Karlsruher Institut für Technologie

Algorithmic Graph Theory

Proofs

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For every graph G we have

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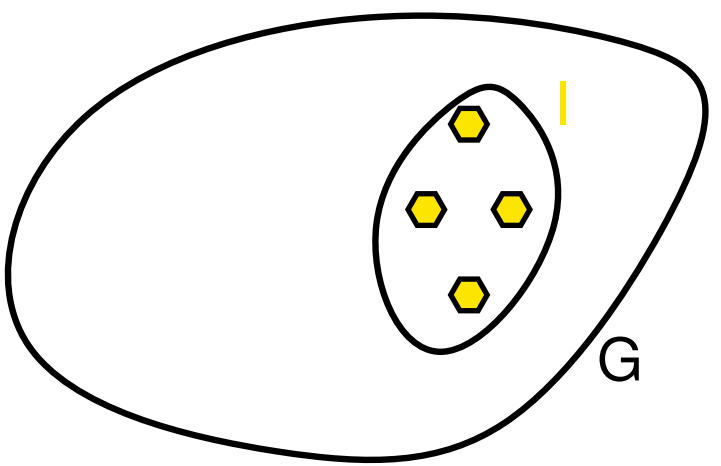
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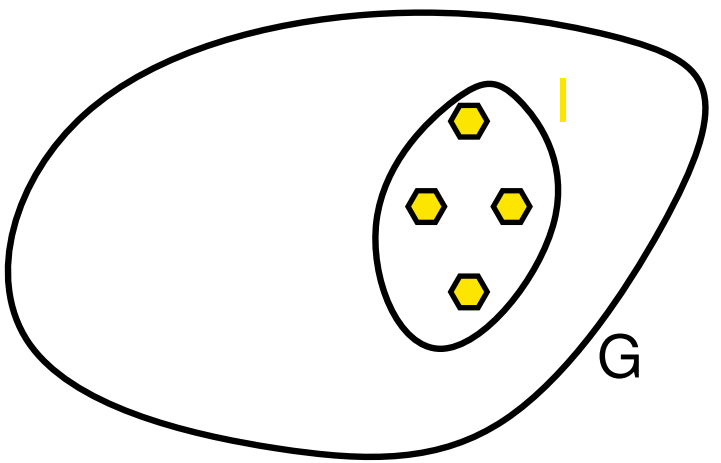
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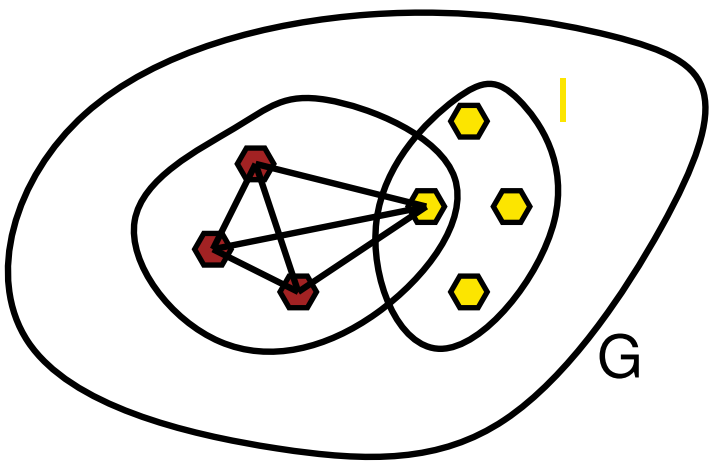
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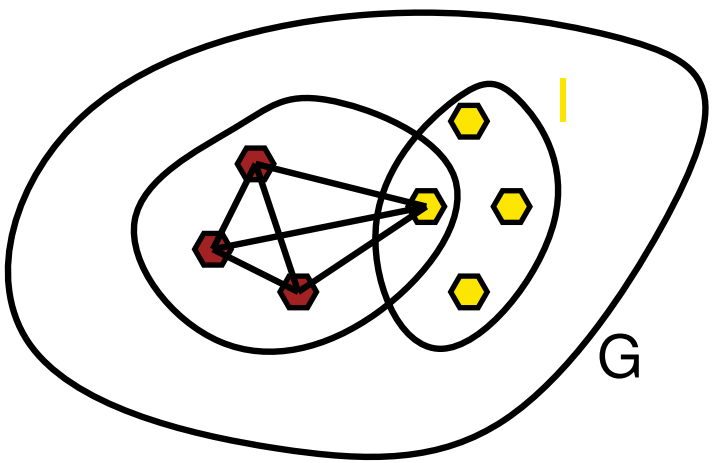
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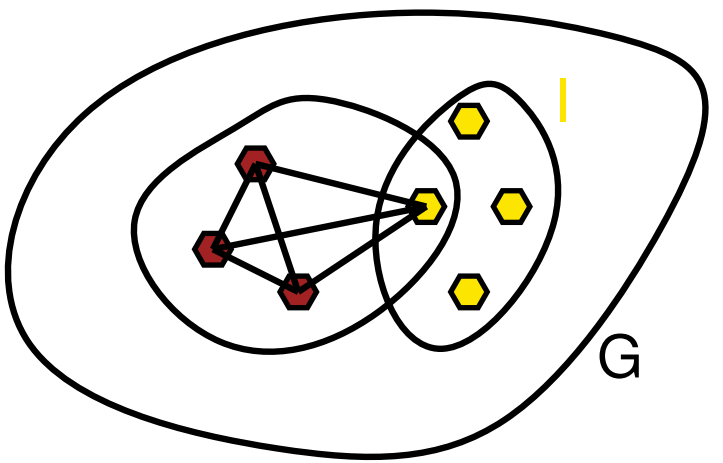
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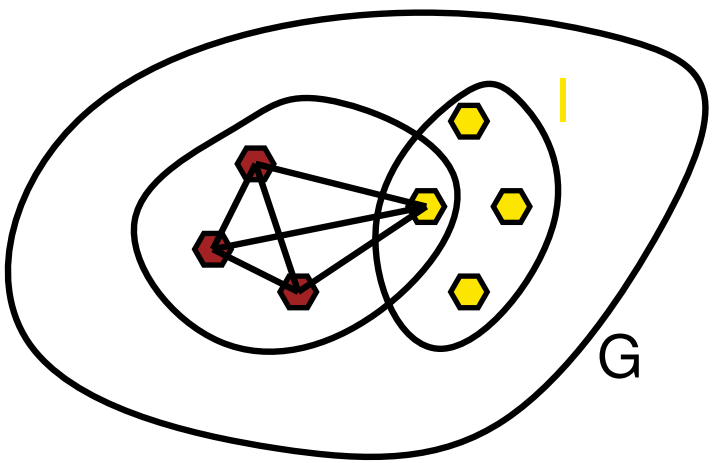
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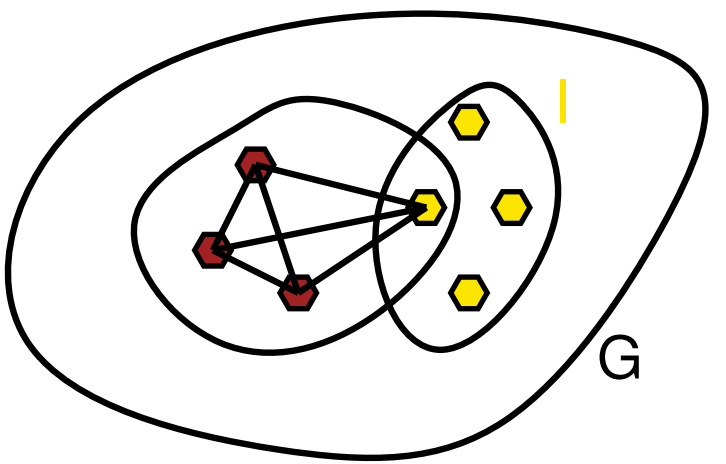
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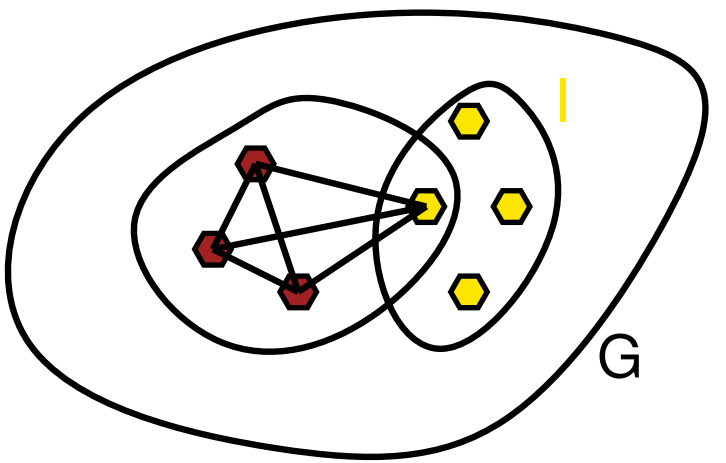
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$|I| = \alpha(G)$ $t \geq |I|$ $\kappa(G) \geq \alpha(G)$



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Weak Perfect Graph Theorem (WPGT)

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Weak Perfect Graph Theorem (WPGT) Equivalent

For every graph G it holds:

$$G \text{ has } (P1) \iff \overline{G} \text{ has } (P1)$$

WPGT: Lemma 2.6

Lemma 2.6

For G and $H = G \circ h$ we have:

- $(P1)$ for $G \implies (P1)$ for H
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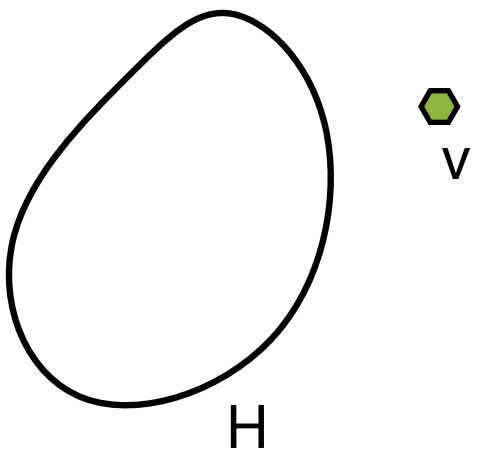
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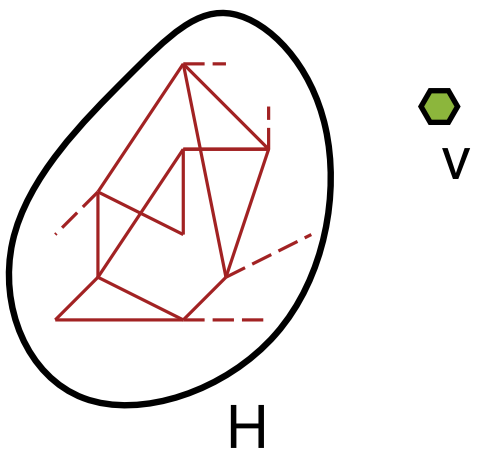
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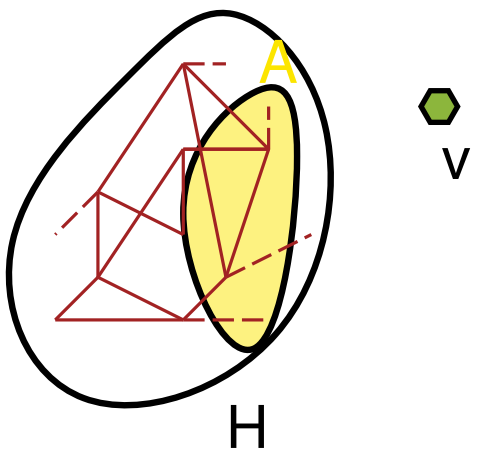
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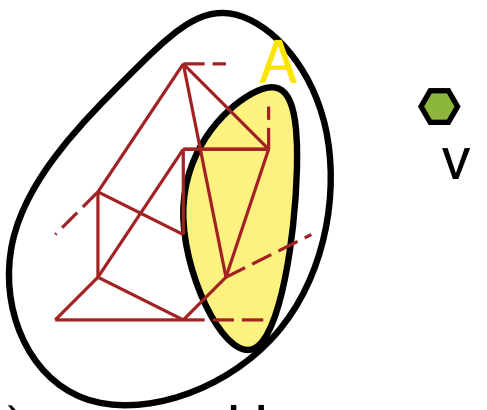
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Since $(P1)$ holds for G, we know that $\chi(H_A) = \chi(G_A) = \omega(G_A) = \omega(H_A)$



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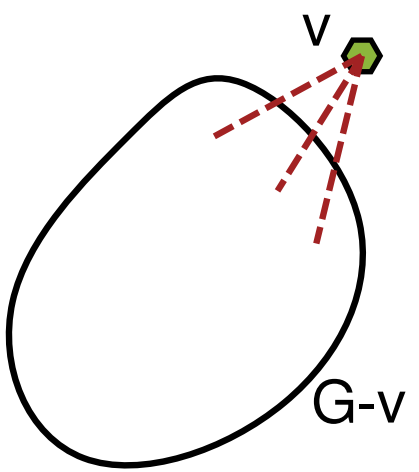
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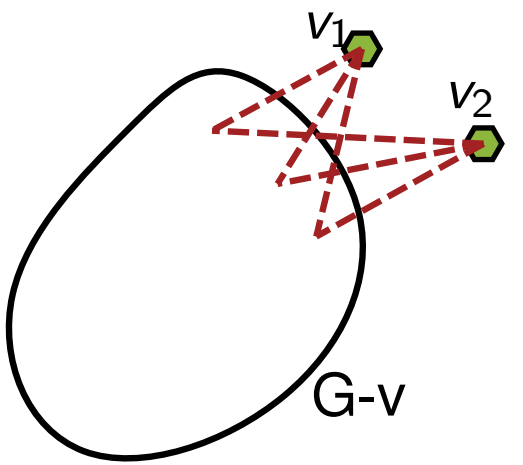
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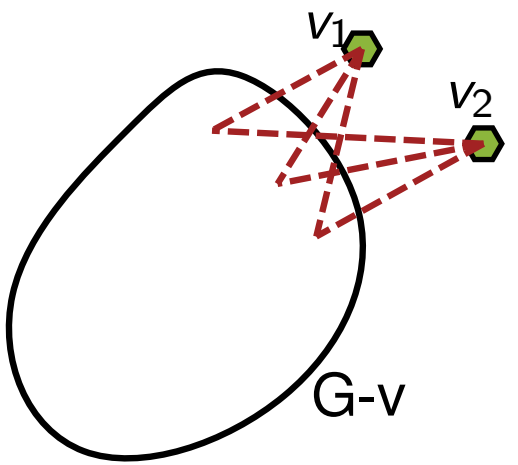
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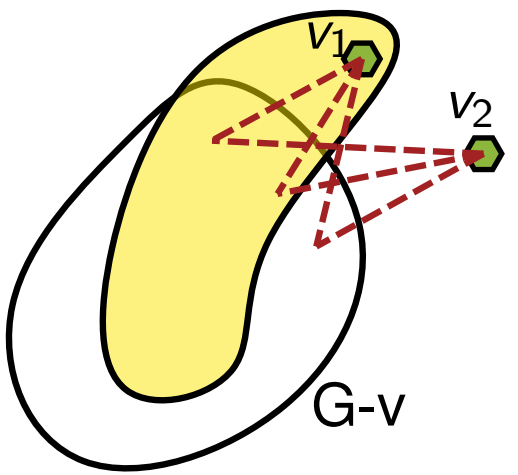
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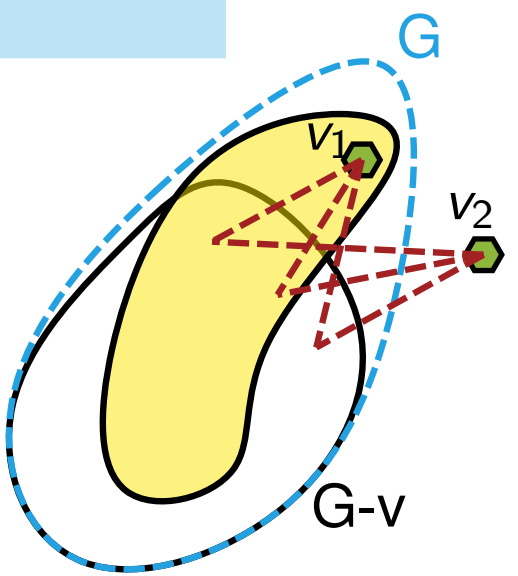
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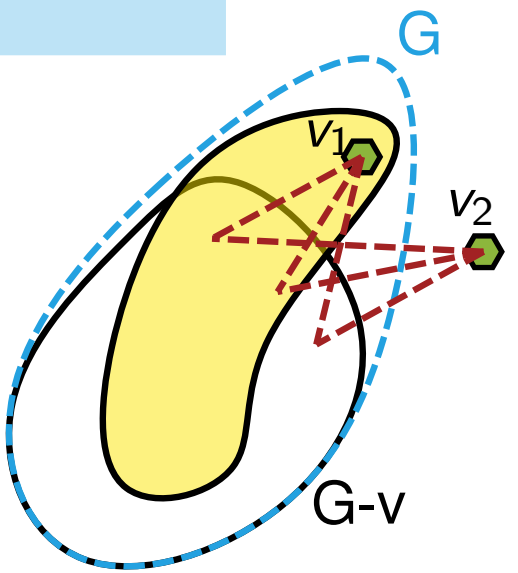
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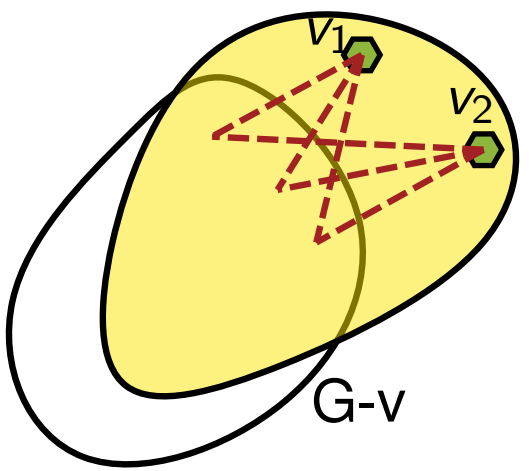
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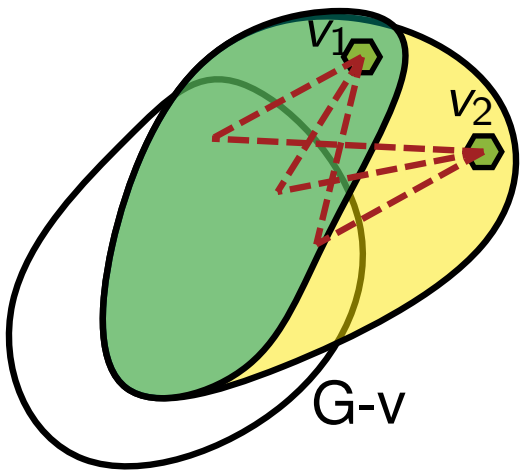
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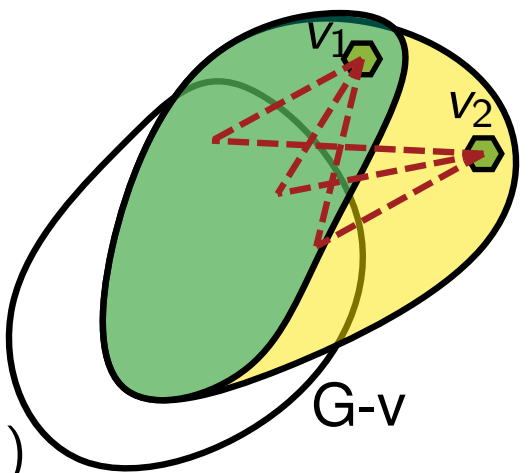
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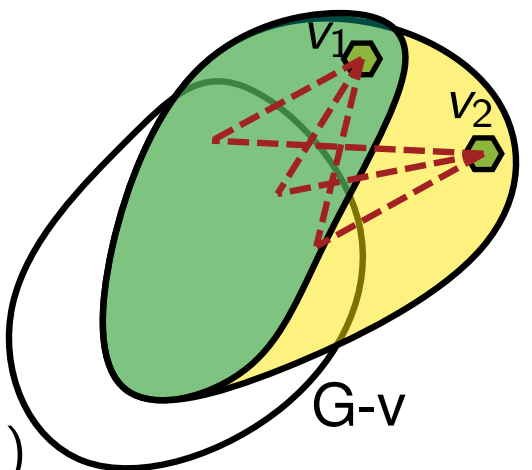
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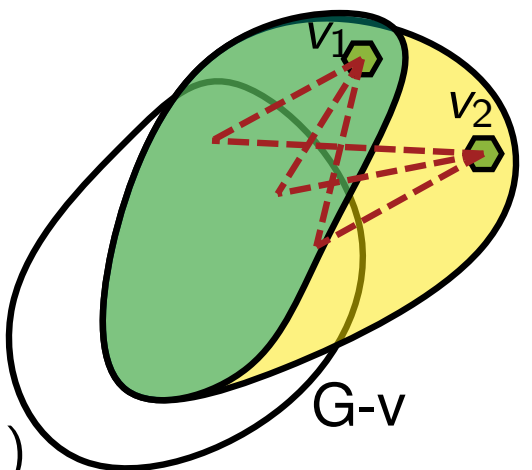
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by (P1) and Observation 1: $\chi(H_A) \leq \chi(G_{A'}) \overset{(P1)}{=} \omega(G_{A'}) \leq \omega(H_A) \overset{\text{Obs.1}}{\leq} \chi(H_A)$



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wlog: $H = G \circ v$ or $H = G - v$ (see problem class 2 ex. 2)

$H = G - v$ works by the same argument as before

WPGT: Lemma 2.6

Lemma 2.6

For G and $H = G \circ h$ we have:

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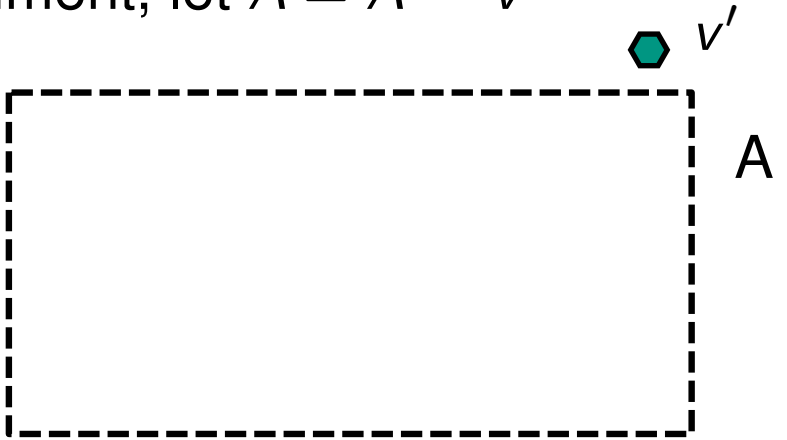
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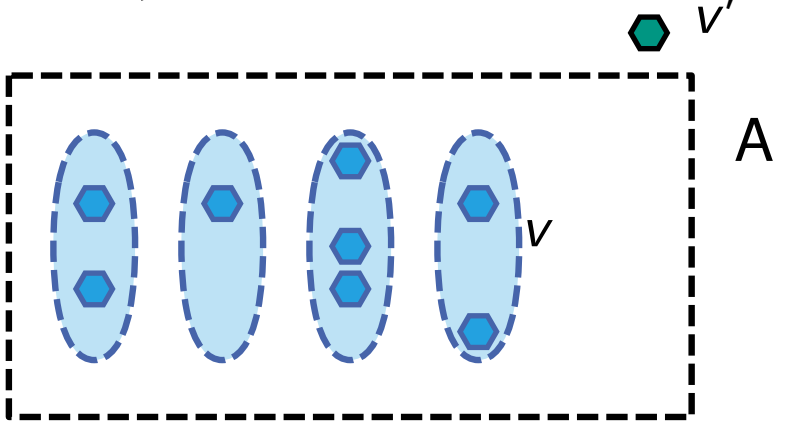
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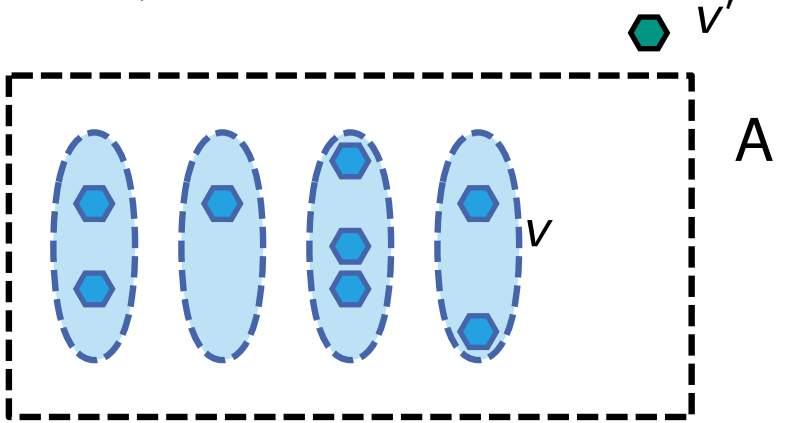
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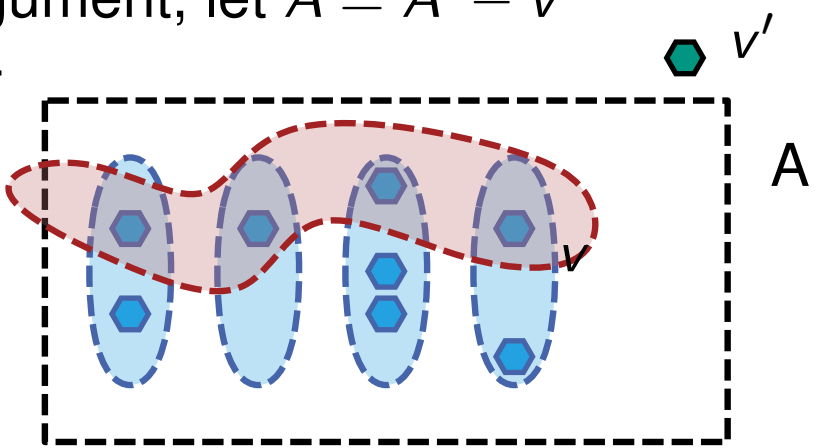
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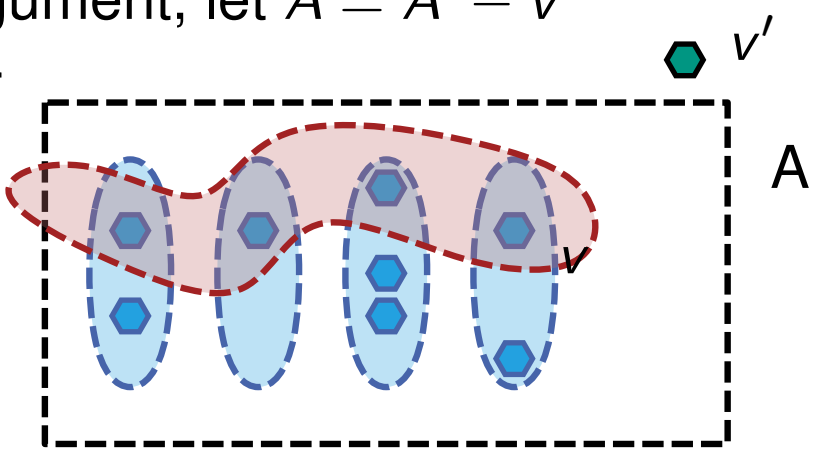
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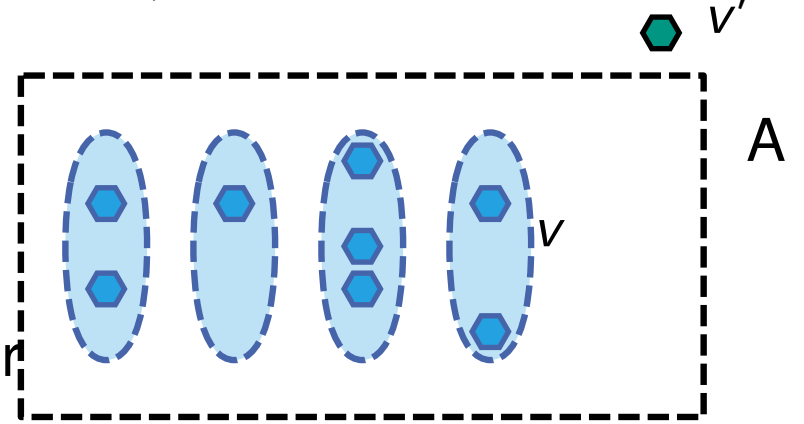
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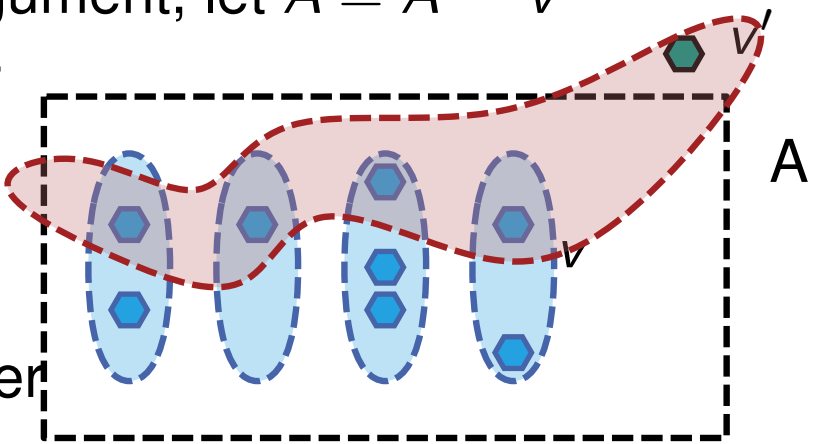
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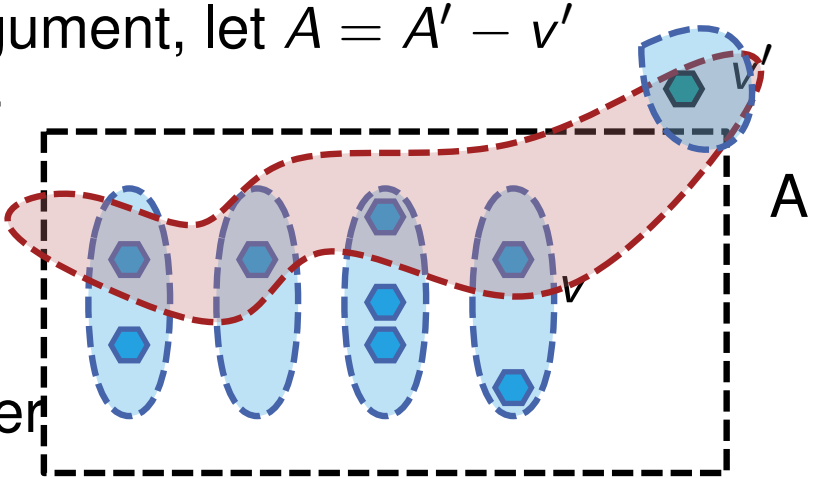
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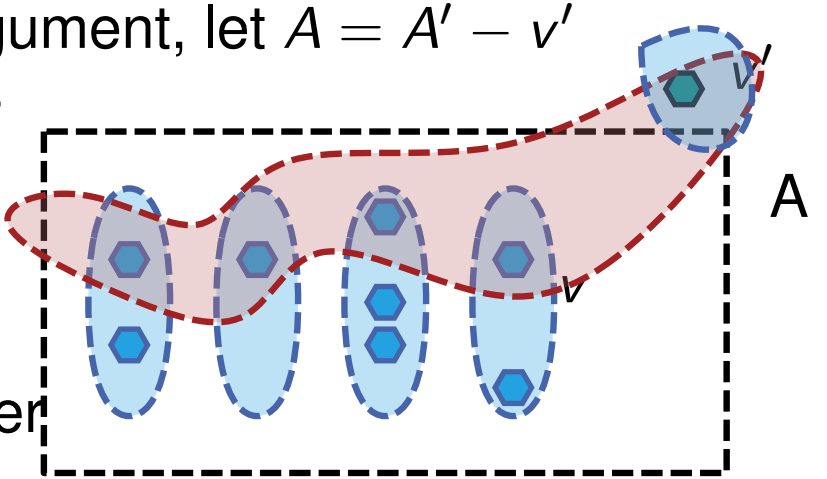
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$$\kappa(H_{A'}) \leq t + 1 \leq \alpha(H_{A'}) \leq \kappa(H_{A'})$$

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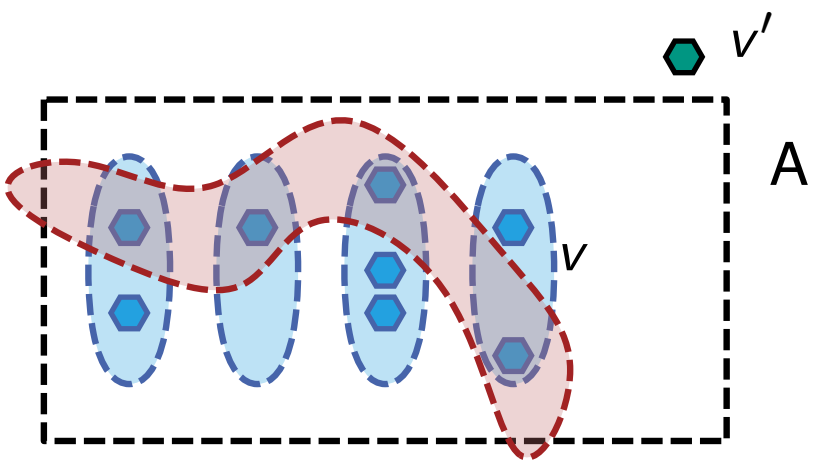
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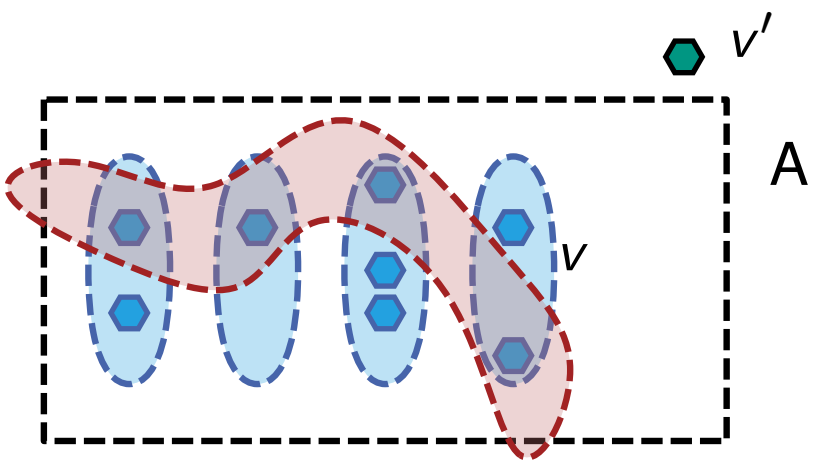
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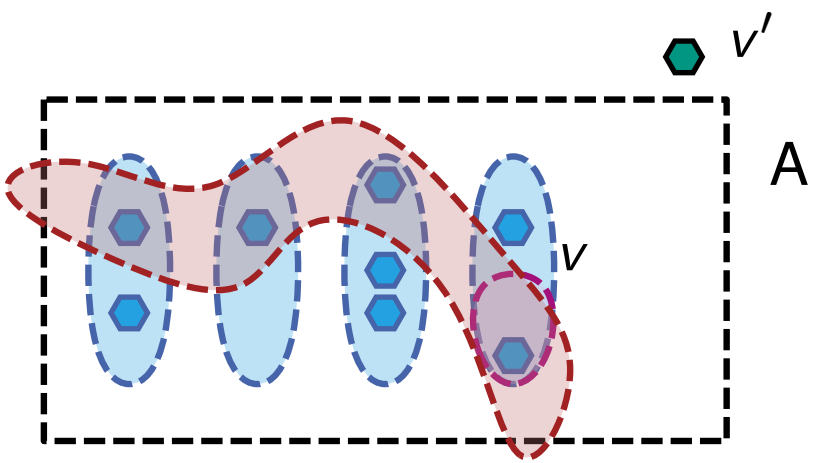
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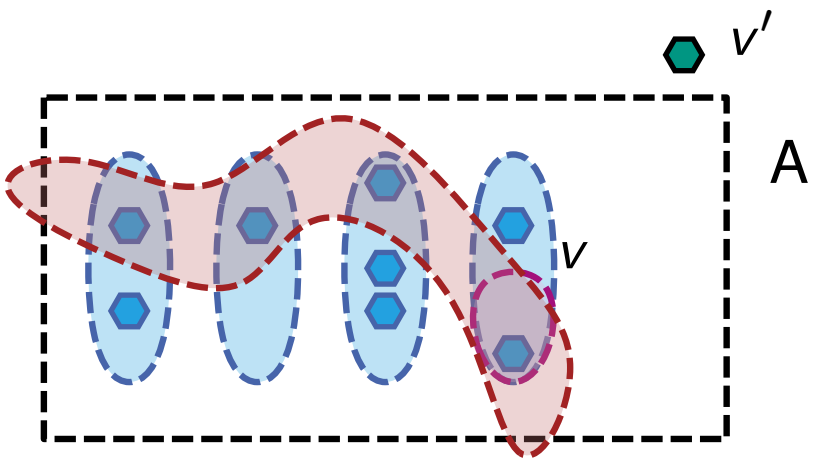
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Since (P2) for G $\exists$ clique-cover $V(G_{A-C})$ $V(H_{A-C}) = W_1 + \ldots + W_s$ mit $s \leq t - 1$



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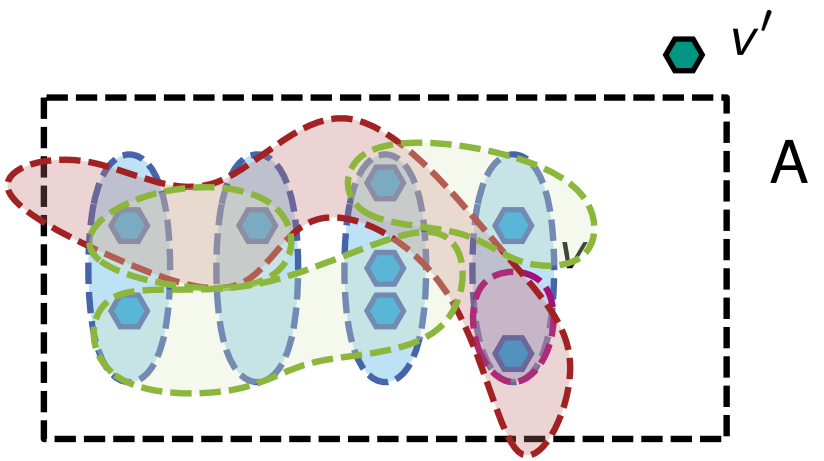
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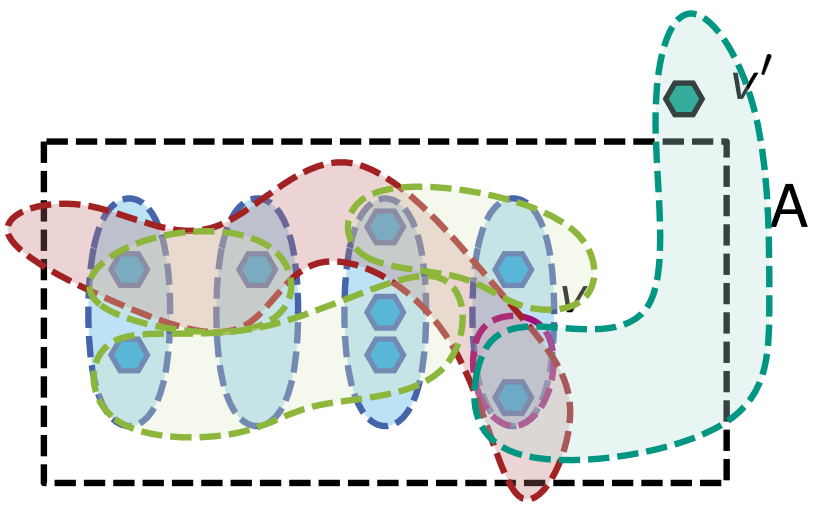
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Since (P2) for G $\exists$ clique-cover $V(G_{A-C}) = V(H_{A-C}) = W_1 + ... + W_s$ mit $s \leq t - 1$

$W_1 + ... + W_s + (C + v')$ is a clique cover of $A \implies \kappa(H_{A'}) \leq t \leq \alpha(H_{A'}) \leq k(H_{A'})$



WPGT: Lemma 2.7

Lemma 2.7

For G and $H = G \circ h$ we have: If G has (P3) and (P2) for all proper induced subgraphs of G then (P3) holds for H.

Proof:

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(ex class 2) $A = G \circ h''$. the "wlog" means that we would have considered $H := A$ before considering the original H.

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Sketch:

- consider $\alpha(H - S)$
- use (P2) to find clique cover
- consider only maximal cliques and s_1
- find i-set (using P3) that hits s_1

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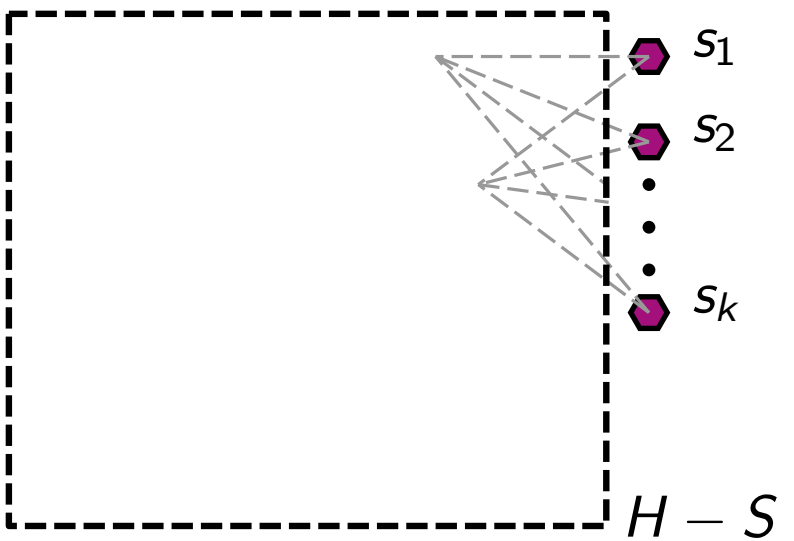
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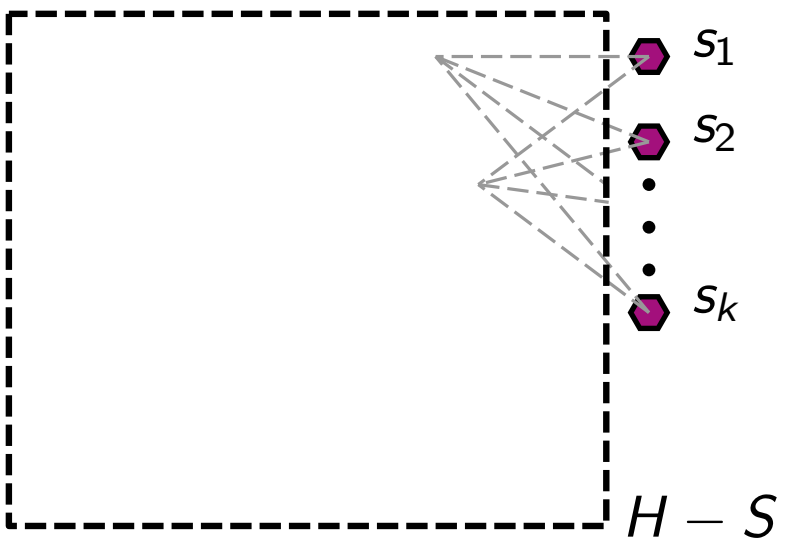
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$H - s_k$ has (P3)



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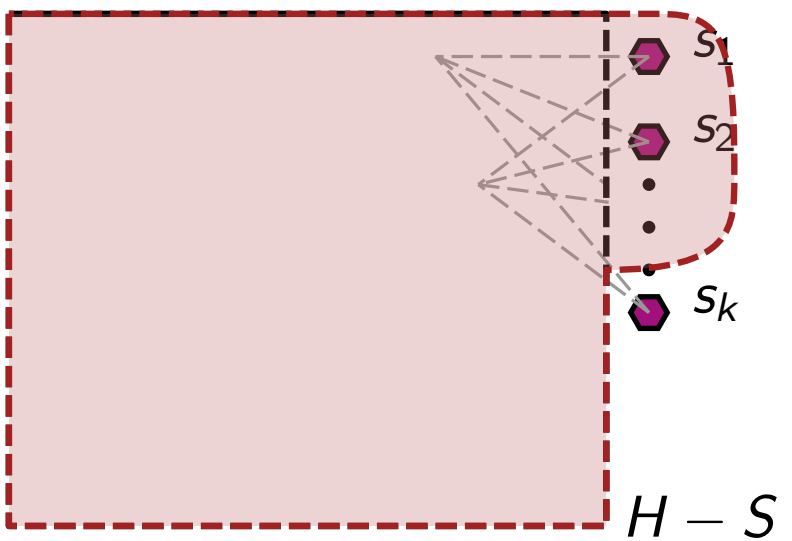
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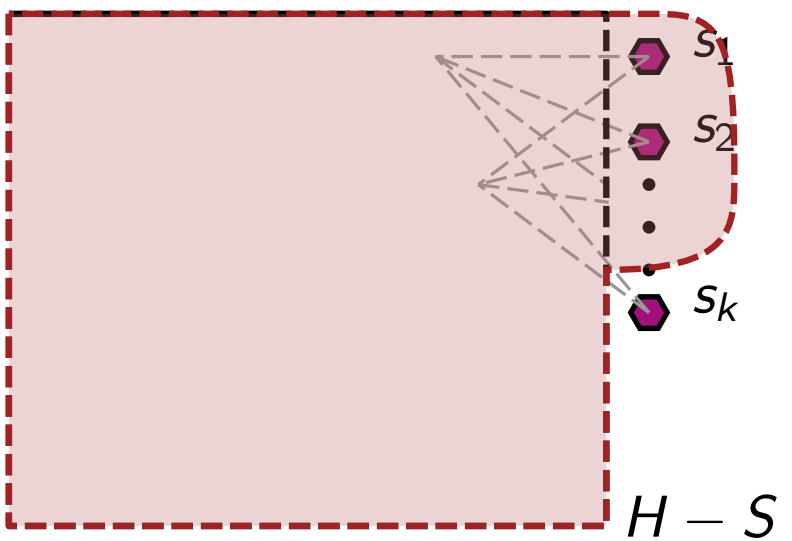
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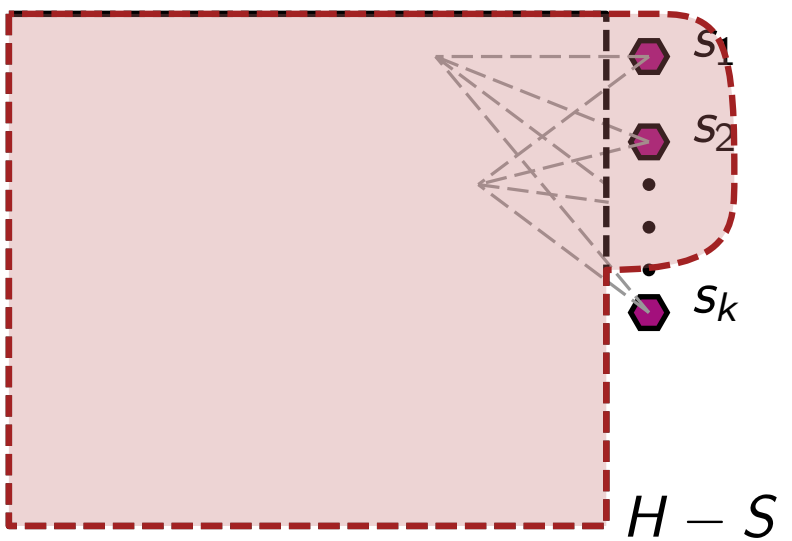
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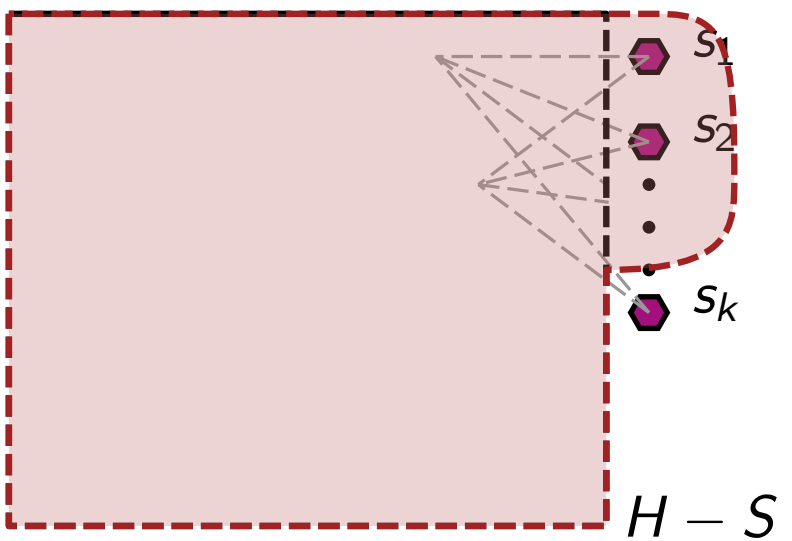
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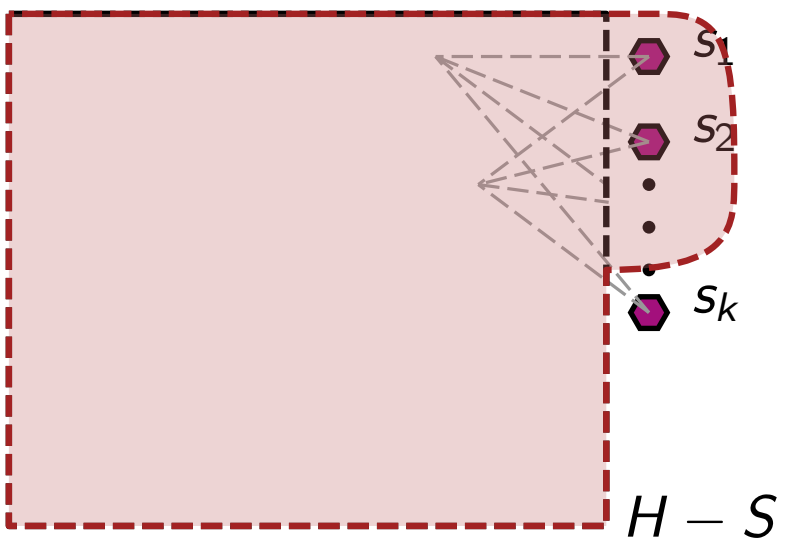
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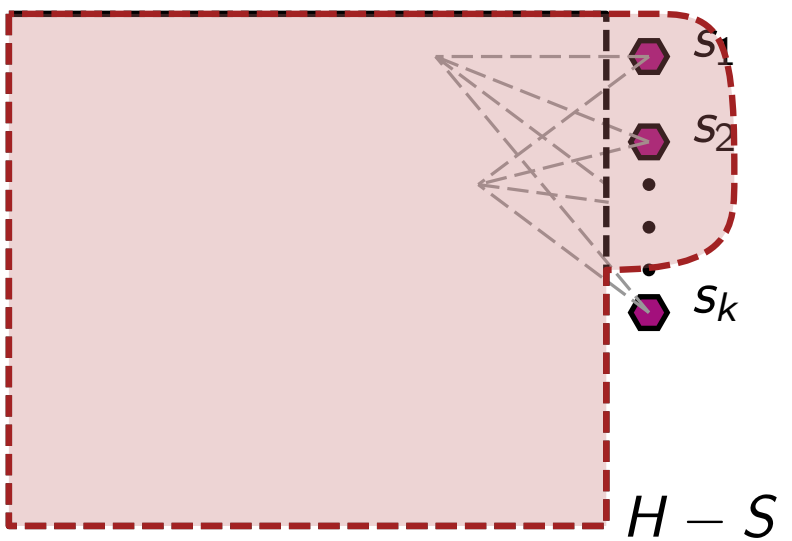
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$\alpha(H - s_k) = \alpha(H)$ and $\omega(H - s_k) = \omega(H)$ since both cant be at most one less and those cases the product cannot be $|V_H| - 1$



