: A range query in a Range Tree takes $O(\log^2 n + k)$ time, where k is the number of reported points.

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Observation: Range queries in a Range Tree perform $O(\log n)$ 1d queries, each taking $O(\log n + k_v)$ time. The query interval $[y, y']$ is always the same!

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Can we do better than two binary searches?

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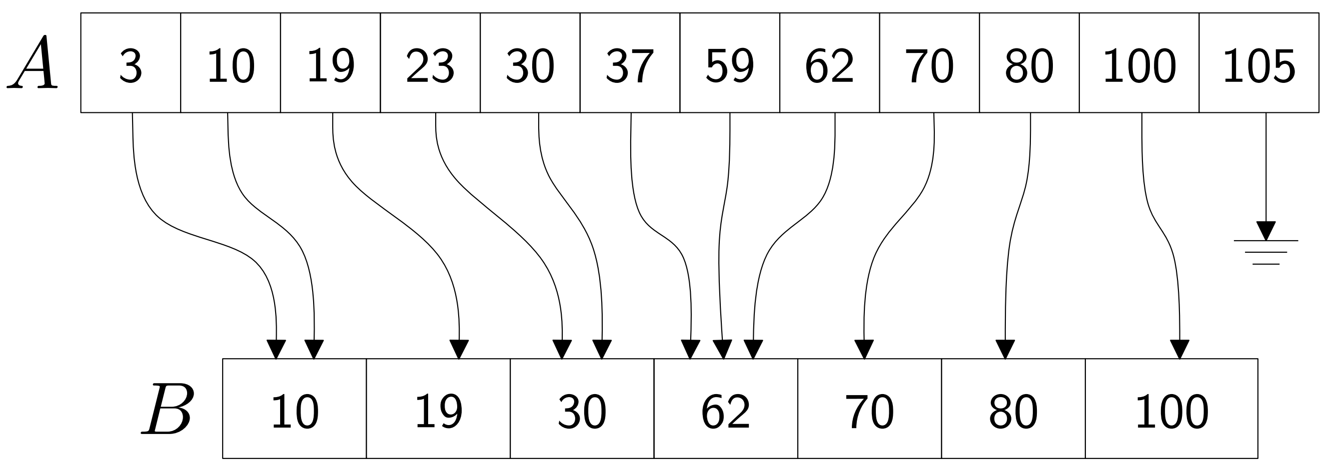

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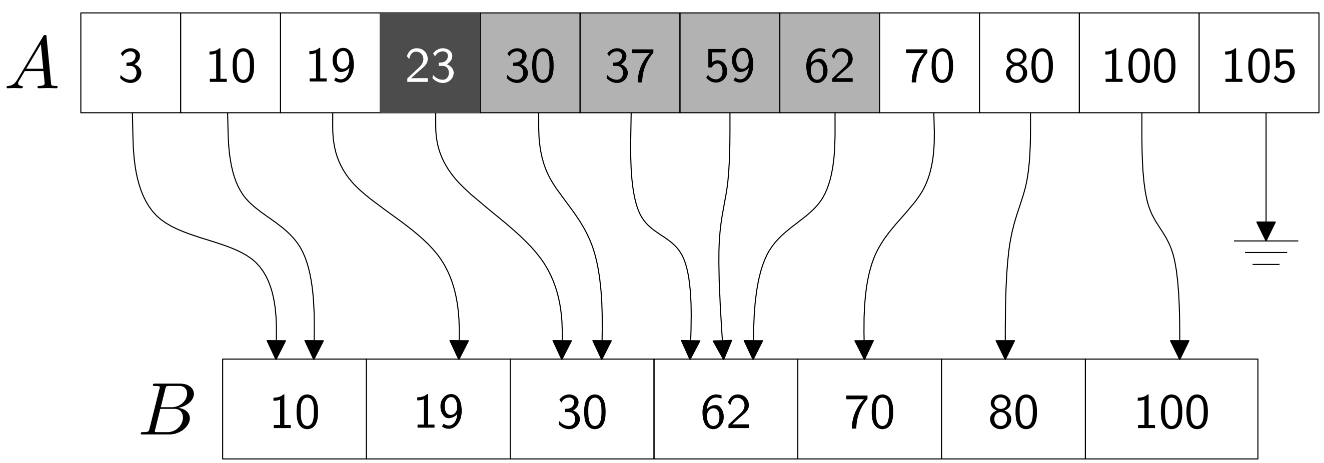

