Karlsruher Institut für Technologie

Vertex Ordering Problems

Lecture · May 14, 2018
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INSTITUT FÜR THEORETISCHE INFORMATIK · LEHRSTUHL ALGORITHMIK

Übersicht

Einführung

- Problemstellung
- Notation

1.Algorithmus: Dynamische Programmierung

- Beweis
- Pseudocode
- Beispiel

2.Algorithmus: Rekursion

- Pseudocode
- Beispiel

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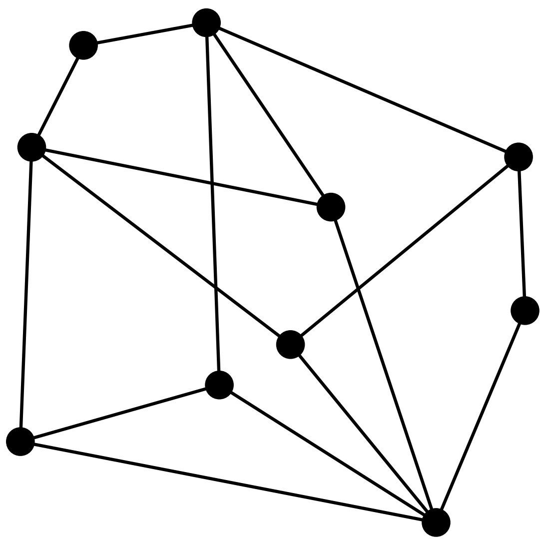
Problemstellung



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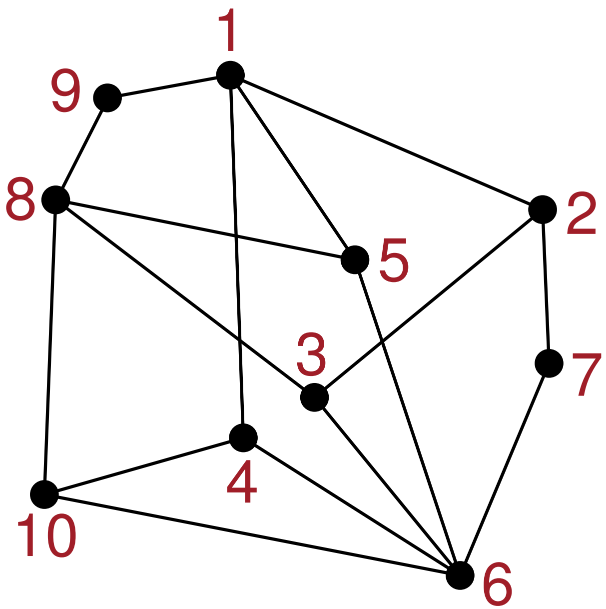
Problemstellung



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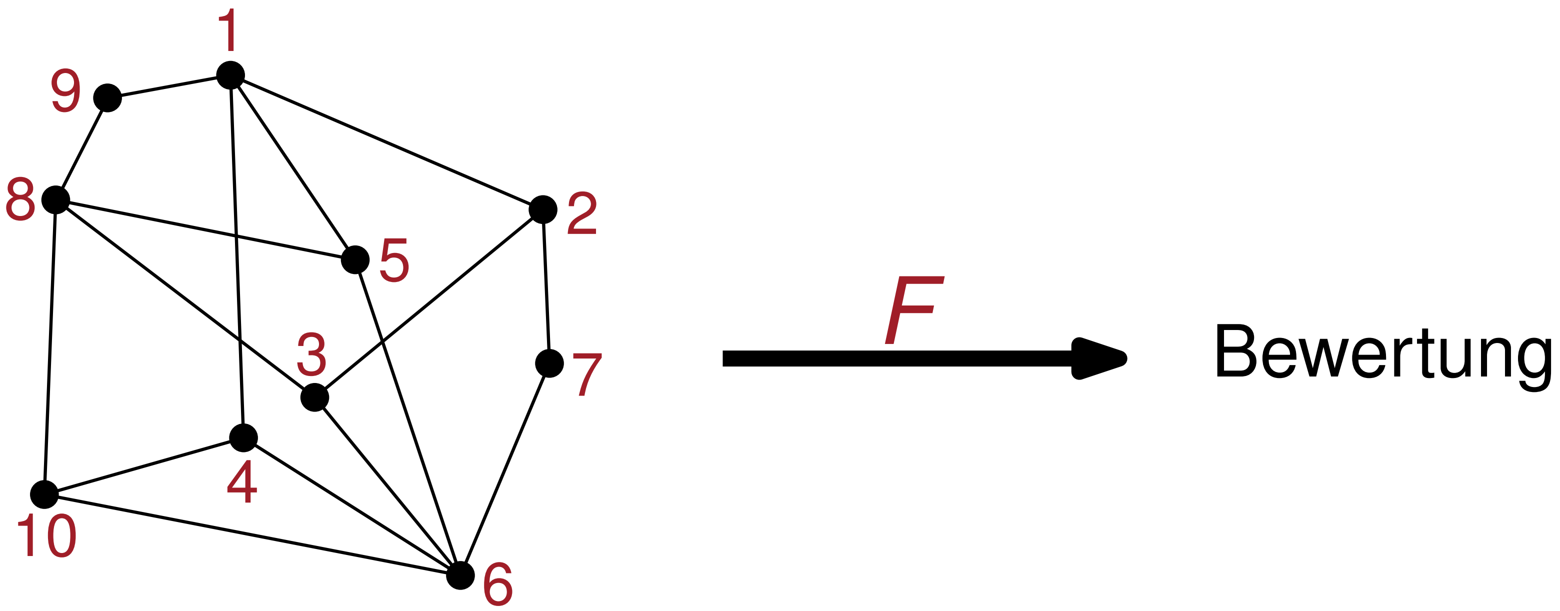
Problemstellung