WPGT: Lemma 2.7

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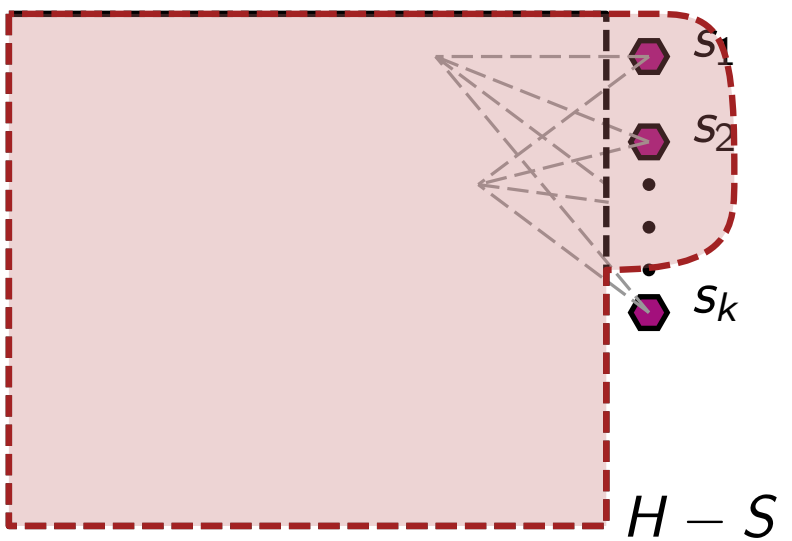
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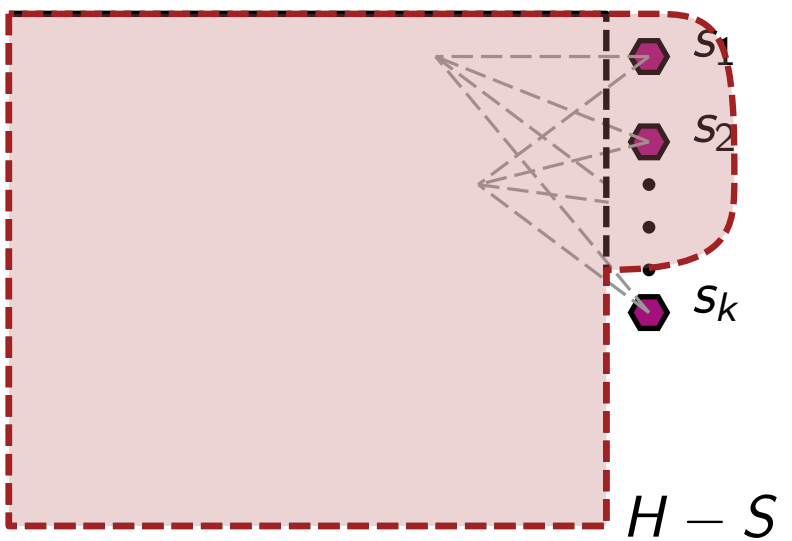
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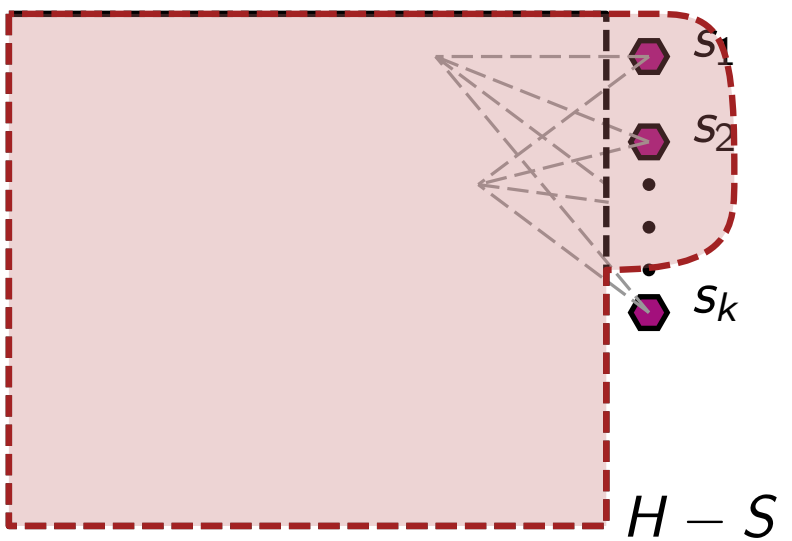
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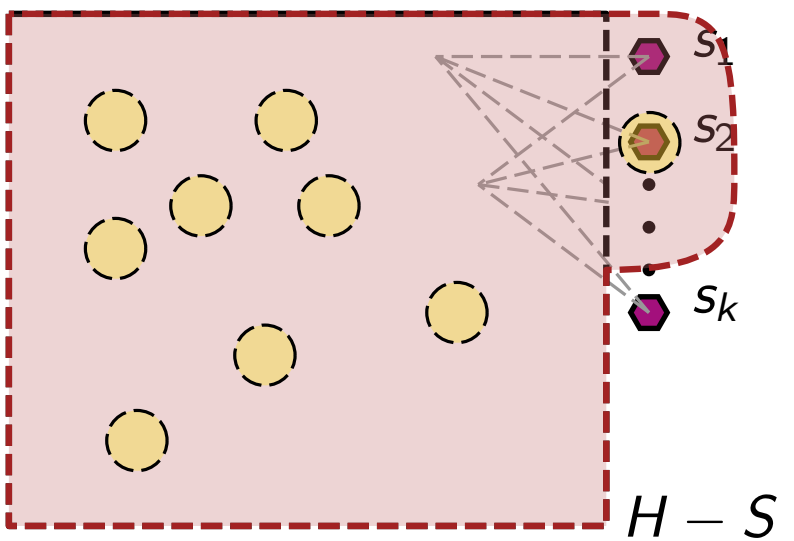
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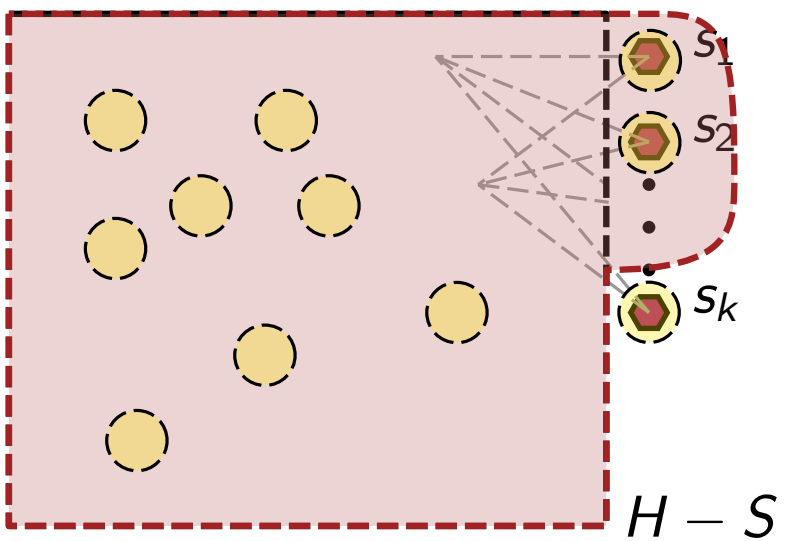
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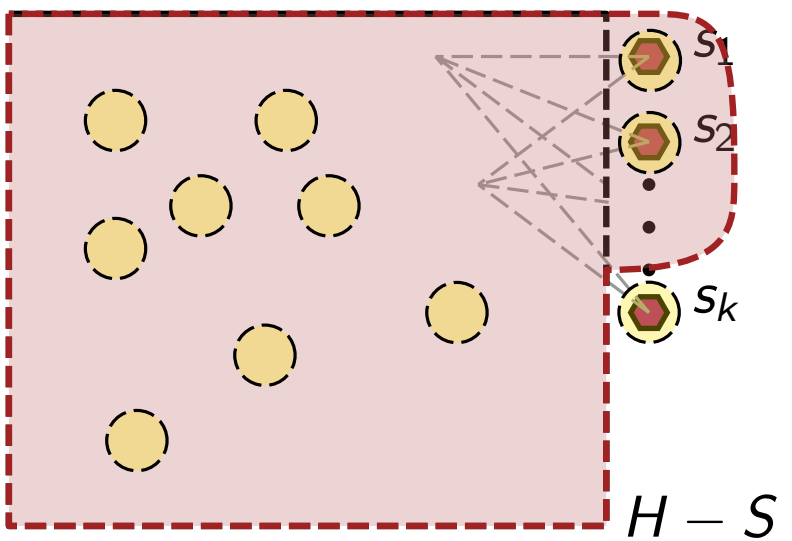
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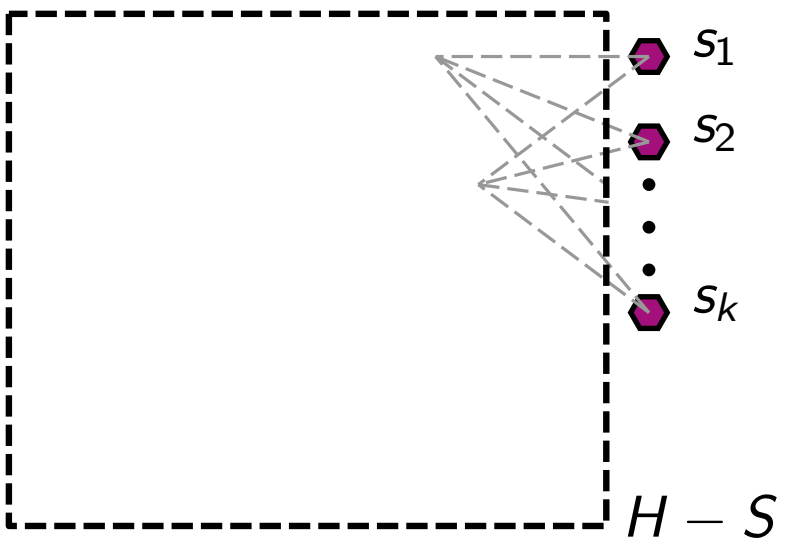
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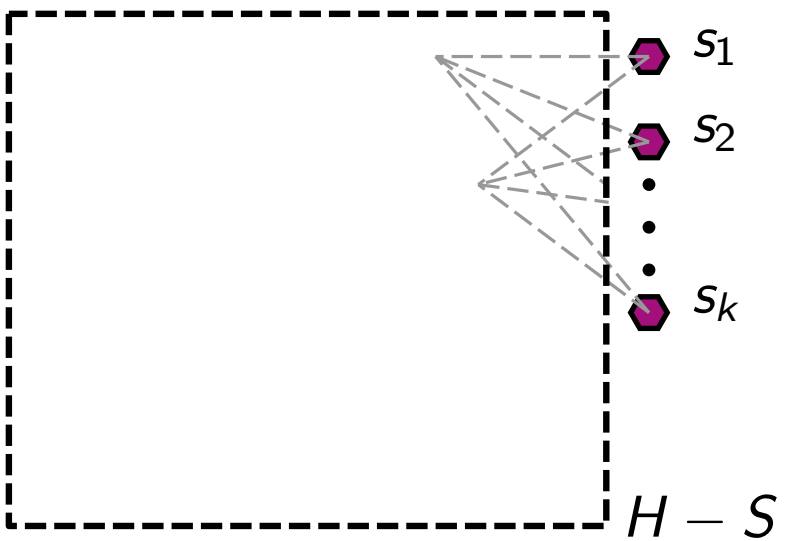
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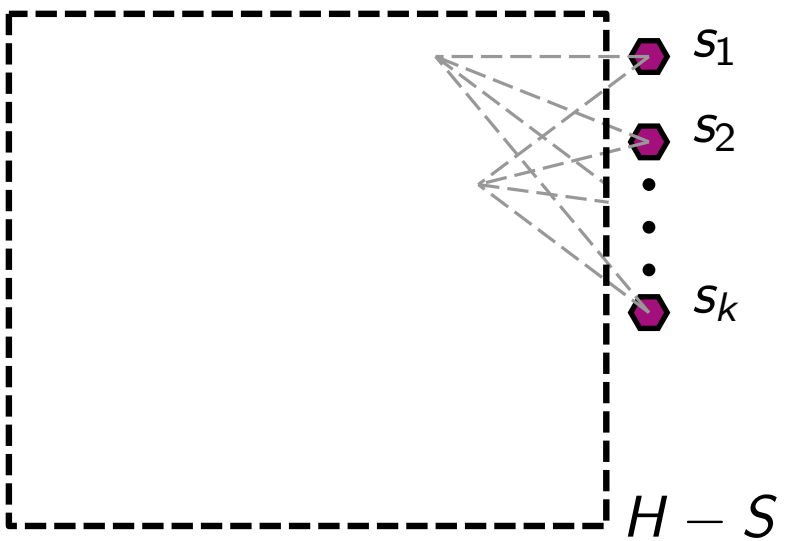
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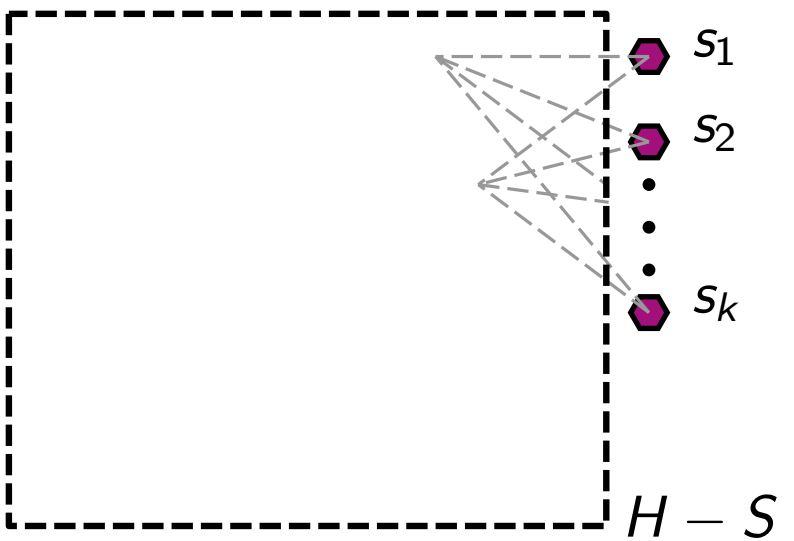
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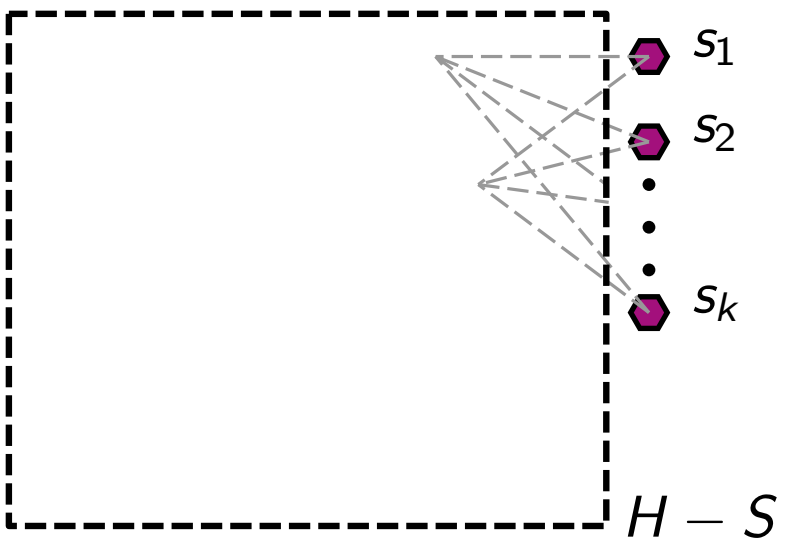
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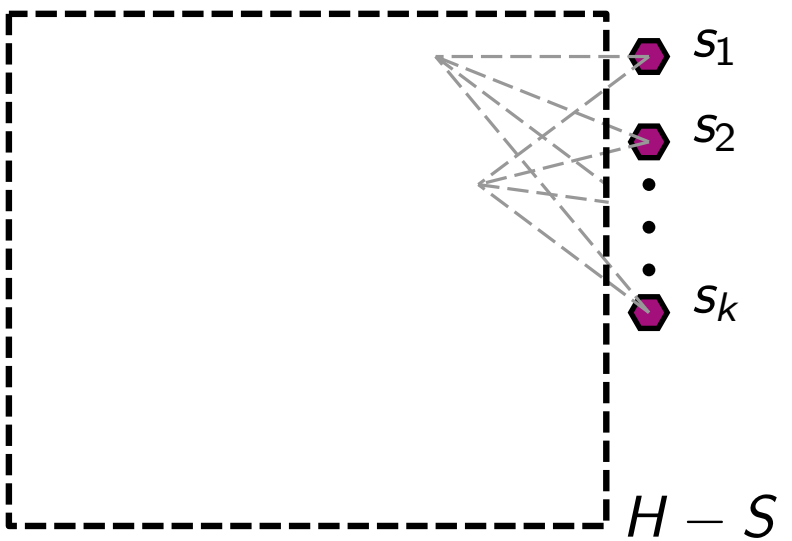
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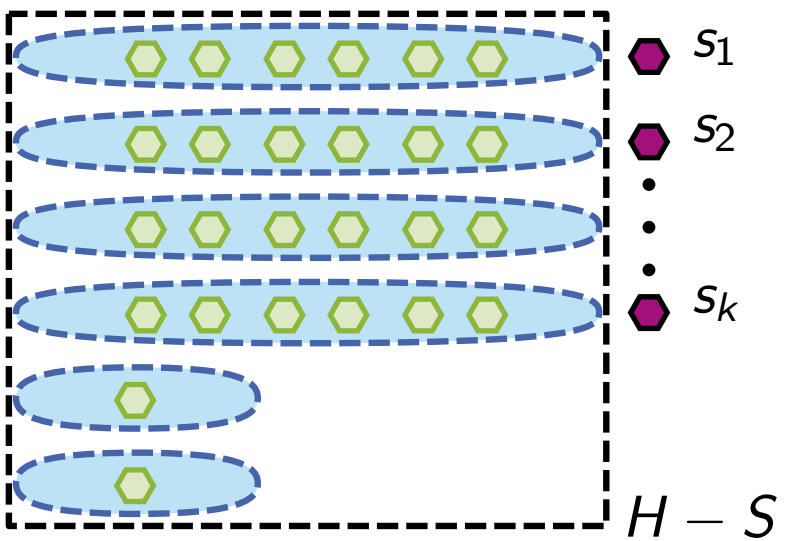
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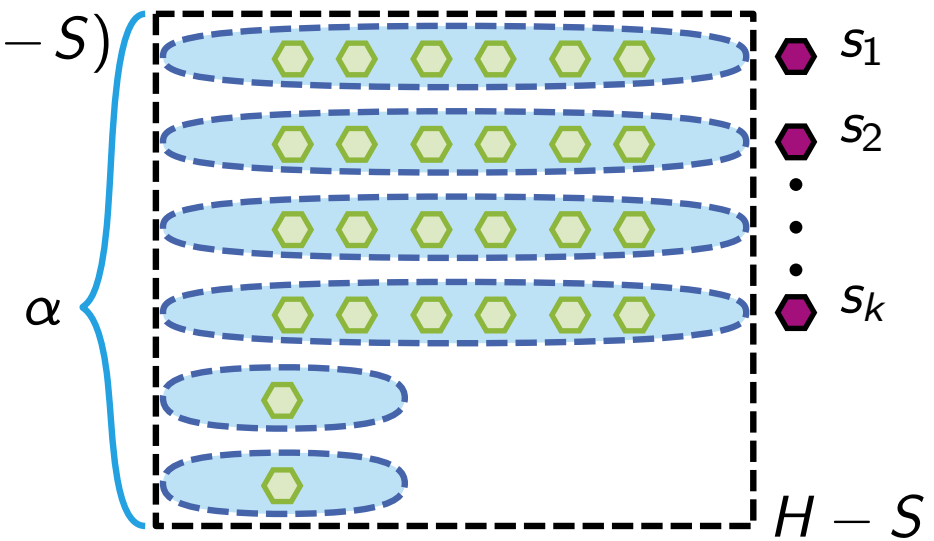
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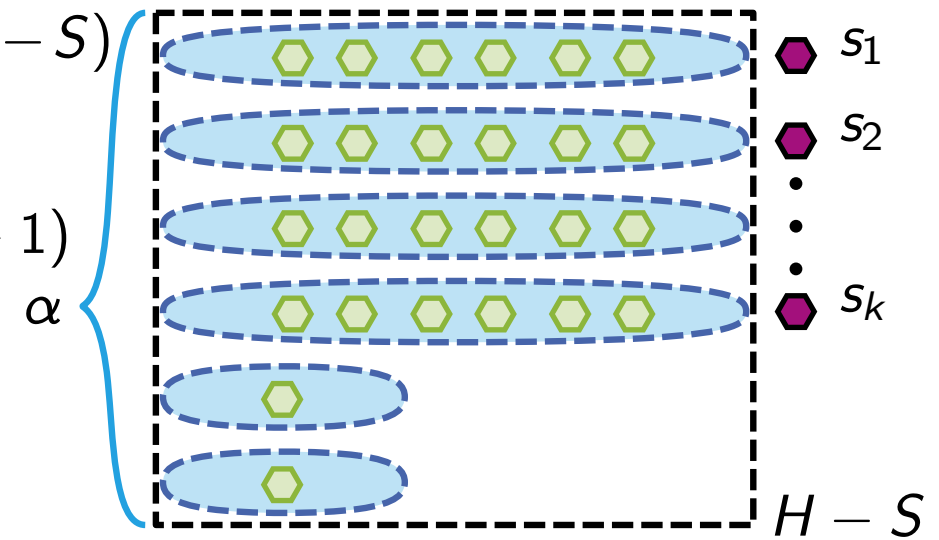
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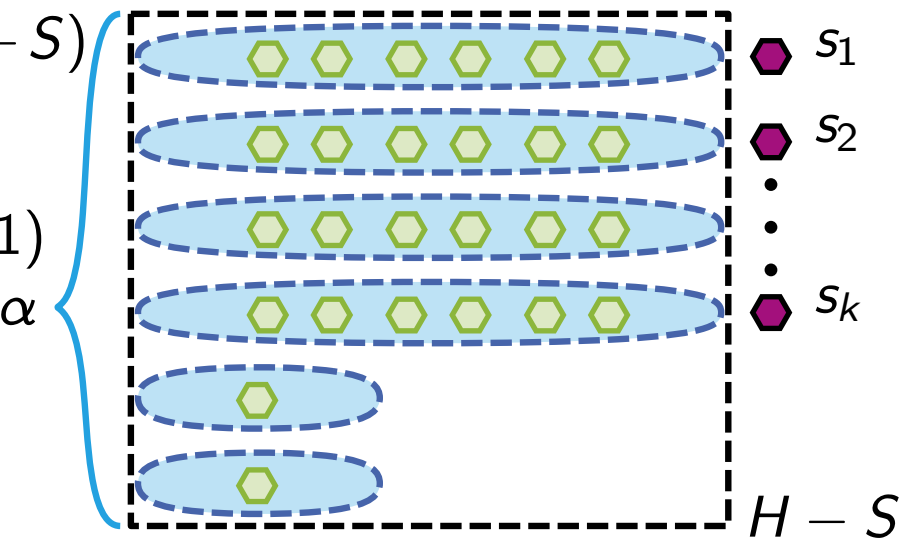
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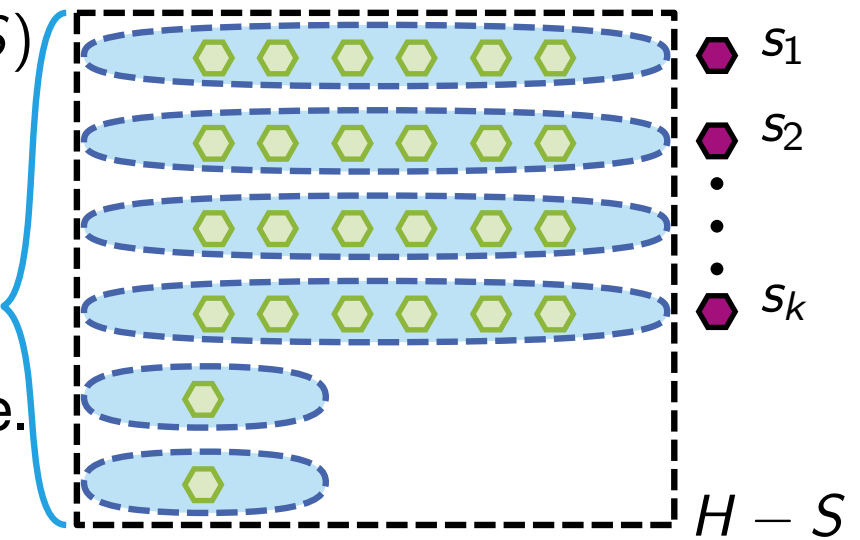
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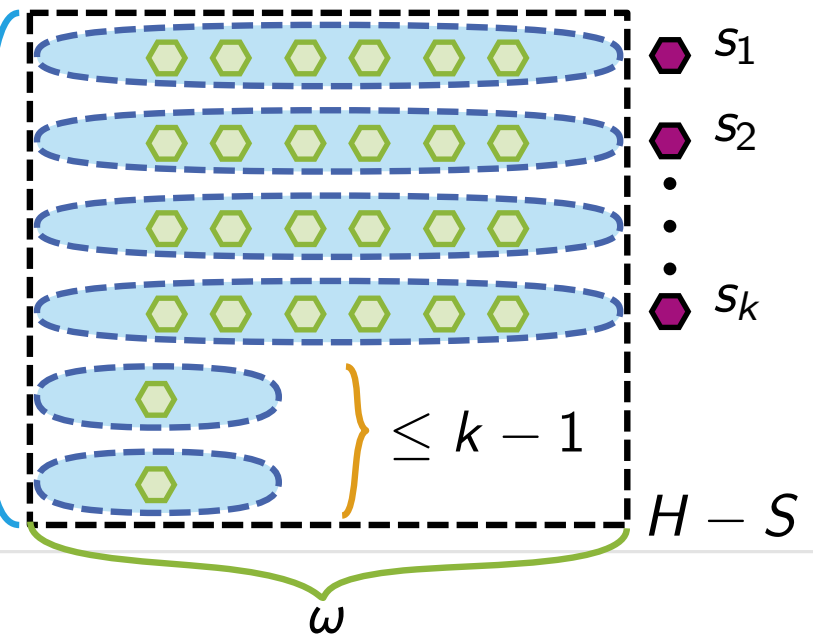
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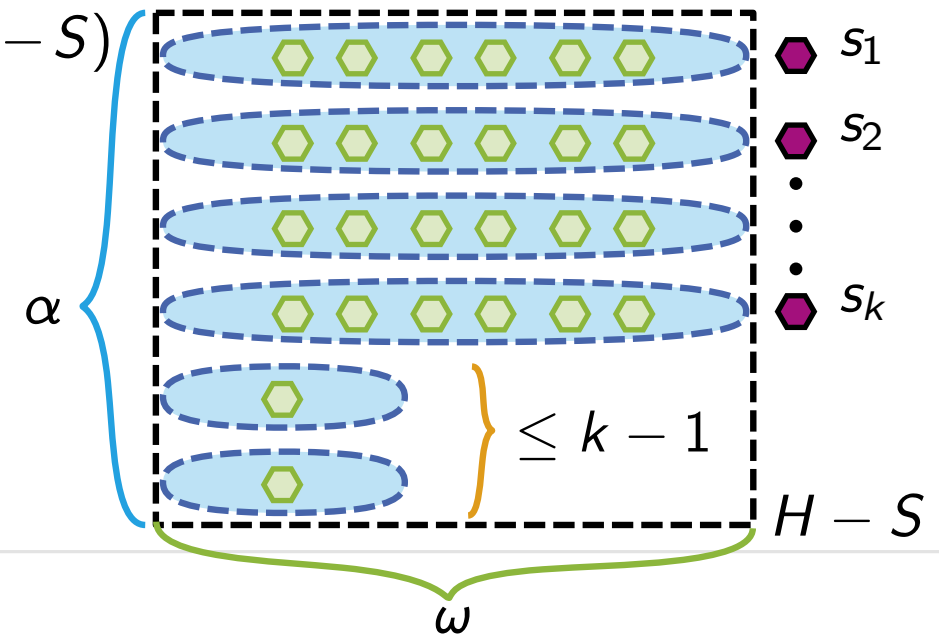
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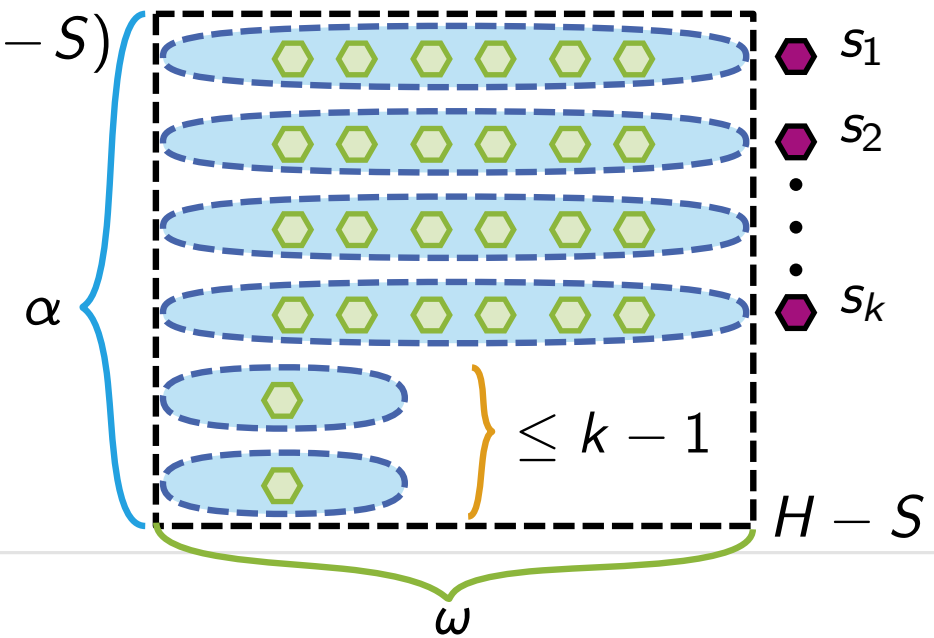
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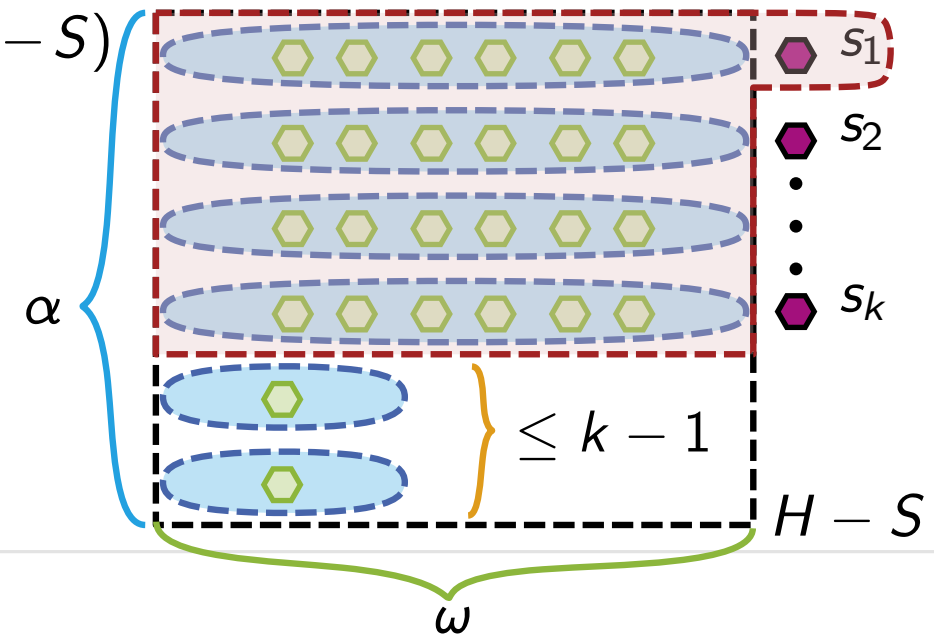
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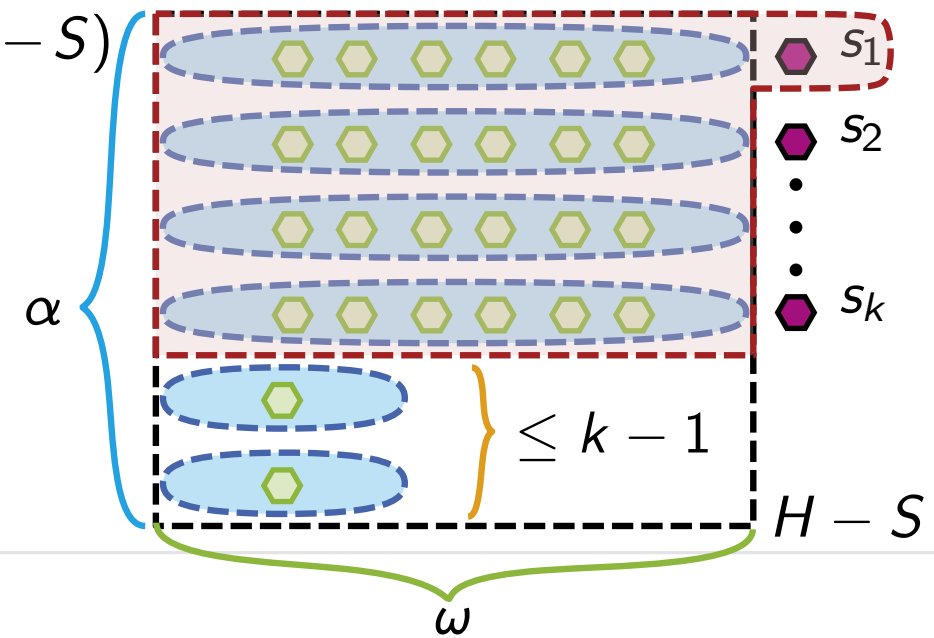
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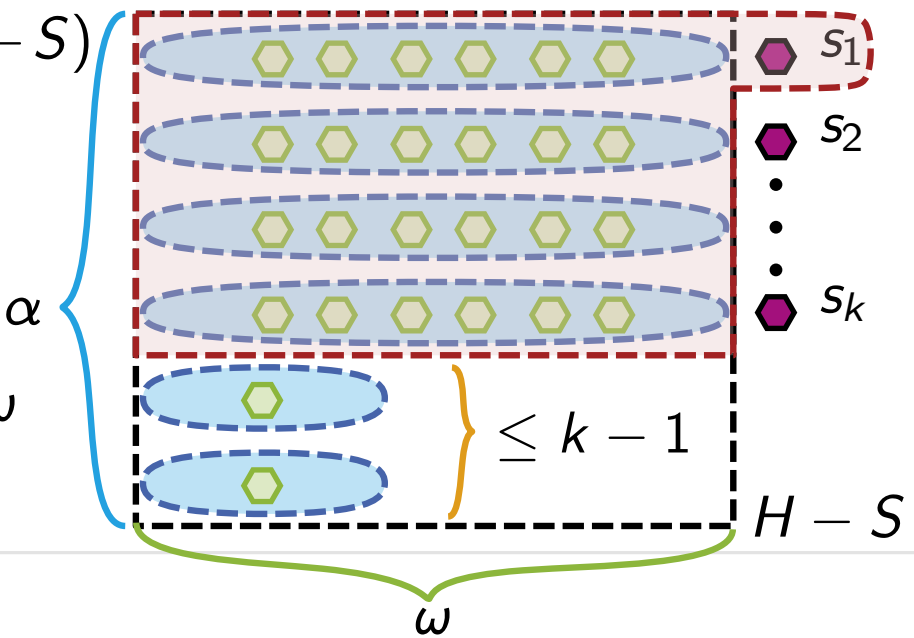
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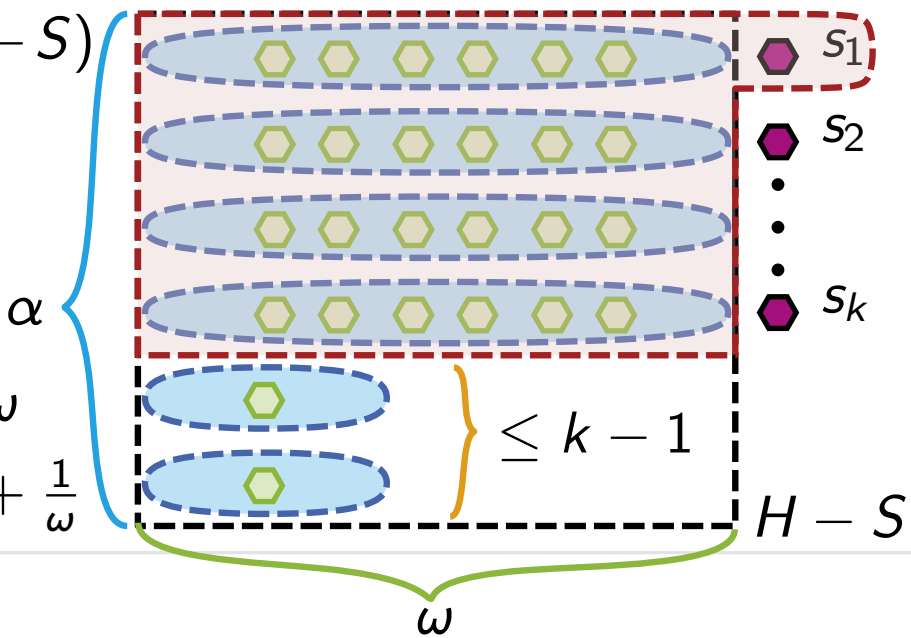
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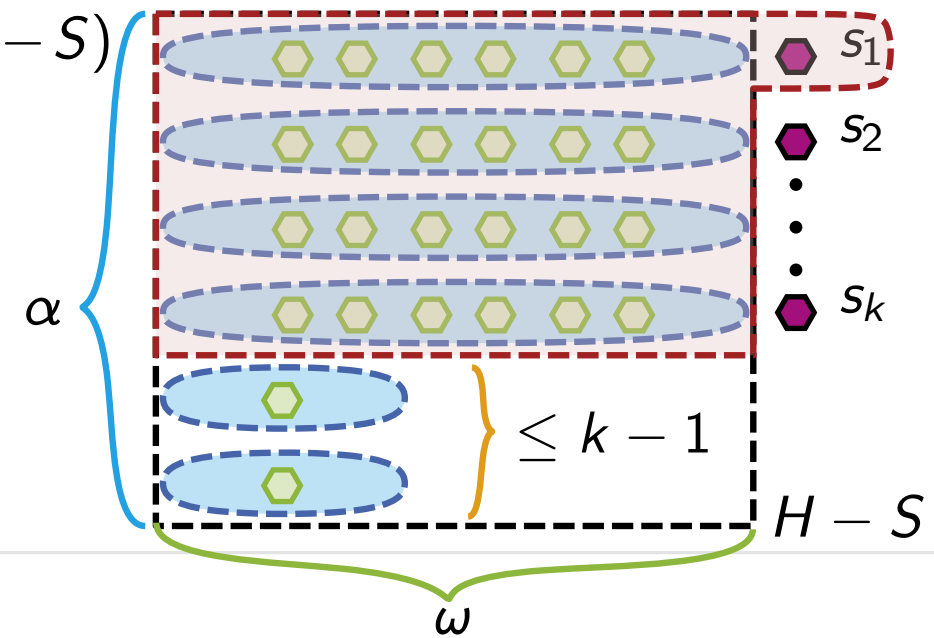
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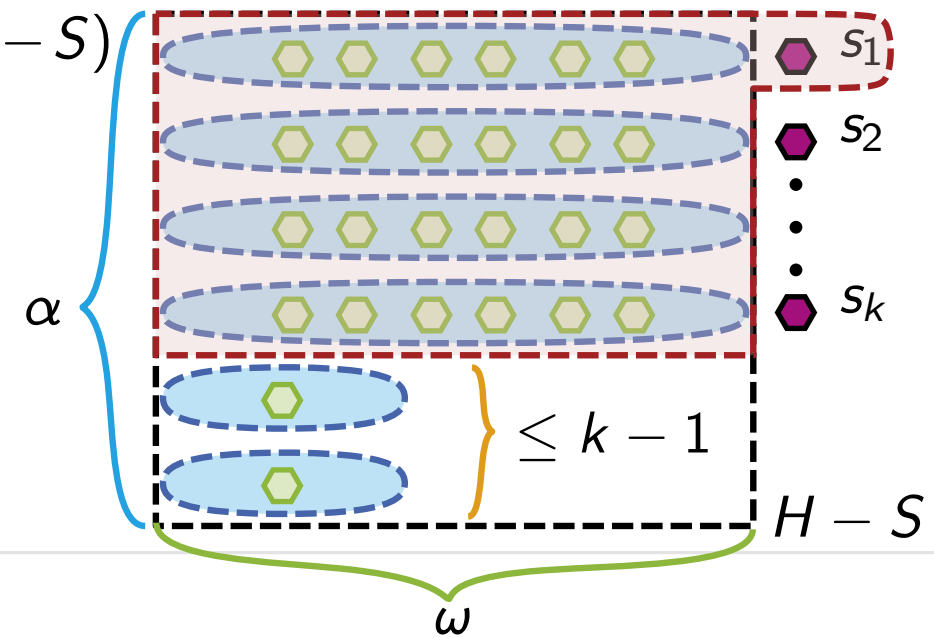
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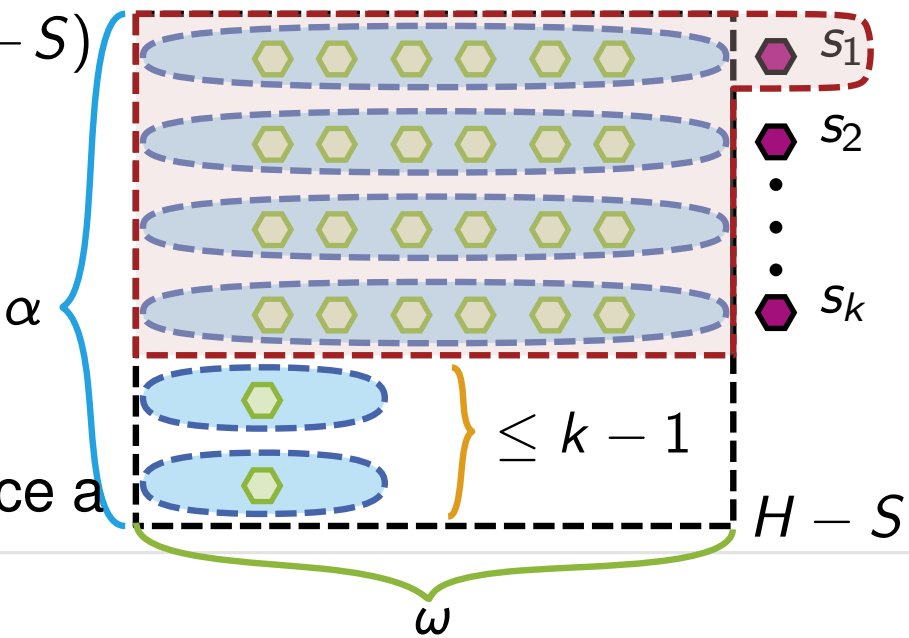
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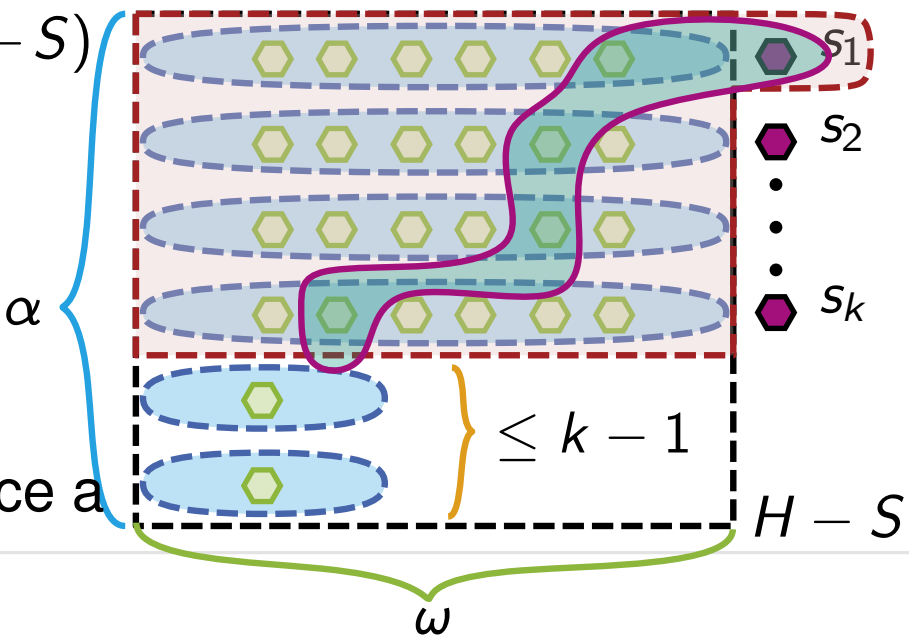
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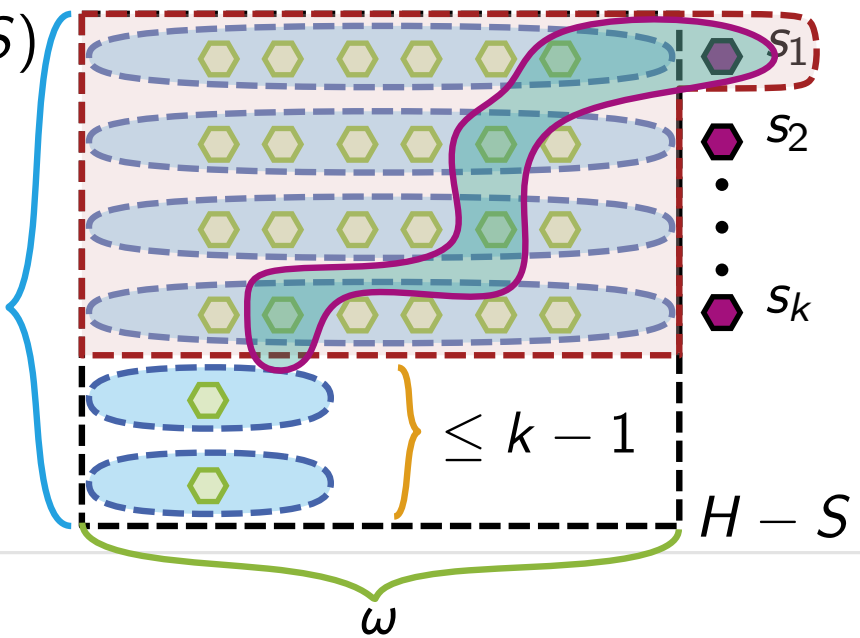
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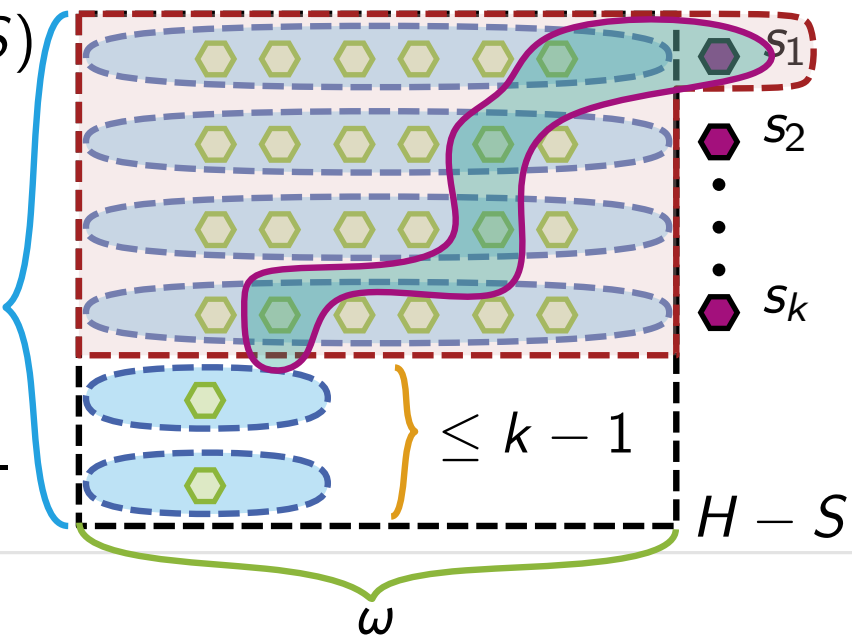
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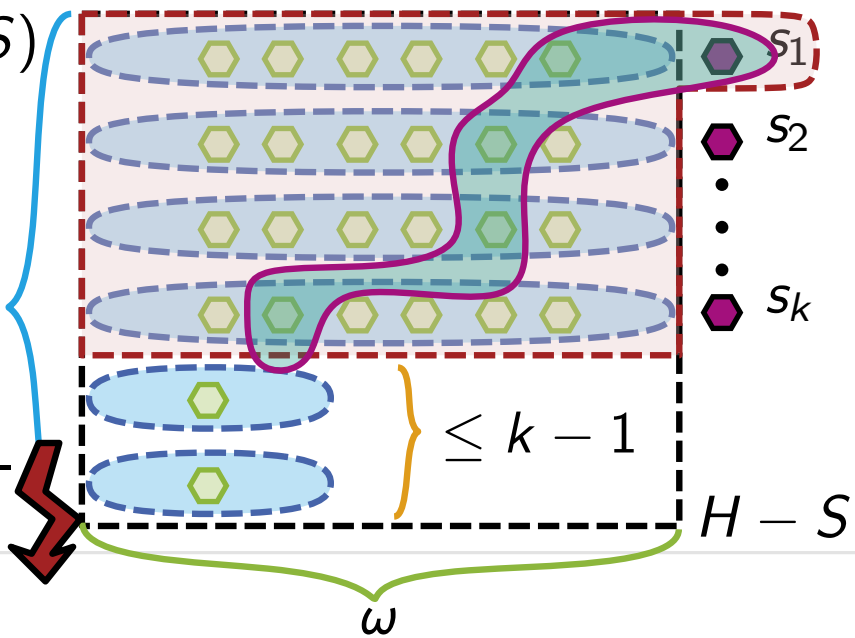
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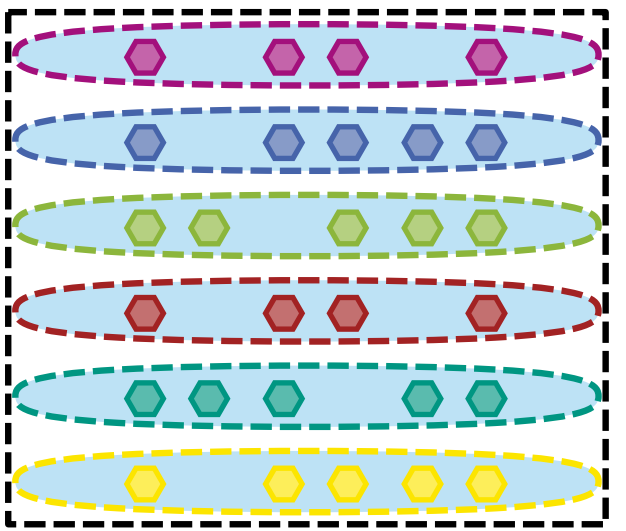
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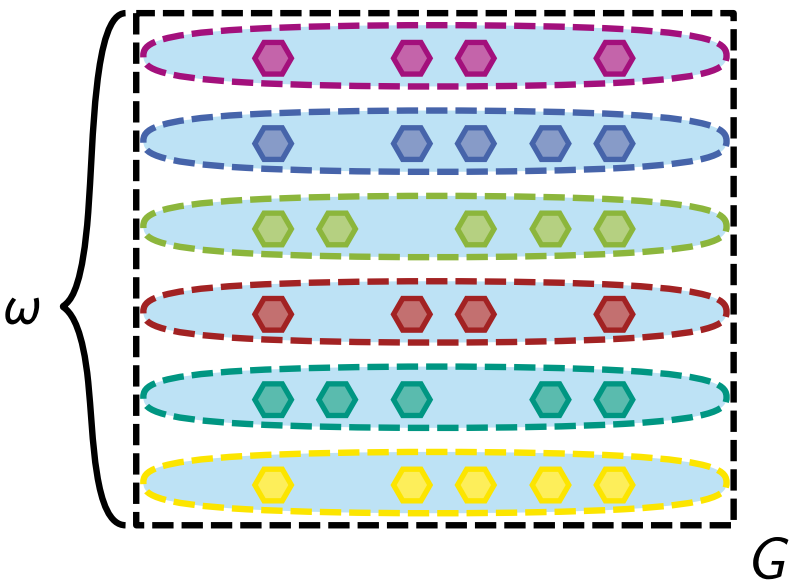
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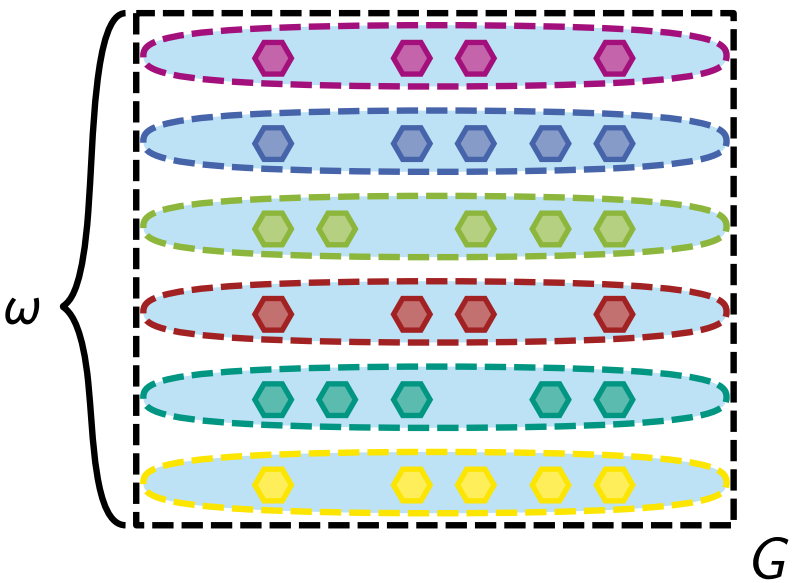
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V_i are i-sets $\implies$ $|V_i| \leq \alpha(G)$



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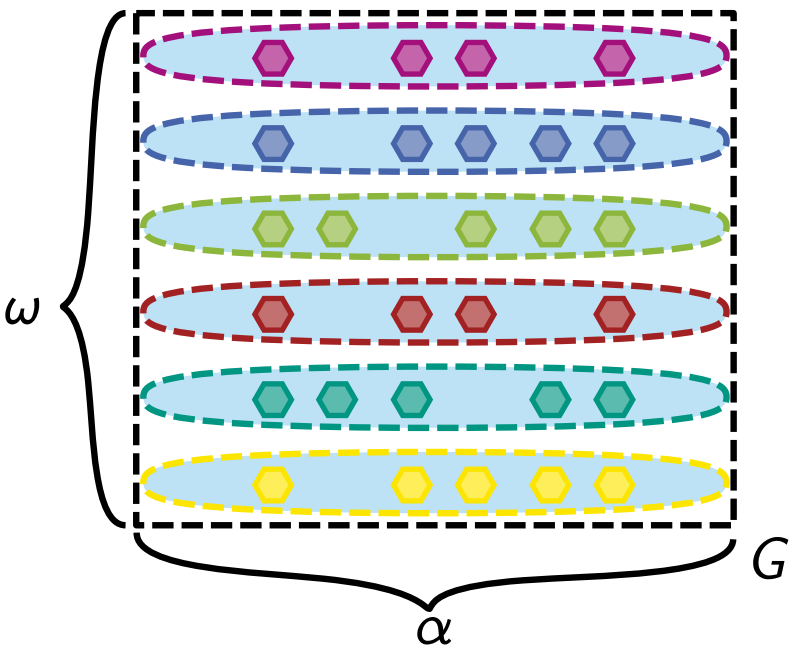
We prove (P1) $\iff$ (P2) $\iff$ (P3) by induction on $|V|$

induction step: "(P1) $\implies$ (P3)"

Let $A \subseteq V(G)$ wlog $A = V(G)$ i.e. we need to show $\omega(G) \cdot \alpha(G) \geq |V(G)|$

(P1) $\implies$ $\exists$ coloring $V_1 + \ldots + V_t$ with $t = \omega(G)$

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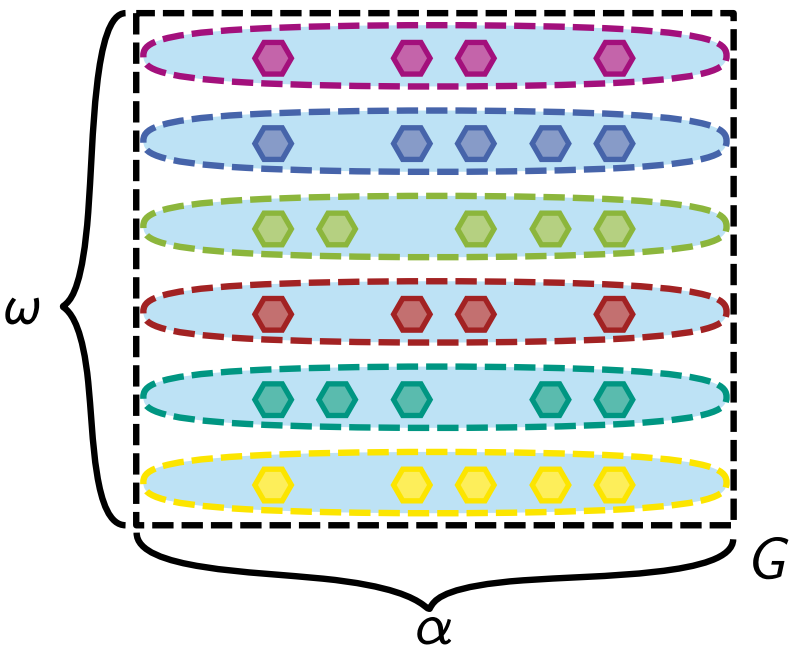
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$|V(G)| = \sum_i^{\omega(G)} |V_i| \leq \omega(G) \cdot \alpha(G)$



WPGT:Proof

We prove (P1) $\iff$ (P2) $\iff$ (P3) by induction on $|V|$

induction step: ”(P3) $\implies$ (P1)”

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by Obs. 1 it is enough to show $\chi(G) \leq \omega(G)$

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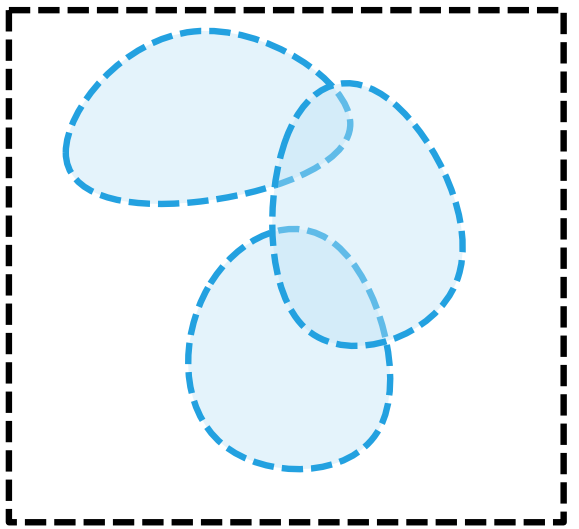
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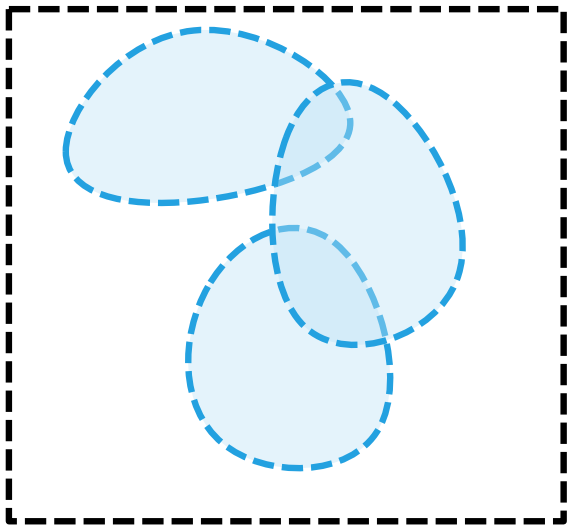
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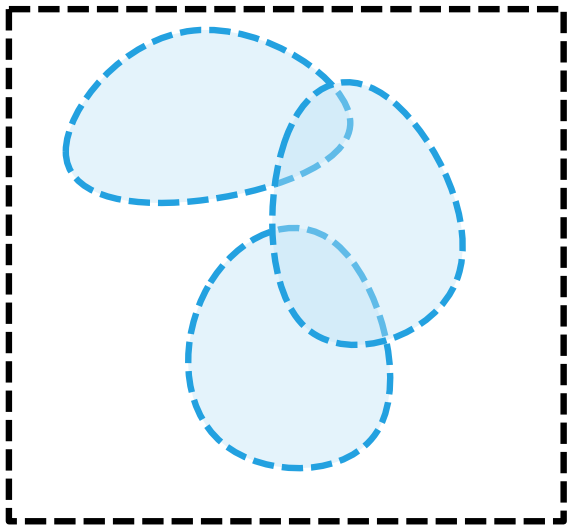
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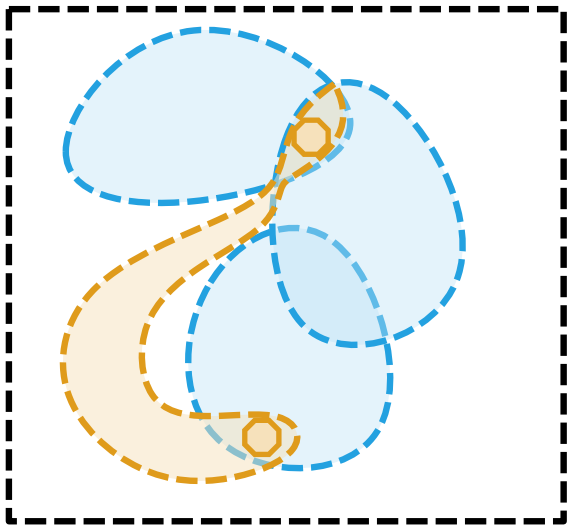
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I hits every biggest clique

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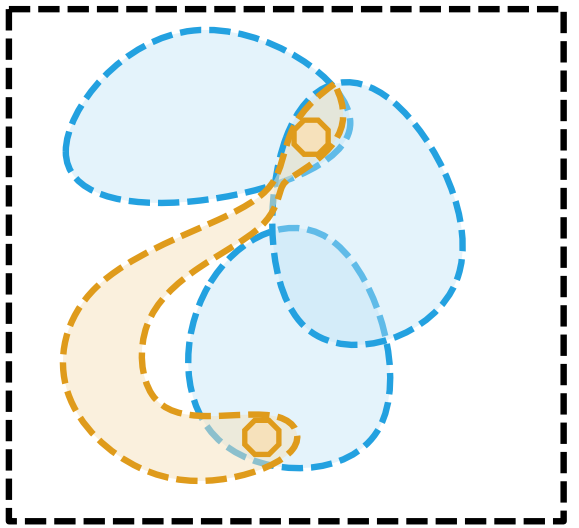
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notice $\omega(G - I) \leq \omega(G) - 1$



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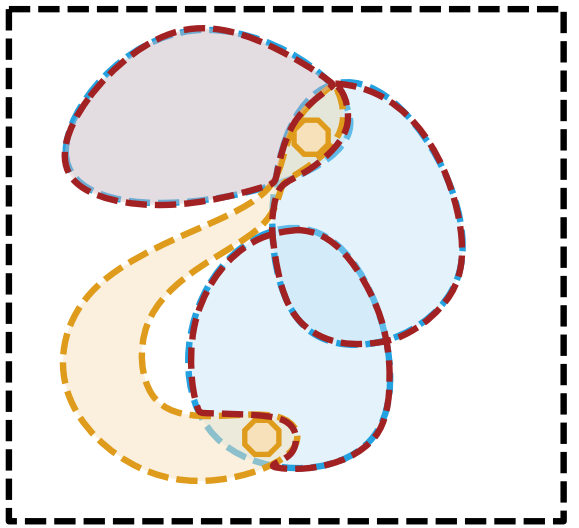
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reduce every biggest clique by at least one



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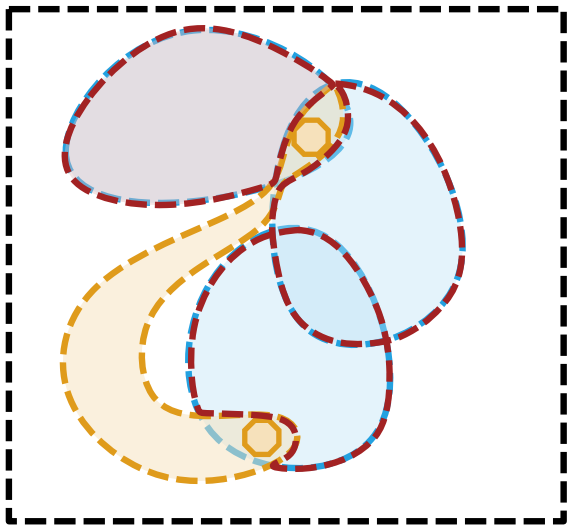
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by induction (P1) holds for $G - I$

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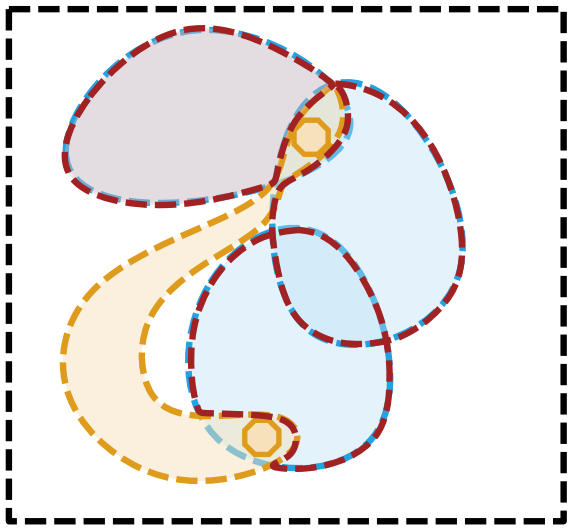
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reduce every biggest clique by at least one



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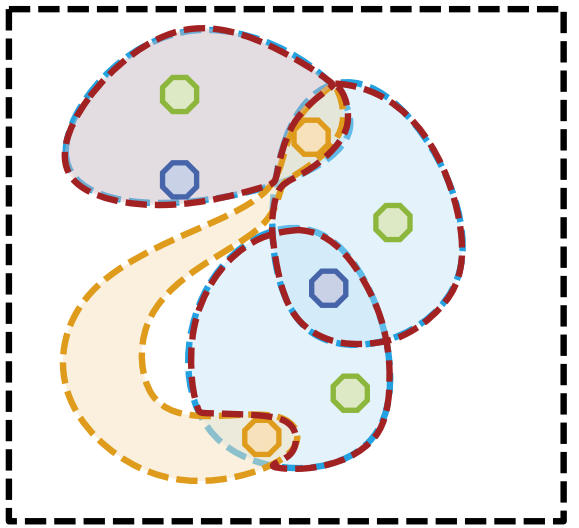
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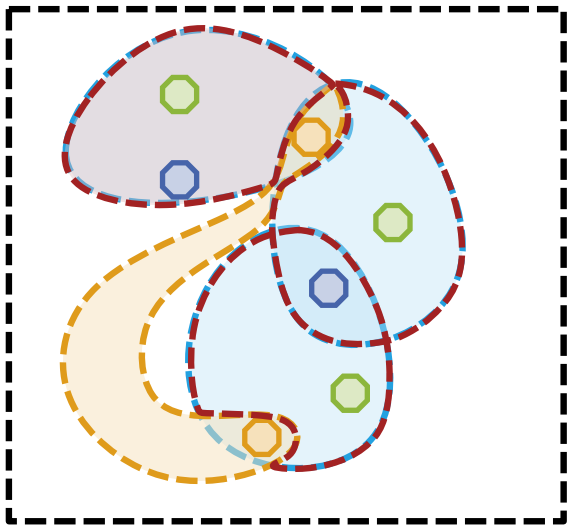
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$V_1 + ... + V_t + I$ coloring of G

reduce every biggest clique by at least one



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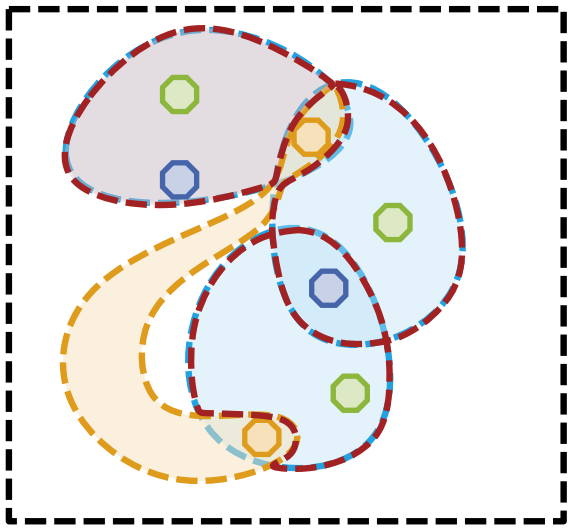
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$V_1 + ... + V_t + I$ coloring of G

$\Rightarrow \chi(G) \leq t + 1 \leq \omega(G) - 1 + 1 = \omega(G)$

reduce every biggest clique by at least one



I hits every biggest clique

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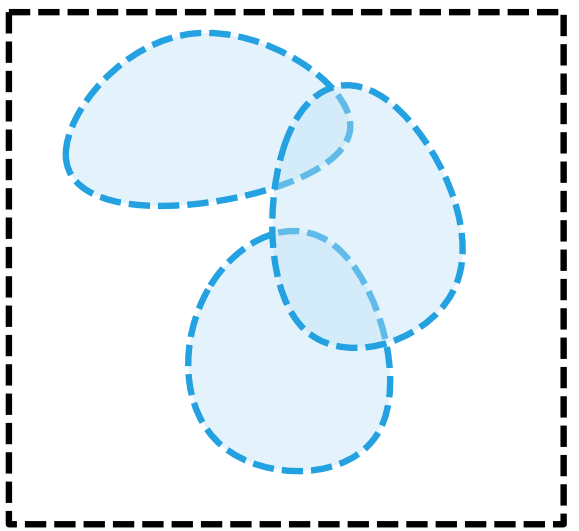
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WPGT:Proof

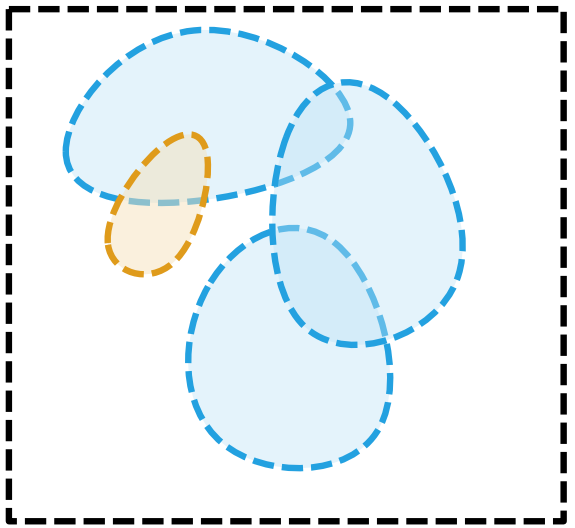
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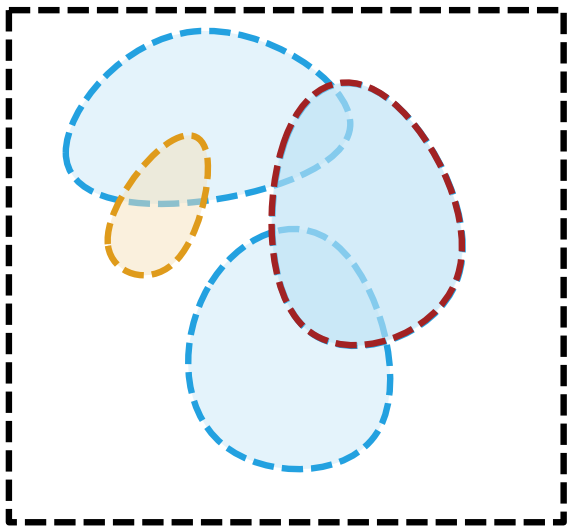
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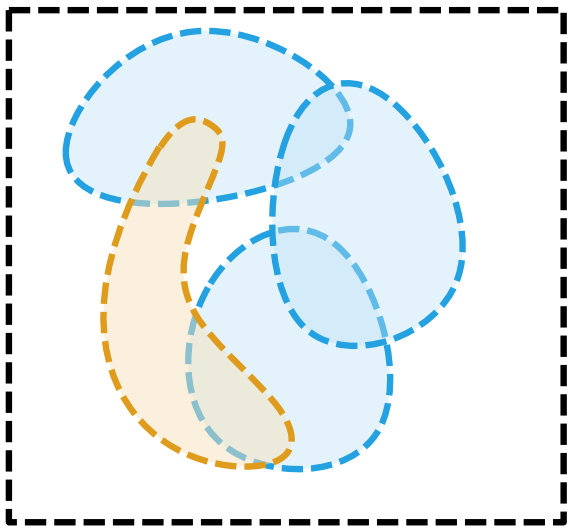
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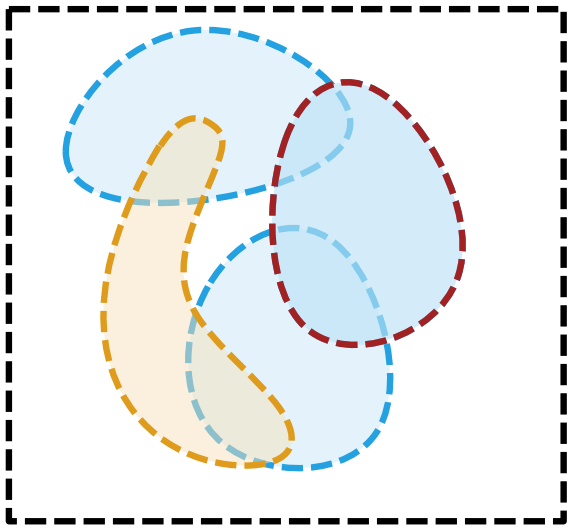
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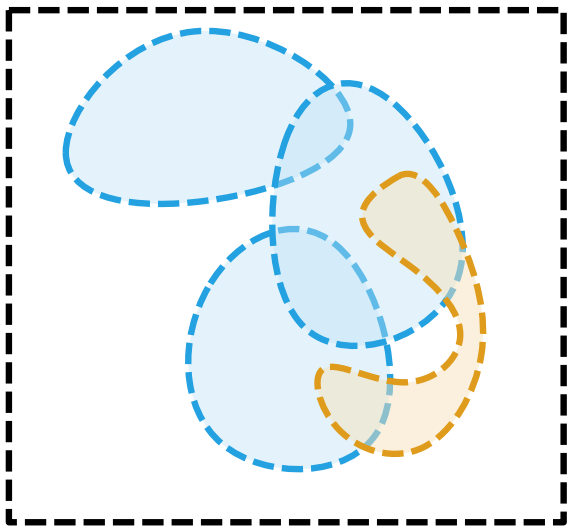
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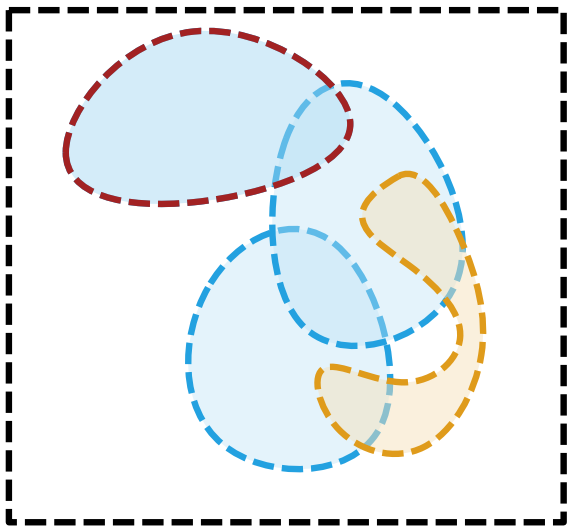
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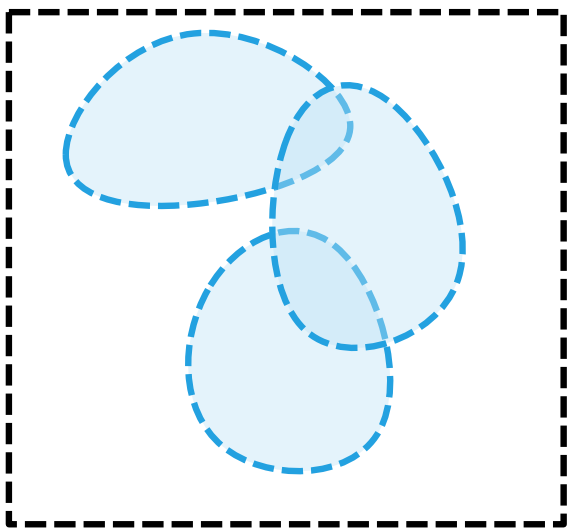
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fix $C(I)$ in place for every $I \in Y :=$ i-sets of G