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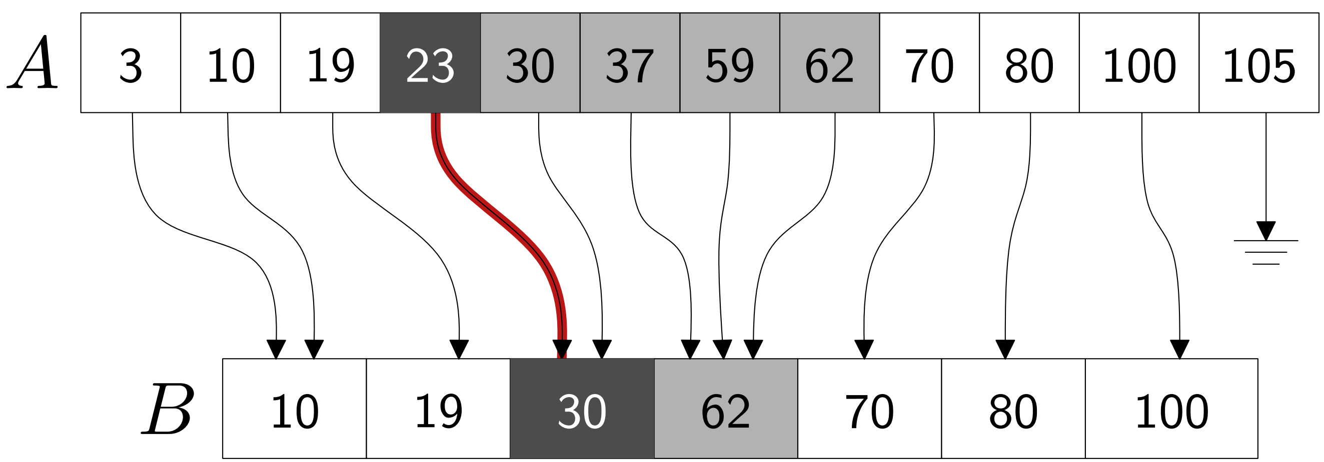

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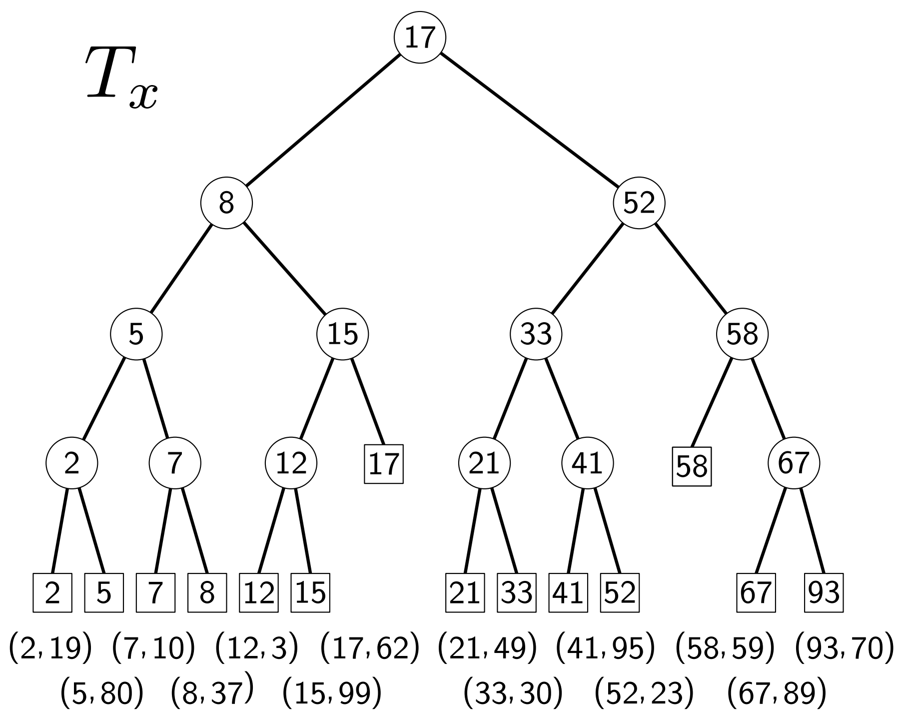
Search interval [20,65]