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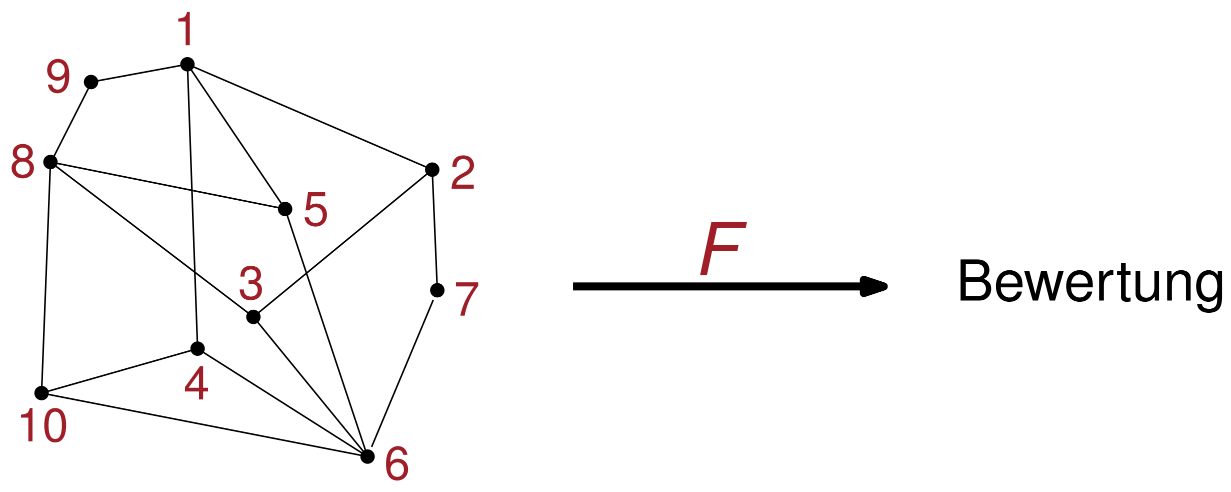
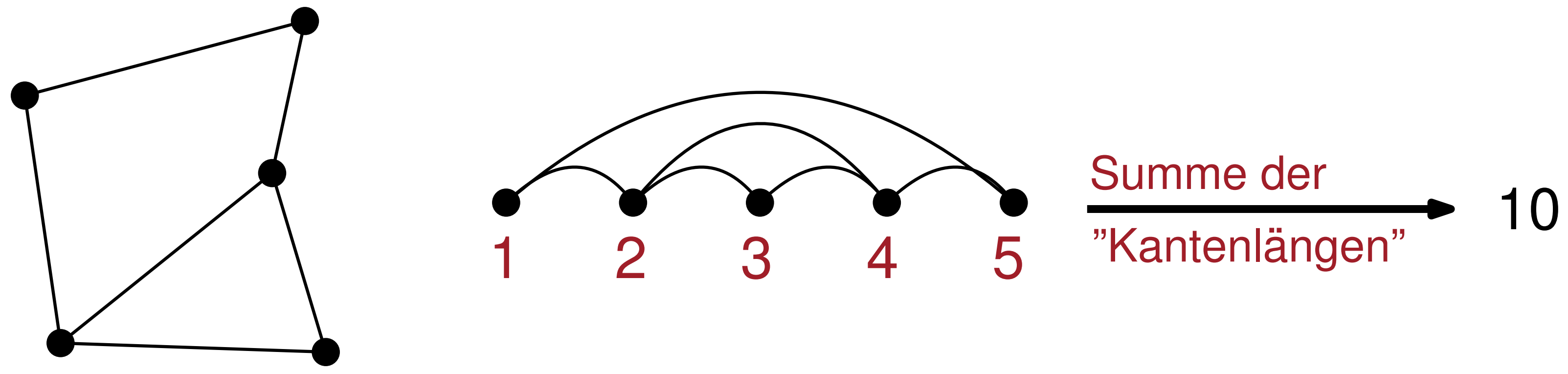
Problemstellung





Notation

Definition: Vertex Ordering/Knotenreihenfolge

Ein Vertex Ordering auf einem Graphen $G = (V, E)$ ist eine Bijektion $\pi : V \to \{1, 2, .., |V|\}$

- $\pi_{<v}$ ist die Menge der Knoten, die vor v in der Reihenfolge vorkommen
- $P(V)$ bezeichne die Menge aller Ordering auf dem Graphen $G = (V, E)$
- Für die Laufzeiten: O^* vernachlässigt polynomielle Faktoren

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Problemstellung (Teil 2)

Problemform

Gegeben ein Graph $G = (V, E)$ und eine Funktion $f : (G, L \subseteq V, v \in V) \mapsto r \in \mathbb{R}$, berechne

$$\min_{\pi \in P(V)} \max_{v \in V} f(G, \pi_{<v}, v)$$

oder

$$\min_{\pi \in P(V)} \sum_{v \in V} f(G, \pi_{<v}, v)$$

- Weitere Bedingung: f ist in Polynomialzeit berechenbar
- Kein Zugriff auf die Reihenfolge π in f, aber Wissen über Vorgänger und Nachfolger

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1. Algorithmus
Dynamische Programmierung

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Beweis

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Definition

$$F_G(S) = \min_{\pi \in P(S)} \max_{v \in S} f(G, \pi_{<v}, v),$$

$$F_G(\emptyset) = -\infty$$

- Wir wollen $F_G(V)$ berechnen

Beweis

Definition

$$F_G(S) = \min_{\pi \in P(S)} \max_{v \in S} f(G, \pi_{<v}, v),$$
$$F_G(\varnothing) = -\infty$$

Lemma

Sei $G = (V, E)$ und $S \subseteq V$, $S \neq \varnothing$, dann gilt

$$F_G(S) = \min_{v \in S} \max\{f(G, S \setminus \{v\}, v), F_G(S \setminus \{v\})\}$$

- Wahl des ”besten” letzten Knoten für die Reihenfolge
- Beweis: Betrachte $\geq$ und $\leq$ separat

Beweis

Definition

$$F_G(S) = \min_{\pi \in P(S)} \max_{v \in S} f(G, \pi_{<v}, v),$$
$$F_G(\emptyset) = -\infty$$

$$F_G(S) = \min_{v \in S} \max\{f(G, S \setminus \{v\}, v), F_G(S \setminus \{v\})\}$$

$\geq$:

- Wähle $\pi \in P(S)$ so, dass $F_G = \max_{v \in S} f(G, \pi_{<v}, v)$
- Sei w der letzte Knoten in der Reihenfolge, also $\pi(w) = |S|$, dann ist $\pi_{<w} = S \setminus \{w\}$

Beweis

Definition

$$F_G(S) = \min_{\pi \in P(S)} \max_{v \in S} f(G, \pi_{<v}, v),$$
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$$F_G(S) = \max_{v \in S} f(G, \pi_{<v}, v)$$

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Beweis

Definition

$$F_G(S) = \min_{\pi \in P(S)} \max_{v \in S} f(G, \pi_{<v}, v),$$
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- Sei w der letzte Knoten in der Reihenfolge, also $\pi(w) = |S|$, dann ist $\pi_{<w} = S \setminus \{w\}$

$$\begin{aligned} F_G(S) &= \max_{v \in S} f(G, \pi_{<v}, v) \\ &= \max\{f(G, \pi_{<w}, w), \max_{v \in S \setminus \{w\}} f(G, \pi_{<v}, v)\} \end{aligned}$$

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Beweis

Definition

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$$F_G(S) = \min_{v \in S} \max\{f(G, S \setminus \{v\}, v), F_G(S \setminus \{v\})\}$$

$\geq$:

- Wähle $\pi \in P(S)$ so, dass $F_G = \max_{v \in S} f(G, \pi_{<v}, v)$
- Sei w der letzte Knoten in der Reihenfolge, also $\pi(w) = |S|$, dann ist $\pi_{<w} = S \setminus \{w\}$

$$F_G(S) = \max_{v \in S} f(G, \pi_{<v}, v)$$
$$= \max\{f(G, \pi_{<w}, w), \max_{v \in S \setminus \{w\}} f(G, \pi_{<v}, v)\}$$

$$F_G(S \setminus \{w\}) = \min_{\pi' \in P(S \setminus \{w\})} \max_{v \in S \setminus \{w\}} f(G, \pi'_{<v}, v) \leq \max_{v \in S \setminus \{w\}} f(G, \pi_{<v}, v)$$

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Beweis

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$$F_G(S) = \min_{v \in S} \max\{f(G, S \setminus \{v\}, v), F_G(S \setminus \{v\})\}$$

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- Wähle $\pi \in P(S)$ so, dass $F_G = \max_{v \in S} f(G, \pi_{<v}, v)$
- Sei w der letzte Knoten in der Reihenfolge, also $\pi(w) = |S|$, dann ist $\pi_{<w} = S \setminus \{w\}$

$$\begin{aligned} F_G(S) &= \max_{v \in S} f(G, \pi_{<v}, v) \\ &= \max\{f(G, \pi_{<w}, w), \max_{v \in S \setminus \{w\}} f(G, \pi_{<v}, v)\} \\ &\geq \max\{f(G, S \setminus \{w\}, w), F_G(S \setminus \{w\})\} \end{aligned}$$

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Beweis

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$$F_G(S) = \min_{v \in S} \max\{f(G, S \setminus \{v\}, v), F_G(S \setminus \{v\})\}$$