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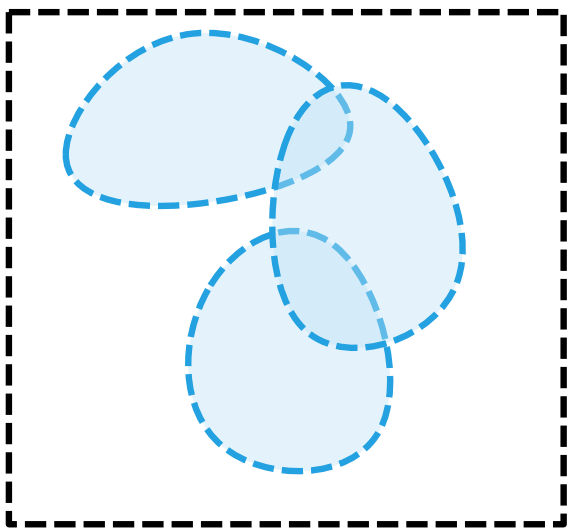
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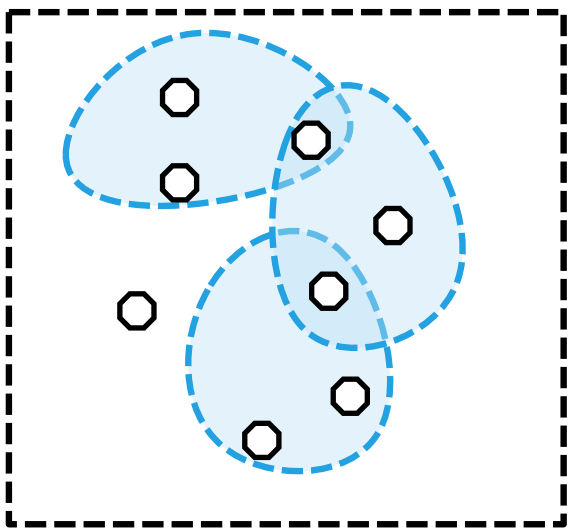
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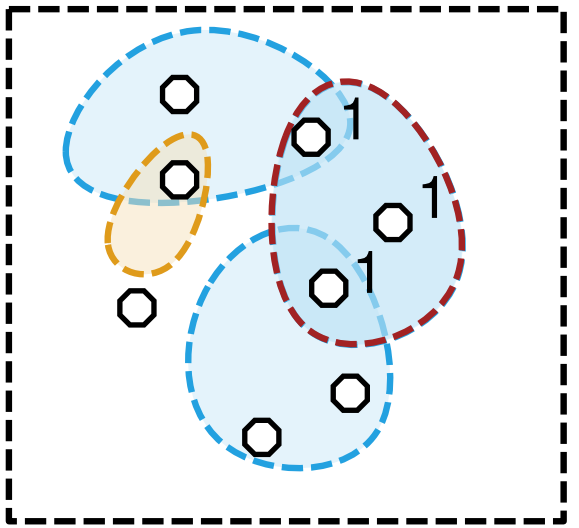
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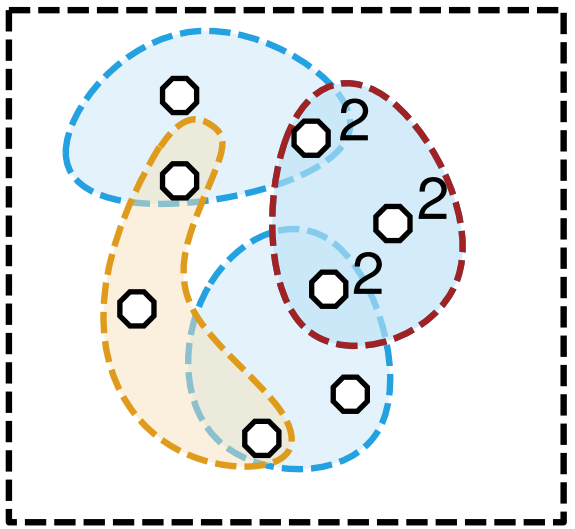
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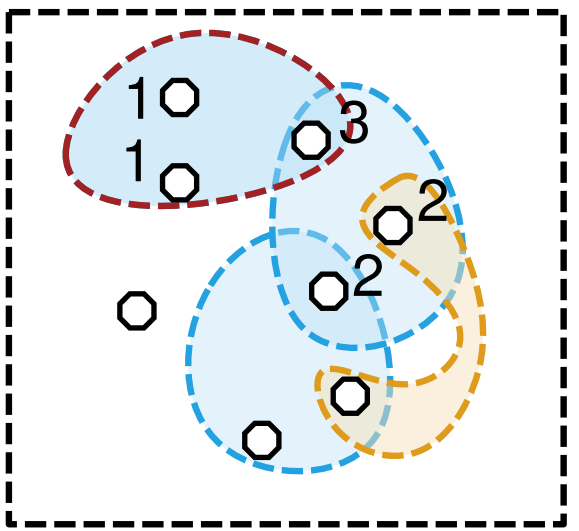
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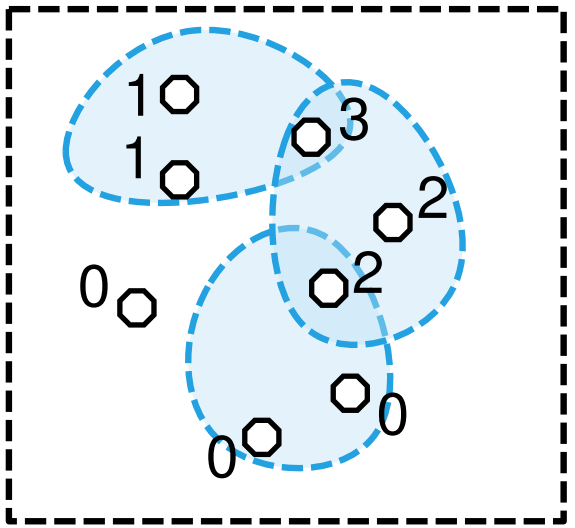
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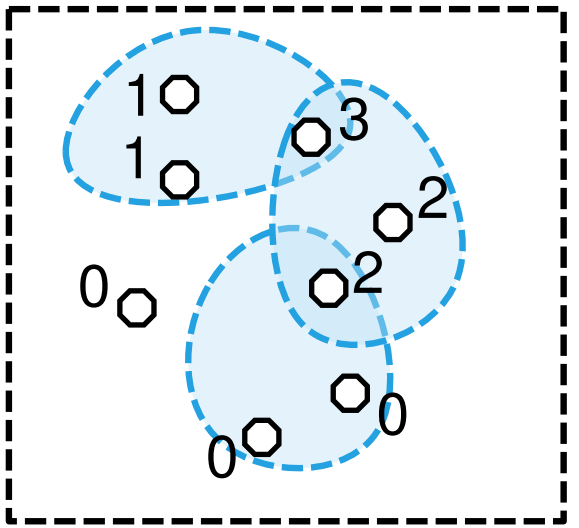
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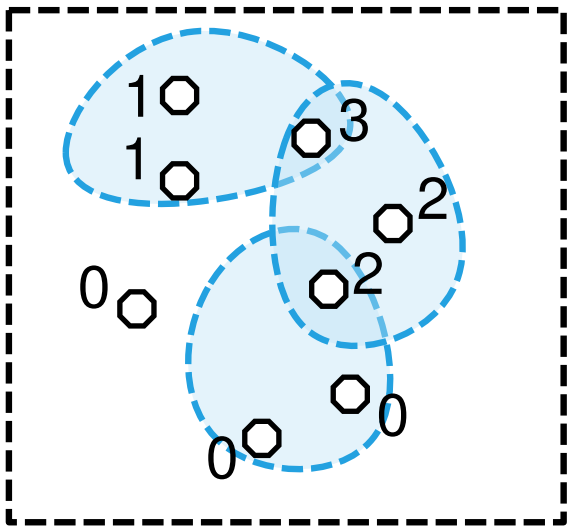
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Since (P3) holds for G and by induction (P2) for G_A

$\forall A \subsetneq V(G) \overset{\text{Lem.2.7}}{\implies}$ (P3) for H



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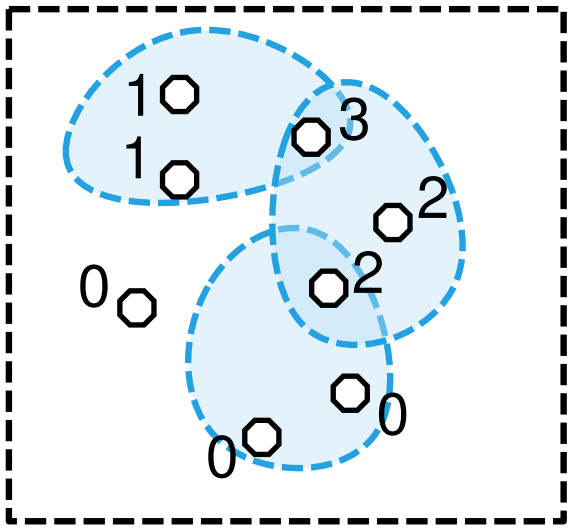
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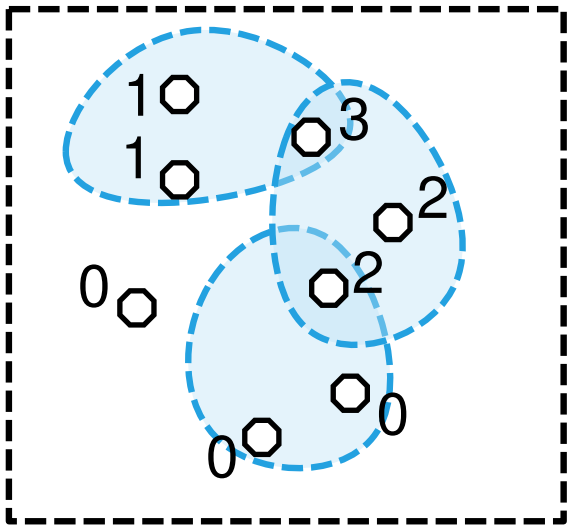
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$\omega(H) \cdot \alpha(H) \geq |V(G)|$



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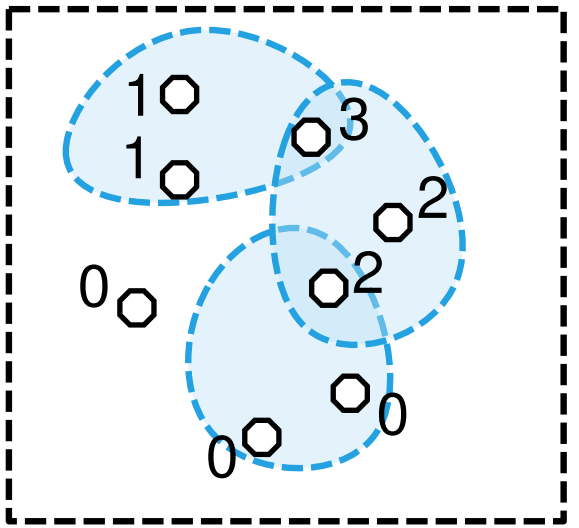
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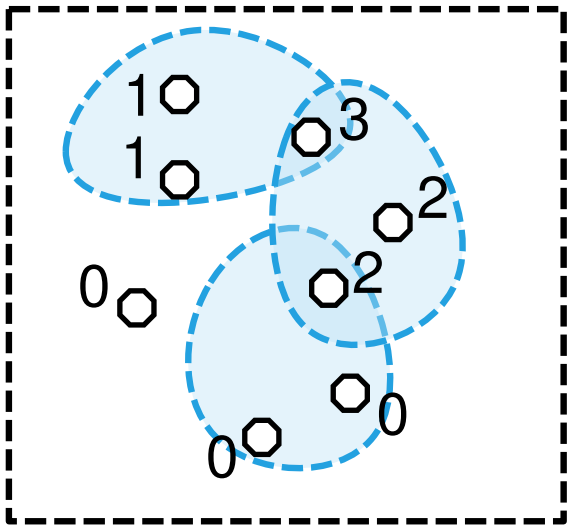
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für jedes Element in Y wird bei $C(I)$ um 1 hochgezählt

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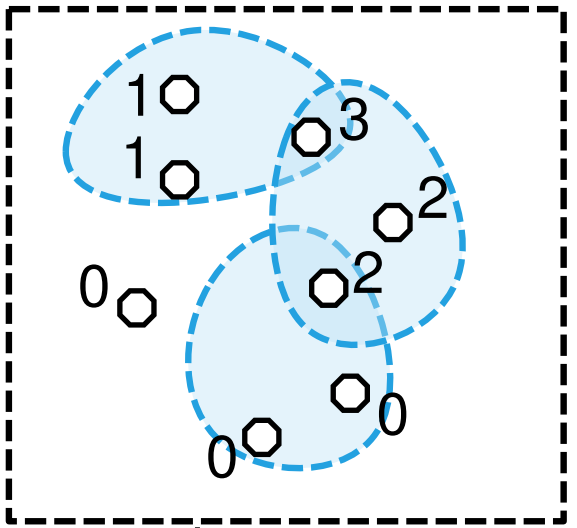
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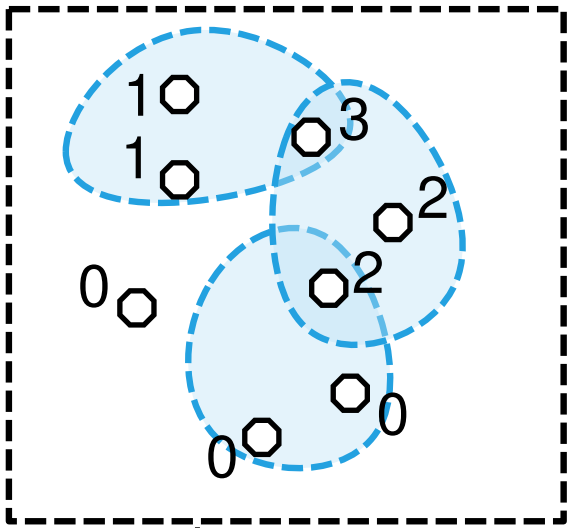
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$\alpha(H) = \max_{I \in Y} \sum_{v \in I} h(v)$ since duplicates are independent $\rightsquigarrow$ project every i-set to Y

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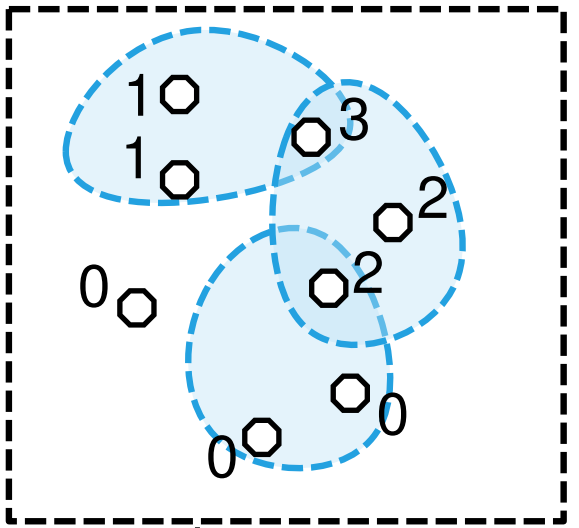
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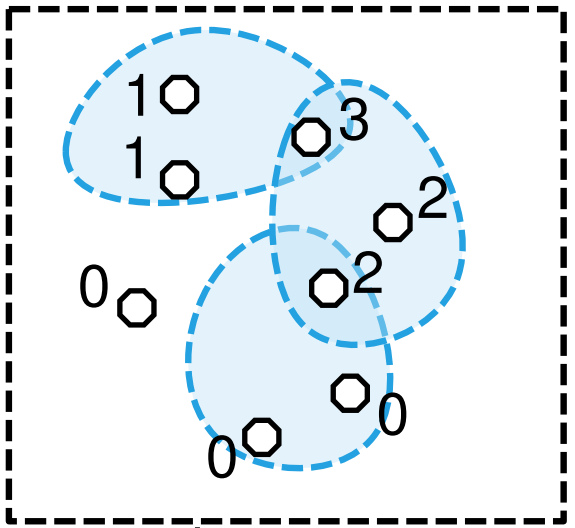
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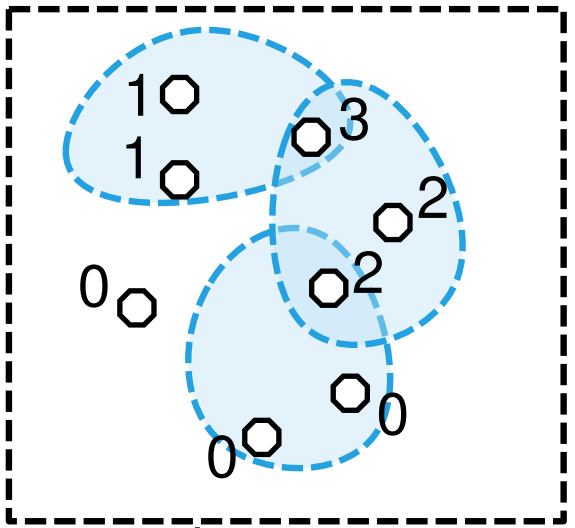
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at most 1 element, $C(I) \cap I = \varnothing$

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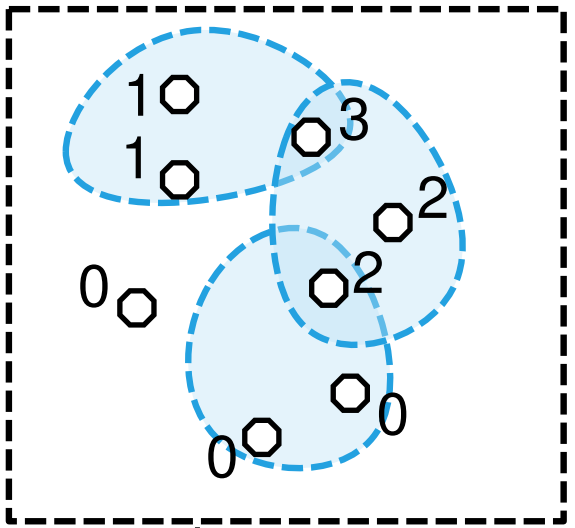
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$\omega(G_A) = \alpha(\overline{G}_A)$ and $\alpha(G_A) = \omega(\overline{G}_A)$, so the product stays the same

Intervall Graphs: t-holes

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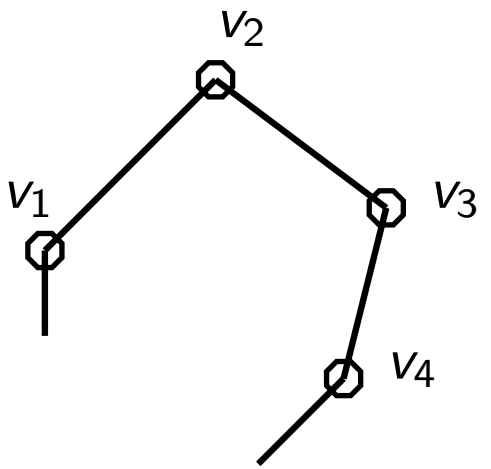
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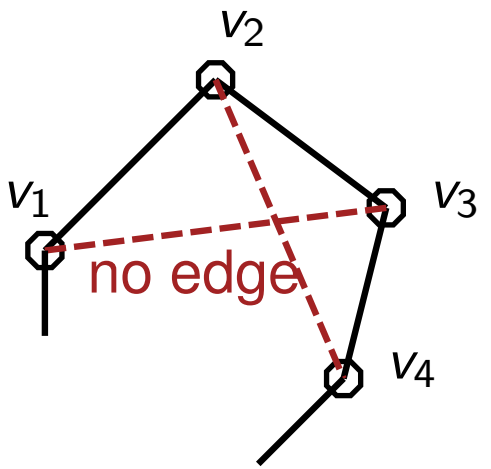
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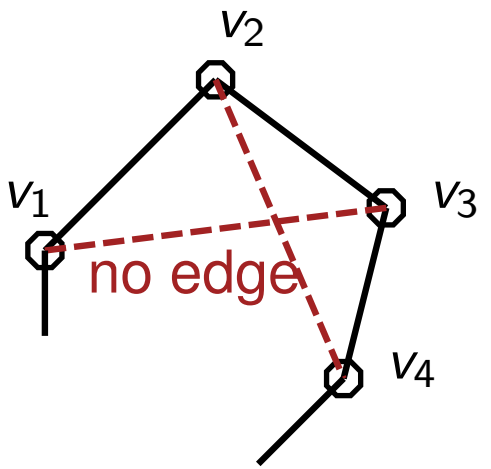
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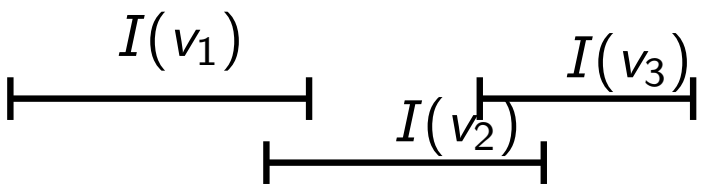
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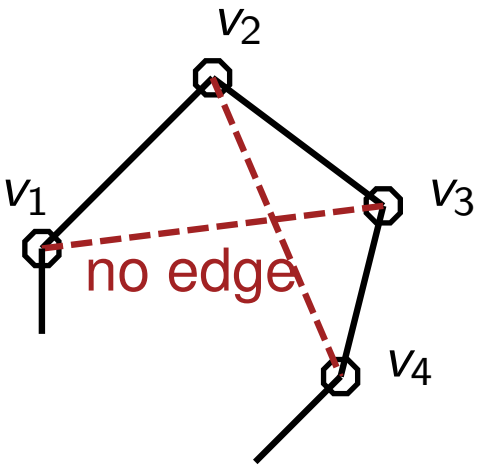
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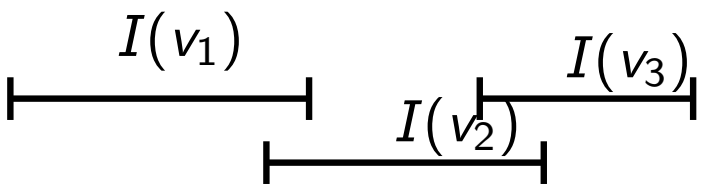
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$I(v_1)$ and $I(v_3)$ must over distinct points of $I(v_3)$
or else $v_1 v_3 \in E(G)$





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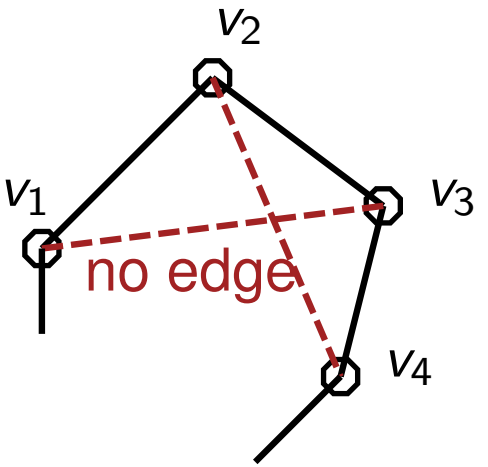
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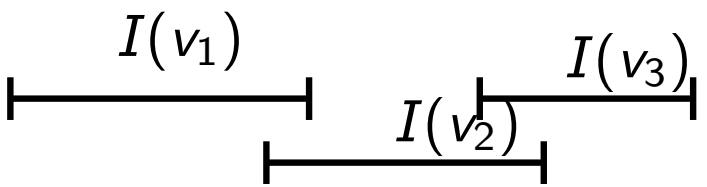
intervall representation $\mathcal{I} = \{I(v) = [l_v, r_v] \mid v \in V\}$

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or else $v_1 v_3 \in E(G)$ (v_{i-})





Intervall Graphs: t-holes

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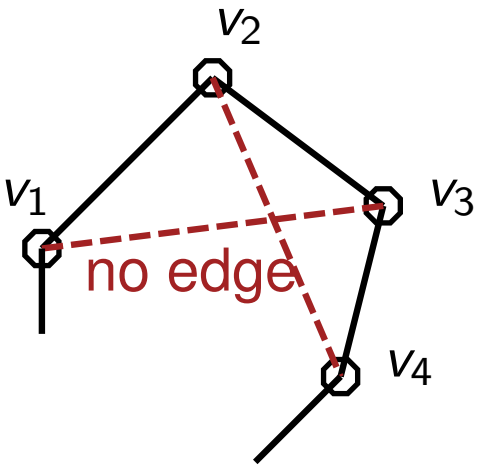
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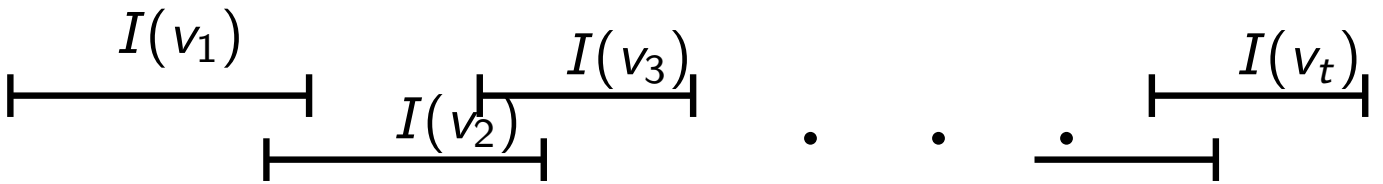
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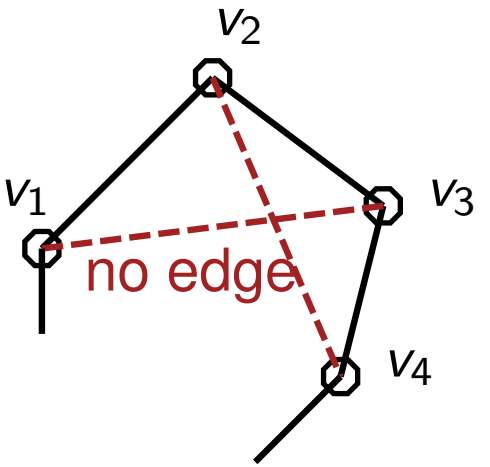
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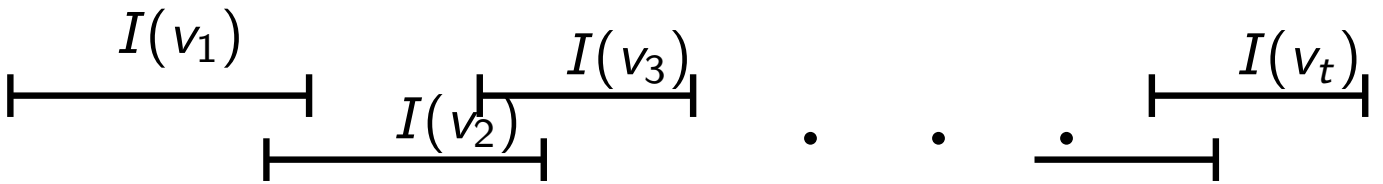
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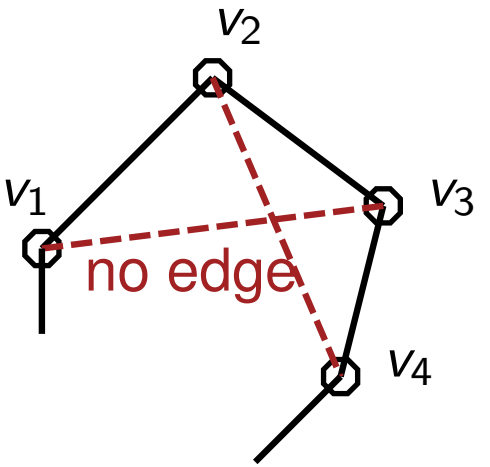
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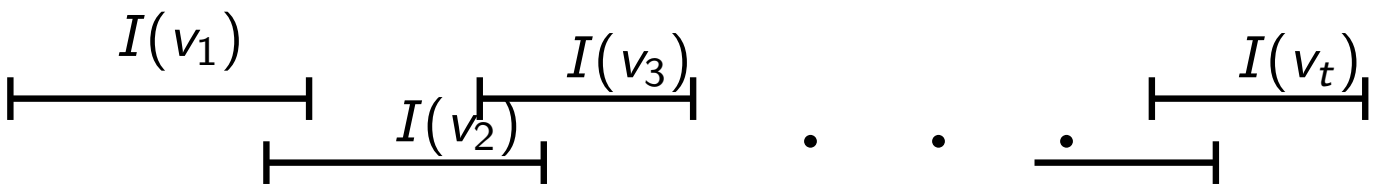
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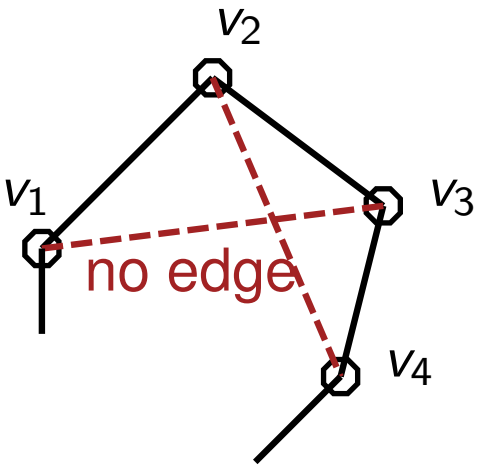
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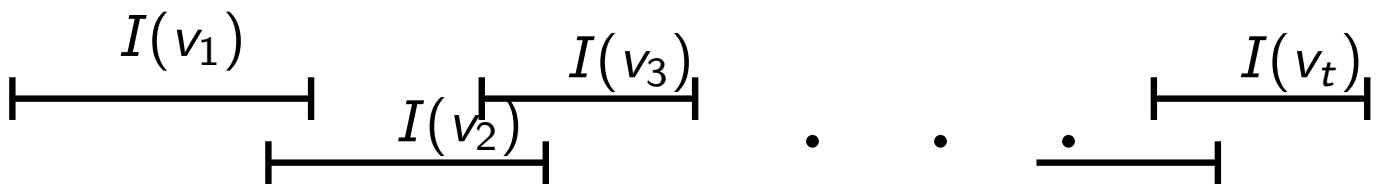
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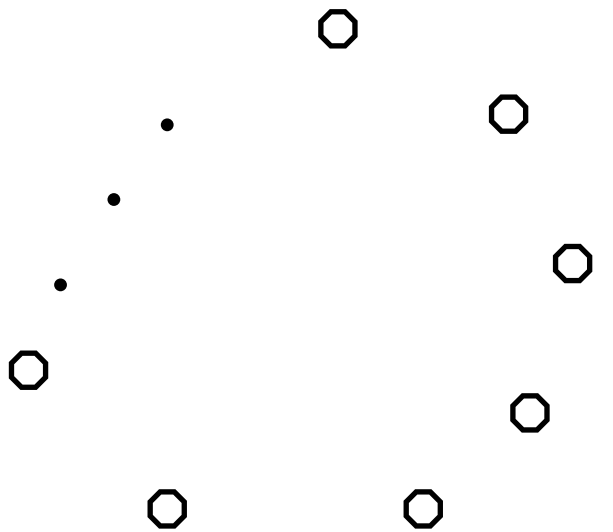
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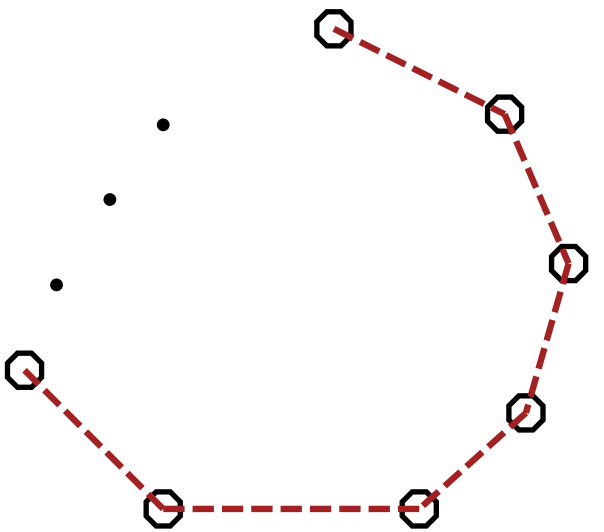
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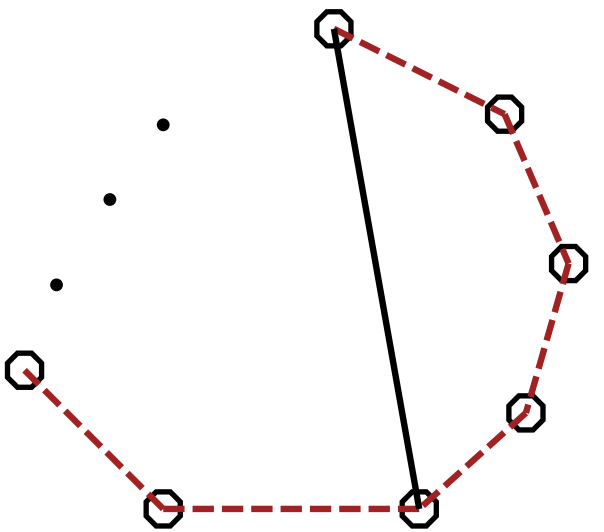
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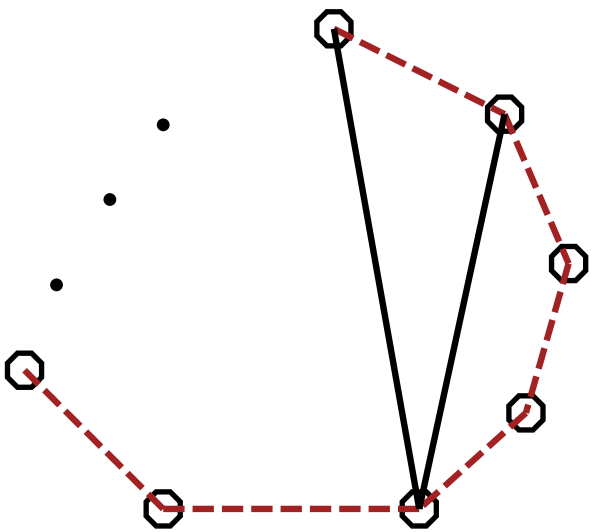
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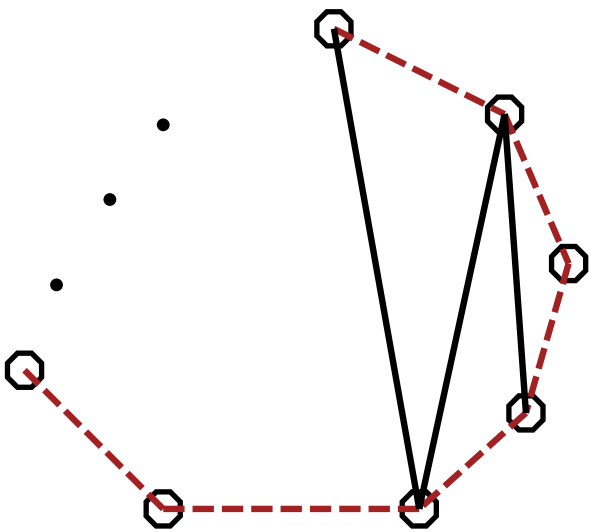
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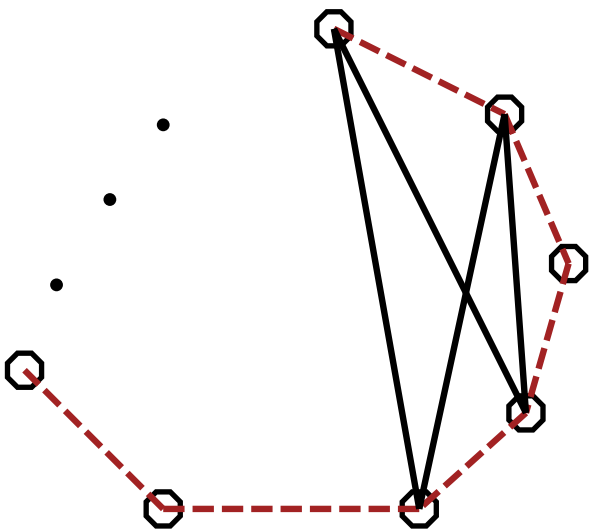
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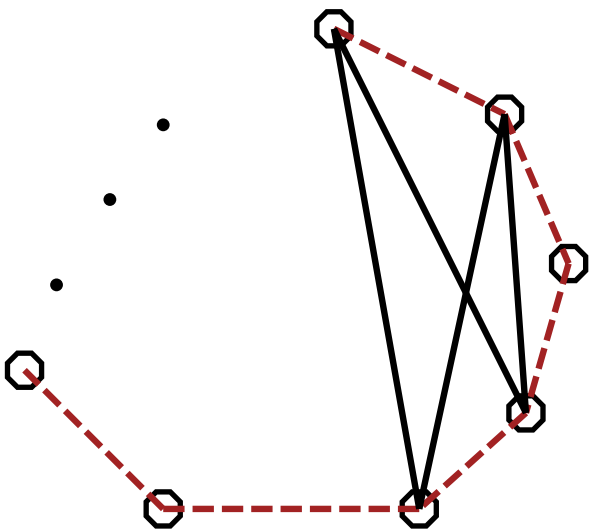
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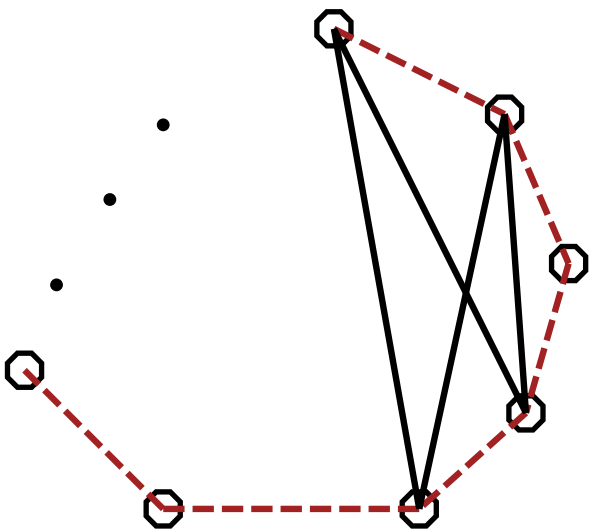
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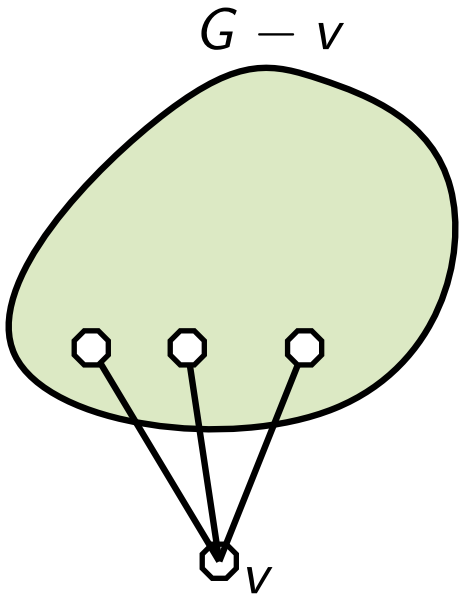
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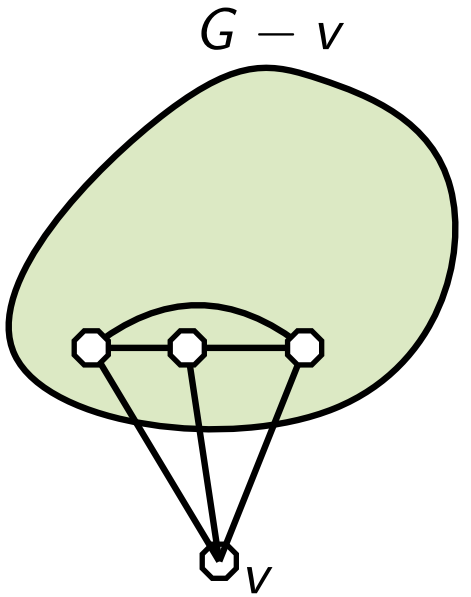


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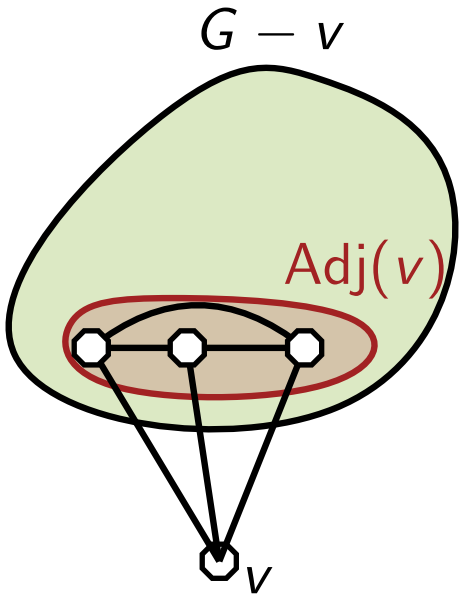


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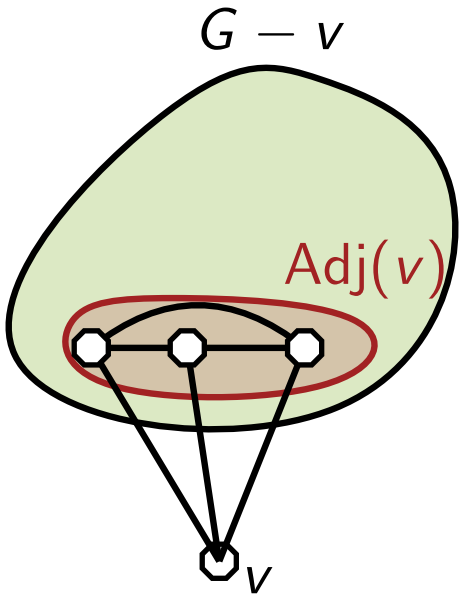
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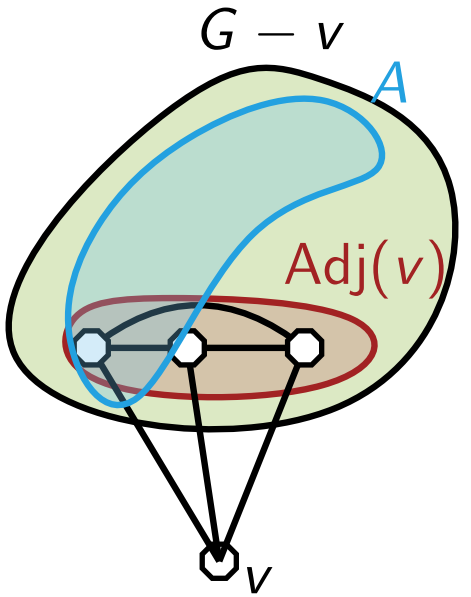
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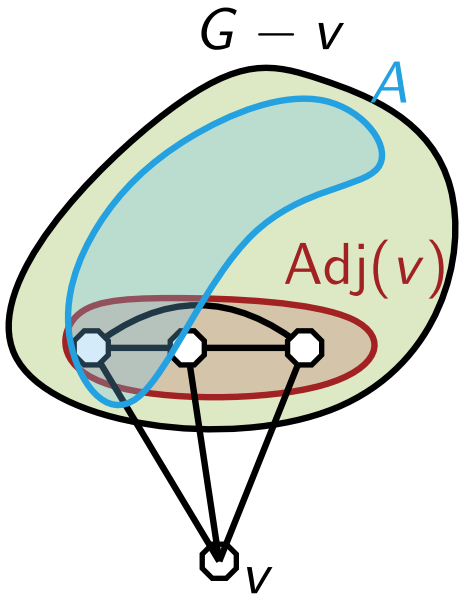
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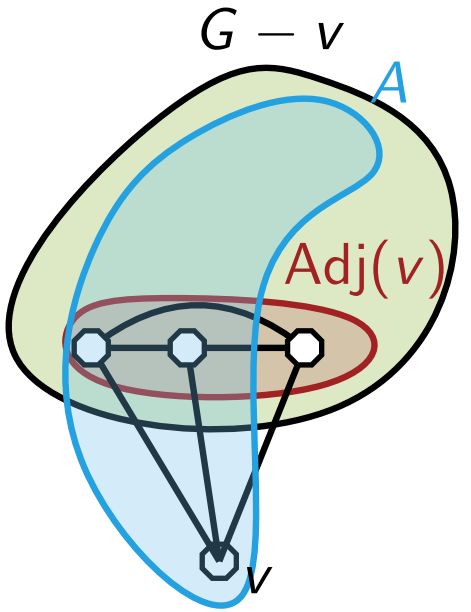
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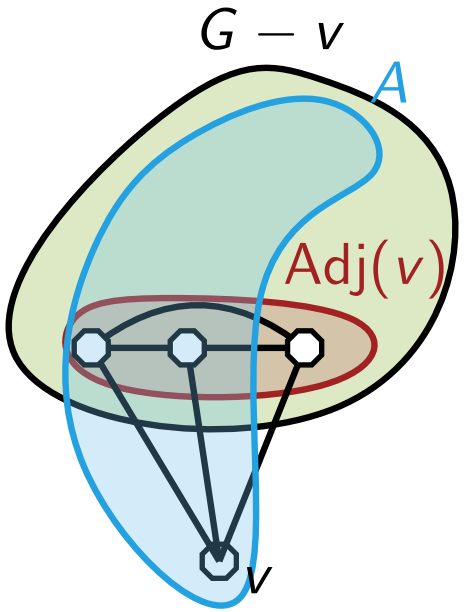
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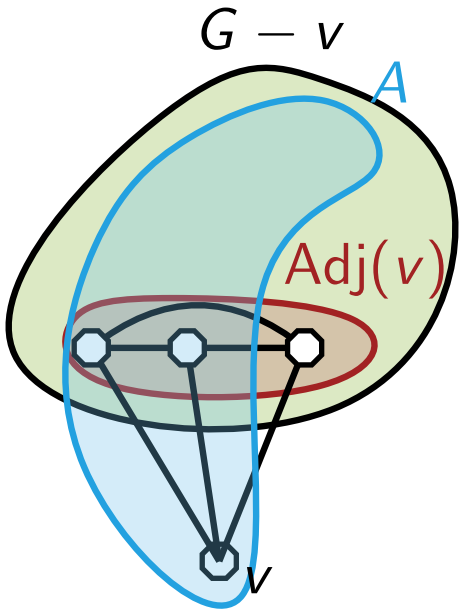
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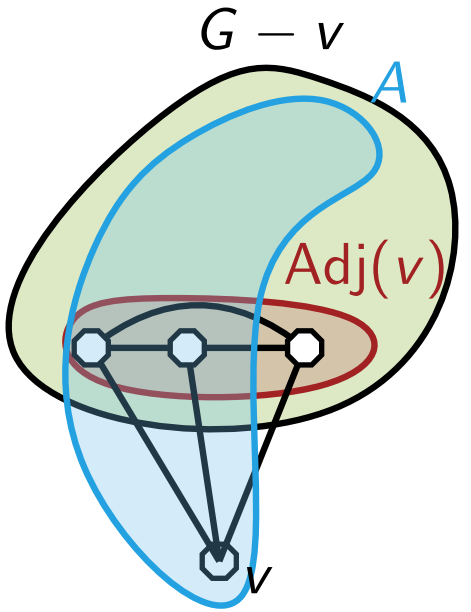
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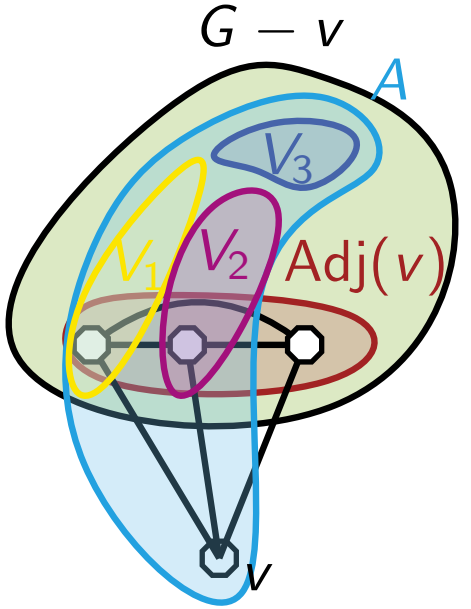
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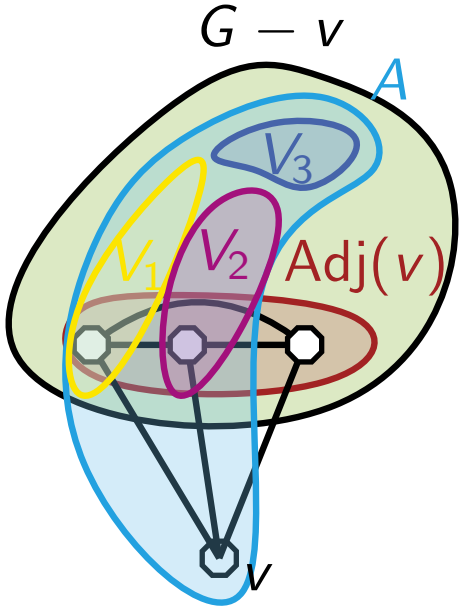
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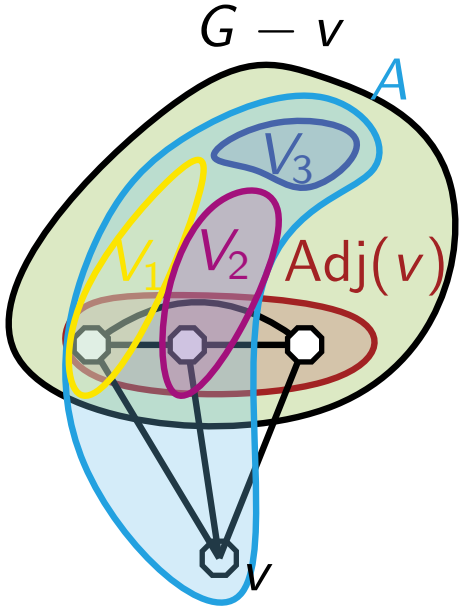
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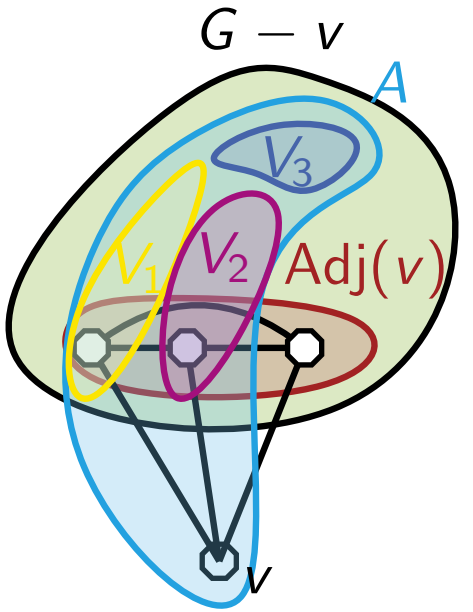
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$V_i' = V_i + v$ extends coloring since all neighbours have different color



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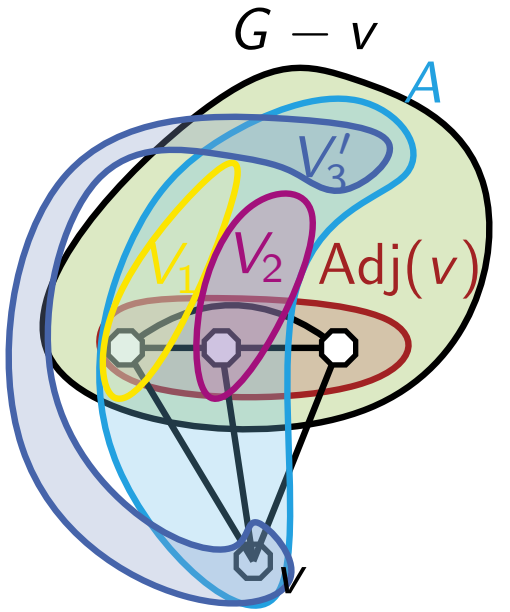
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$V_i' = V_i + v$ extends coloring since all neighbours have different color



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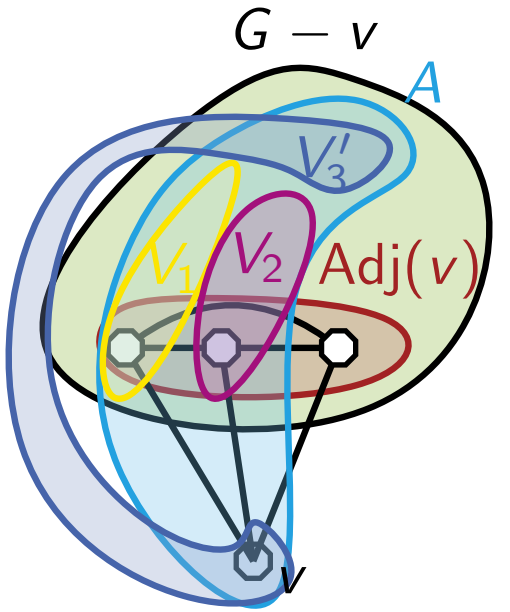
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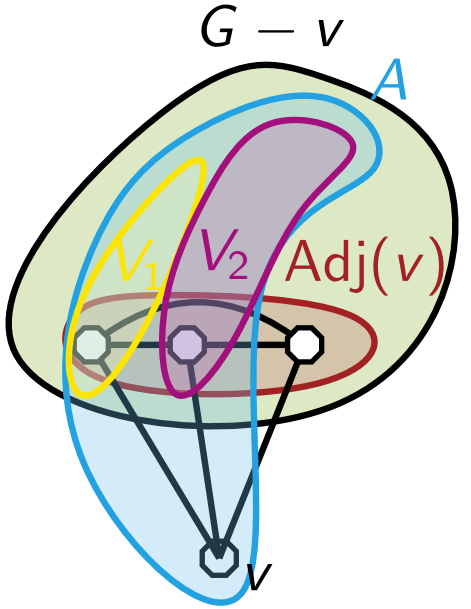
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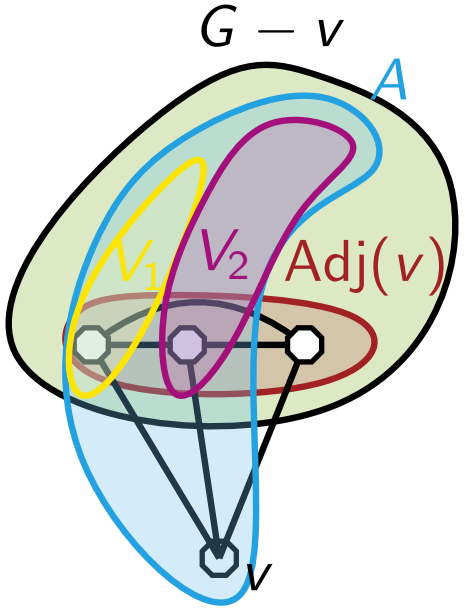
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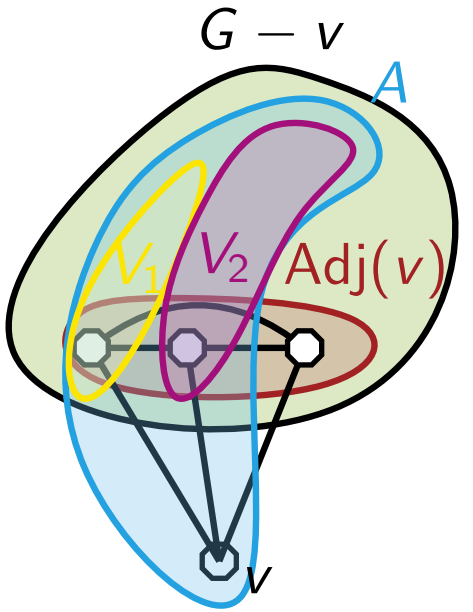
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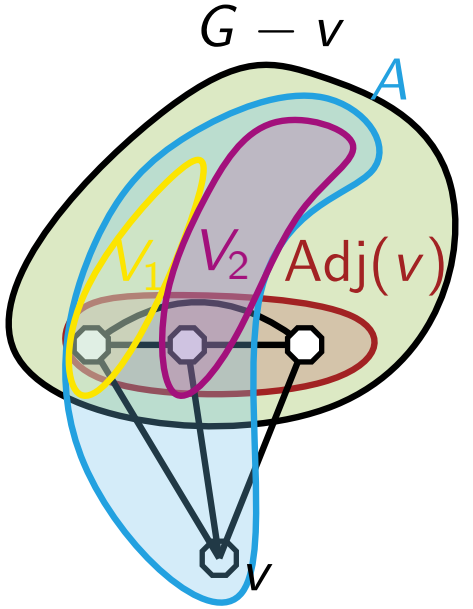
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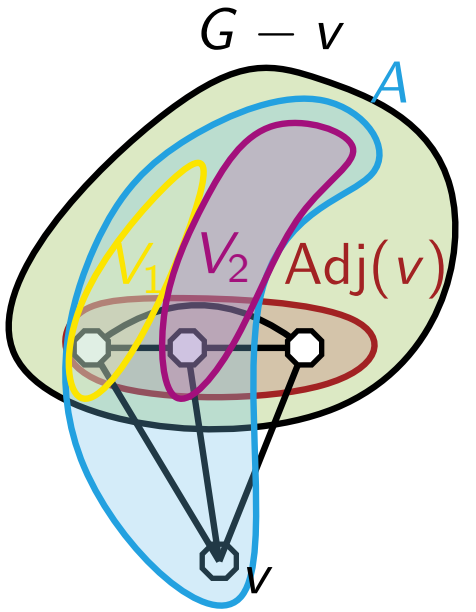
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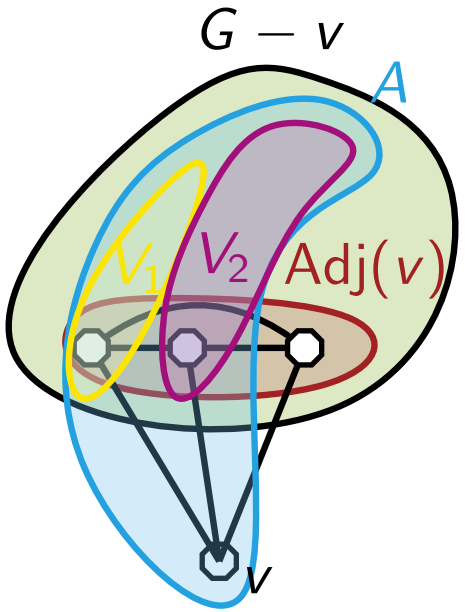
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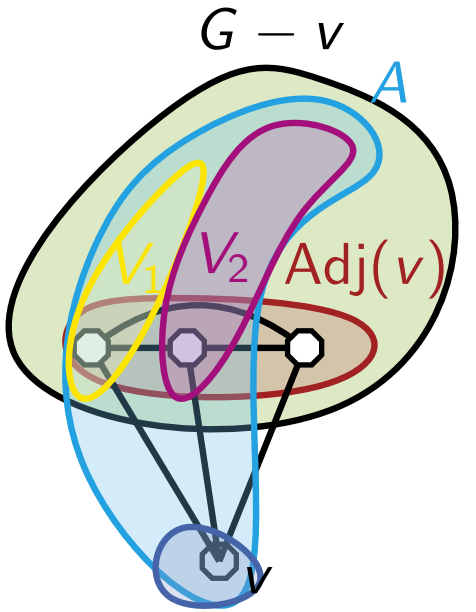
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$$\left.\begin{array}{l} G = (V, E) \text{ chordal}, a \neq b \in V, ab \notin E \\ S \subseteq G \text{ is an inclusion minimal } a, b - \text{seperator} \end{array}\right\} \Longrightarrow S \text{ is a clique.}$$