Pointer yields starting point for second search in $O(1)$ time

Speed-up with Fractional Cascading

- In Range Trees we have $P(\mathrm{lc}(v)) \subseteq P(v)$ and $P(\mathrm{rc}(v)) \subseteq P(v)$ as the canonical sets.



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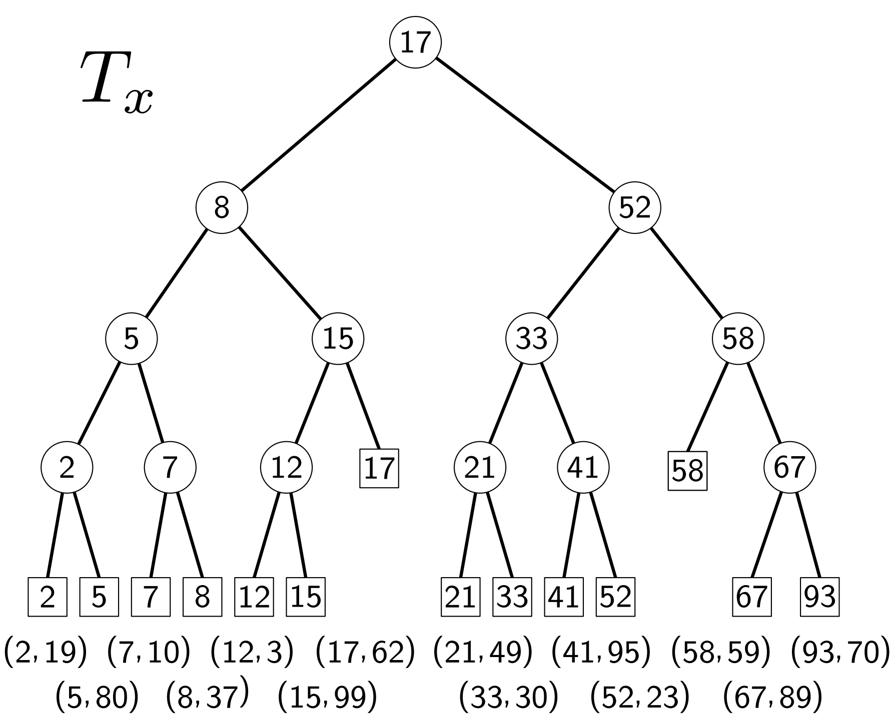