$\leq$:

- Wähle $w \in S$ so, dass $\max\{f(G, S \setminus \{w\}, w), F_G(S \setminus \{w\})\}$ minimal ist
- Wähle $\pi' \in P(S \setminus \{w\})$ so, dass $F_G(S \setminus \{w\}) = \max_{v \in S \setminus \{w\}} f(G, \pi'_{<v}, v)$
- Sei π die Reihenfolge, die mit π' anfängt und mit w endet

$$\pi(v) = \begin{cases} \pi'(v) & v \neq w \\ |S| & v = w \end{cases}$$

Beweis

Definition

$$F_G(S) = \min_{\pi \in P(S)} \max_{v \in S} f(G, \pi_{<v}, v),$$
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$$F_G(S) = \min_{v \in S} \max\{f(G, S \setminus \{v\}, v), F_G(S \setminus \{v\})\}$$

$\leq$:

$$F_G(S) \leq \max_{v \in S} f(G, \pi_{<v}, v)$$

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Beweis

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$$F_G(S) = \min_{\pi \in P(S)} \max_{v \in S} f(G, \pi_{<v}, v),$$
$$F_G(\emptyset) = -\infty$$

$$F_G(S) = \min_{v \in S} \max\{f(G, S \backslash \{v\}, v),$$
$$F_G(S \backslash \{v\})\}$$

$\leq$:

$$F_G(S) \leq \max_{v \in S} f(G, \pi_{<v}, v)$$
$$= \max\{f(G, \pi_{<w}, w), \max_{v \in S \backslash \{w\}} f(G, \pi_{<v}, v)\}$$

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$$F_G(S) \leq \max_{v \in S} f(G, \pi_{<v}, v)$$
$$= \max\{f(G, \pi_{<w}, w), \max_{v \in S \setminus \{w\}} f(G, \pi_{<v}, v)\}$$
$$= \max\{f(G, S \setminus \{w\}, w), \max_{v \in S \setminus \{w\}} f(G, \pi'_{<v}, v)\}$$

$$\pi(v) = \begin{cases} \pi'(v) & v \neq w \\ |S| & v = w \end{cases}$$

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$$F_G(S) = \min_{\pi \in P(S)} \max_{v \in S} f(G, \pi_{<v}, v),$$
$$F_G(\varnothing) = -\infty$$

$$F_G(S) = \min_{v \in S} \max\{f(G, S \setminus \{v\}, v), F_G(S \setminus \{v\})\}$$

$\leq$:

$$\begin{aligned}
F_G(S) &\leq \max_{v \in S} f(G, \pi_{<v}, v) \\
&= \max\{f(G, \pi_{<w}, w), \max_{v \in S \setminus \{w\}} f(G, \pi_{<v}, v)\} \\
&= \max\{f(G, S \setminus \{w\}, w), \max_{v \in S \setminus \{w\}} f(G, \pi'_{<v}, v)\} \\
&= \max\{f(G, S \setminus \{w\}, w), F_G(S \setminus \{w\})\}
\end{aligned}$$

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&= \max\{f(G, \pi_{<w}, w), \max_{v \in S \setminus \{w\}} f(G, \pi_{<v}, v)\} \\
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&= \max\{f(G, S \setminus \{w\}, w), F_G(S \setminus \{w\})\} \\
&= \min_{w \in S} \max\{f(G, S \setminus \{w\}, w), F_G(S \setminus \{w\})\}
\end{aligned}$$

Pseudocode

Algorithm 1: Dynamic-Programming-Algorithm ($G = (V, E)$)

$F_G(\emptyset) \coloneqq -\infty$
for $i = 1..n$ **do**
  for *each* $S \subseteq V, |S| = i$ **do**
    $F_G(S) \coloneqq \min_{w \in S} \max\{f(G, S \setminus \{w\}, w), F_G(S \setminus \{w\})\}$
  end for
end for
return $F_G(V)$

- Laufzeit: $O^*(2^n)$
- Speicherverbrauch: $O^*(2^n)$

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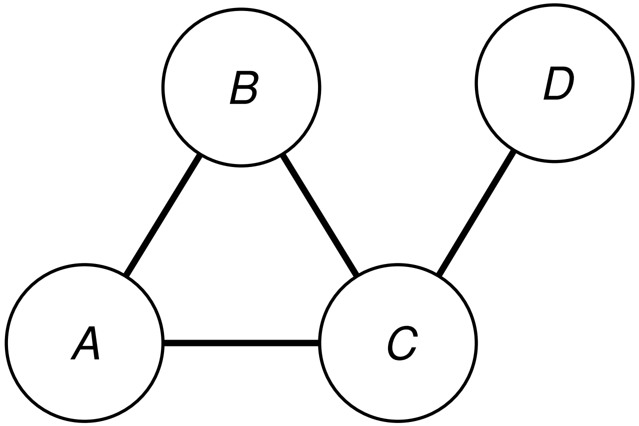
Beispiel



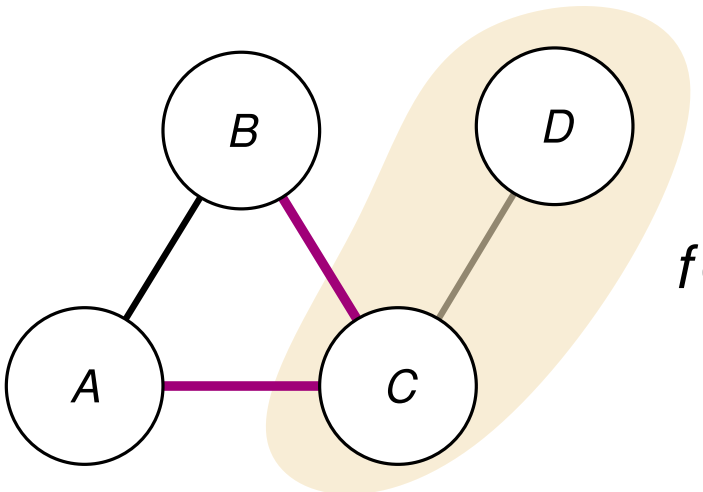
Optimal Linear Arrangement

Minimiere $\sum_{\{x,y\}\in E} |\pi(x) - \pi(y)| = \sum_{v\in V} |\{\{x,y\} \in E | \pi(x) \leq \pi(v) < \pi(y)\}|$,

also:

$f(G, L, v) = |\{\{x,y\} \in E | x \in L \cup \{v\}, y \in V \setminus (L \cup \{v\})\}|$

Beispiel



$f(G, \{C\}, D) = 2$

Optimal Linear Arrangement

Minimiere $\sum_{\{x,y\}\in E} |\pi(x) - \pi(y)| = \sum_{v \in V} |\{\{x,y\} \in E | \pi(x) \leq \pi(v) < \pi(y)\}|,$

also:

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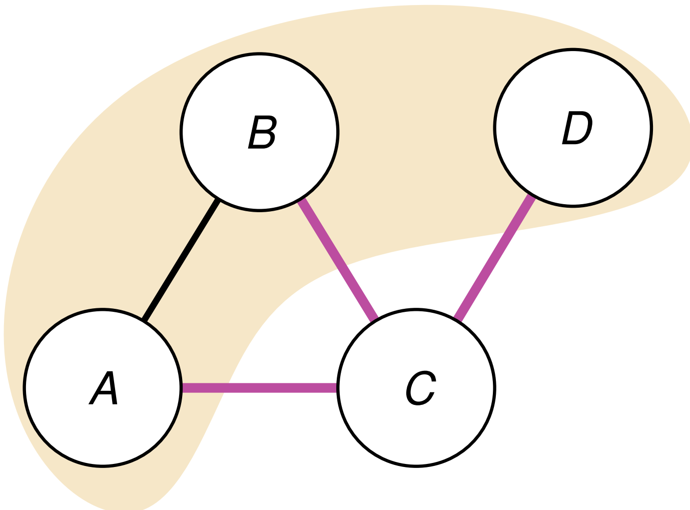
Beispiel