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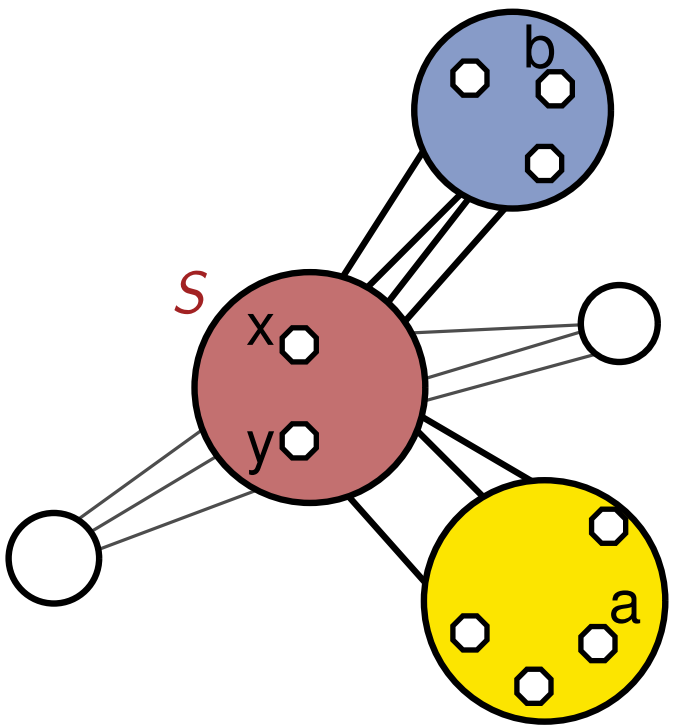
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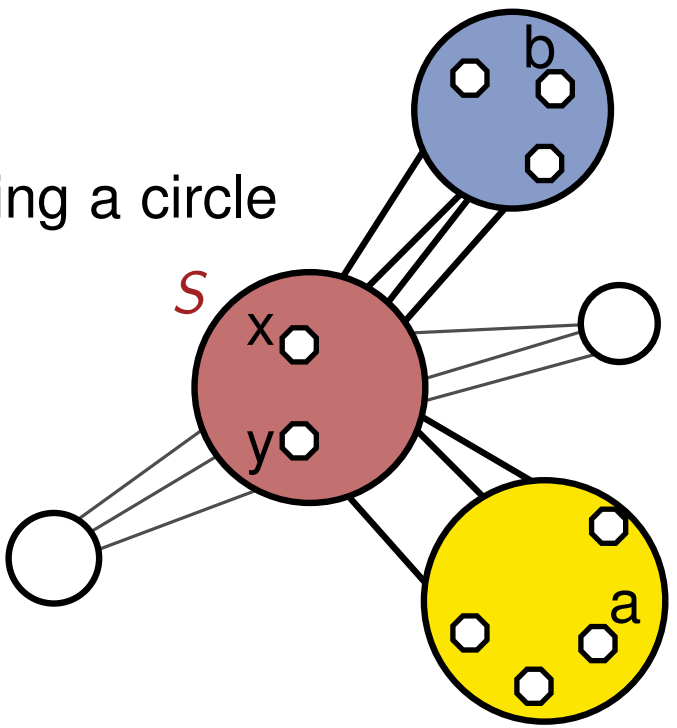
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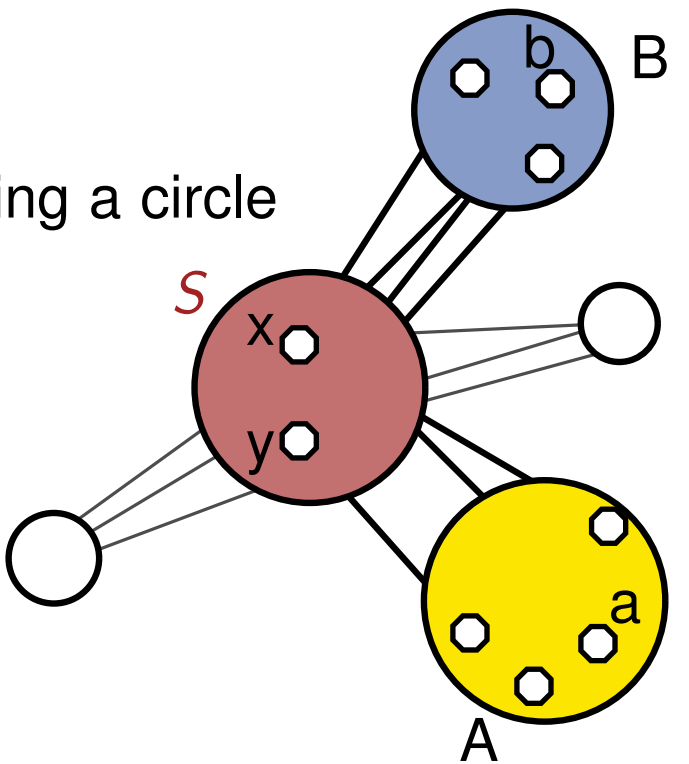
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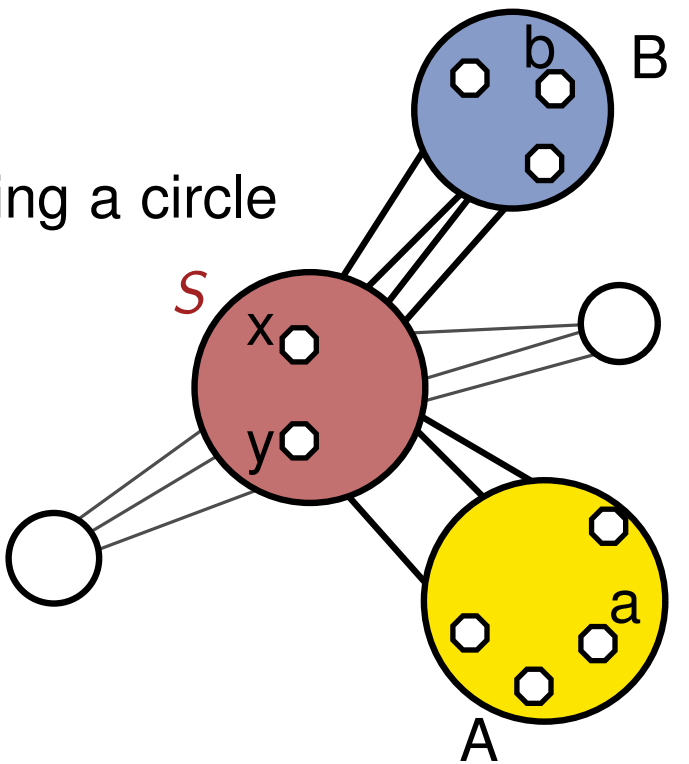
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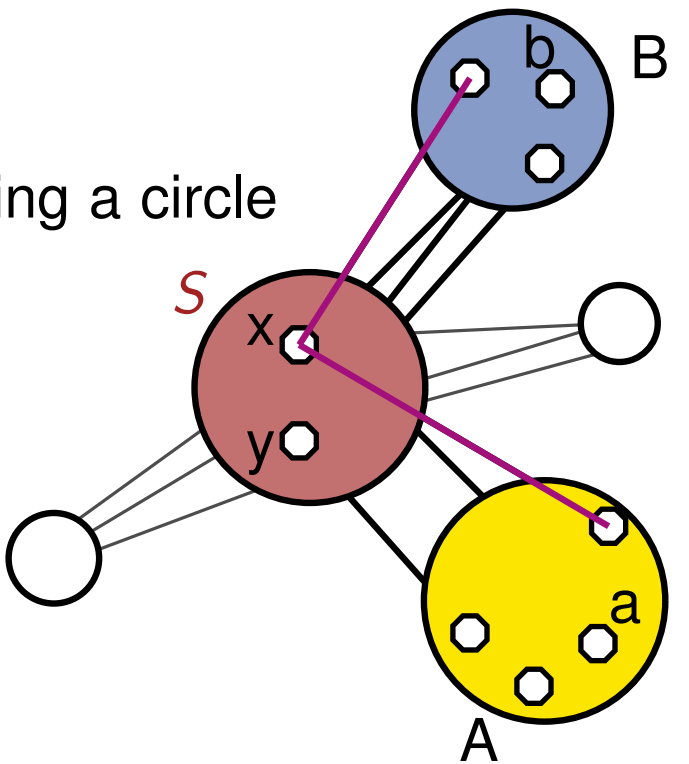
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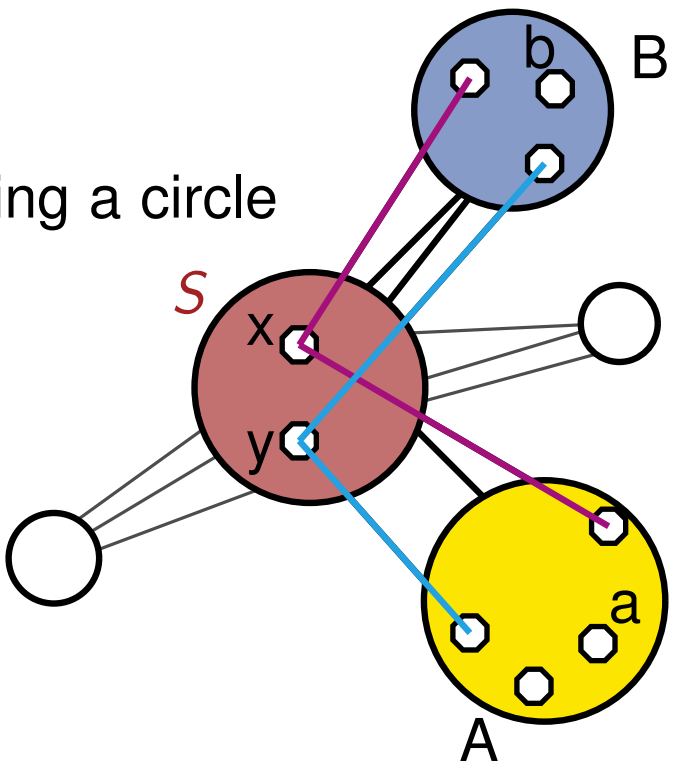
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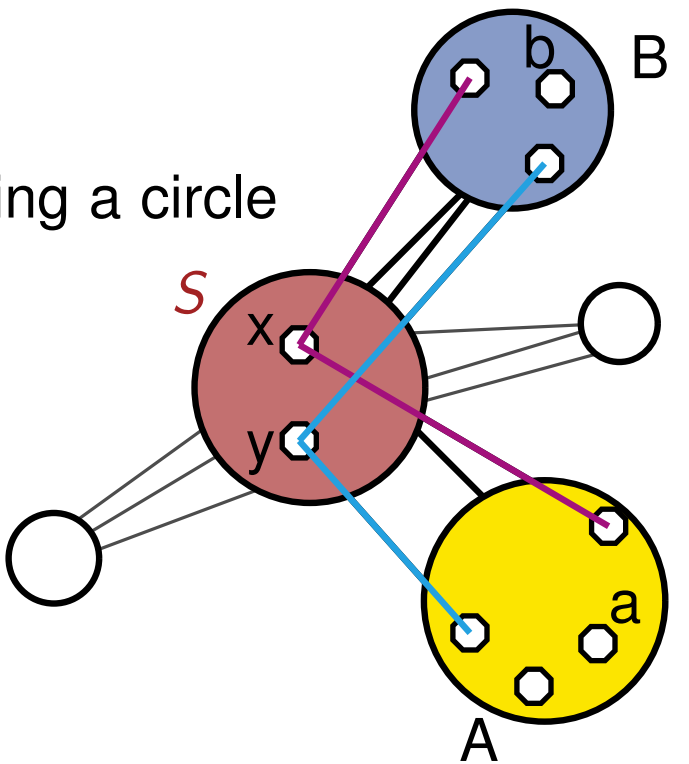
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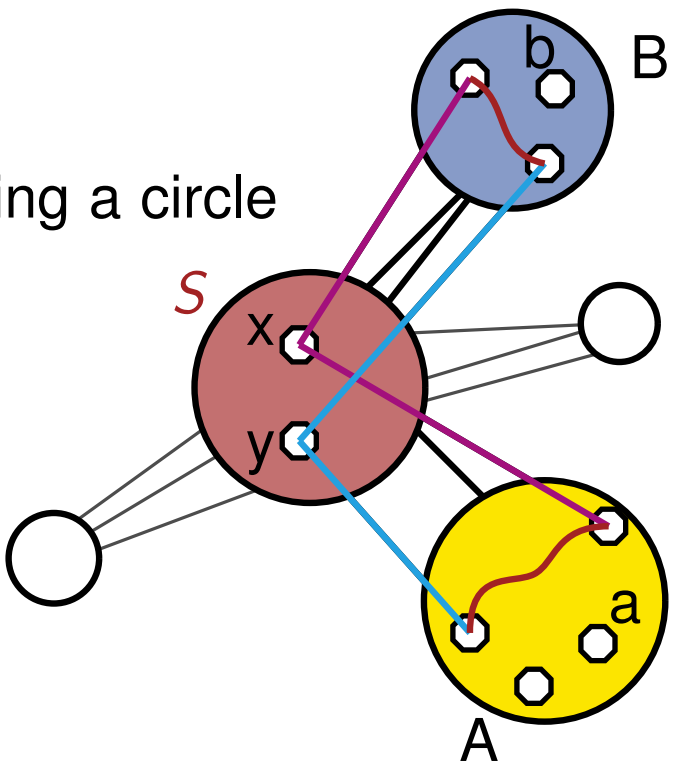
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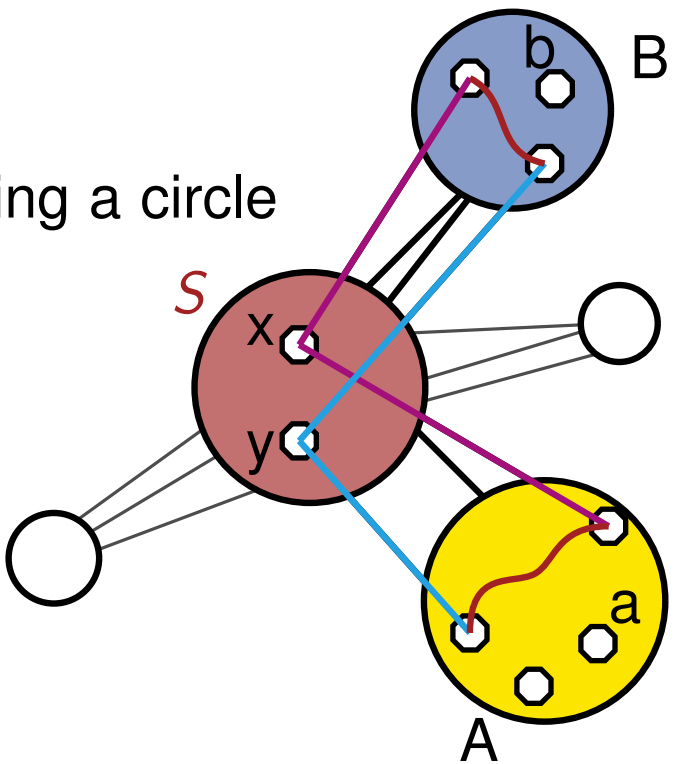
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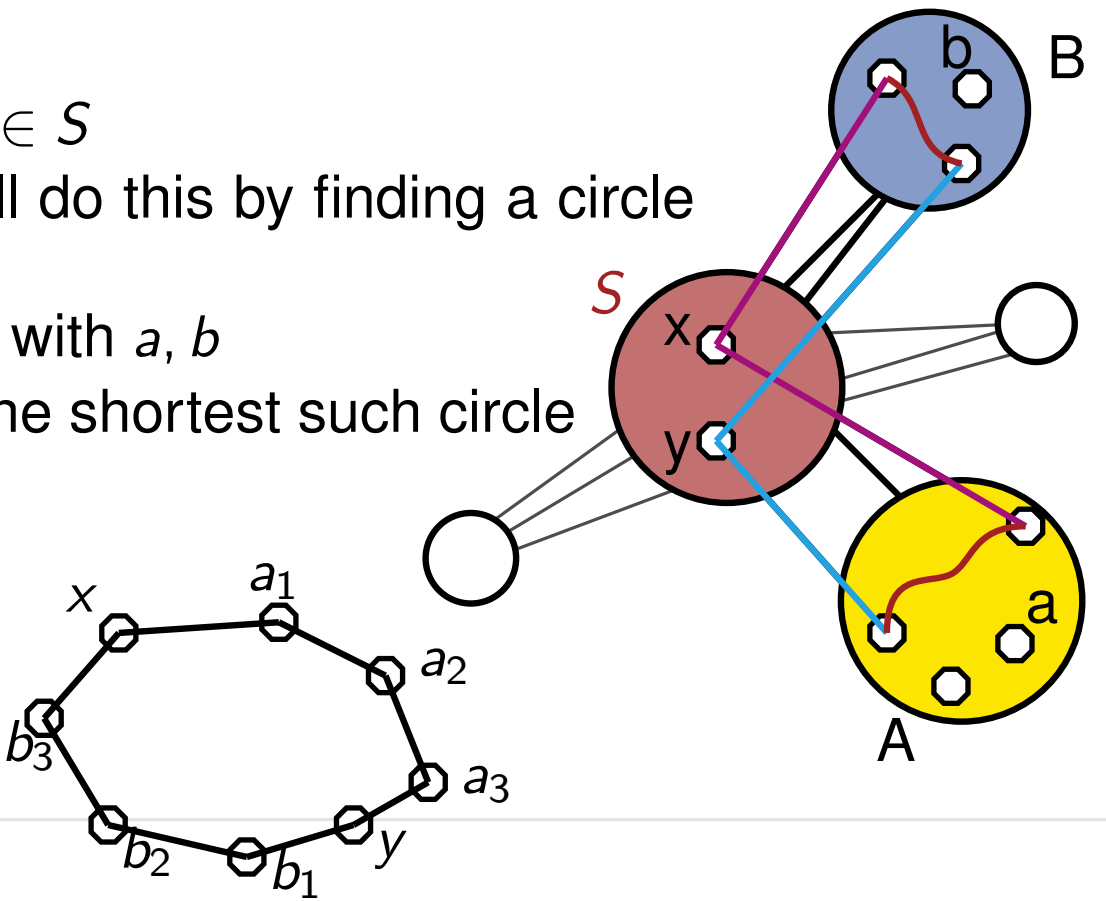
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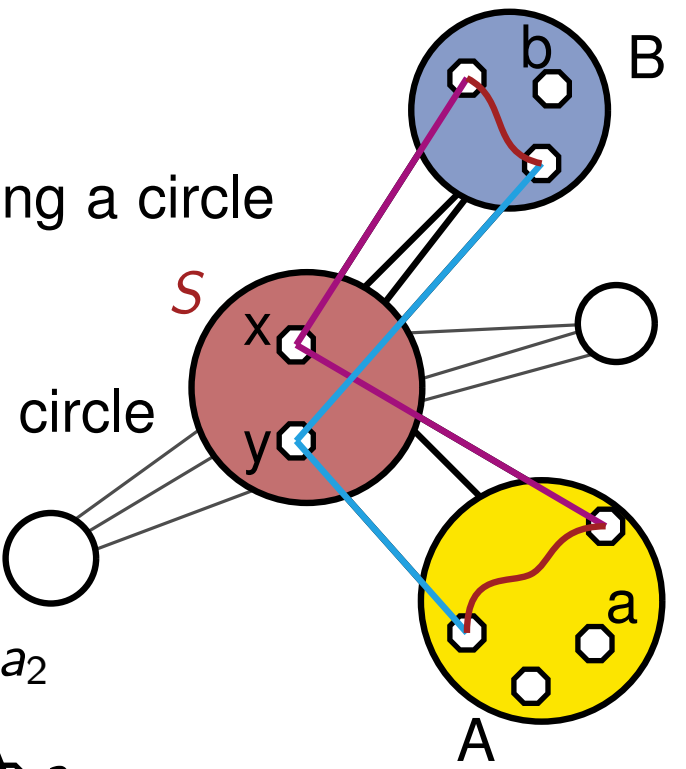
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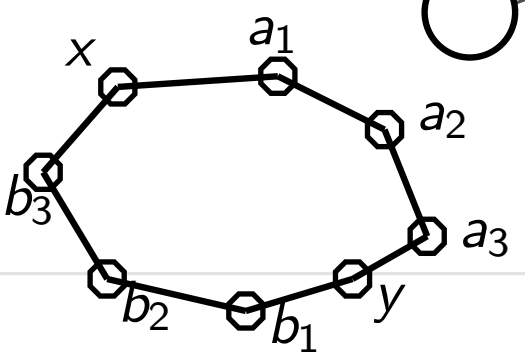
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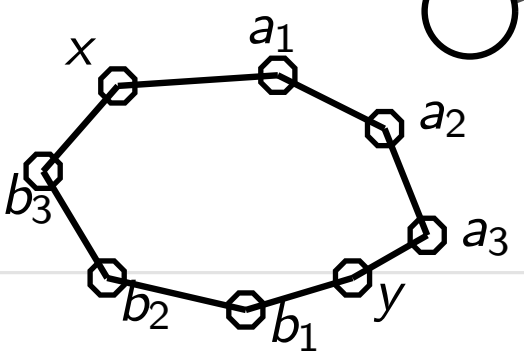
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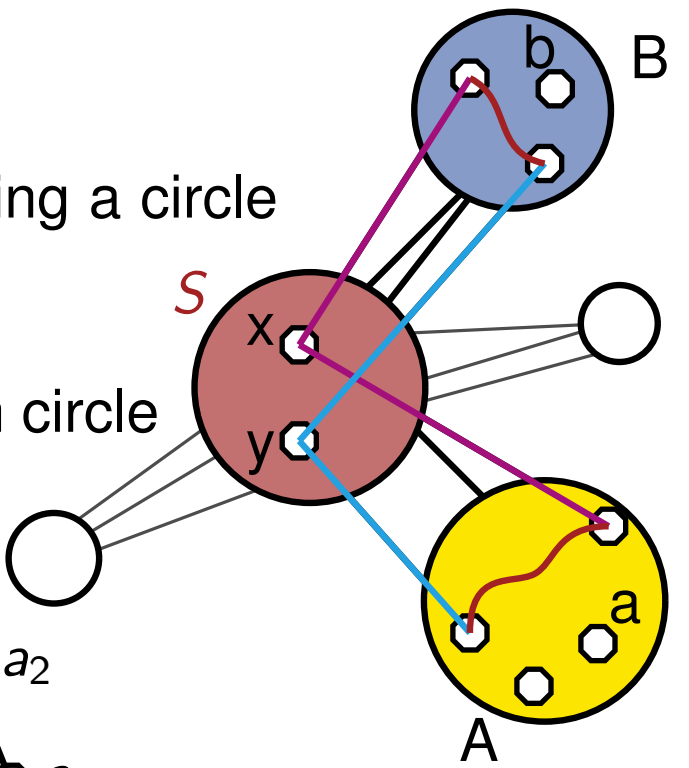
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Where is e?





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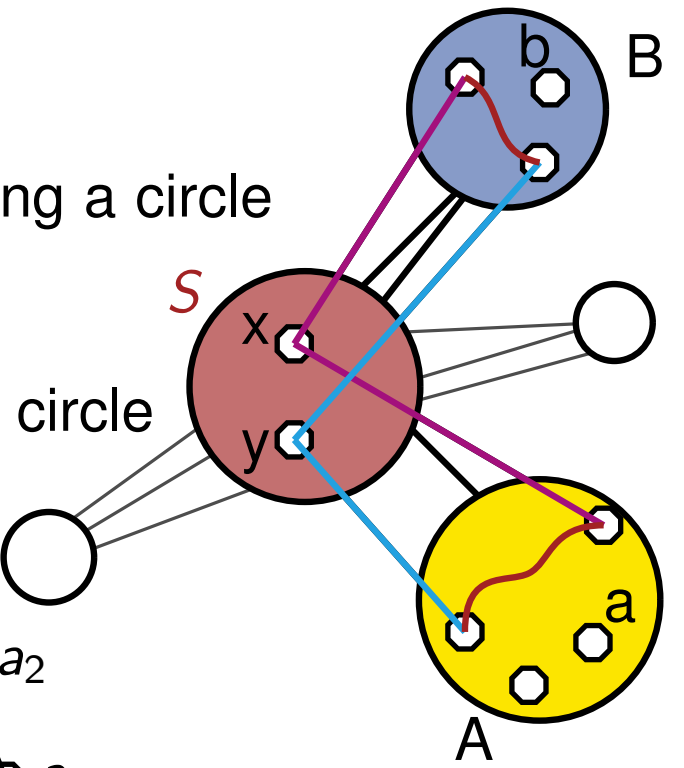
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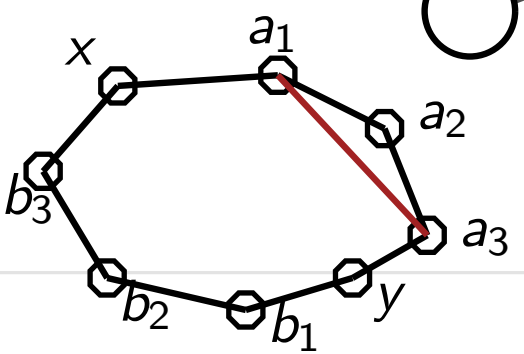
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C has at least 4 vertices $\Longrightarrow$ C has a chord $e \in G$

Where is e?

$e = a_i a_j$?





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If $|S| \leq 1$ then S clique

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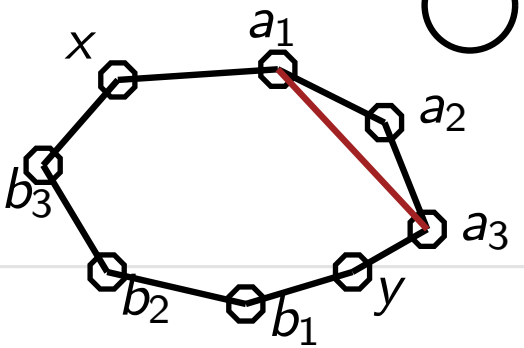
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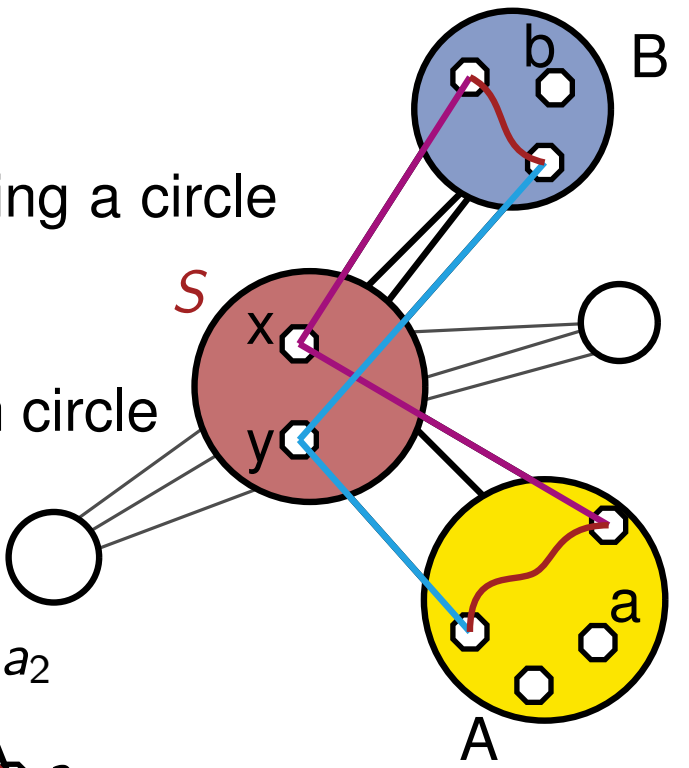
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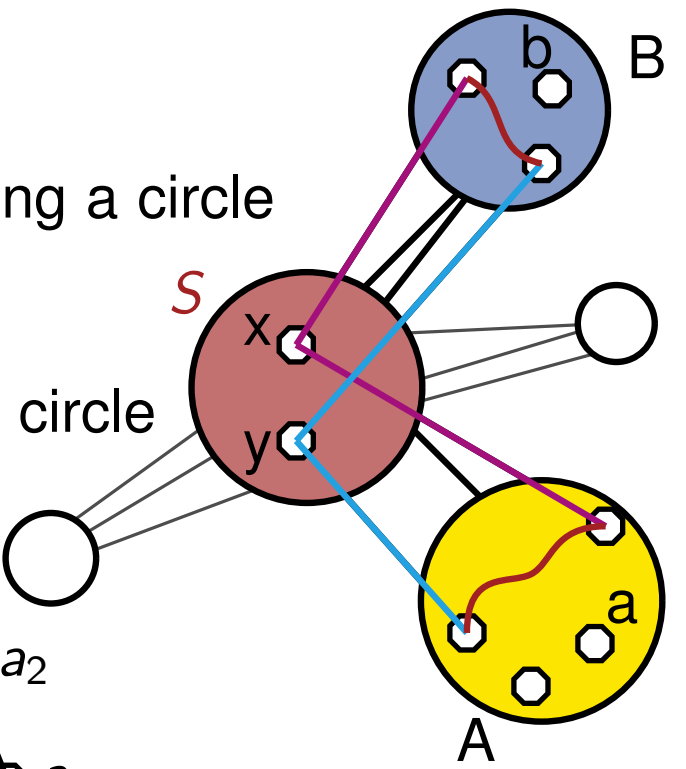
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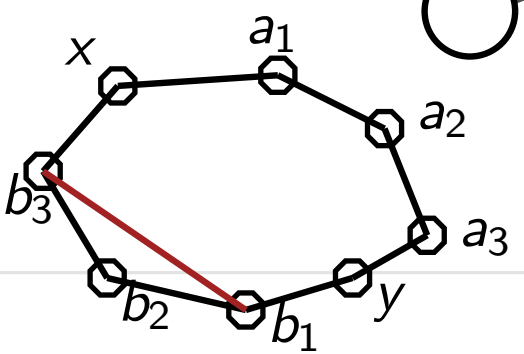
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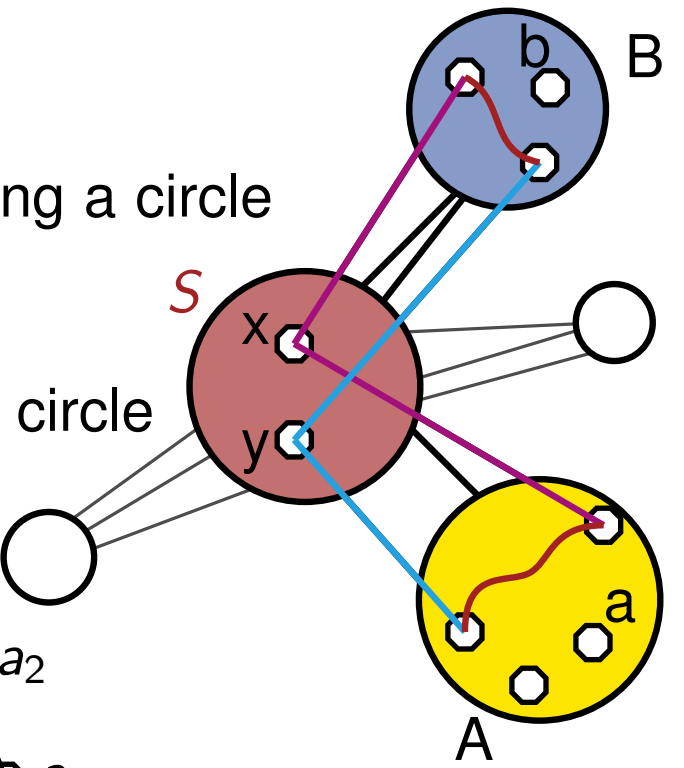
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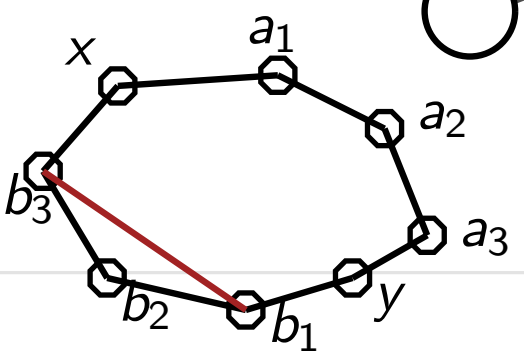
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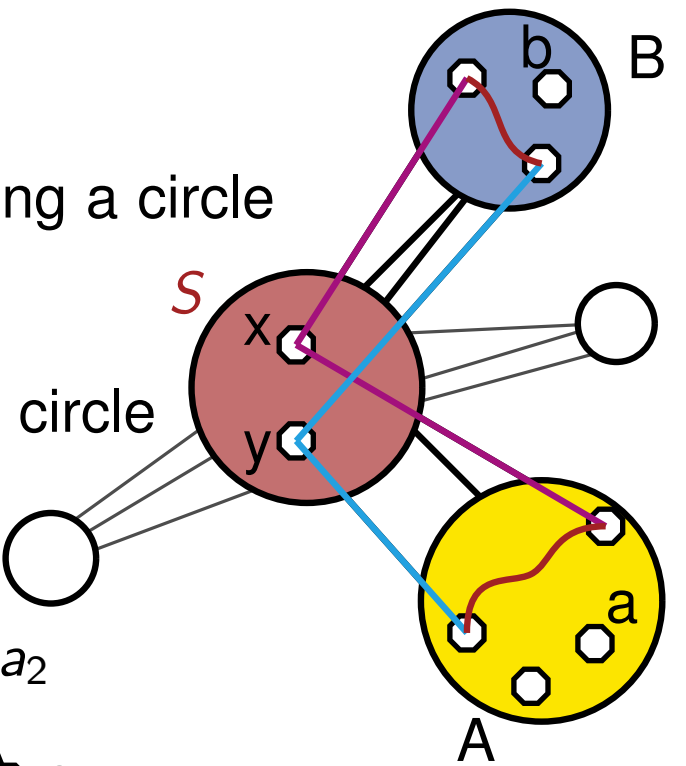
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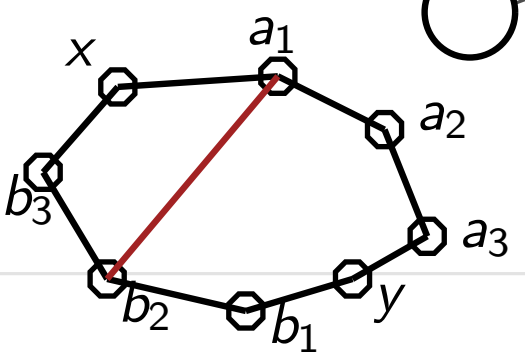
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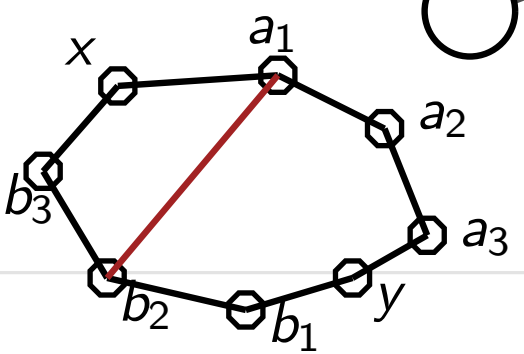
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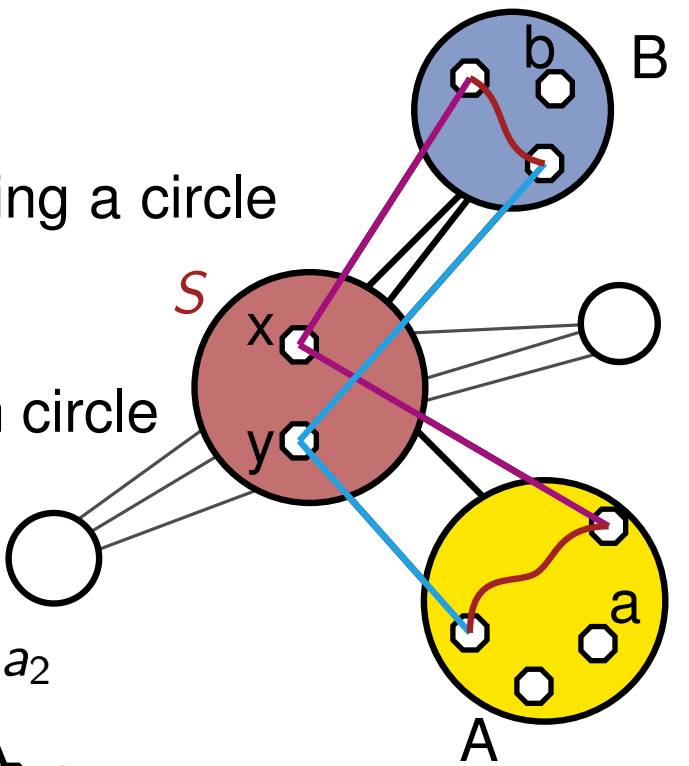
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No A and B are distinct components in $G - S$





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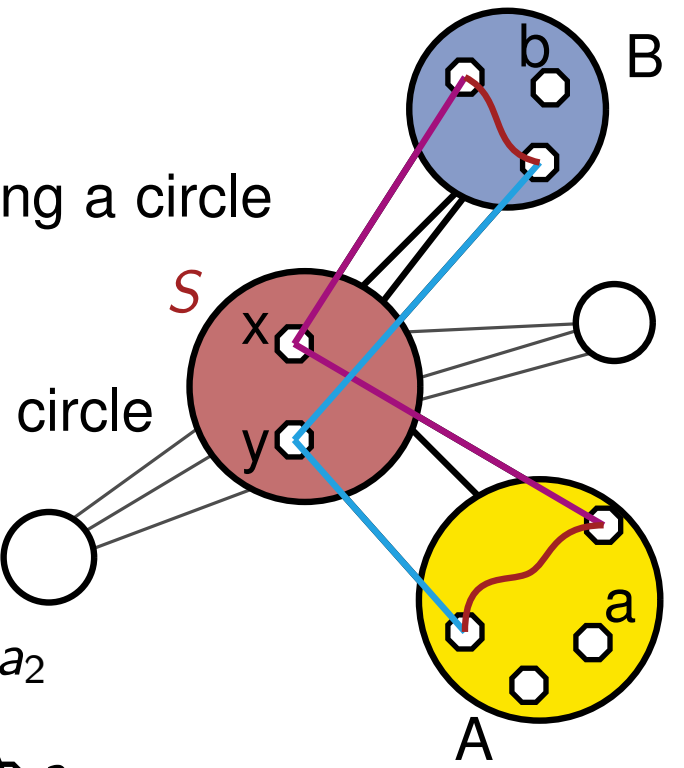
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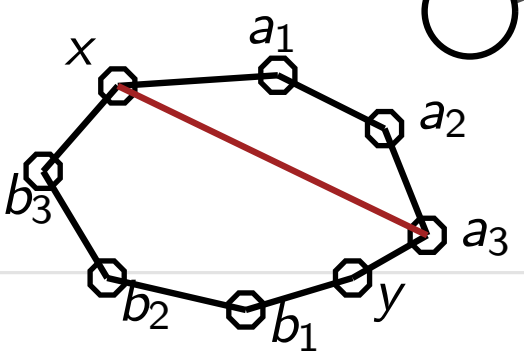
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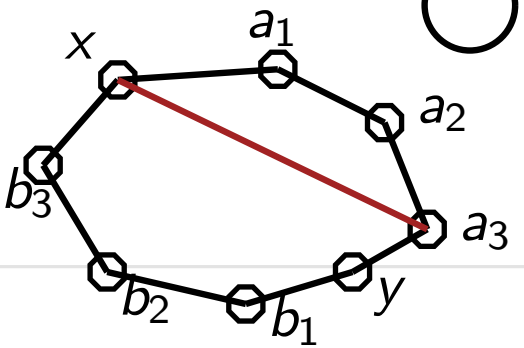
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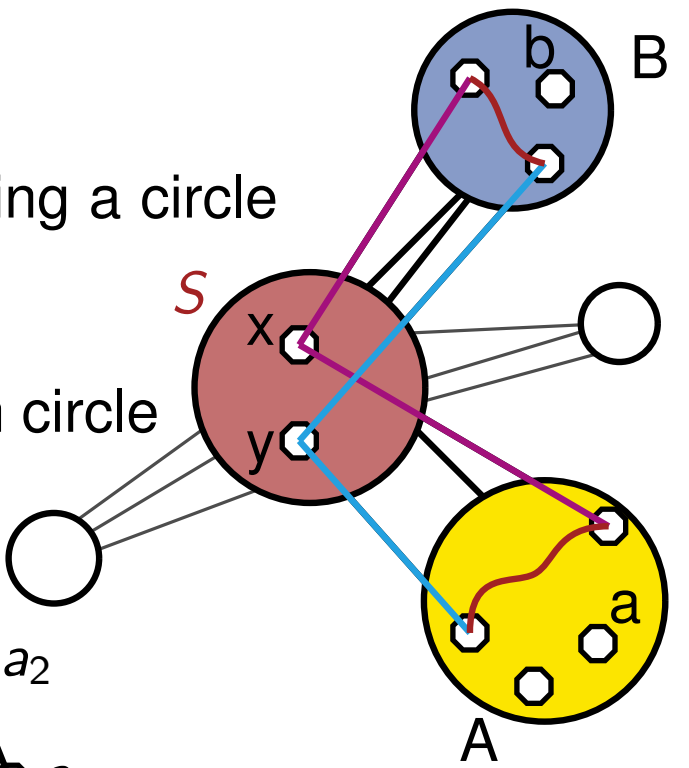
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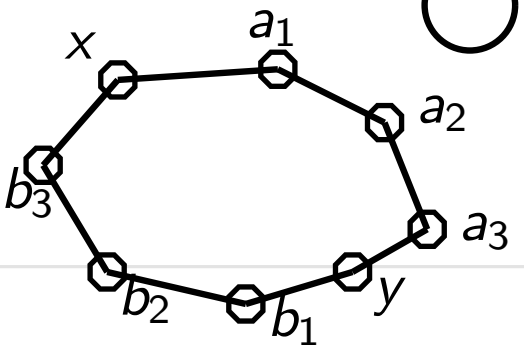
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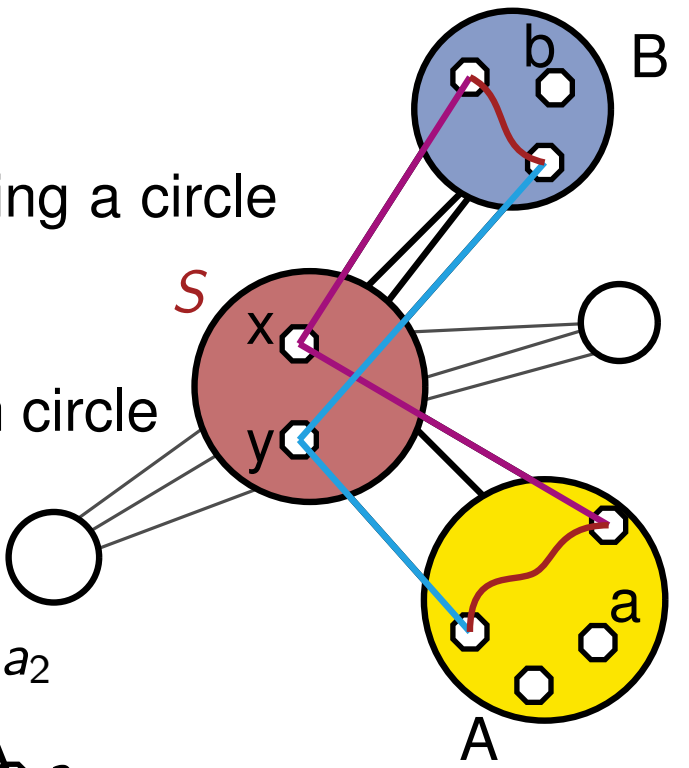
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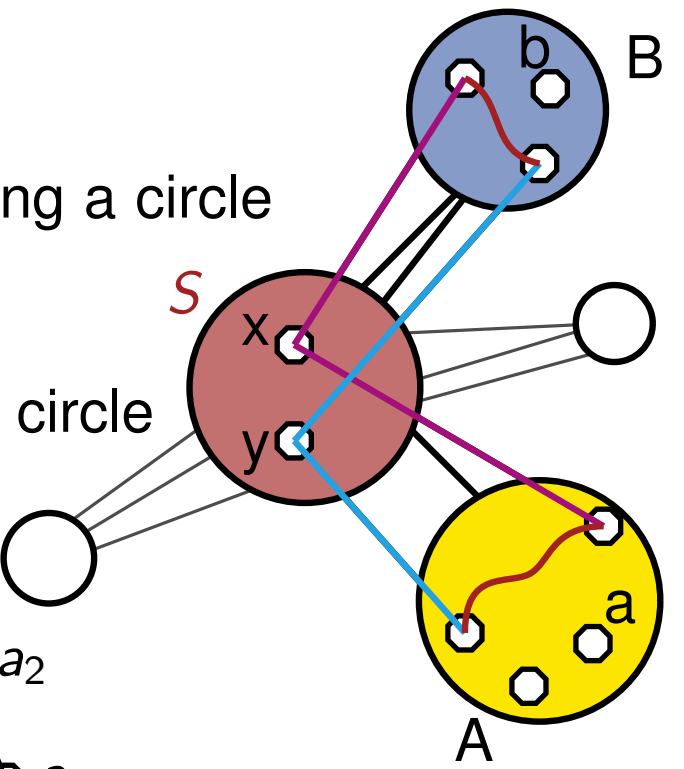
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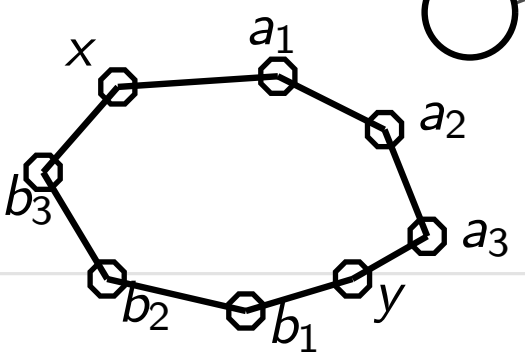
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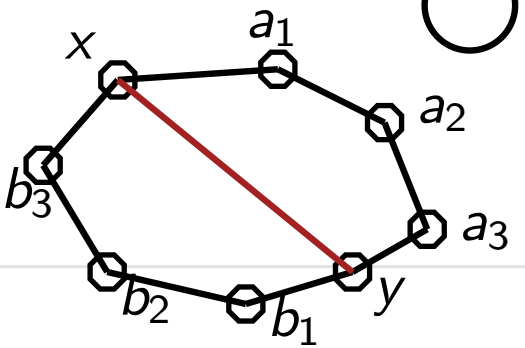
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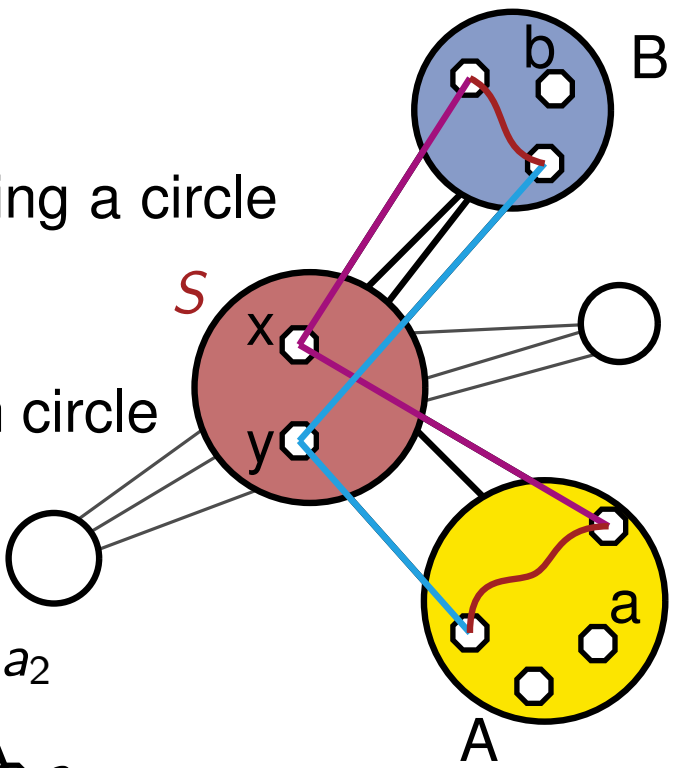
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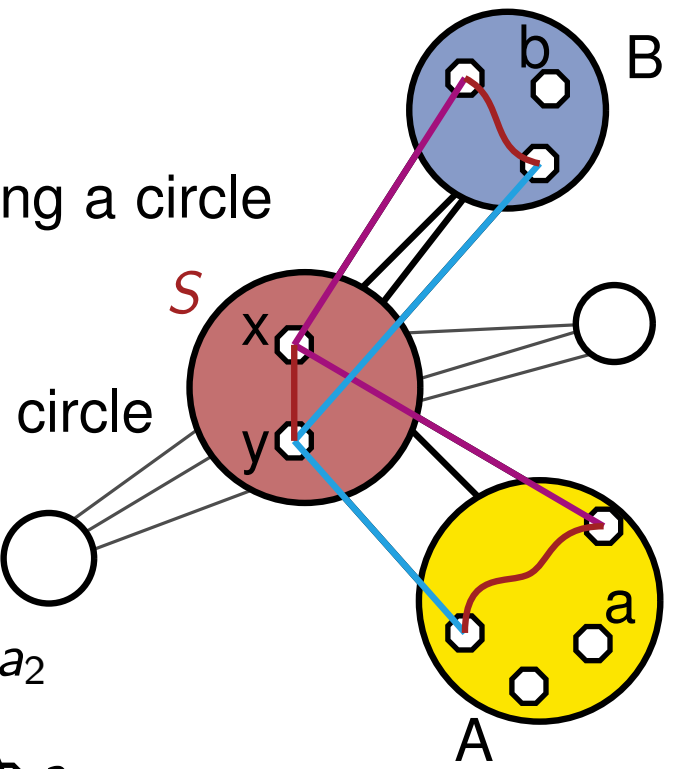
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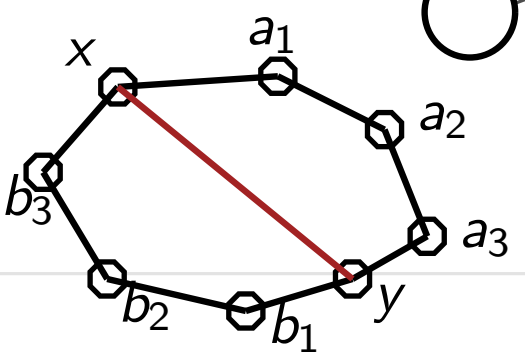
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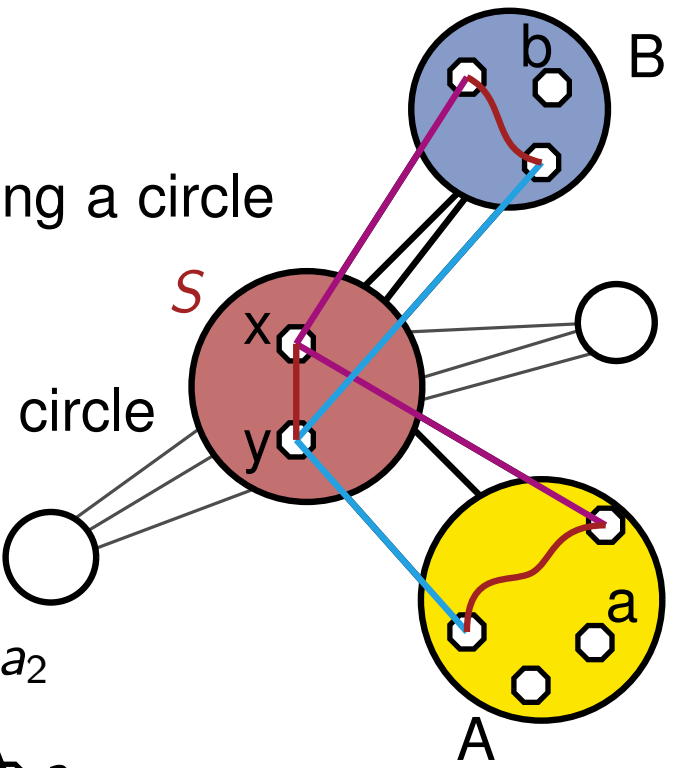
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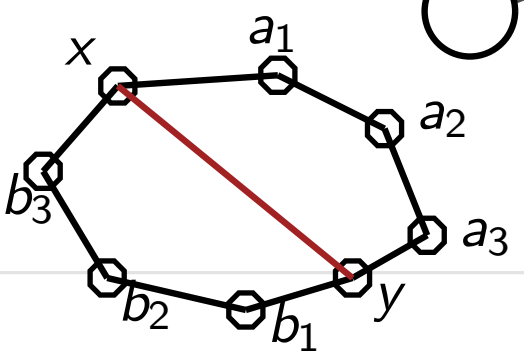
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Where is e?

e must be $xy \in E$

since this holds for all $x \neq y \in S$, S must be a clique





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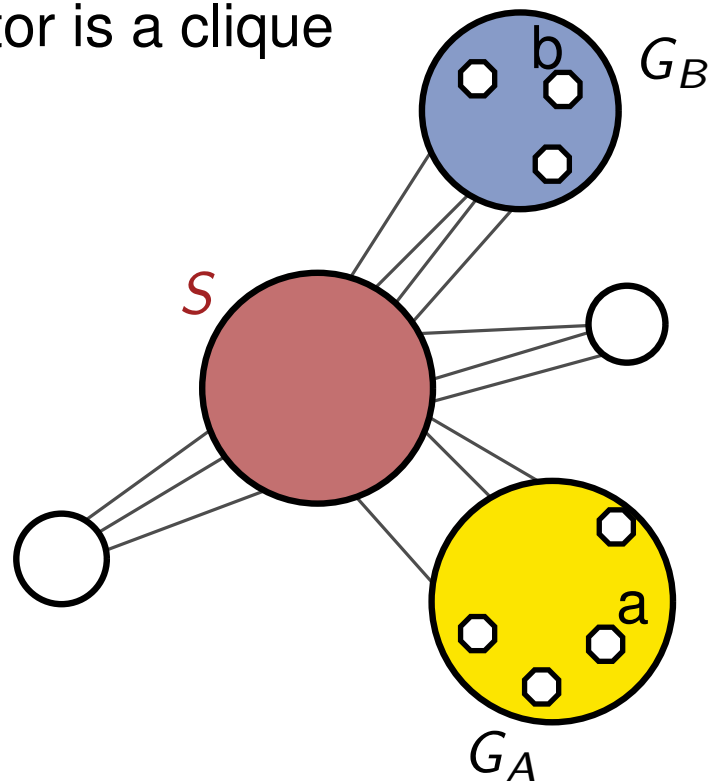
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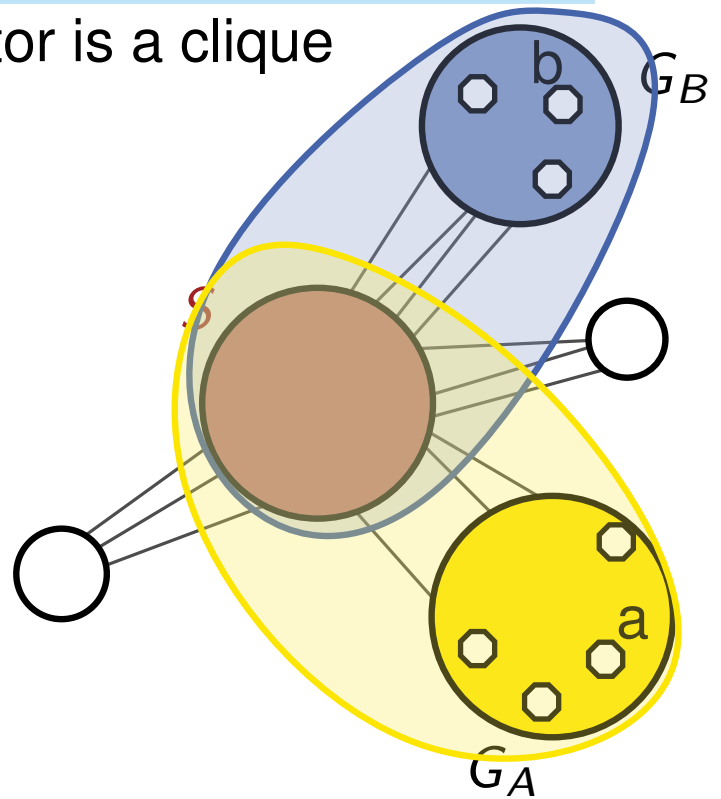
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and $a \in G_A$, $b \in G_B$ two components in $G - S$

apply induction on G_{S+a} and G_{S+B}



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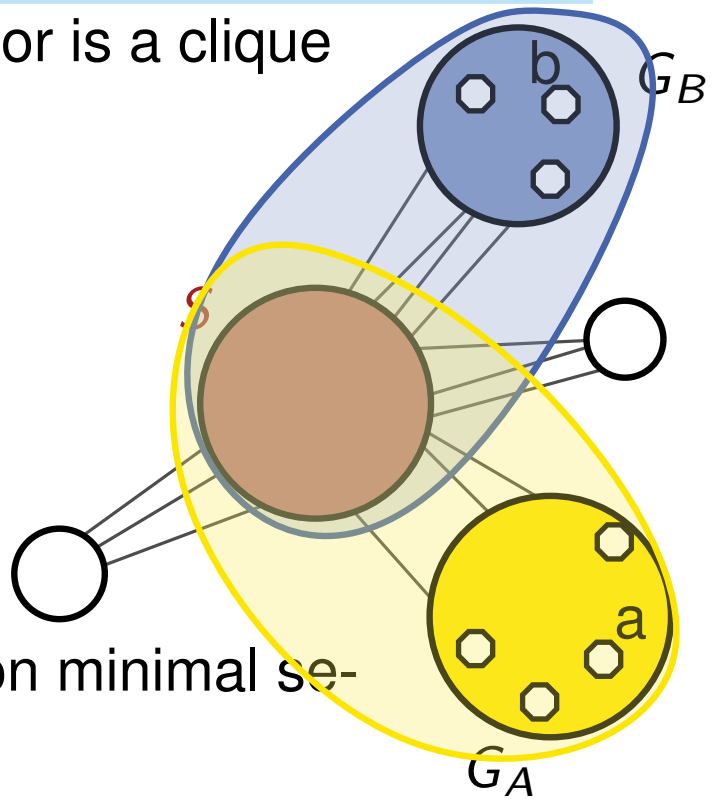
case $G \ncong K_n$:

Let $a \neq b \in V(G)$ with $ab \in E(G)$

Let S be an inclusion minimal a, b seperator

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apply induction on G_{S+a} and G_{S+B}



any inclusion minimal seperator in G_X is also an inclusion minimal seperator in G and a clique

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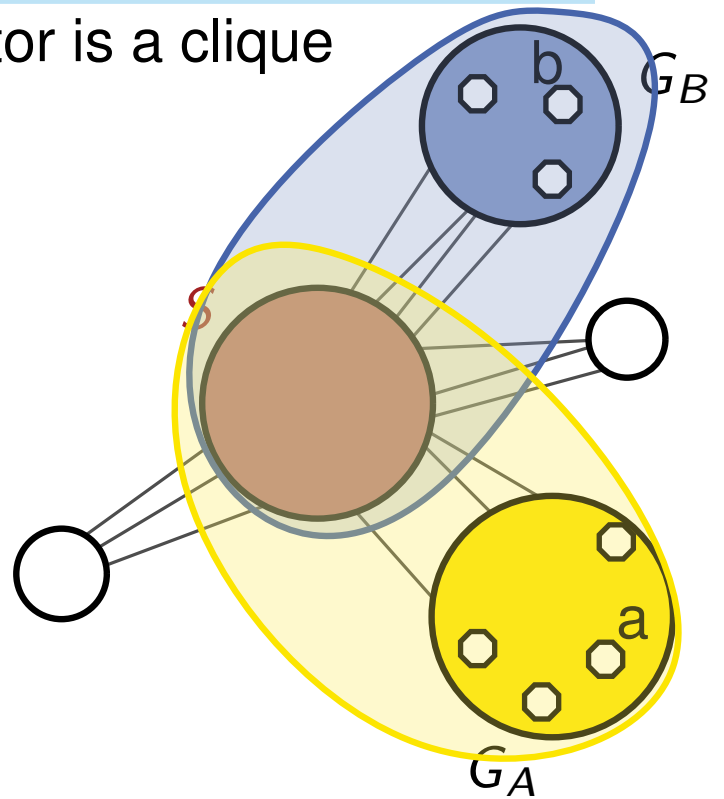
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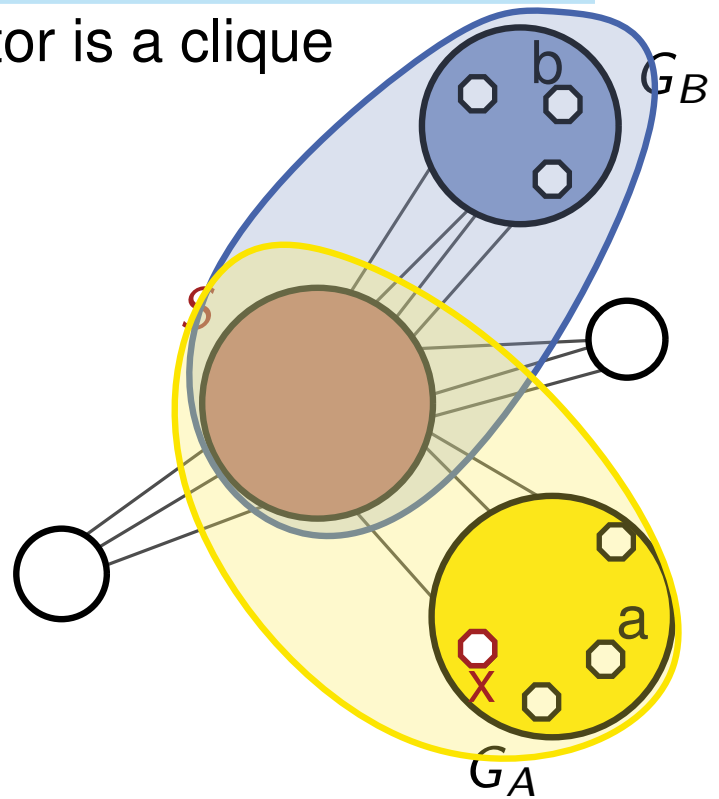
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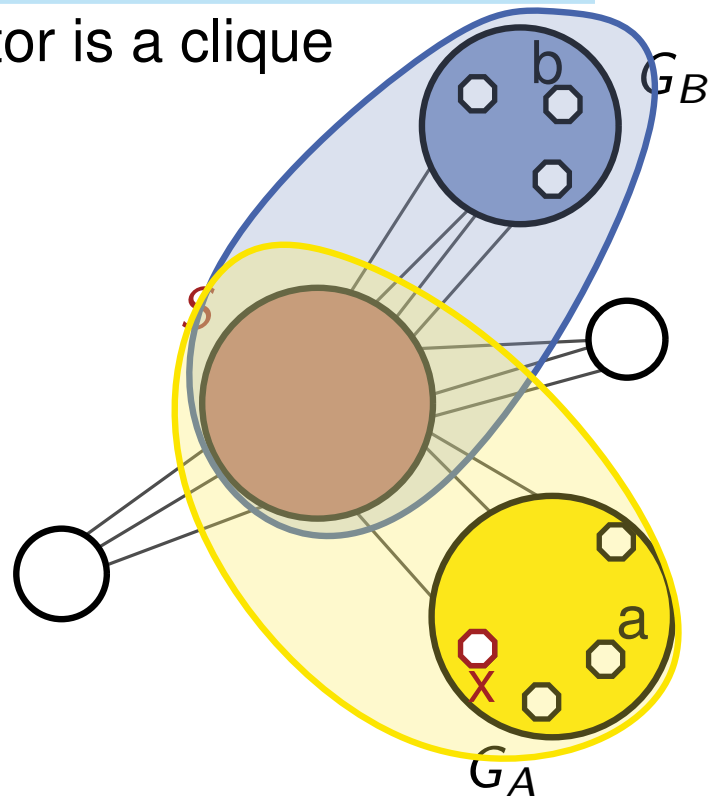
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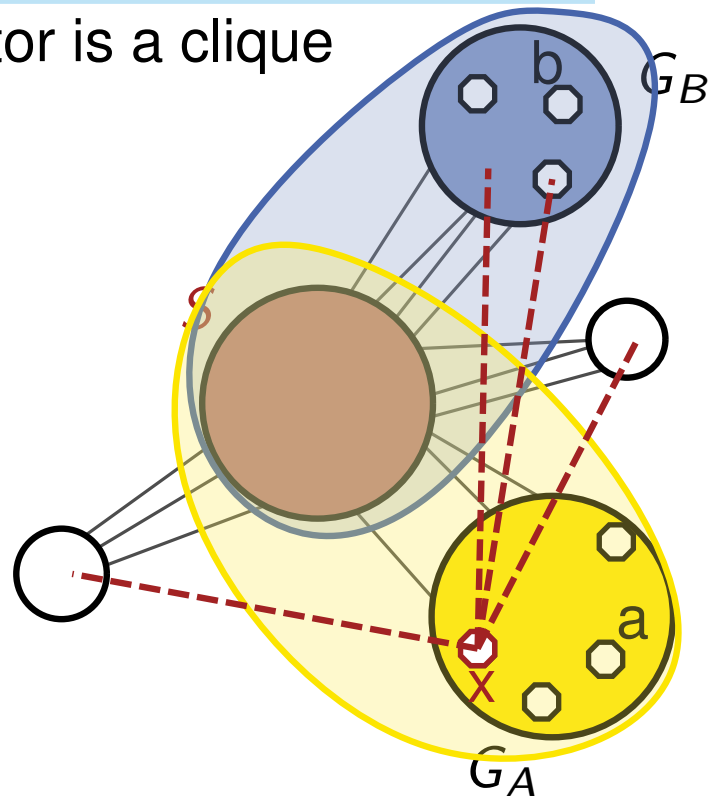
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no edges between different components in $G - S$ possible



