Computational Geometry – Problem Session

Convex Hull & Line Segment Intersection

LEHRSTUHL FÜR ALGORITHMIK · INSTITUT FÜR THEORETISCHE INFORMATIK · FAKULTÄT FÜR INFORMATIK

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04.05.2018

Modus Operandi

To register for the oral exam we expect you to present an original solution for at least one problem in the exercise session.

- this is about working *together*
- don't worry if your idea doesn't work!

Outline

Convex Hull

Line Segment Intersection

Definition of Convex Hull

Def: A region $S \subseteq \mathbb{R}^2$ is called **convex**, when for two points $p, q \in S$ then line $\overline{pq} \in S$.
The **convex hull** $CH(S)$ of S is the smallest convex region containing S.

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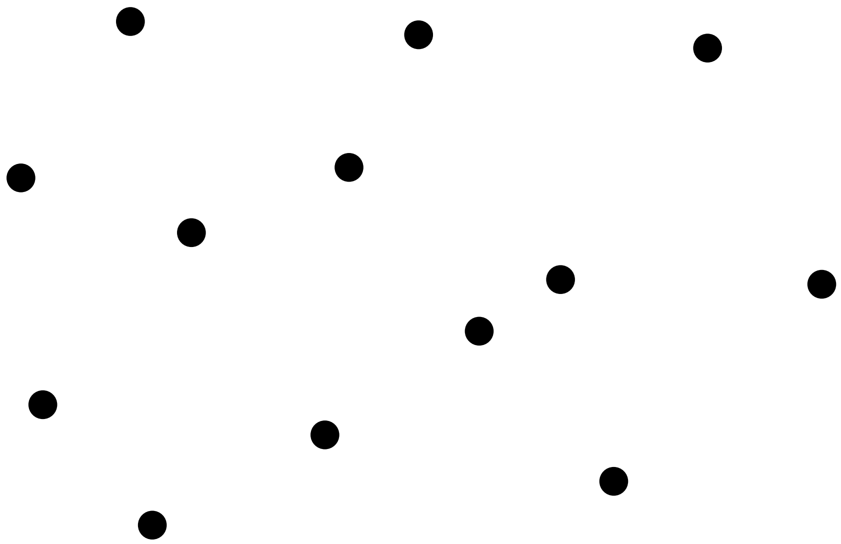

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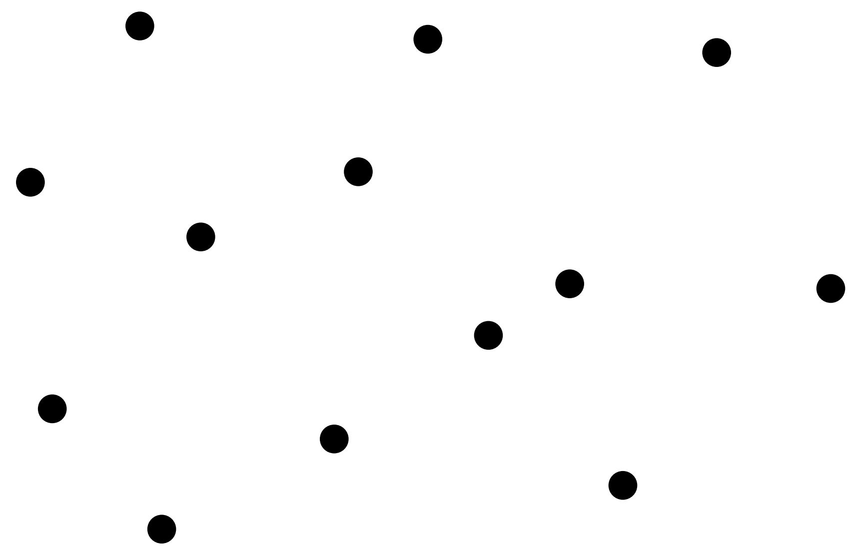

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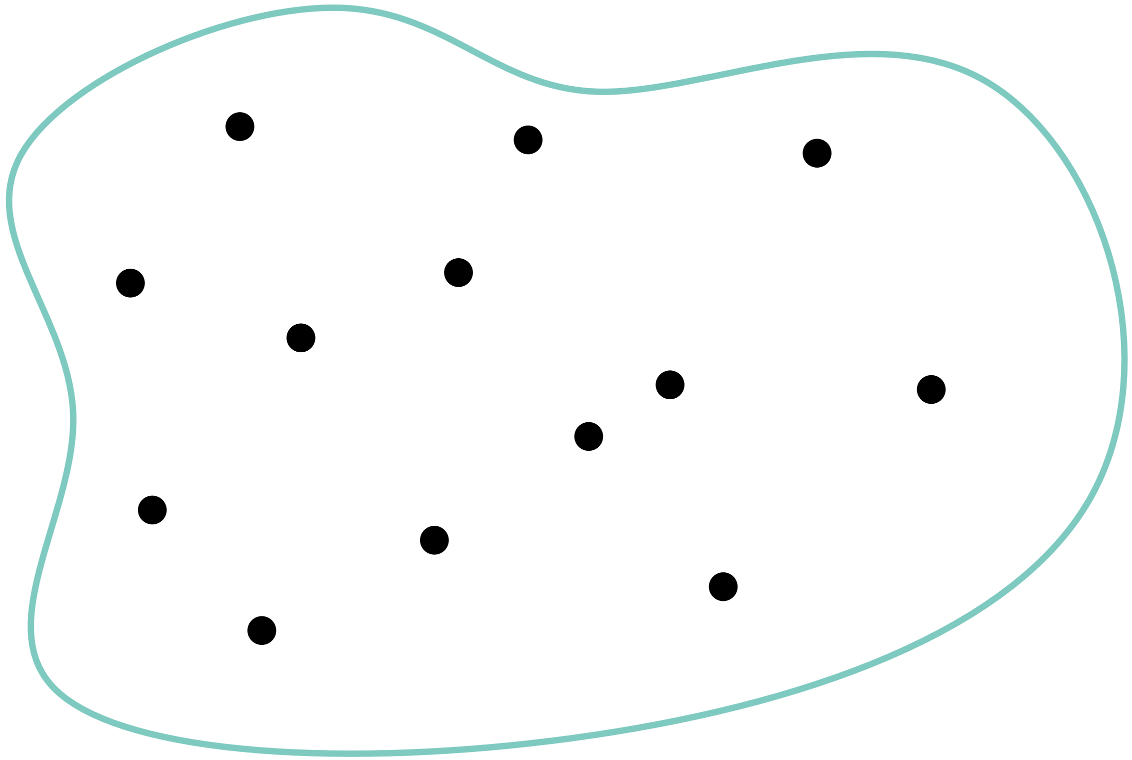

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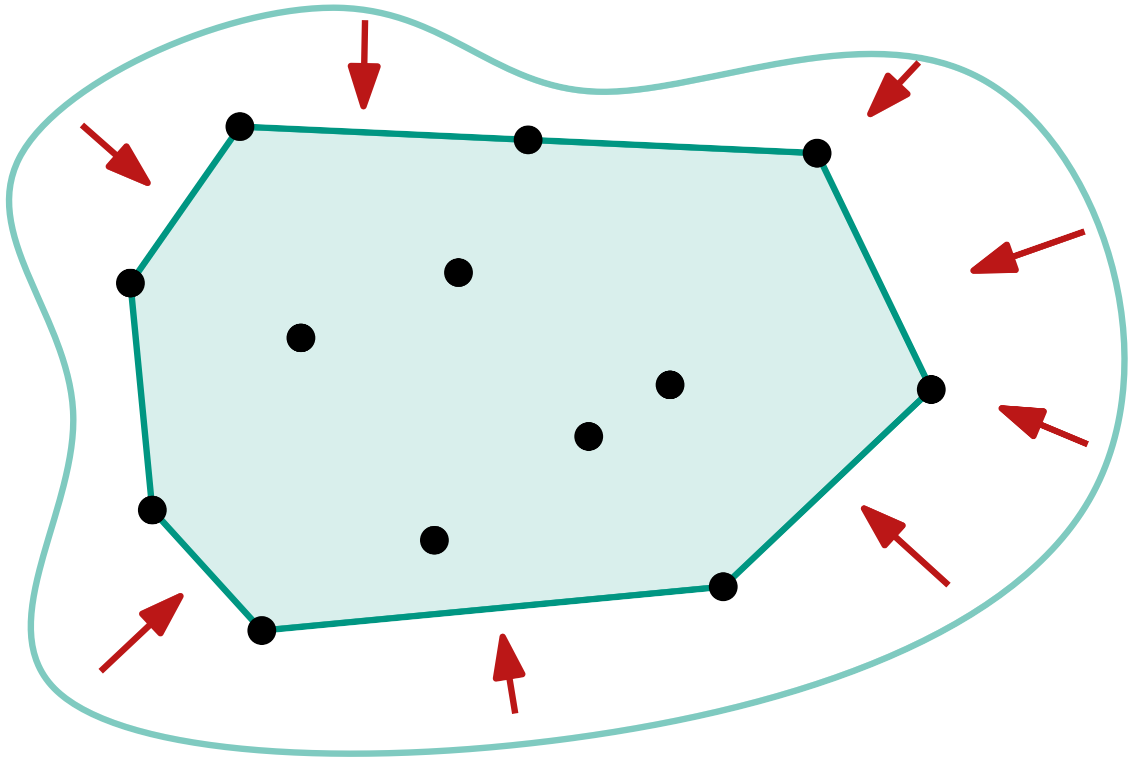

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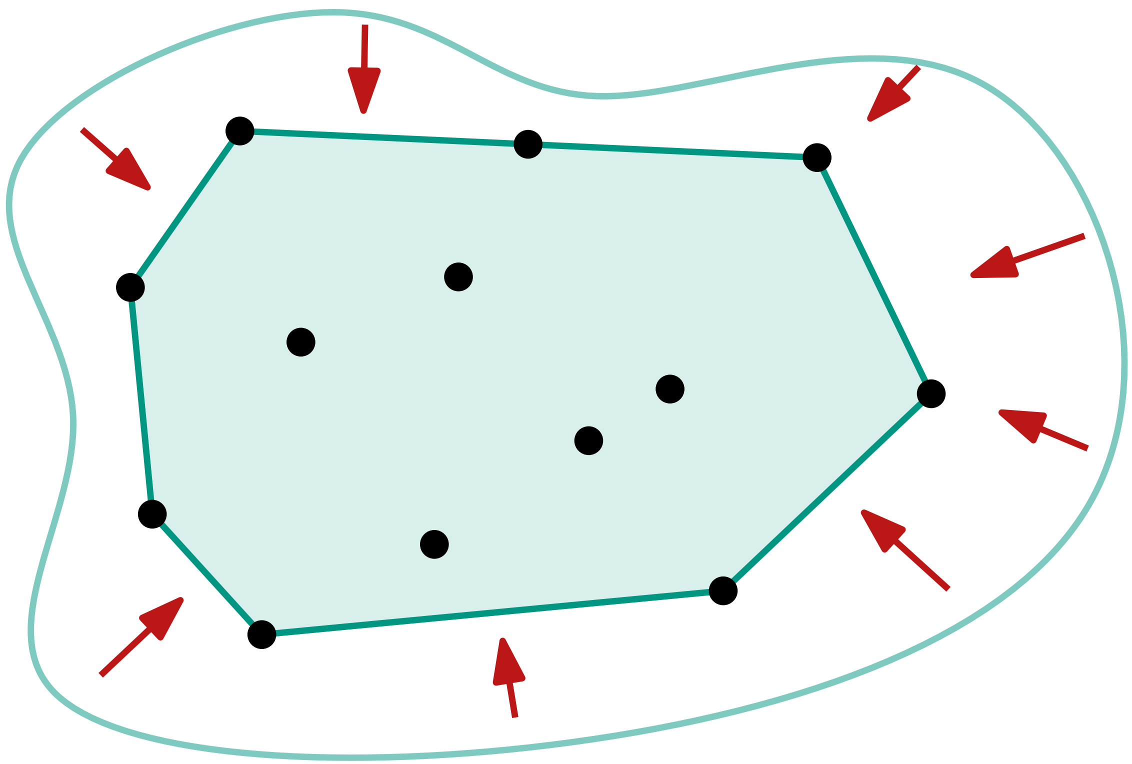

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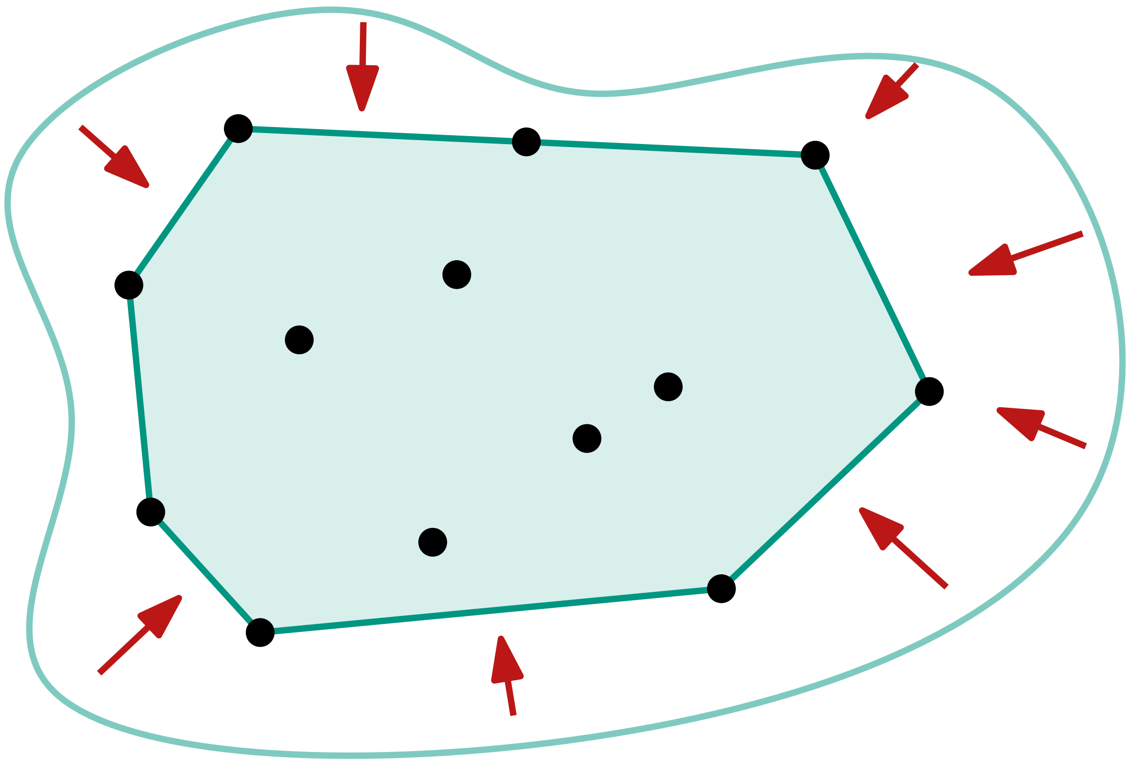


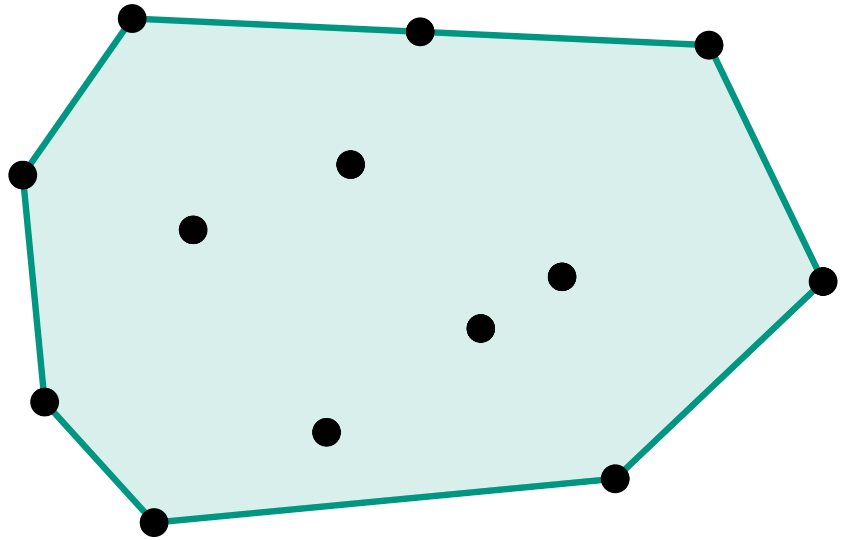
In mathematics:

- define $CH(S) = \bigcap_{C \supseteq S:\ C \text{ convex}} C$
- does not help :-(

Algorithmic Approach

Lemma:

For a set of points $P \subseteq \mathbb{R}^2$, $CH(P)$ is a convex polygon that contains P and whose vertices are in P.



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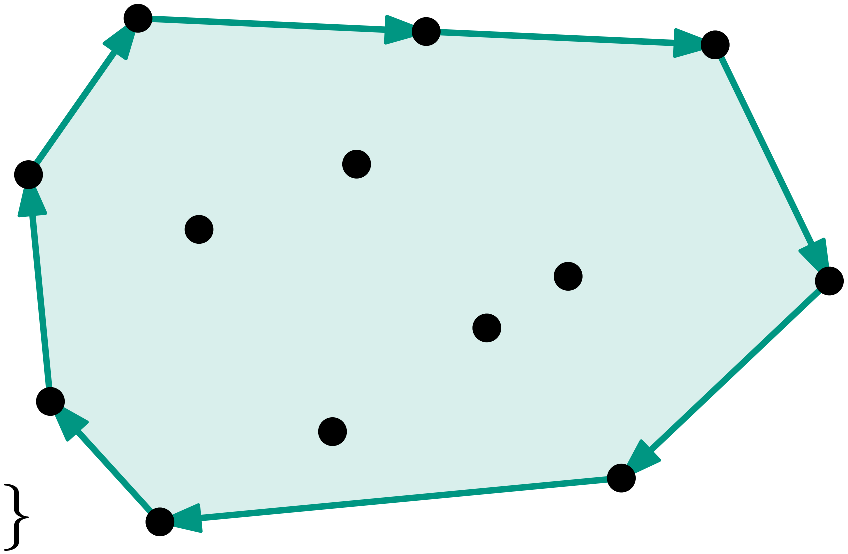

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Output: List of nodes of $CH(P)$ in clockwise order

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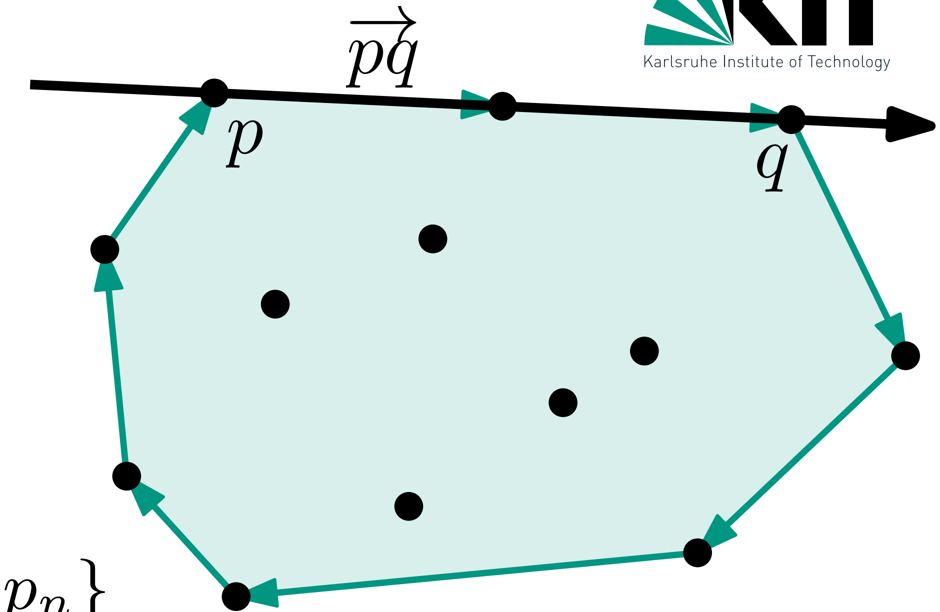

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Output: List of nodes of $CH(P)$ in clockwise order

Observation:

(p, q) is an edge of $CH(P) \Leftrightarrow$ each point $r \in P \setminus \{p, q\}$

- strictly right of the oriented line $\overrightarrow{pq}$ or
- on the line segment $\overline{pq}$

Running Time Analysis

FirstConvexHull(P)

$E \leftarrow \emptyset$

foreach $(p, q) \in P \times P$ with $p \neq q$ **do** $(n^2 - n)\cdot$

  valid ← *true*

  foreach $r \in P$ **do**

    if not (r is strictly right of $\overrightarrow{pq}$ **or** $r \in \overline{pq}$) **then**

      valid ← *false*

  if *valid* **then**

    $E \leftarrow E \cup \{(p, q)\}$

$\Theta(1)$ $\Theta(n)$ $\Theta(n^3)$

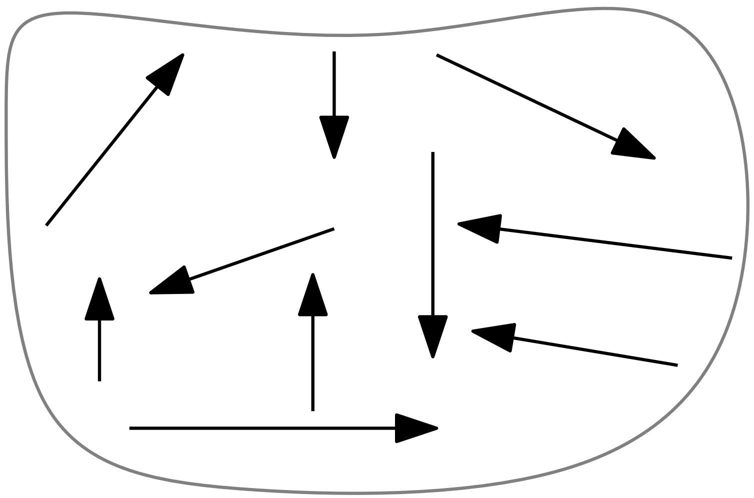
construct the sorted node list L from $CH(P)$ out of E

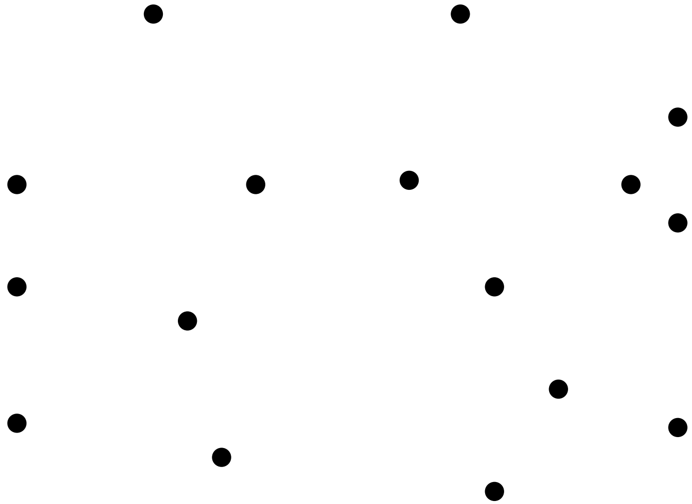
return L

Question: How do we implement this?

Solution

Set of edges.





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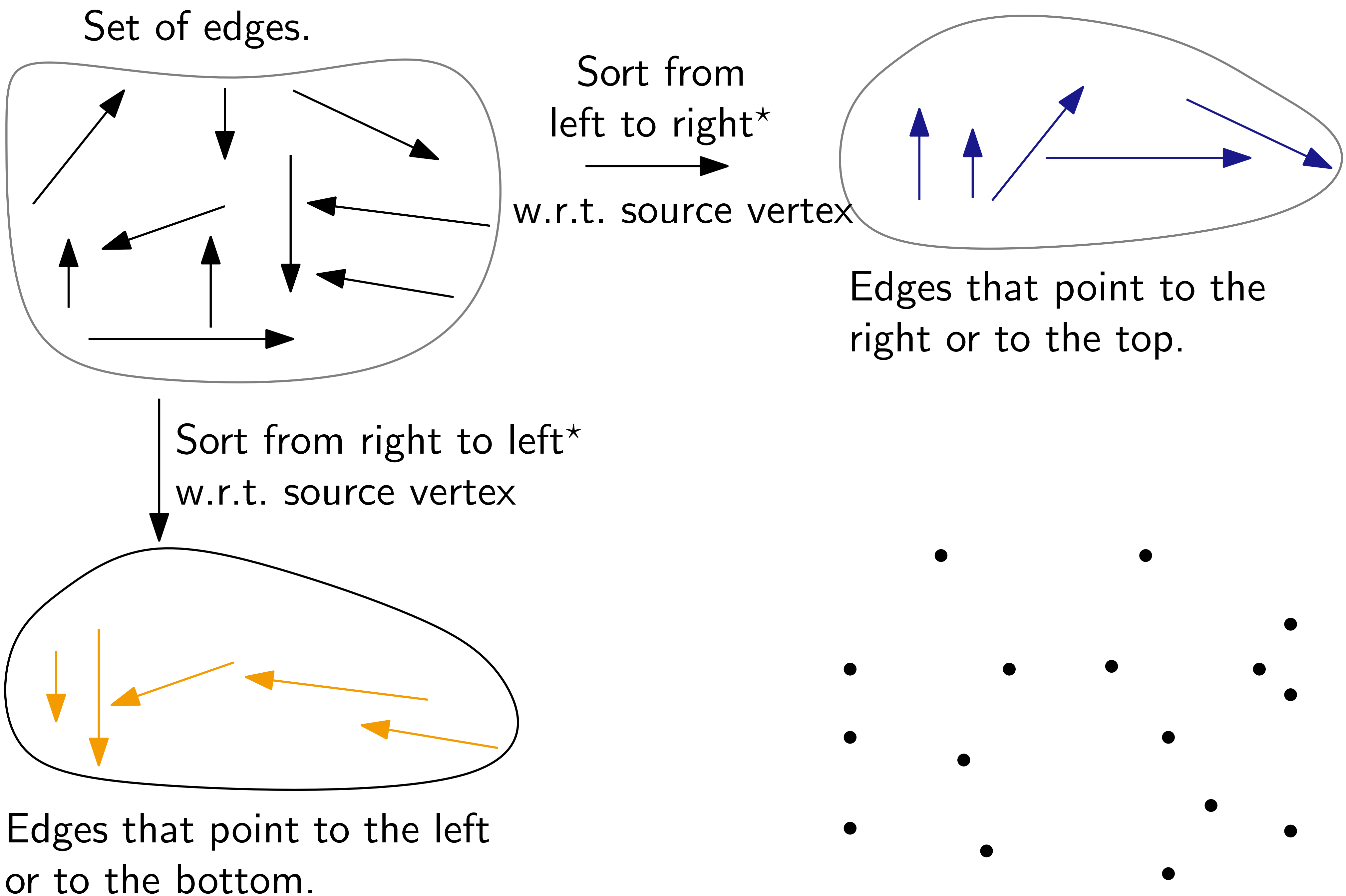
Set of edges.
Sort from
left to right*
w.r.t. source vertex
Edges that point to the
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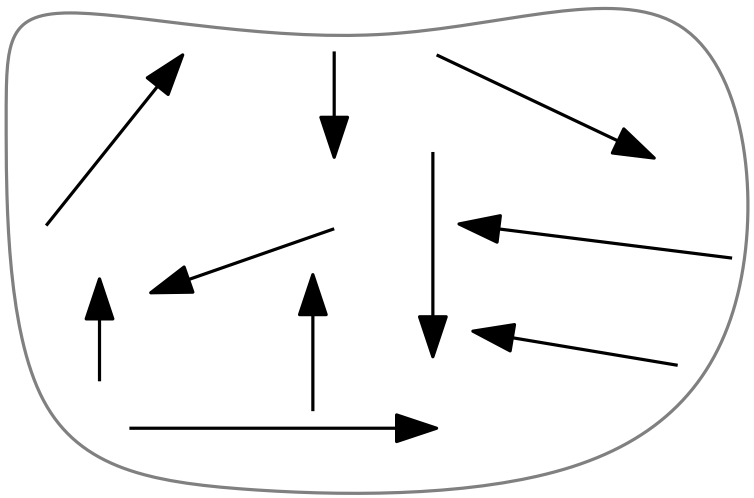

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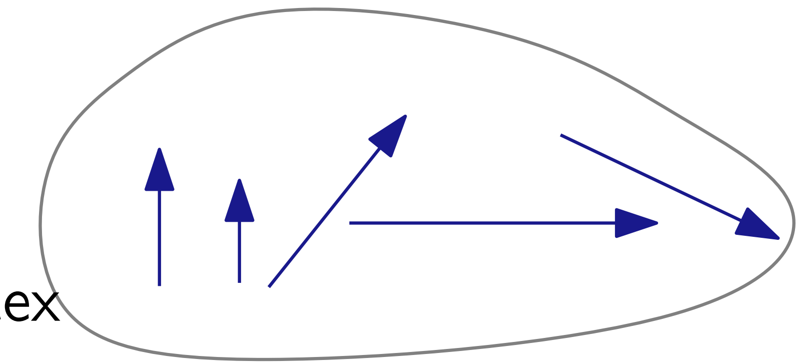
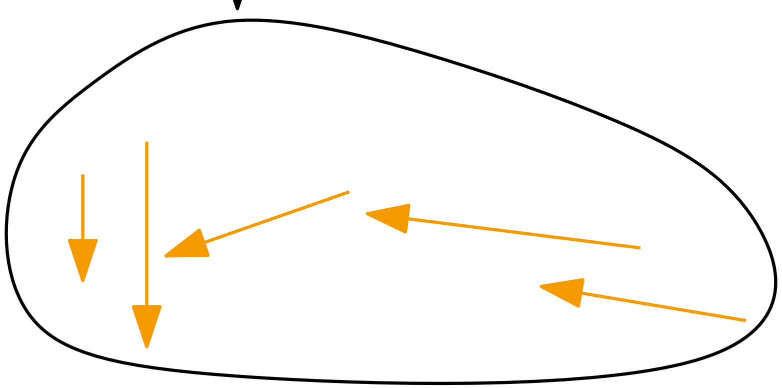


Edges that point to the right or to the top.

Sort from right to left* w.r.t. source vertex

*if not unique:
from bottom to top.
from top to bottom.



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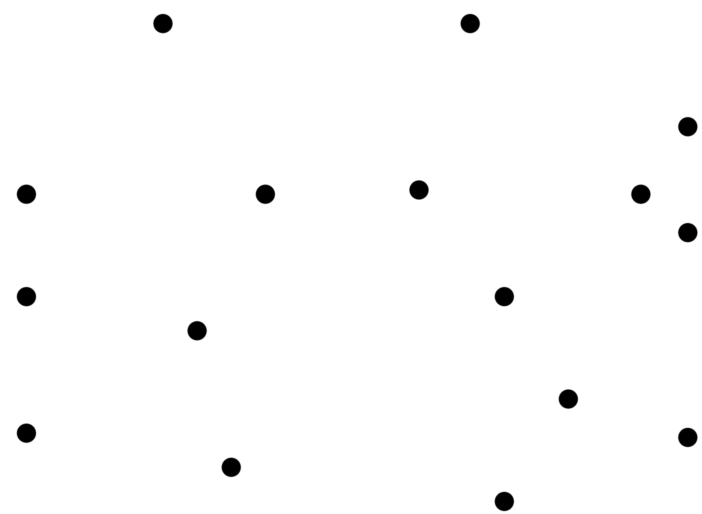

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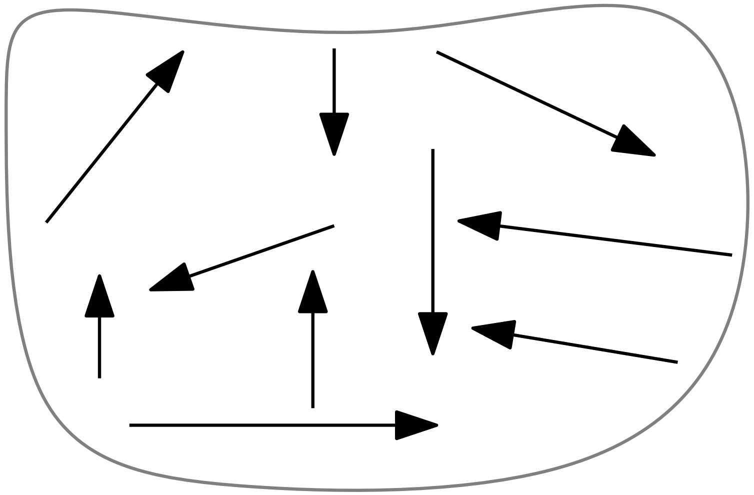

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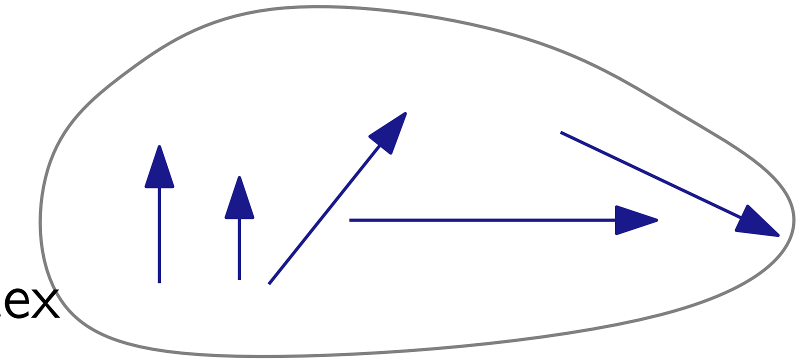

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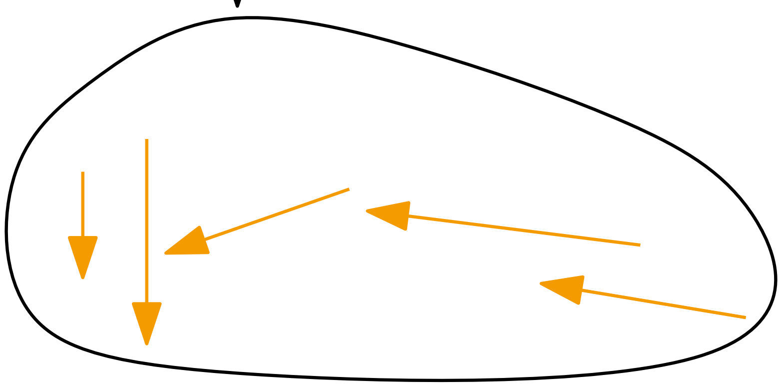

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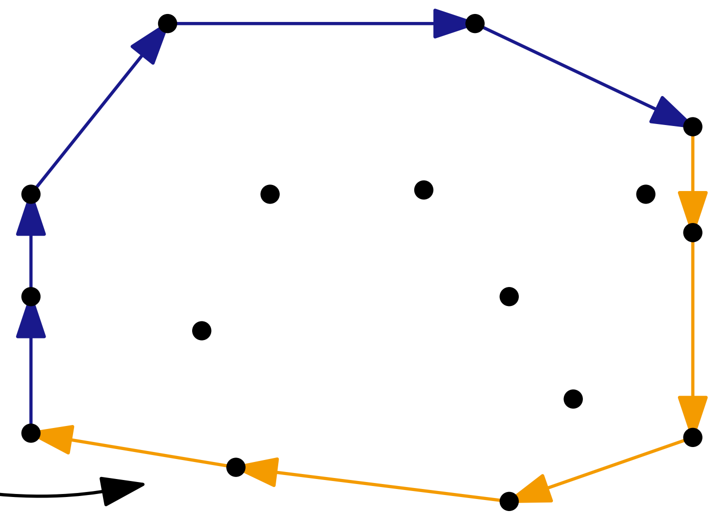

Alternative: Gift Wrapping

Idea: Begin with a point p_1 of $CH(P)$, then find the next edge of $CH(P)$ in clockwise order.