$f(G, \{A, D\}, B) = 3$

Optimal Linear Arrangement

Minimiere $\sum_{\{x,y\}\in E} |\pi(x) - \pi(y)| = \sum_{v \in V} |\{\{x, y\} \in E | \pi(x) \leq \pi(v) < \pi(y)\}|$,

also:

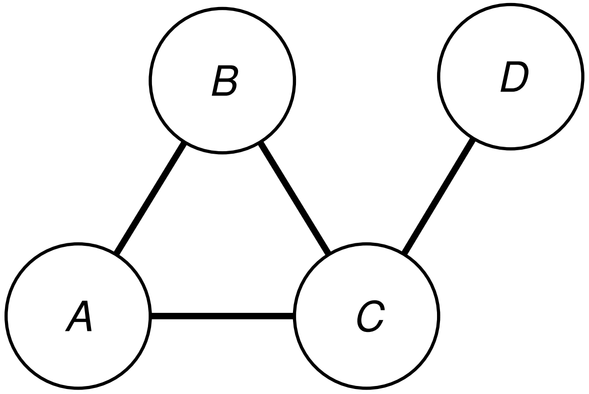
$f(G, L, v) = |\{\{x, y\} \in E | x \in L \cup \{v\}, y \in V \setminus (L \cup \{v\})\}|$

Beispiel

- $i = 1$

S	$F(S)$
$\{A\}$	2
$\{B\}$	2
$\{C\}$	3
$\{D\}$	1

$F(\emptyset) := 0$
for $i = 1..n$ **do**
  for *each* $S \subseteq V, |S| = i$ **do**
    $A(S) := \min_{w \in S} f(S \setminus \{w\}, w) + F(S \setminus \{w\})$
  end for
end for
return $F(V)$



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Beispiel

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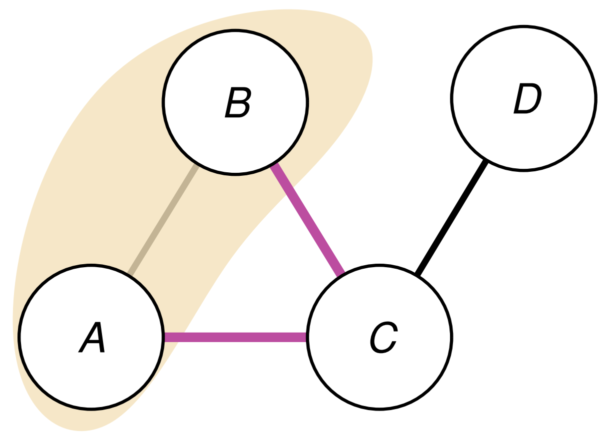
$\{A\}$	2
$\{B\}$	2
$\{C\}$	3
$\{D\}$	1

$F(\emptyset) \coloneqq 0$
for $i = 1..n$ **do**
  for *each* $S \subseteq V, |S| = i$ **do**
    $A(S) \coloneqq \min_{w \in S} f(S \setminus \{w\}, w) + F(S \setminus \{w\})$
  end for
end for
return $F(V)$

- $i = 2$

S	w	$f(S \setminus \{w\}, w) + F(S \setminus \{w\})$
$\{A, B\}$	B	$f(\{A\}, B) + F(\{A\})$
	A	$f(\{B\}, A) + F(\{B\})$



Beispiel

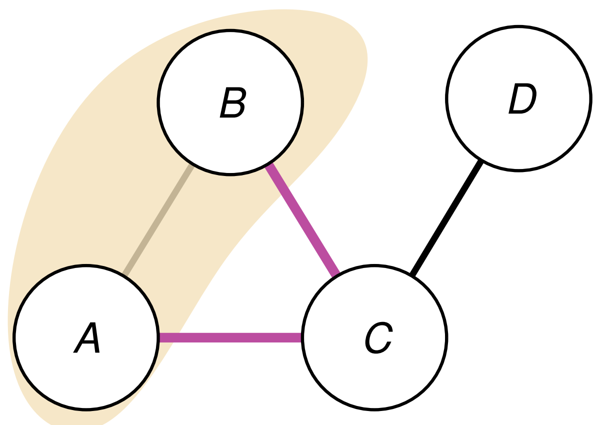
- $i = 1$

S	$F(S)$
$\{A\}$	2
$\{B\}$	2
$\{C\}$	3
$\{D\}$	1

$F(\emptyset) :=\ 0$
for $i = 1..n$ **do**
  for *each* $S \subseteq V, |S| = i$ **do**
    $A(S) := \min_{w \in S} f(S \setminus \{w\}, w) + F(S \setminus \{w\})$
  end for
end for
return $F(V)$

- $i = 2$

S	w	$f(S \setminus \{w\}, w) + F(S \setminus \{w\})$
$\{A, B\}$	B	4
	A	$f(\{B\}, A) + F(\{B\})$



Beispiel

- $i = 1$

S	$F(S)$
$\{A\}$	2
$\{B\}$	2
$\{C\}$	3
$\{D\}$	1

$F(\emptyset) := 0$

for $i = 1..n$ **do**

  for *each* $S \subseteq V, |S| = i$ **do**

    $A(S) := \min_{w \in S} f(S \setminus \{w\}, w) + F(S \setminus \{w\})$

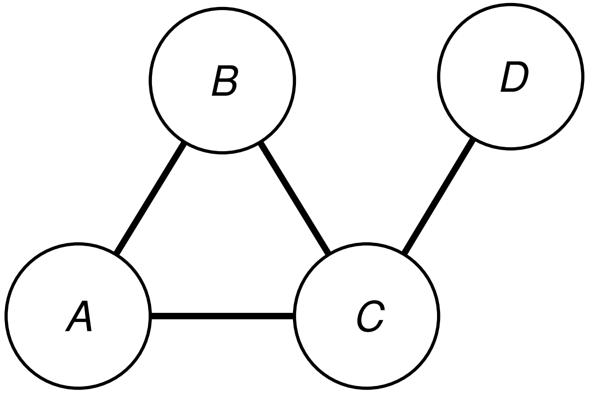
  end for

end for

return $F(V)$

- $i = 2$

S	w	$f(S\setminus\{w\}, w) + F(S\setminus\{w\})$
$\{A, B\}$	B	4
	A	4



Beispiel

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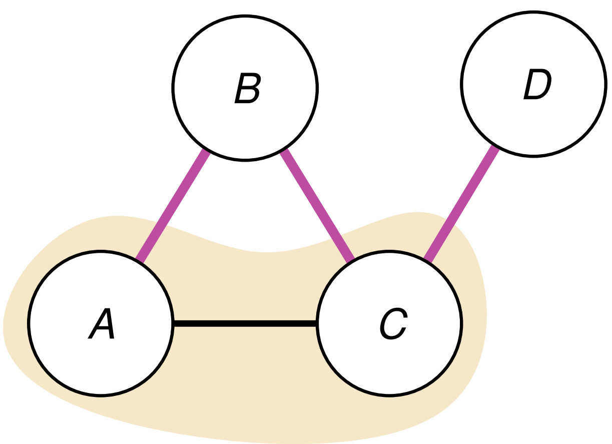
    $A(S) := \min_{w \in S} f(S \setminus \{w\}, w) + F(S \setminus \{w\})$
  end for
end for
return $F(V)$