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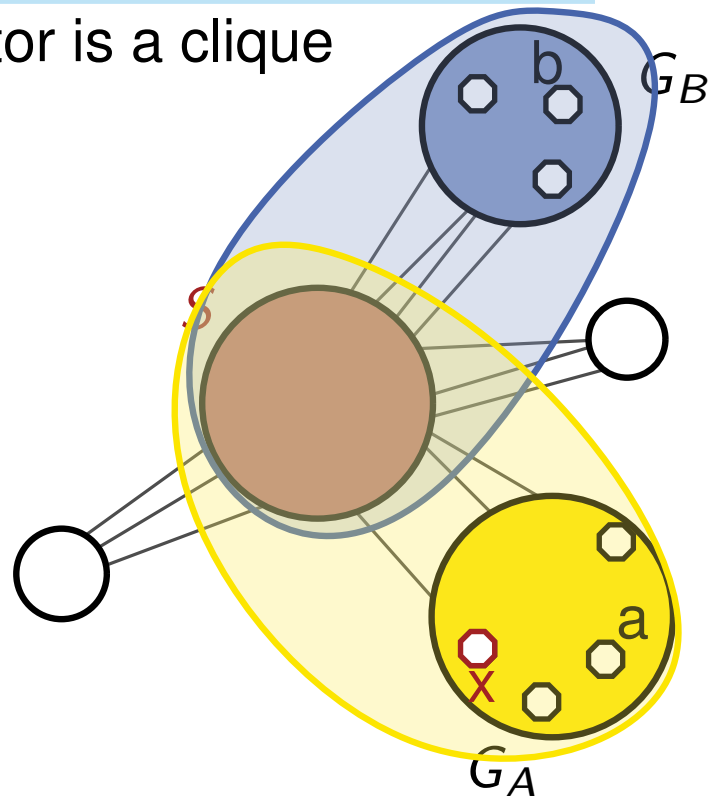
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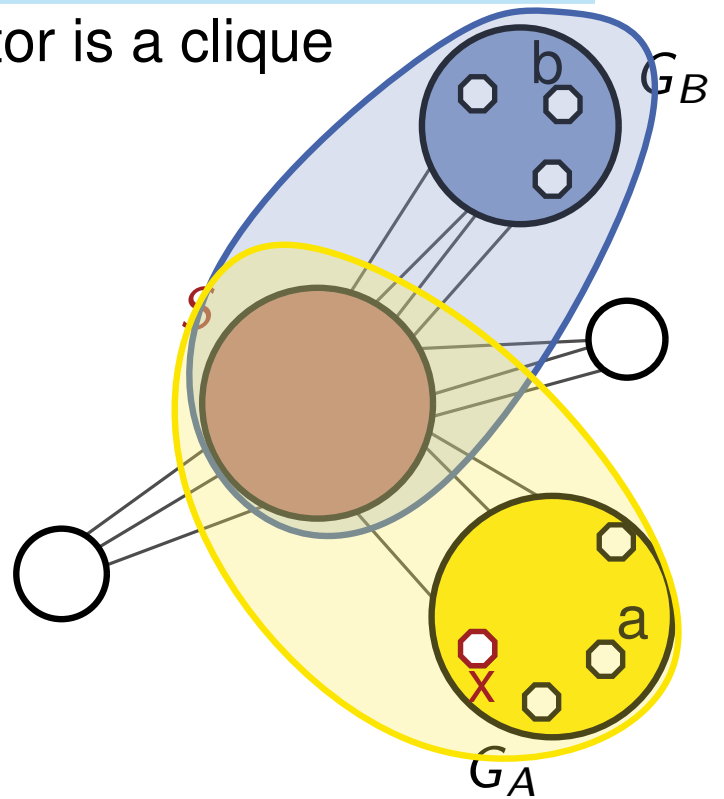
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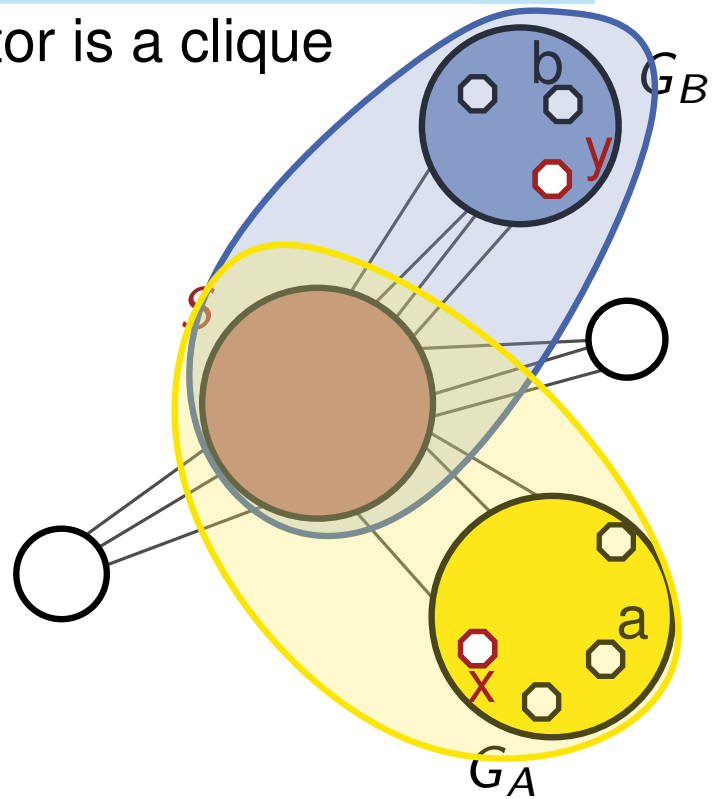
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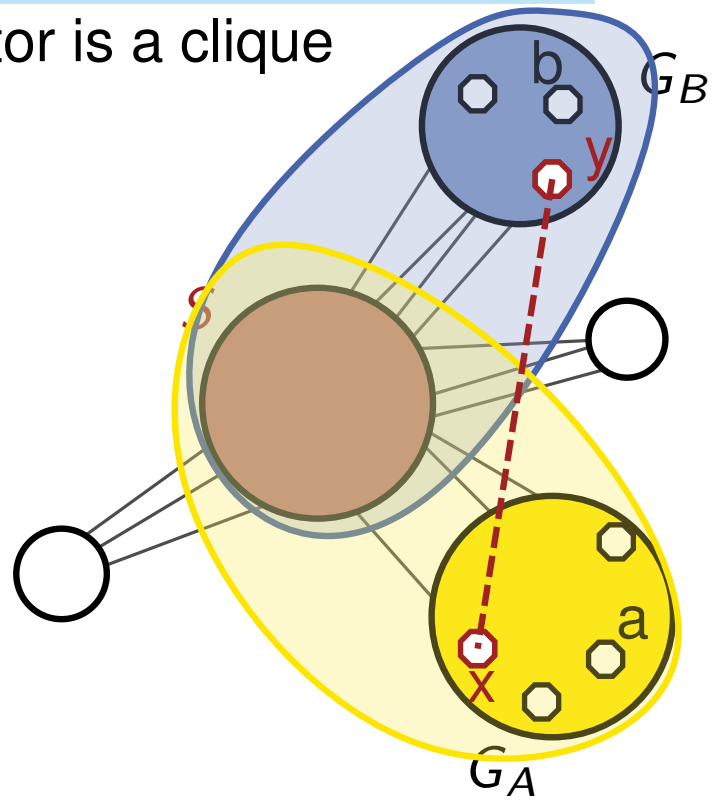
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$xy \notin E(G)$ since otherwise in same component



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until now we have proven

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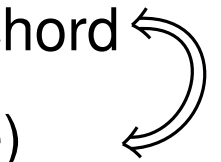
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Lemma 3.4

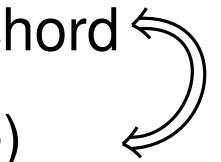
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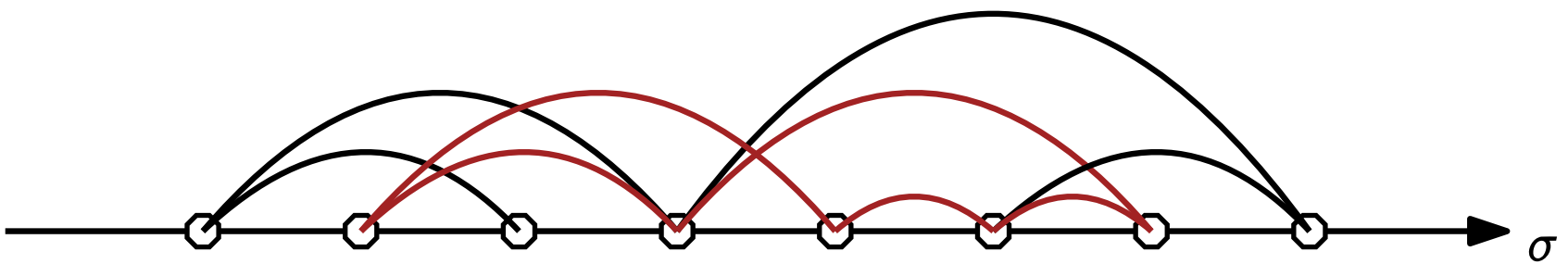
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Proof of $(*)$

G graph, σ PES, C cycle of length ≥ 4

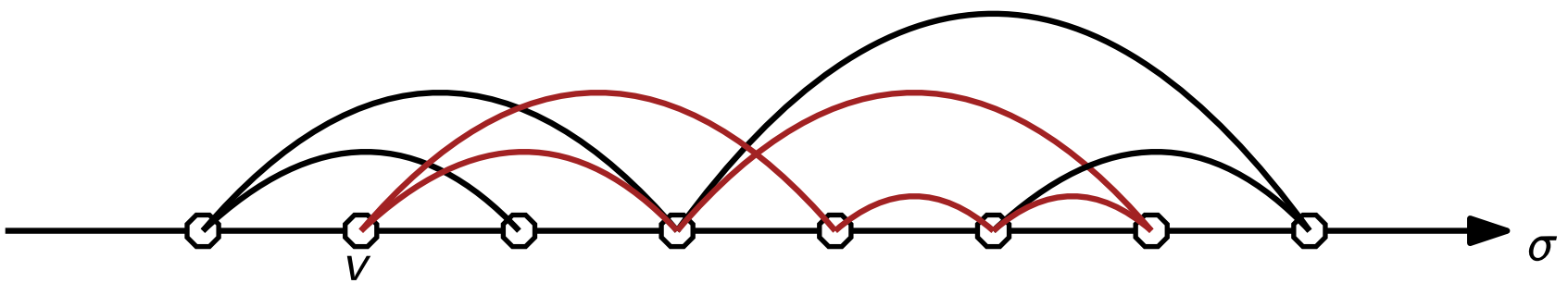
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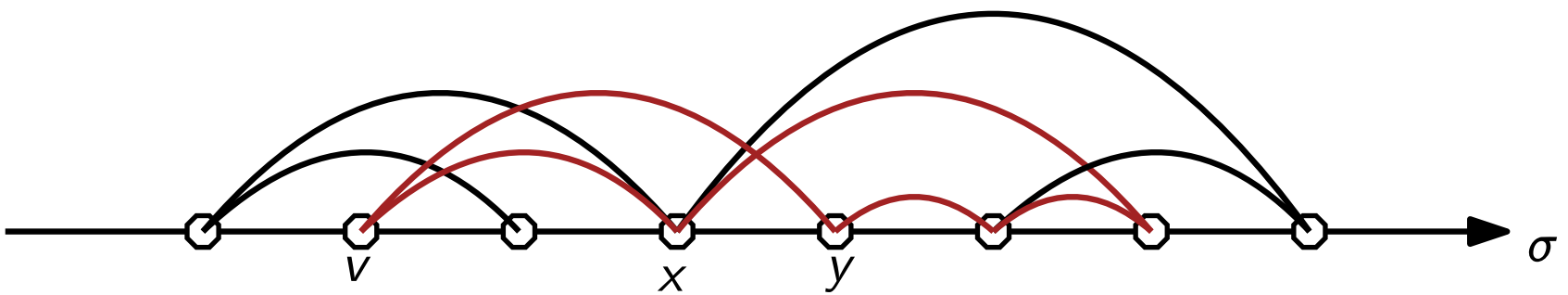
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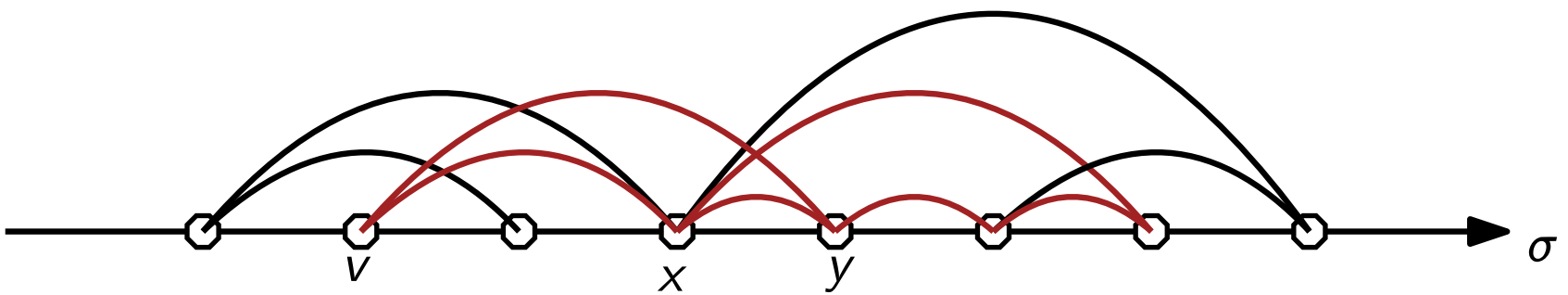
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since σ PES $xy \in E(G)$. We found a chord since not every pair x, y can be after each other in the circle



Structure of LexBFS results (L3)

Lemma

Let $\sigma \leftarrow \text{LexBFS}(G)$.

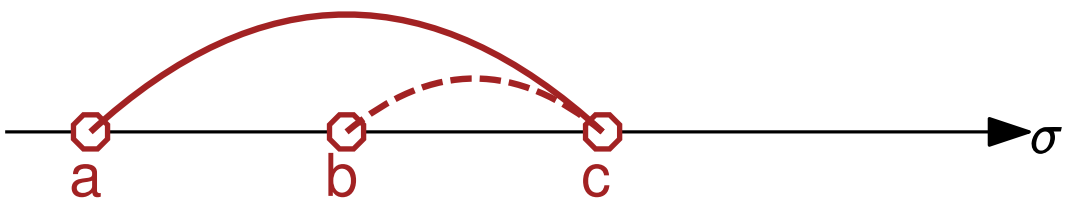
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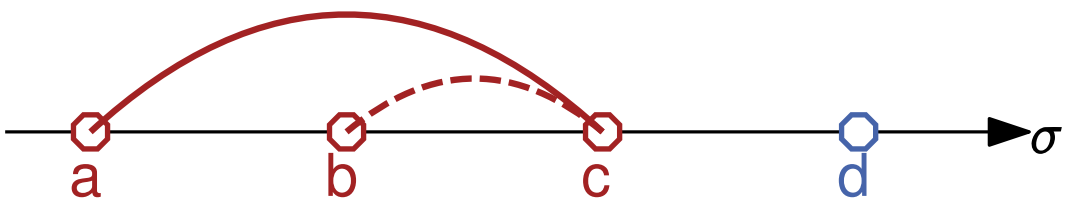


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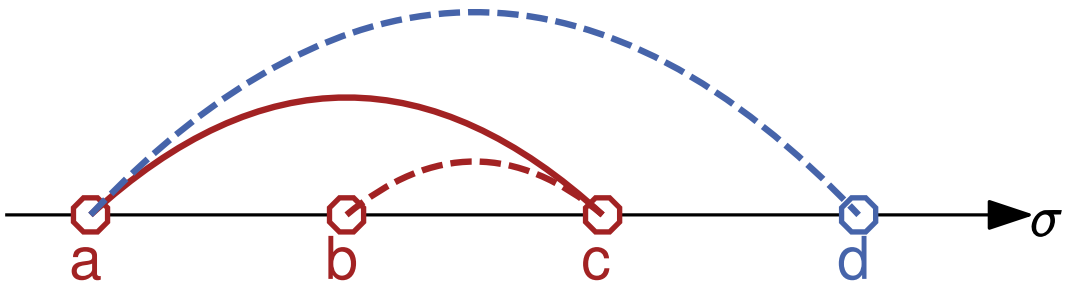


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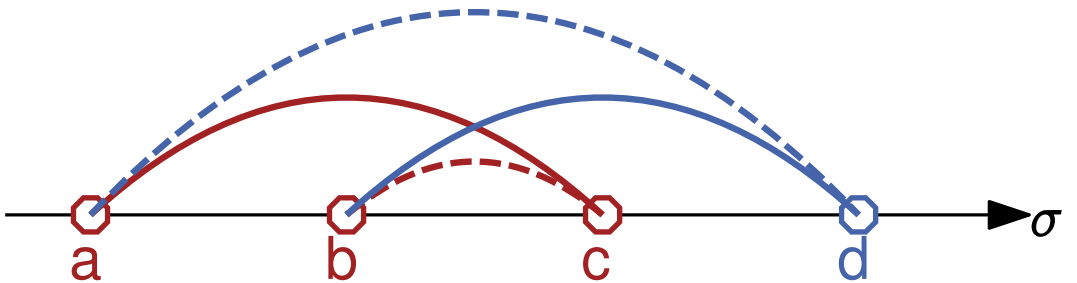


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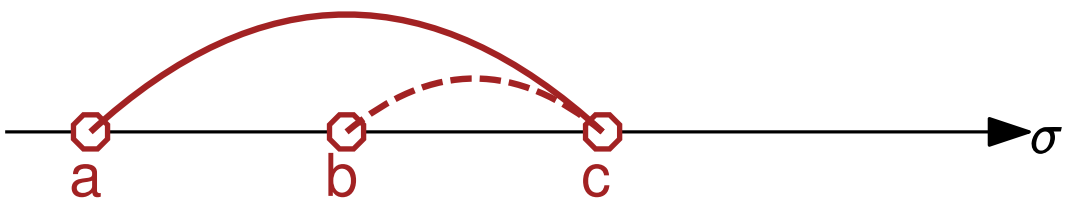


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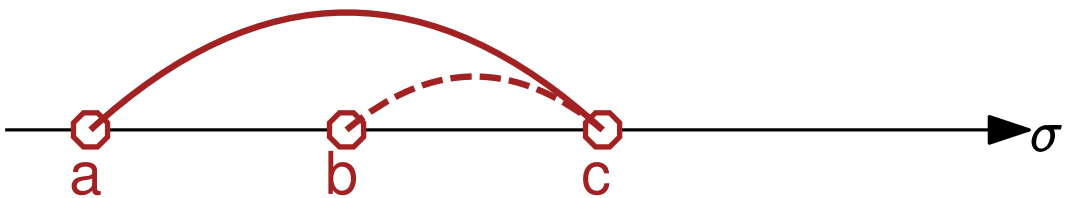
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consider the iteration i of LexBFS where c is added to σ



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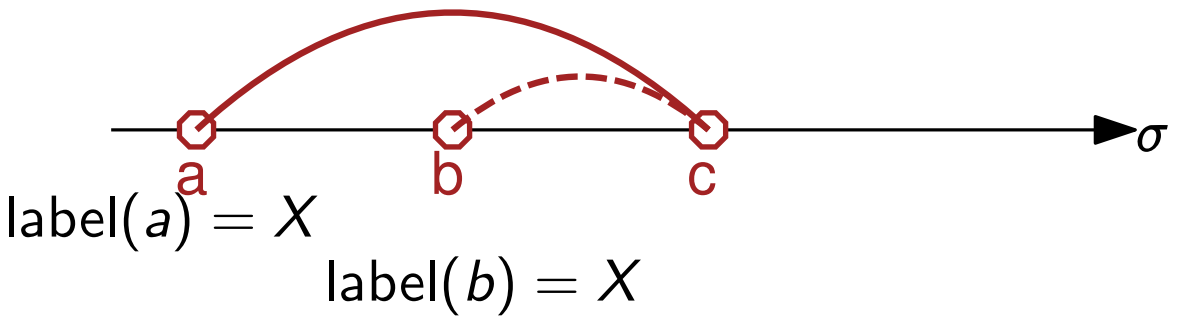
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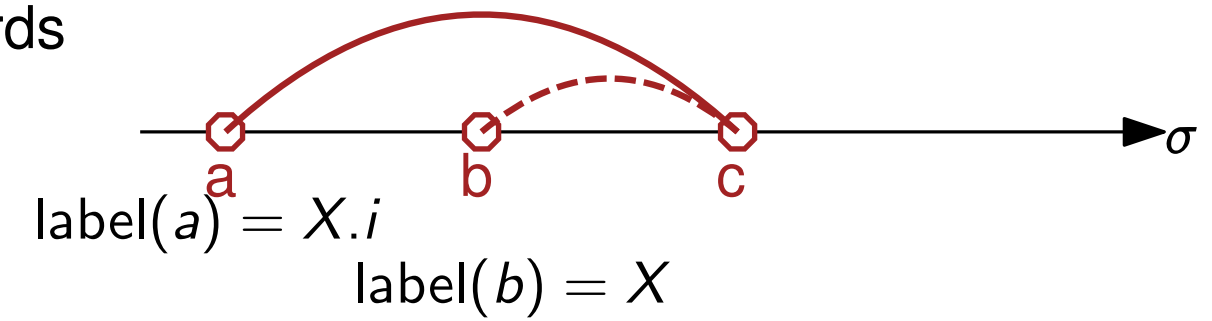
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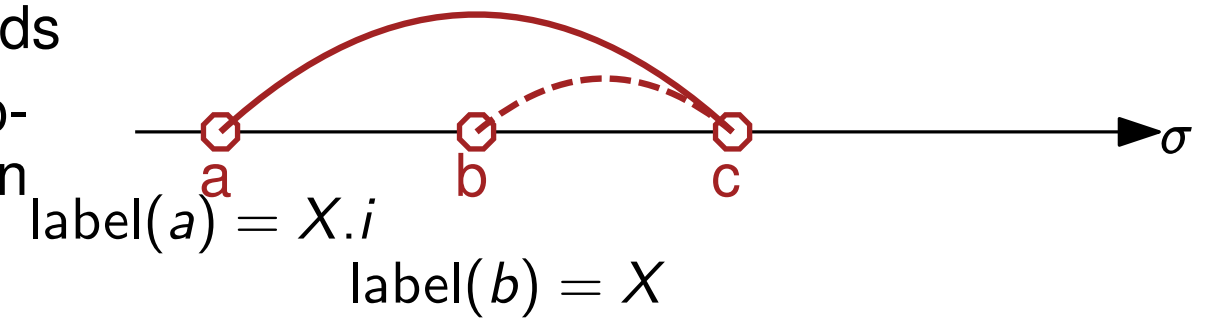
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this would still hold when b is processed since only smaller i's can be appended afterwards



Structure of LexBFS results (L3)

Lemma

Let $\sigma \leftarrow \text{LexBFS}(G)$.

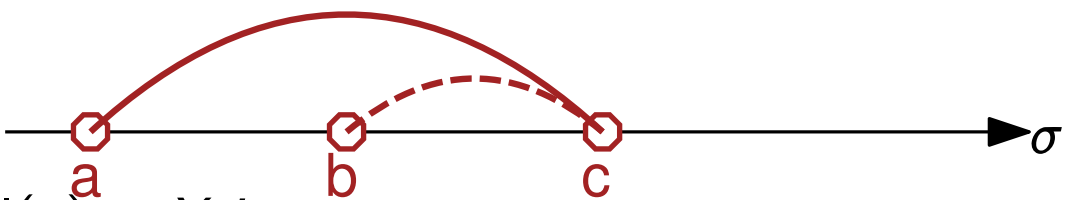
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Zeichen in w & $v < i$

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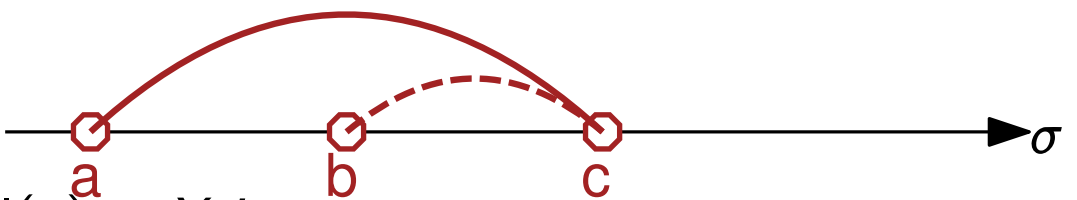
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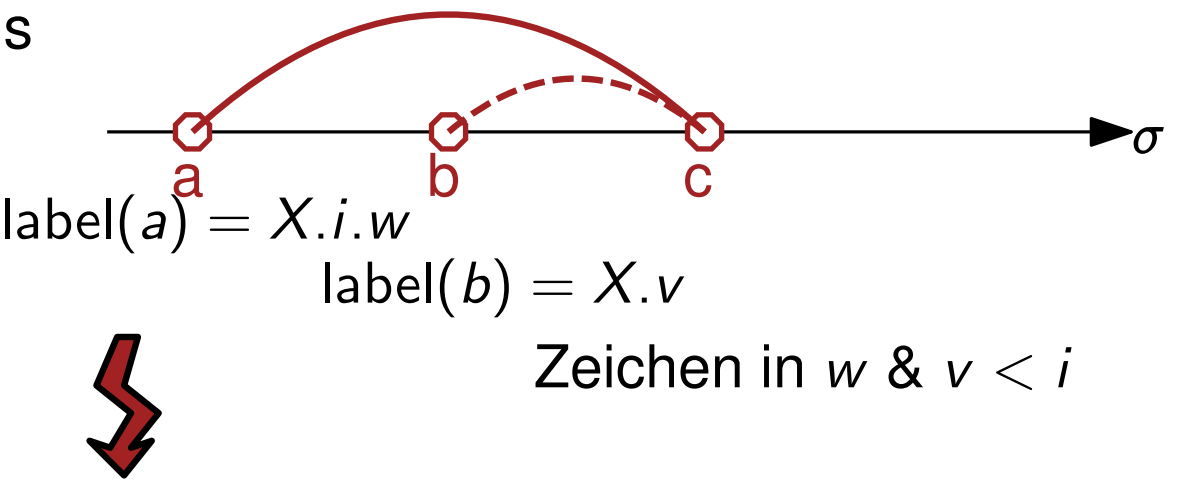
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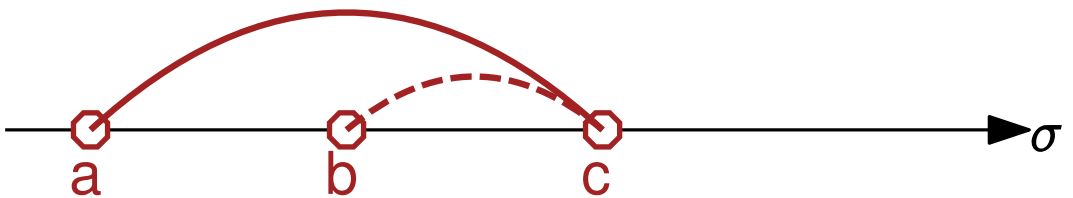
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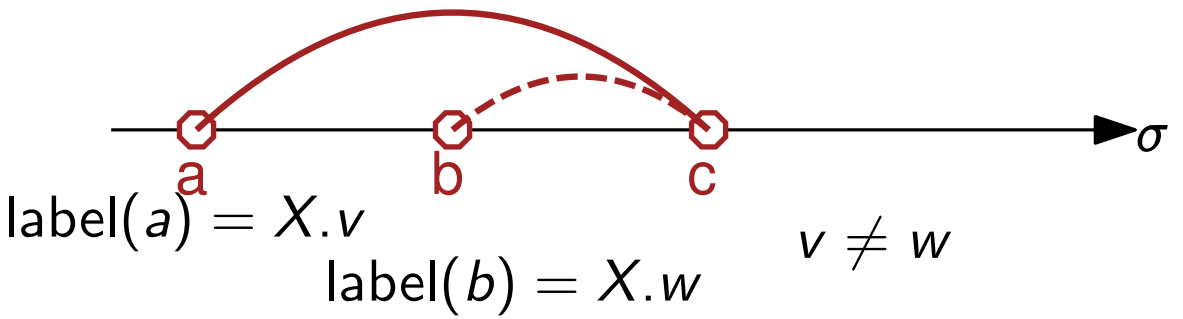
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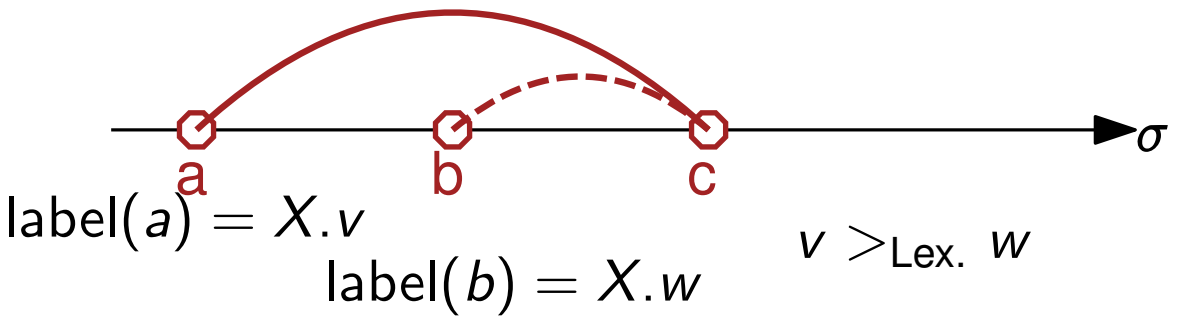
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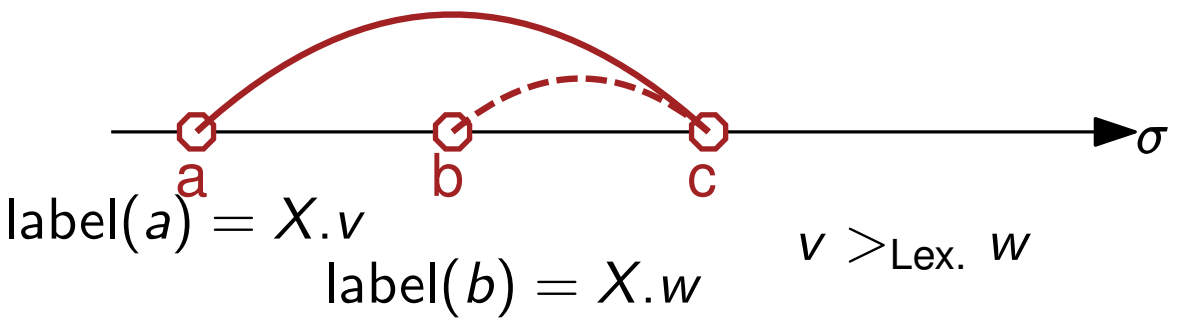
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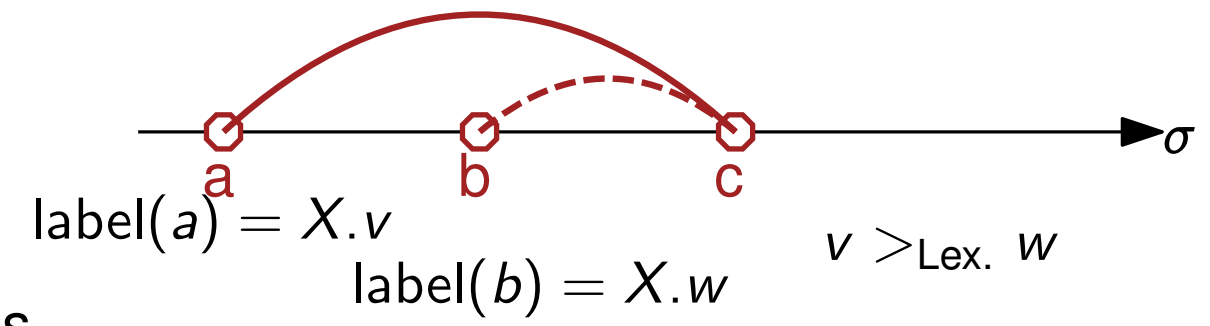
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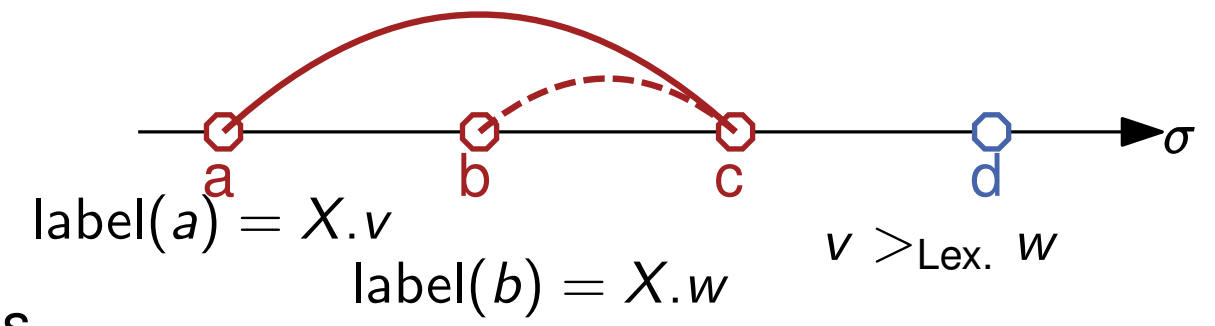
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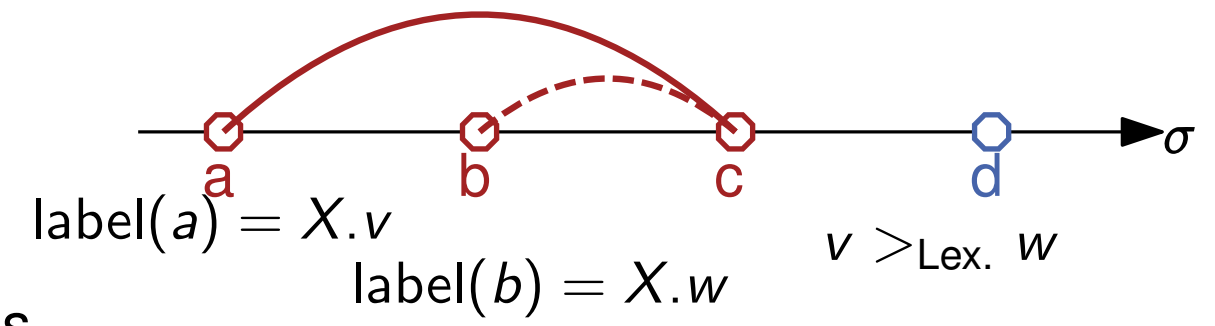
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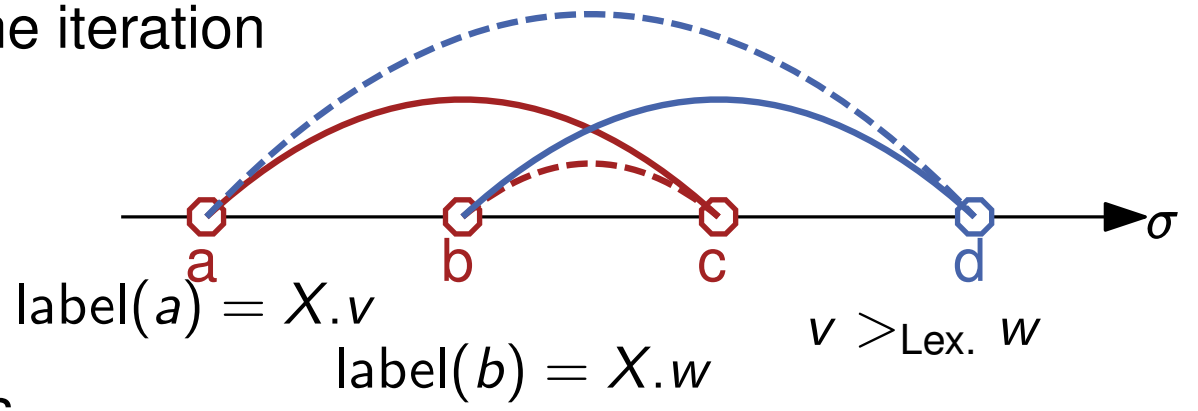
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Theorem 3.9

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G is chordal $\iff$ LexBFS outputs perfect elimination scheme (PES)

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"$\Leftarrow$"

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"$\Leftarrow$" clear since, G is chordal $\iff$ PES exists

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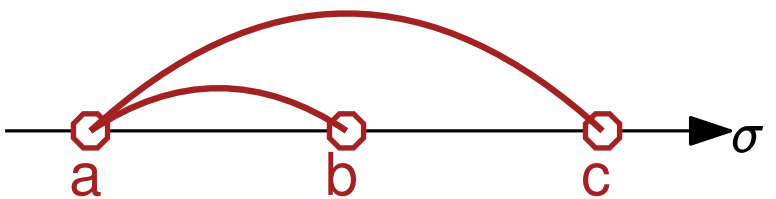
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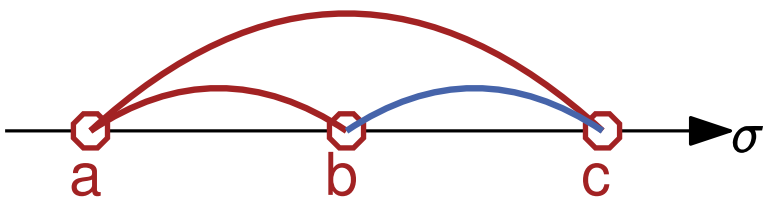
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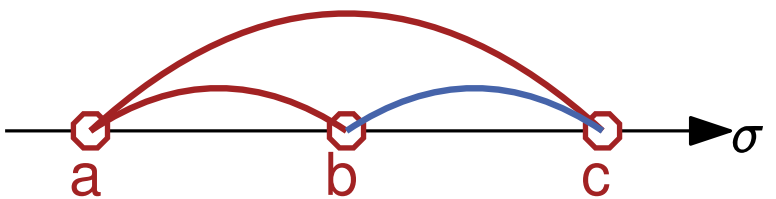
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this is just a reformulation of the definition that every right neighbourhood must be a clique

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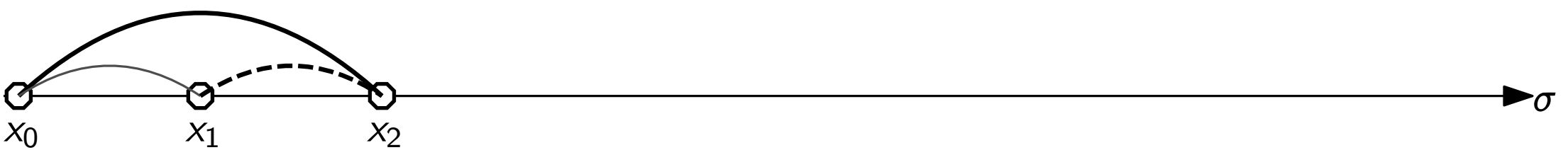
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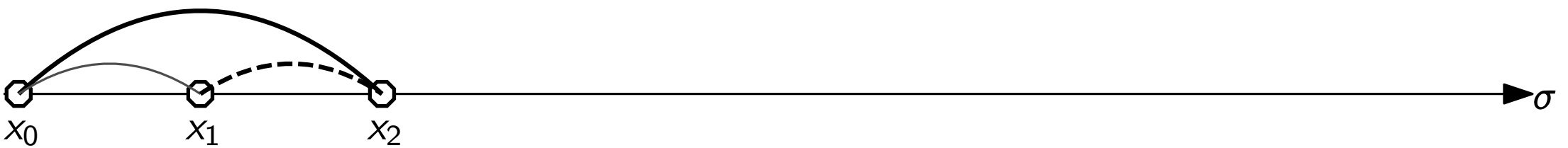
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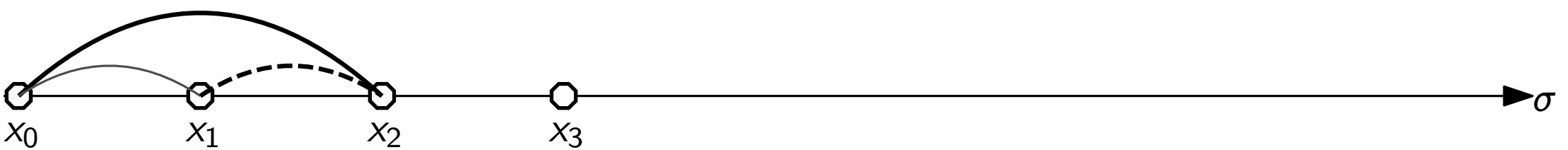
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G is chordal $\iff$ LexBFS outputs perfect elimination scheme (PES)

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choose right-most x_2

since σ is LexBFS result (L3) holds and there is an x_3:

choose right-most such x_3

- $x_2 <_\sigma x_3$
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$x_3x_1x_0x_2$ induzieren P_4



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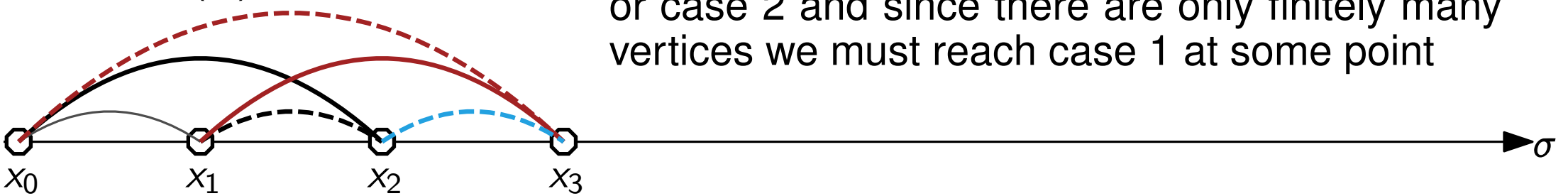
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we will now inductively step through this argument. In each step we must be either in case 1 or case 2 and since there are only finitely many vertices we must reach case 1 at some point



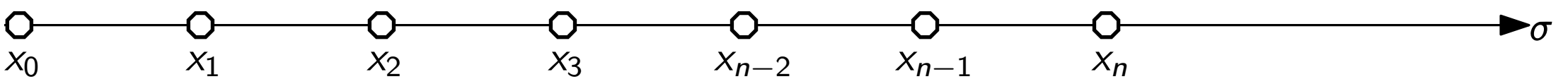
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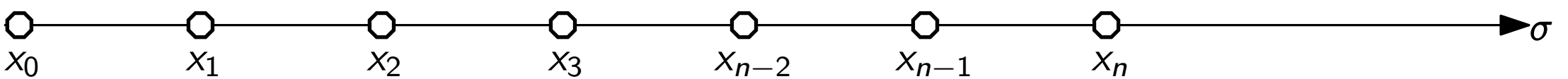
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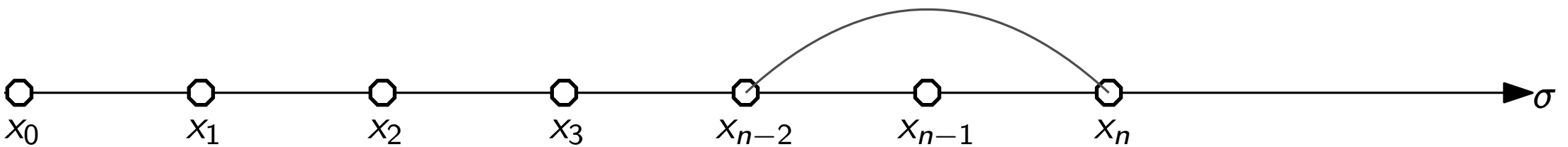
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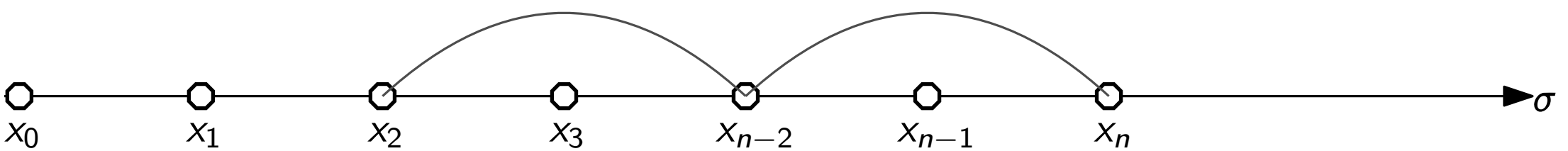
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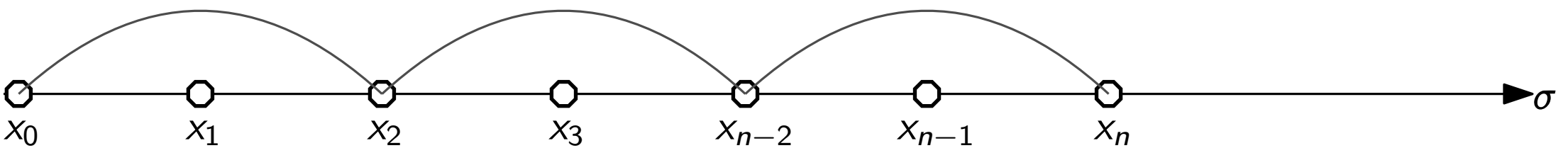
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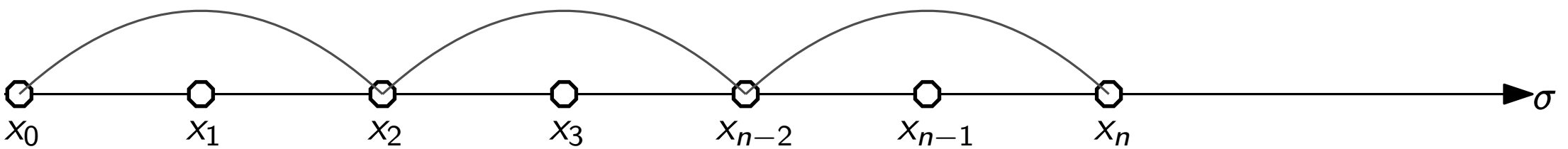
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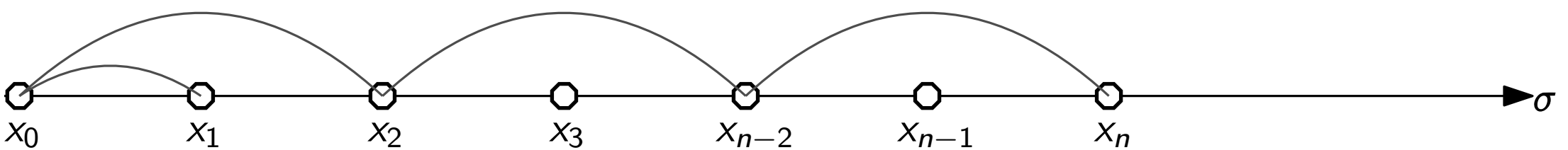
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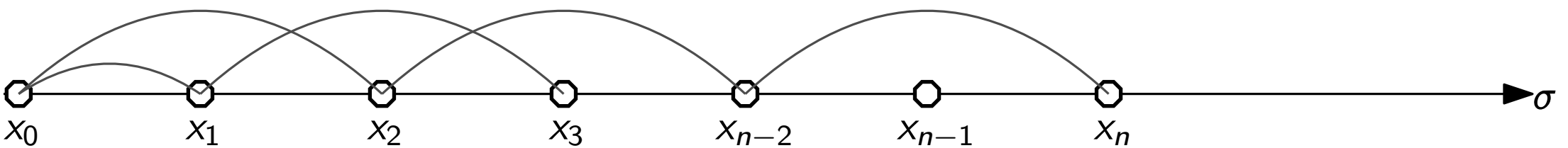
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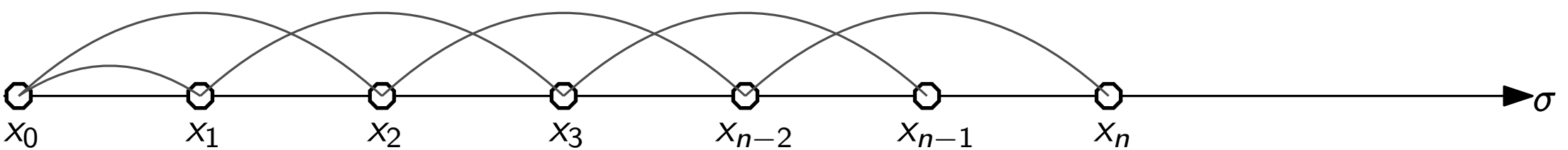
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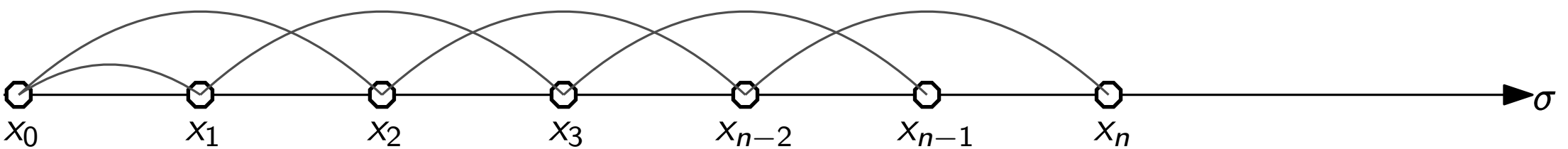
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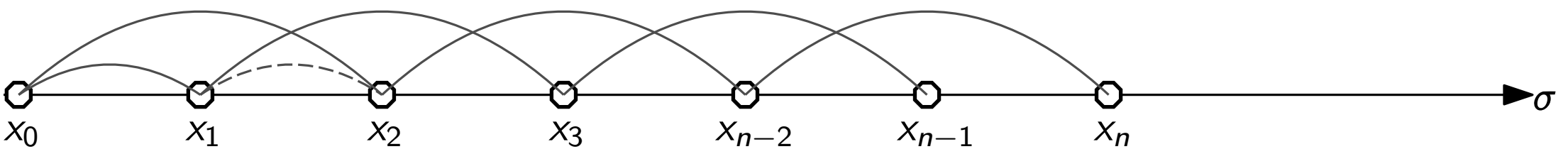
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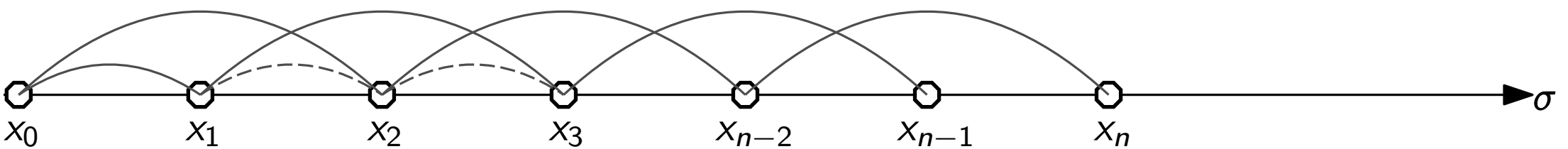
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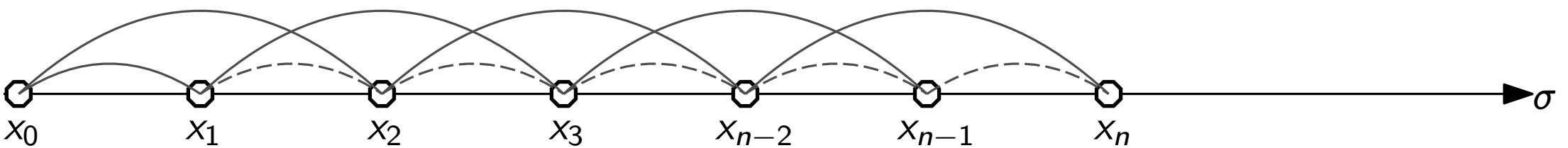
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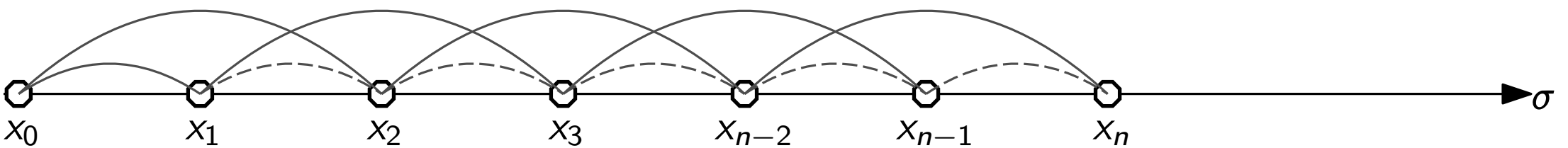
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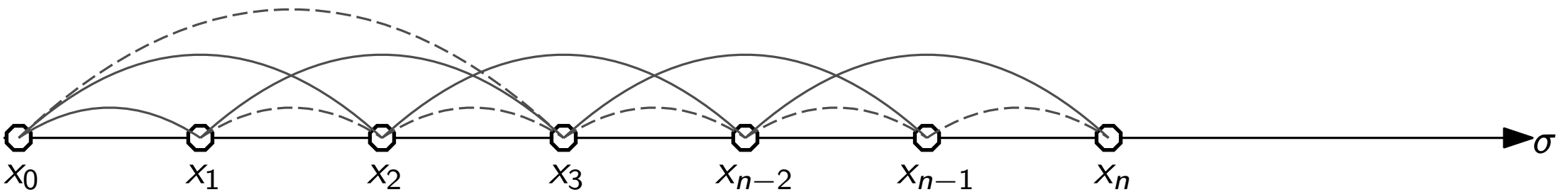
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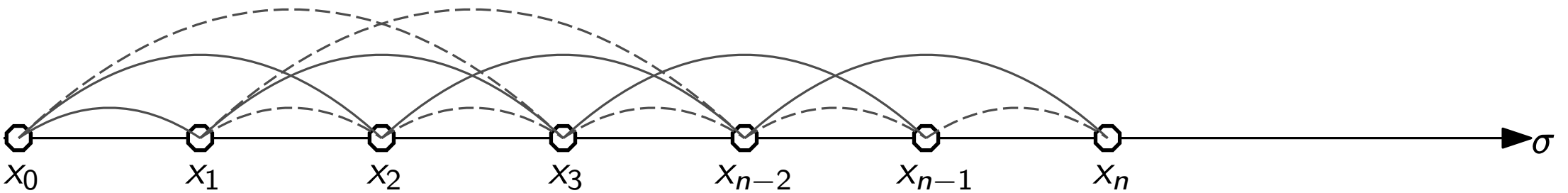
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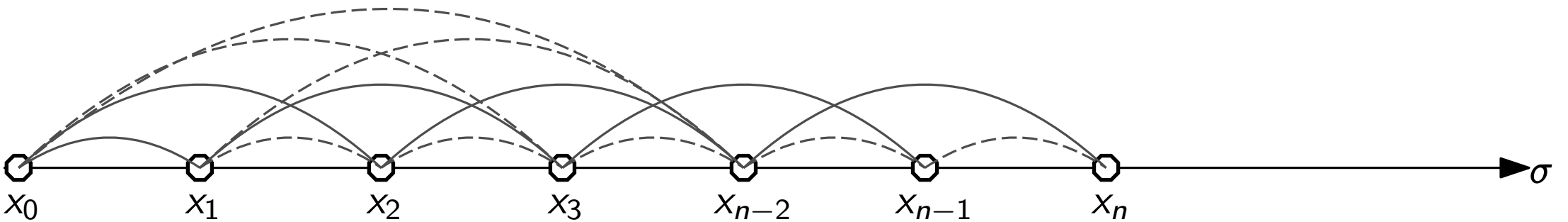
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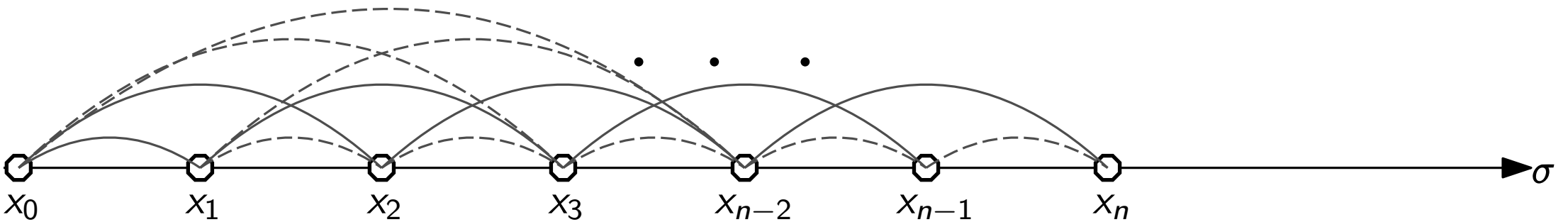
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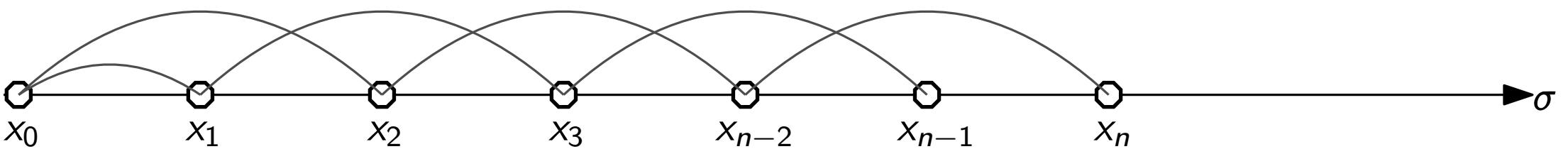
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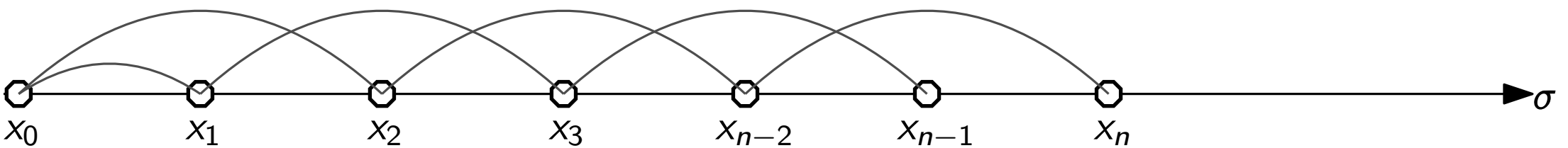
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Theorem 3.9

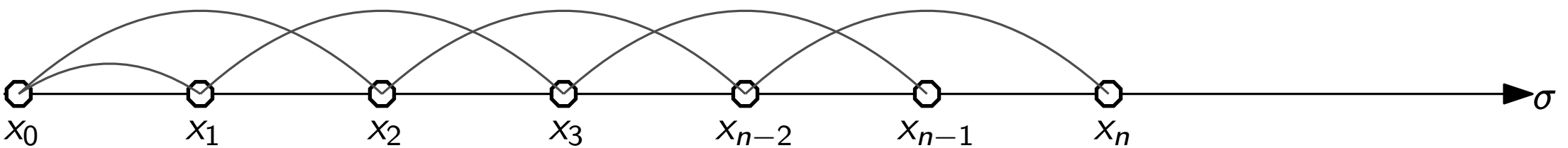
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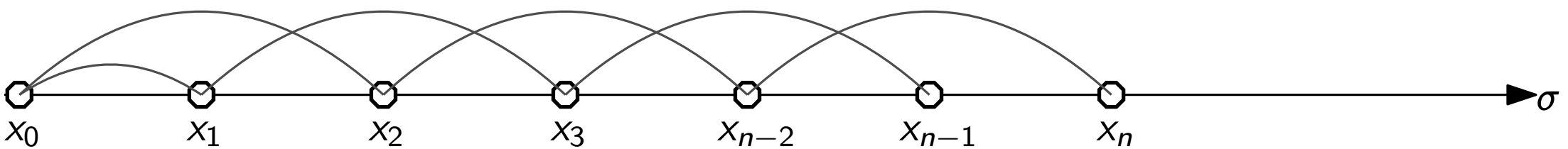
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we will show that an x_{n+1} exists s.t. $x_1, \ldots, x_{n+1}$ induces P_{n+1} like above or induces C_{n+1}