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GiftWrapping(P)

$p_1 = (x_1, y_1) \leftarrow$ rightmost point in P; $p_0 \leftarrow (x_1, \infty)$; $j \leftarrow 1$
while true **do**
  $p_{j+1} \leftarrow \arg\max\{\angle p_{j-1}, p_j, q \mid q \in P \setminus \{p_{j-1}, p_j\}\}$
  if $p_{j+1} = p_1$ **then** break **else** $j \leftarrow j + 1$
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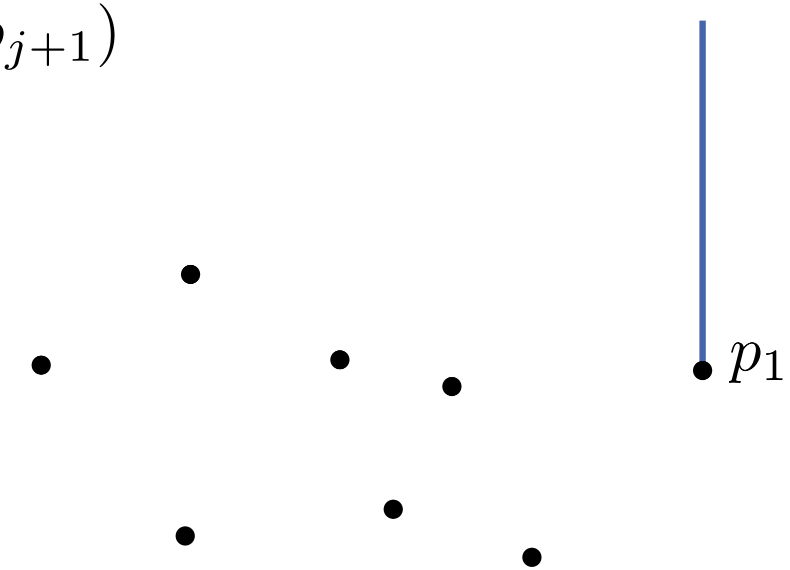

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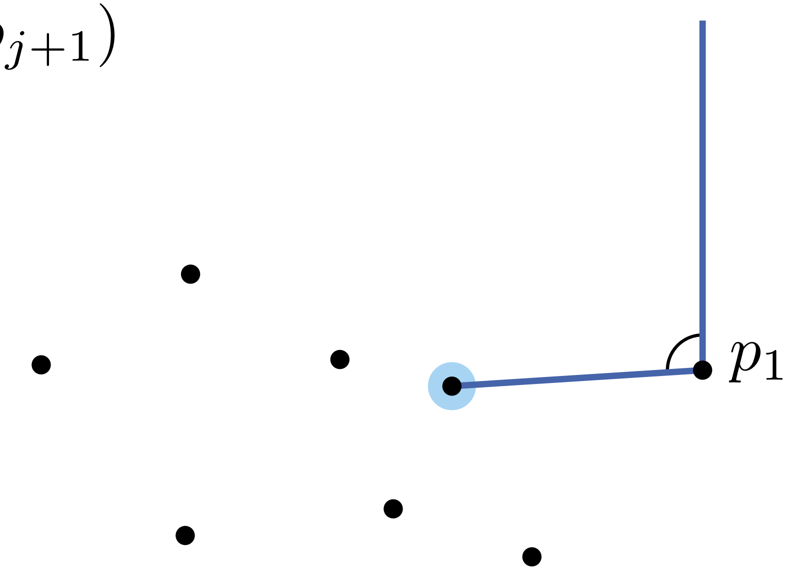

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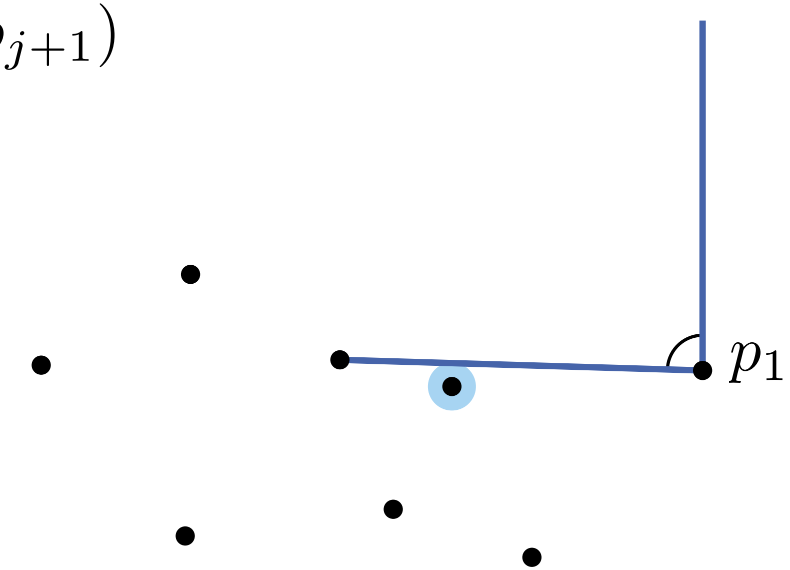

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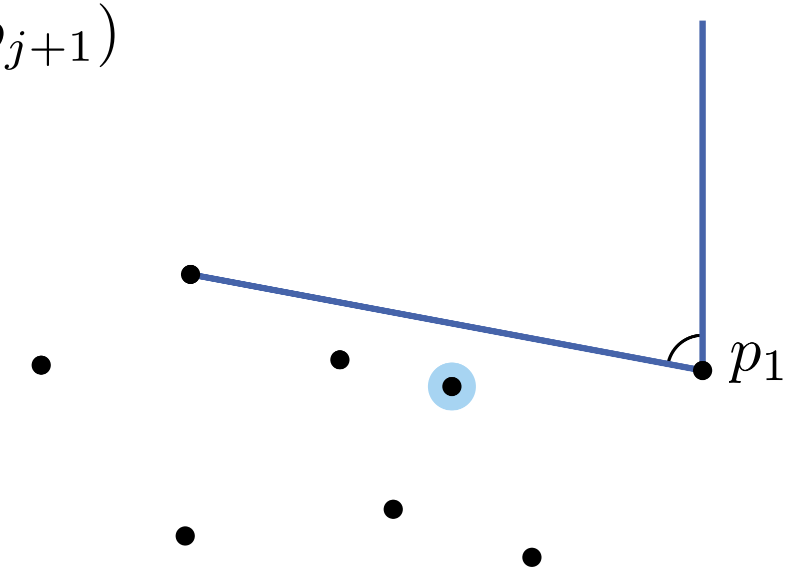

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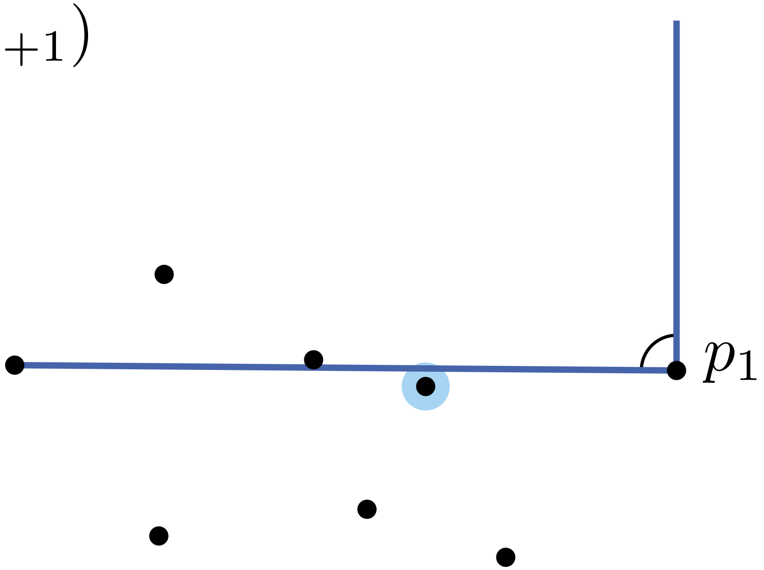

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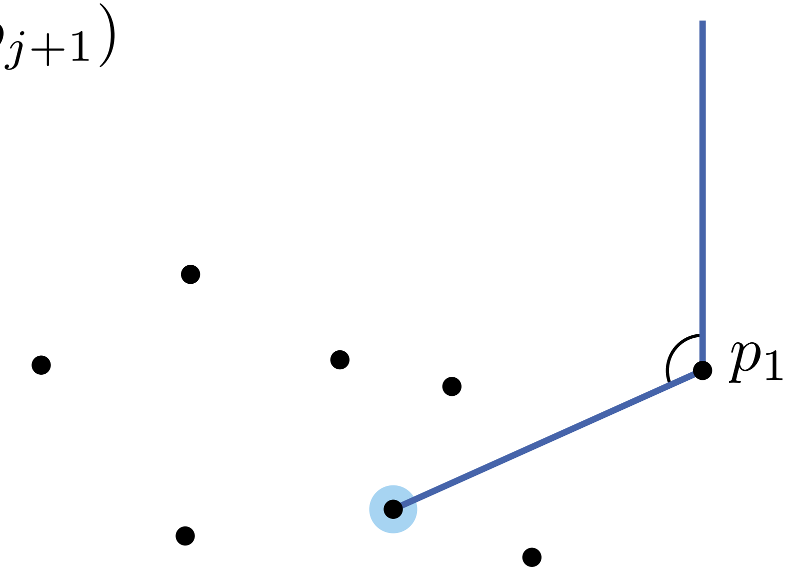

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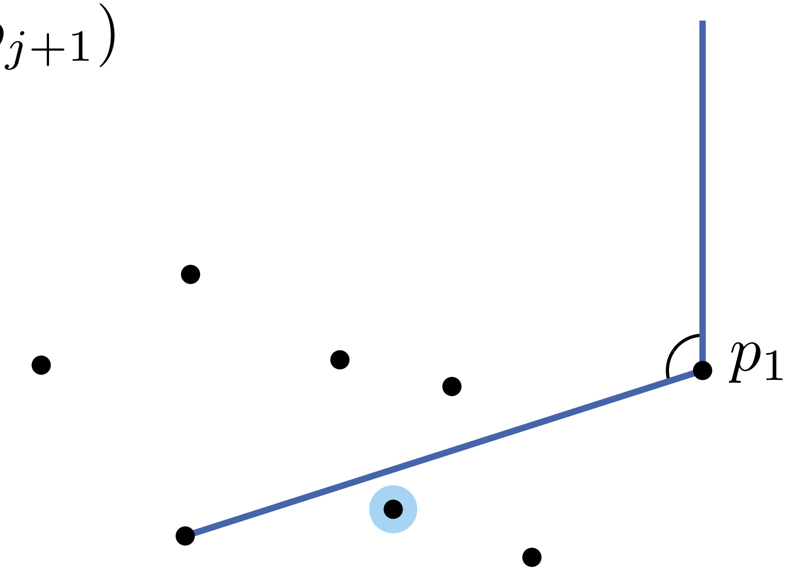

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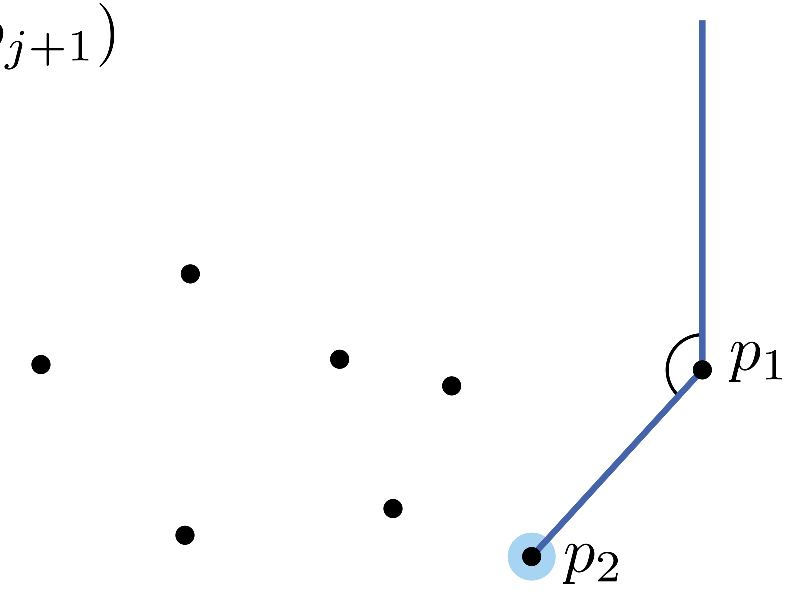

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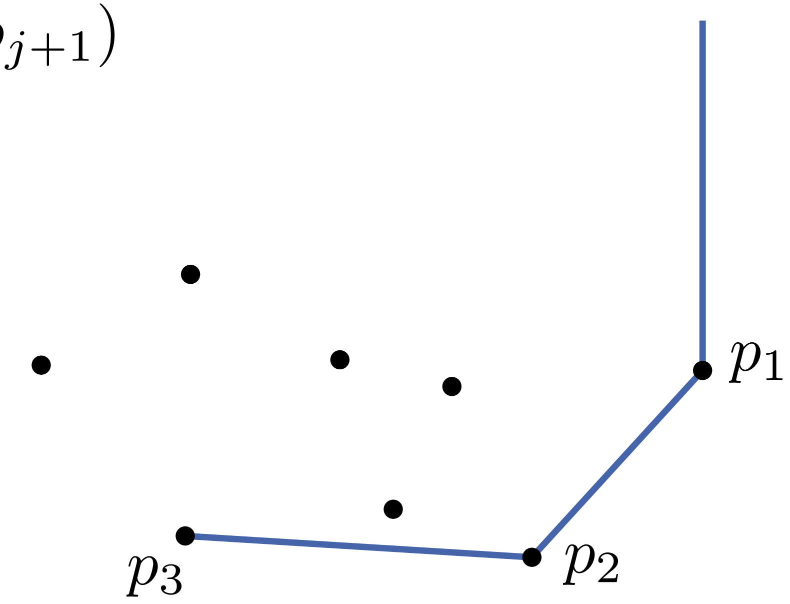

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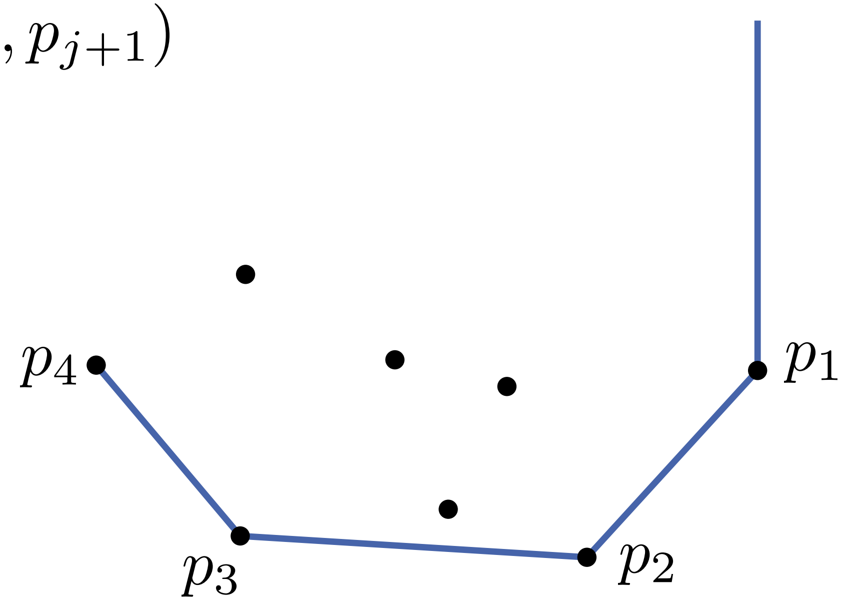

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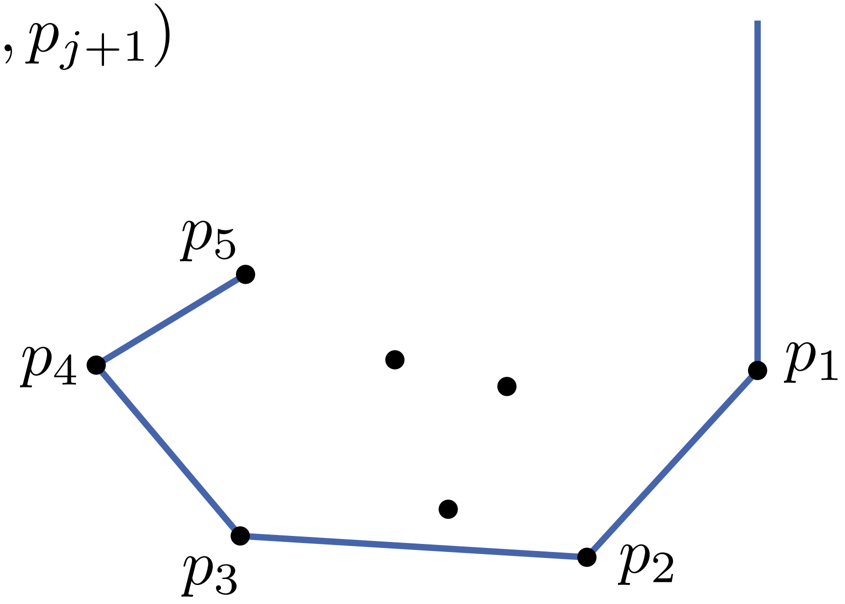

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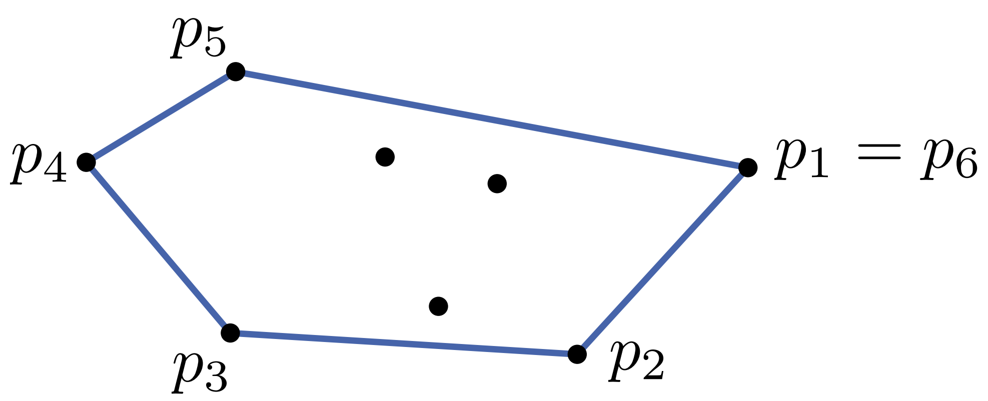

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Correctness (ideas):

- **Base Case**: p_1 lies on convex hull.

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return $(p_1, \ldots, p_{j+1})$

Correctness (ideas):

- **Base Case**: p_1 lies on convex hull.
- **Assumption**: First i points belong to convex hull CH(P)

Alternative: Gift Wrapping

Idea: Begin with a point p_1 of $CH(P)$, then find the next edge of $CH(P)$ in clockwise order.

GiftWrapping(P)

$p_1 = (x_1, y_1) \leftarrow$ rightmost point in P; $p_0 \leftarrow (x_1, \infty)$; $j \leftarrow 1$
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Correctness (ideas):

- **Base Case**: p_1 lies on convex hull.
- **Assumption**: First i points belong to convex hull CH(P)
- **Step**: By assump. p_{i+1} lies to the right of line $\overrightarrow{p_{i-1}p_i}$ $\Rightarrow$ 'right bend'
  By the chosen angle: all points lie to the right of line $\overrightarrow{p_i p_{i+1}}$

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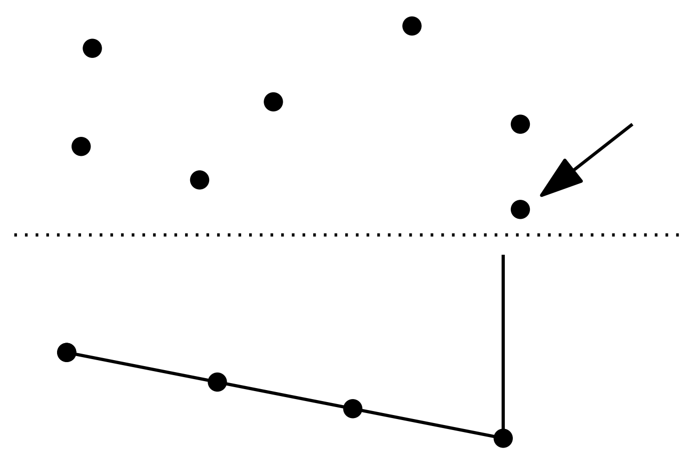
Degenerated cases:

1. Choice of p_1 is not unique.
   Choose the bottommost rightmost point.

2. Choice of p_{j+1} is not unique.
   Choose the point of largest distances.



Computation of Tangents

Given: convex polygon P (clockswise) and point p outside of P

Find: *right* tangent at P through p in $O(\log n)$ time.

Right tangent means polygon lies left to tangent.

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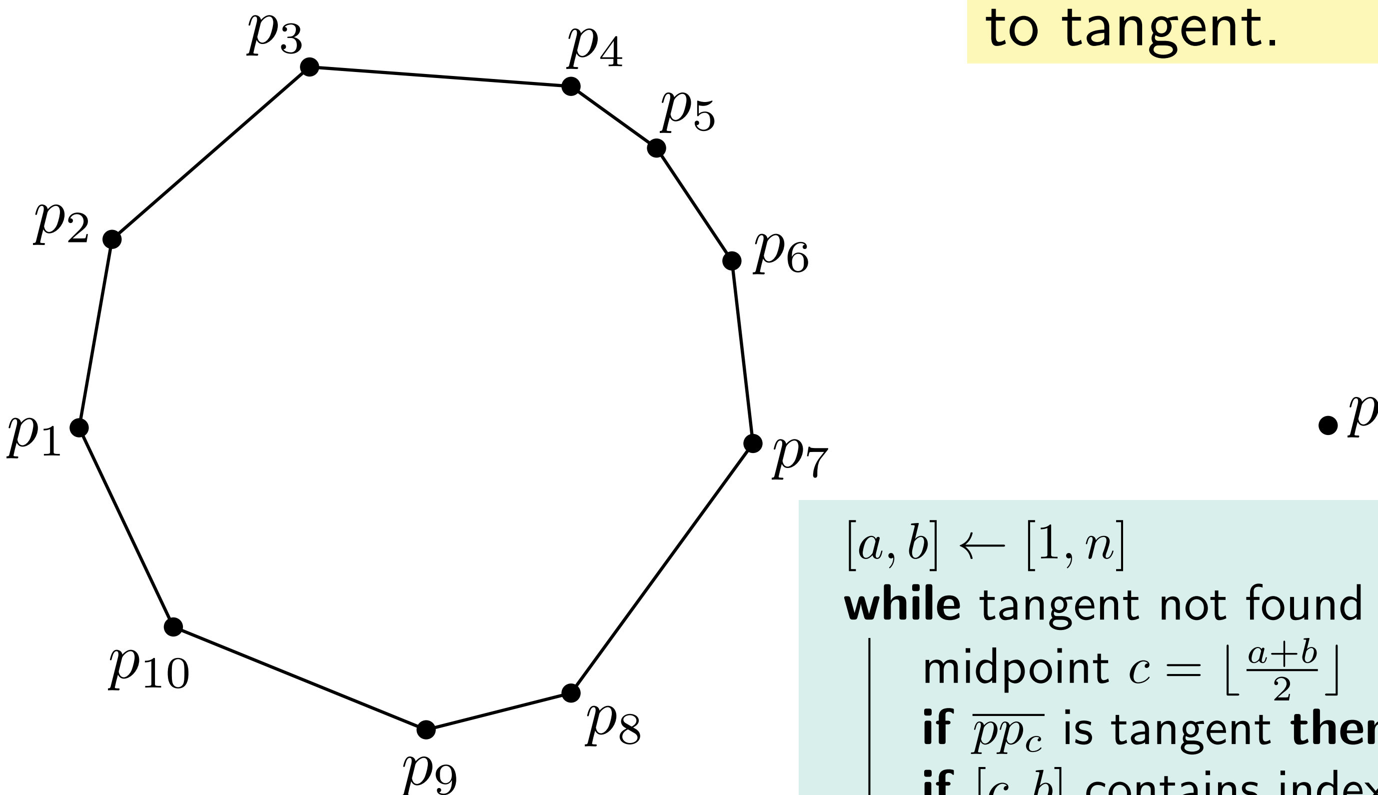

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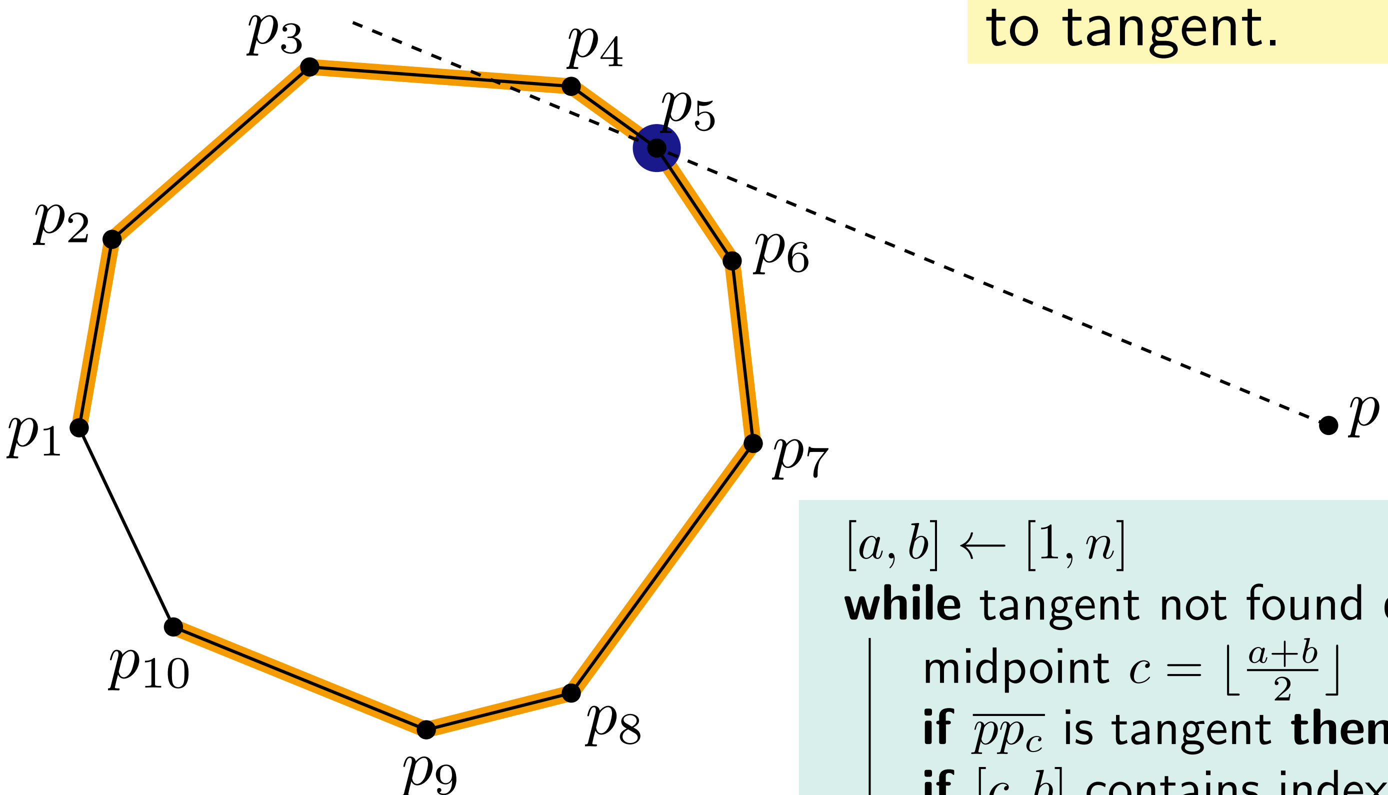

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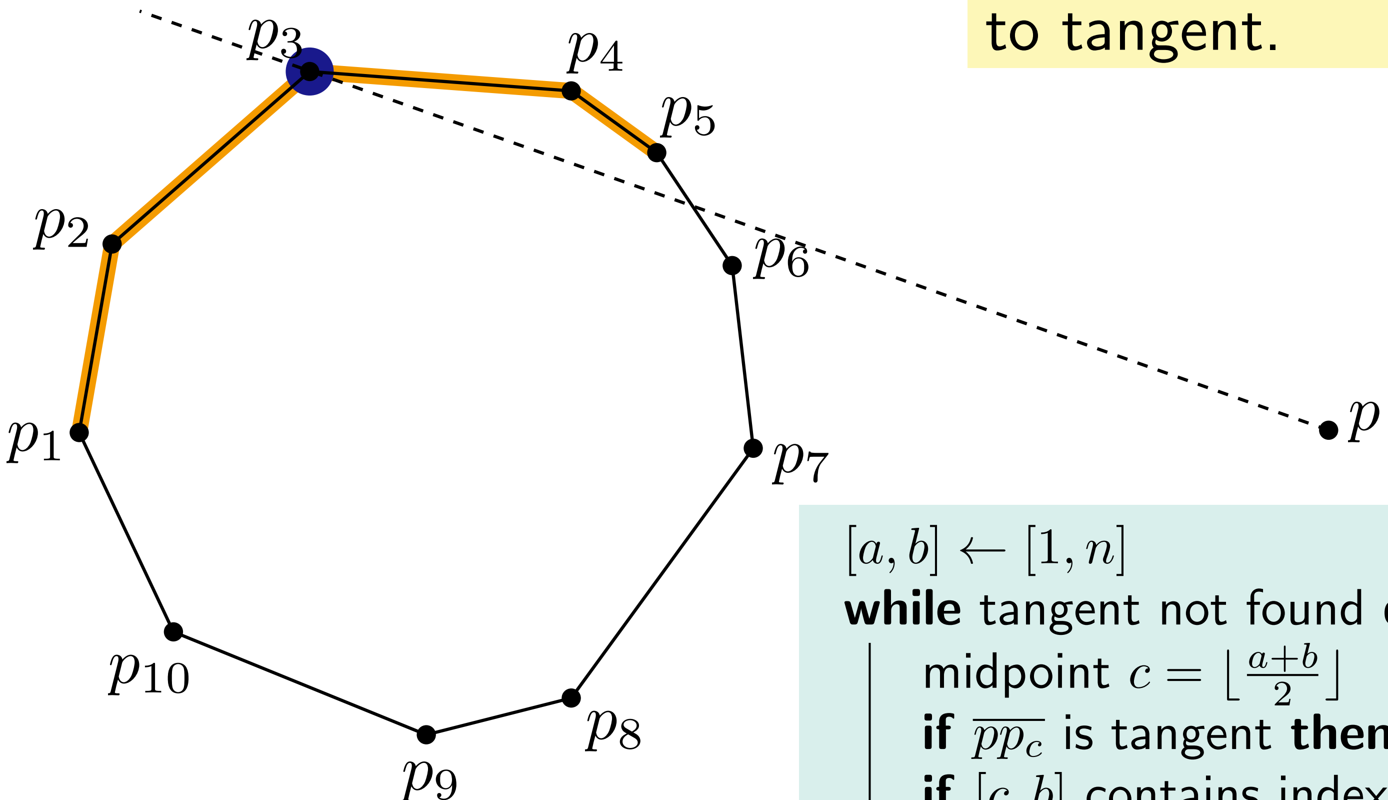