- $i = 2$

S	$F(S)$		
$\{A, B\}$	4		
$\{A, C\}$		w	$f(S \setminus \{w\}, w) + F(S \setminus \{w\})$
		C	5
		A	6



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Beispiel

- $i = 1$

S	$F(S)$
$\{A\}$	2
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$\{C\}$	3
$\{D\}$	1

- $i = 2$

S	$F(S)$
$\{A, B\}$	4
$\{A, C\}$	5
$\{A, D\}$	5
$\{B, C\}$	5
$\{B, D\}$	4
$\{C, D\}$	3

$F(\emptyset) := 0$
for $i = 1..n$ **do**
  for *each* $S \subseteq V, |S| = i$ **do**
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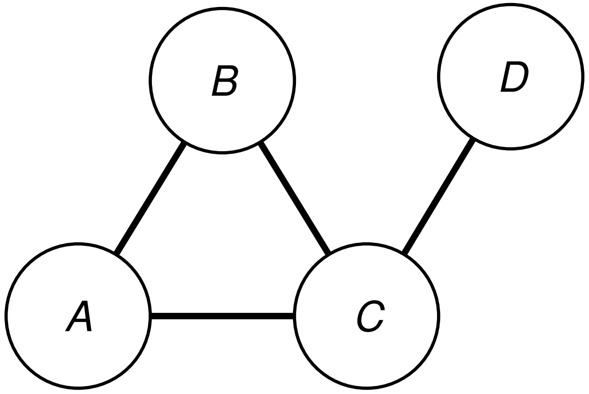
  end for
end for
return $F(V)$

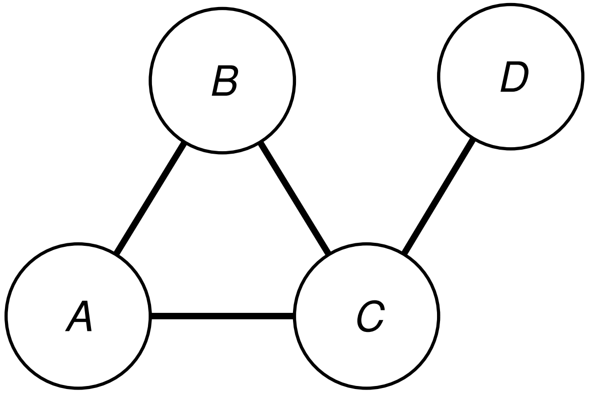


Beispiel

- $i = 3$

S	$F(S)$
$\{A, B, C\}$	5
$\{A, B, D\}$	7
$\{A, C, D\}$	5
$\{B, C, D\}$	5

> $F(\emptyset) := 0$
> **for** $i = 1..n$ **do**
>   **for** *each* $S \subseteq V, |S| = i$ **do**
> $A(S) := \min_{w \in S} f(S \setminus \{w\}, w) + F(S \setminus \{w\})$
>   **end for**
> **end for**
> **return** $F(V)$



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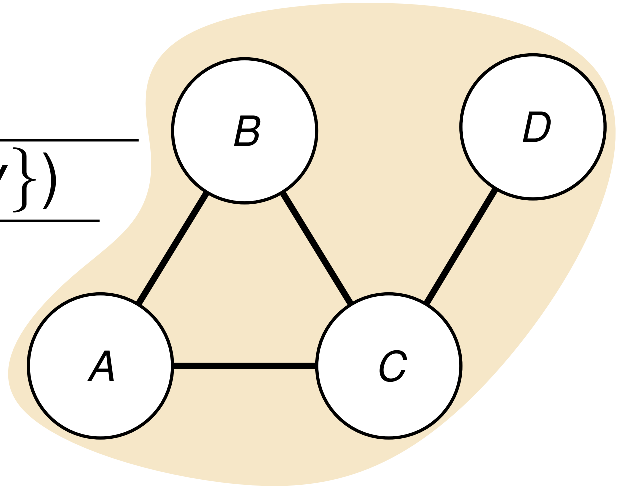
for $i = 1..n$ **do**
  for *each* $S \subseteq V, |S| = i$ **do**
    $A(S) := \min_{w \in S} f(S \setminus \{w\}, w) + F(S \setminus \{w\})$
  end for
end for
return $F(V)$

- $i = 4$

S	w	$f(S\setminus\{w\}, w) + F(S\setminus\{w\})$
$\{A, B, C, D\}$	D	5
	C	7
	B	5
	A	5



Beispiel

- $i = 3$

S	$F(S)$
$\{A, B, C\}$	5
$\{A, B, D\}$	7
$\{A, C, D\}$	5
$\{B, C, D\}$	5

- $i = 4$

S	$F(S)$
$\{A, B, C, D\}$	5

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for $i = 1..n$ **do**
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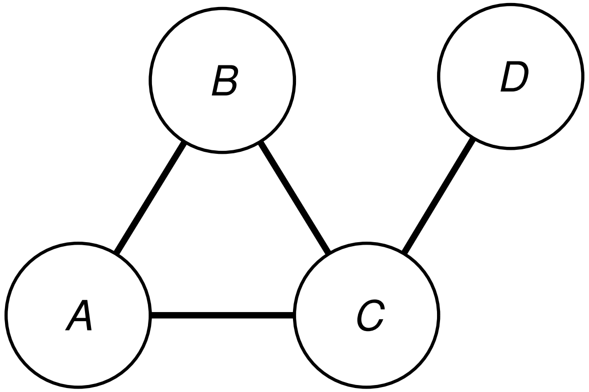
end for
return $F(V)$



2. Algorithmus
Rekursion

Pseudocode

Algorithm 2: Recursive(Graph G, vertex set L, vertex set R)

```
if R = {v} then
    return f(G, L, v)
end if
opt := ∞
for each R' ⊆ R, |R'| = ⌊|R|/2⌋ do
    v1 := Recursive(G, L, R')
    v2 := Recursive(G, L ∪ R', R \ R')
    opt := min{opt, max{v1, v2}}
end for
return opt
```

- Laufzeit: $O^*(4^n)$
- Speicherverbrauch: polynomiell

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Laufzeit

Laufzeit: $O^*(4^n)$:

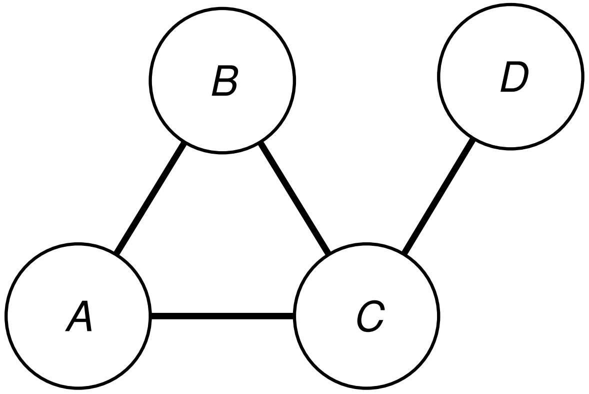
- Rekursive Aufrufe:

$$T(k) \leq \binom{k}{\left\lfloor \frac{k}{2} \right\rfloor} \left(T\left(\left\lfloor \frac{k}{2} \right\rfloor \right) + T\left(\left\lceil \frac{k}{2} \right\rceil \right) + 2 \right)$$

- Es folgt: $T(k) \leq 4^k$

$\text{opt} := \infty$
for *each* $R' \subseteq R, |R'| = \lfloor |R|/2 \rfloor$ **do**
$\quad v_1 := \text{Recursive}(L, R')$
$\quad v_2 := \text{Recursive}(L \cup R', R \setminus R')$
$\quad \text{opt} := \min\{\text{opt}, v_1 + v_2\}$
end for