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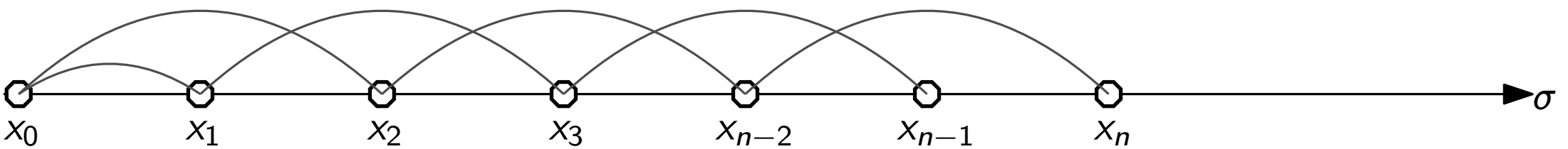
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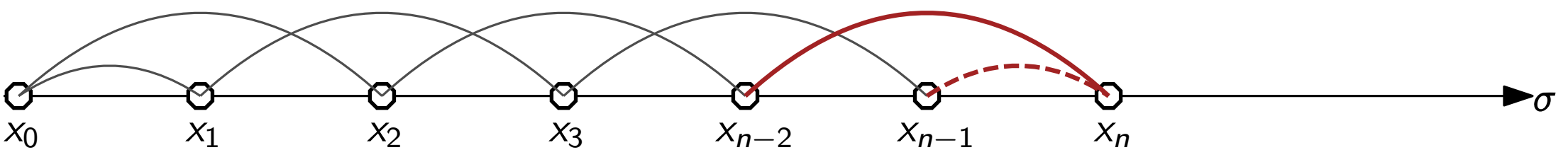
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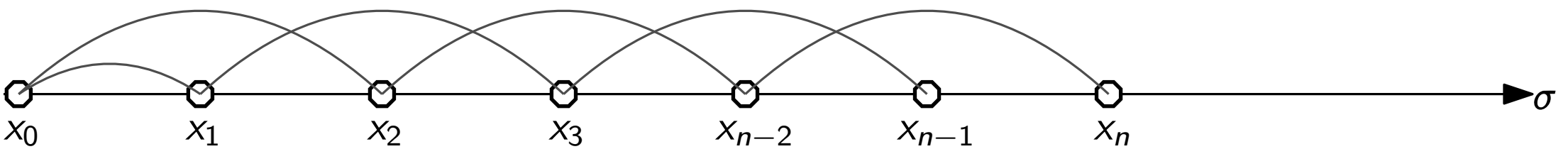
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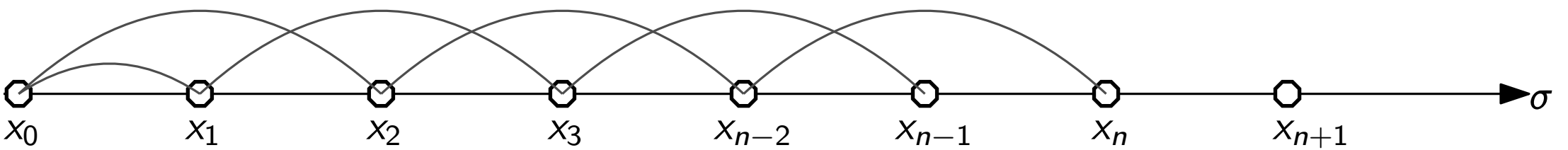
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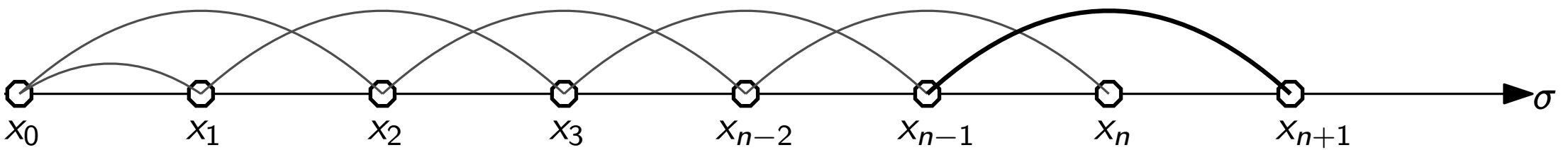
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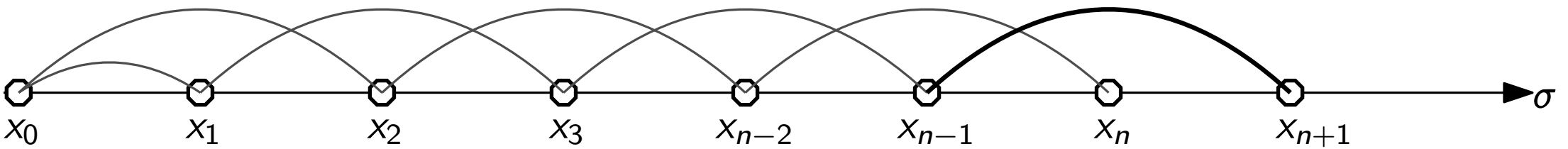
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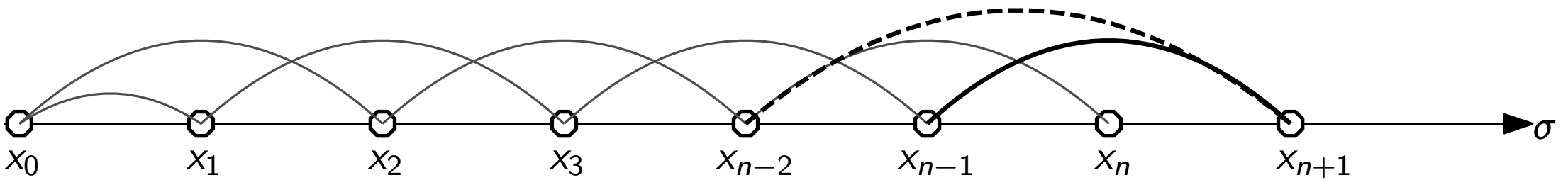
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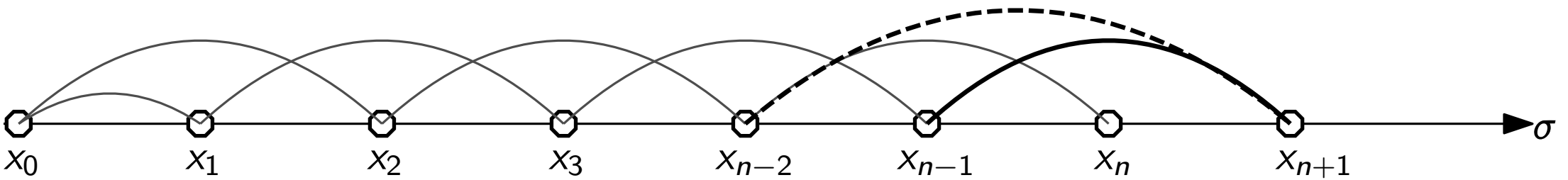
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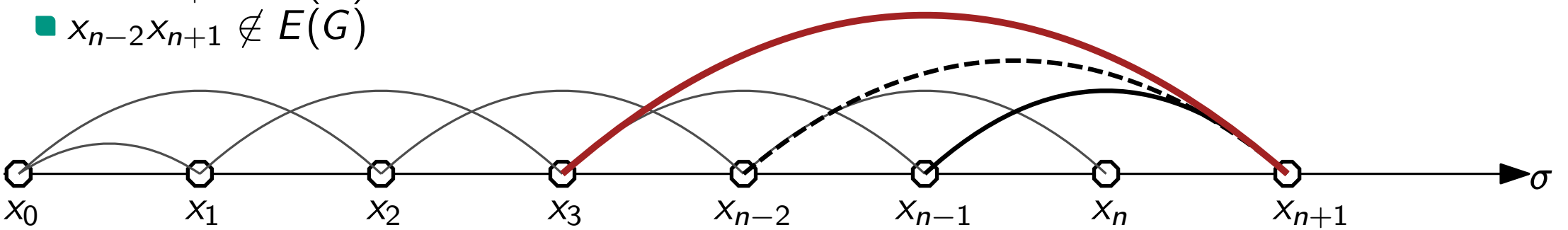
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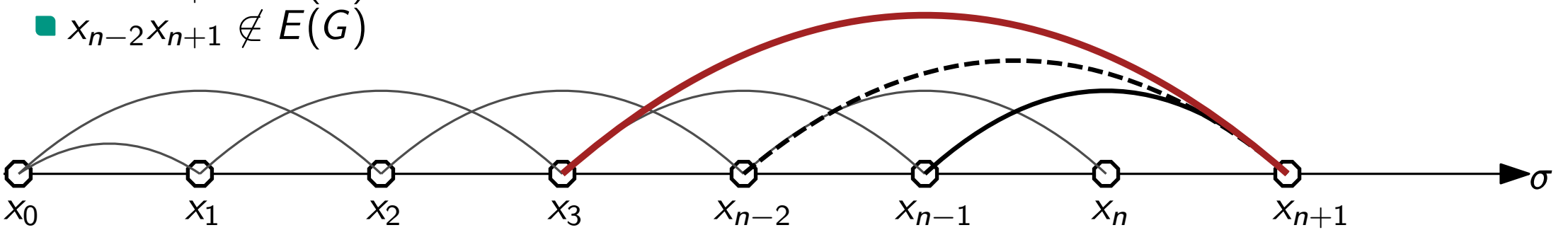
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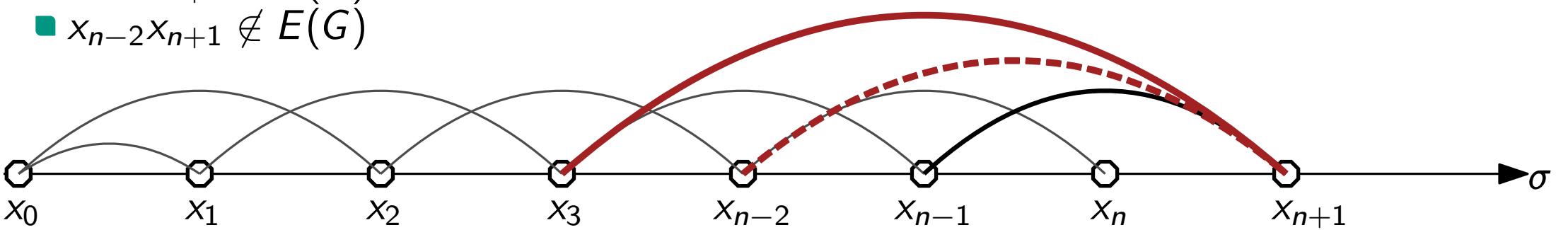
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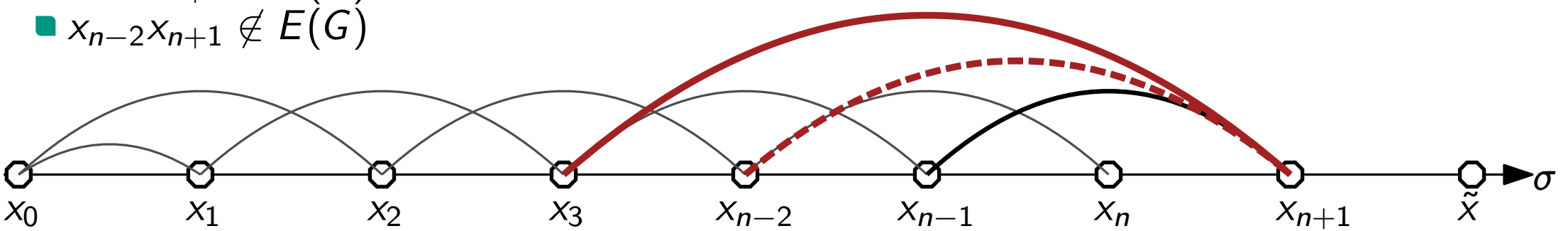
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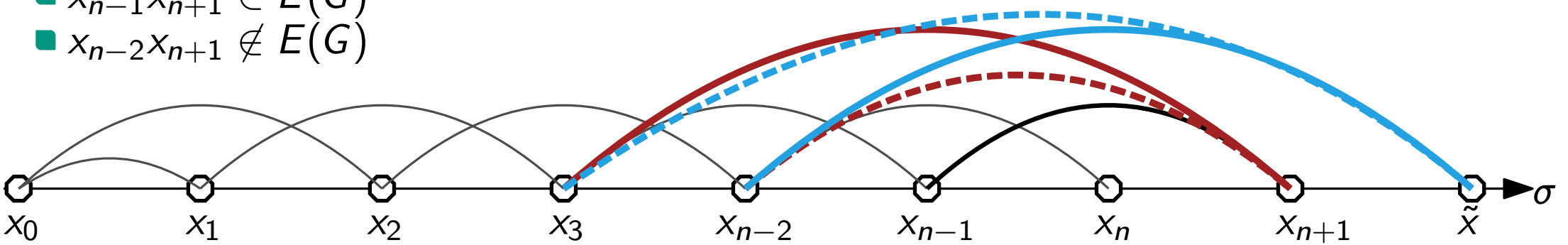
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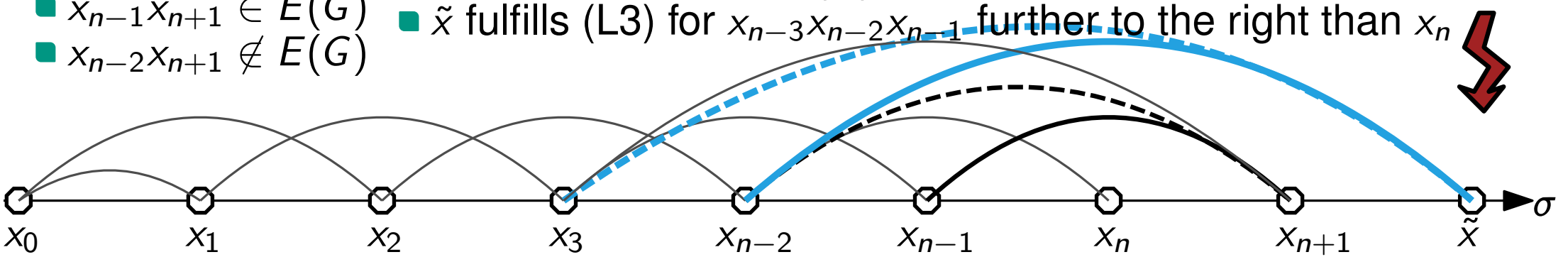
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- $\tilde{x}$ fulfills (L3) for $x_{n-3}x_{n-2}x_{n-1}$ further to the right than x_n



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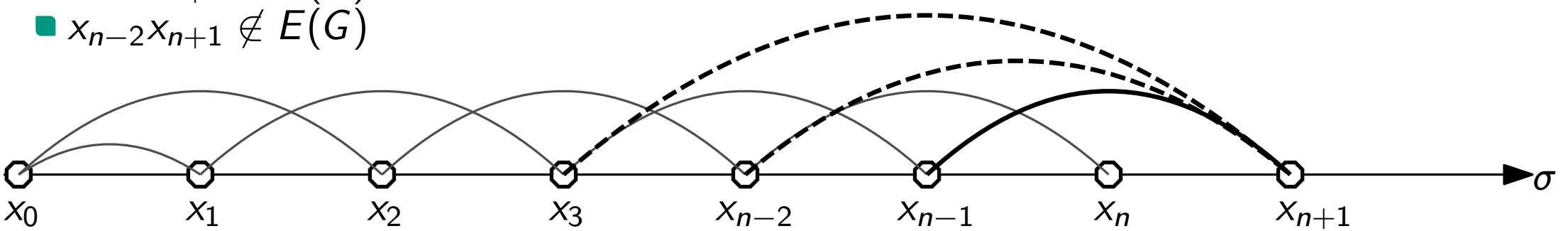
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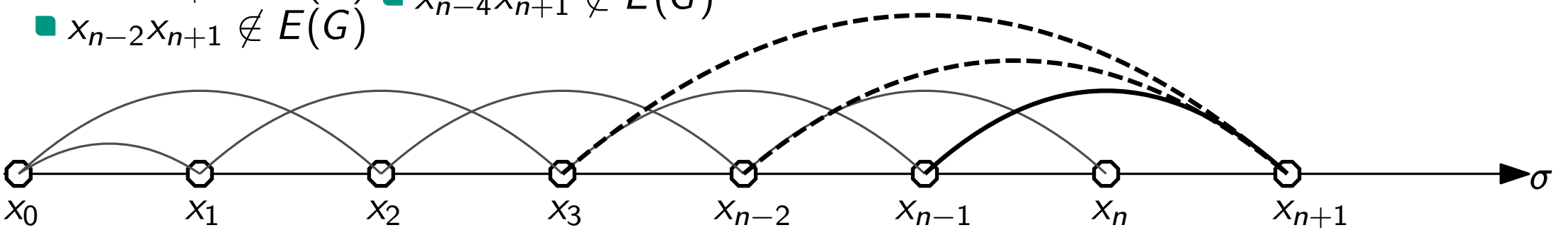
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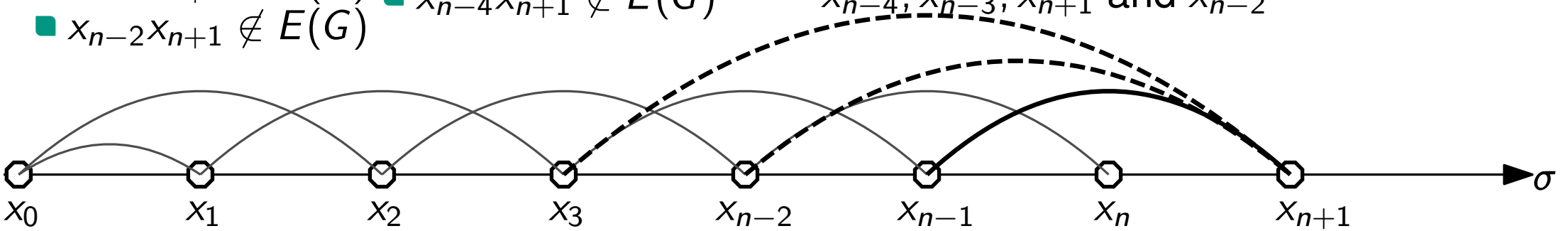
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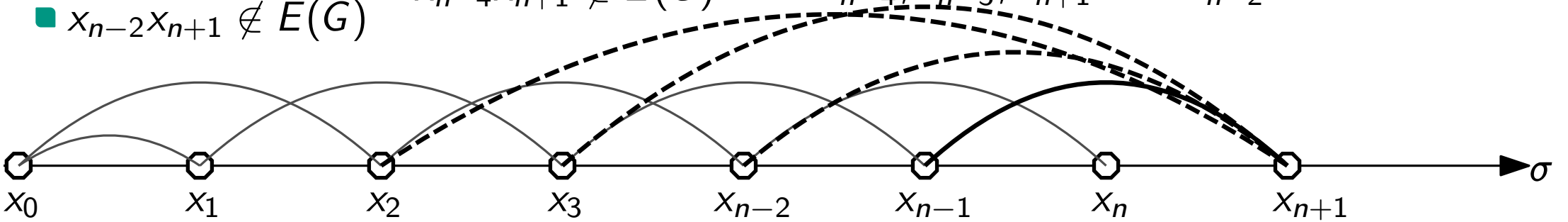
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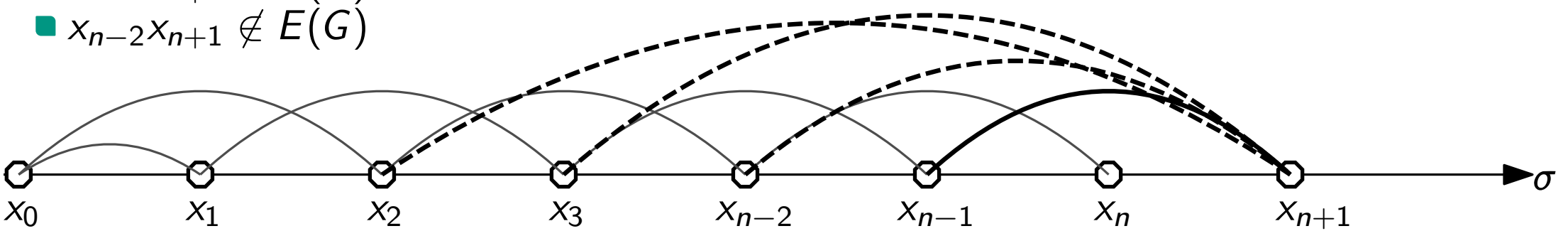
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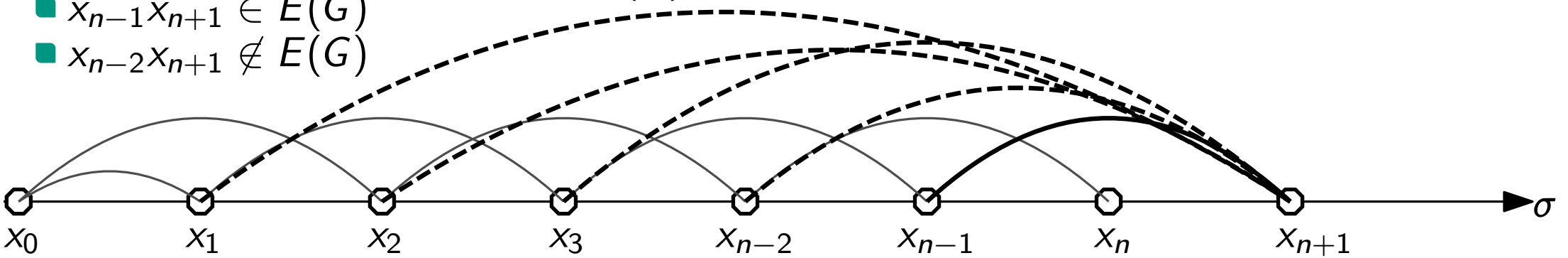
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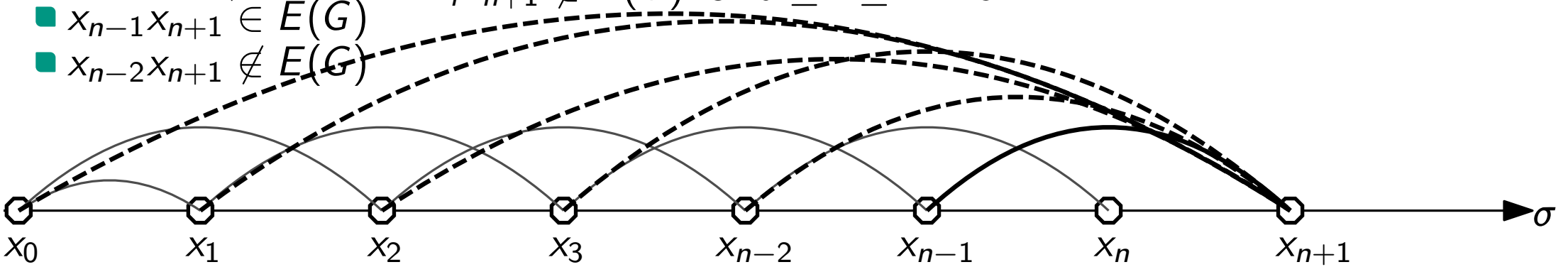
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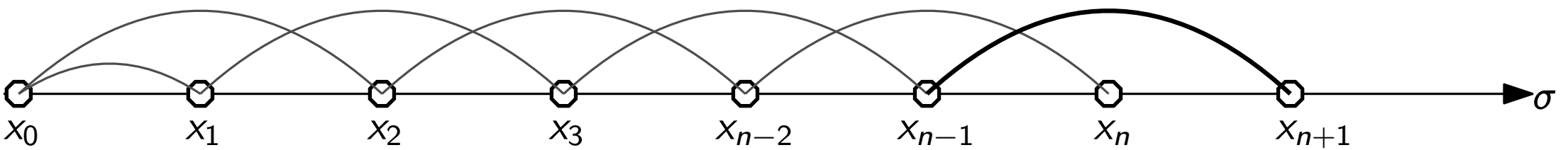
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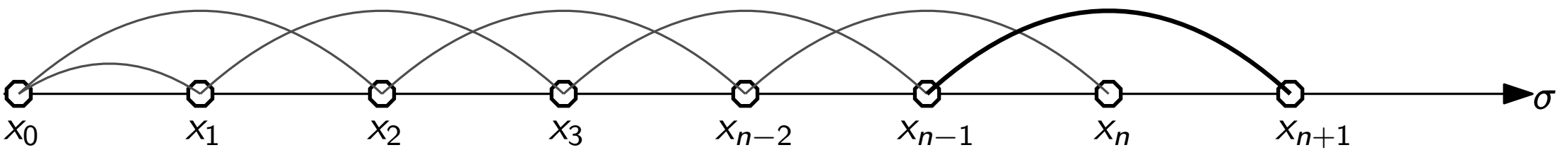
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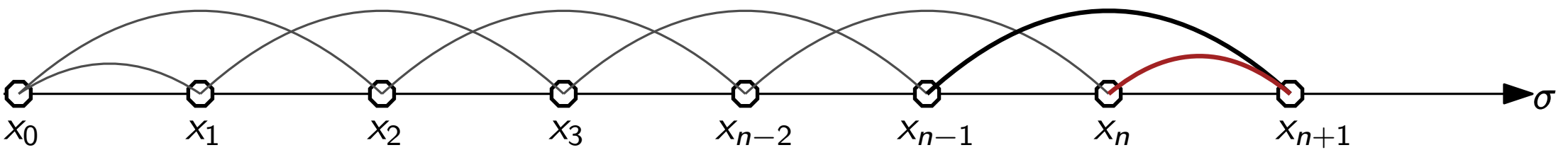
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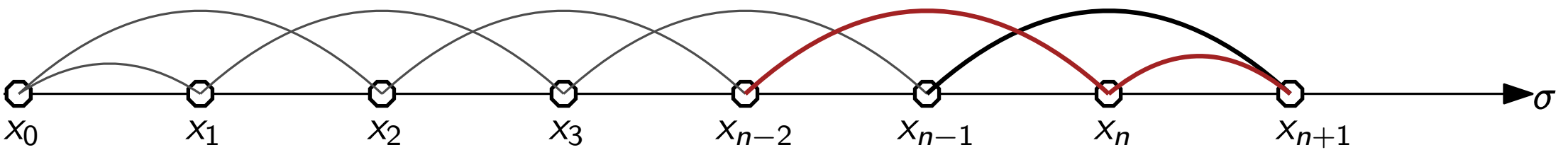
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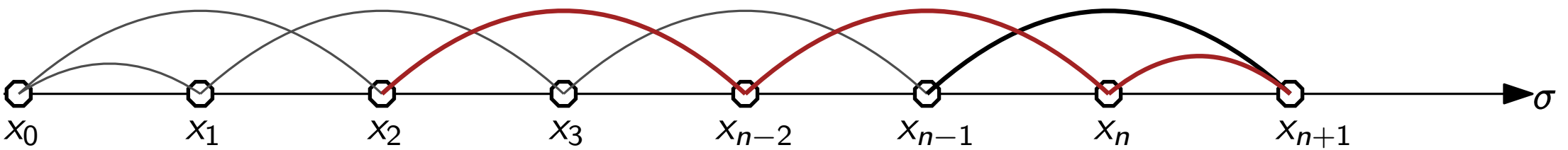
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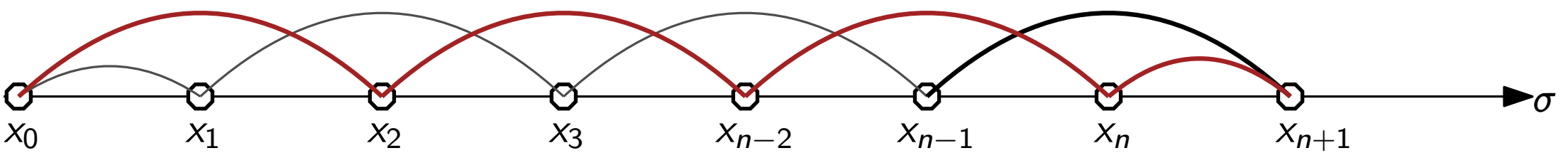
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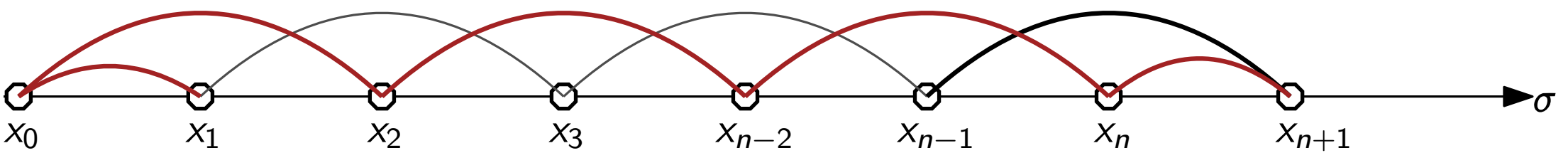
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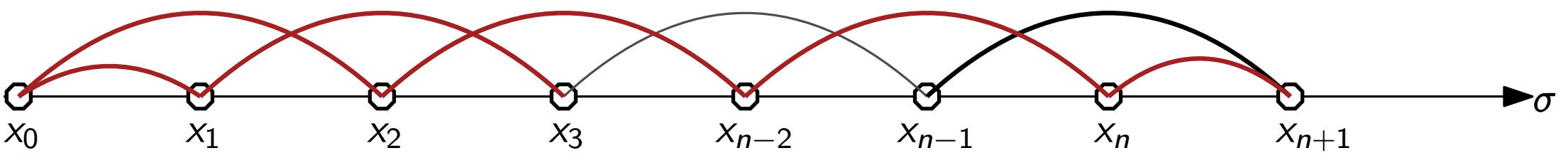
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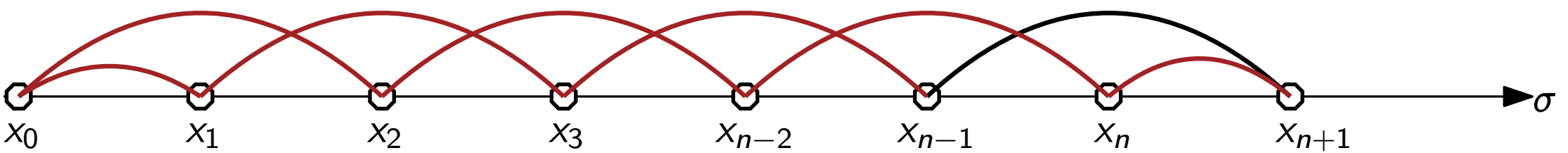
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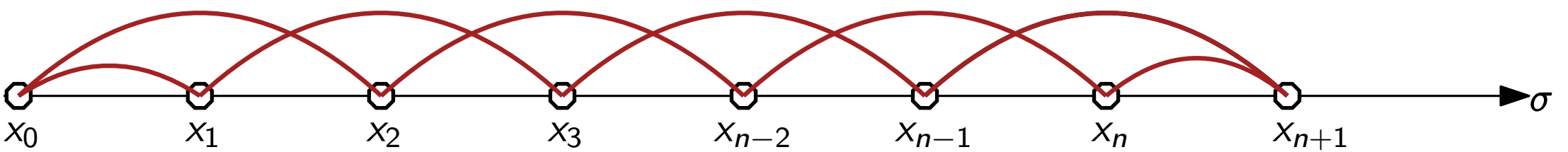
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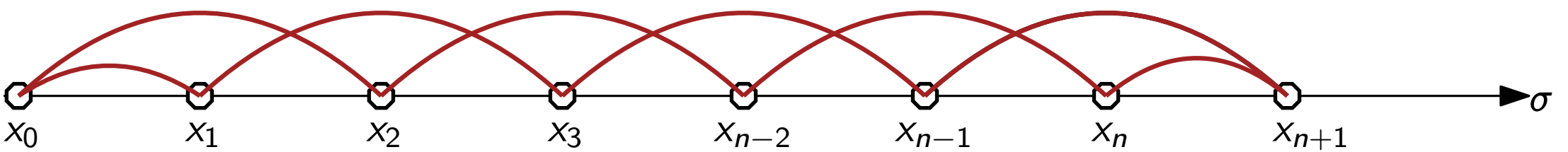
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G is not chordal



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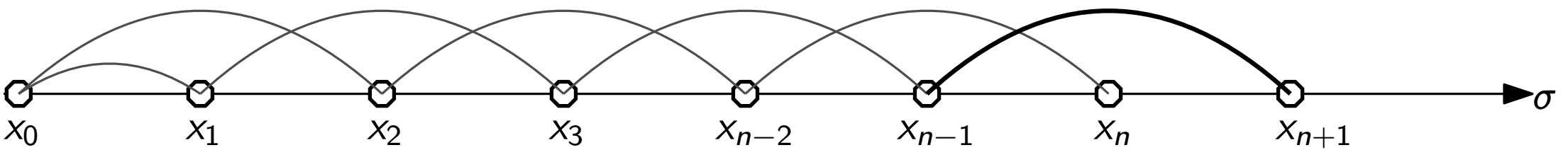
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if $x_n x_n + 1 \notin E(G)$ we obtain induced P_{n+1} and continue the argument



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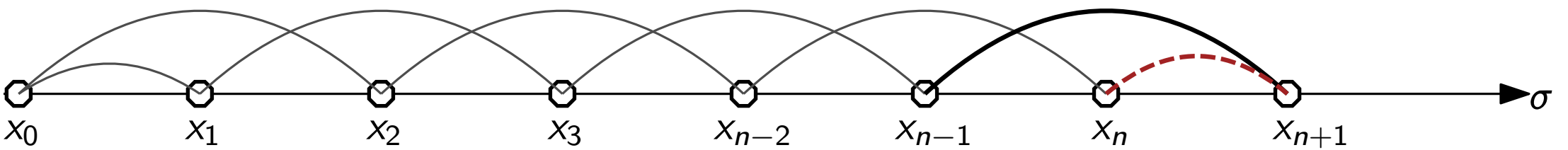
Given $x_0 <_\sigma \ldots <_\sigma x_n$, $n \geq 2$ that induce $P_n = x_n x_{n-2} \ldots x_{n \bmod 2} x_{n+1 \bmod 2} x_{(n+1 \bmod 2)+2} \ldots$ i.e.

x_2 rightmost one s.t. x_0, x_1, x_2 counterexample to PES.

x_i, $i \geq 3$ rightmost one obtained by (L3) on $x_{i-3}, x_{i-2}, x_{i-1}$

use $x_{n-2}x_{n-1}x_n$ for (L3) we obtain $\exists x_{n+1}$ (pick rightmost one) with

if $x_n x_n + 1 \notin E(G)$ we obtain induced P_{n+1} and continue the argument



Algorithm 3 is correct

Theorem

Algorithm 3 returns TRUE $\iff$ σ is a PES of G.

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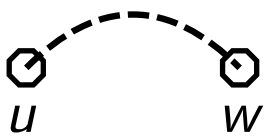
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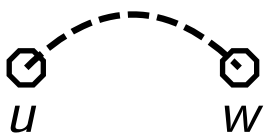
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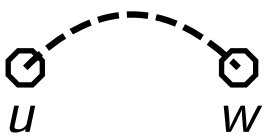
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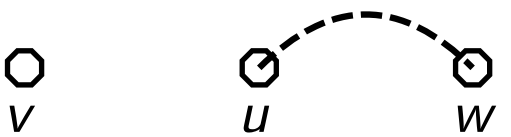
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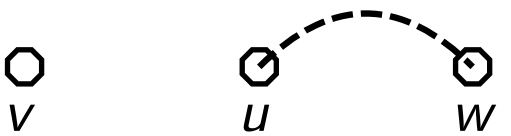
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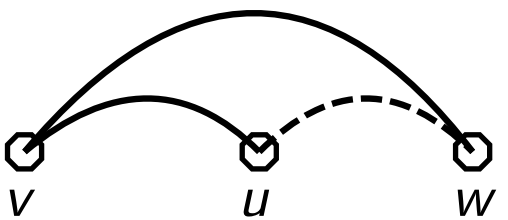
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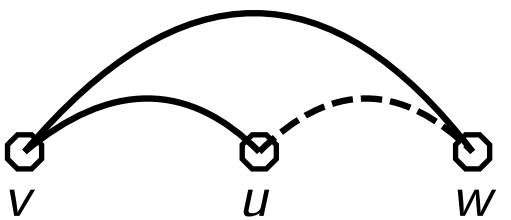
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bad triple. σ not PES



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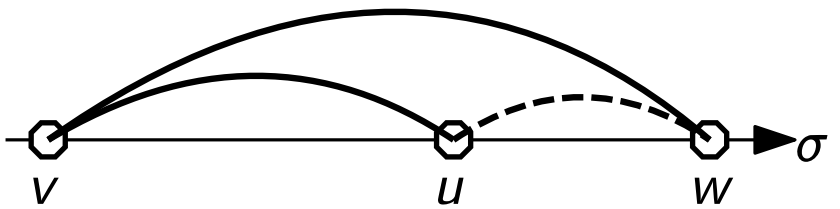
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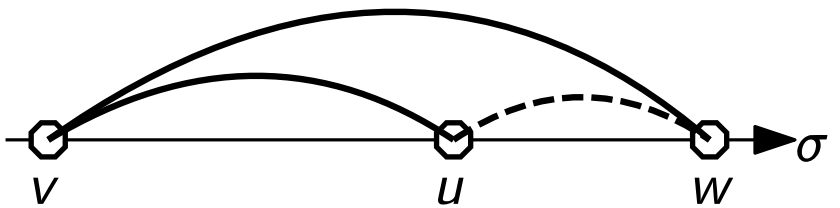


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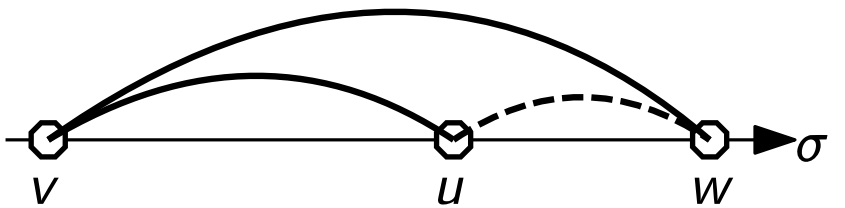
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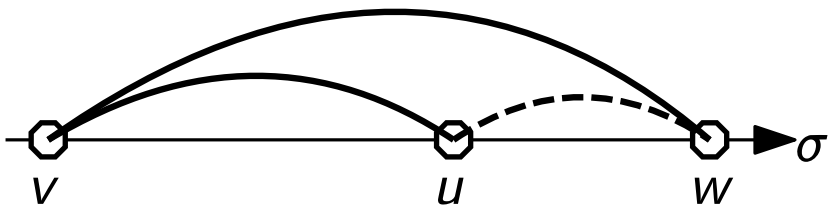
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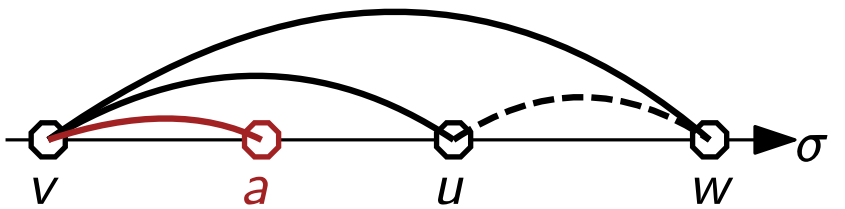
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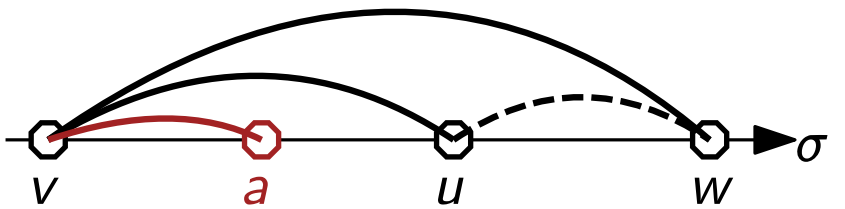
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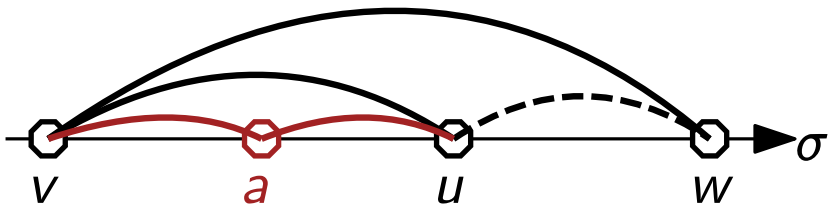
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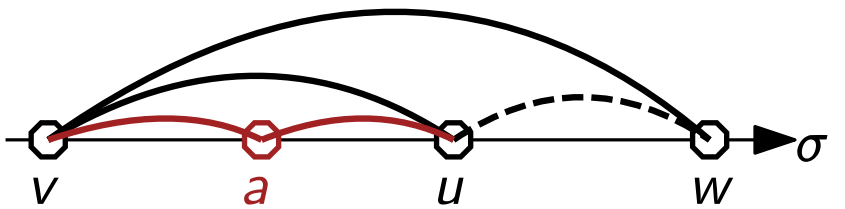
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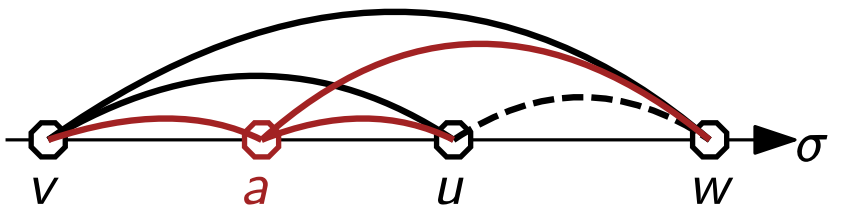
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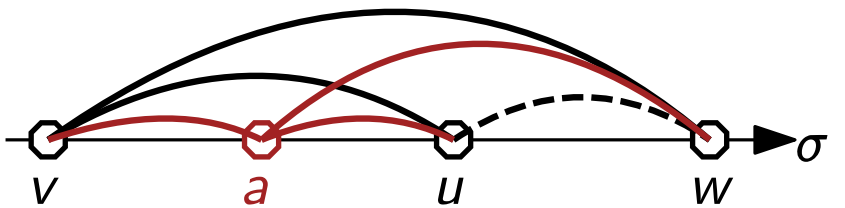
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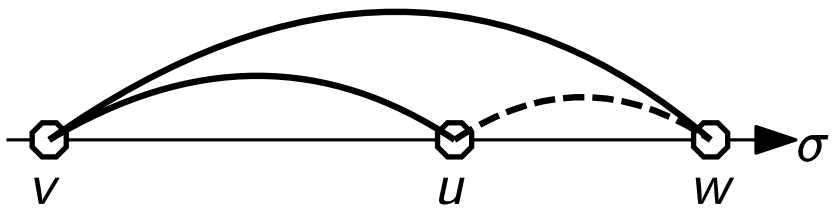
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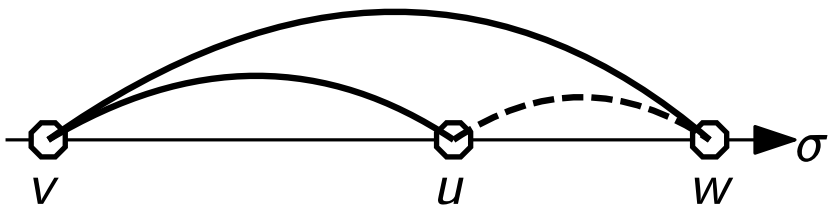
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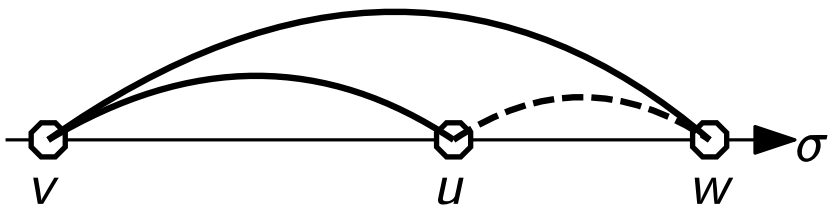
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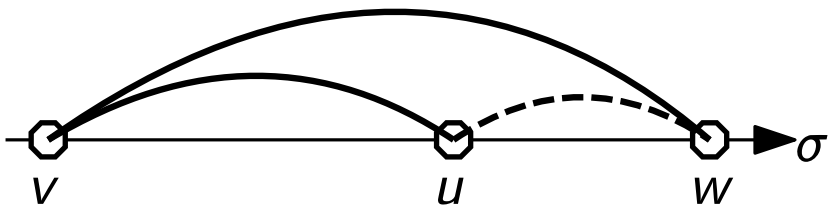
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Input : graph $G = (V, E)$, vertex ordering σ.
Output : true, if σ PES, false otherwise.

```
1  for each vertex v do A(v) ← ∅;
2  for i ← 1 to n − 1 do
3  |  v ← σ(i);
4  |  X ← {x ∈ Adj(v) | σ(v) < σ(x)};
5  |  if X = ∅ then go to line 8;
6  |  u ← argmin{σ(x) | x ∈ X};
7  |  add X − {u} to A(u);
8  |  if A(v) − Adj(v) ≠ ∅ then
9  |  |  return false;
10 |  end if
11 end for
12 return true;
```

Algorithm 3 : Test for perfect elimination scheme

Algorithm 3 is fast

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Algorithm 3 can be done in $\mathcal{O}(|V| + |E|)$.

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	1	**for** each vertex v **do** $A(v) \leftarrow \emptyset$;
$\mathcal{O}(\|V\|)$	2	**for** $i \leftarrow 1$ **to** $n - 1$ **do**
	3	$v \leftarrow \sigma(i)$;
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	5	**if** $X = \emptyset$ **then** go to line **8**;
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8	**if** $A(v) - \text{Adj}(v) \neq \emptyset$ **then**	
9	**return** false;	
10	**end if**	
11	**end for**	
12	**return** true;	

Algorithm 3 : Test for perfect elimination scheme

Algorithm 3 is fast

Theorem

Algorithm 3 can be done in $\mathcal{O}(|V| + |E|)$.

Input : graph $G = (V, E)$, vertex ordering σ.
Output : true, if σ PES, false otherwise.

	for each vertex v **do** $A(v) \leftarrow \emptyset$;	$\mathcal{O}(\|V\|)$
2	**for** $i \leftarrow 1$ **to** $n - 1$ **do**	
3	$v \leftarrow \sigma(i)$;	$\mathcal{O}(1)$
4	$X \leftarrow \{x \in \text{Adj}(v) \mid \sigma(v) < \sigma(x)\}$;	$\mathcal{O}(\|\text{Adj}(v)\|)$
5	**if** $X = \emptyset$ **then** go to line **8**;	$\mathcal{O}(1)$
6	$u \leftarrow \text{argmin}\{\sigma(x) \mid x \in X\}$;	$\mathcal{O}(\|\text{Adj}(v)\|)$
7	add $X - \{u\}$ to $A(u)$;	
8	**if** $A(v) - \text{Adj}(v) \neq \emptyset$ **then**	
9	**return** false;	
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$\mathcal{O}(\lvert V\rvert)$	**1**	**for** each vertex v **do** $A(v) \leftarrow \emptyset$;
	2	**for** $i \leftarrow 1$ **to** $n-1$ **do**
$\mathcal{O}(1)$	**3**	$v \leftarrow \sigma(i)$;
$\mathcal{O}(\lvert \mathrm{Adj}(v)\rvert)$	**4**	$X \leftarrow \{x \in \mathrm{Adj}(v) \mid \sigma(v) < \sigma(x)\}$;
$\mathcal{O}(1)$	**5**	**if** $X = \emptyset$ **then** go to line **8**;
$\mathcal{O}(\lvert \mathrm{Adj}(v)\rvert)$	**6**	$u \leftarrow \mathrm{argmin}\{\sigma(x) \mid x \in X\}$;
	7	add $X - \{u\}$ to $A(u)$; **don't check for duplicates just append!**
	8	**if** $A(v) - \mathrm{Adj}(v) \neq \emptyset$ **then**
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$\mathcal{O}(\lvert\mathrm{Adj}(v)\rvert)$	**4**	$X \leftarrow \{x \in \mathrm{Adj}(v) \mid \sigma(v) < \sigma(x)\}$;
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$\mathcal{O}(\lvert\mathrm{Adj}(v)\rvert)$	**6**	$u \leftarrow \mathrm{argmin}\{\sigma(x) \mid x \in X\}$;
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	9	**return** false;
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	11	**end for**
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Algorithm 3 : Test for perfect elimination scheme

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Theorem

Algorithm 3 can be done in $\mathcal{O}(|V| + |E|)$.

Input : lists $\mathrm{Adj}(v)$, $A(v)$.
Output : true, if $A(v) - \mathrm{Adj}(v) \neq \emptyset$, false otherwise.

```
1  for w ∈ Adj(v) do Test(w) ← true;
2  for w ∈ A(v) do
3  |  if Test(w) = false then
4  |  |  return true;
5  |  end if
6  end for
7  for w ∈ Adj(v) do Test(w) ← false;
8  return false;
```

Algorithm 4 : Test for $A(v) - \mathrm{Adj}(v) \neq \emptyset$ in line **8**

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Input : lists $\mathrm{Adj}(v)$, $A(v)$.
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$\mathcal{O}(|\mathrm{Adj}(v)|)$

```
1  for w ∈ Adj(v) do Test(w) ← true;
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4  |  |  return true;
5  |  end if
6  end for
7  for w ∈ Adj(v) do Test(w) ← false;
8  return false;
```

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Theorem

Algorithm 3 can be done in $\mathcal{O}(|V| + |E|)$.

Input : lists $\mathrm{Adj}(v)$, $A(v)$.
Output : true, if $A(v) - \mathrm{Adj}(v) \neq \emptyset$, false otherwise.

$\mathcal{O}(\lvert\mathrm{Adj}(v)\rvert)$	1	**for** $w \in \mathrm{Adj}(v)$ **do** $\mathrm{Test}(w) \leftarrow$ true;
	2	**for** $w \in A(v)$ **do**
$\mathcal{O}(1)$	3	**if** $\mathrm{Test}(w) =$ false **then**
	4	**return** true;
	5	**end if**
	6	**end for**
	7	**for** $w \in \mathrm{Adj}(v)$ **do** $\mathrm{Test}(w) \leftarrow$ false;
	8	**return** false;

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	6	**end for**
	7	**for** $w \in \mathrm{Adj}(v)$ **do** $\mathrm{Test}(w) \leftarrow$ false;
	8	**return** false;

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Output : true, if $A(v) - \mathrm{Adj}(v) \neq \emptyset$, false otherwise.

$\mathcal{O}(|\mathrm{Adj}(v)|)$

$\mathcal{O}(|A(v)|)$

```
1  for w ∈ Adj(v) do Test(w) ← true;
2  for w ∈ A(v) do
O(1) 3  |  if Test(w) = false then
O(1) 4  |  |  return true;
5  |  end if
6  end for
7  for w ∈ Adj(v) do Test(w) ← false;
8  return false;
```

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Output : true, if $A(v) - \mathrm{Adj}(v) \neq \emptyset$, false otherwise.

$\mathcal{O}(\lvert\mathrm{Adj}(v)\rvert)$	1	**for** $w \in \mathrm{Adj}(v)$ **do** $\mathrm{Test}(w) \leftarrow$ true;
	2	**for** $w \in A(v)$ **do**
$\mathcal{O}(\lvert A(v)\rvert)$	$\mathcal{O}(1)$ 3	**if** $\mathrm{Test}(w) =$ false **then**
	$\mathcal{O}(1)$ 4	return true;
	5	**end if**
	6	**end for**
$\mathcal{O}(\lvert\mathrm{Adj}(v)\rvert)$	7	**for** $w \in \mathrm{Adj}(v)$ **do** $\mathrm{Test}(w) \leftarrow$ false;
	8	**return** false;

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$\mathcal{O}(1)$	**3**	$v \leftarrow \sigma(i)$;
$\mathcal{O}(\lvert\mathrm{Adj}(v)\rvert)$	**4**	$X \leftarrow \{x \in \mathrm{Adj}(v) \mid \sigma(v) < \sigma(x)\}$;
$\mathcal{O}(1)$	**5**	**if** $X = \emptyset$ **then** go to line **8**;
$\mathcal{O}(\lvert\mathrm{Adj}(v)\rvert)$	**6**	$u \leftarrow \mathrm{argmin}\{\sigma(x) \mid x \in X\}$;
$\mathcal{O}(\lvert\mathrm{Adj}(v)\rvert)$	**7**	add $X - \{u\}$ to $A(u)$; **don't check for duplicates just append!**
$\mathcal{O}(\lvert\mathrm{Adj}(v)\rvert + \lvert A(v)\rvert)$	**8**	**if** $A(v) - \mathrm{Adj}(v) \neq \emptyset$ **then**
	9	**return** false;
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2	**for** $i \leftarrow 1$ **to** $n - 1$ **do**	
3	$v \leftarrow \sigma(i)$;	$\mathcal{O}(1)$
4	$X \leftarrow \{x \in \text{Adj}(v) \mid \sigma(v) < \sigma(x)\}$;	$\mathcal{O}(\lvert\text{Adj}(v)\rvert)$
5	**if** $X = \emptyset$ **then** go to line **8**;	$\mathcal{O}(1)$
6	$u \leftarrow \text{argmin}\{\sigma(x) \mid x \in X\}$;	$\mathcal{O}(\lvert\text{Adj}(v)\rvert)$
7	add $X - \{u\}$ to $A(u)$;	$\mathcal{O}(\lvert\text{Adj}(v)\rvert)$
8	**if** $A(v) - \text{Adj}(v) \neq \emptyset$ **then**	$\mathcal{O}(\lvert\text{Adj}(v)\rvert + \lvert A(v)\rvert)$
9	**return** false;	
10	**end if**	$\mathcal{O}(1)$
11	**end for**	
12	**return** true;	

don't check for duplicates just append!

Algorithm 3 : Test for perfect elimination scheme

Algorithm 3 is fast

Theorem

Algorithm 3 can be done in $\mathcal{O}(|V| + |E|)$.

$\mathcal{O}(|V|)$

$$\left.\begin{array}{l} \mathcal{O}(1) \\ \mathcal{O}(|\mathrm{Adj}(v)|) \\ \mathcal{O}(1) \\ \mathcal{O}(|\mathrm{Adj}(v)|) \\ \mathcal{O}(|\mathrm{Adj}(v)|) \\ \mathcal{O}(|\mathrm{Adj}(v)| + |A(v)|) \\ \mathcal{O}(1) \end{array}\right\} \quad \mathcal{O}\left(\sum_{v \in V} |A(v)| + \sum_{v \in V} |\mathrm{Adj}(v)|\right)$$

Algorithm 3 is fast

Theorem

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$\mathcal{O}(|V|)$

$$\left.\begin{array}{l}\mathcal{O}(1)\\ \mathcal{O}(|\mathrm{Adj}(v)|)\\ \mathcal{O}(1)\\ \mathcal{O}(|\mathrm{Adj}(v)|)\\ \mathcal{O}(|\mathrm{Adj}(v)|)\\ \mathcal{O}(|\mathrm{Adj}(v)| + |A(v)|)\\ \mathcal{O}(1)\end{array}\right\} \quad \mathcal{O}\Big(\sum_{v\in V} |A(v)| + \sum_{v\in V} |\mathrm{Adj}(v)|\Big)$$

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$\mathcal{O}(|V|)$

$\mathcal{O}(1)$
$\mathcal{O}(|\text{Adj}(v)|)$
$\mathcal{O}(1)$
$\mathcal{O}(|\text{Adj}(v)|)$
$\mathcal{O}(|\text{Adj}(v)|)$
$\mathcal{O}(|\text{Adj}(v)| + |A(v)|)$
$\mathcal{O}(1)$

$\Big\}$ $\mathcal{O}(\sum_{v \in V} |A(v)| + \sum_{v \in V} |\text{Adj}(v)|)$

for every vertex u we add $\text{Adj}(u)$ at most to one $A(v)$

Algorithm 3 is fast

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$\mathcal{O}(|V|)$

$$\left.\begin{array}{l} \mathcal{O}(1) \\ \mathcal{O}(|\mathrm{Adj}(v)|) \\ \mathcal{O}(1) \\ \mathcal{O}(|\mathrm{Adj}(v)|) \\ \mathcal{O}(|\mathrm{Adj}(v)|) \\ \mathcal{O}(|\mathrm{Adj}(v)| + |A(v)|) \\ \mathcal{O}(1) \end{array}\right\} \quad \mathcal{O}\left(\sum_{v \in V} |\mathrm{Adj}(v)| + \sum_{v \in V} |\mathrm{Adj}(v)|\right)$$

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$$\left.\begin{array}{l} \mathcal{O}(1) \\ \mathcal{O}(|\mathrm{Adj}(v)|) \\ \mathcal{O}(1) \\ \mathcal{O}(|\mathrm{Adj}(v)|) \\ \mathcal{O}(|\mathrm{Adj}(v)|) \\ \mathcal{O}(|\mathrm{Adj}(v)| + |A(v)|) \\ \mathcal{O}(1) \end{array}\right\} \quad \mathcal{O}\left(\sum_{v \in V} |\mathrm{Adj}(v)|\right)$$

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$\mathcal{O}(|V|)$

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$\mathcal{O}(|\text{Adj}(v)|)$
$\mathcal{O}(1)$
$\mathcal{O}(|\text{Adj}(v)|)$
$\mathcal{O}(|\text{Adj}(v)|)$
$\mathcal{O}(|\text{Adj}(v)| + |A(v)|)$
$\mathcal{O}(1)$

$\Big\}$ $\mathcal{O}(\sum_{v \in V} |\text{Adj}(v)|)$

Handshake Lemma: visit every edge twice

Algorithm 3 is fast

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$\mathcal{O}(|V|)$

$$\left.\begin{array}{l} \mathcal{O}(1) \\ \mathcal{O}(|\mathrm{Adj}(v)|) \\ \mathcal{O}(1) \\ \mathcal{O}(|\mathrm{Adj}(v)|) \\ \mathcal{O}(|\mathrm{Adj}(v)|) \\ \mathcal{O}(|\mathrm{Adj}(v)| + |A(v)|) \\ \mathcal{O}(1) \end{array}\right\} \mathcal{O}(|V| + |E|)$$

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$\mathcal{O}(|V|)$

$$\left.\begin{array}{l} \mathcal{O}(1) \\ \mathcal{O}(|\mathrm{Adj}(v)|) \\ \mathcal{O}(1) \\ \mathcal{O}(|\mathrm{Adj}(v)|) \\ \mathcal{O}(|\mathrm{Adj}(v)|) \\ \mathcal{O}(|\mathrm{Adj}(v)| + |A(v)|) \\ \mathcal{O}(1) \end{array}\right\} \mathcal{O}(|V| + |E|)$$

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$$\left.\begin{array}{l} \mathcal{O}(1) \\ \mathcal{O}(|\mathrm{Adj}(v)|) \\ \mathcal{O}(1) \\ \mathcal{O}(|\mathrm{Adj}(v)|) \\ \mathcal{O}(|\mathrm{Adj}(v)|) \\ \mathcal{O}(|\mathrm{Adj}(v)| + |A(v)|) \\ \mathcal{O}(1) \end{array}\right\} \mathcal{O}(|V| + |E|)$$

$$\Longrightarrow \text{ Gesamtlaufzeit } \mathcal{O}(|V| + |V| + |E|) = \mathcal{O}(|V| + |E|)$$

Algorithm 5 is correct

Theorem

Algorithm 5 computes a clique C and a coloring Φ with $|C| = \omega(G)$ and $\max_{v \in V} \Phi(v) = \chi(G)$.

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Why is C a clique?

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σ is PES

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σ is PES
right neighbourhood X_v must be clique

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$X_v + \{v\}$ must be clique

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Why is C a clique?

σ is PES

right neighbourhood X_v must be clique

$X_v + \{v\}$ must be clique

$\Longrightarrow \max_{v \in V}(|X_v| + 1) = |C| = \omega(G)$

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we set color $\Phi(v)$ once and never change it

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we set color $\Phi(v)$ once and never change it

$\Phi(v) \geq 1$

Algorithm 5 is correct

Theorem

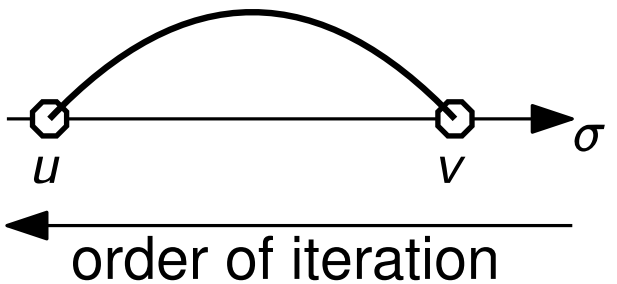
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$\Phi(v) \geq 1$

Let $uv \in E(G)$ wlog $u <_\sigma v$



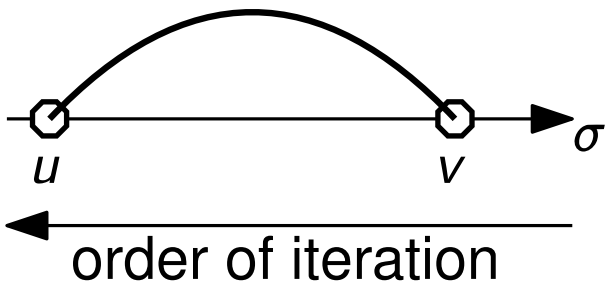
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- we set color $\Phi(v)$ once and never change it
- $\Phi(v) \geq 1$
- Let $uv \in E(G)$ wlog $u <_\sigma v$
- then $u \in X_v$



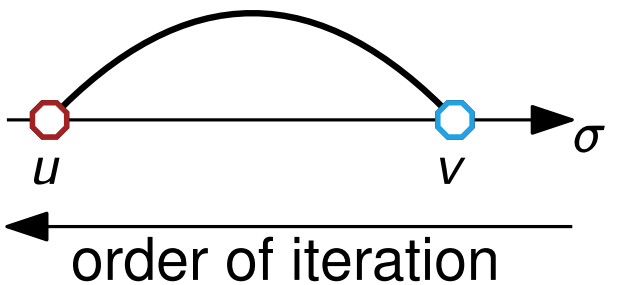
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Why is Φ a coloring?

- we set color $\Phi(v)$ once and never change it
- $\Phi(v) \geq 1$
- Let $uv \in E(G)$ wlog $u <_\sigma v$
- then $u \in X_v$
- $\Phi(u)$ is chosen differently from $\Phi(v)$



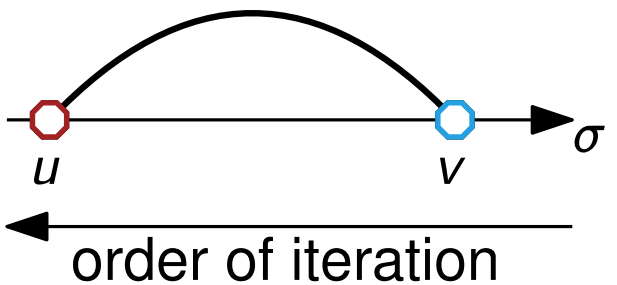
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- we set color $\Phi(v)$ once and never change it
- $\Phi(v) \geq 1$
- Let $uv \in E(G)$ wlog $u <_\sigma v$
- then $u \in X_v$
  $\Phi(u)$ is chosen differently from $\Phi(v)$
- coloring goes through every number in $1 \ldots \max_{v \in V} \Phi(v)$



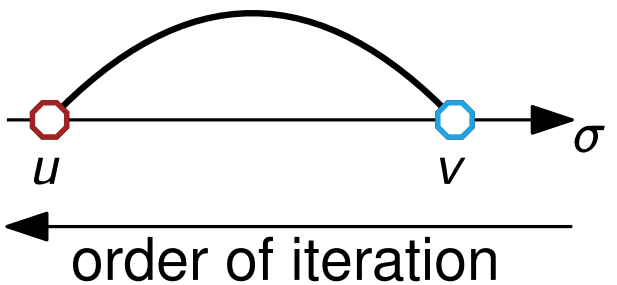
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- we set color $\Phi(v)$ once and never change it
- $\Phi(v) \geq 1$
- Let $uv \in E(G)$ wlog $u <_\sigma v$
- then $u \in X_v$
  $\Phi(u)$ is chosen differently from $\Phi(v)$
- coloring goes through every number in $1 \ldots \max_{v \in V} \Phi(v)$
  $\implies \chi(G) \leq \max_{v \in V} \Phi(v)$



Algorithm 5 is correct

Theorem

Algorithm 5 computes a clique C and a coloring Φ with $|C| = \omega(G)$ and $\max_{v \in V} \Phi(v) = \chi(G)$.

Why are C and Φ optimal?

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$$\chi(G) = \max_{v \in V} \Phi(v) = \max_{v \in V} (|X_v| + 1) = \omega(G) = \chi(G)$$

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runtime of Algo 5 using handshake lemma in $\mathcal{O}(|V| + |E|)$

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invariant: $\forall w \in U : v >_\sigma w$ with $\Psi(v) = 0 \implies vw \notin E(G)$

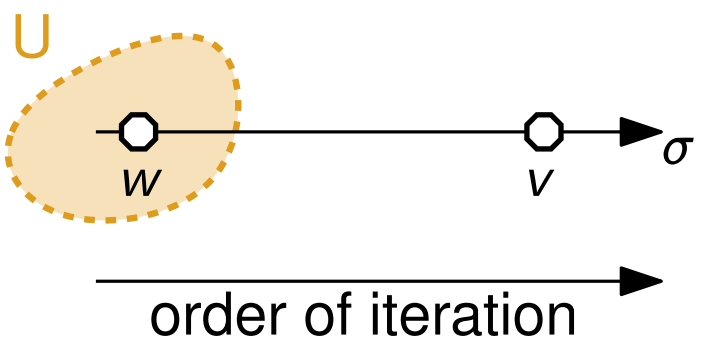
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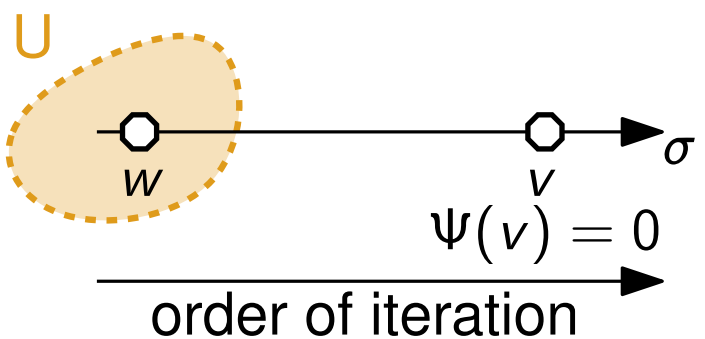
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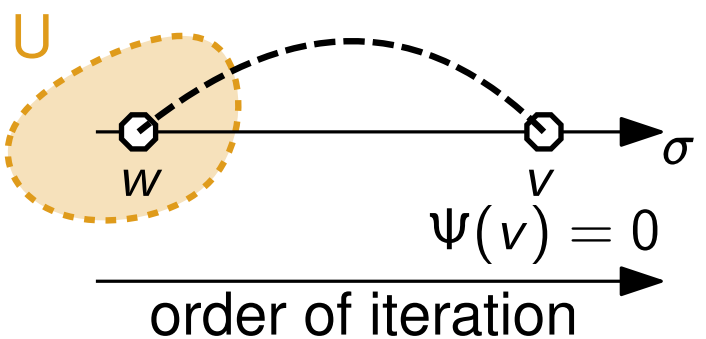
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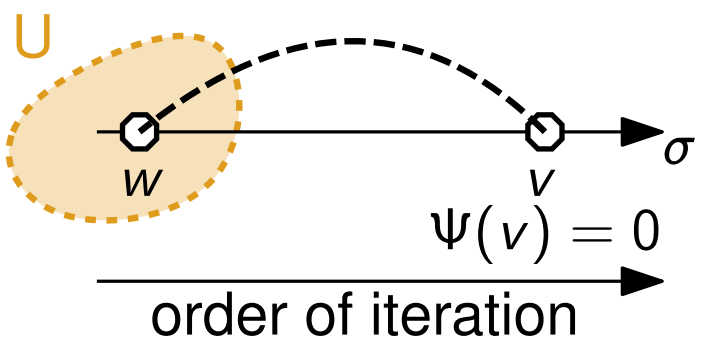
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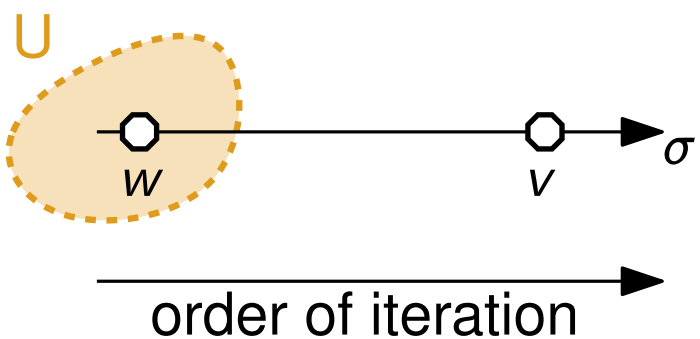
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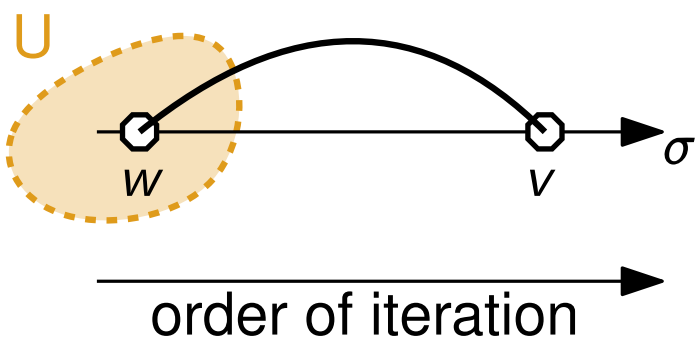
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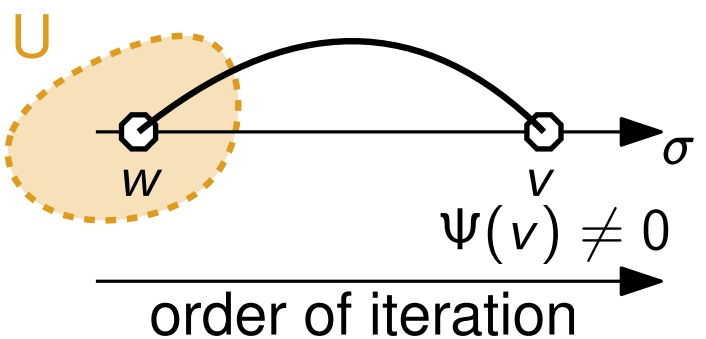
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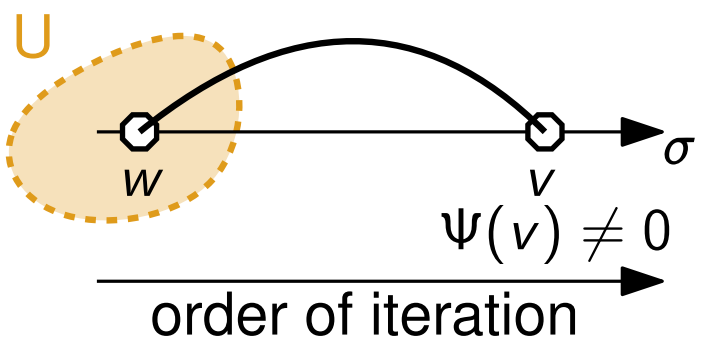
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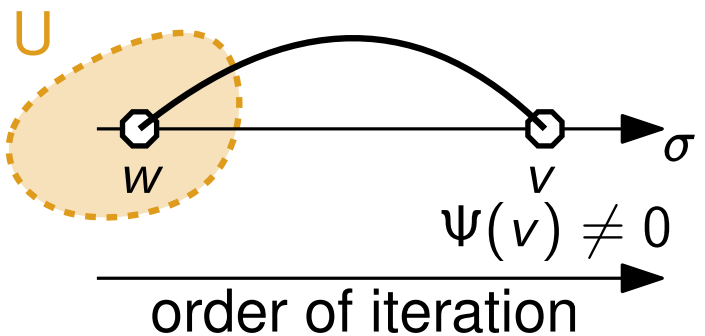
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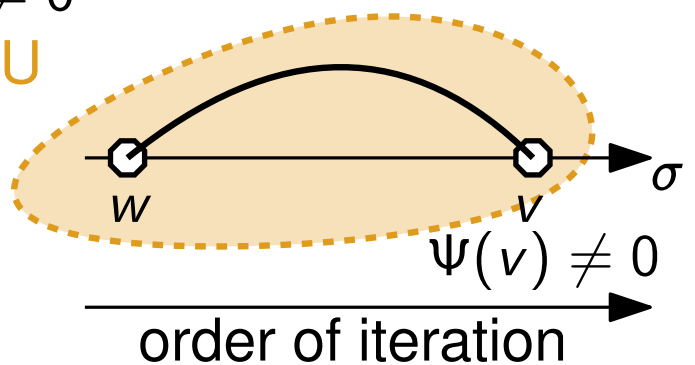
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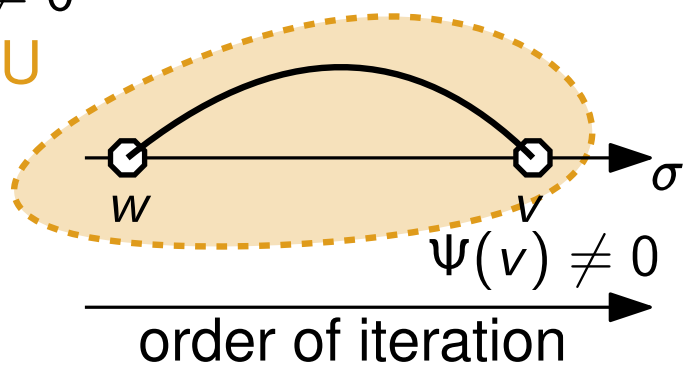
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$\implies |U| \leq \alpha(G)$



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Why is Ψ a clique cover?