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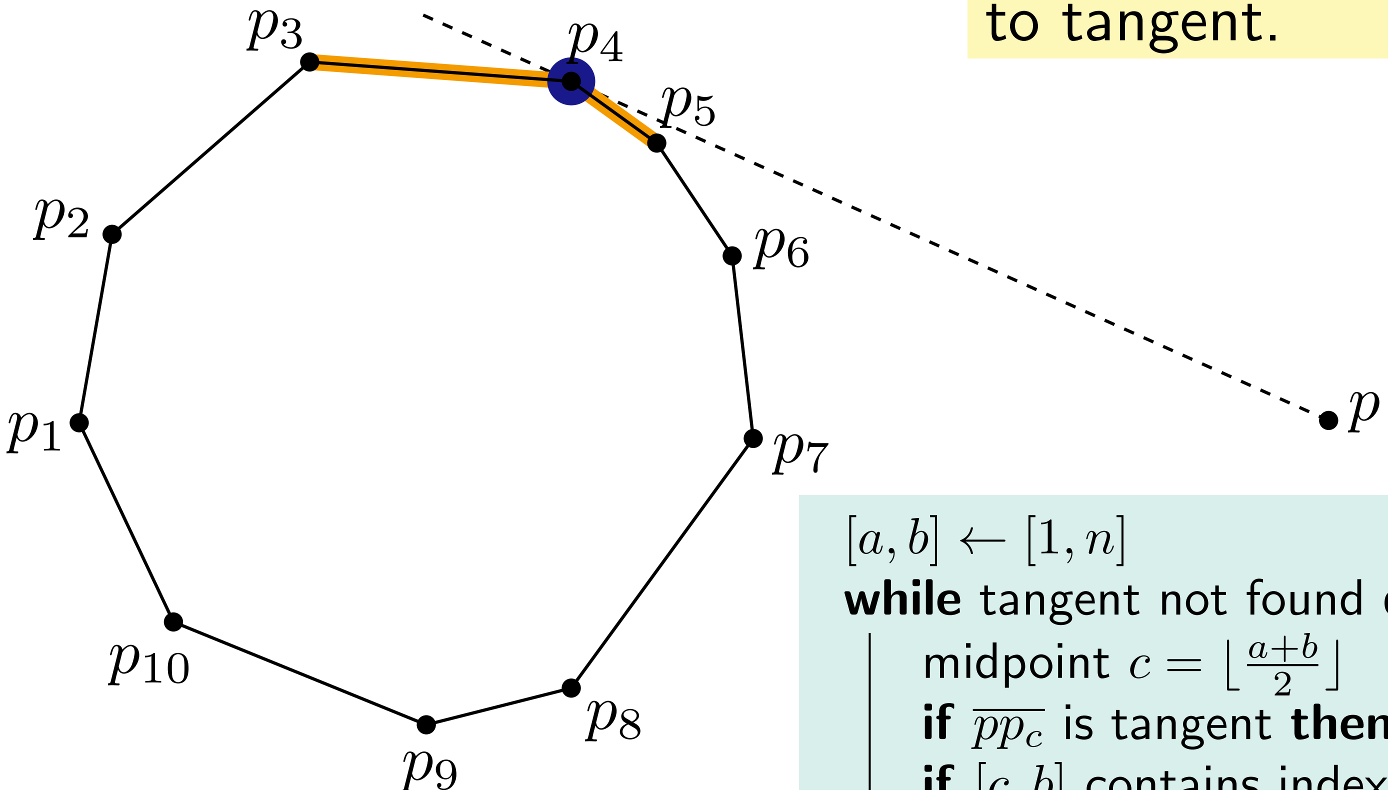


```
[a, b] ← [1, n]
while tangent not found do
    midpoint c = ⌊(a+b)/2⌋
    if pp_c is tangent then return p_c
    if [c, b] contains index of contact point then
        [a, b] ← [c, b]
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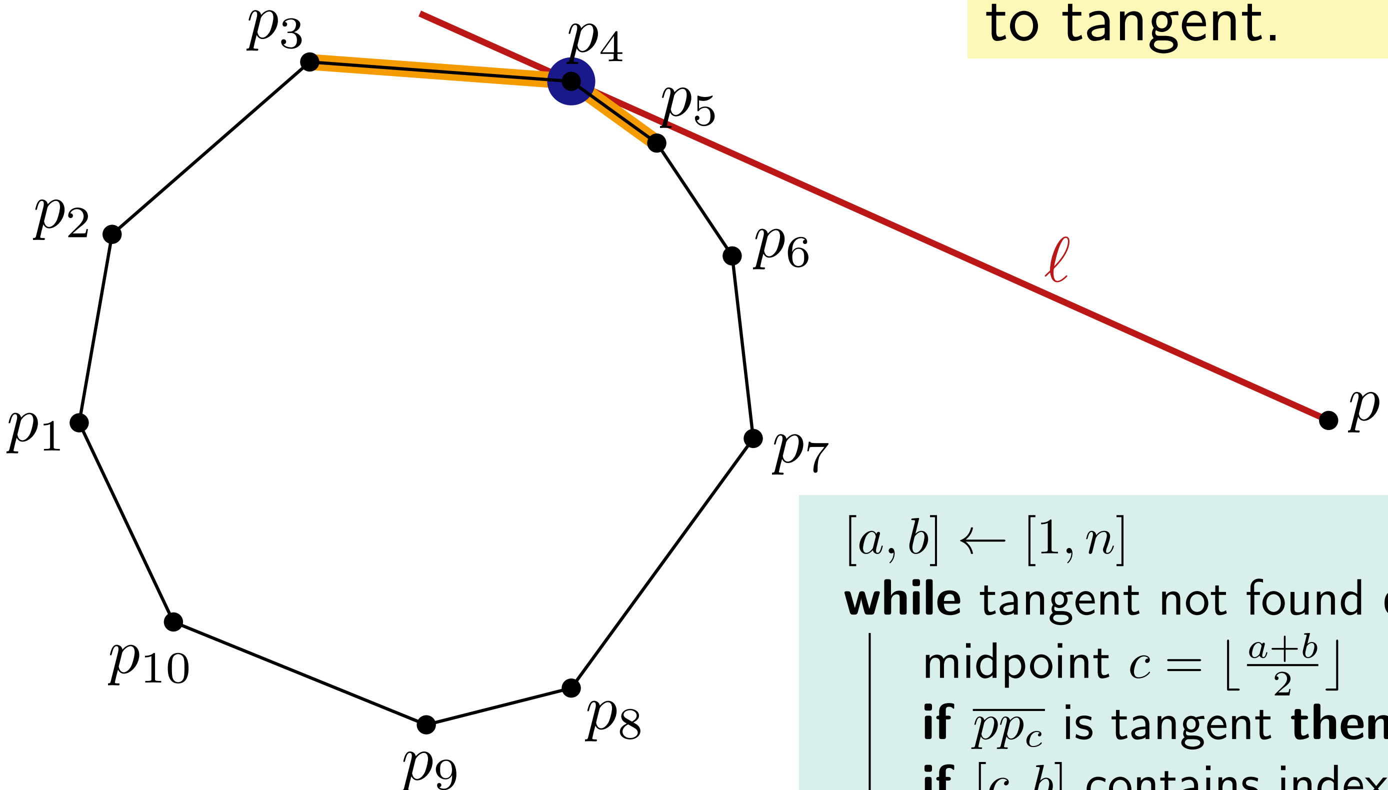

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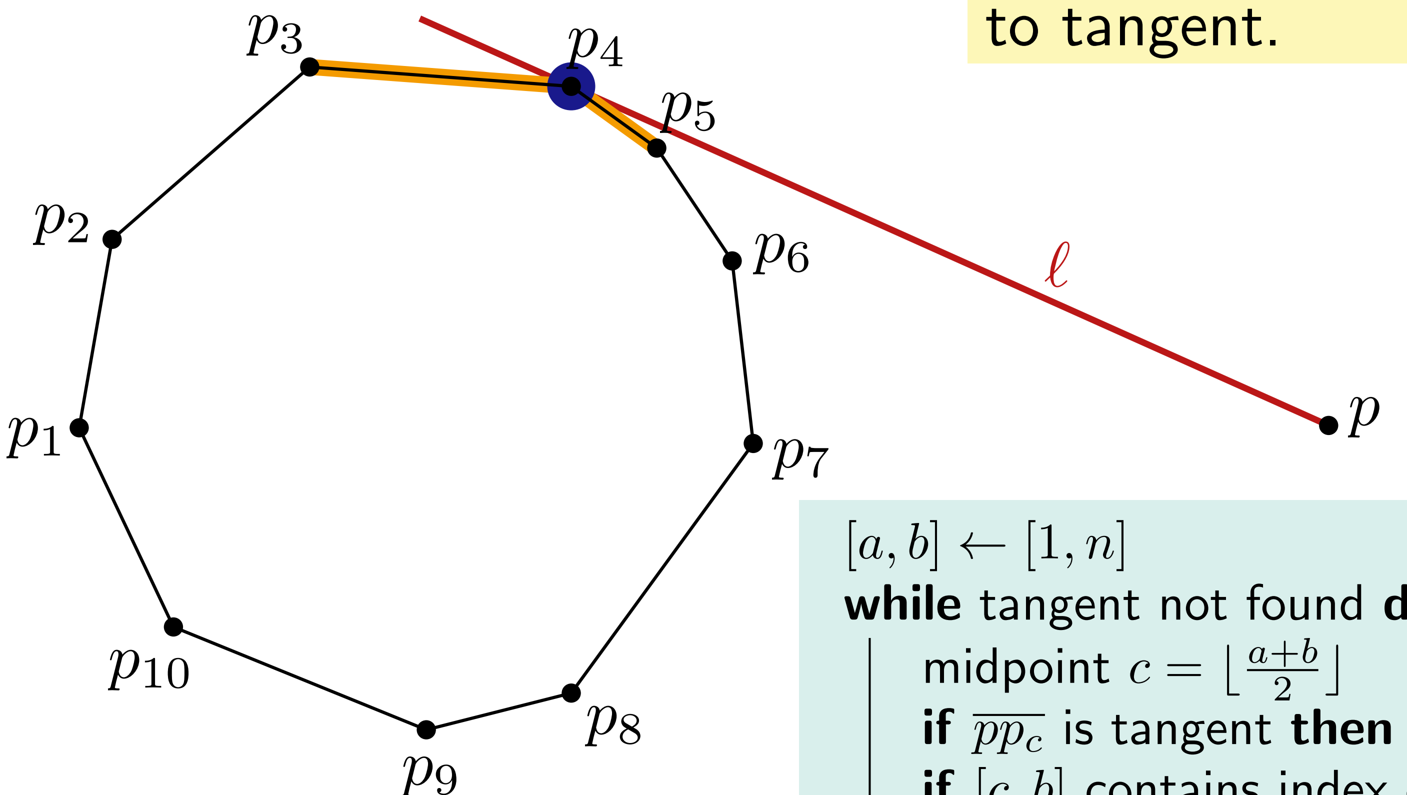

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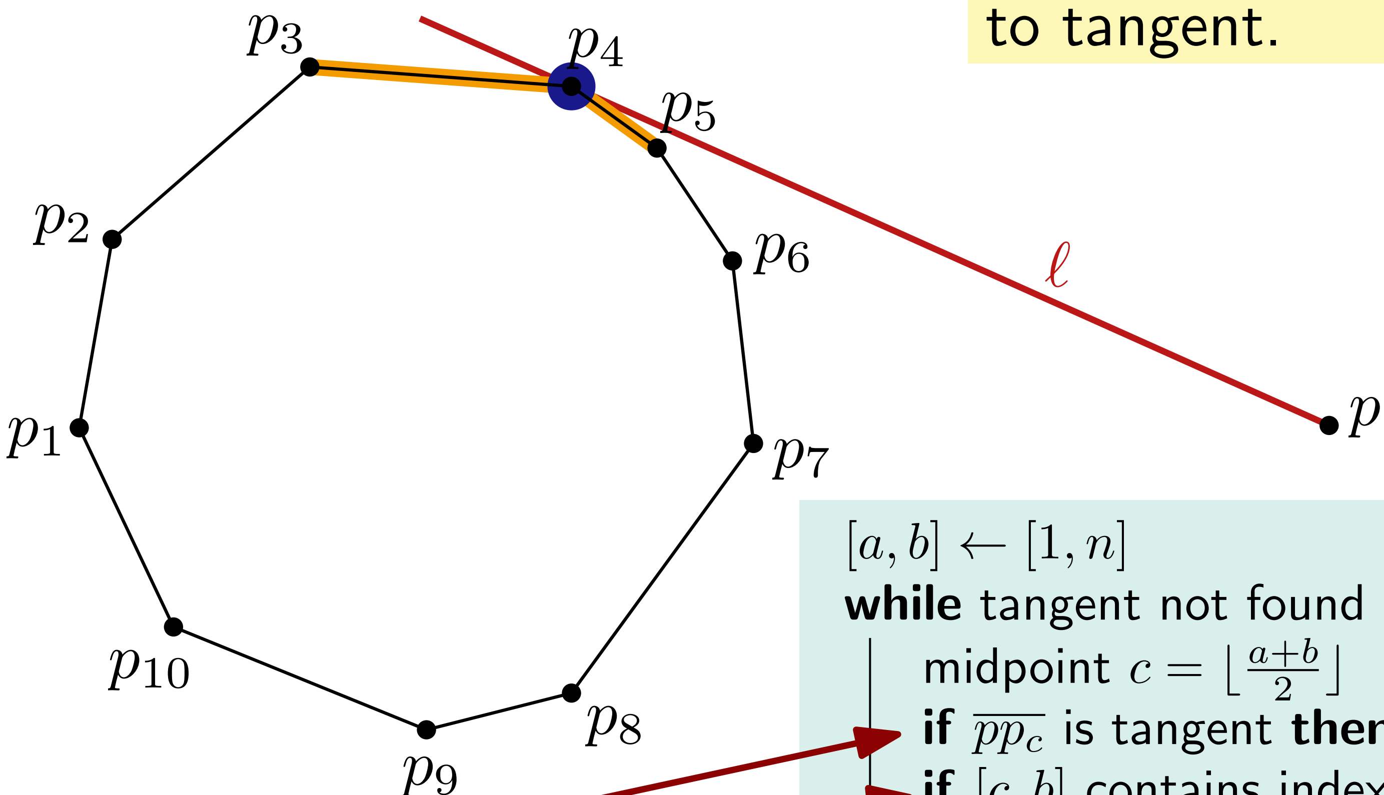

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How to test in constant time?

Computation of Tangents

$[a, b] \leftarrow [1, n]$
while tangent not found **do**
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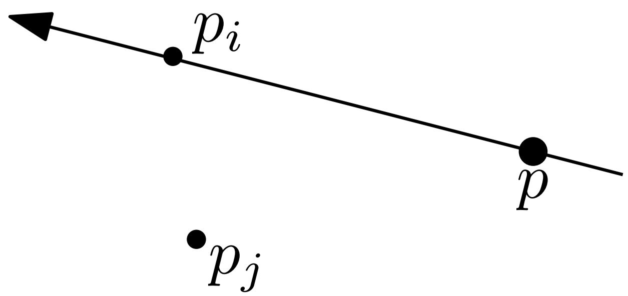

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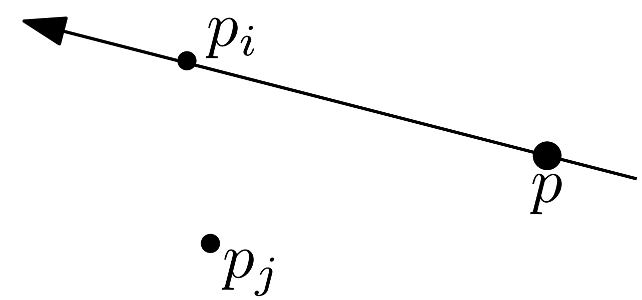

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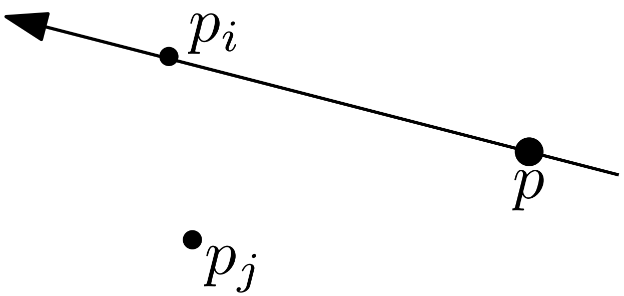

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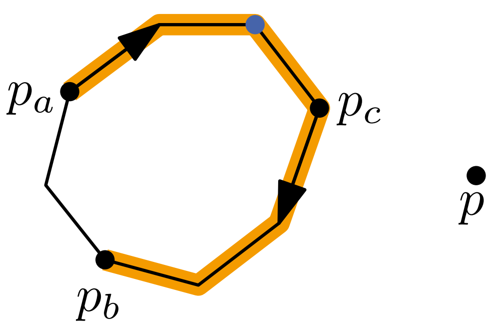
p_{a+1} above p_a:

p_c above p_{c+1}



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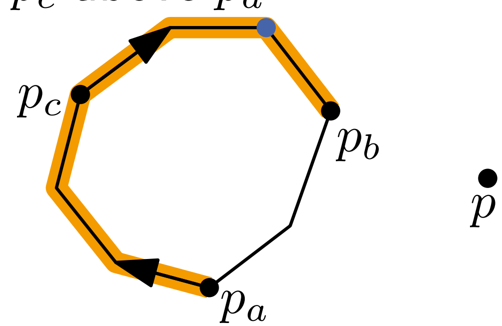

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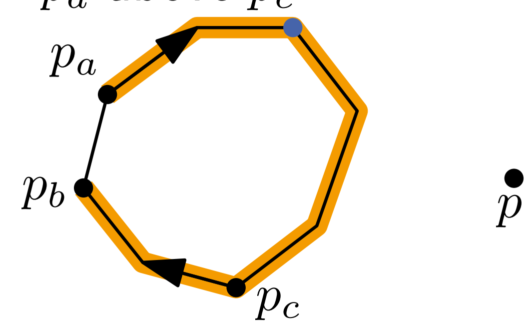
p_a above p_c



$[a, b] \leftarrow [a, c]$

p_a above p_{a+1}: Analogous statements.

Lower Bound

We require that any algorithm computing the convex hull of a given set of points returns the convex hull vertices as a clockwise sorted list of points.

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We require that any algorithm computing the convex hull of a given set of points returns the convex hull vertices as a clockwise sorted list of points.

1. Show that any algorithm for computing the convex hull of n points has a worst case running time of $\Omega(n \log n)$ and thus *Graham Scan* is worst-case optimal.

2. Why is the running time of the *gift wrapping* algorithm not in contradiction to part (a)?

Outline

Convex Hull

Line Segment Intersection

Problem Formulation

Given: Set $S = \{s_1, \ldots, s_n\}$ of line segments in the plane