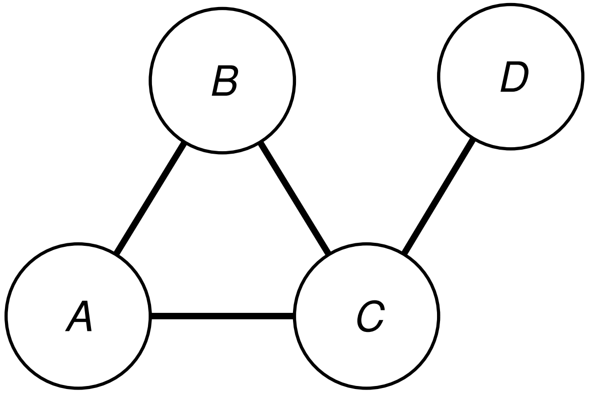
$\text{Recursive}(\{\}, \{A, B, C, D\})$:



$$\text{Recursive}(\{\}, \{A, B, C, D\})\text{: opt} = \infty$$

$$R' = \{A, B\}$$

opt $:= \infty$
for *each* $R' \subseteq R, |R'| = \lfloor |R|/2 \rfloor$ **do**
  $v_1 :=$ Recursive(L, R')
  $v_2 :=$ Recursive$(L \cup R', R \setminus R')$
  opt $:= \min\{\text{opt}, v_1 + v_2\}$
end for

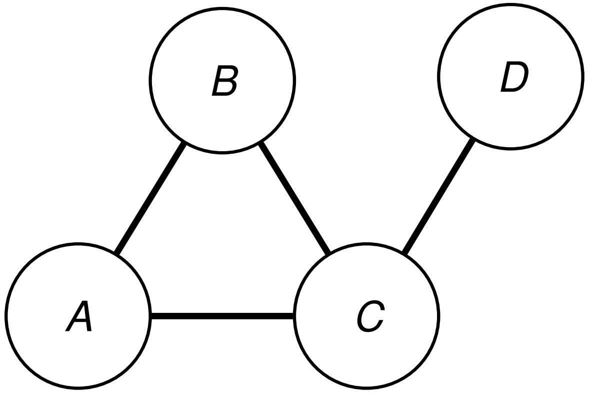


$$\begin{array}{l}
\text{opt} := \infty \\
\textbf{for } \textit{each } R' \subseteq R, |R'| = \lfloor |R|/2 \rfloor \textbf{ do} \\
\quad v_1 := \text{Recursive}(L, R') \\
\quad v_2 := \text{Recursive}(L \cup R', R \setminus R') \\
\quad \text{opt} := \min\{\text{opt}, v_1 + v_2\} \\
\textbf{end for}
\end{array}$$

Recursive($\{\}, \{A, B, C, D\}$): opt $= \infty$

$R' = \{A, B\}$

$v_1 = $ Recursive($\{\}, \{A, B\}$):



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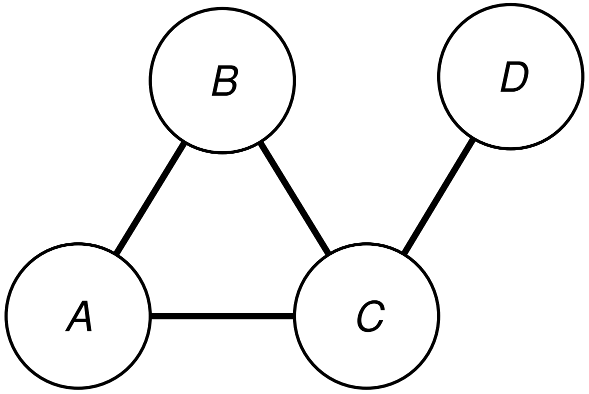
$$
\begin{aligned}
&\text{opt} \coloneqq \infty \\
&\textbf{for } \textit{each } R' \subseteq R, |R'| = \lfloor |R|/2 \rfloor \textbf{ do} \\
&\quad v_1 \coloneqq \text{Recursive}(L, R') \\
&\quad v_2 \coloneqq \text{Recursive}(L \cup R', R \setminus R') \\
&\quad \text{opt} \coloneqq \min\{\text{opt}, v_1 + v_2\} \\
&\textbf{end for}
\end{aligned}
$$

Recursive($\{\}, \{A, B, C, D\}$): opt $= \infty$

$R' = \{A, B\}$

$v_1 =$ Recursive($\{\}, \{A, B\}$): opt $= \infty$

$R' = \{A\}$



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$$
\begin{array}{l}
\text{opt} := \infty \\
\textbf{for } \textit{each } R' \subseteq R, |R'| = \lfloor |R|/2 \rfloor \textbf{ do} \\
\quad v_1 := \text{Recursive}(L, R') \\
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\quad \text{opt} := \min\{\text{opt}, v_1 + v_2\} \\
\textbf{end for}
\end{array}
$$

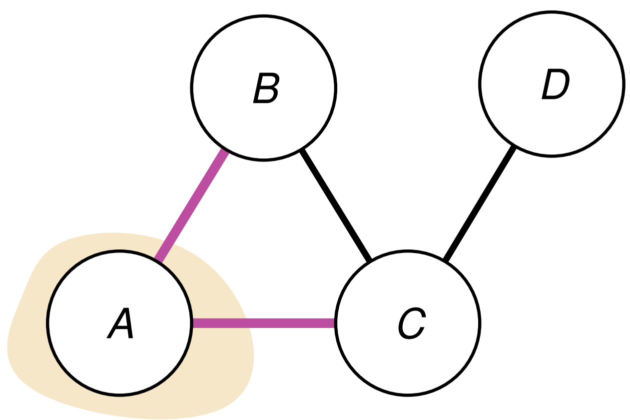
Recursive($\{\}, \{A, B, C, D\}$): opt $= \infty$

$R' = \{A, B\}$

$v_1 =$ Recursive($\{\}, \{A, B\}$): opt $= \infty$

$R' = \{A\}$

$v_1 =$ Recursive($\{\}, \{A\}$):



$$
\begin{aligned}
&\text{opt} \coloneqq \infty \\
&\textbf{for } \textit{each } R' \subseteq R, |R'| = \lfloor |R|/2 \rfloor \textbf{ do} \\
&\quad v_1 \coloneqq \text{Recursive}(L, R') \\
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&\quad \text{opt} \coloneqq \min\{\text{opt}, v_1 + v_2\} \\
&\textbf{end for}
\end{aligned}
$$

Recursive($\{\}, \{A, B, C, D\}$): opt $= \infty$

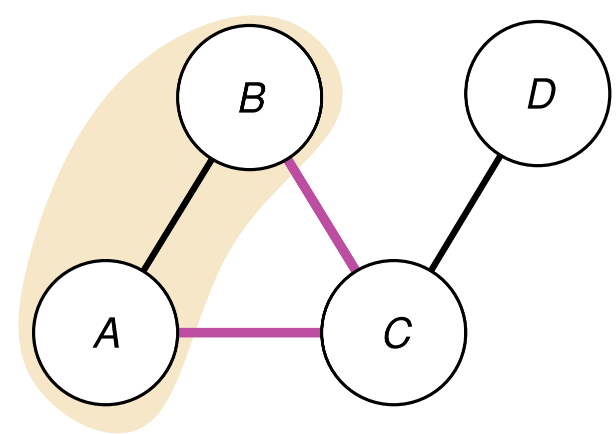
$R' = \{A, B\}$

$v_1 =$ Recursive($\{\}, \{A, B\}$): opt $= \infty$

$R' = \{A\}$

$v_1 =$ Recursive($\{\}, \{A\}$): 2

$v_2 =$ Recursive($\{A\}, \{B\}$):