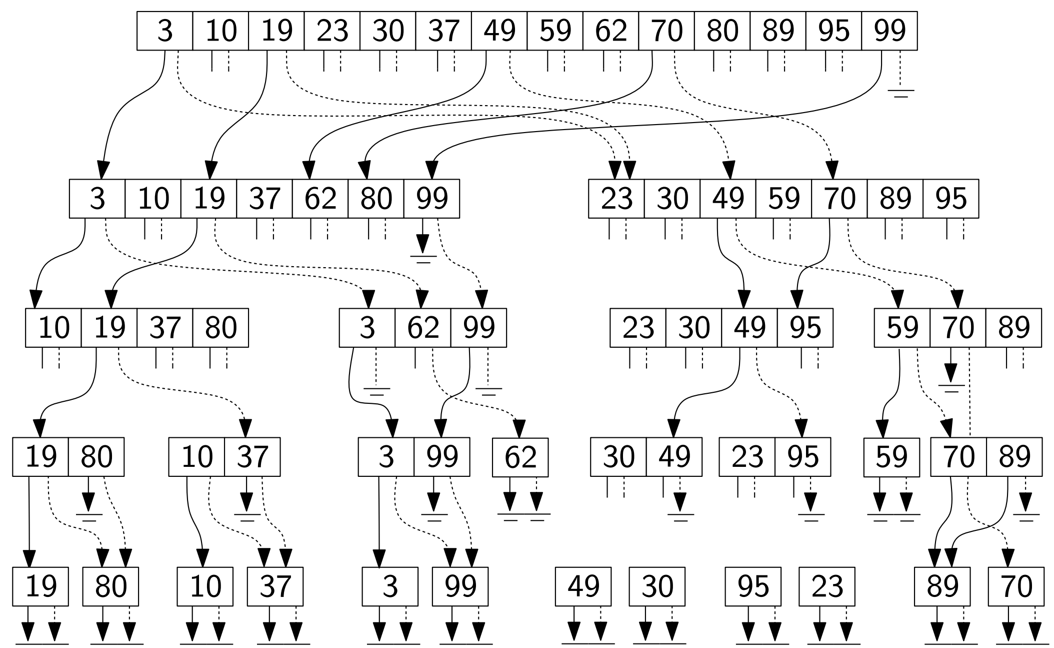

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Karlsruhe Institute of Technology

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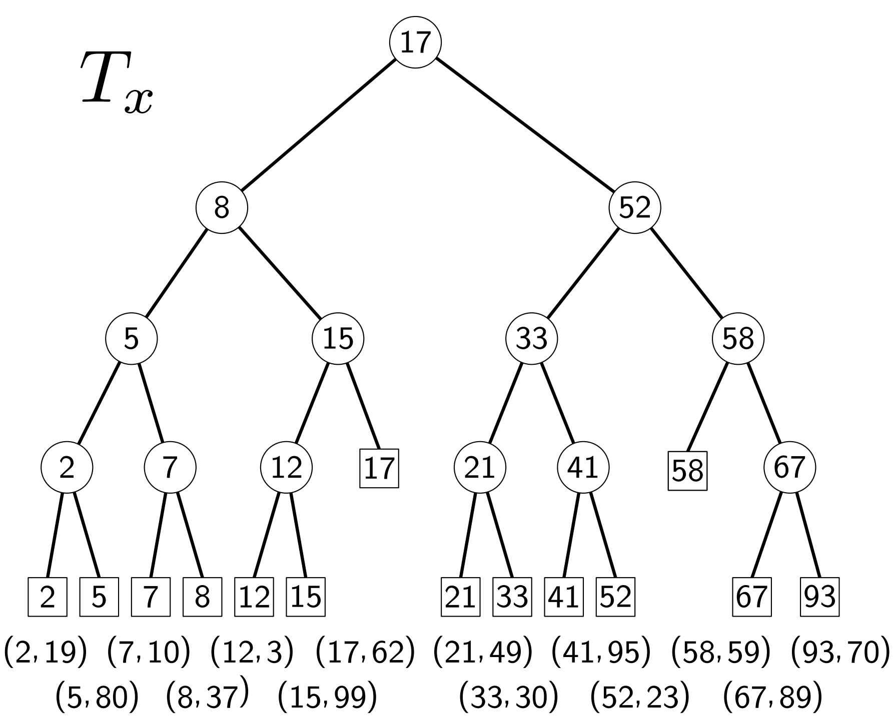