Output:
- all intersections of two or more line segments
- for each intersection, the line segments involved.

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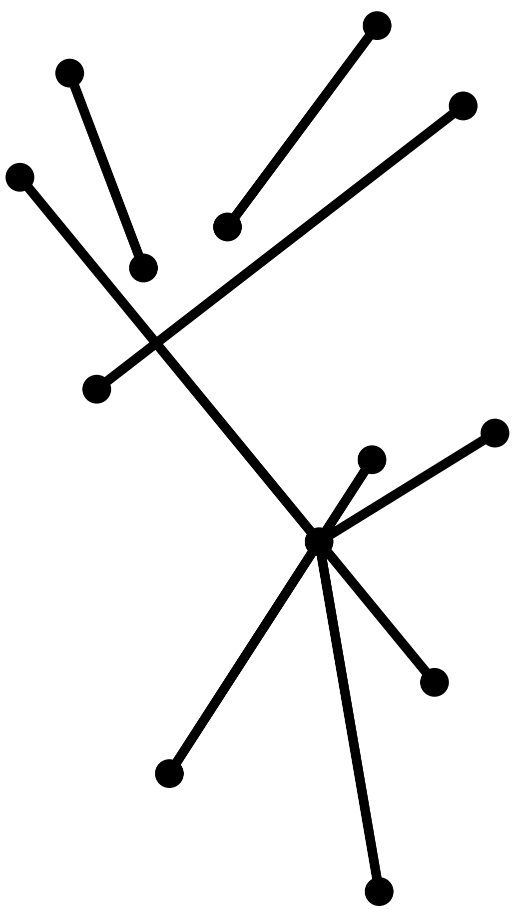
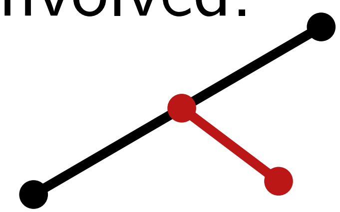
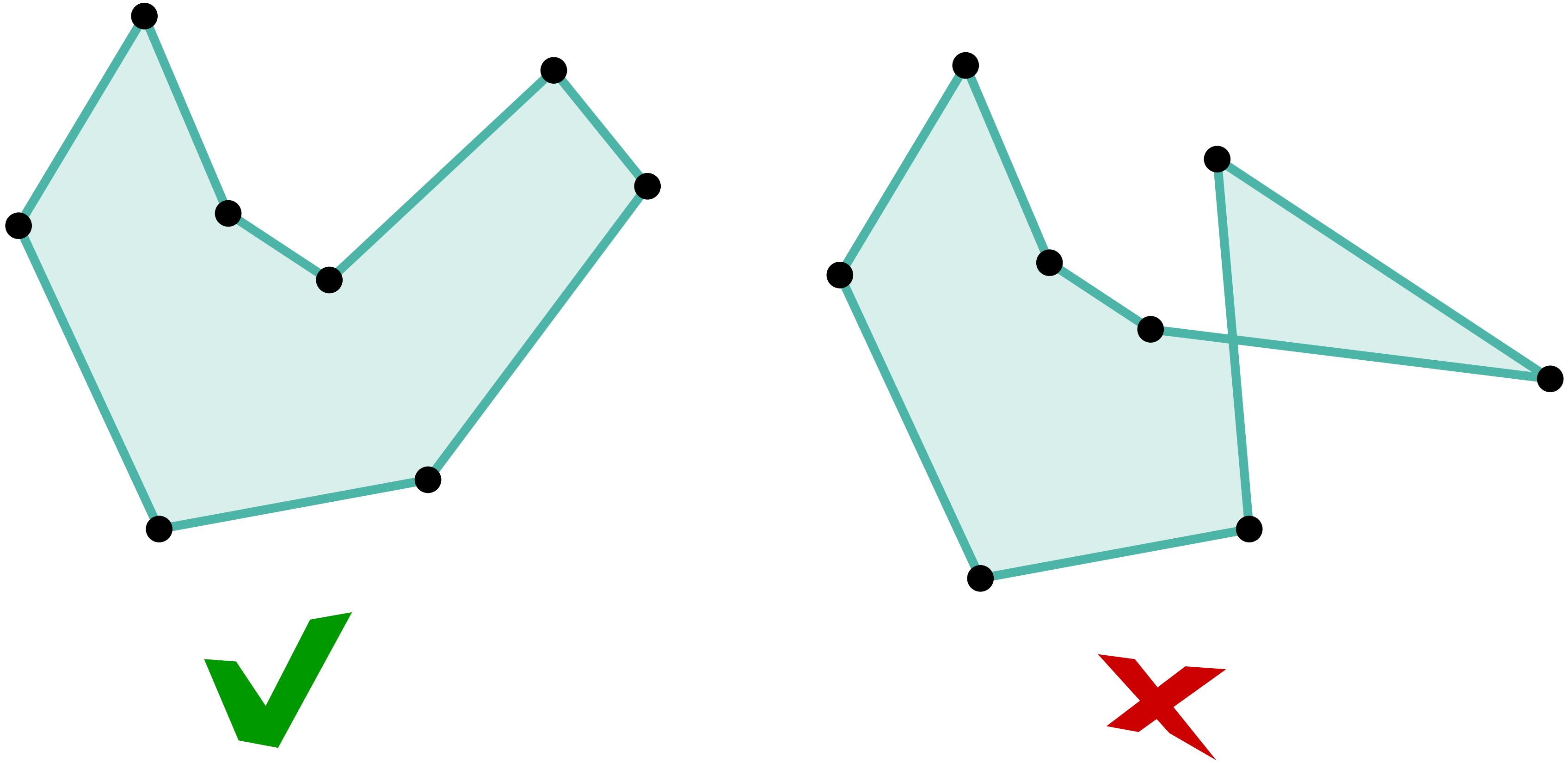




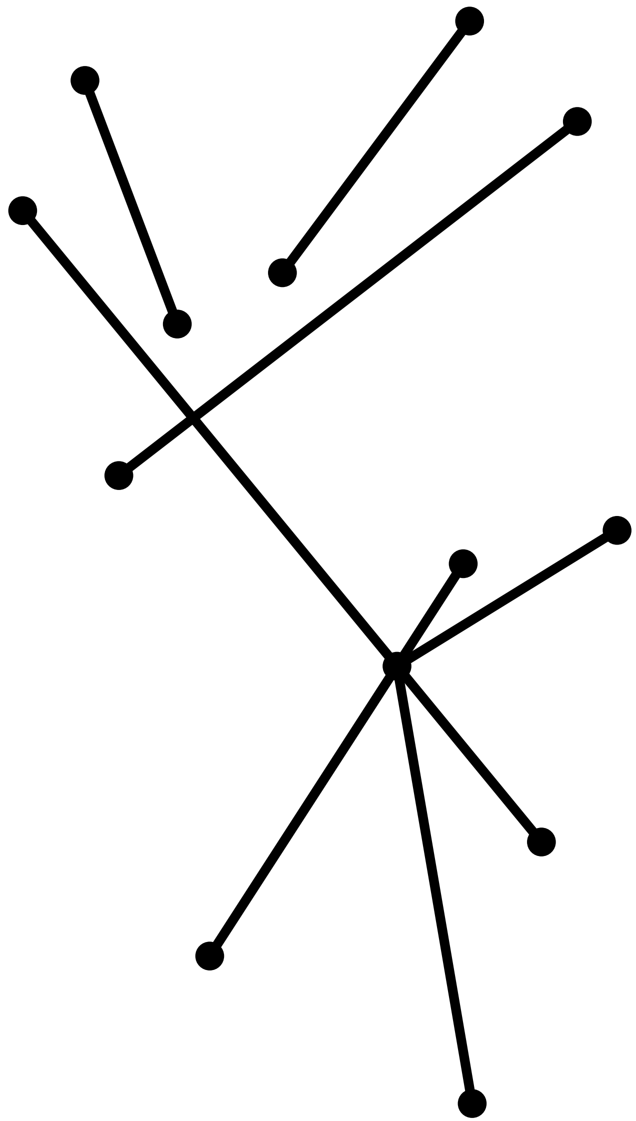
Warm Up

Find:

Algorithm that determines whether a polygon has no self-intersection using $\mathcal{O}(n \log n)$ running time.



Sweep-Line: Example



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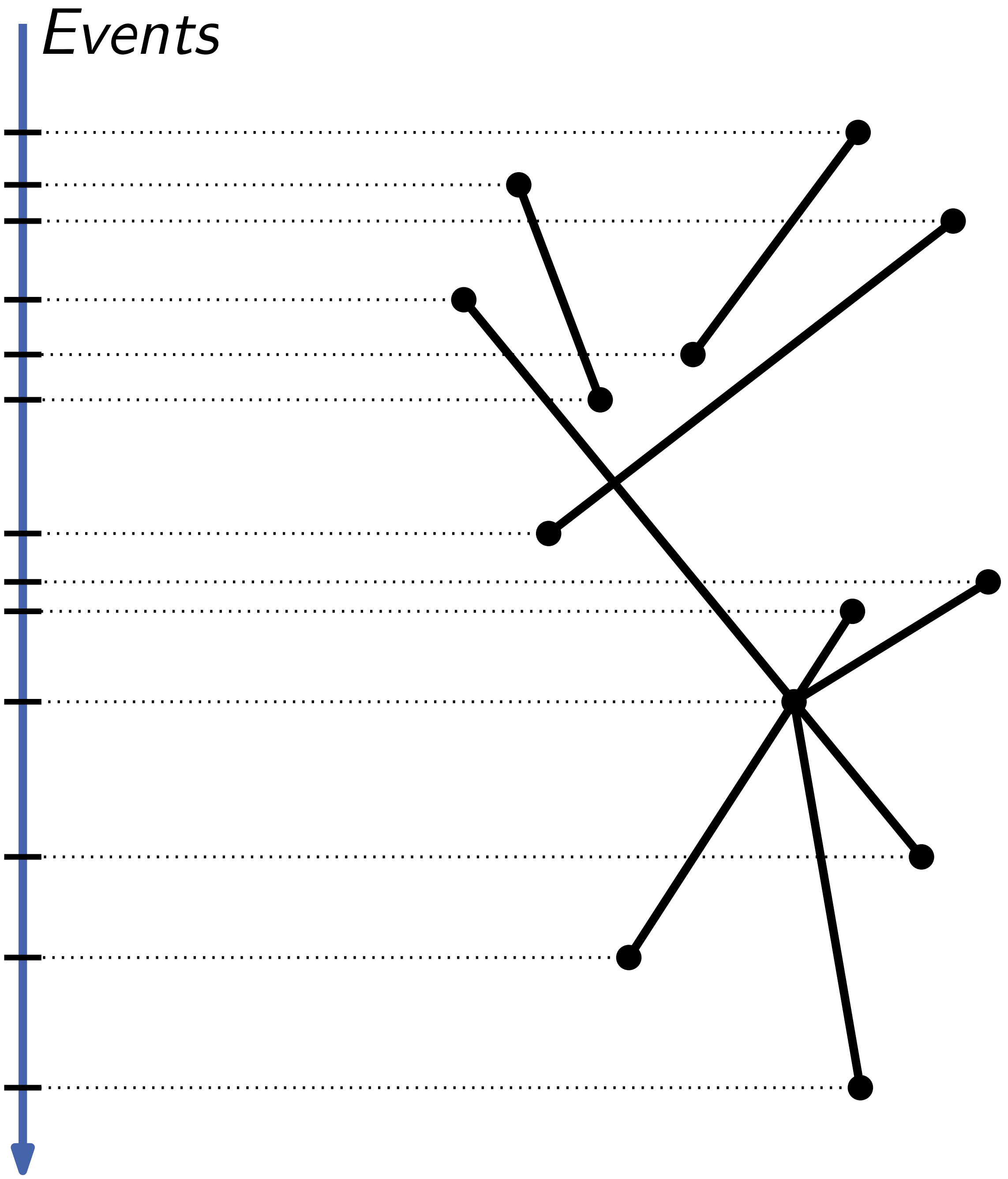

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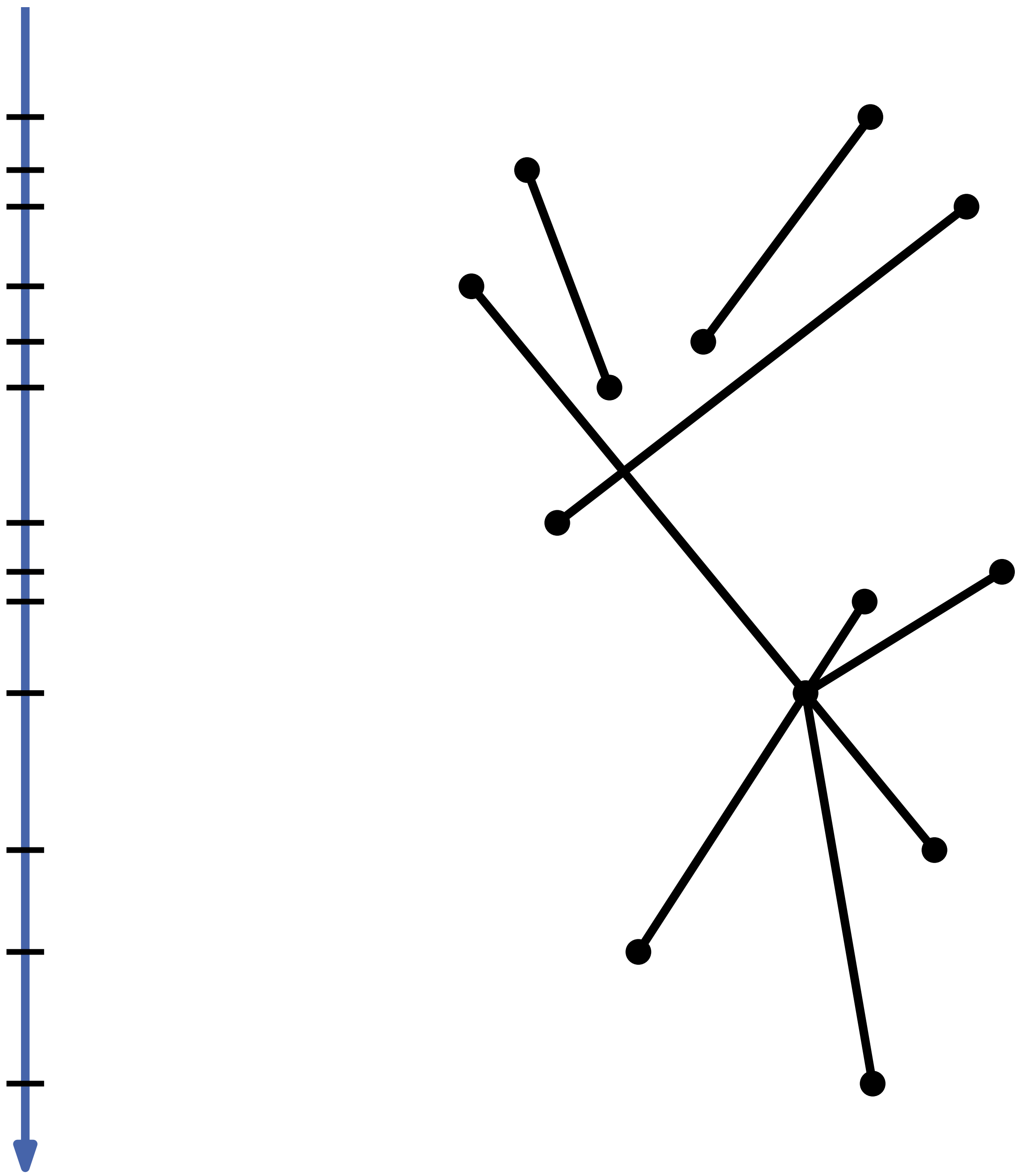

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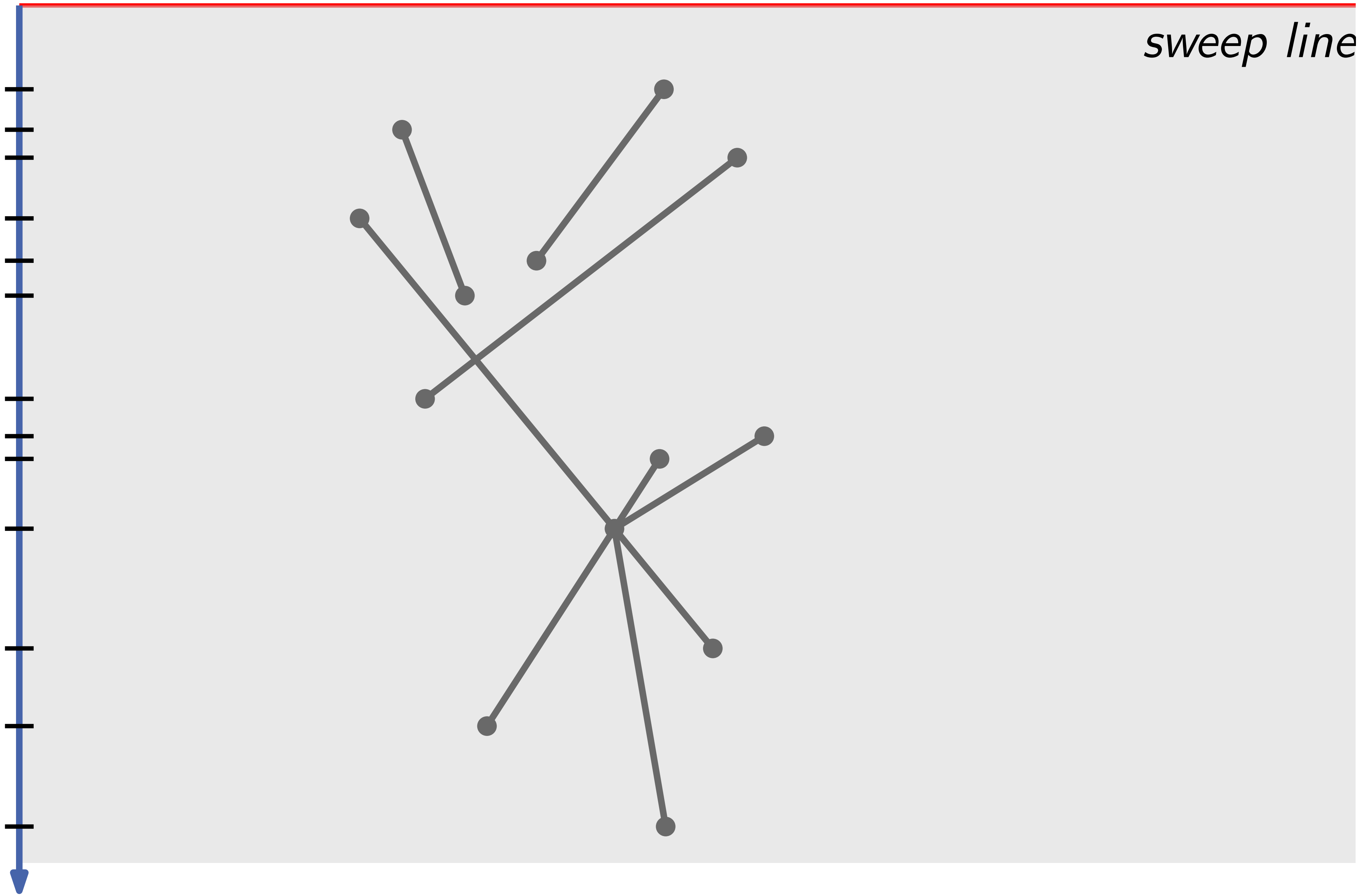

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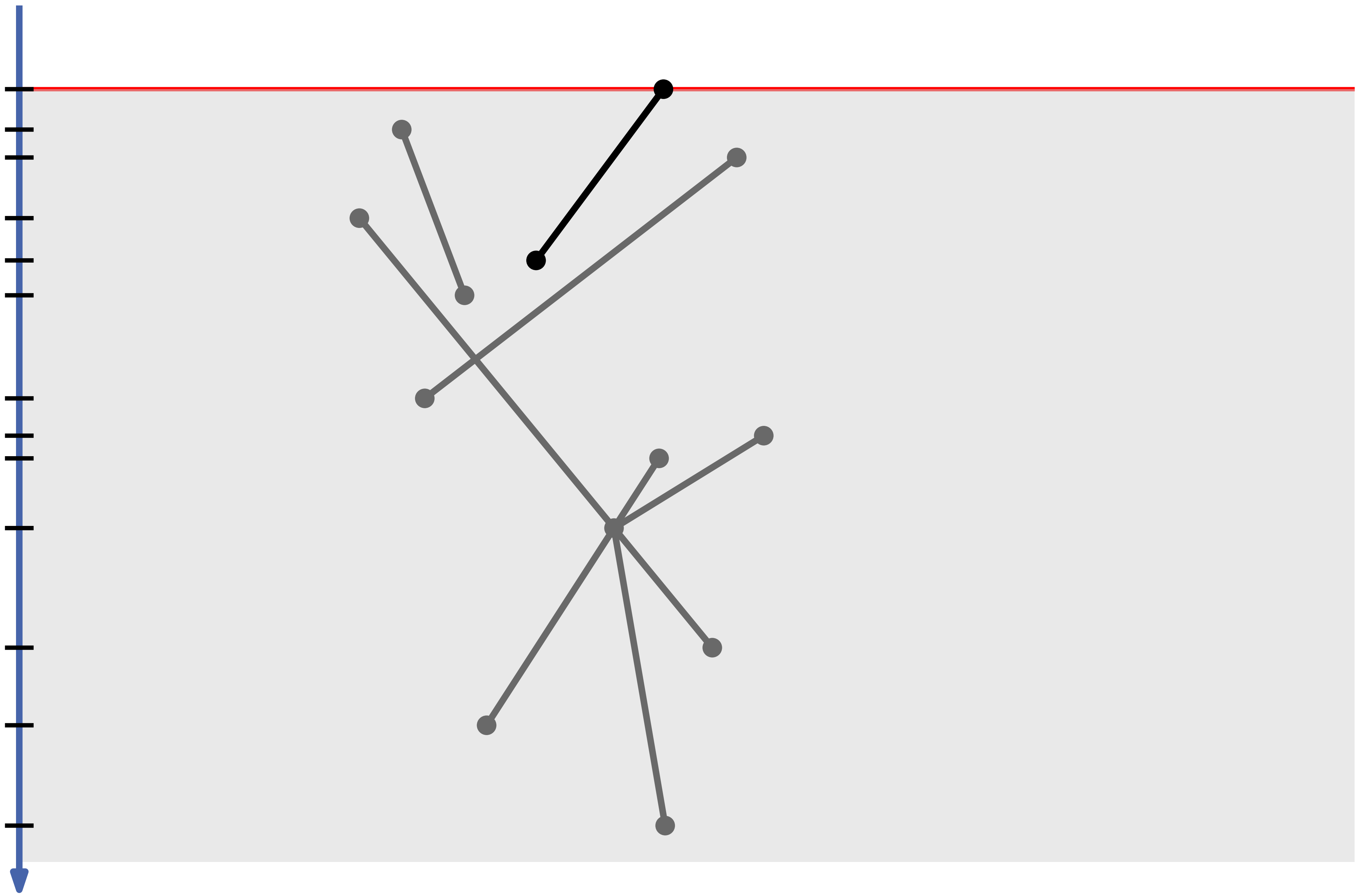

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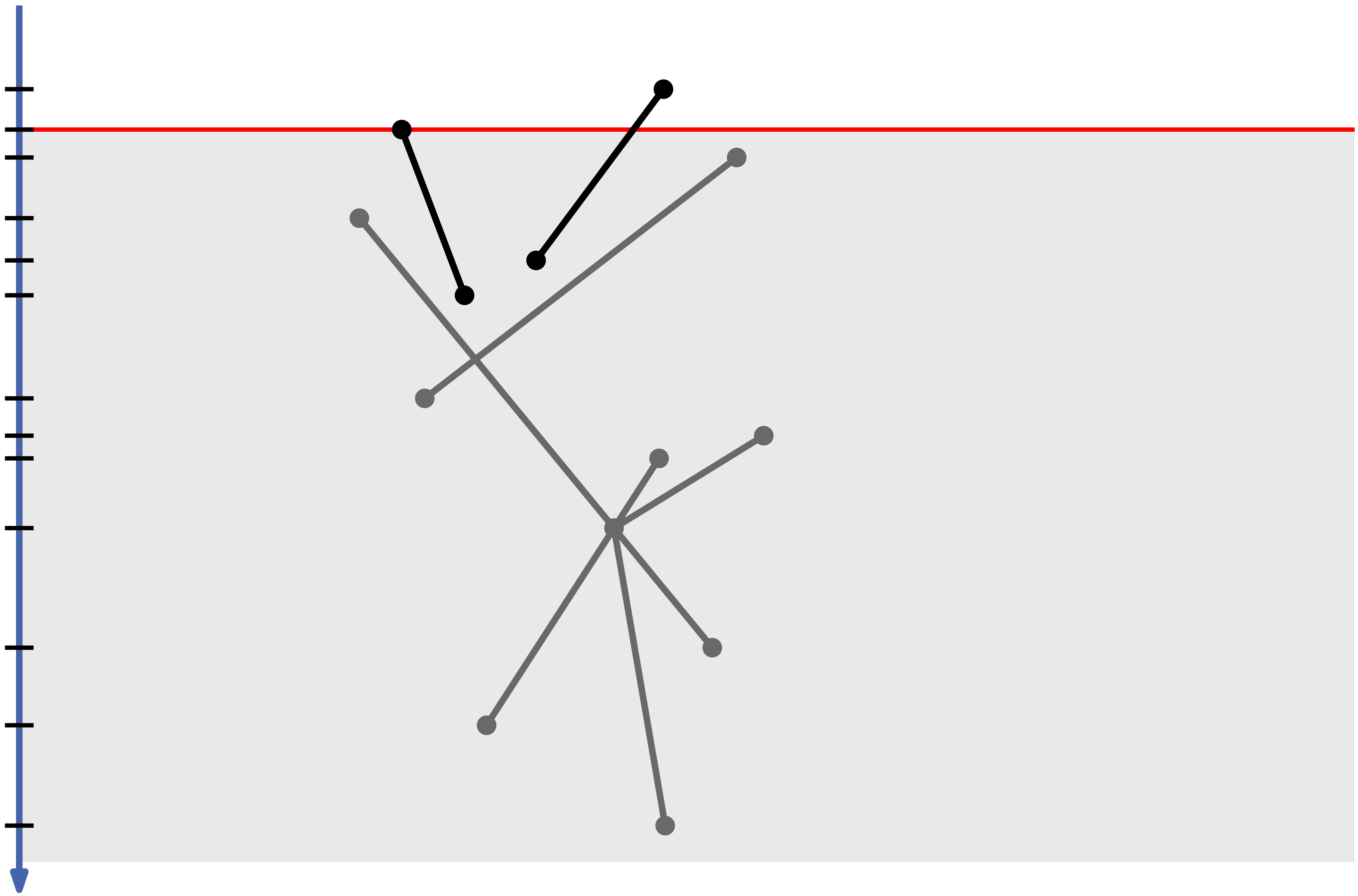

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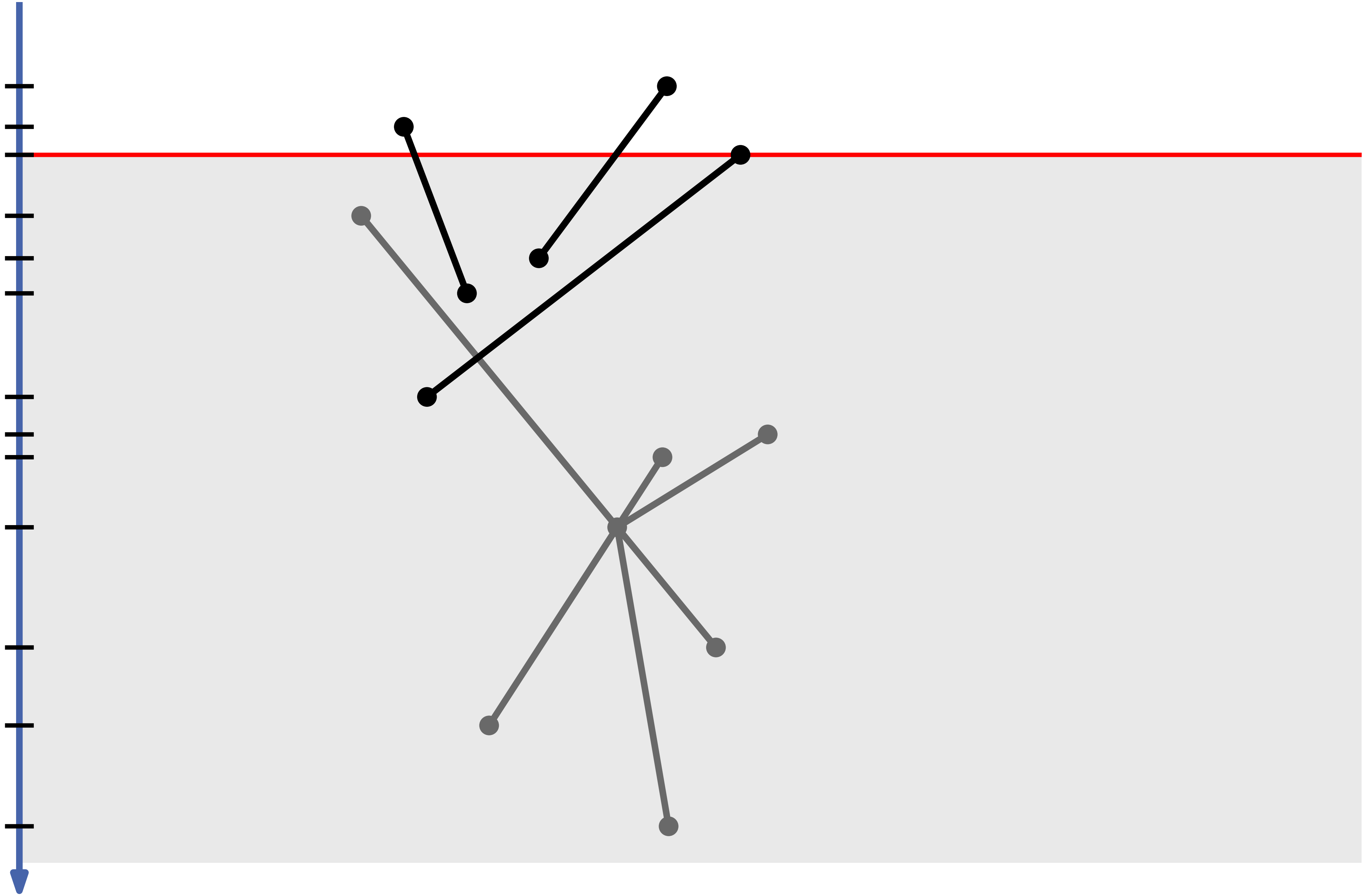

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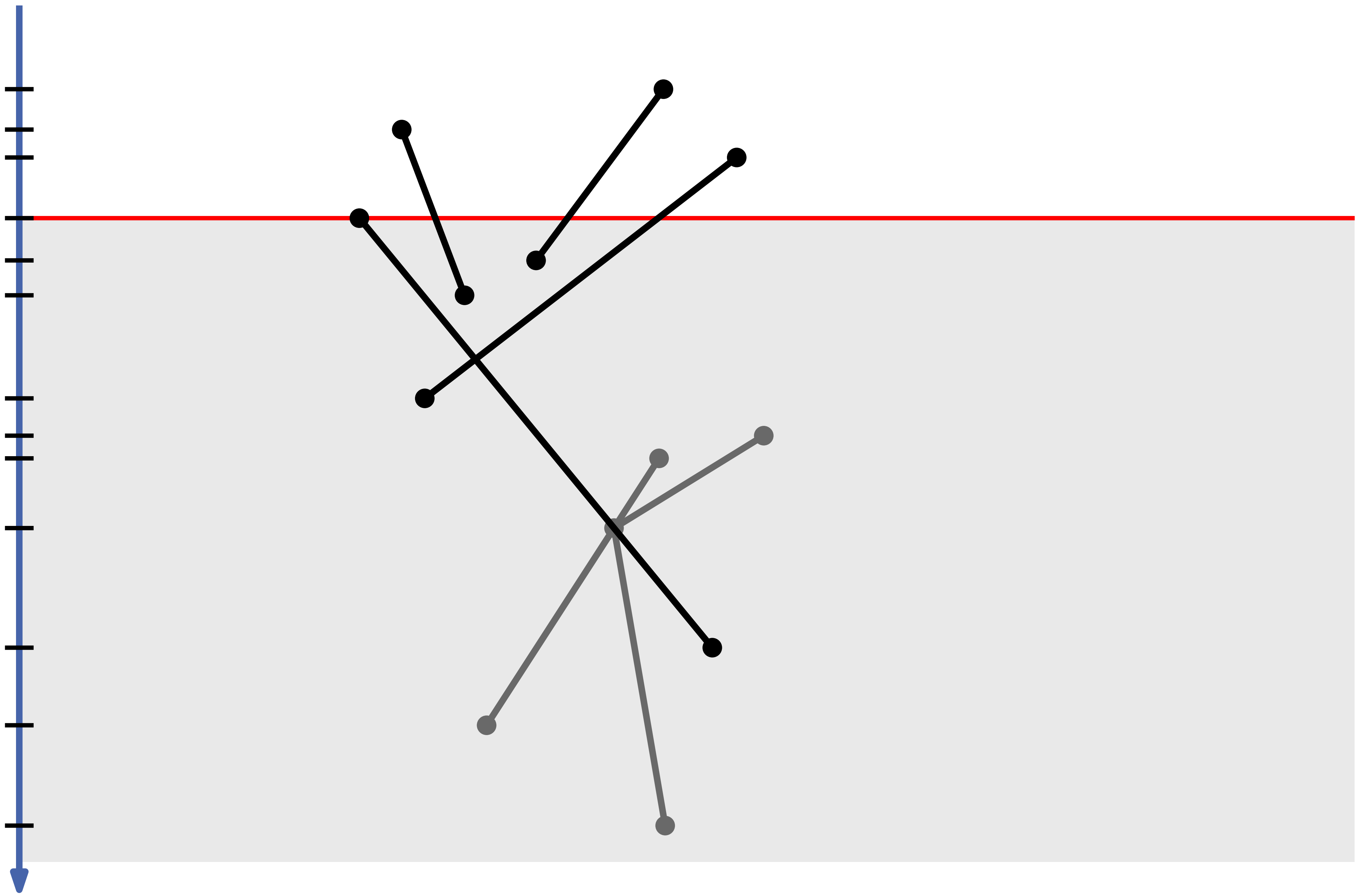

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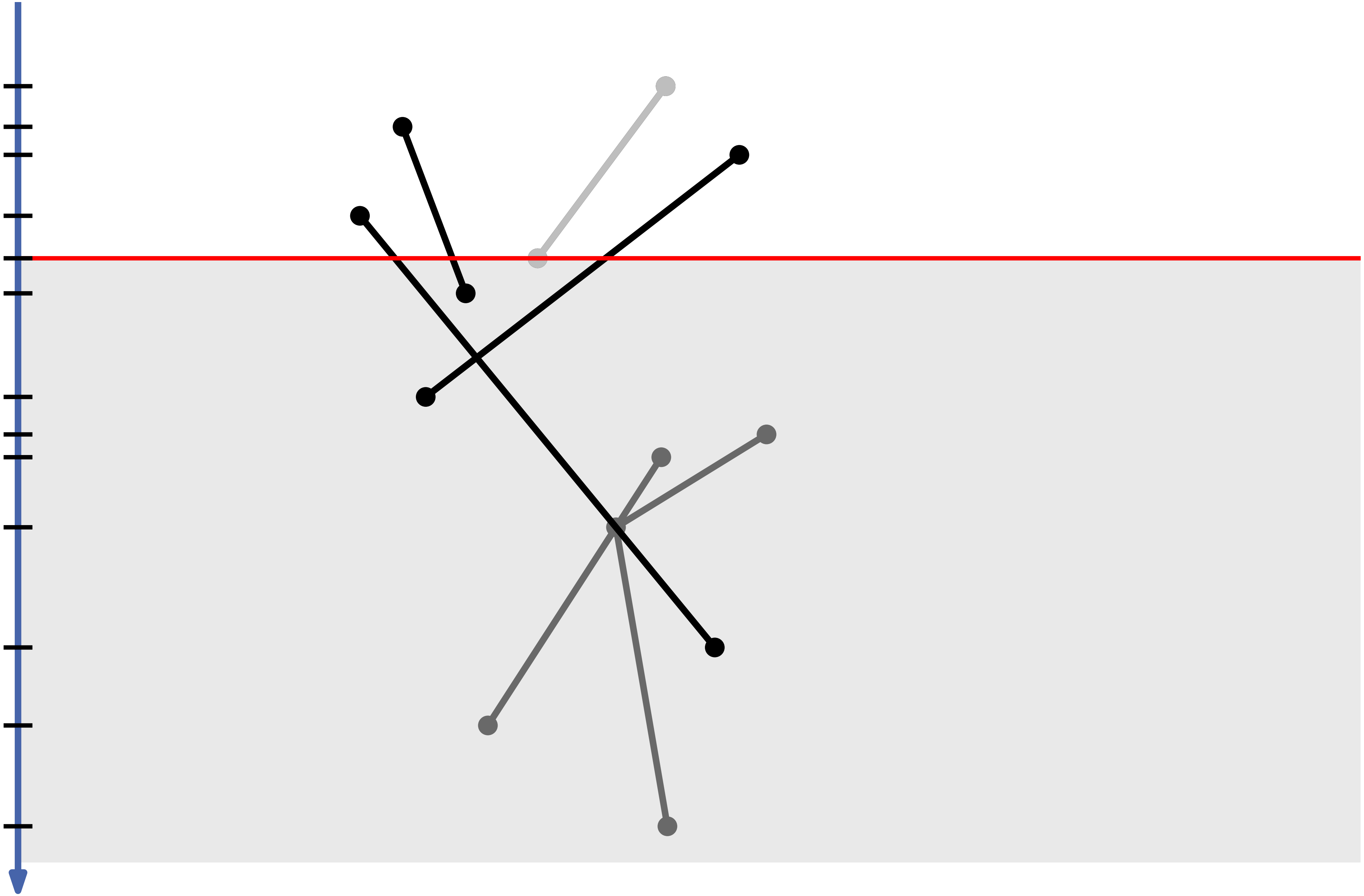

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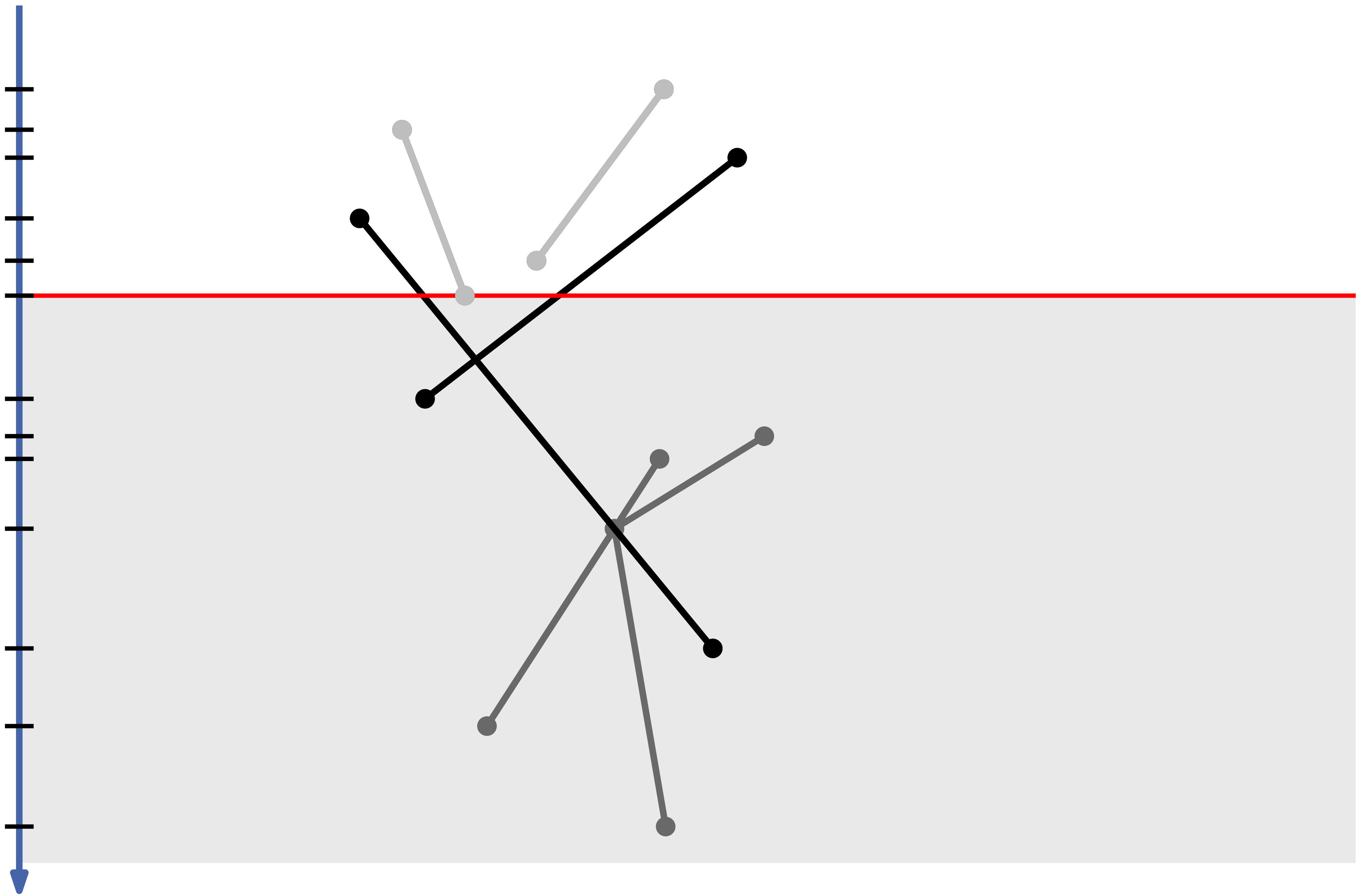

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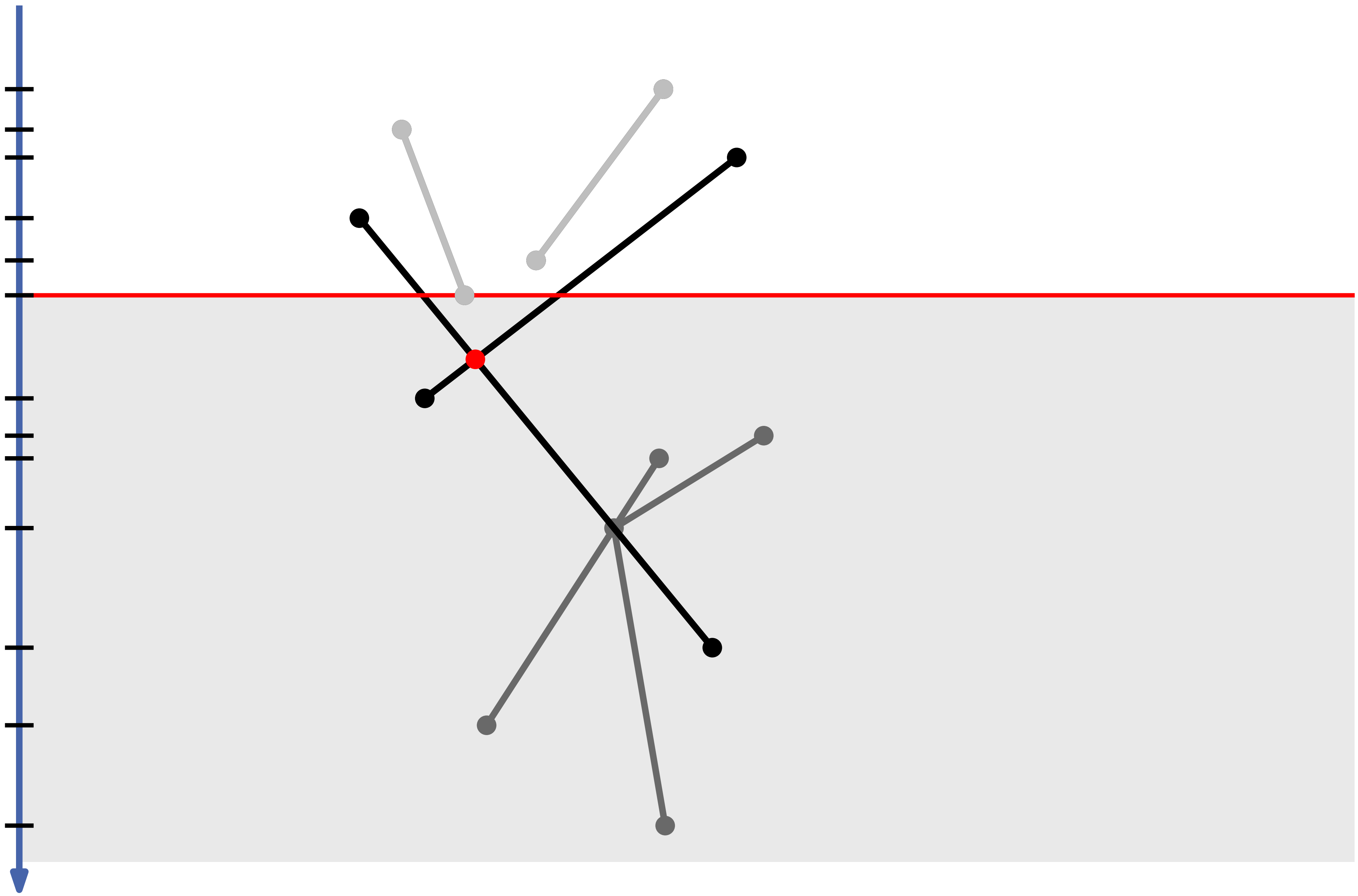

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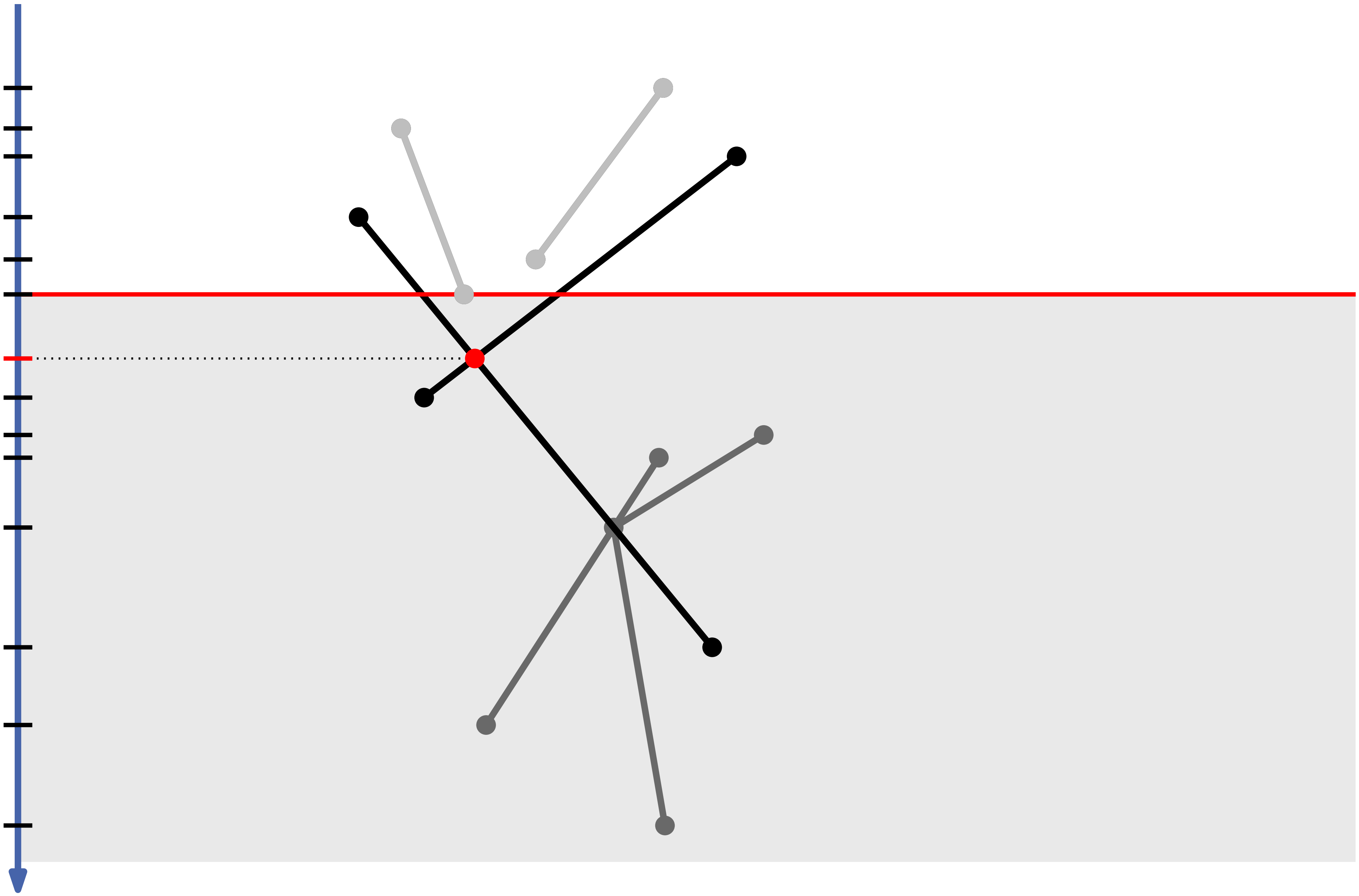

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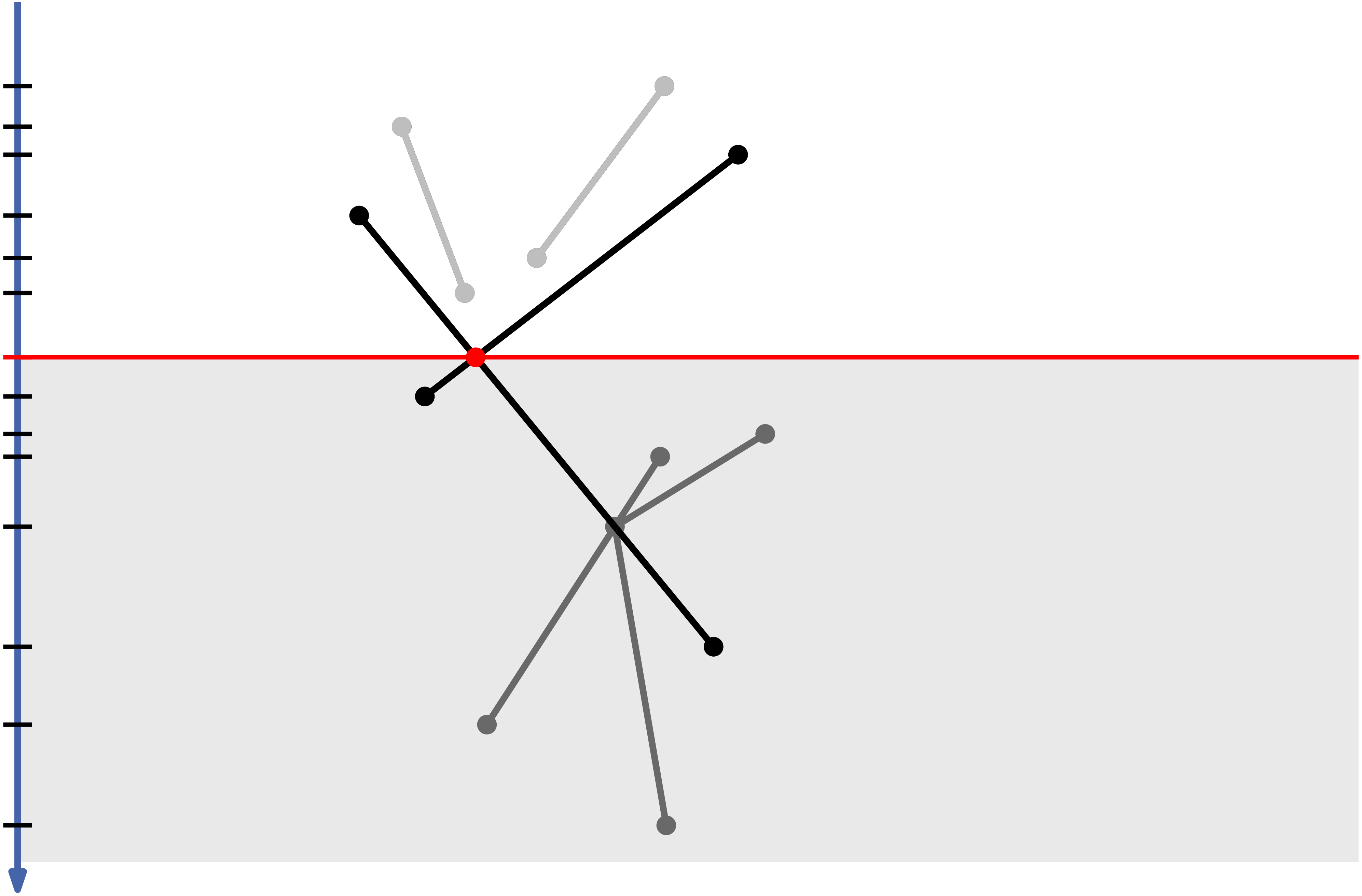

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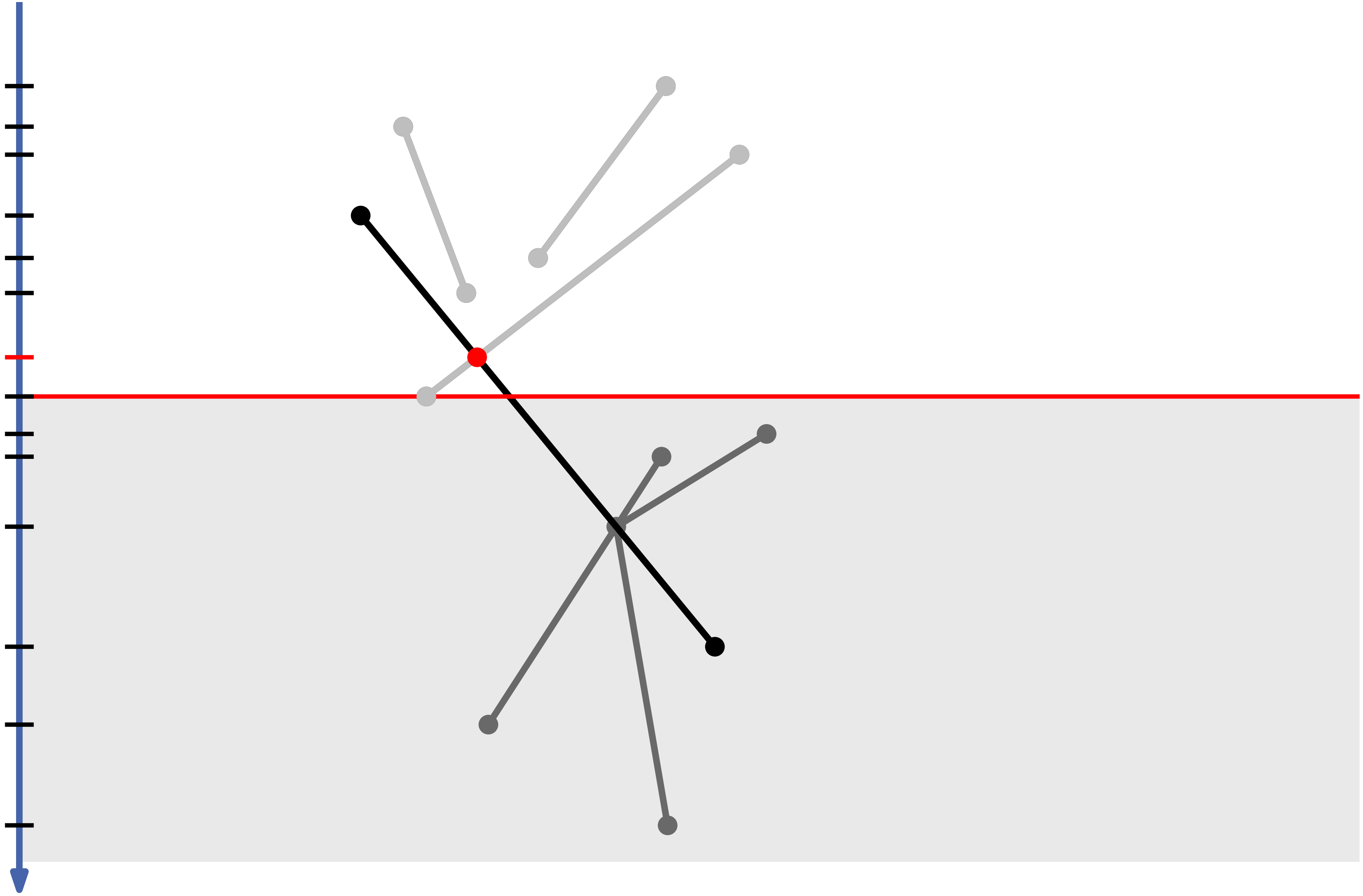

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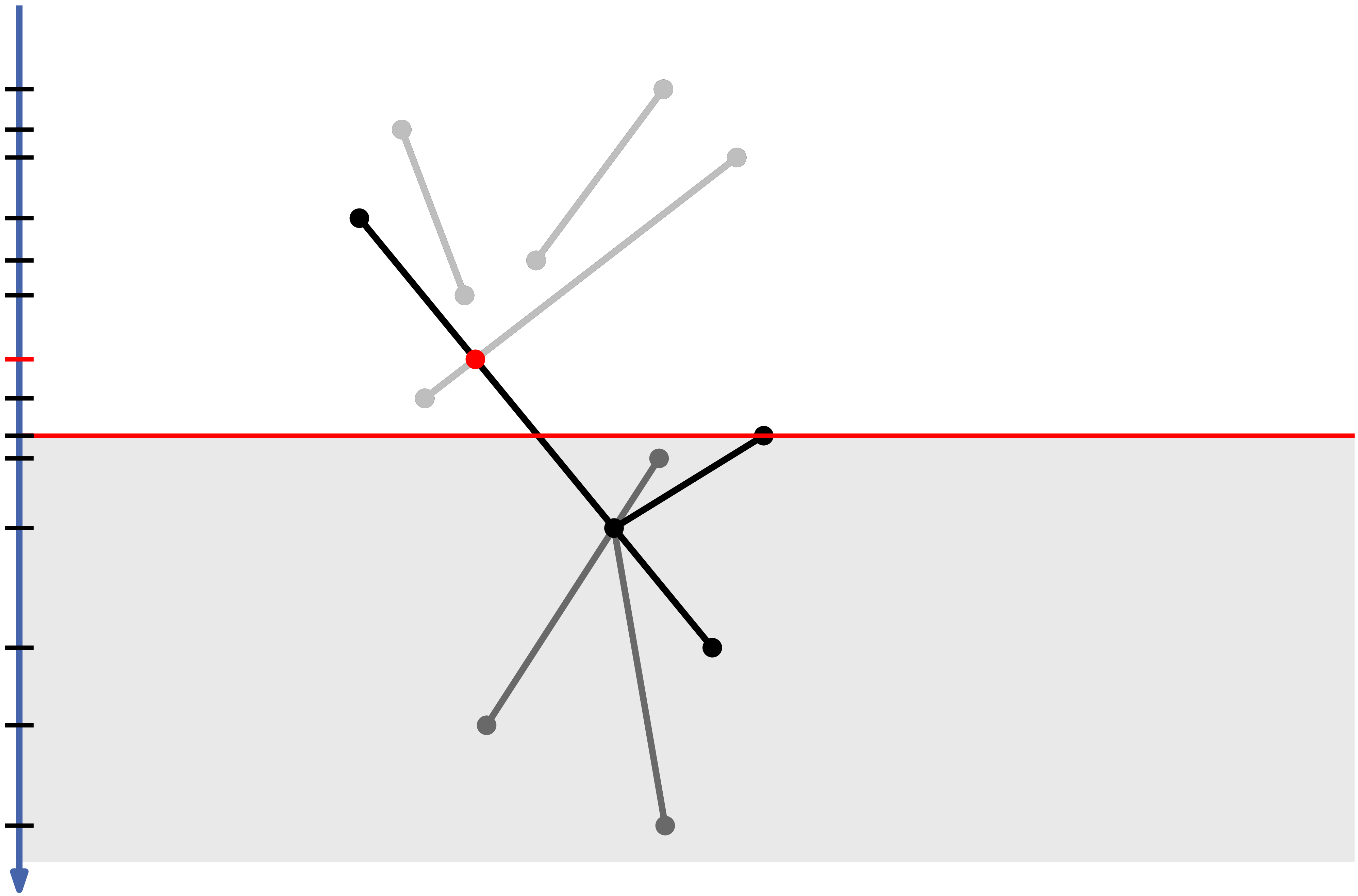

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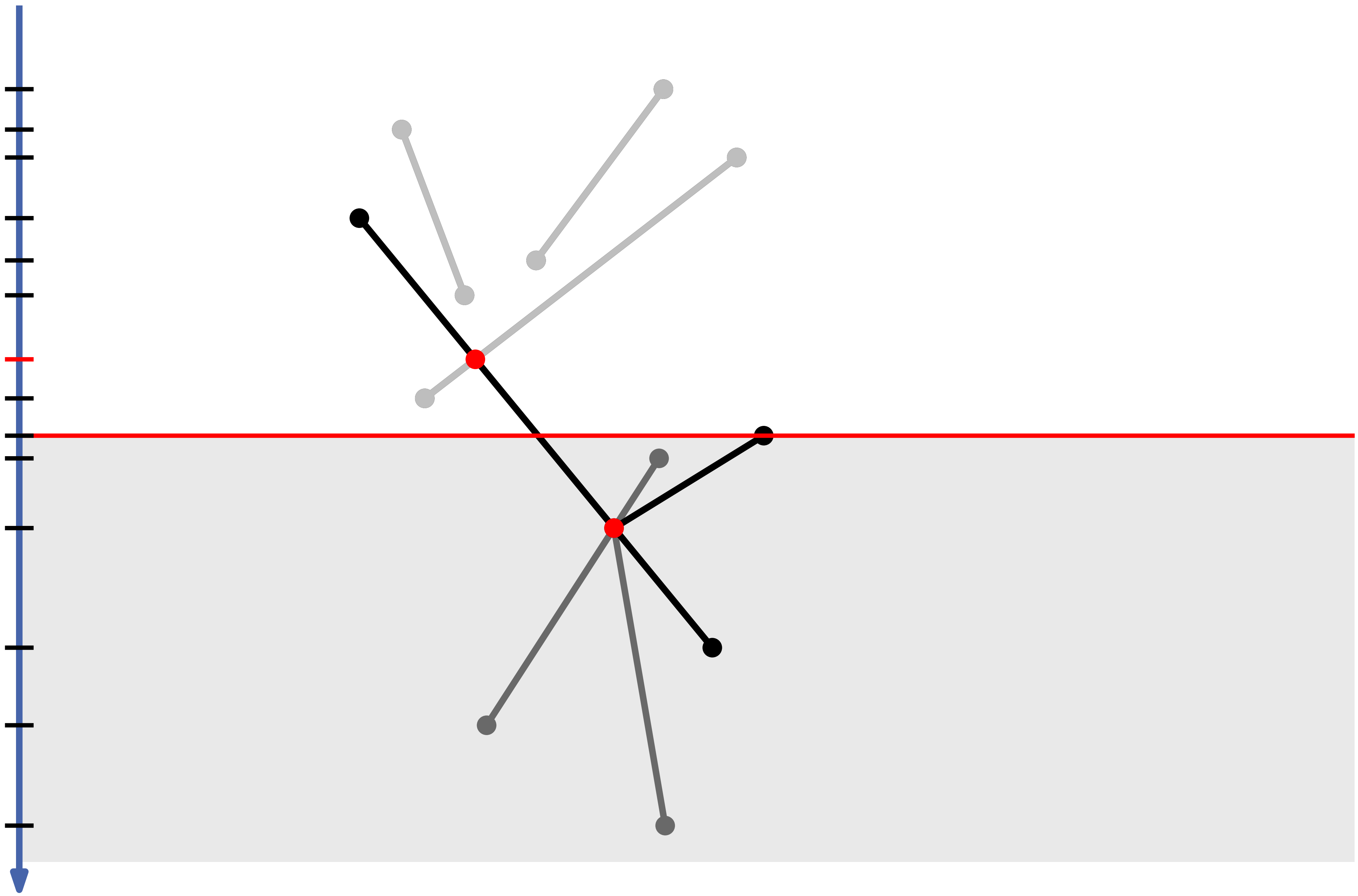

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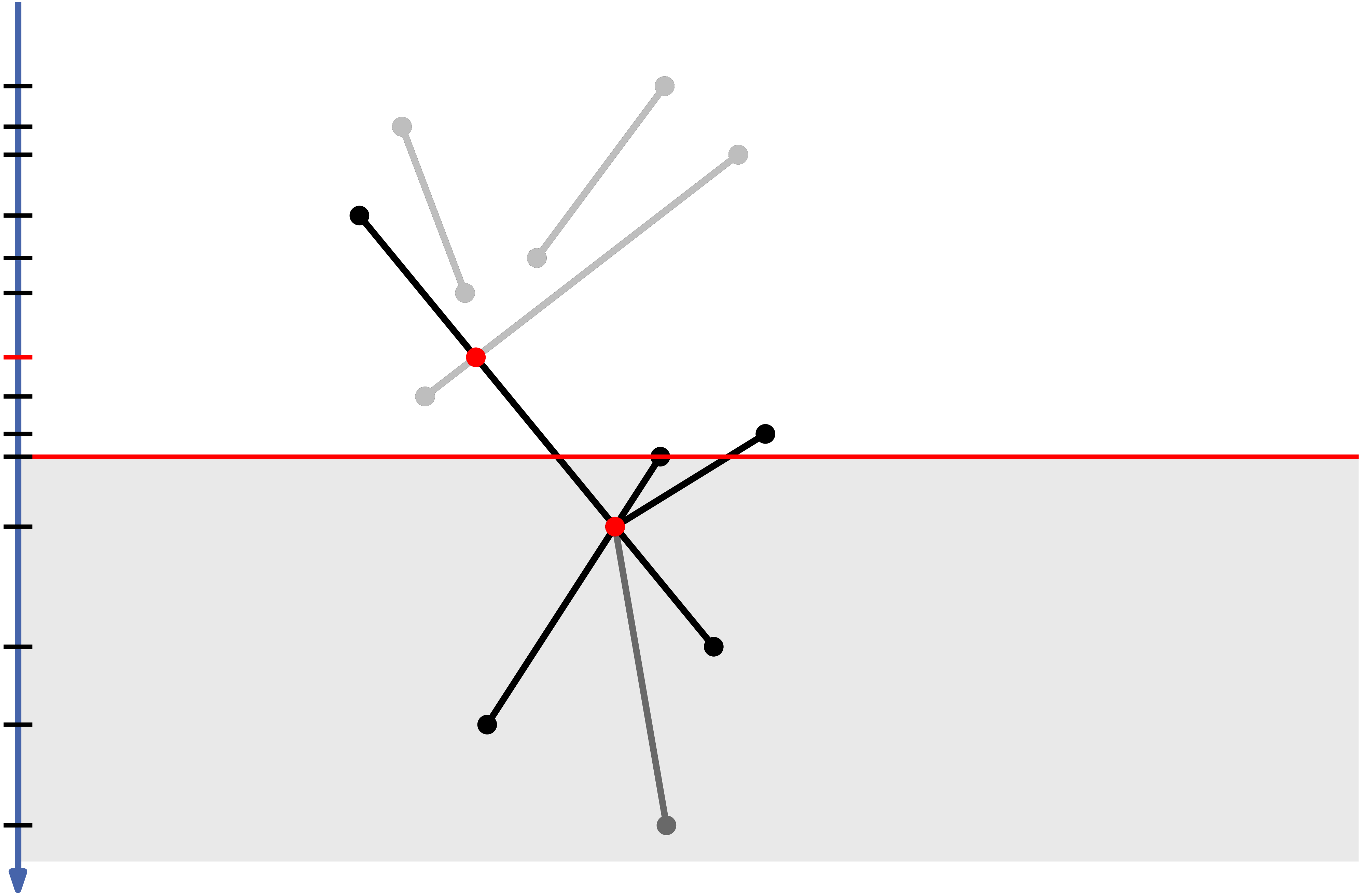

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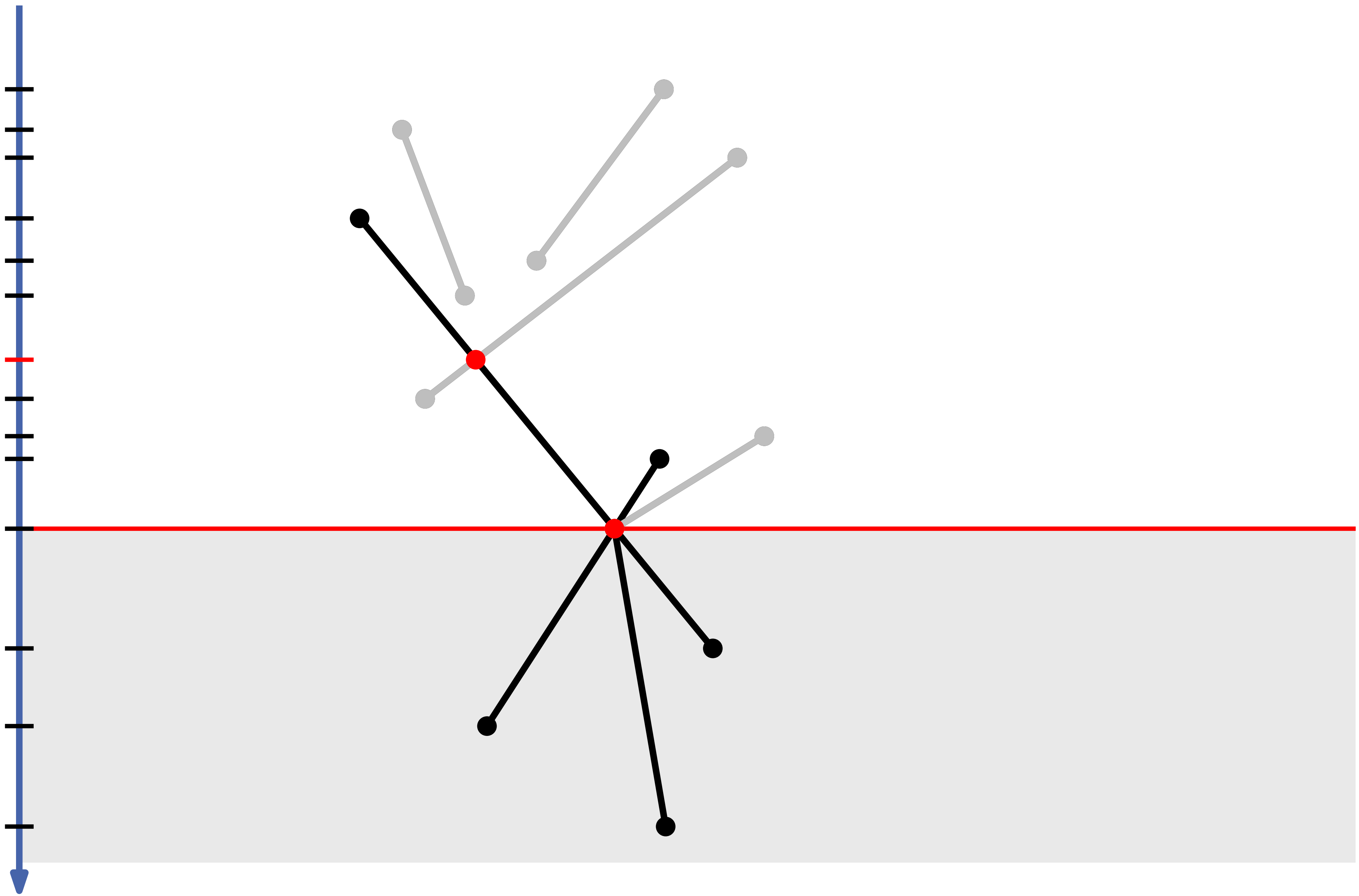