$\mathrm{opt} := \infty$
for *each* $R' \subseteq R, |R'| = \lfloor |R|/2 \rfloor$ **do**
  $v_1 := \mathrm{Recursive}(L, R')$
  $v_2 := \mathrm{Recursive}(L \cup R', R \setminus R')$
  $\mathrm{opt} := \min\{\mathrm{opt}, v_1 + v_2\}$
end for

$\mathrm{Recursive}(\{\}, \{A, B, C, D\})$: $\mathrm{opt} = \infty$

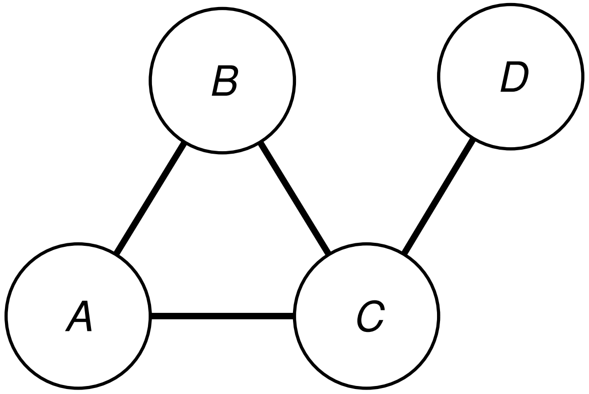
$R' = \{A, B\}$

$v_1 = \mathrm{Recursive}(\{\}, \{A, B\})$: $\mathrm{opt} = \infty$

$R' = \{A\}$ $\quad v_1 + v_2 = 4$

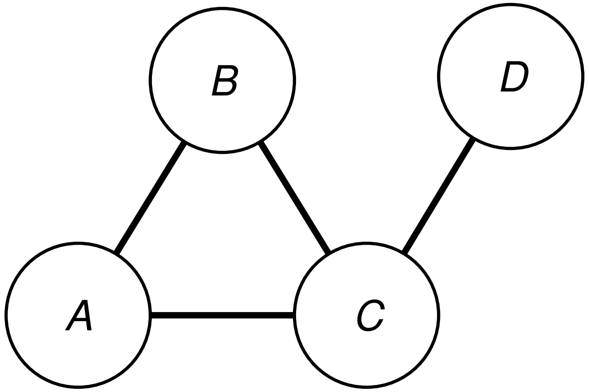
$v_1 = \mathrm{Recursive}(\{\}, \{A\})$: 2

$v_2 = \mathrm{Recursive}(\{A\}, \{B\})$: 2



$\text{opt} \coloneqq \infty$
for *each* $R' \subseteq R, |R'| = \lfloor |R|/2 \rfloor$ **do**
$\quad v_1 \coloneqq \text{Recursive}(L, R')$
$\quad v_2 \coloneqq \text{Recursive}(L \cup R', R \backslash R')$
$\quad \text{opt} \coloneqq \min\{\text{opt}, v_1 + v_2\}$
end for

$\text{Recursive}(\{\}, \{A, B, C, D\})$: $\text{opt} = \infty$
$\quad R' = \{A, B\}$
$\quad\quad v_1 = \text{Recursive}(\{\}, \{A, B\})$: $\text{opt} = 4$
$\quad\quad\quad R' = \{B\}$



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$\text{opt} \coloneqq \infty$
for *each* $R' \subseteq R, |R'| = \lfloor |R|/2 \rfloor$ **do**
$v_1 \coloneqq \text{Recursive}(L, R')$
$v_2 \coloneqq \text{Recursive}(L \cup R', R \setminus R')$
$\text{opt} \coloneqq \min\{\text{opt}, v_1 + v_2\}$
end for

$\text{Recursive}(\{\}, \{A, B, C, D\})$: $\text{opt} = \infty$

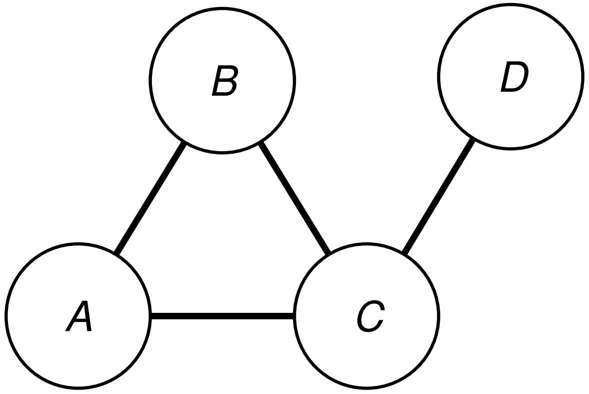
$R' = \{A, B\}$

$v_1 = \text{Recursive}(\{\}, \{A, B\})$: $\text{opt} = 4$

$R' = \{B\}$ $\quad v_1 + v_2 = 4$

$v_1 = \text{Recursive}(\{\}, \{B\})$: 2

$v_2 = \text{Recursive}(\{B\}, \{A\})$: 2



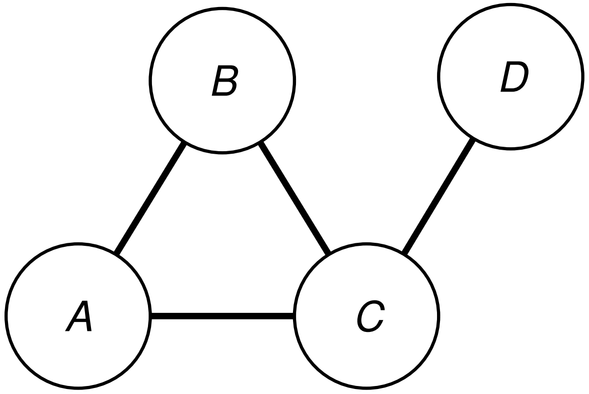
$\text{opt} \coloneqq \infty$
for *each* $R' \subseteq R, |R'| = \lfloor |R|/2 \rfloor$ **do**
$v_1 \coloneqq \text{Recursive}(L, R')$
$v_2 \coloneqq \text{Recursive}(L \cup R', R \setminus R')$
$\text{opt} \coloneqq \min\{\text{opt}, v_1 + v_2\}$
end for

$\text{Recursive}(\{\}, \{A, B, C, D\})\text{: opt} = \infty$

$R' = \{A, B\}$

$v_1 = \text{Recursive}(\{\}, \{A, B\})\text{: } 4$

$v_2 = \text{Recursive}(\{A, B\}, \{C, D\})\text{:}$



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$\text{opt} := \infty$
for *each* $R' \subseteq R, |R'| = \lfloor |R|/2 \rfloor$ **do**
$v_1 := \text{Recursive}(L, R')$
$v_2 := \text{Recursive}(L \cup R', R \setminus R')$
$\text{opt} := \min\{\text{opt}, v_1 + v_2\}$
end for

$\text{Recursive}(\{\}, \{A, B, C, D\})$: $\text{opt} = \infty$

$R' = \{A, B\}$

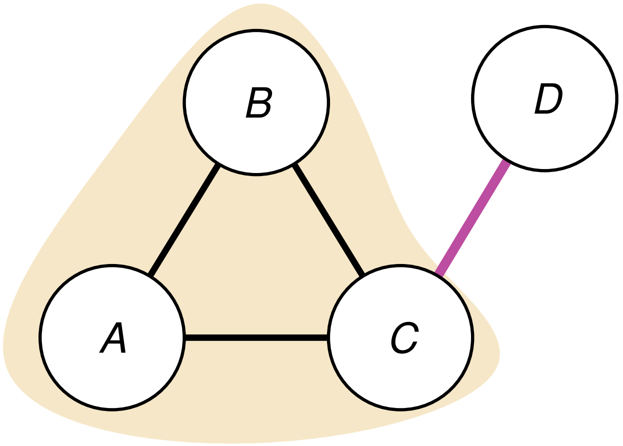
$v_1 = \text{Recursive}(\{\}, \{A, B\})$: 4