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final $\Psi : \{w \in V \mid \Psi(w) = \Psi(v)\} \subseteq \{v\} \cup X_v$ so the set is a clique

also $\Psi(v) \geq 1$ so $\max_{v \in V} \Psi(v) \geq \kappa(G)$

Algorithm 6 is correct

Theorem

Algorithm 6 computes a i-set U and a clique-cover Ψ with $|U| = \alpha(G)$ and $\max_{v \in V} \Psi(v) = \kappa(G)$

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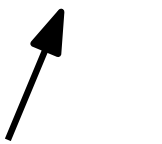
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Input : chordal graph $G = (V, E)$.
Output : independent set U and clique cover ψ.

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1  compute with LexBFS a PES σ of G;
2  U ← ∅, ψ ← 0;
3  for i ← 1 to n do
4  |  v ← σ(i), X_v ← Adj(v) ∩ {σ(i+1), ..., σ(n)};
5  |  if ψ(v) = 0 then
6  |  |  U ← U + {v};
7  |  |  for w ∈ X_v + {v} do
8  |  |  |  ψ(w) ← |U|;
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by runtime of LexBFS and handshake lemma we are done

Theorem 3.14

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For every graph G, the following are equivalent:

- G is chordal.
- There exists a tree $T = (V_T, E_T)$ and a family of subtrees
$$\{T_v \subseteq T \mid v \in V(G),\ T_v \text{ is a subtree}\}$$
such that
$$uv \in E(G) \iff T_u \cap T_v \neq \emptyset,$$
i.e., G is the intersection graph of subtrees of a tree.
- There exists a tree $T = (V_T, E_T)$ such that
$$V_T = \{X \subseteq V \mid X \text{ is an inclusion-maximal clique in } G\},$$
and for every $v \in V$, the set
$$K_v = \{X \in V_T \mid v \in X\}$$
induces a subtree.

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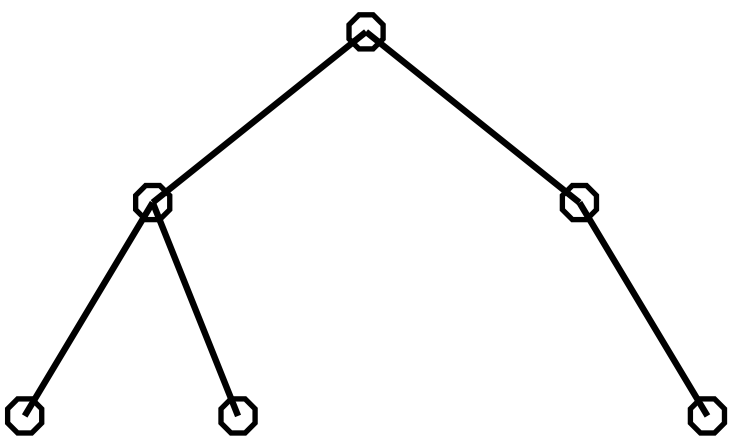
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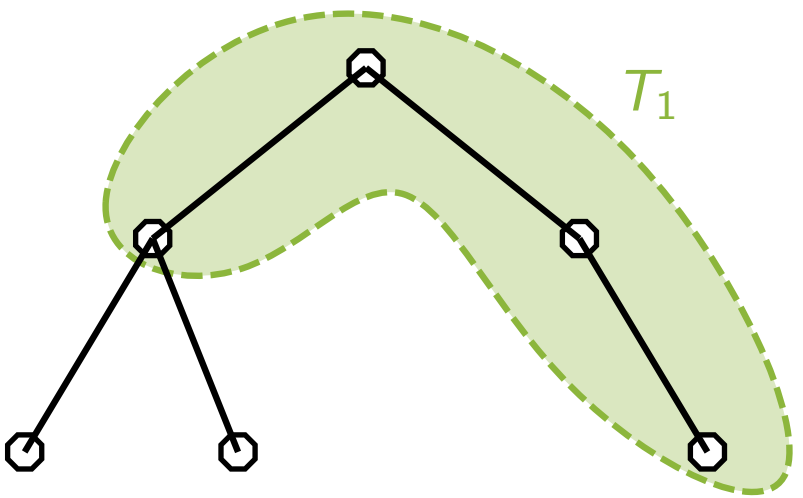
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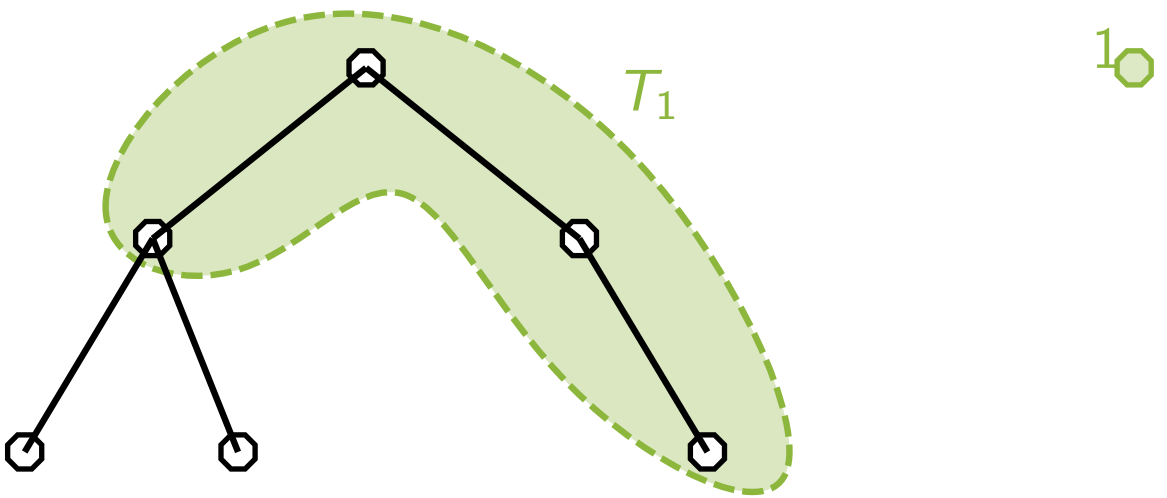
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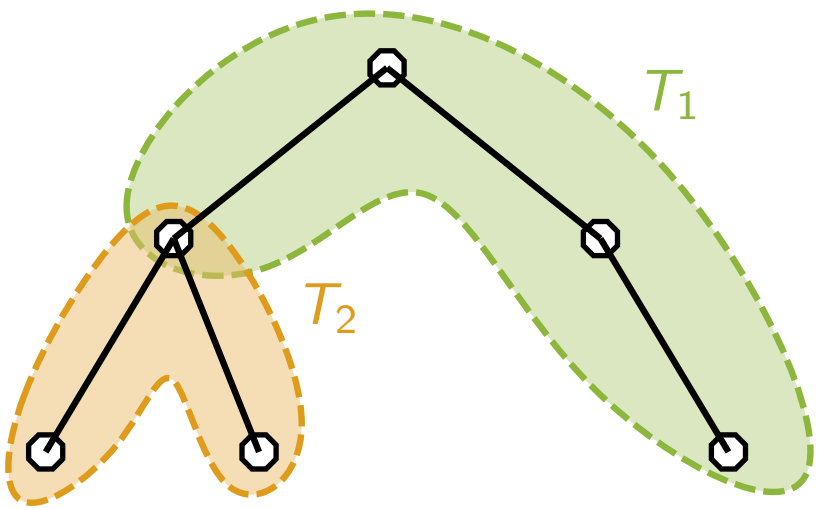
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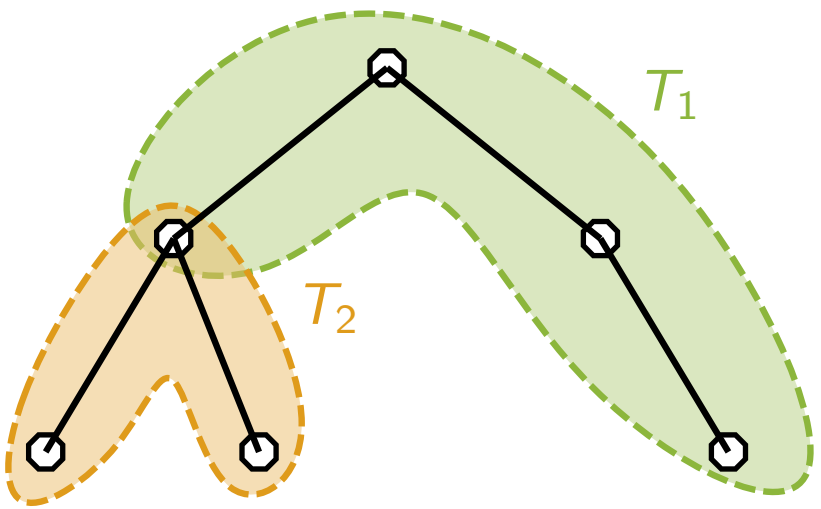
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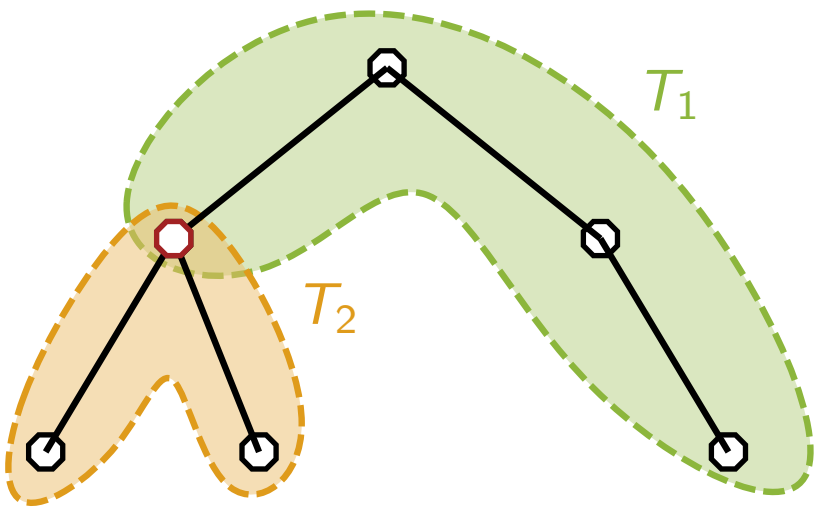
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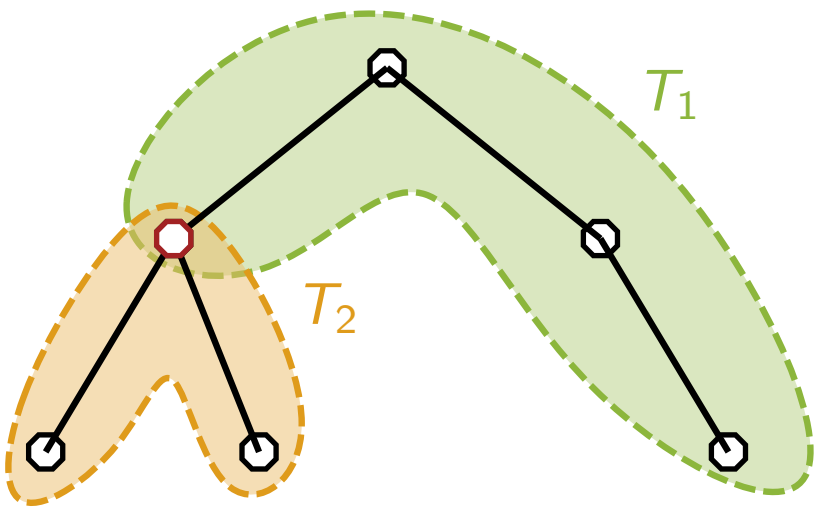
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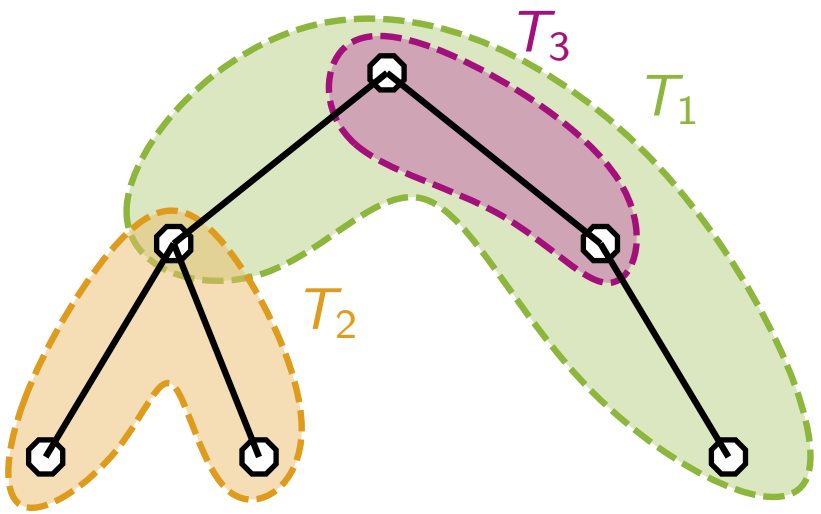
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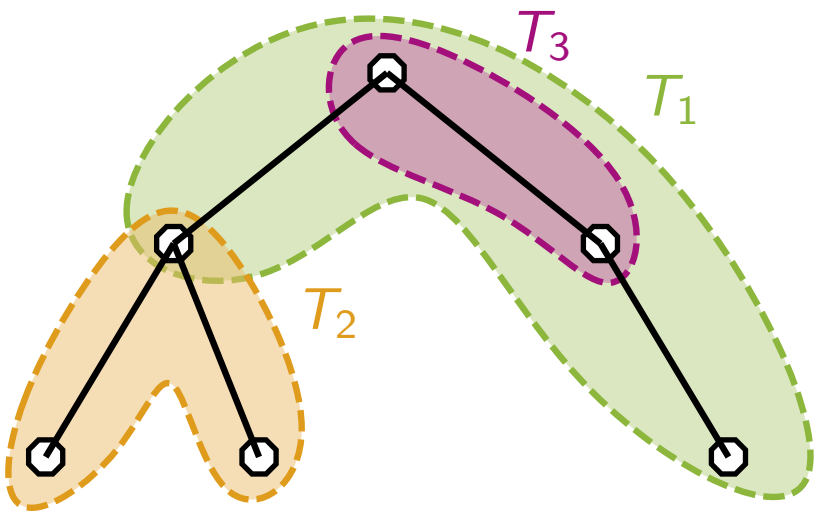
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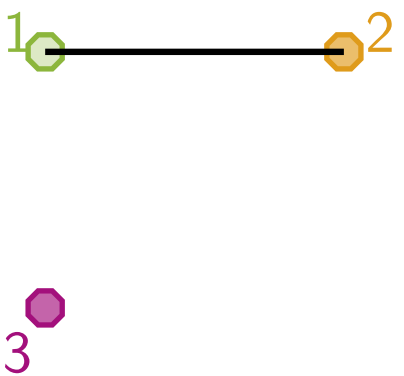
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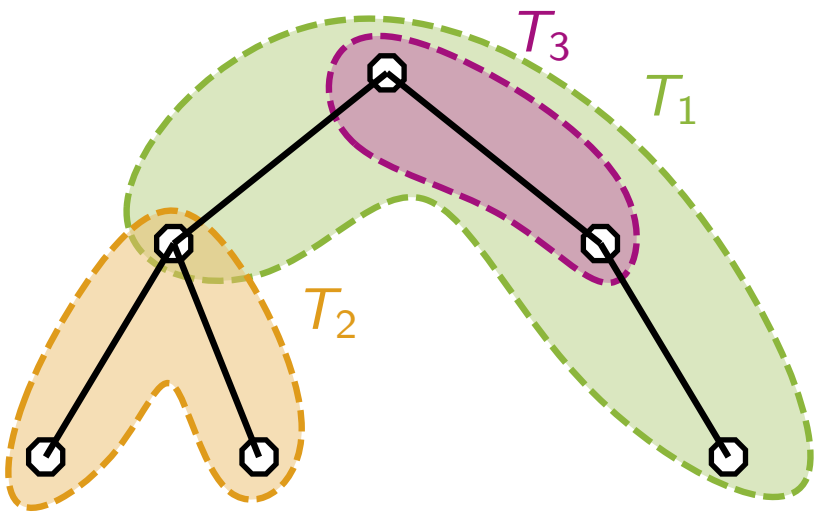
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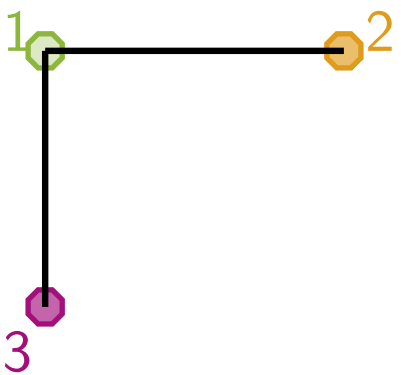
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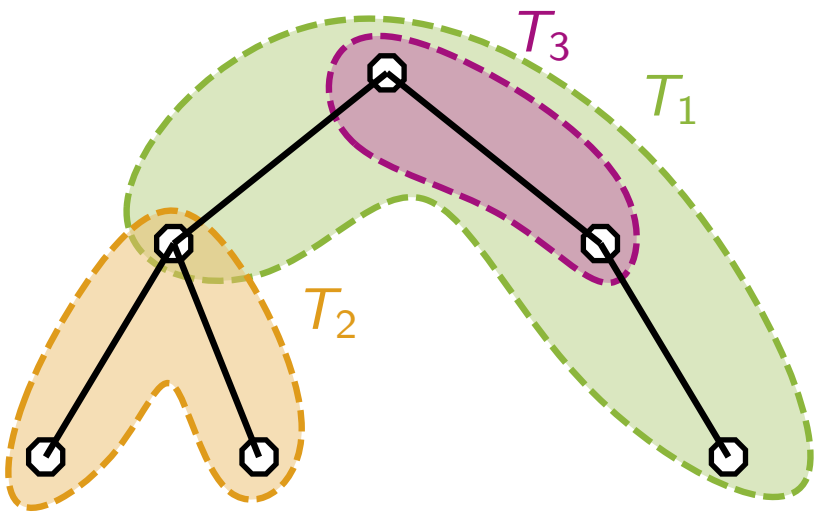
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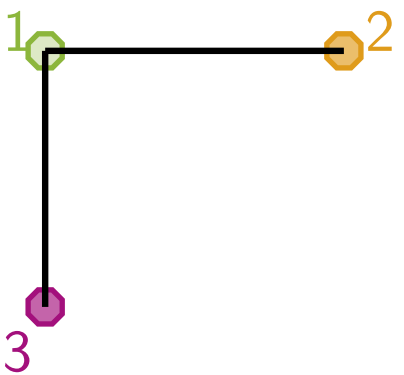
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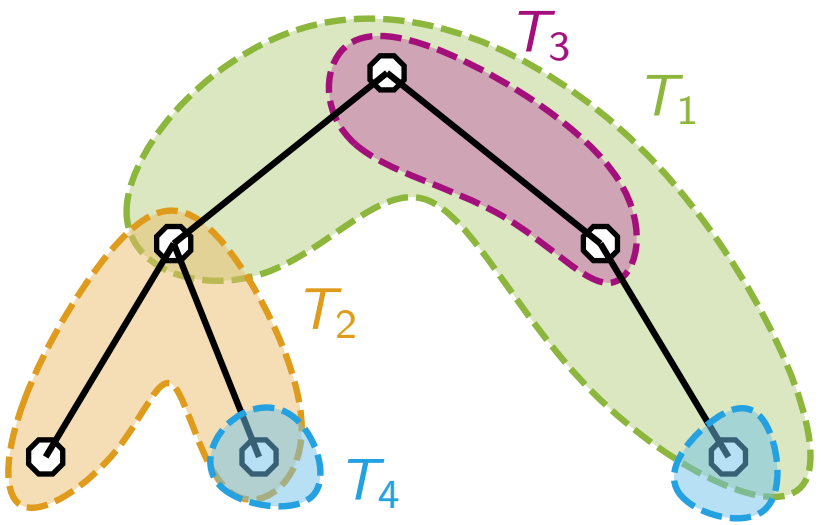
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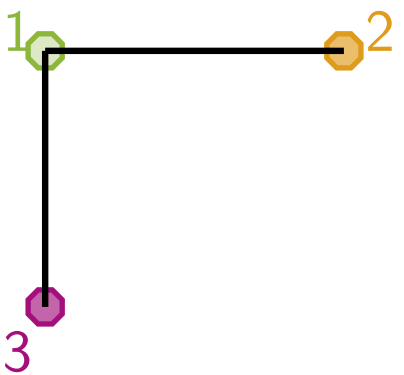
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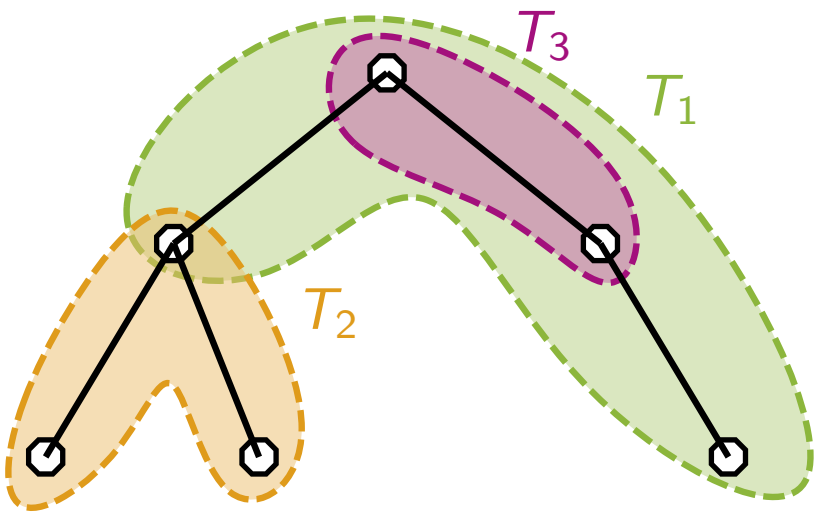
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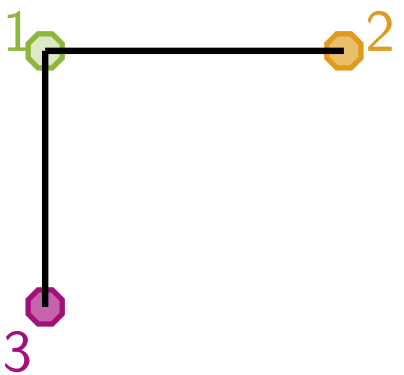
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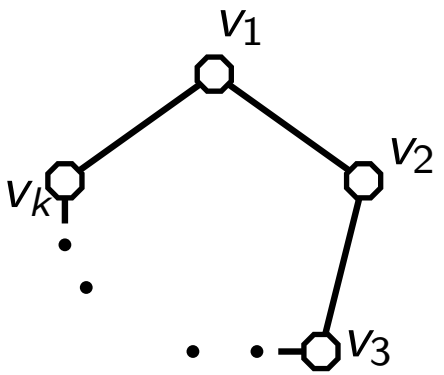
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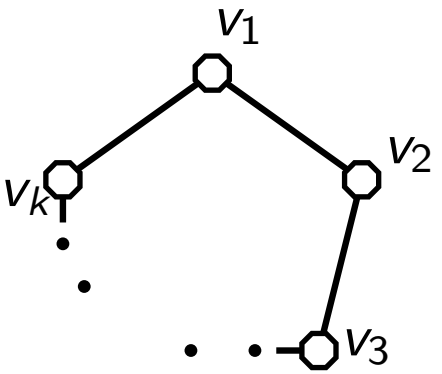


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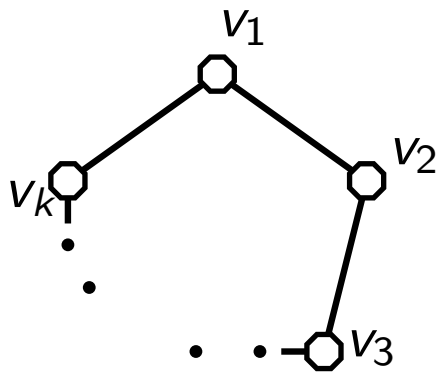


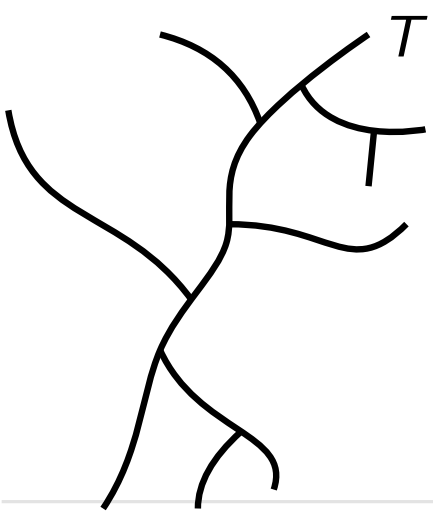
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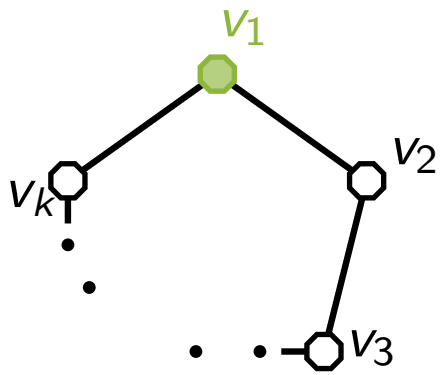


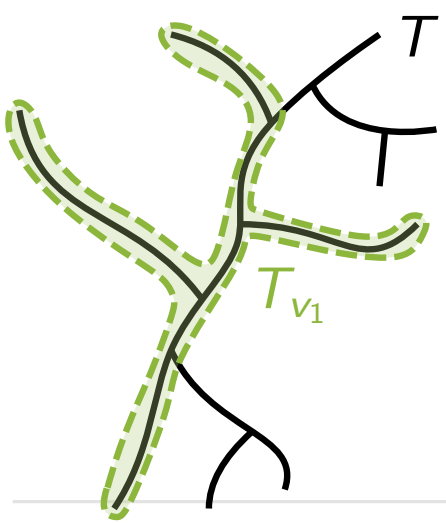
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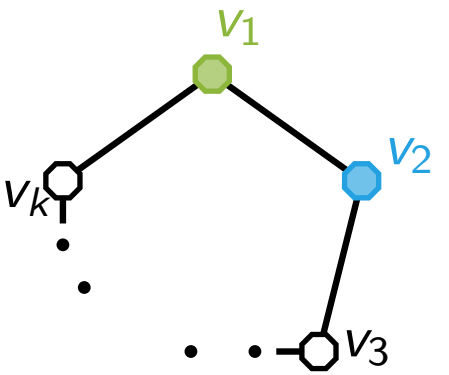


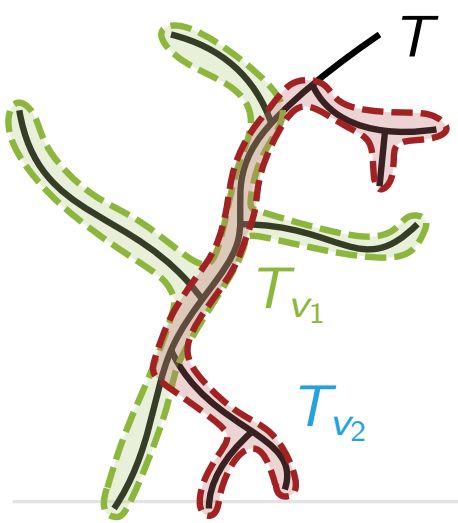
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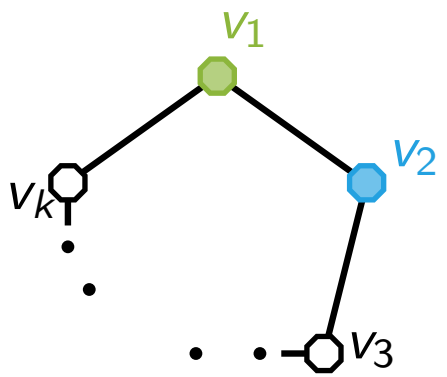


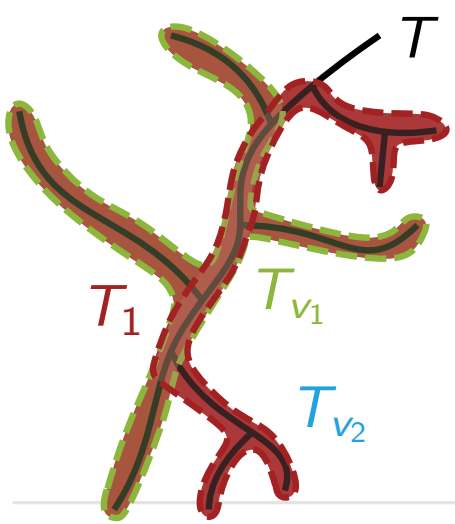
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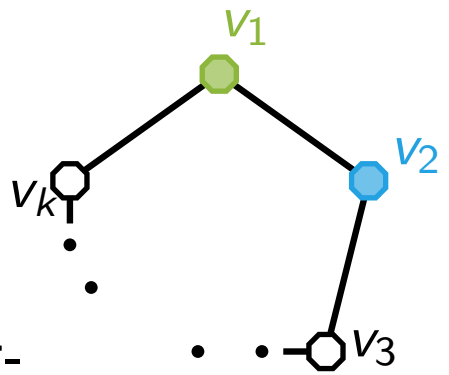


Theorem 3.14 Proof

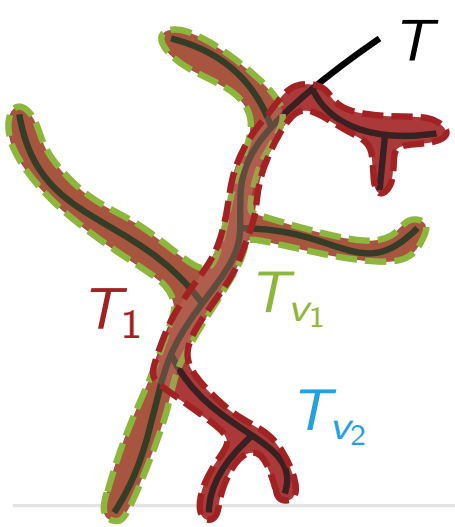
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$\rightsquigarrow$ connected since they have non-empty intersection, since they share an edge

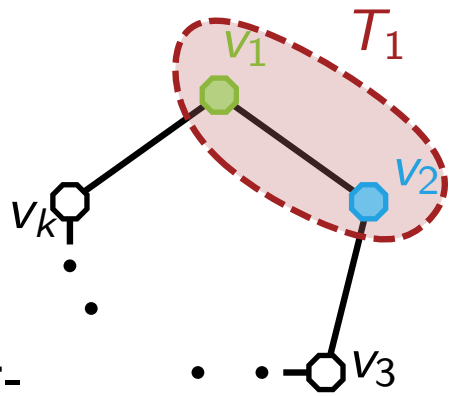


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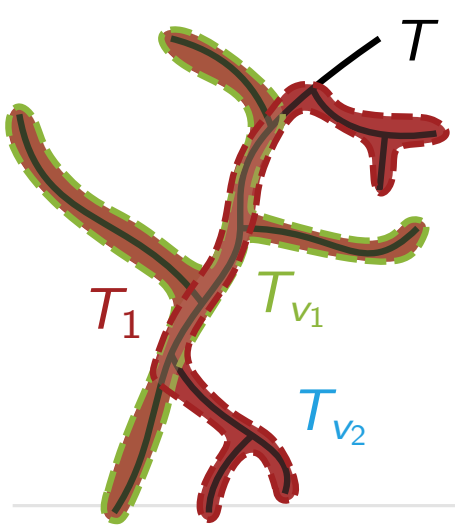
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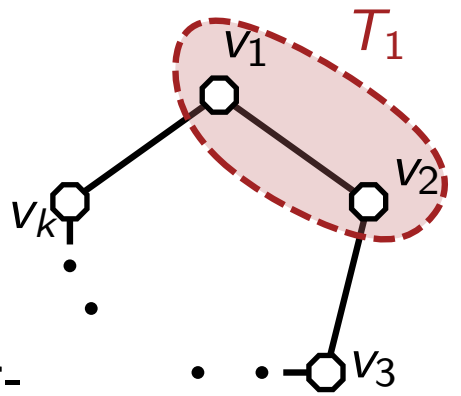


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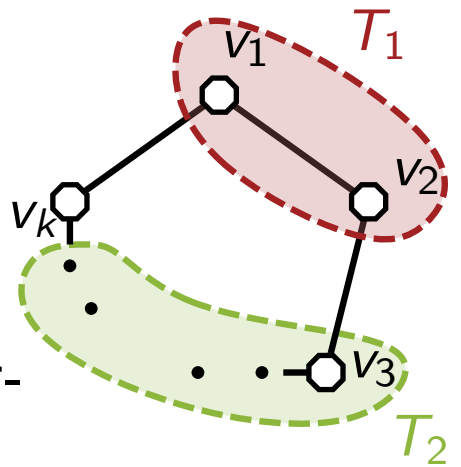
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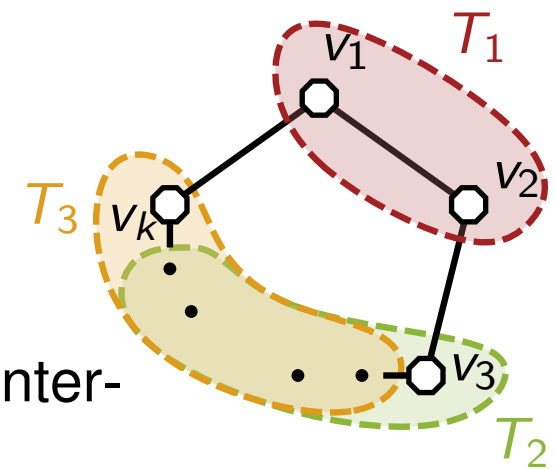
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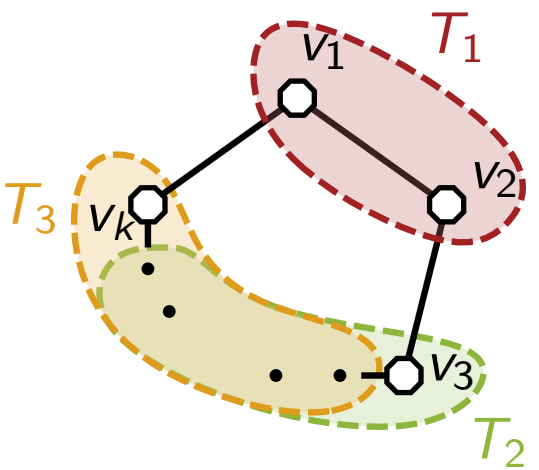
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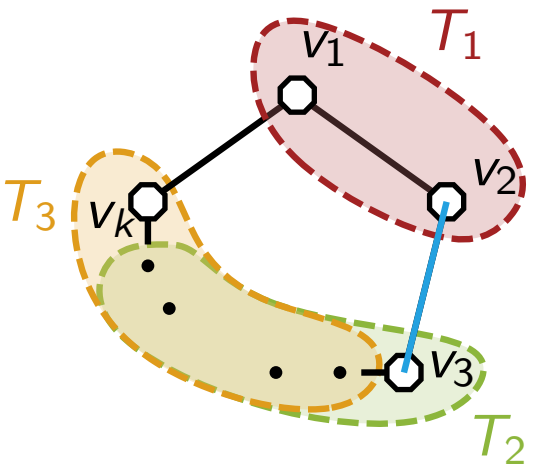
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$T_1 \cap T_2 \neq \varnothing$ since $v_2 v_3 \in E(G)$

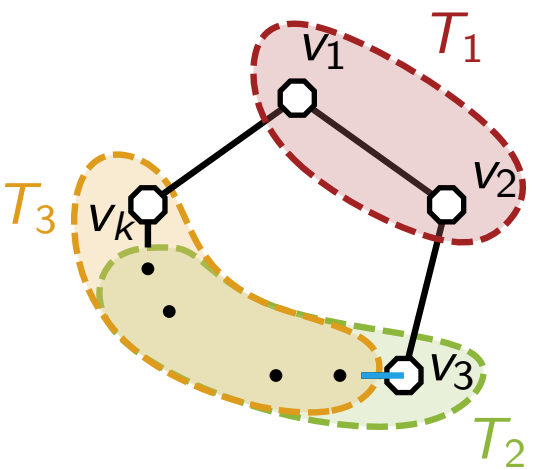


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- $T_1 := T_{v_1} \cup T_{v_2}$ is a subtree of T
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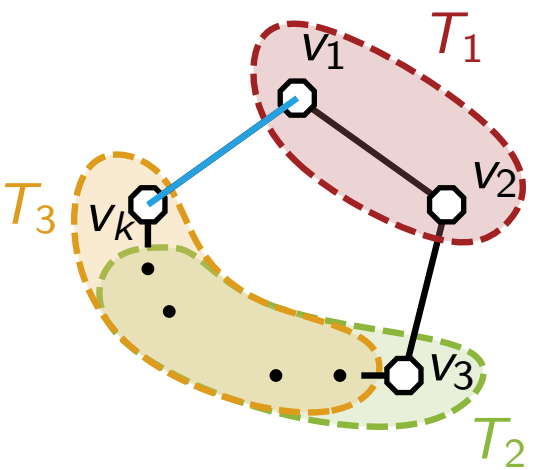
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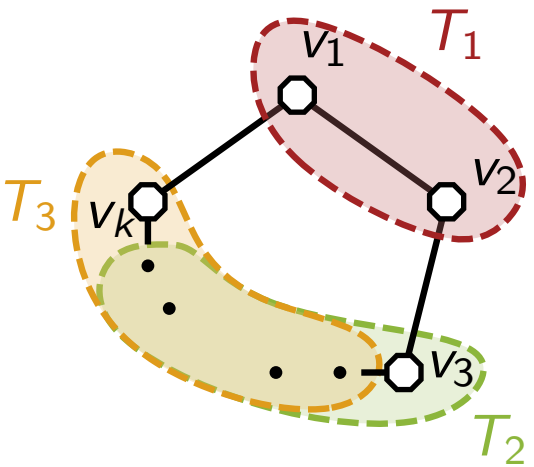
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by Helly property (sh. 4 ex. 5) we get $x \in T_1 \cap T_2 \cap T_3$. Especially $x \in T_{v_1} \cup T_{v_2}$

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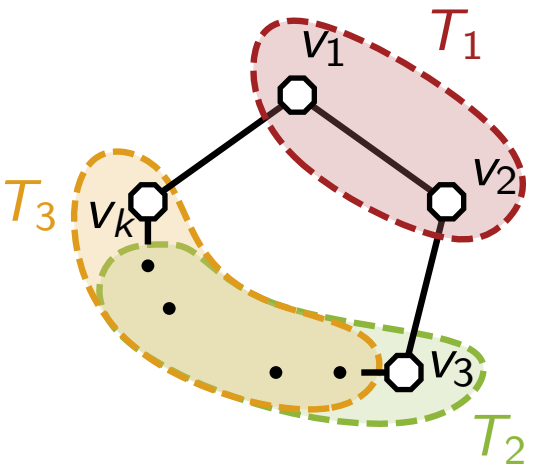
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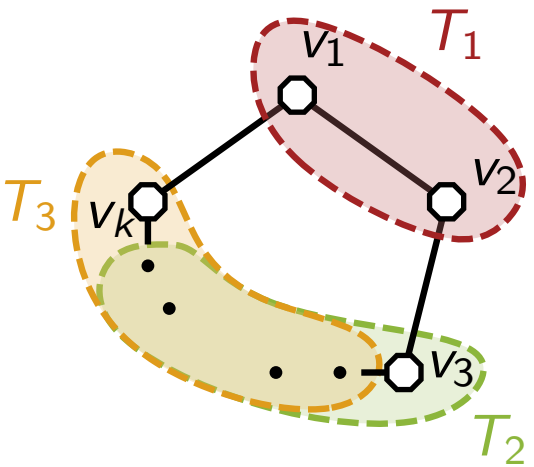
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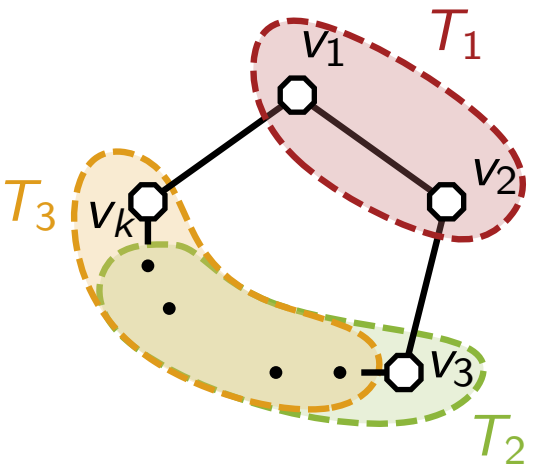
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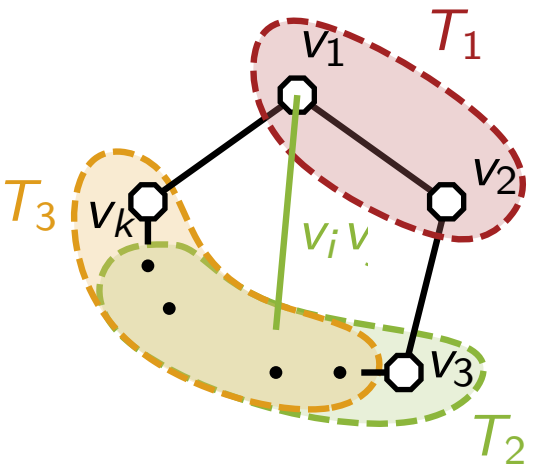
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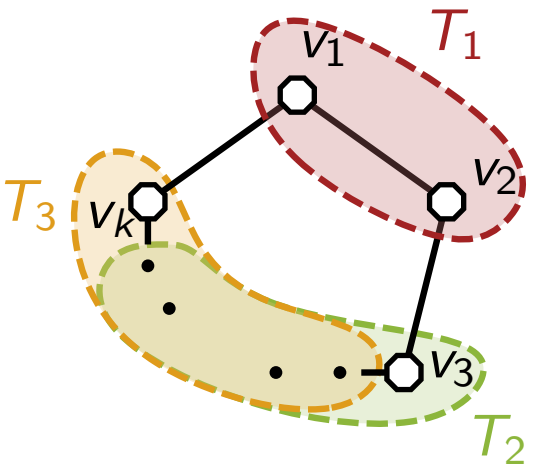
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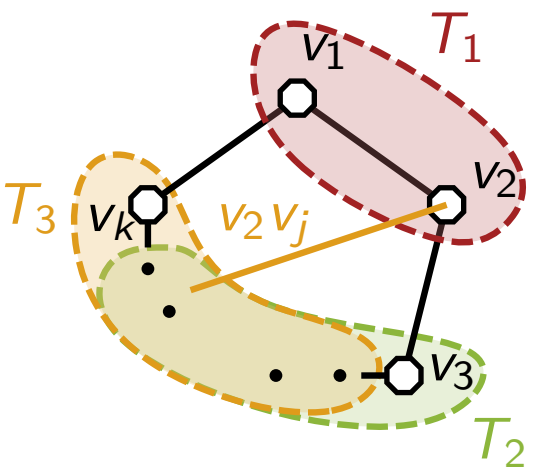
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Theorem 3.14 Proof

$(i) \implies (iii)$

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$(i) \implies (iii)$

(iii):There exists a tree $T = (V_T, E_T)$ such that

$$V_T = \{X \subseteq V \mid X \text{ is an inclusion-maximal clique in } G\},$$

and for every $v \in V$, the set

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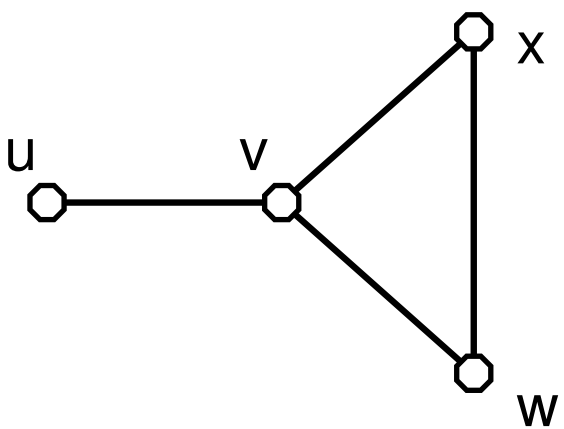
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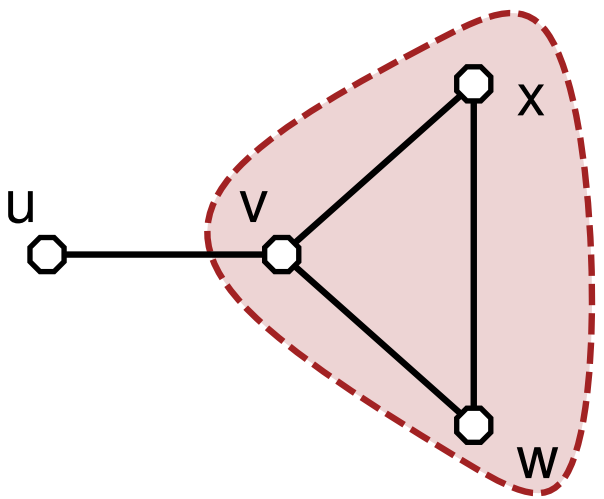
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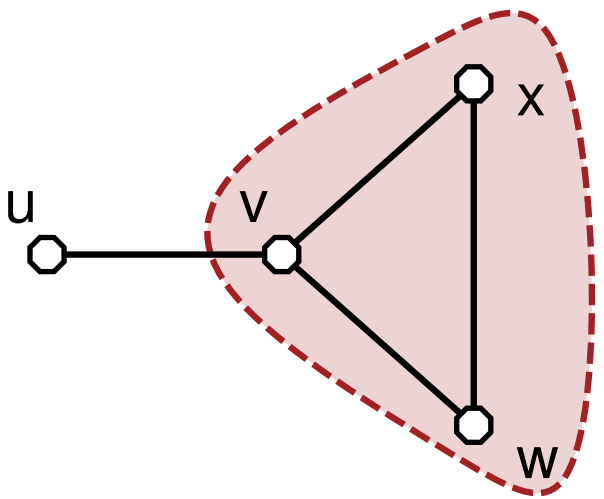
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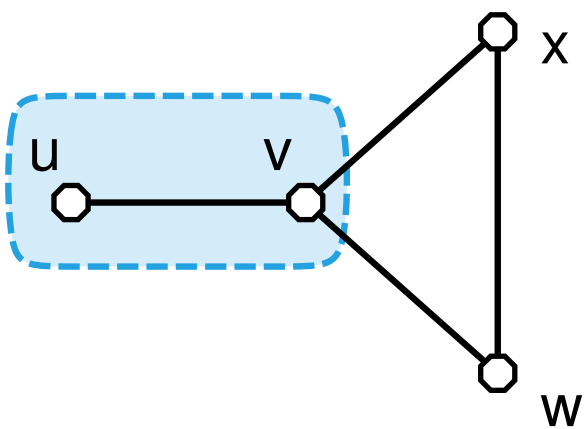
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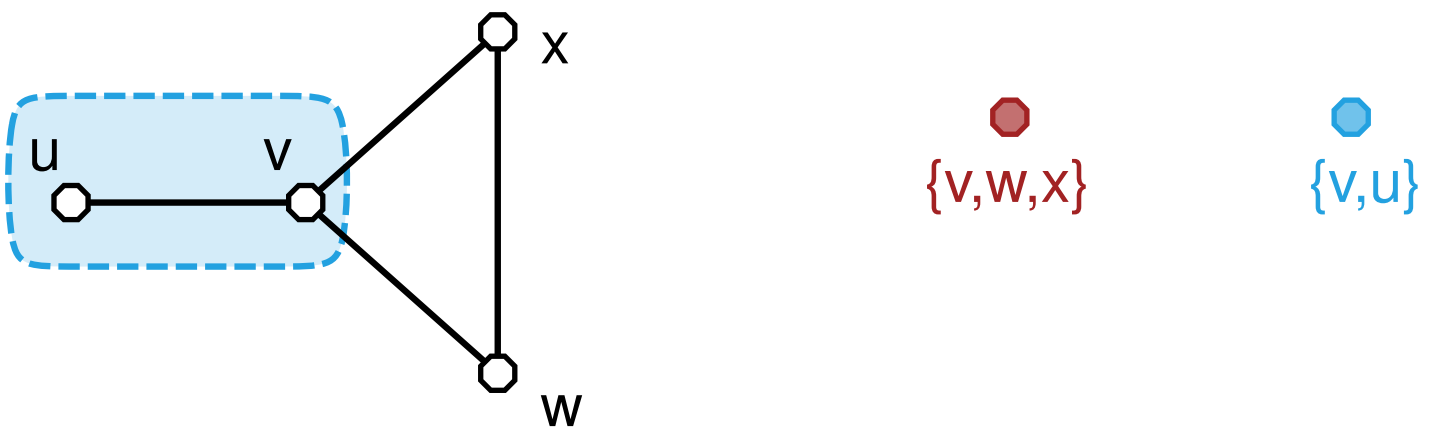
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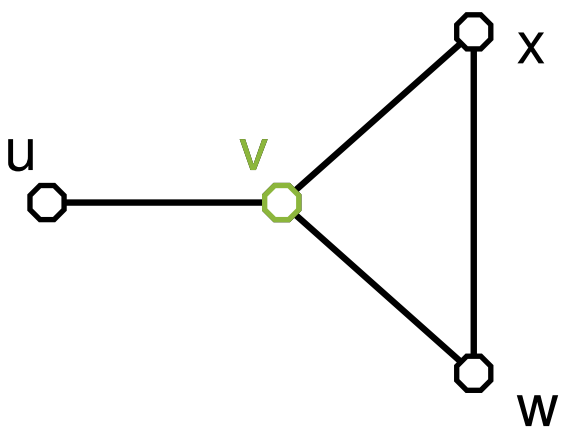
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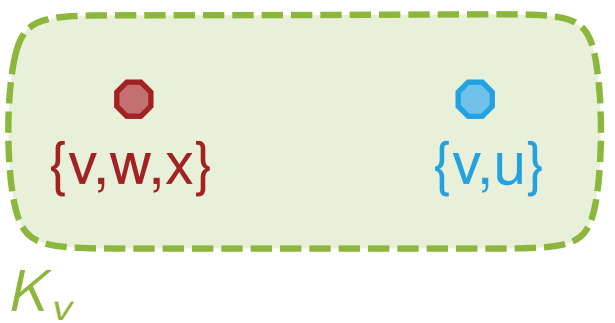
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Theorem 3.14 Proof

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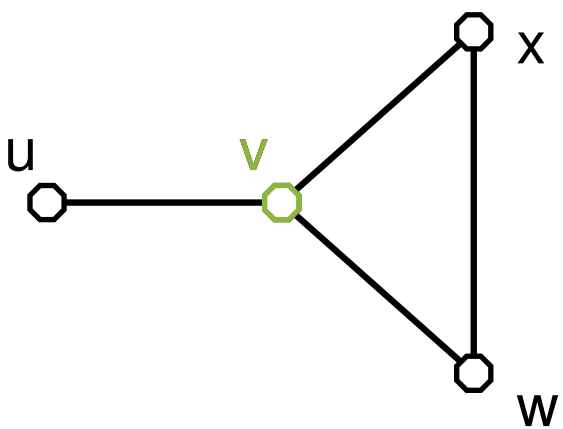
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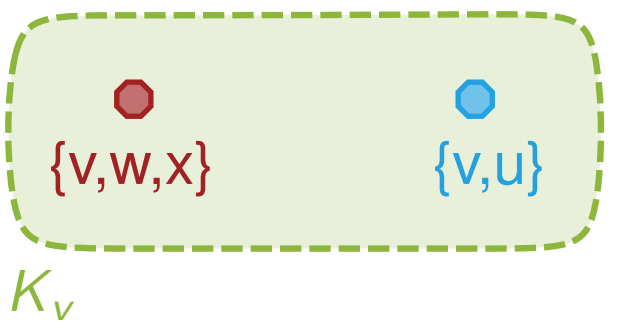
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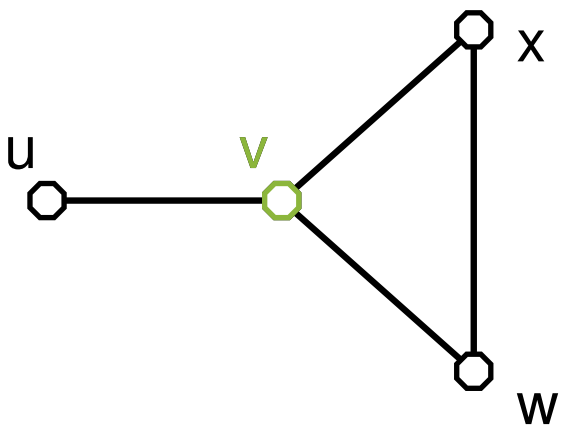
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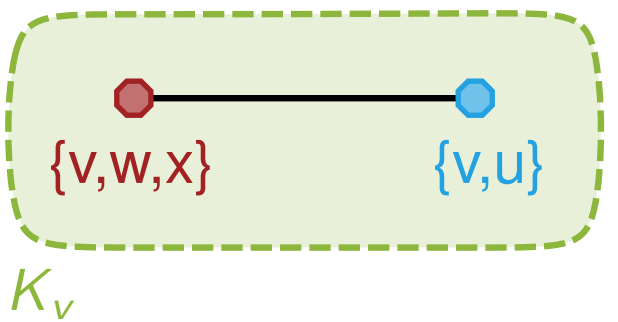
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$(i) \implies (iii)$

Let G be chordal

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Base case: $|V|(G) = \{v\}, V_T = K(G) = \{\{v\}\}$

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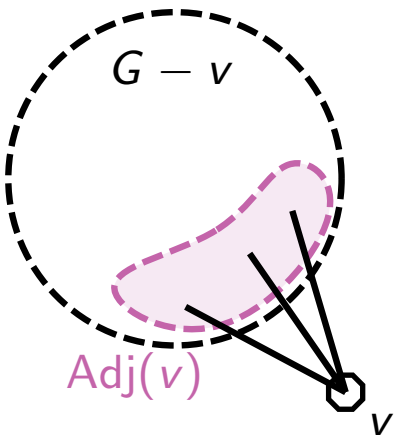
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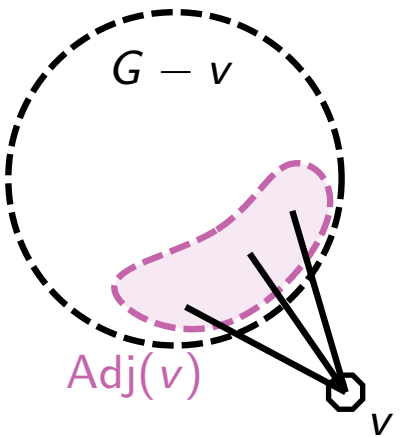
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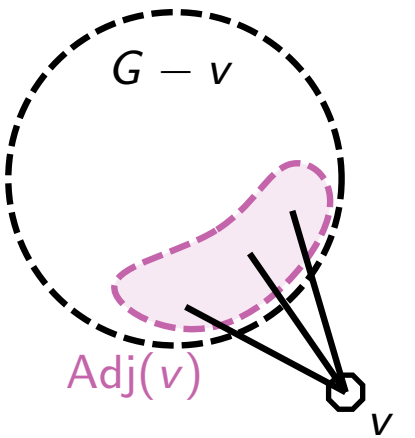
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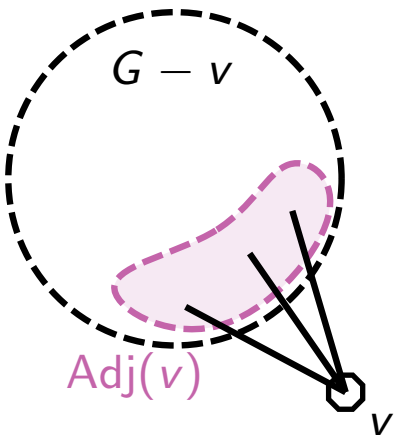
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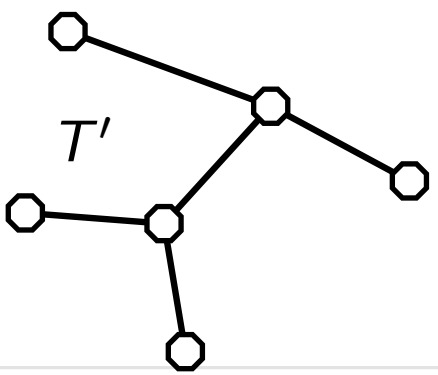
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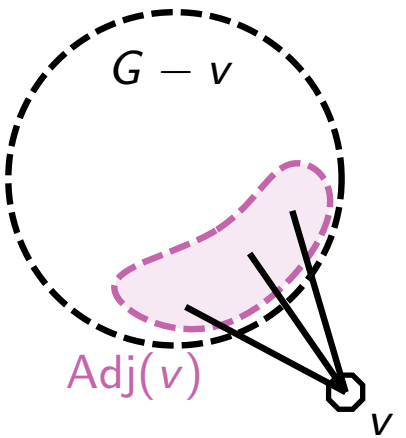
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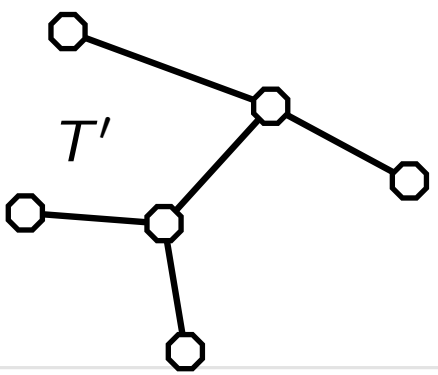
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$\mathrm{Adj}(v)$ is a clique but not necessarily inclusion maximal in $G - v$





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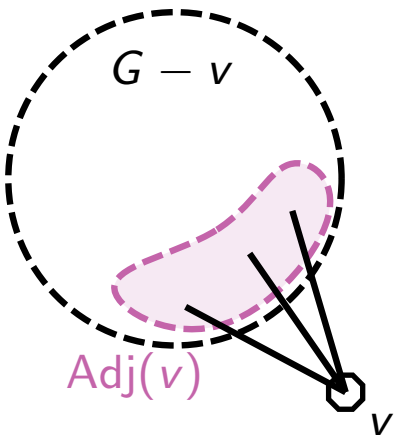
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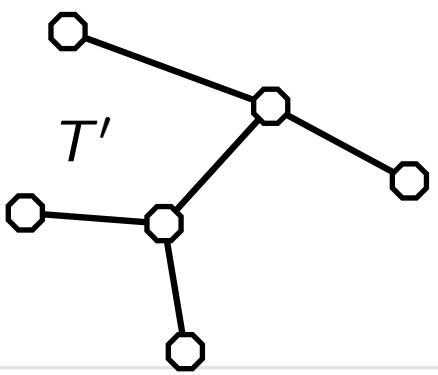
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case: $\text{Adj}(v) \in K(G - v)$





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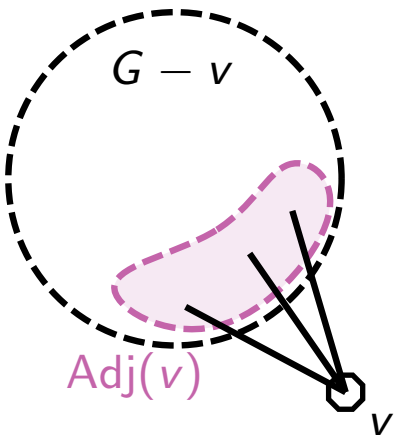
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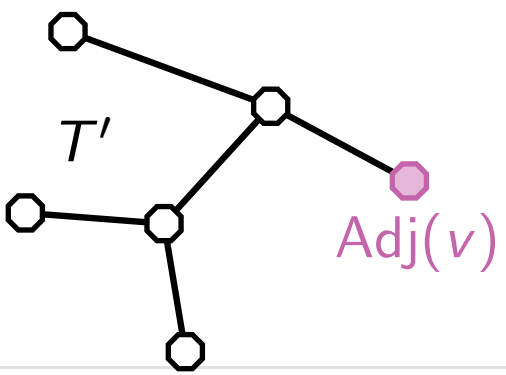
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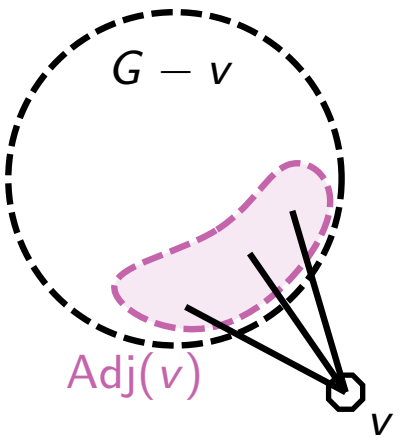
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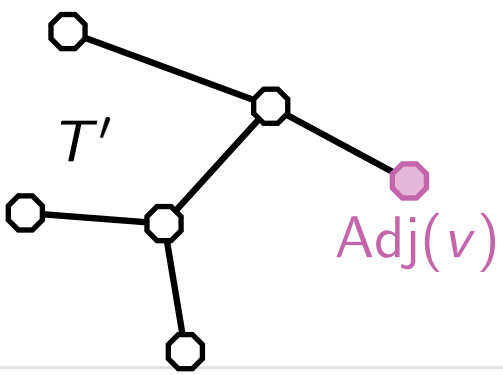
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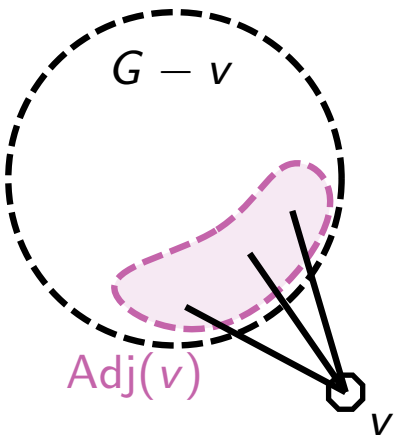
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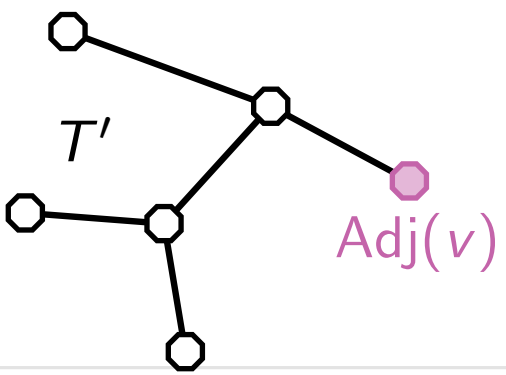
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v can't be in a different inclusion maximal clique in G