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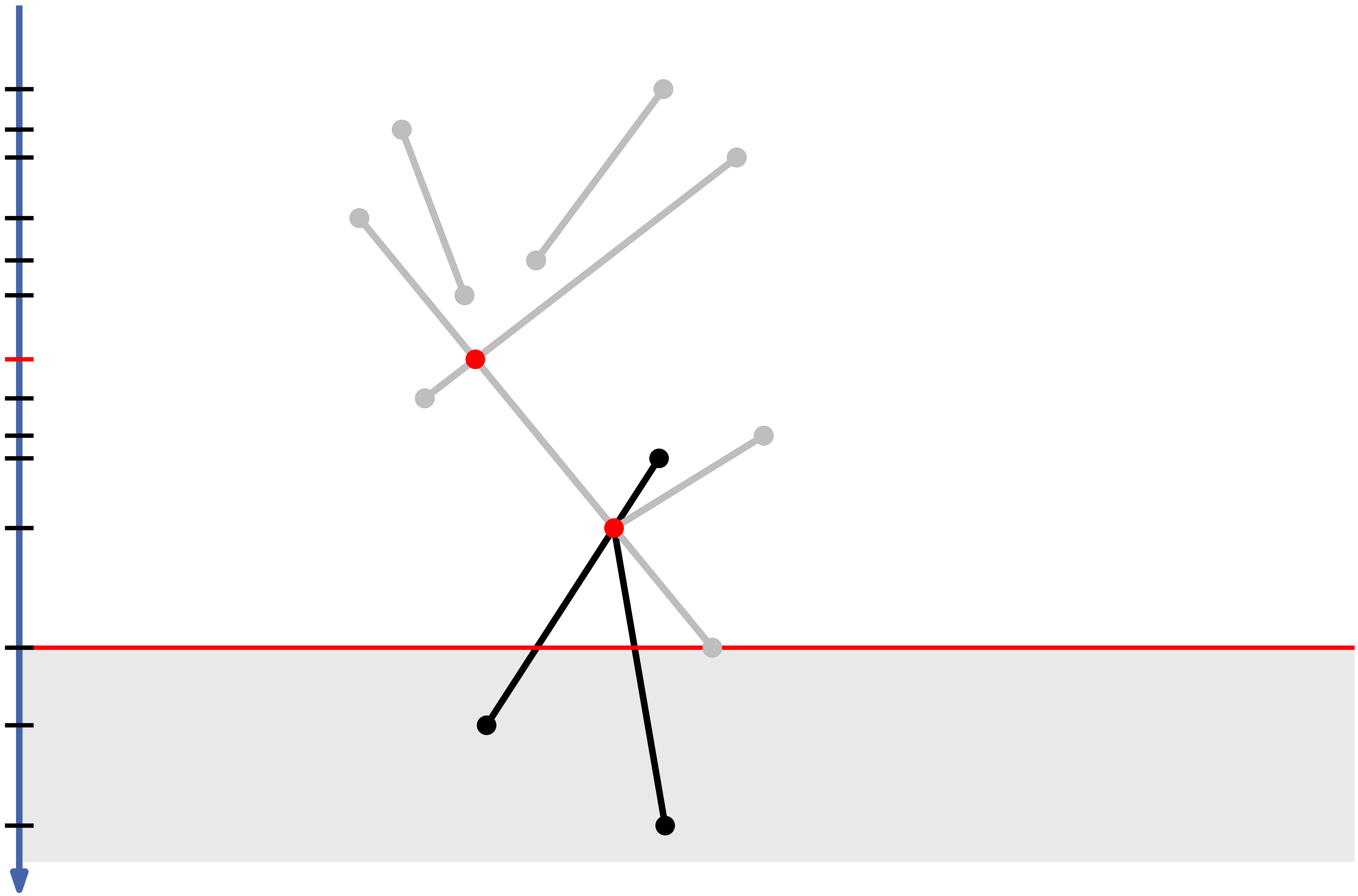

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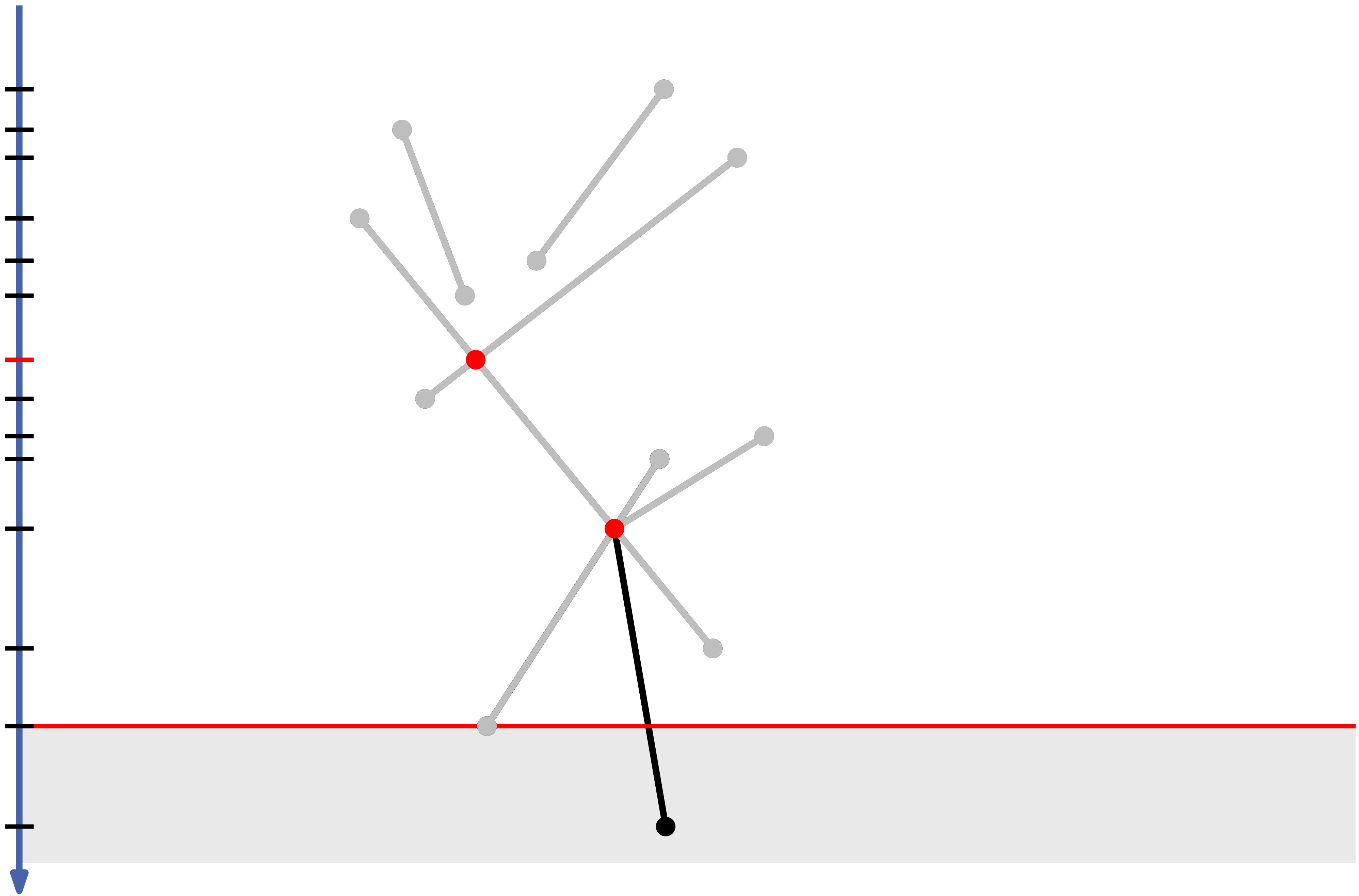

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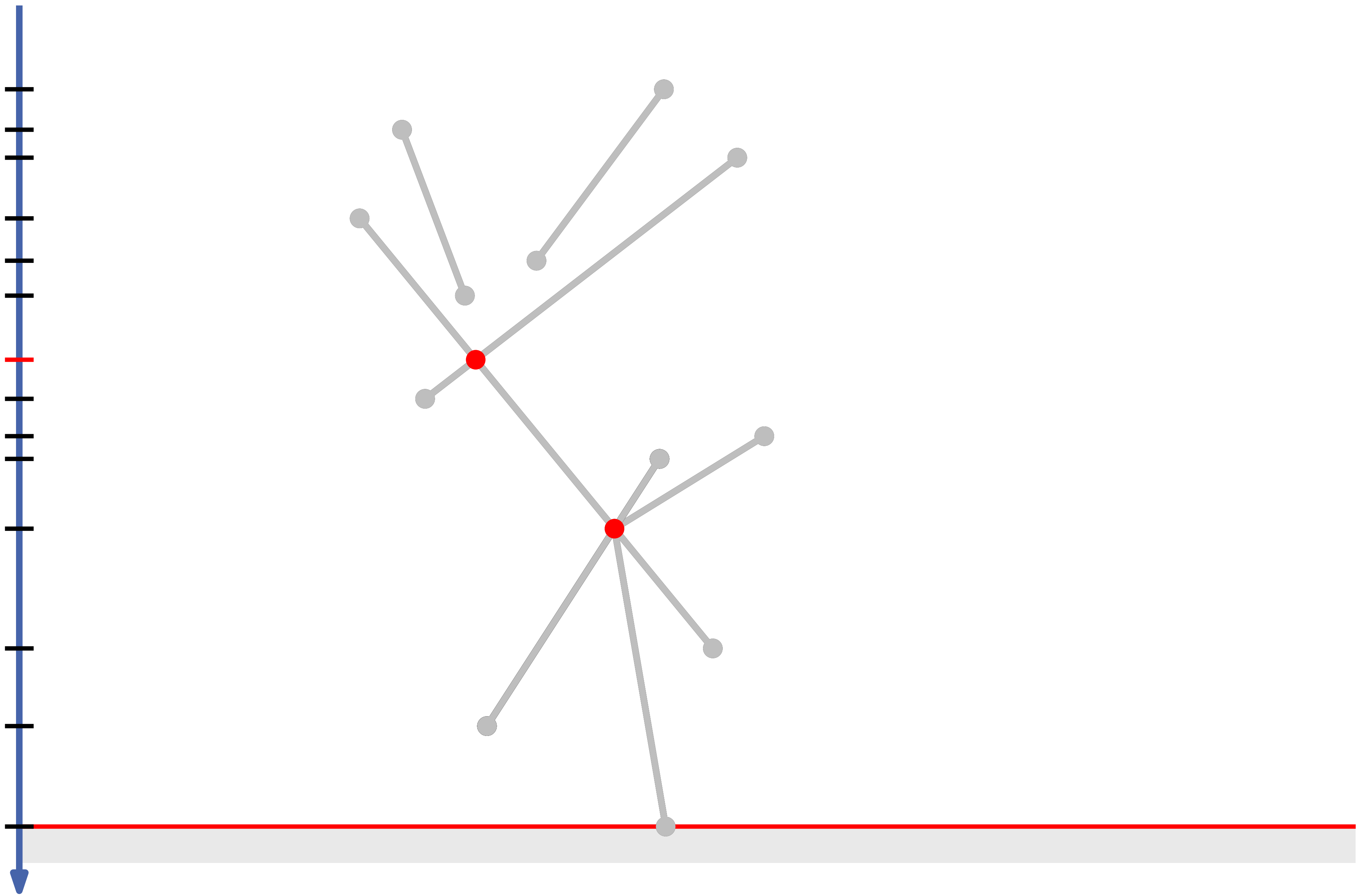

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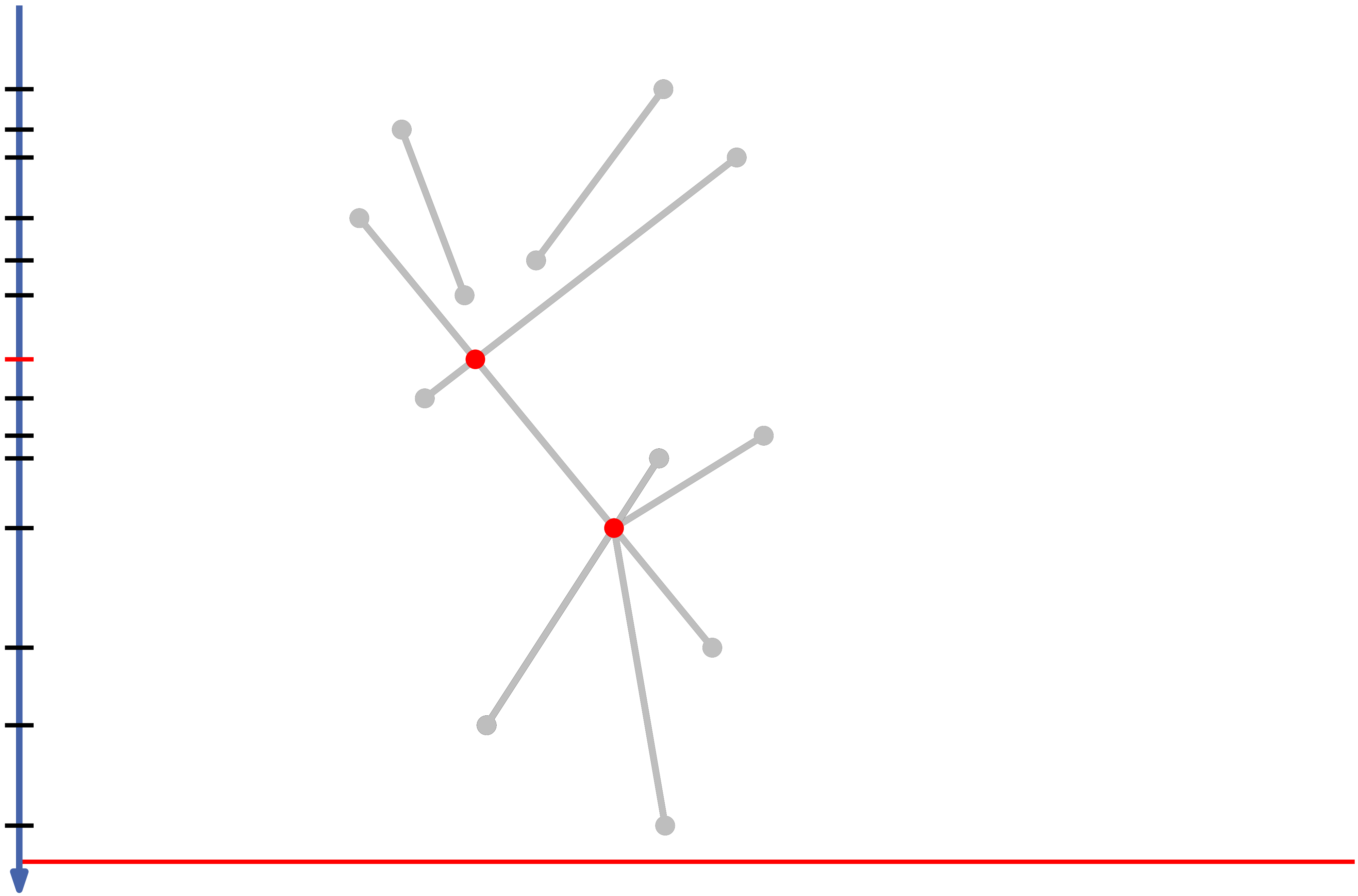
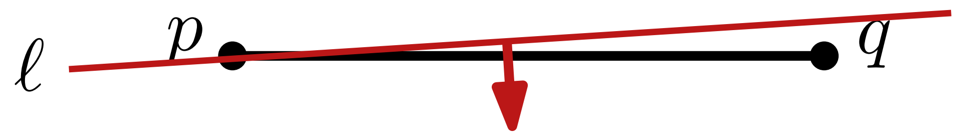


Data Structures

Karlsruhe Institute of Technology

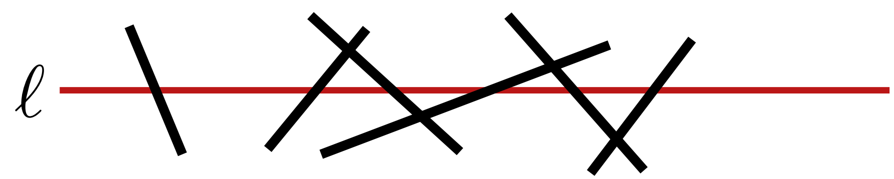
1.) Event Queue $\mathcal{Q}$

- define $p \prec q \quad \Leftrightarrow_{\text{def.}} \quad y_p > y_q \vee (y_p = y_q \wedge x_p < x_q)$



- Store events by $\prec$ in a **balanced binary search tree**
  $\rightarrow$ e.g., AVL tree, red-black tree, . . .
- Operations `insert`, `delete` and `nextEvent` in $O(\log |\mathcal{Q}|)$ time

2.) Sweep-Line Status $\mathcal{T}$



- Stores ℓ cut lines ordered from left to right
- Required operations `insert`, `delete`, `findNeighbor`
- This is also a balanced binary search tree with line segments stored in the leaves!

Algorithm

FindIntersections(S)

Input: Set S of line segments

Output: Set of all intersection points and the line segments involved

$\mathcal{Q} \leftarrow \emptyset; \quad \mathcal{T} \leftarrow \emptyset$

foreach $s \in S$ **do**

- $\mathcal{Q}$.insert(upperEndPoint(s))
- $\mathcal{Q}$.insert(lowerEndPoint(s))

while $\mathcal{Q} \neq \emptyset$ **do**

- $p \leftarrow \mathcal{Q}$.nextEvent()
- $\mathcal{Q}$.deleteEvent(p)
- handleEvent(p)

Algorithm

handleEvent(p)

$U(p) \leftarrow$ Line segments with p as upper endpoint
$L(p) \leftarrow$ Line segments with p as lower endpoint
$C(p) \leftarrow$ Line segments with p as interior point
if $|U(p) \cup L(p) \cup C(p)| > 1$ **then**
    report p and $U(p) \cup L(p) \cup C(p)$
remove $L(p) \cup C(p)$ from $\mathcal{T}$
add $U(p) \cup C(p)$ to $\mathcal{T}$
if $U(p) \cup C(p) = \emptyset$ **then** $//s_l$ and s_r, neighbors of p in $\mathcal{T}$
    $\mathcal{Q} \leftarrow$ check if s_l and s_r intersect below p
else $//s'$ and s'' left- and rightmost line segment in $U(p) \cup C(p)$
    $\mathcal{Q} \leftarrow$ check if s_l and s' intersect below p