$v_2 = \text{Recursive}(\{A, B\}, \{C, D\})$: $\text{opt} = \infty$

$R' = \{C\}$ $\quad v_1 + v_2 = 1$

$v_1 = \text{Recursive}(\{A, B\}, \{C\})$: 1

$v_2 = \text{Recursive}(\{A, B, C\}, \{D\})$: 0



$$
\begin{array}{l}
\text{opt} := \infty \\
\textbf{for } \textit{each } R' \subseteq R, |R'| = \lfloor |R|/2 \rfloor \textbf{ do} \\
\quad v_1 := \text{Recursive}(L, R') \\
\quad v_2 := \text{Recursive}(L \cup R', R \setminus R') \\
\quad \text{opt} := \min\{\text{opt}, v_1 + v_2\} \\
\textbf{end for}
\end{array}
$$

$\text{Recursive}(\{\}, \{A, B, C, D\})$: opt $= \infty$

$R' = \{A, B\}$

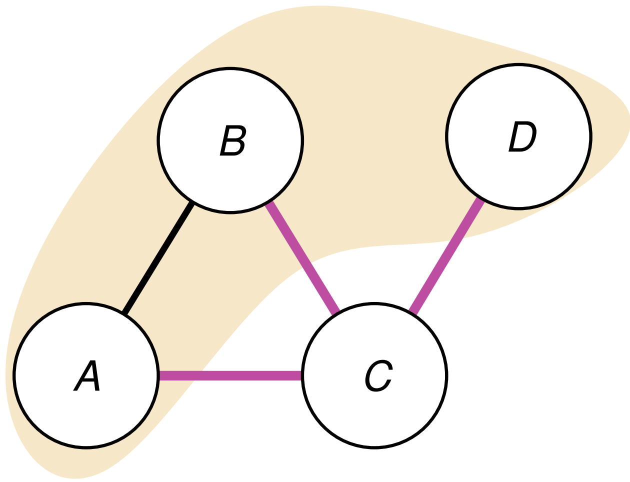
$v_1 = \text{Recursive}(\{\}, \{A, B\})$: 4

$v_2 = \text{Recursive}(\{A, B\}, \{C, D\})$: opt $= 1$

$R' = \{D\}$ $\qquad v_1 + v_2 = 3$

$v_1 = \text{Recursive}(\{A, B\}, \{D\})$: 3

$v_2 = \text{Recursive}(\{A, B, D\}, \{C\})$: 0



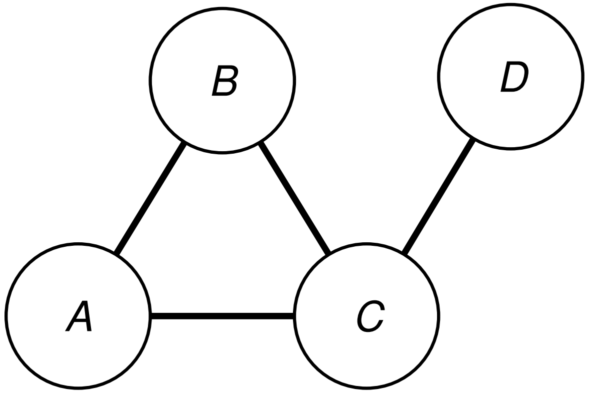
```
opt := ∞
for each R' ⊆ R, |R'| = ⌊|R|/2⌋ do
    v_1 := Recursive(L, R')
    v_2 := Recursive(L ∪ R', R\R')
    opt := min{opt, v_1 + v_2}
end for
```

Recursive($\{\}, \{A, B, C, D\}$): opt $= \infty$

$R' = \{A, B\}$ $\qquad v_1 + v_2 = 5$

$v_1 =$ Recursive($\{\}, \{A, B\}$): 4

$v_2 =$ Recursive($\{A, B\}, \{C, D\}$): 1



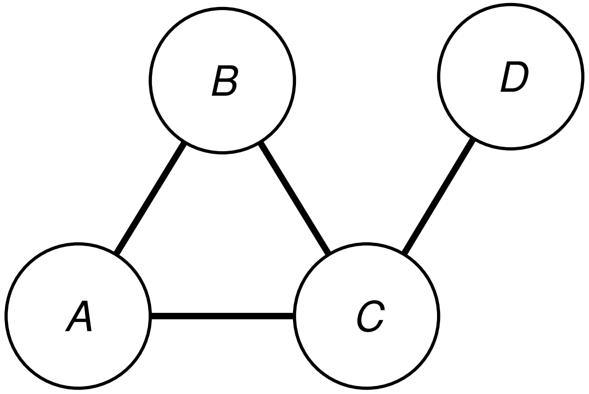
$$\text{Recursive}(\{\}, \{A, B, C, D\})\text{: opt} = 5$$

$R' = \{A, B\}$ $\qquad v_1 + v_2 = 5$

$v_1 = \text{Recursive}(\{\}, \{A, B\})\text{: } 4$

$v_2 = \text{Recursive}(\{A, B\}, \{C, D\})\text{: } 1$

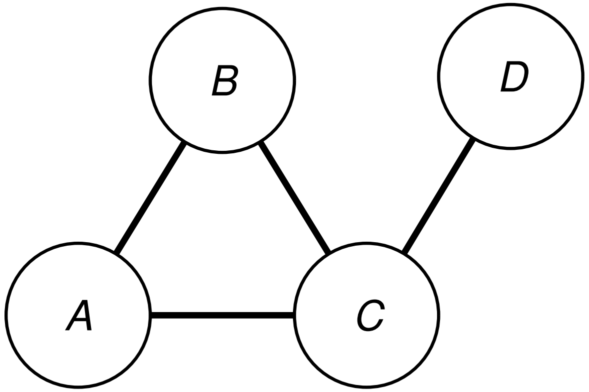
opt $:= \infty$
for *each* $R' \subseteq R, |R'| = \lfloor |R|/2 \rfloor$ **do**
    $v_1 := \text{Recursive}(L, R')$
    $v_2 := \text{Recursive}(L \cup R', R \setminus R')$
    opt $:= \min\{\text{opt}, v_1 + v_2\}$
end for



$\text{opt} := \infty$
for *each* $R' \subseteq R, |R'| = \lfloor |R|/2 \rfloor$ **do**
$v_1 := \text{Recursive}(L, R')$
$v_2 := \text{Recursive}(L \cup R', R \setminus R')$
$\text{opt} := \min\{\text{opt}, v_1 + v_2\}$
end for

$\text{Recursive}(\{\}, \{A, B, C, D\})$: $\text{opt} = 5$

$R' = \{A, C\}$



Abschließendes

Karlsruher Institut für Technologie

- 4^n wächst schnell
- verschiedene interessante Probleme können als Vertex Ordering Probleme formuliert werden:
  TREEWIDTH, MINIMUM FILL-IN, PATHWIDTH, SUM CUT, MINIMUM INTERVAL GRAPH COMPLETION, CUTWIDTH, OPTIMAL LINEAR ARRANGEMENT, DIRECTED FEEDBACK ARC SET

Danke fürs Zuhören