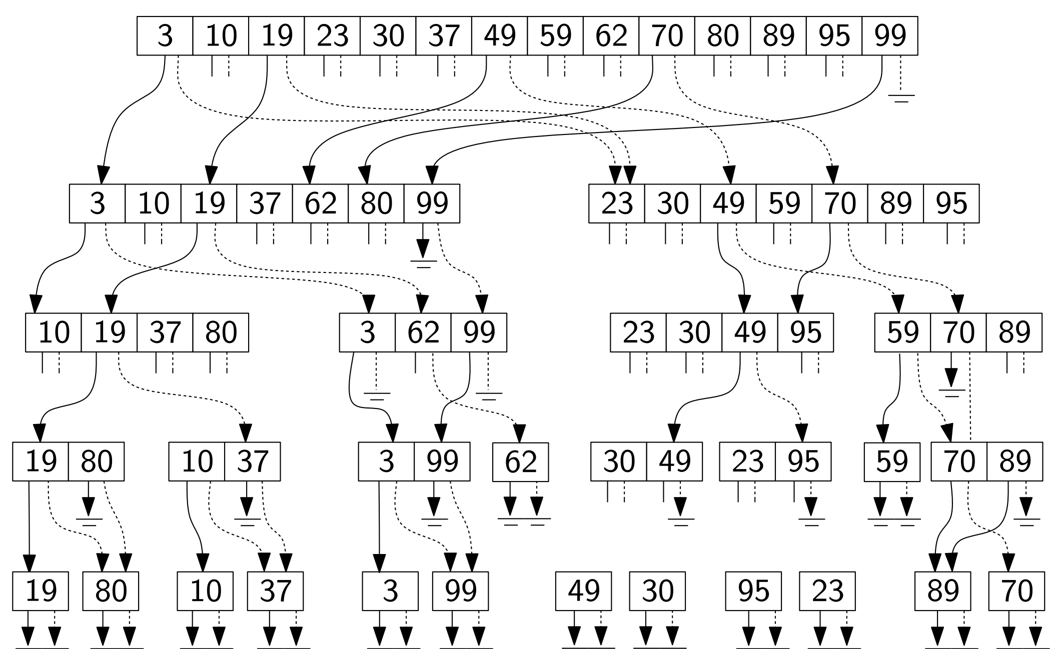

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Theorem 2: A Layered Range Tree on n points in $\mathbb{R}^2$ can be constructed in $O(n \log n)$ time and space. Range queries take $O(\log n + k)$ time, where k is the number of reported points.

Arbitrary Point Sets

So far: Points in general position, where no two points have the same x- or y-coordinate

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Rectangle $R = [x, x'] \times [y, y']$

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Idea: Instead of $\mathbb{R}$, use pairs of numbers $(a|b)$ with total order $\leftrightarrow$ lexicographic order

$$p = (p_x, p_y) \longrightarrow \hat{p} = \big((p_x|p_y),\ (p_y|p_x)\big)$$

unique coord.

Rectangle $R = [x, x'] \times [y, y']$

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Then: $p \in R \ \Leftrightarrow\ \hat{p} \in \hat{R}$

Summary

Given: Set P of n points in $\mathbb{R}^2$

Construct: Data structures with efficient range queries of the form $R = [x, x'] \times [y, y']$

$\rightarrow$ We have seen two alternatives

	kd-Tree	Range Tree
Preprocessing		
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- Range Trees can be built recursively: the auxiliary search tree on the first coordinate is a $(d-1)$-dimensional Range Tree. The construction and space takes $O(n \log^{d-1} n)$ time; a query takes $O(\log^d n + k)$ time, and with fractional cascading, $O(\log^{d-1} n + k)$ time.

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Yes, we can transform any polygon into a point in 4d space (exercise) or we can use windowing queries (comes in a later lecture).

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Question: Can we adapt these data structures for dynamic point sets?

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2) **Augmented dynamic range trees** [Mehlhorn, Näher '90]
support updates in $O(\log n \log\log n)$ time and queries in $O(\log n \log\log n + k)$ time