    $\mathcal{Q} \leftarrow$ check if s_r and s'' intersect below p

Space Consumption

Karlsruhe Institute of Technology

Lecture:
Running time: $\mathcal{O}((n+I)\log n)$
Storage: $\mathcal{O}(n+I)$

Find:
Find algorithm that needs linear space.

Question:
Which data structure may use more than linear space?

Space Consumption

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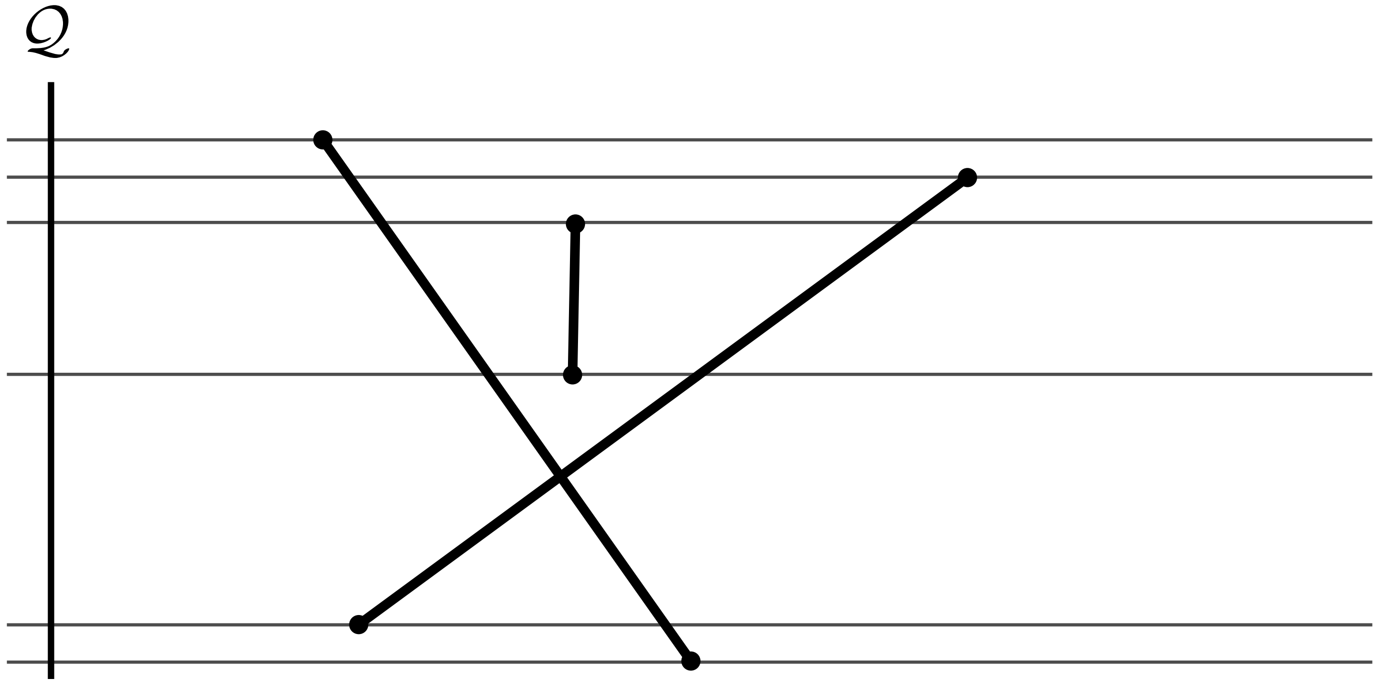
Which data structure may use more than linear space?

Event-Queue may contain $2n + I$ many events,
where $I \in \Omega(n^2)$ in the worst case.

Space Consumption

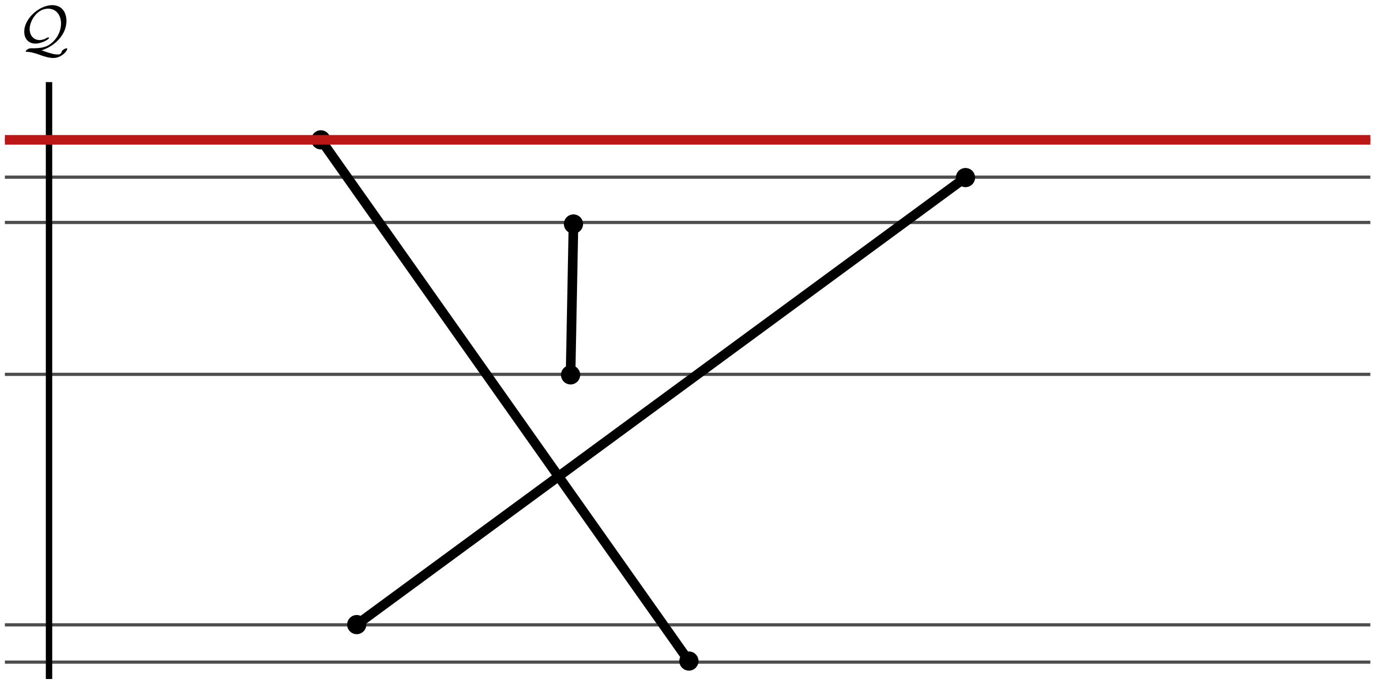


Idea: Store only intersection points that are **currently** adjacent in $\mathcal{T}$.

Obs.: At each point in time there are $O(n)$ many such intersection points.

Procedure: If line segments loose their adjacency in $\mathcal{T}$, remove corresponding intersection points in Q.

Space Consumption



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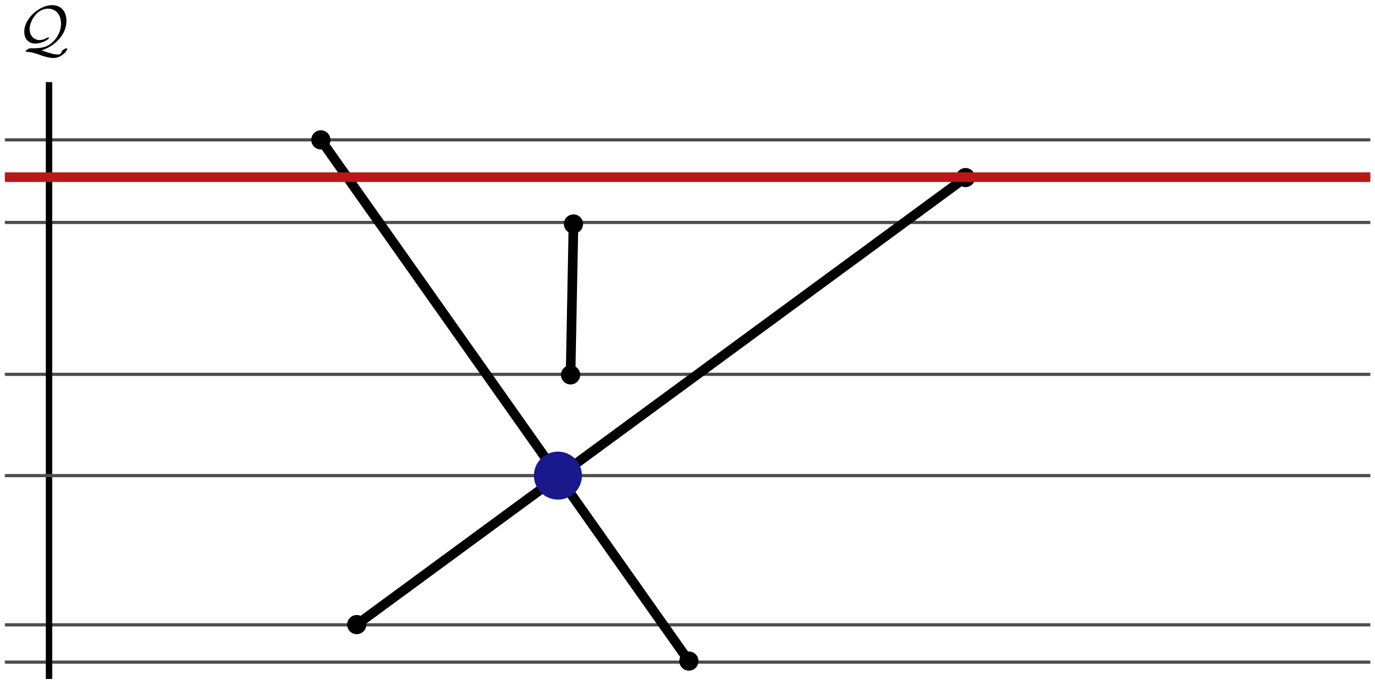

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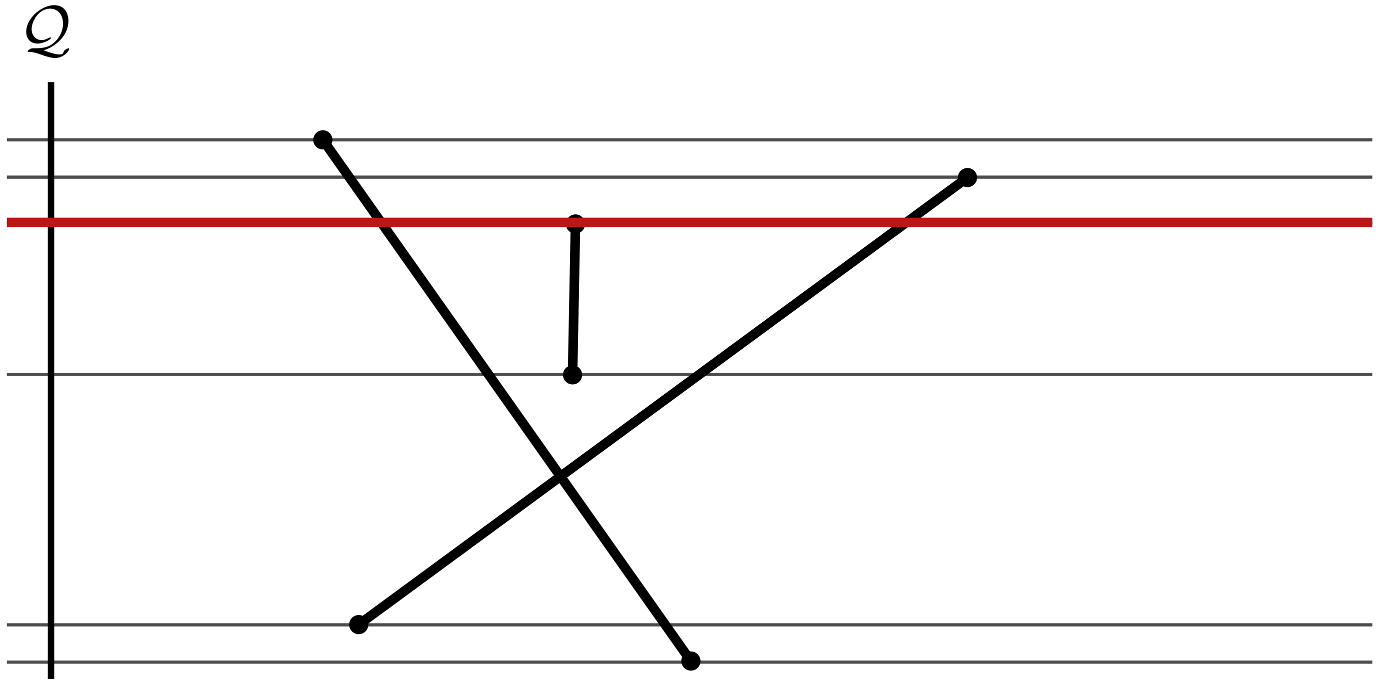

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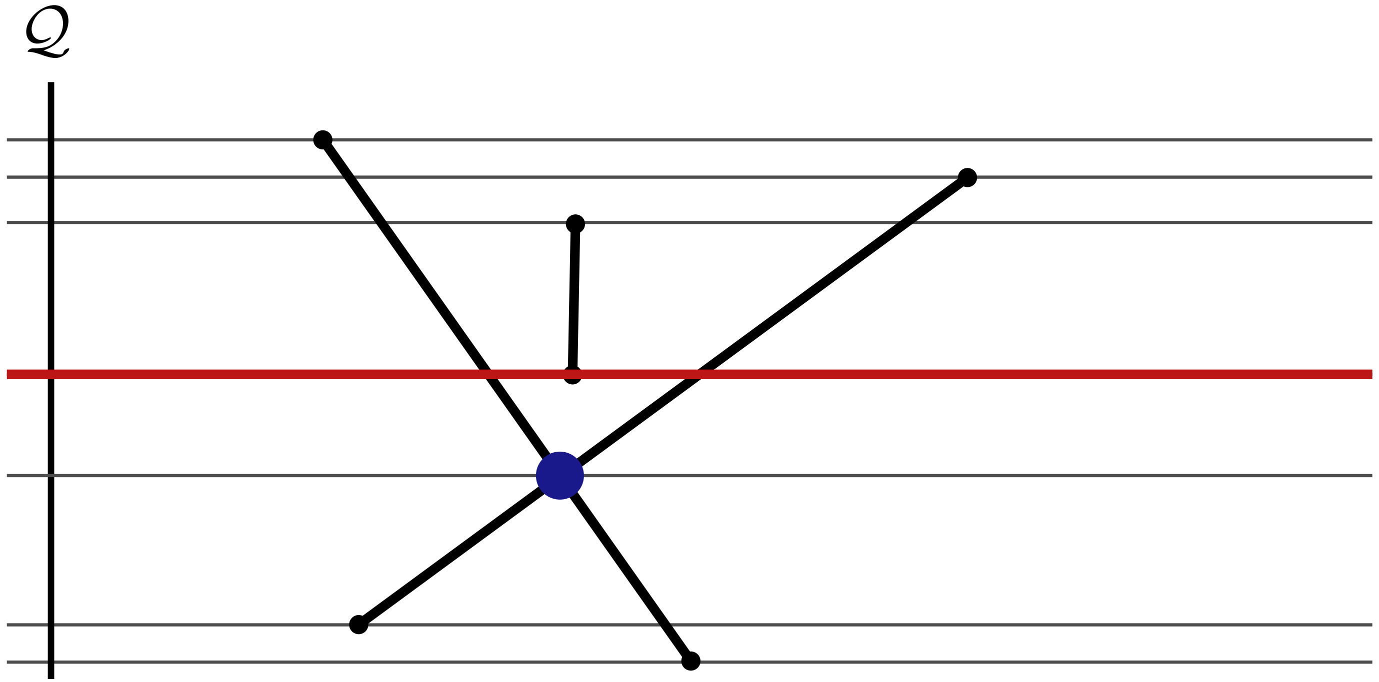

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Largest Top-Right Region

Given: Set P with n points.

Definition:
The *largest top-right region* of a point $p \in P$ is the union of all open axis-aligned squares that touch p with their bottom left corner and contain no other point of P in their interior.

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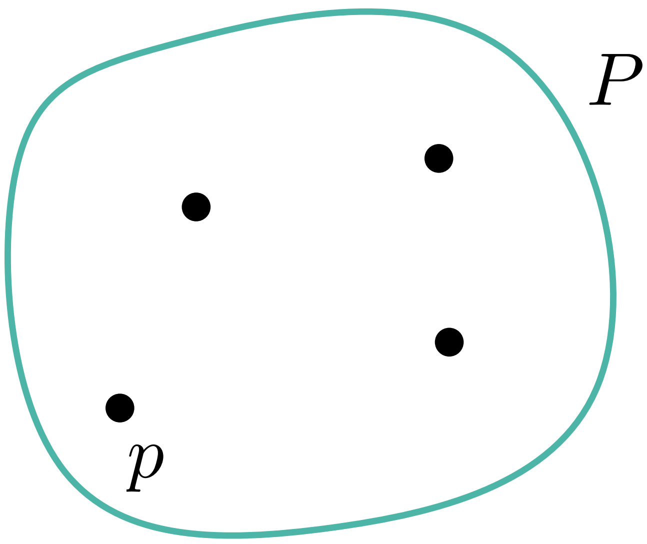

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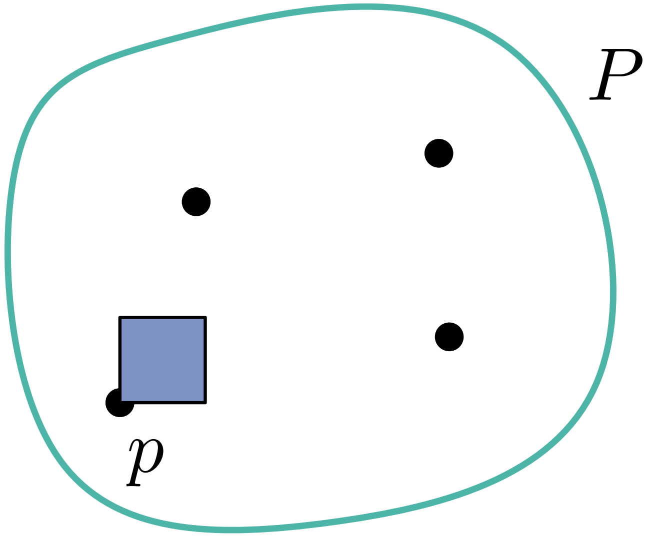

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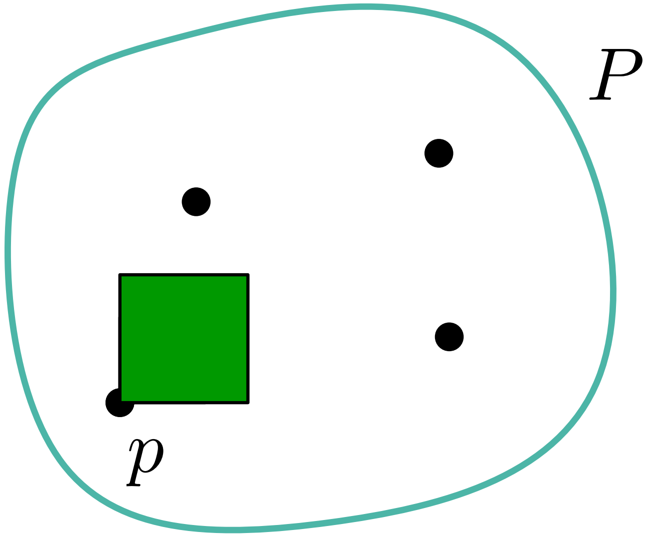

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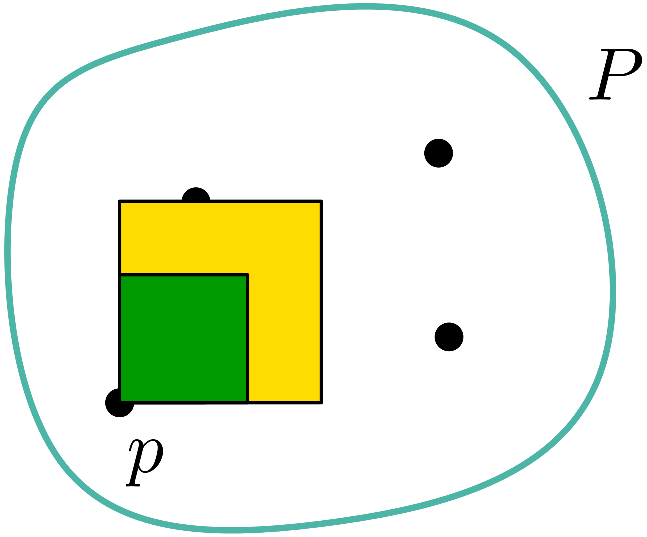

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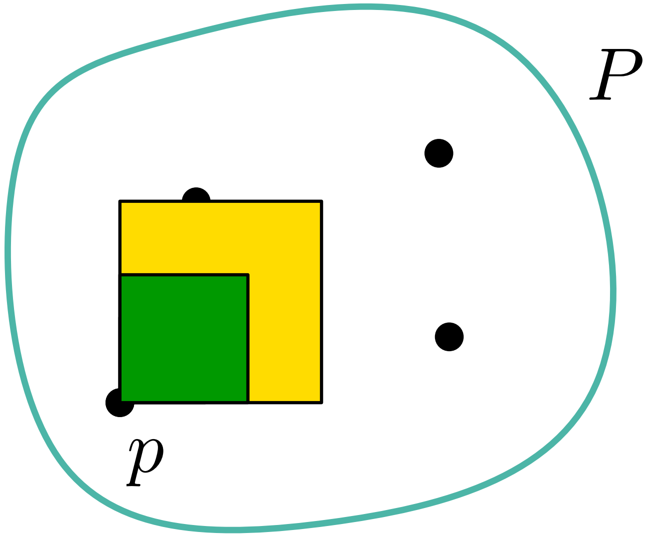

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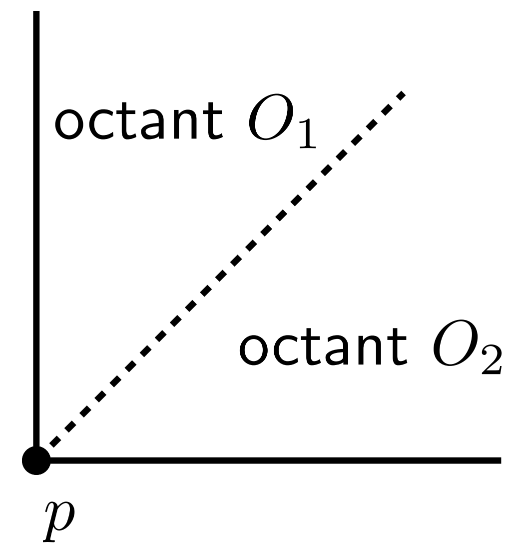
b1) Which point in $O_1 \cap P$ restricts the largest top-right region of p the most?
b2) Which point in $O_2 \cap P$ restricts the largest top-right region of p the most?



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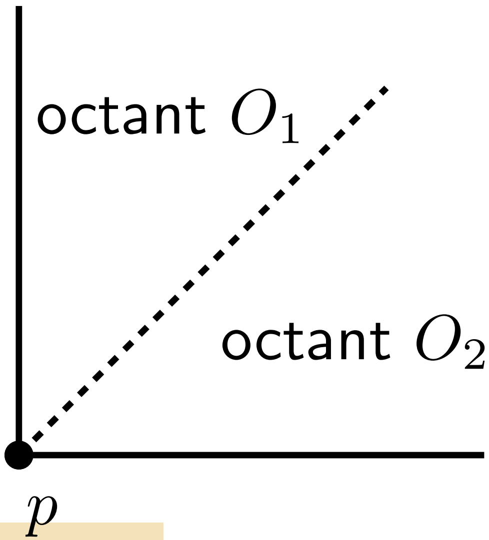

$t(p) \in P$: Point in O_1 with smallest vert. distance to p.

$r(p) \in P$: Point in O_2 with smallest horz. distance to p.

Largest Top-Right Region

Karlsruhe Institute of Technology

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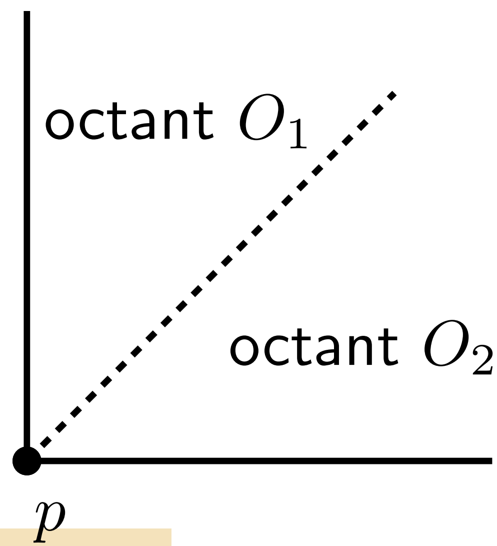

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$t(p) \in P$: Point in O_1 with smallest vert. distance to p.

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c) Largest top-right region for all points in $O(n \log n)$ running time.

Largest Top-Right Region

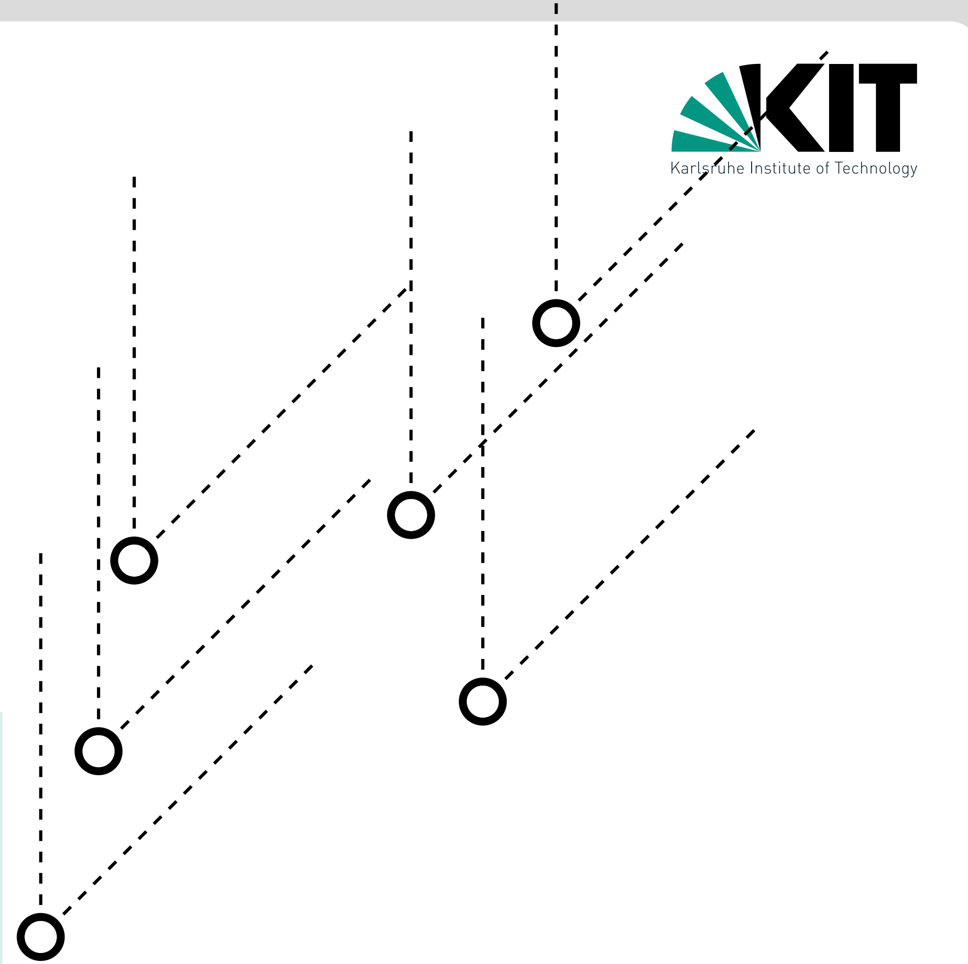
Idea: Determine for each point p the point $t(p)$ (and $r(p)$)

Sweepline: from bottom to top
Events: Points in P

Handling event p

1. Insert p into $\mathcal{T}$.
2. Find point $p' \in \mathcal{T}$ directly left to p:

If p lies in upper octant of p' :
  $t(p') \leftarrow p$, delete p' from $\mathcal{T}$
  repeat step 2



Data structure:
Binary search tree $\mathcal{T}$ over P, where point $p \in \mathcal{T}$, if

1. p lies below the sweep-line
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Initially $\mathcal{T}$ is empty and points in $\mathcal{T}$ are sorted w.r.t. their x-coord.

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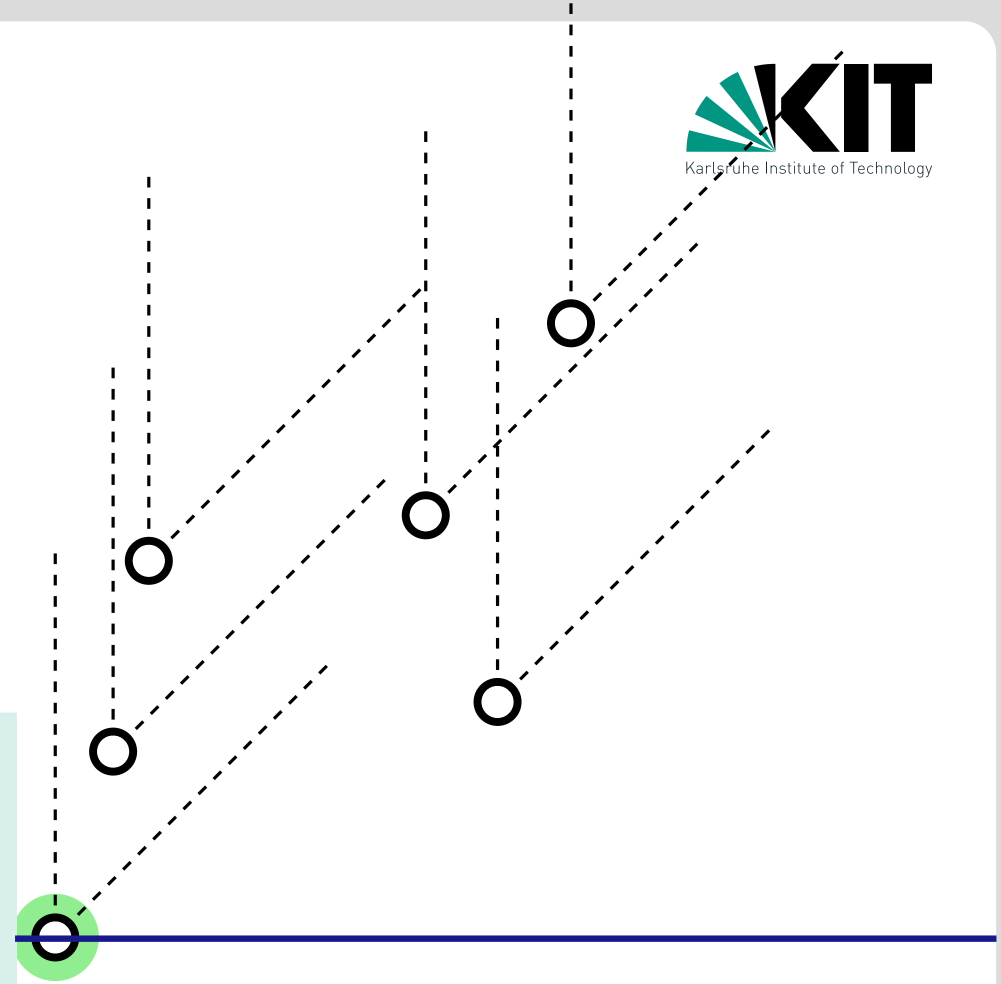

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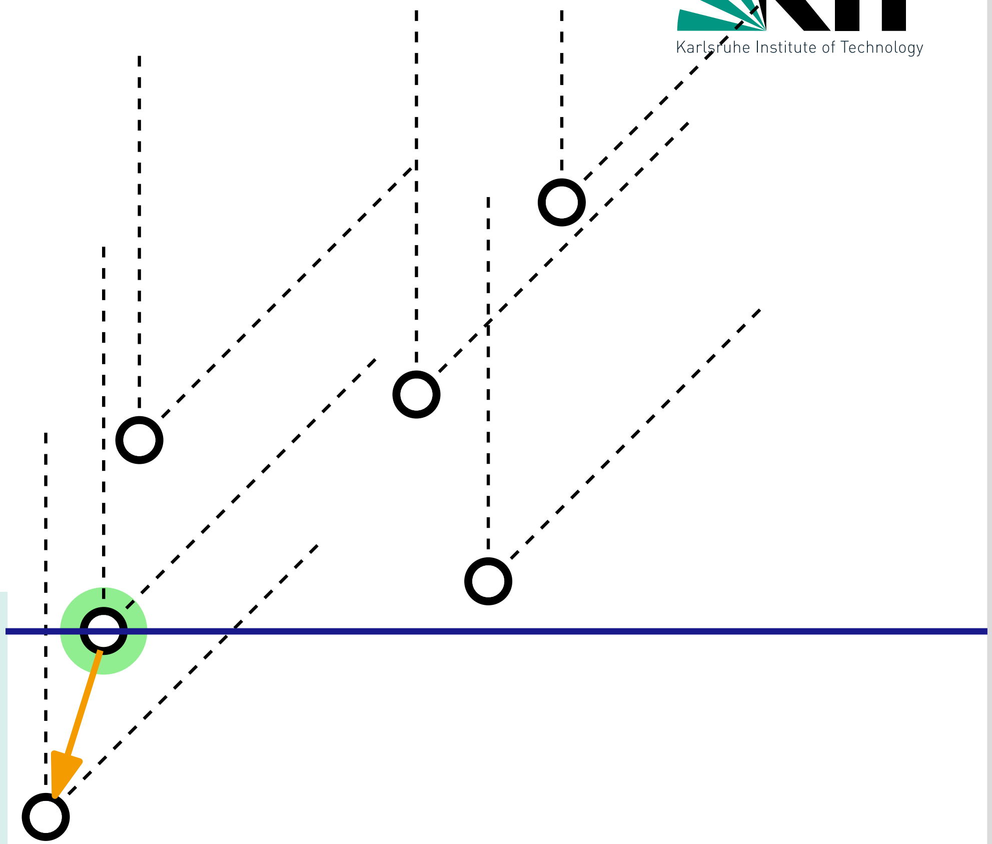

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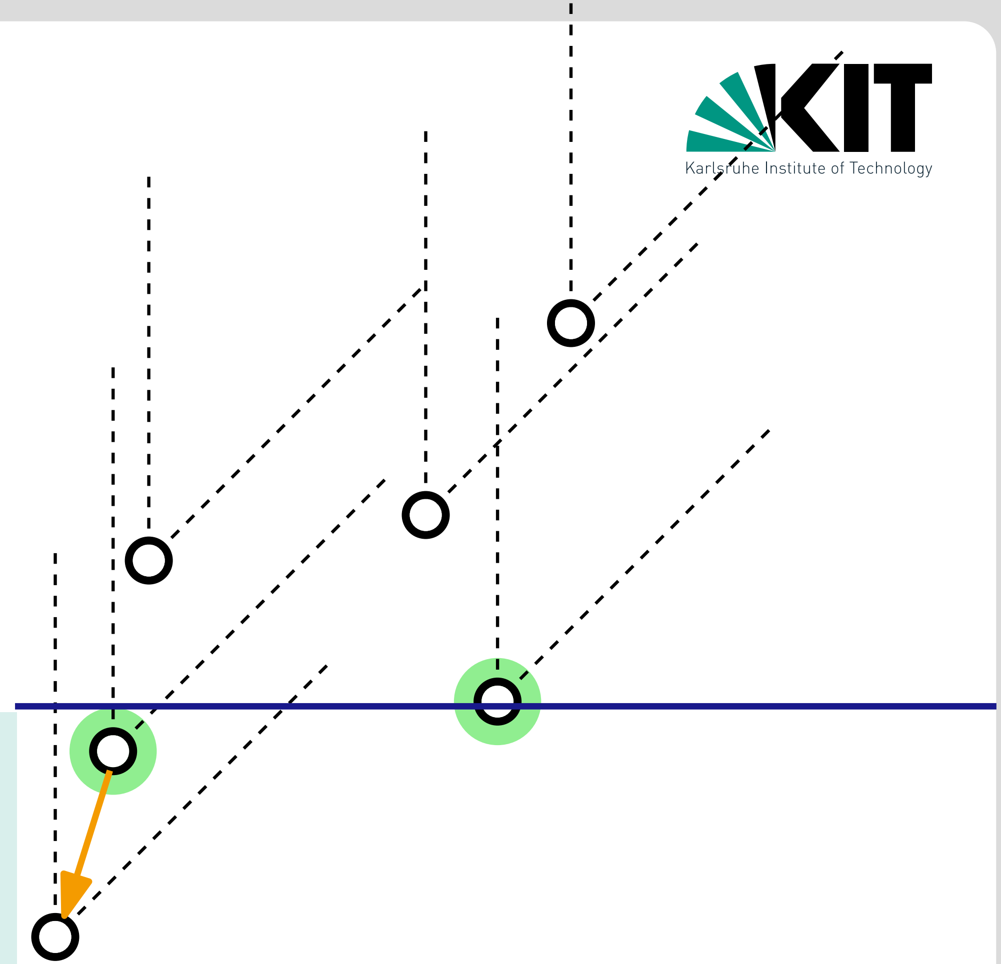

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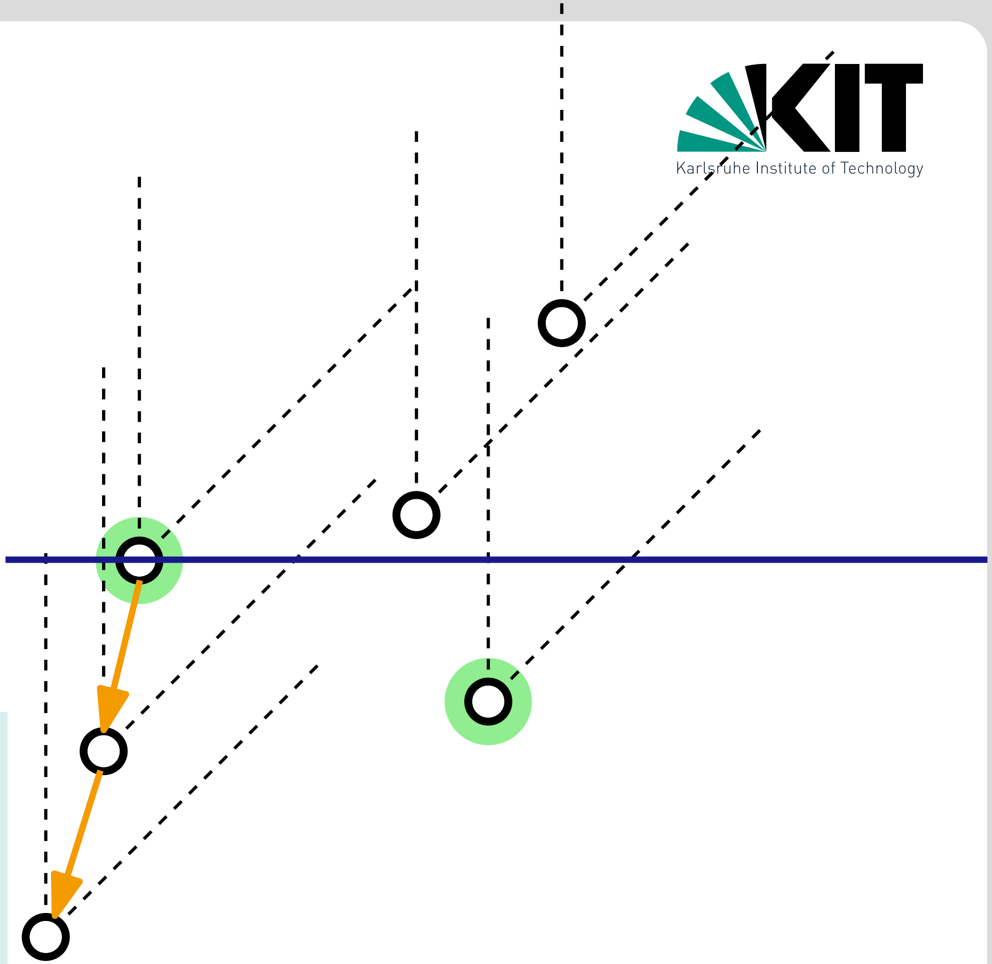

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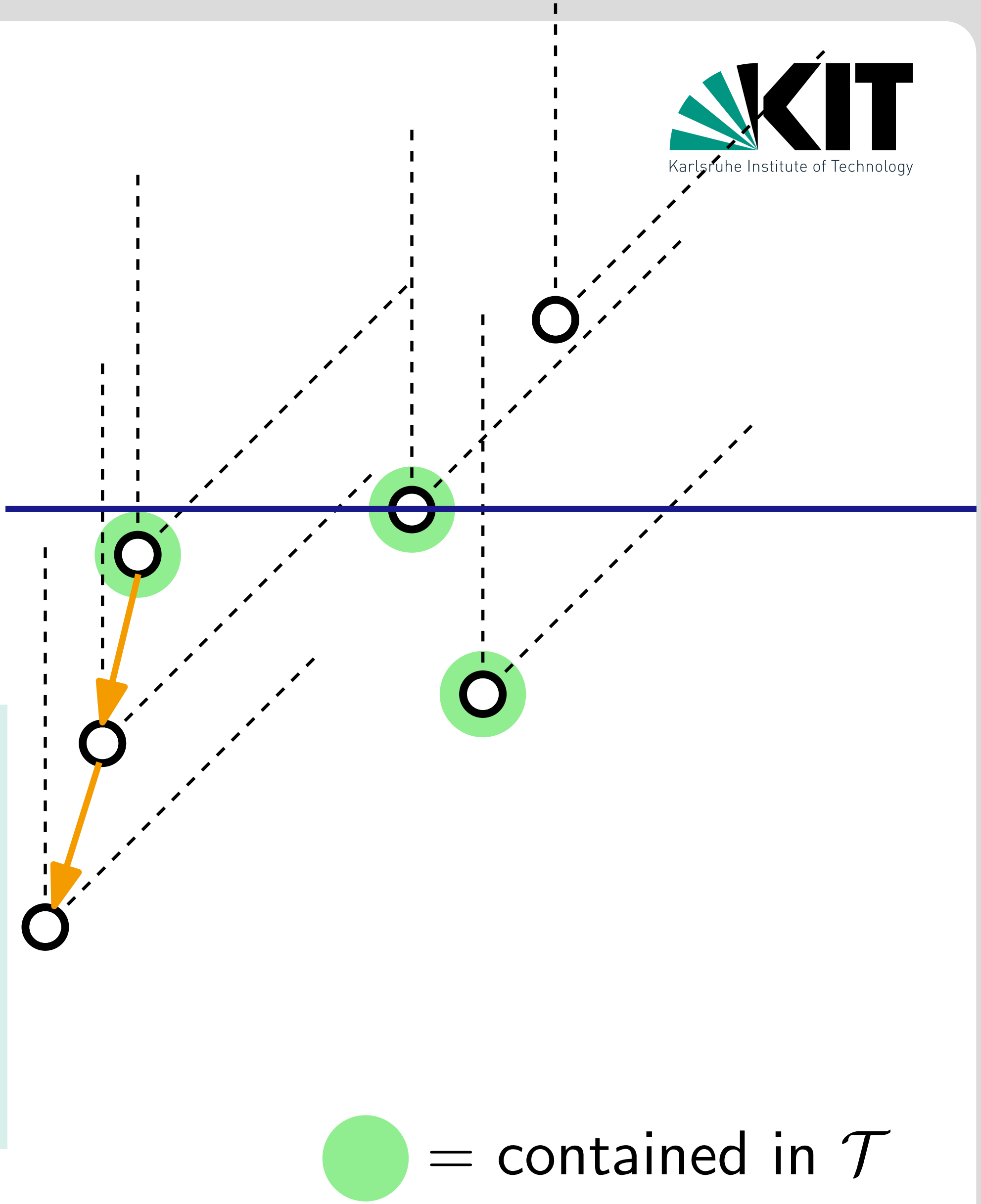

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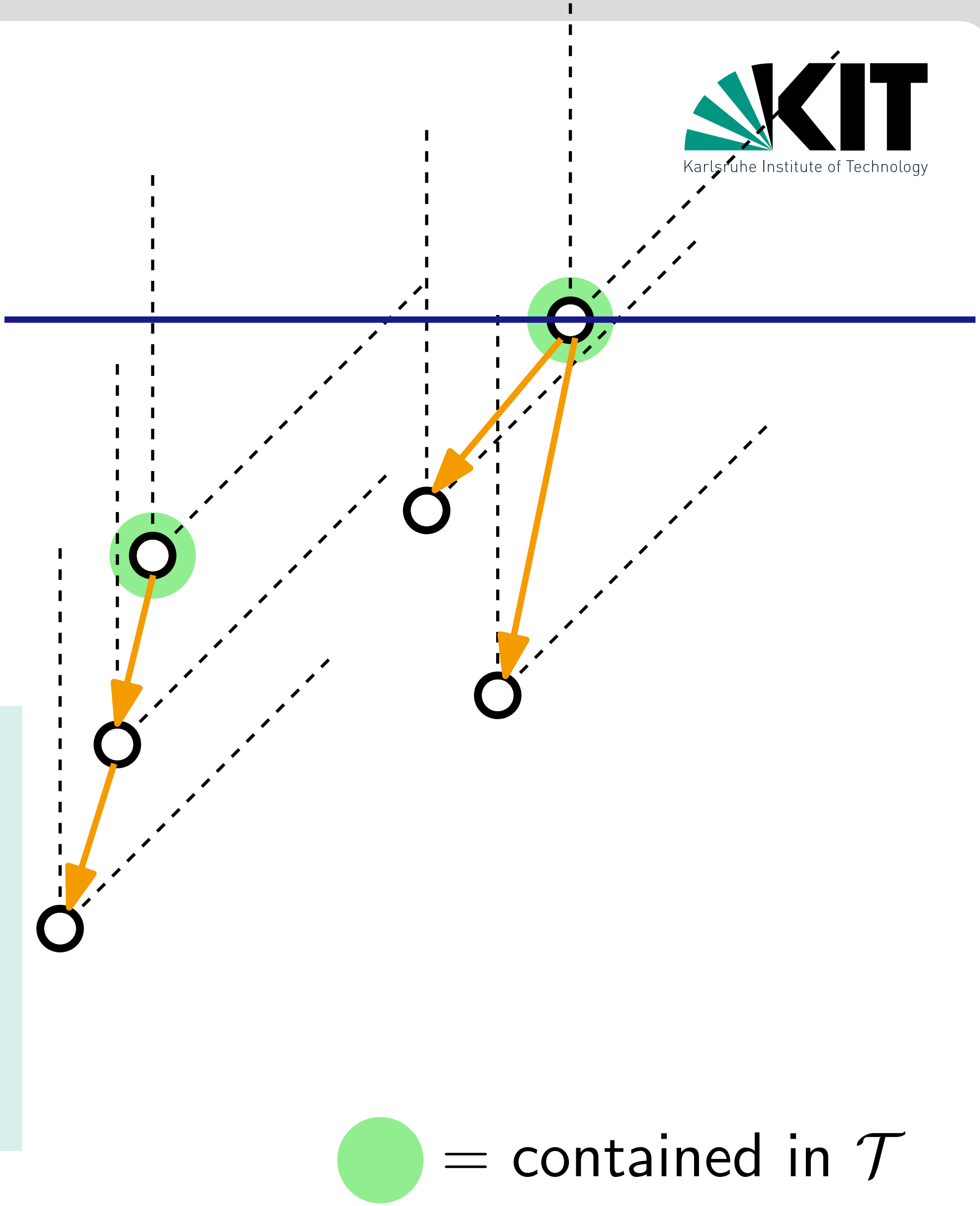

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Subdivision of plane.

Quelle: Google Earth

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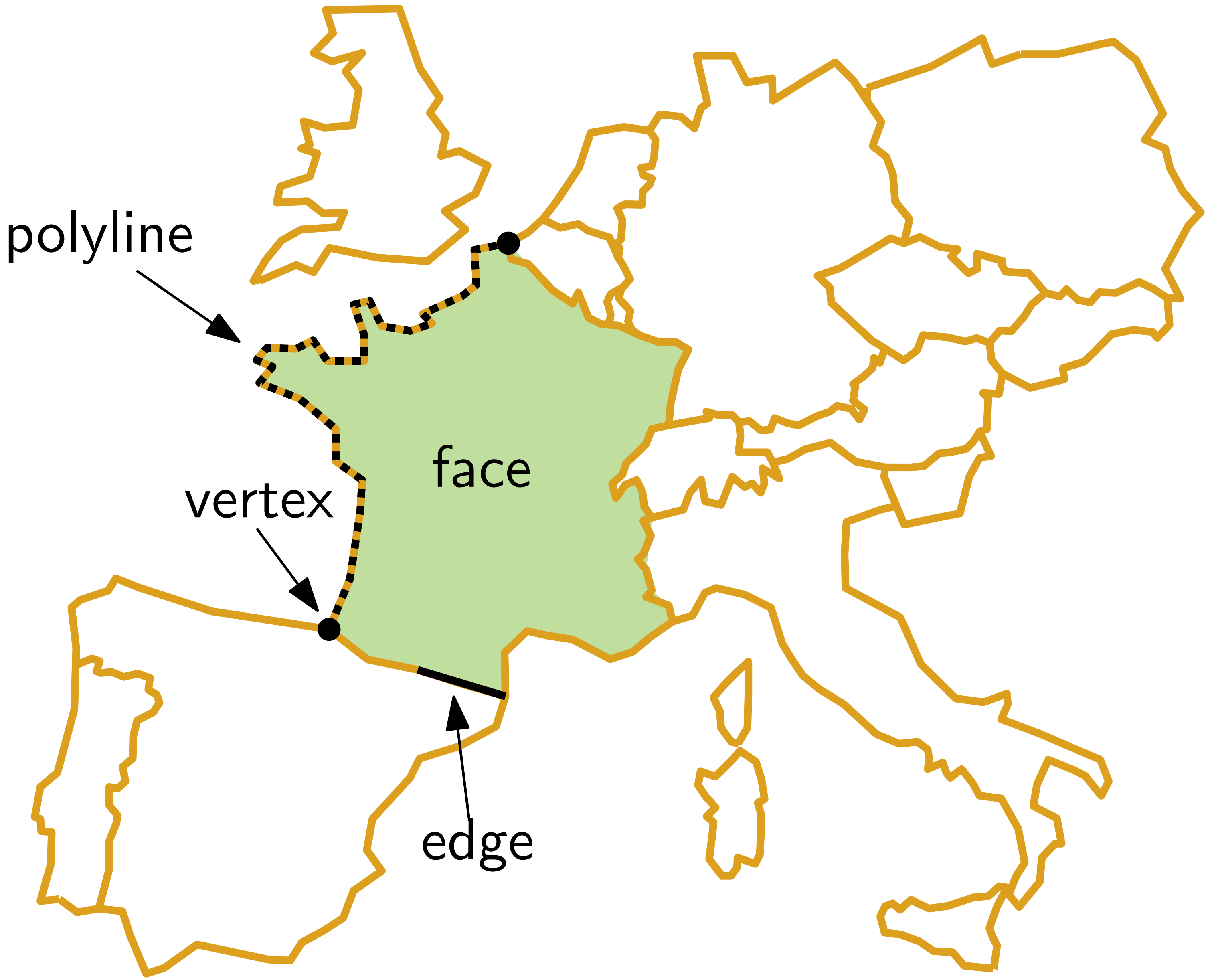

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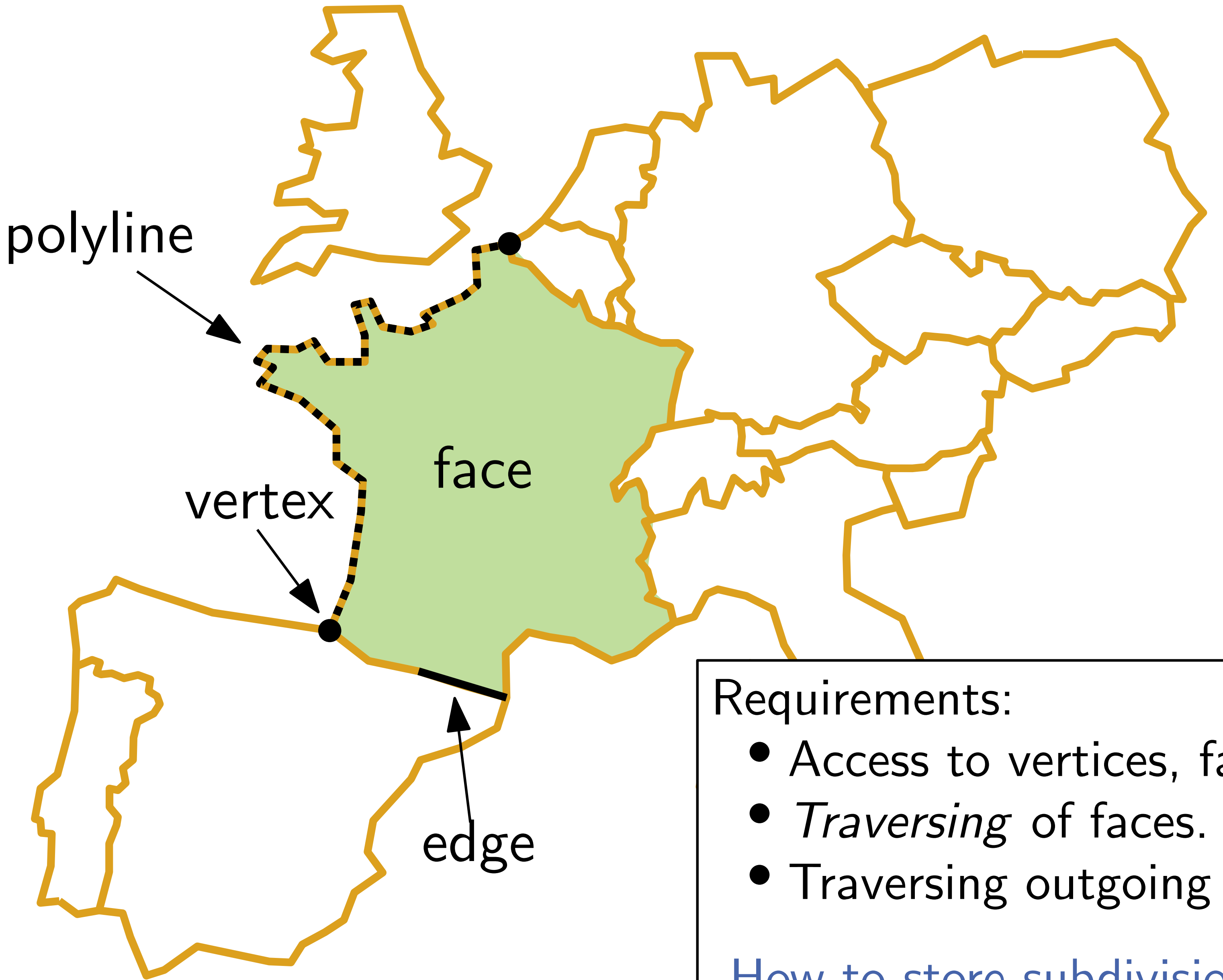
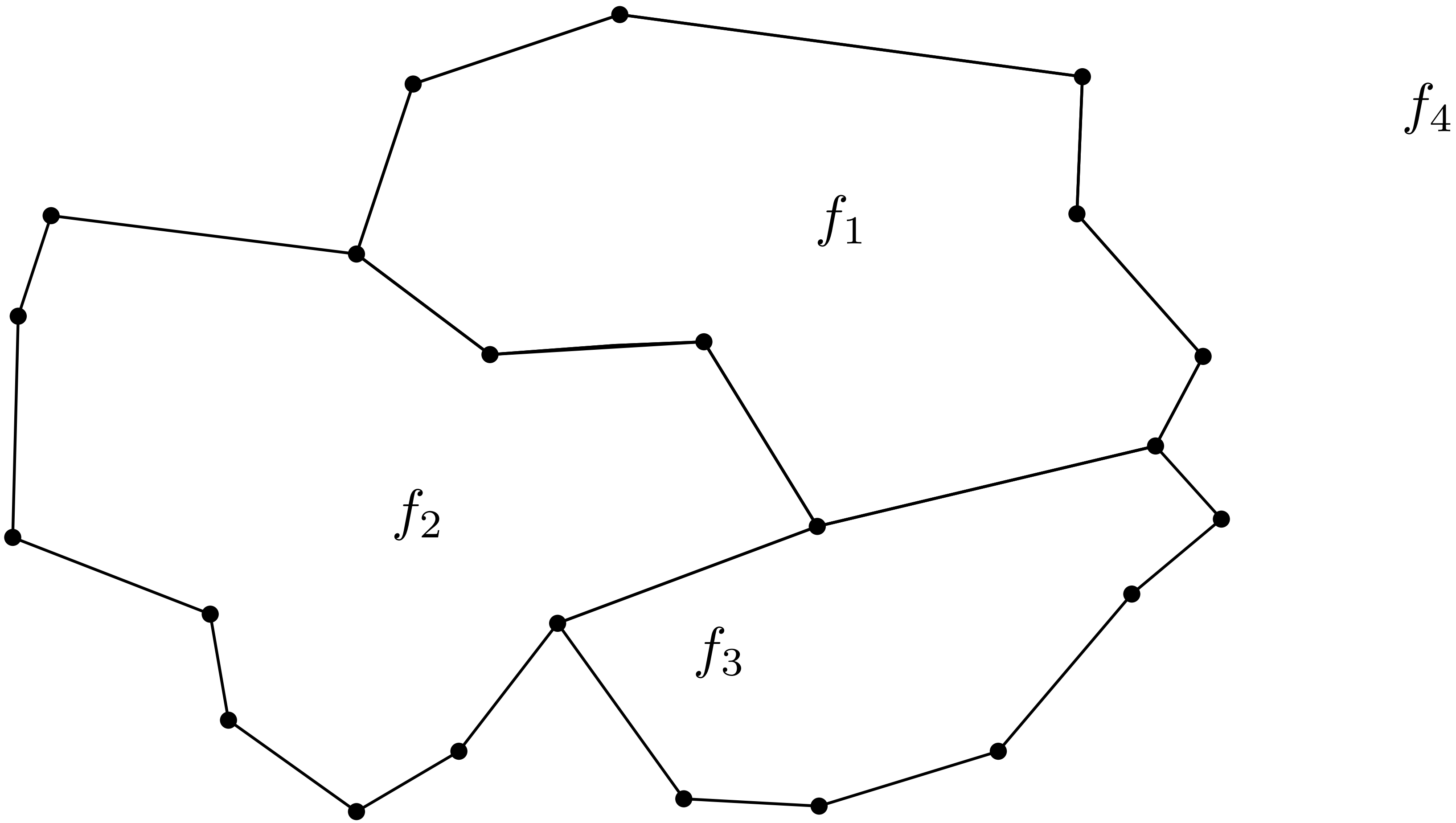


Requirements:

- Access to vertices, faces and edges.
- *Traversing* of faces.
- Traversing outgoing edges.

How to store subdivision efficiently?

Subdivision



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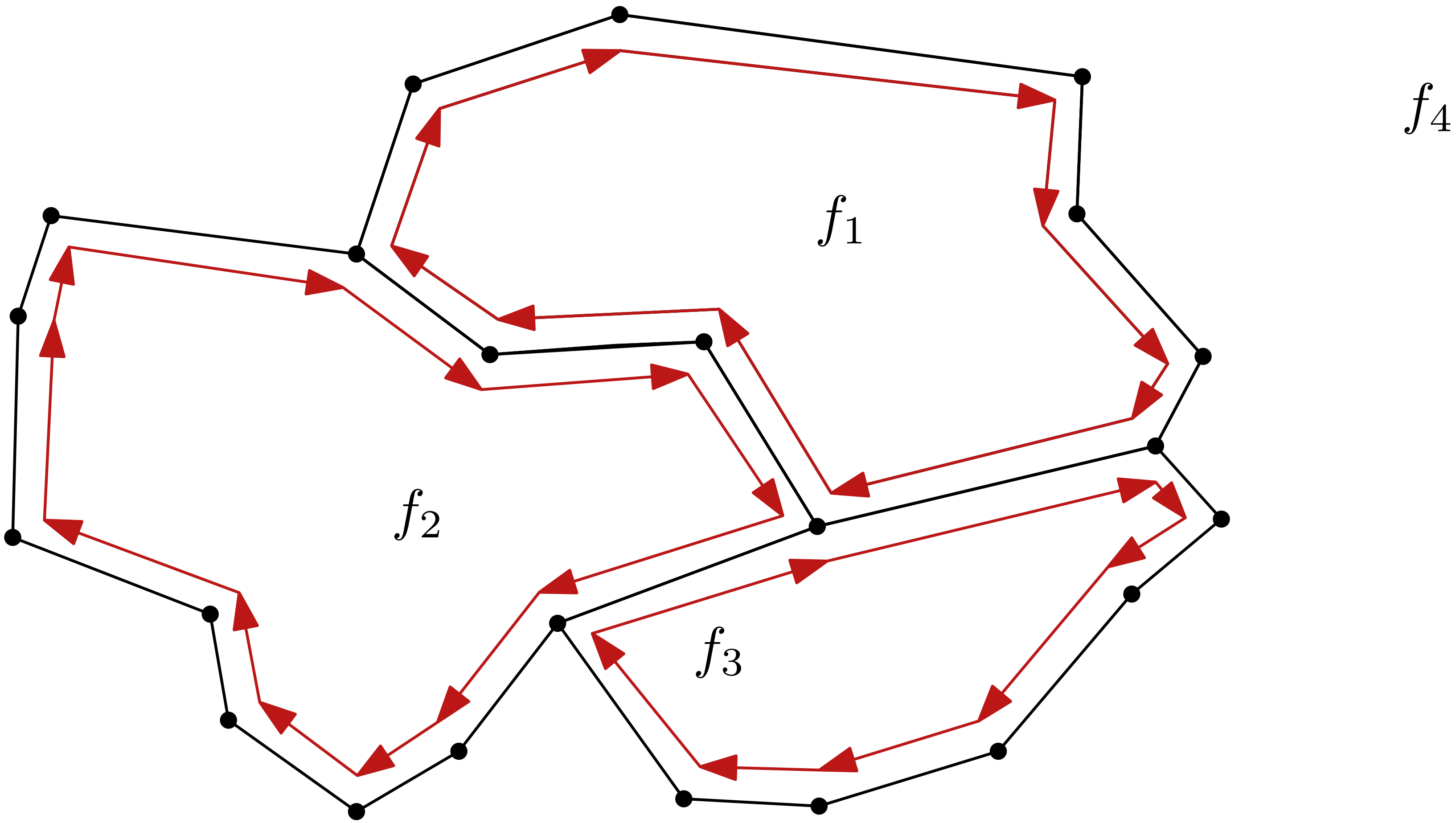

For each edge of internal faces introduce directed half-edge (clockwise)

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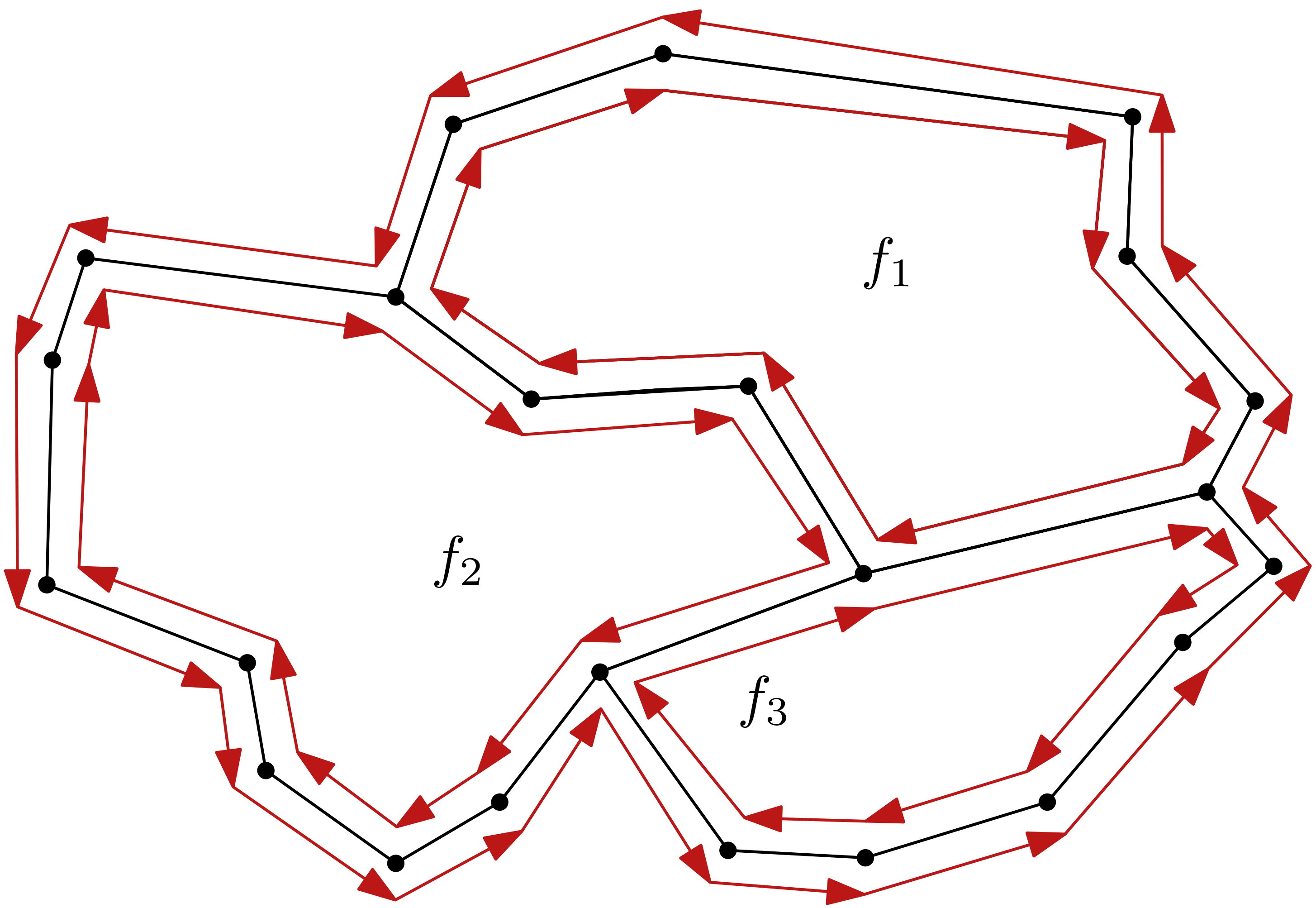
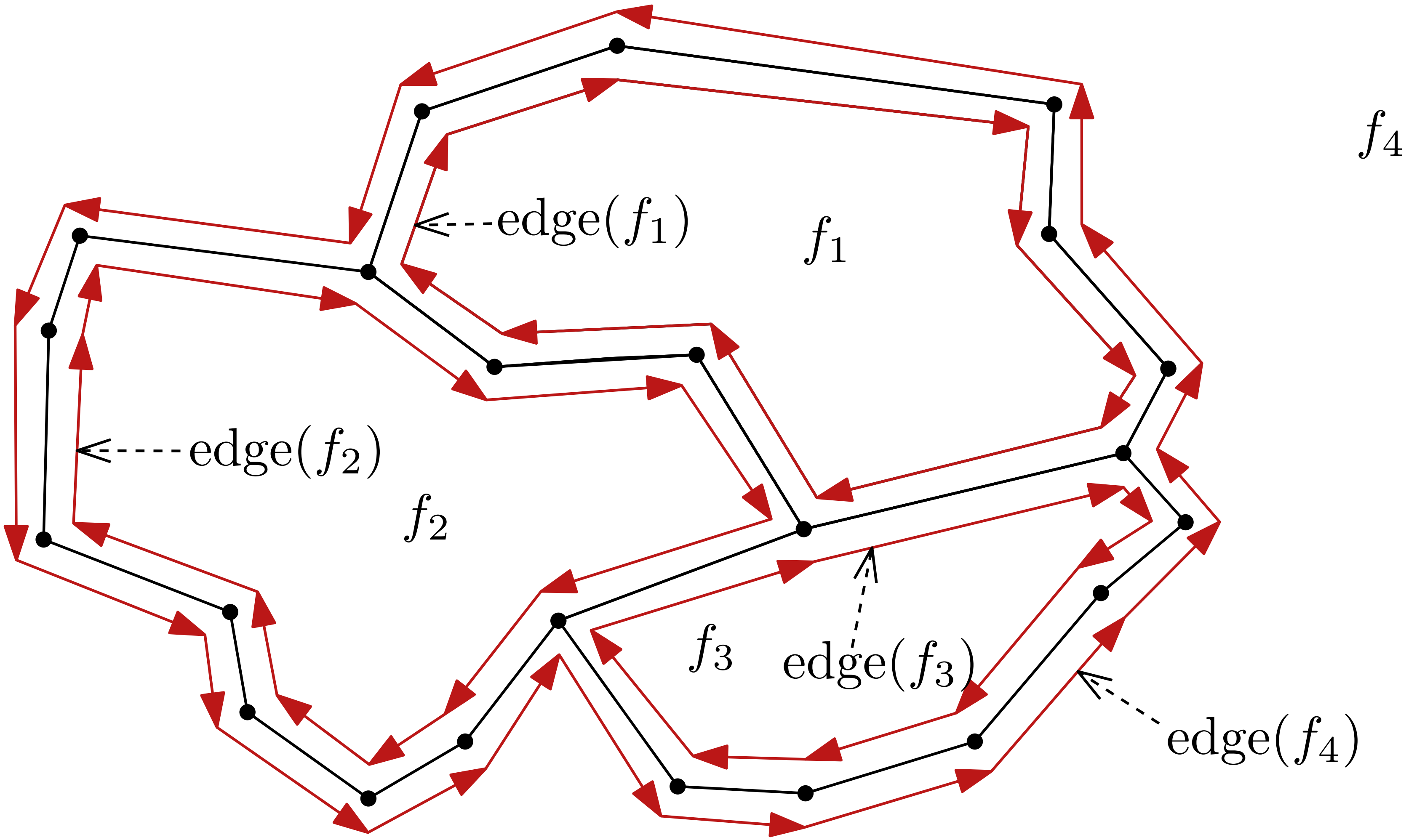


For each edge of internal faces introduce directed half-edge (clockwise)

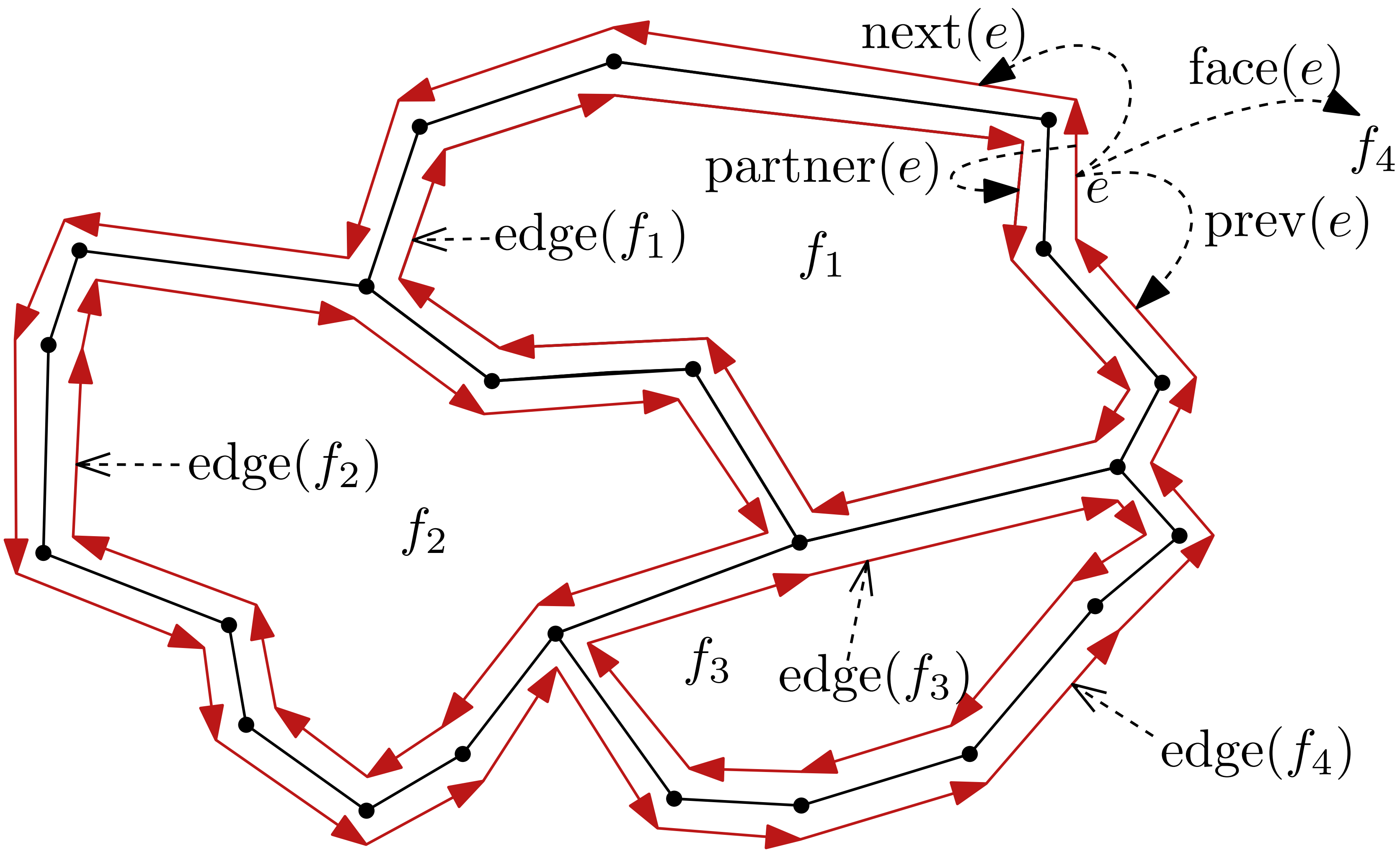
For each edge of external face introduce directed half-edge (counter-clockw.)

Subdivision



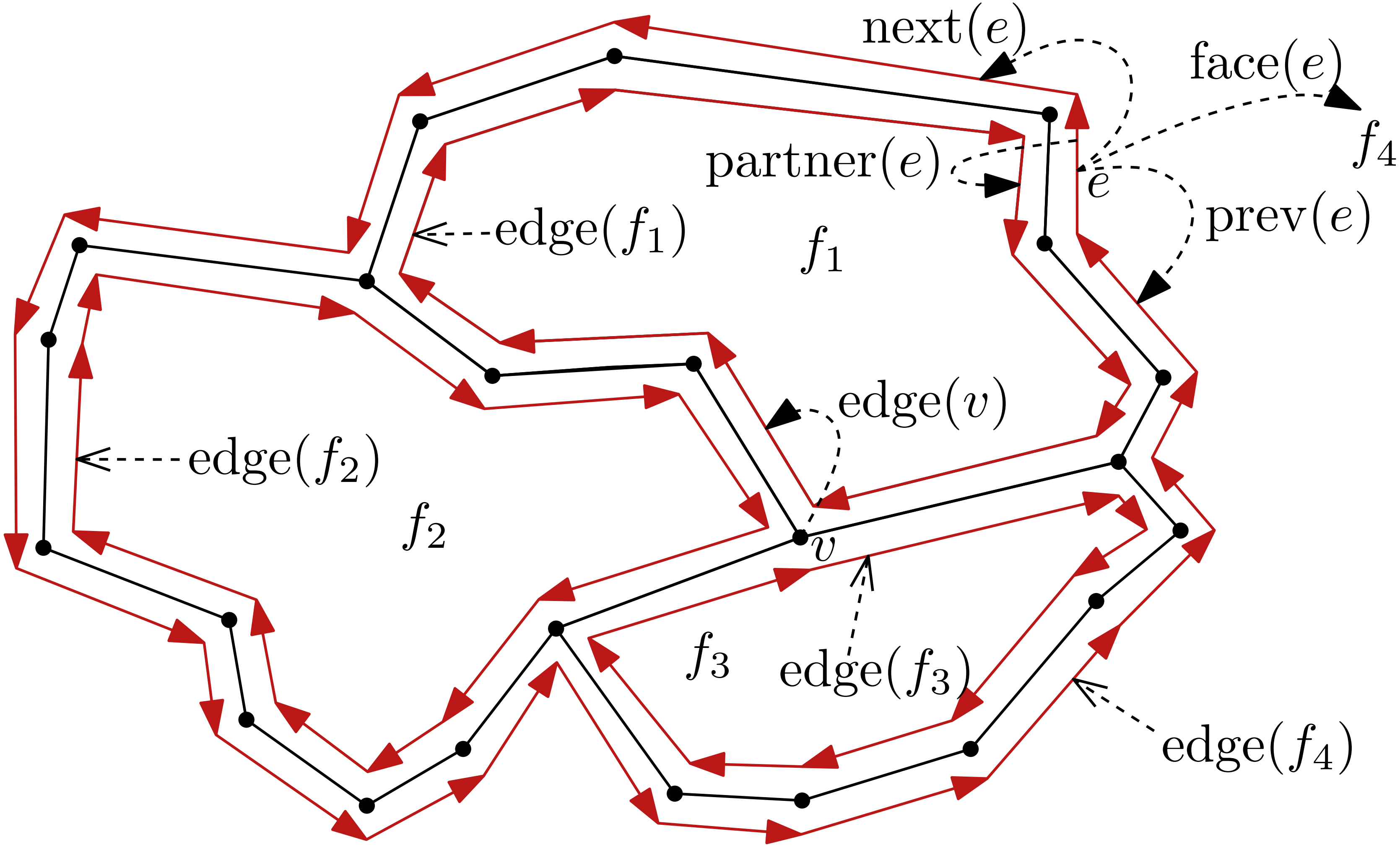
Store for each face arbitrary adjacent half-edge.

Subdivision



Store for each face arbitrary adjacent half-edge.
Store for each half-edge successor/predecessor, the half-edge on the opposite side, and the adjacent face.

Subdivision

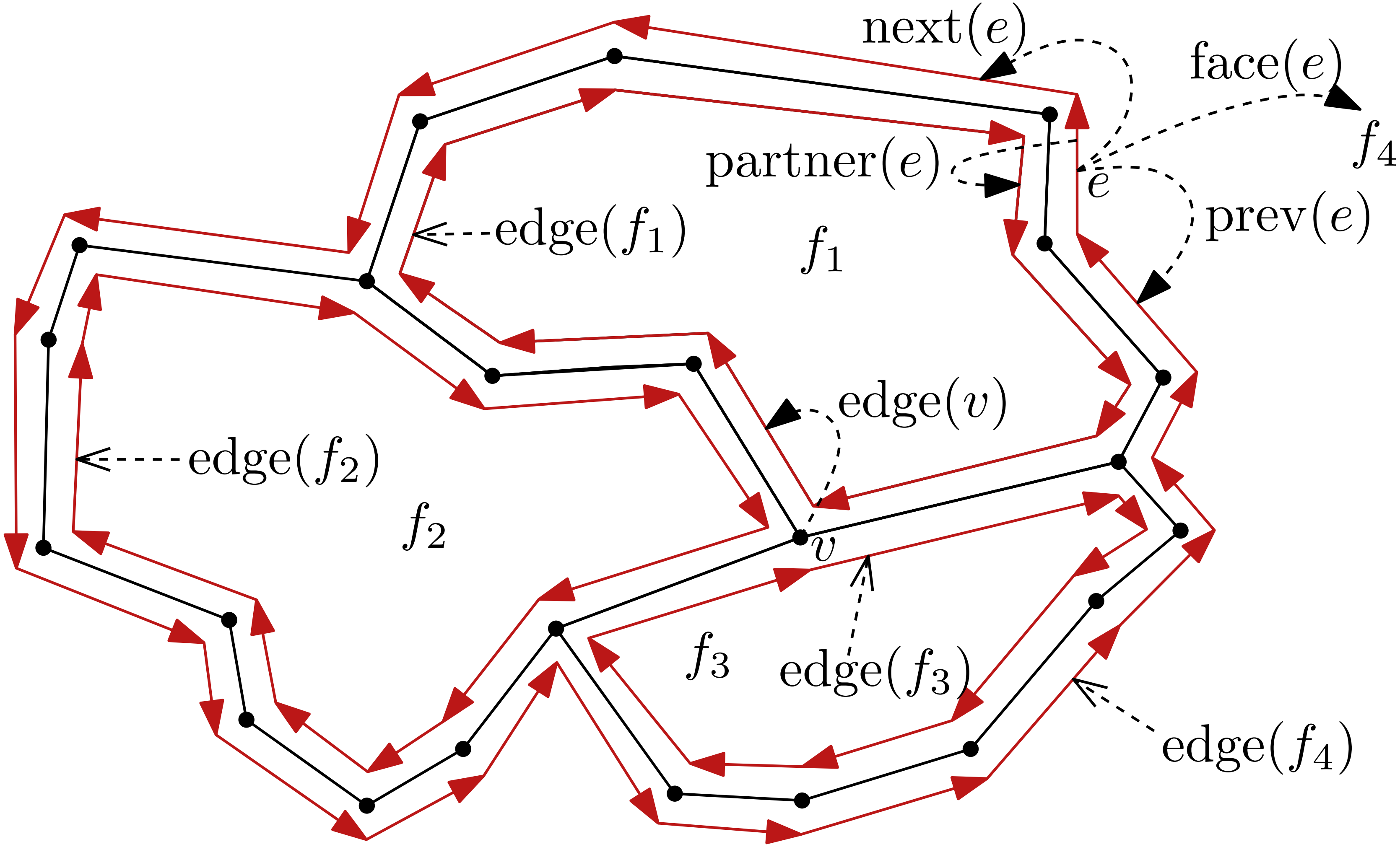


Store for each face arbitrary adjacent half-edge.

Store for each half-edge successor/predecessor, the half-edge on the opposite side, and the adjacent face.

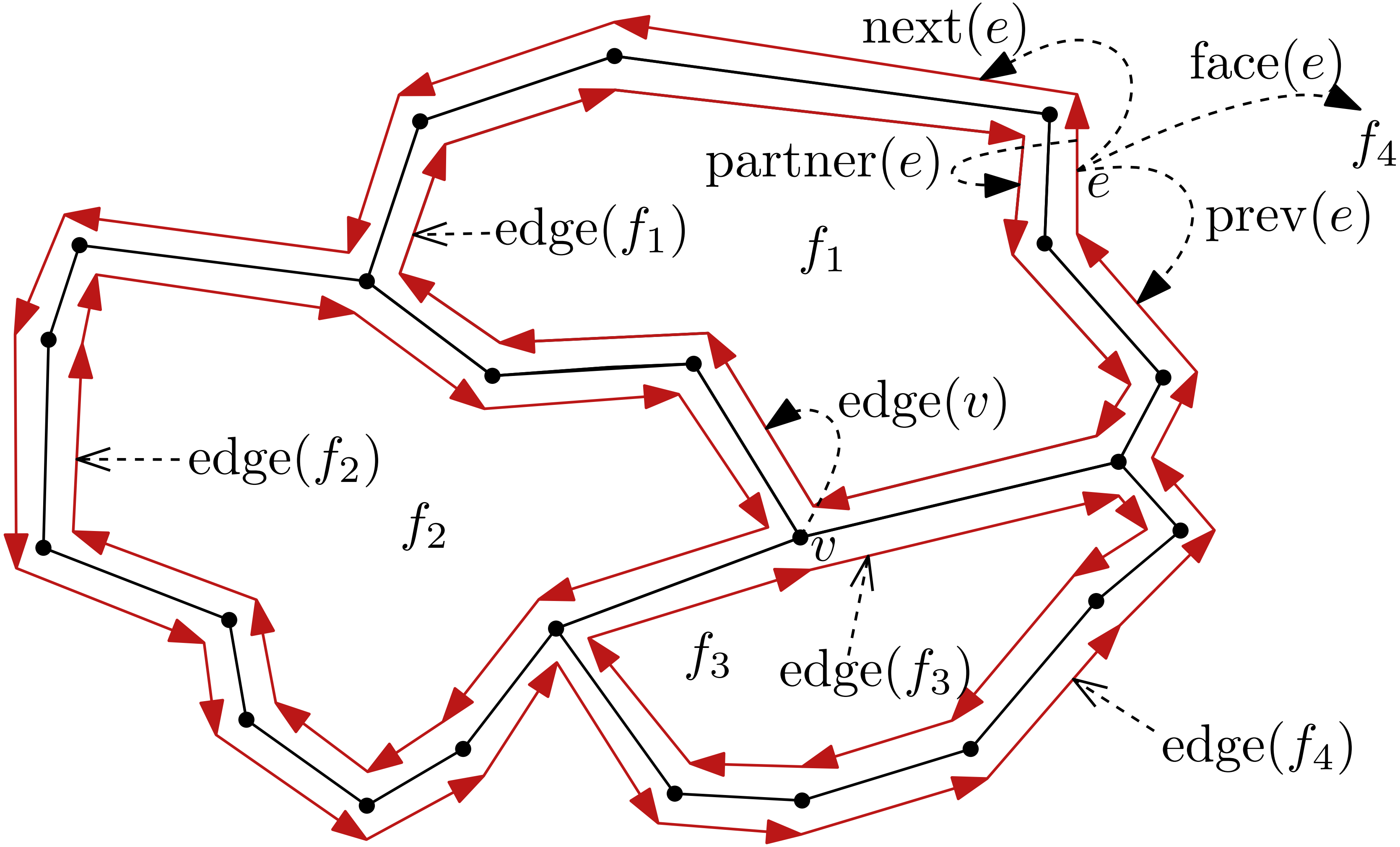
Store for each vertex an arbitrary incident outgoing half-edge.

Subdivision



- Access vertices, faces and edges.
- *Traversing* single faces.
- Traversing outgoing edges of vertex.

Subdivision



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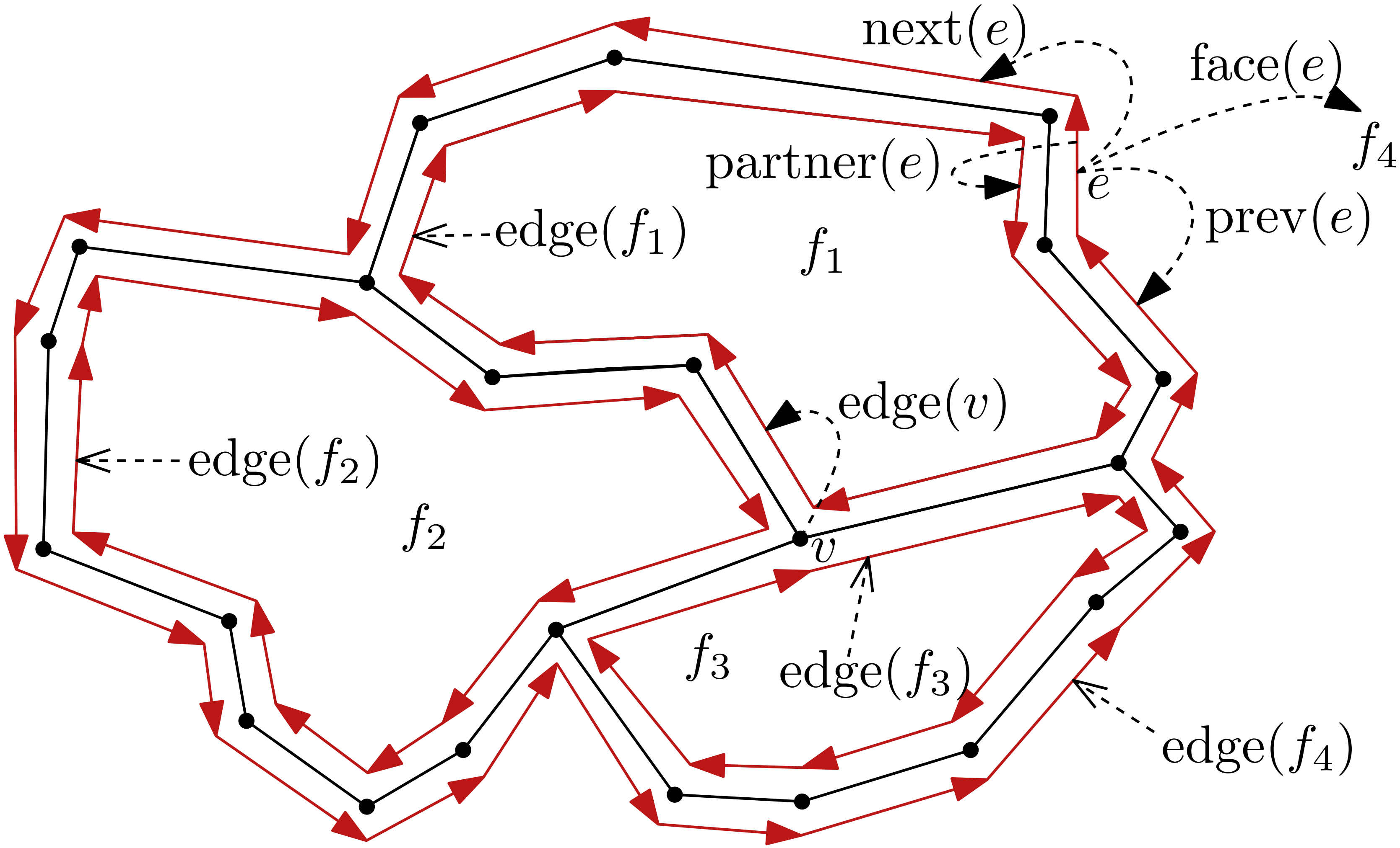

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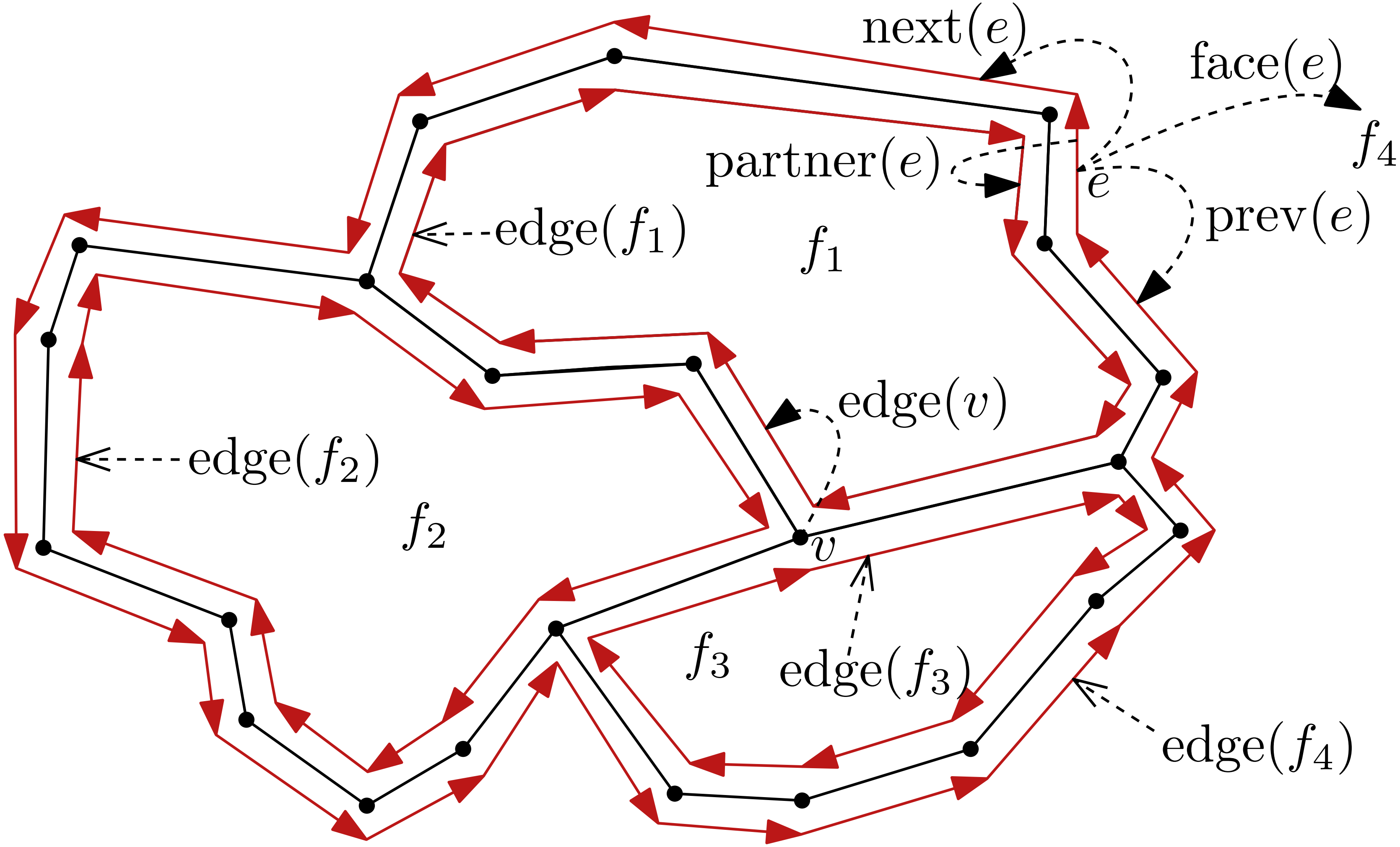

- Access vertices, faces and edges ✓
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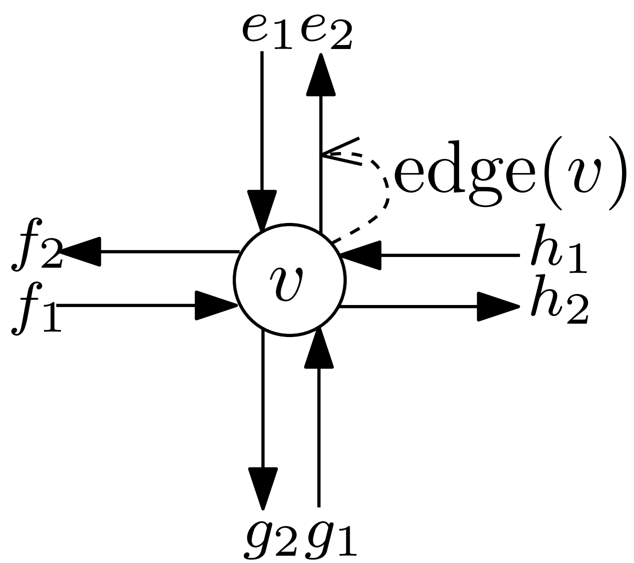
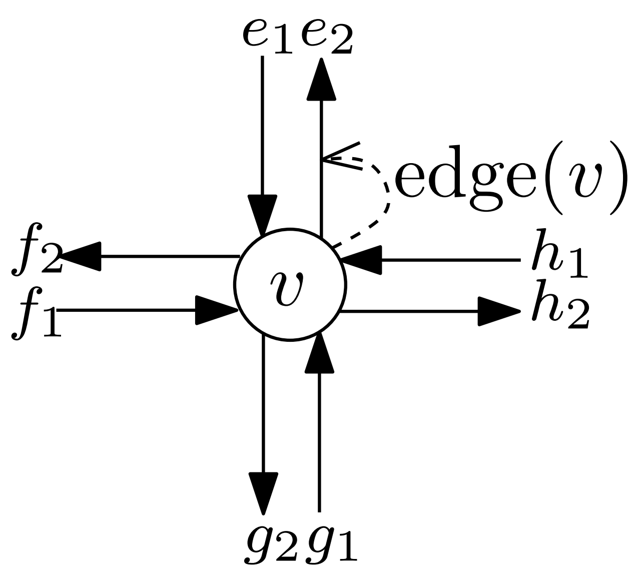
- Traversing outgoing edges of vertex. ?

Traversing incident edges



Traversing incident edges



Traversing in counter clockwise order.

$$f_2 = \text{next}(\text{partner}(e_2))$$
$$g_2 = \text{next}(\text{partner}(f_2))$$
$$h_2 = \text{next}(\text{partner}(g_2))$$
$$e_2 = \text{next}(\text{partner}(h_2))$$