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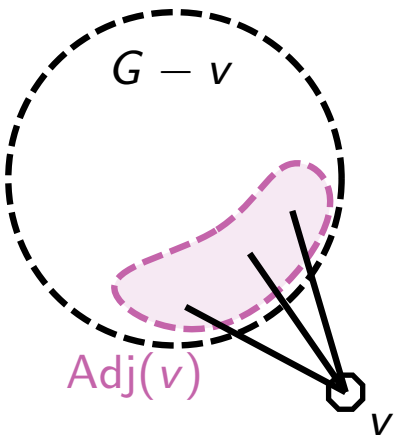
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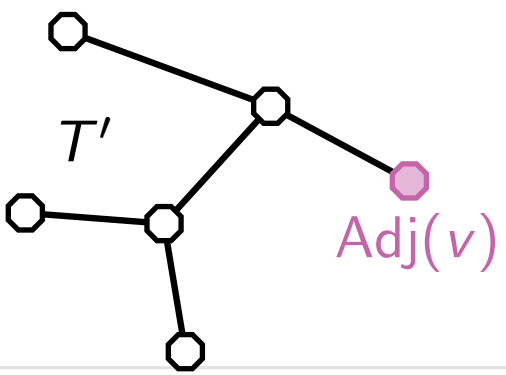
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$K(G - v) - \text{Adj}(v) = K(G) - (\text{Adj}(v) + \{v\})$





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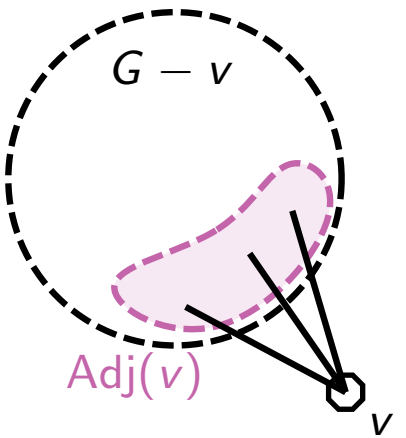
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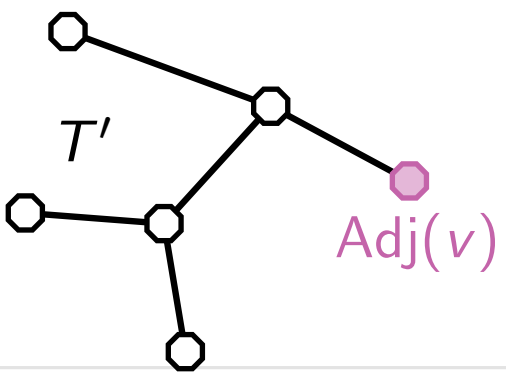
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$K(G - v) - \text{Adj}(v) = K(G) - (\text{Adj}(v) + \{v\})$

relabel $\text{Adj}(v)$ in T' to $Adj(v) + \{v\}$ in T





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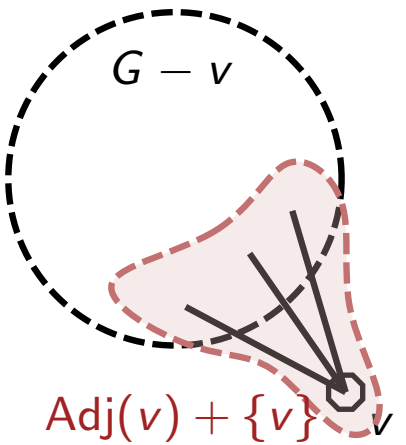
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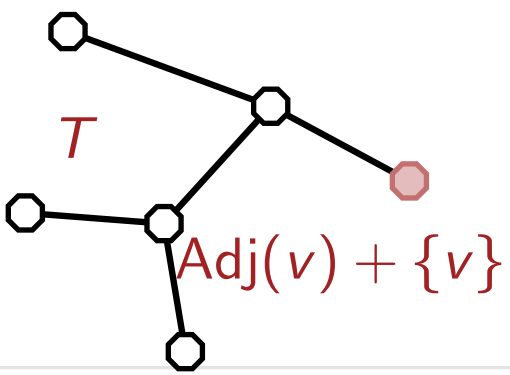
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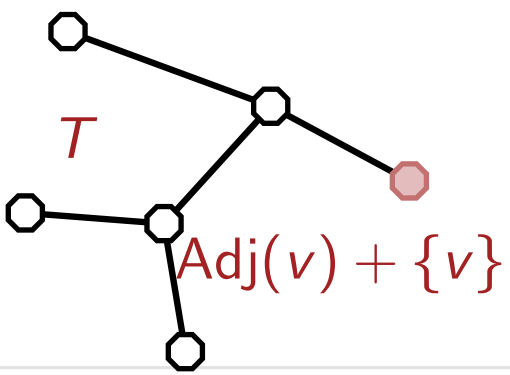
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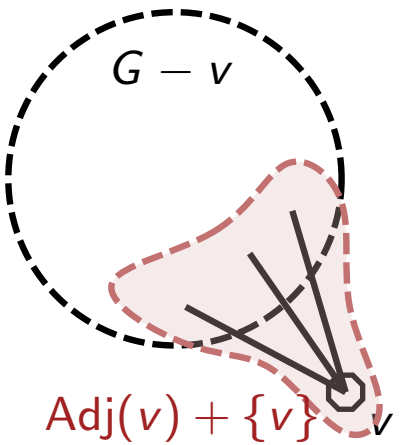
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relabel $\text{Adj}(v)$ in T' to $Adj(v) + \{v\}$ in T

$(*)$ still holds since $\forall w \neq v : K_w$ didn't change and

$K_v = \{\text{Adj}(v) + \{v\}\}$



$$(*) \begin{cases} \forall v \in V : K_v = \{X \in K(G) \mid v \in \\ \text{induces a subtree in T.} \end{cases}$$

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$(i) \implies (iii)$

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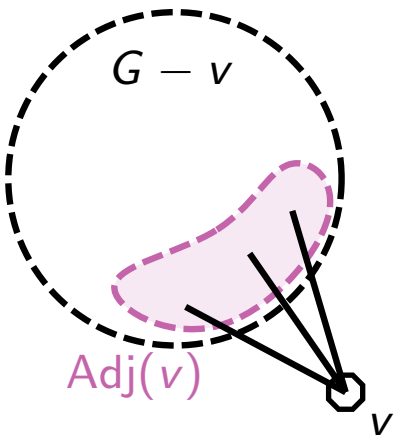
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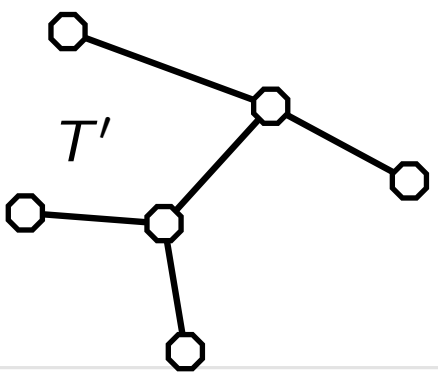
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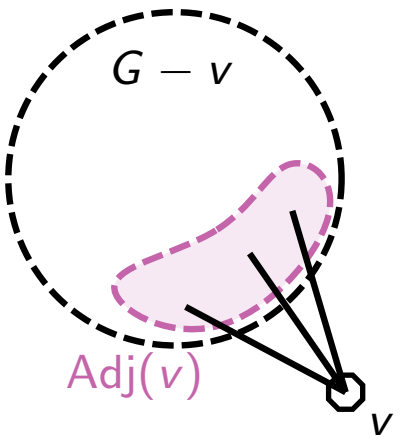
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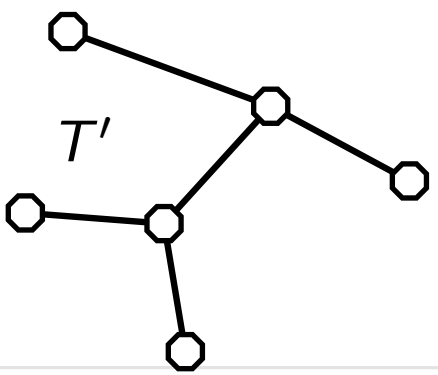
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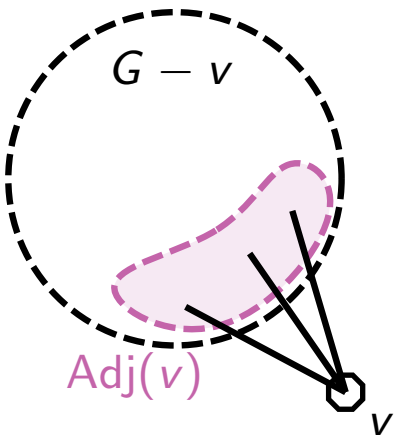
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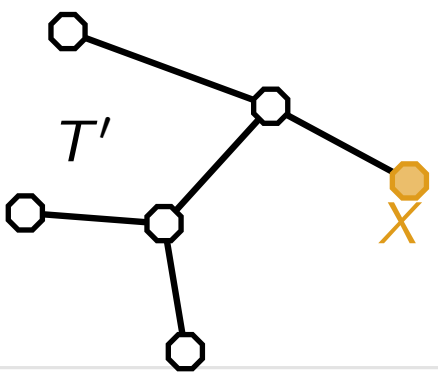
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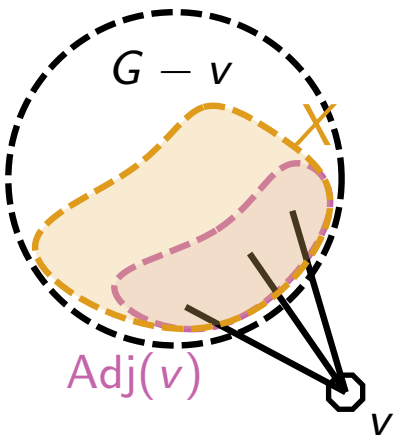
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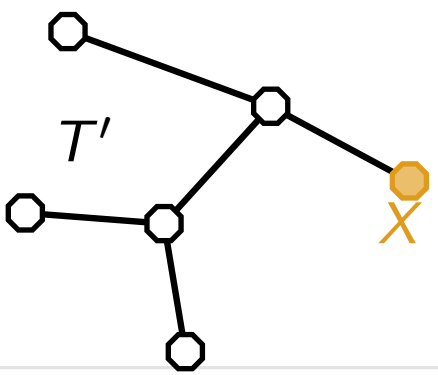
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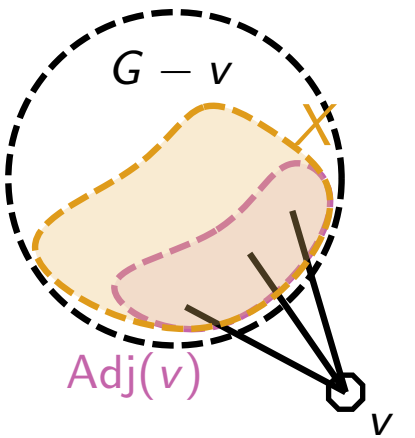
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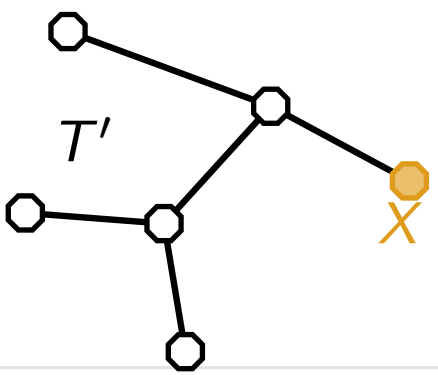
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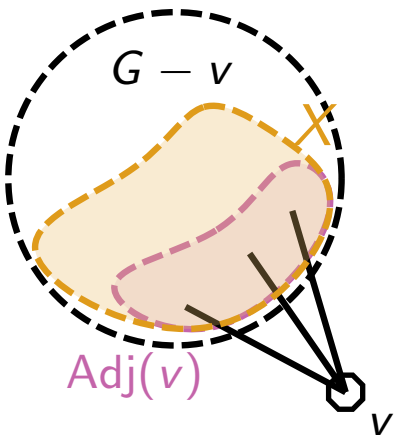
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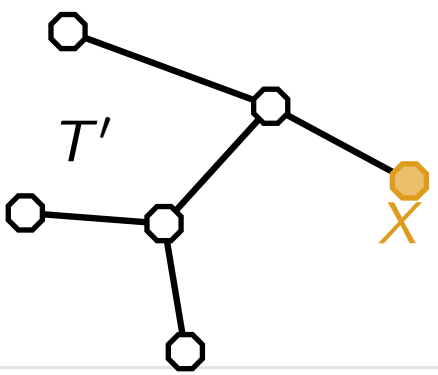
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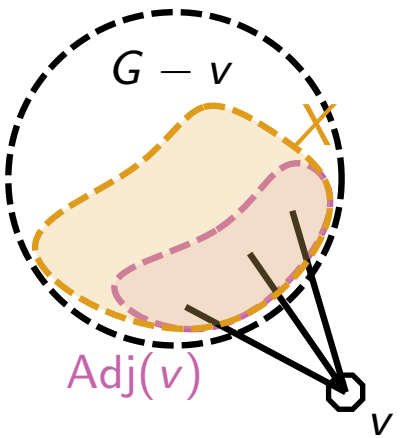
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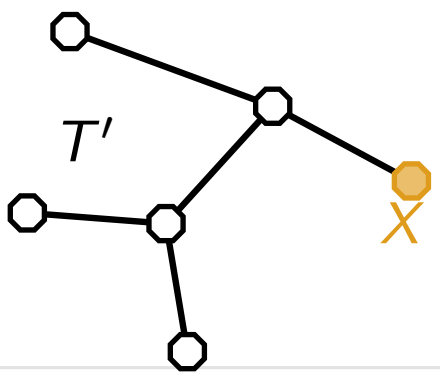
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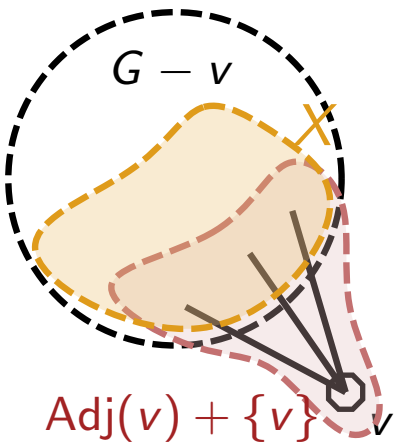
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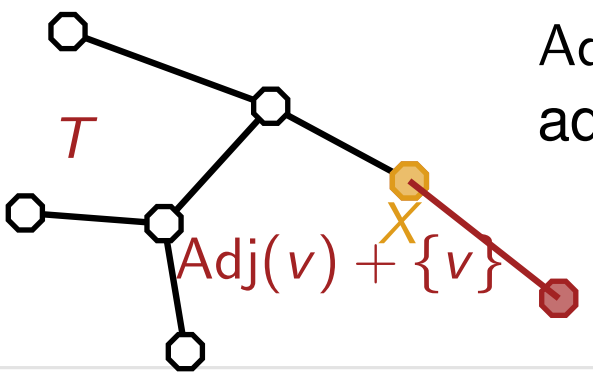
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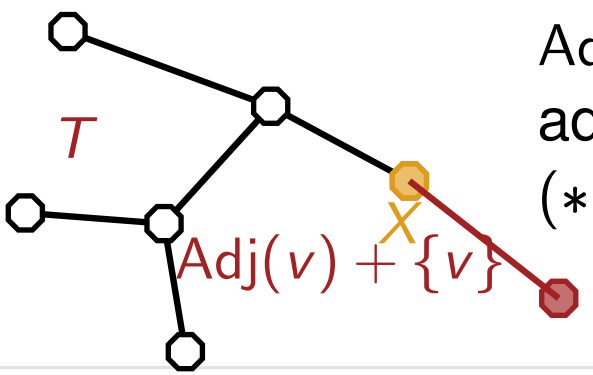
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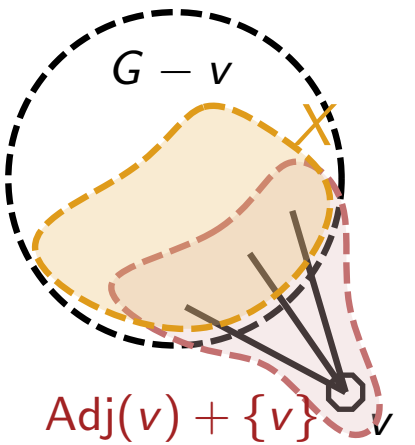
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$(*)$ still holds as $\forall w \in \text{Adj}(v) : w \in x$





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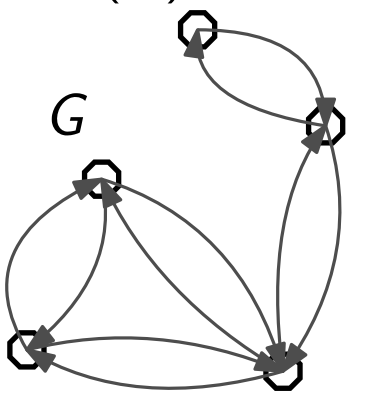
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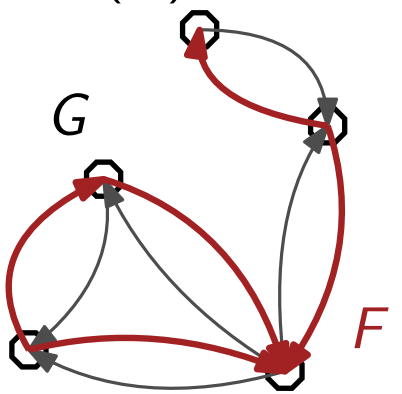
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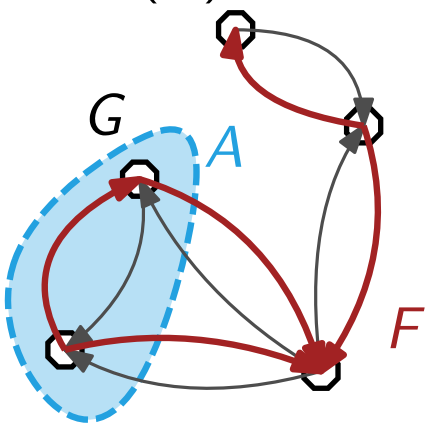
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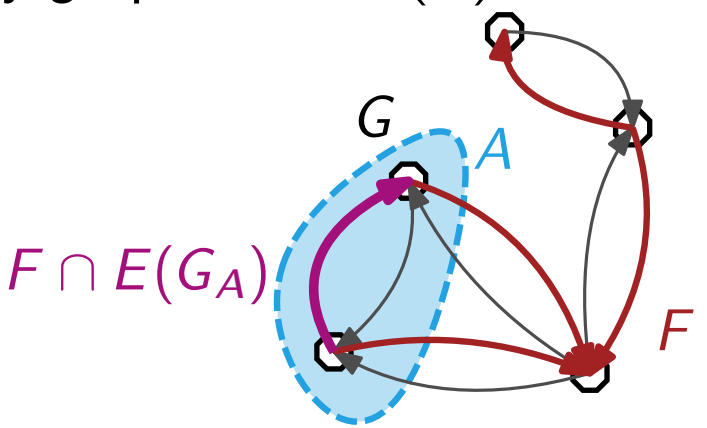
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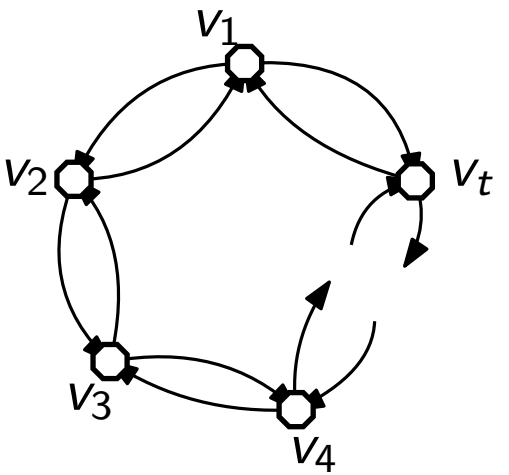
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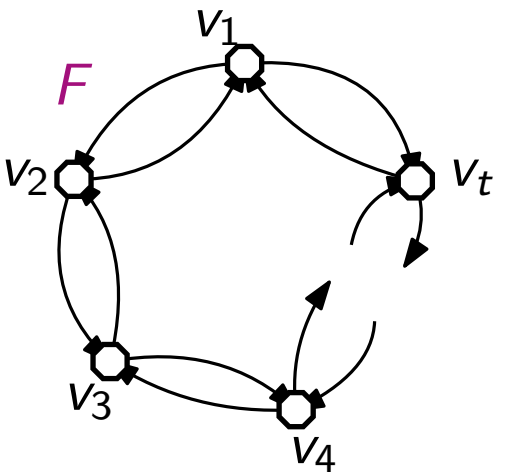
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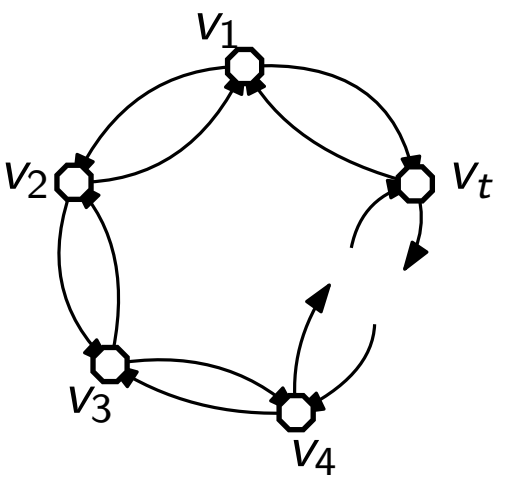
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Comparability Graphs are perfect

Theorem

G is a comparability graph $\implies$ G is perfect

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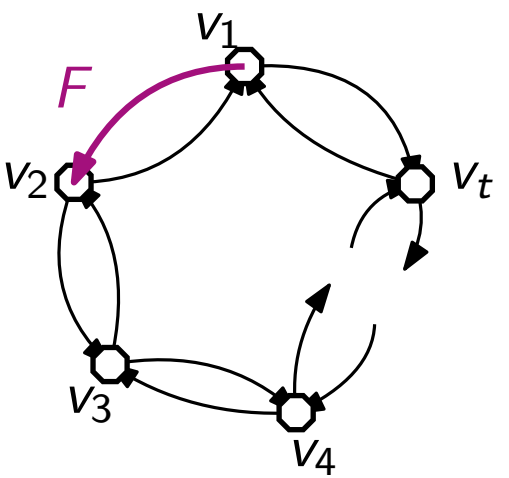
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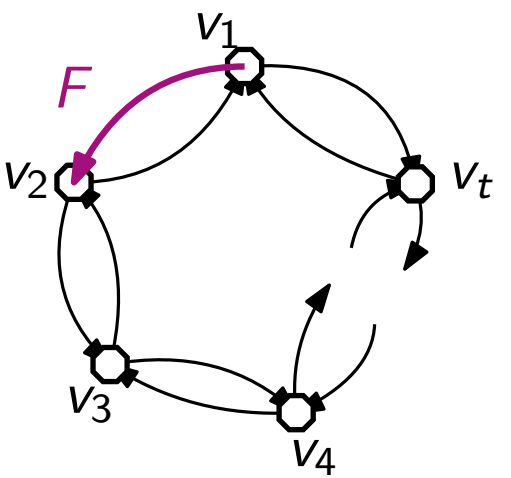
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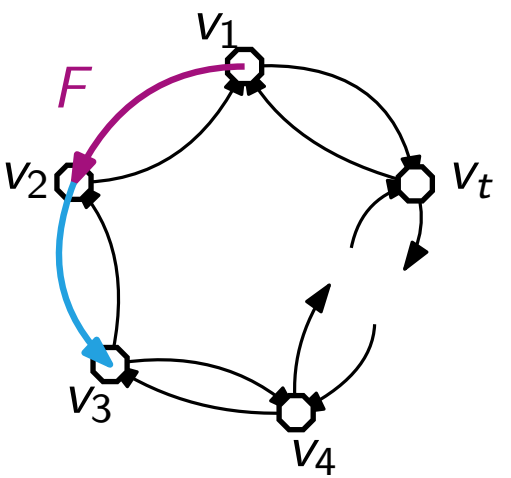
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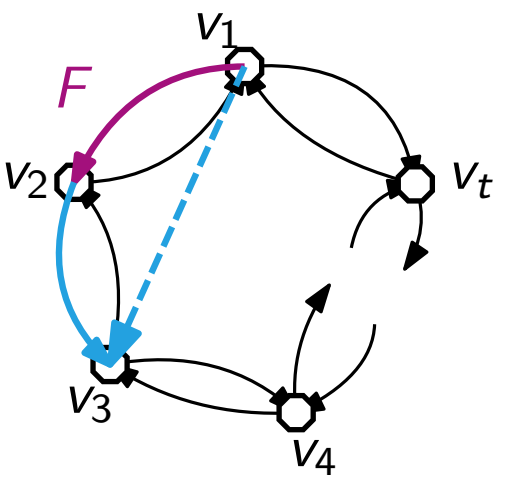
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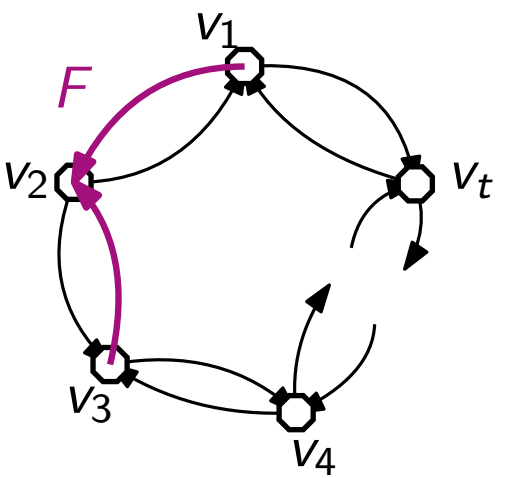
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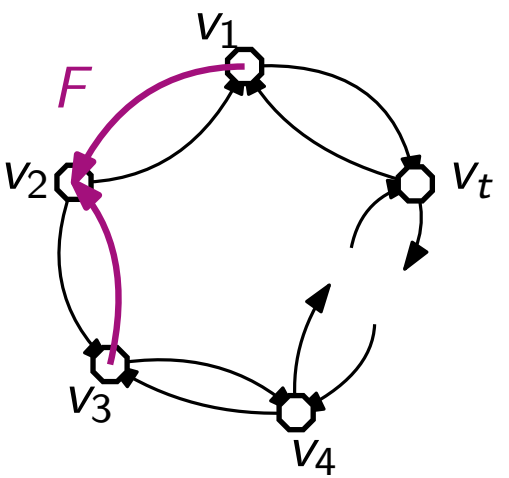
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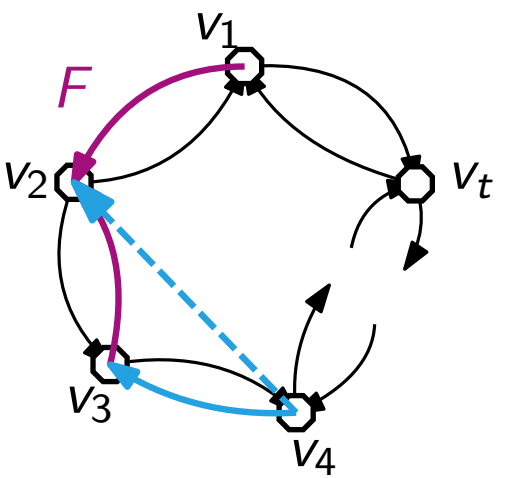
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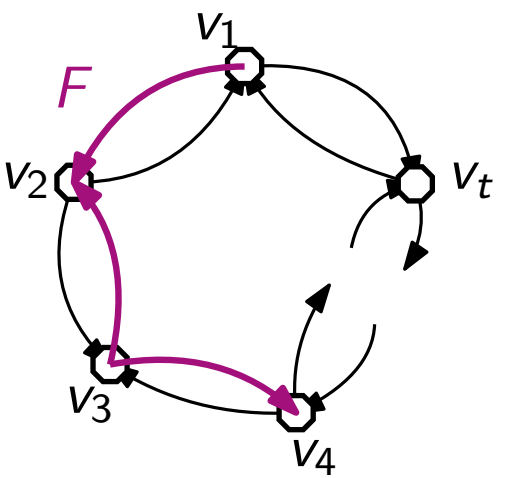
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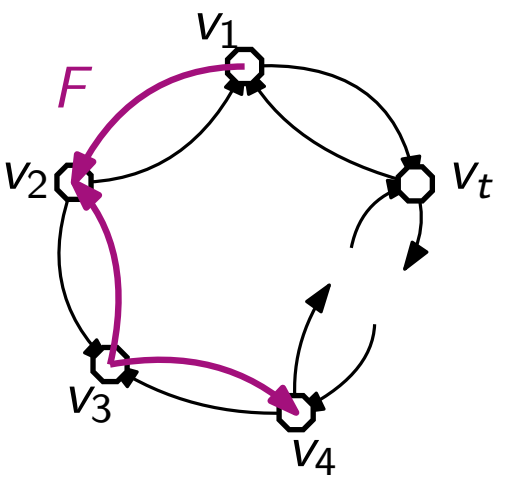
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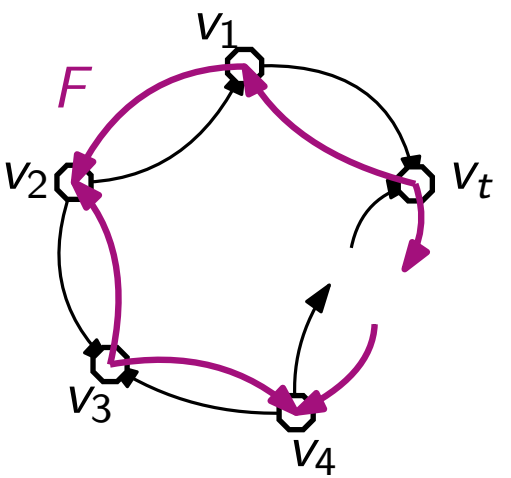
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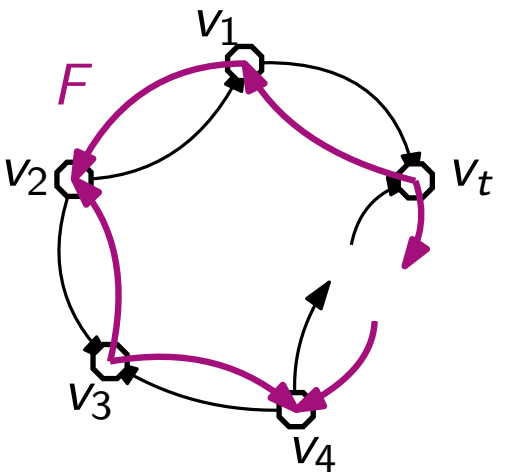
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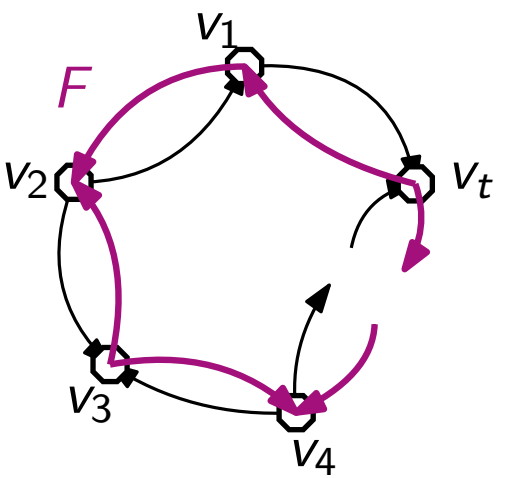
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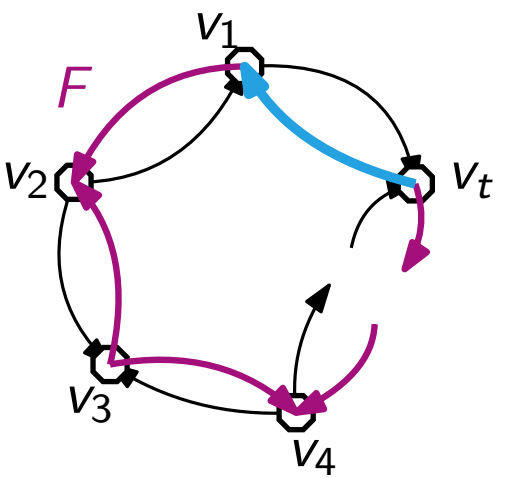
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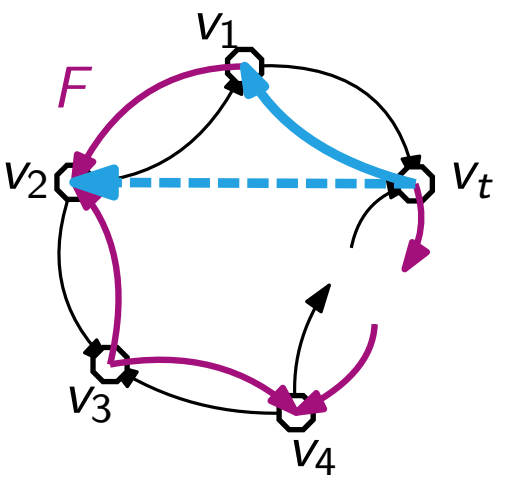
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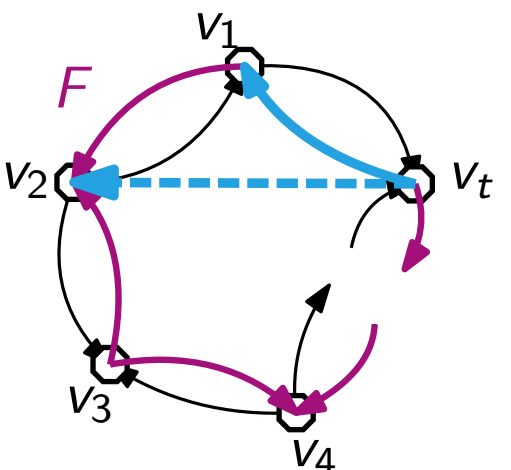
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Comparability Graphs are perfect

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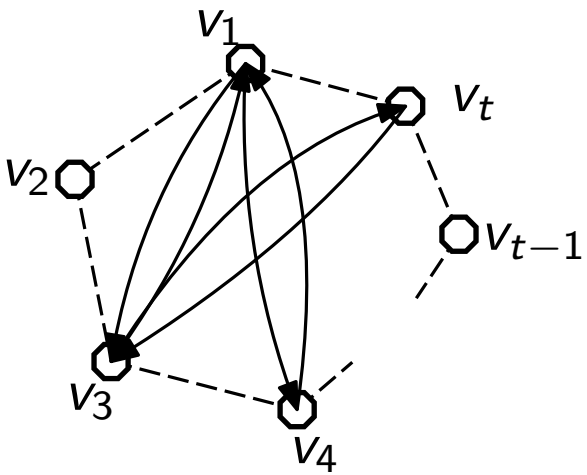
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section of edges of $\overline{C_t}$

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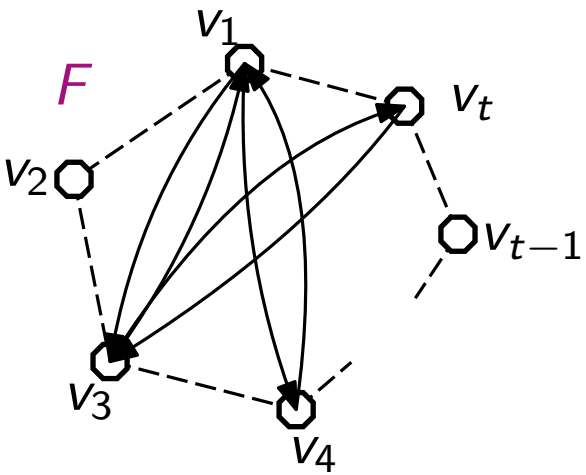
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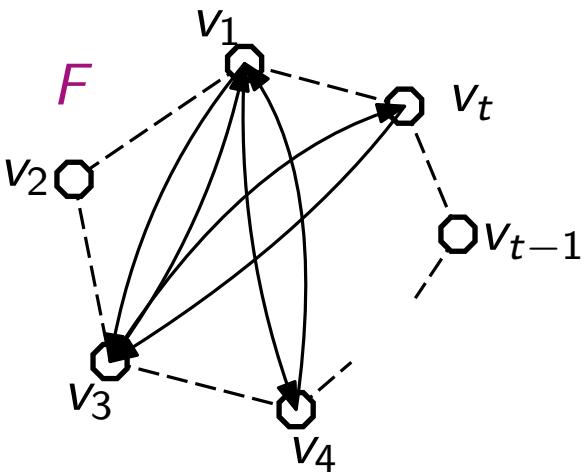
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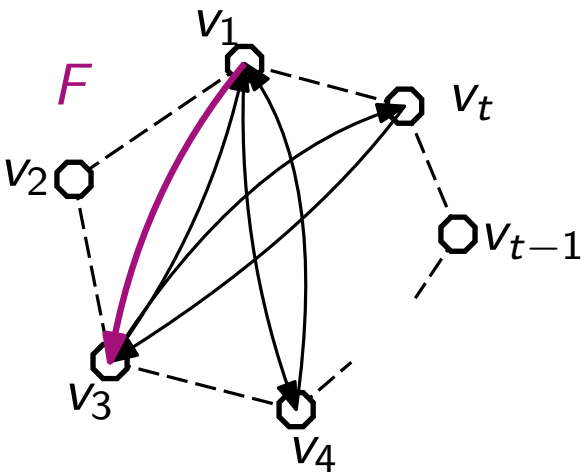
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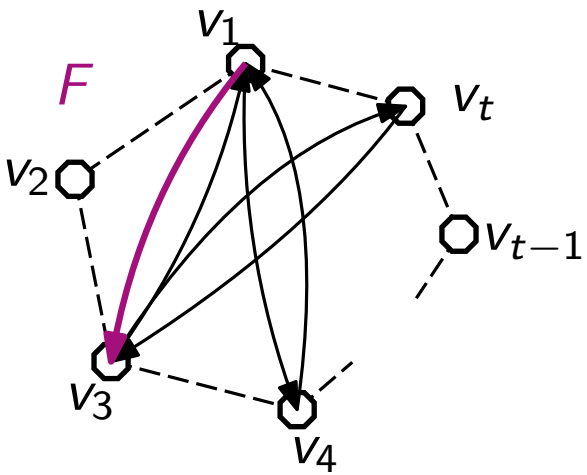
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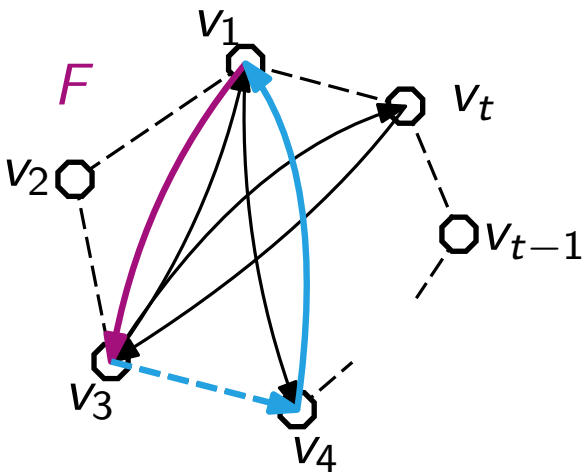
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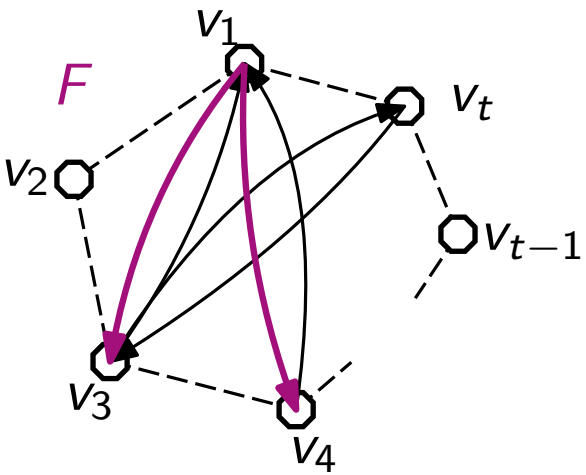
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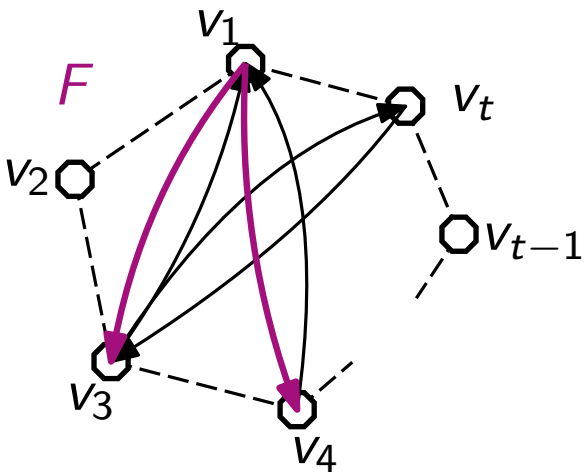
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wlog $v_1 v_3 \in F$

iteratively we can apply the same argument to obtain $v_1 v_i \in F$ for all $3 \leq i \leq t-1$



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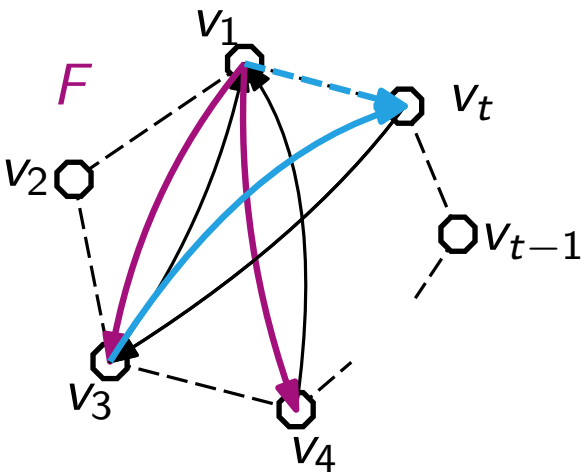
$\overline{C_t}$, $t \geq 6$ odd:

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wlog $v_1 v_3 \in F$

iteratively we can apply the same argument to obtain $v_1 v_i \in F$ for all $3 \leq i \leq t-1$

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section of edges of $\overline{C_t}$

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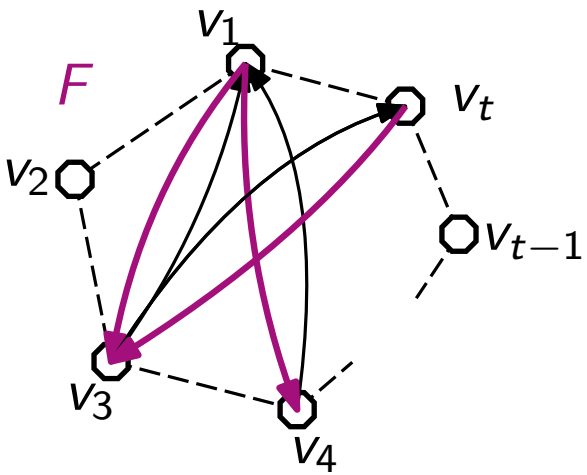
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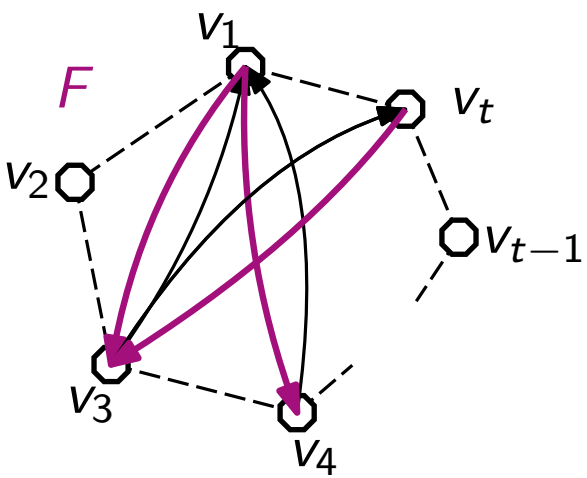
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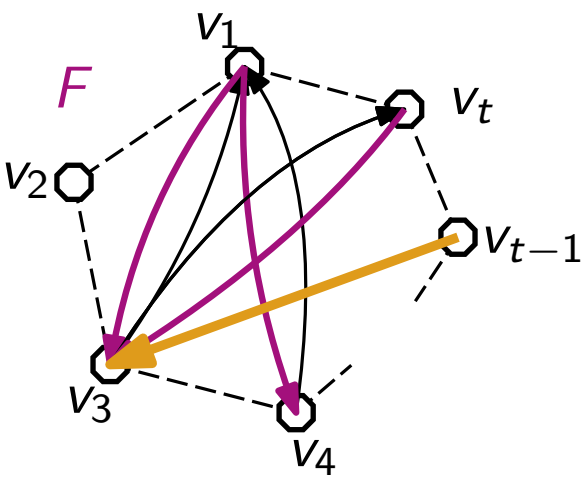
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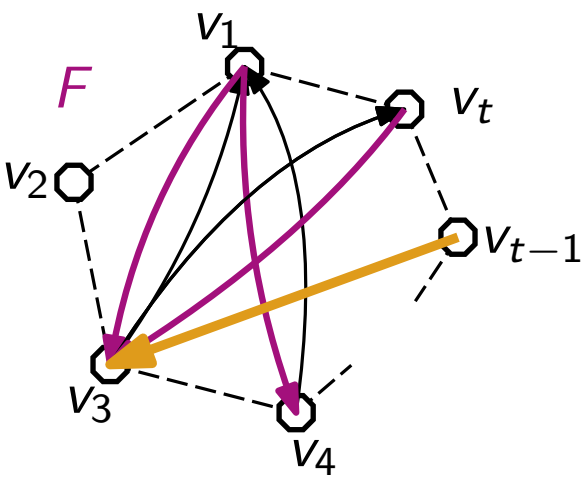
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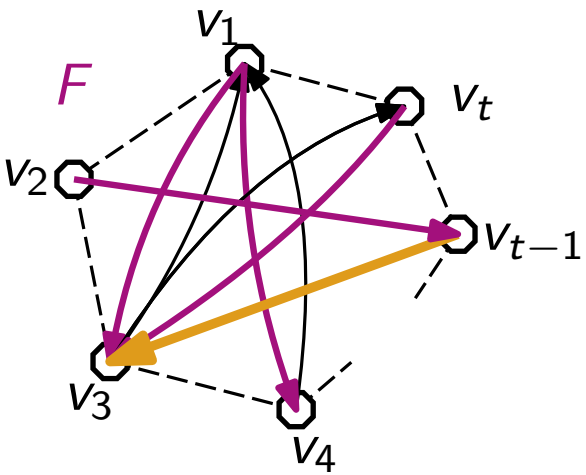
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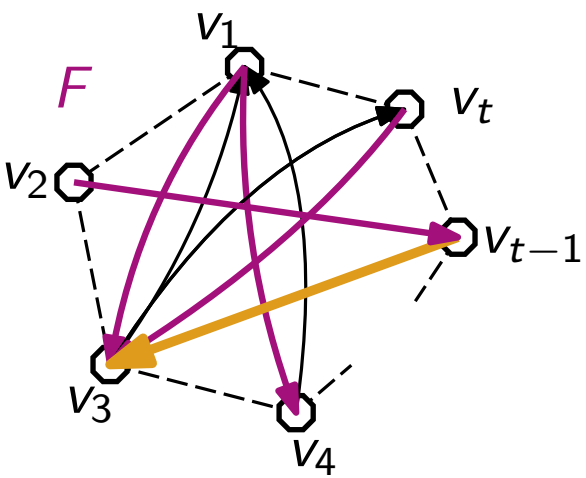
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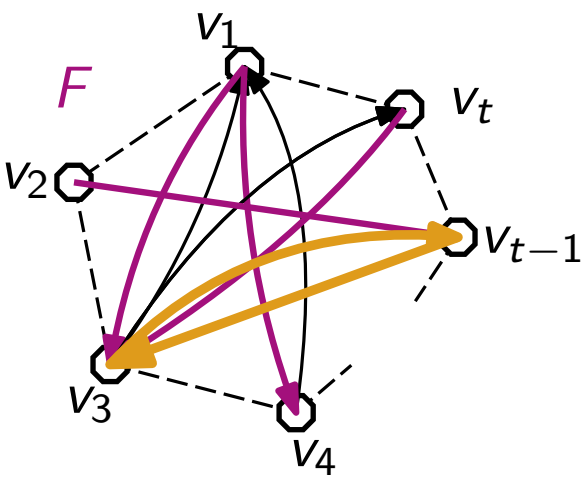
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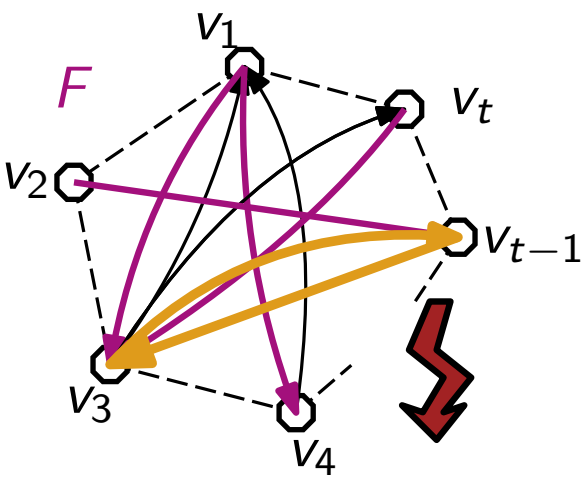
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Observation on implication classes

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Let $ab \in A \iff ba \in A^{-1}$

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Let $ab \in A \iff ba \in A^{-1}$

It follows $ba\Gamma^* yx \iff ab\Gamma^* xy \iff xy \in A \iff yx \in A^{-1}$

Theorem 4.1

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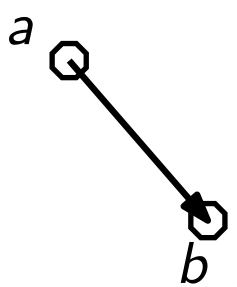
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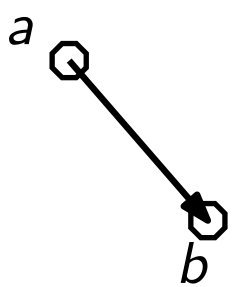


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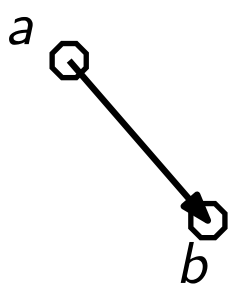


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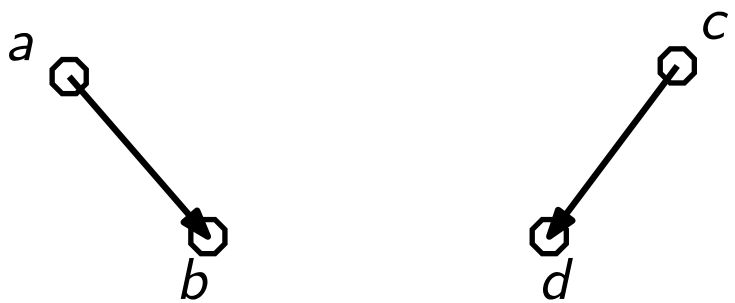
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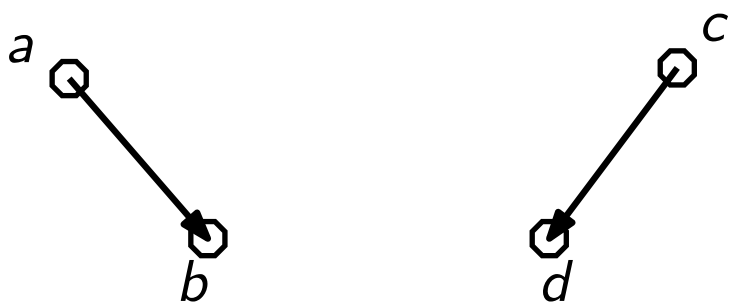
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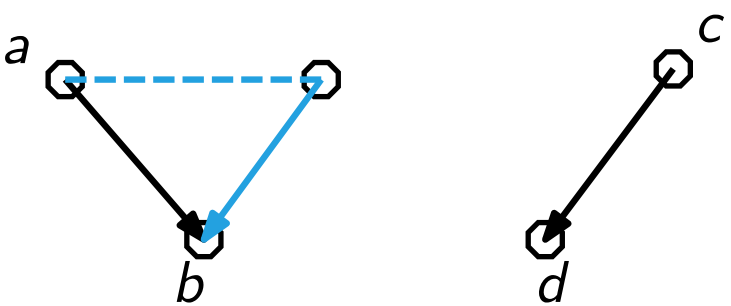
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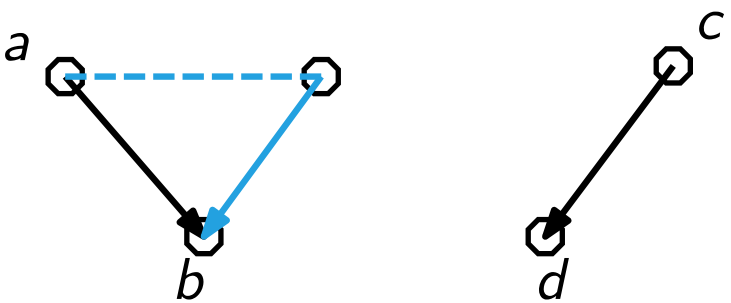
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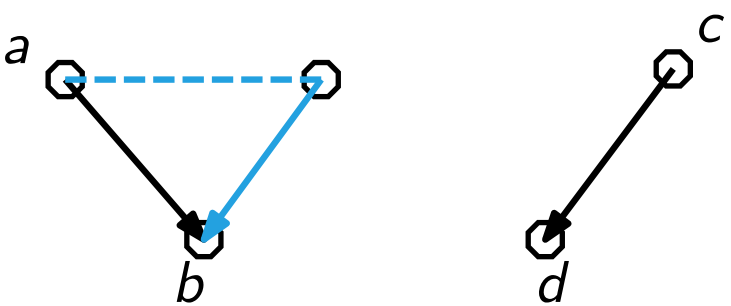
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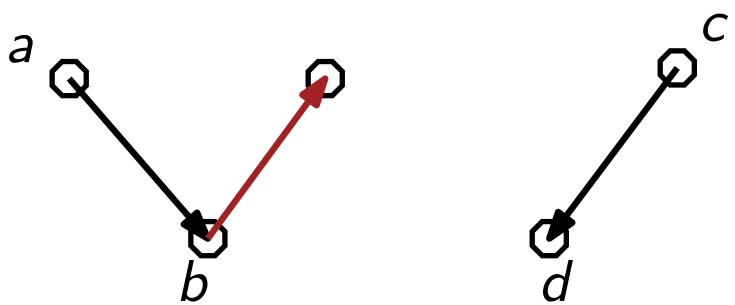
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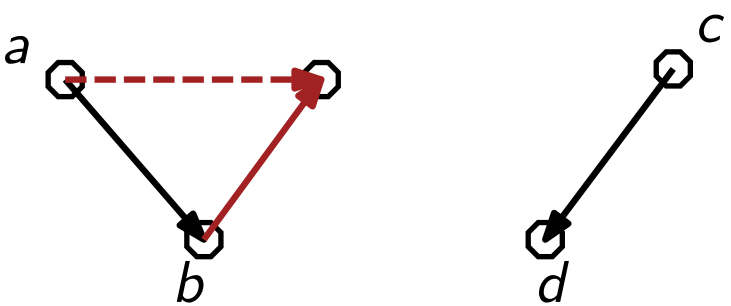
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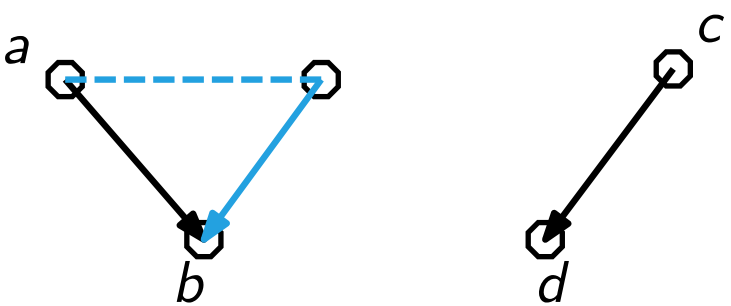
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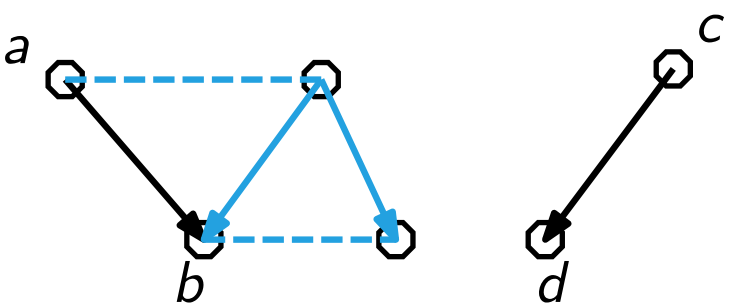
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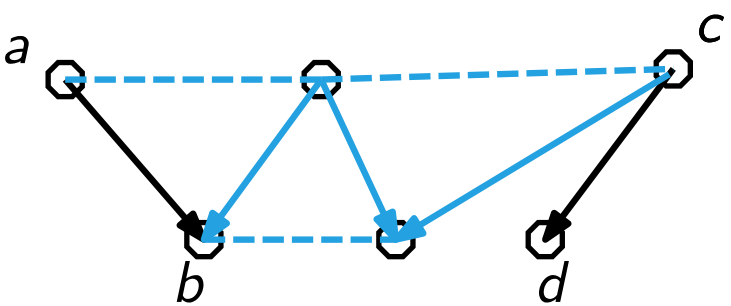
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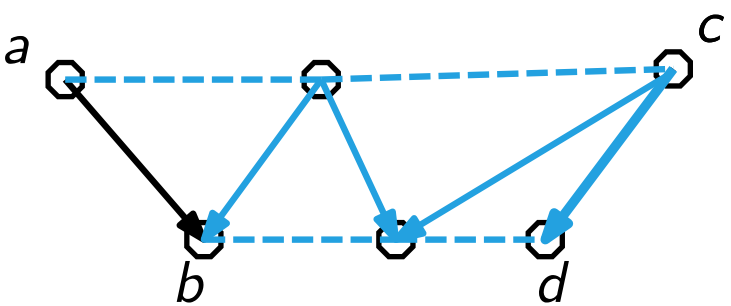
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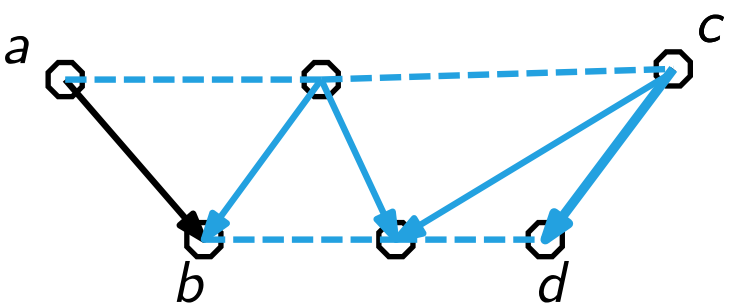
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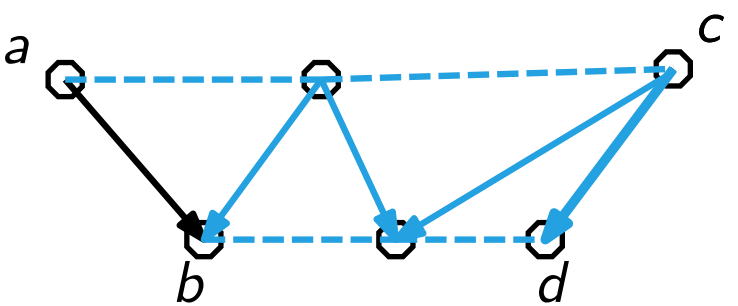
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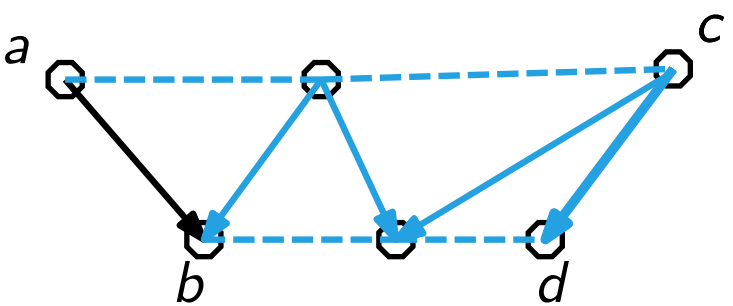
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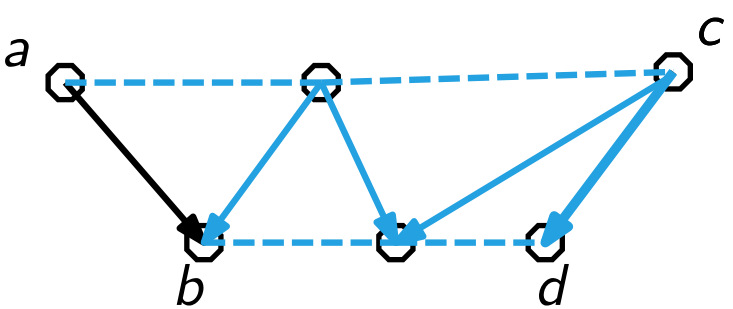
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- by replacing A with A^{-1} in the previous case (and vice versa) we can similarly obtain $F \cap \hat{A} = A^{-1}$

Corollary from Theorem 4.1

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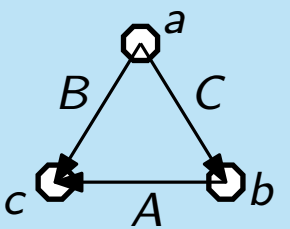
hence $A \cap A^{-1} = \varnothing$

Note: the only to cases can be $A \cap A^{-1} = \varnothing$ and $A = A^{-1}$ since A and A^{-1} are equivalence classes on a equivalence relation

Triangle Lemma

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G undirected graph, $A, B, C \in \mathcal{I}(G)$.
- triangle $abc \in G$ i.e. $ab \in C, ac \in B, bc \in A$
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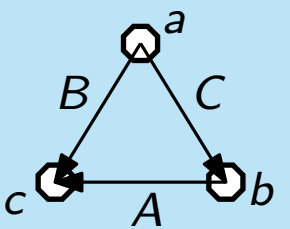
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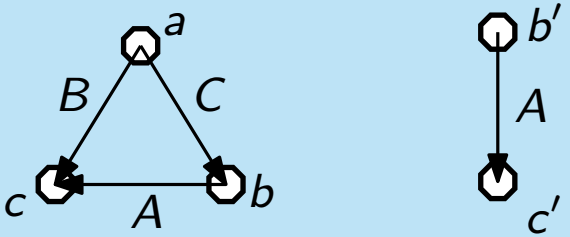
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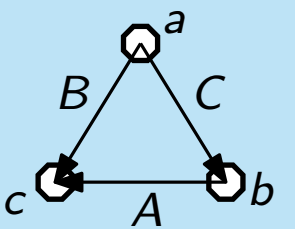


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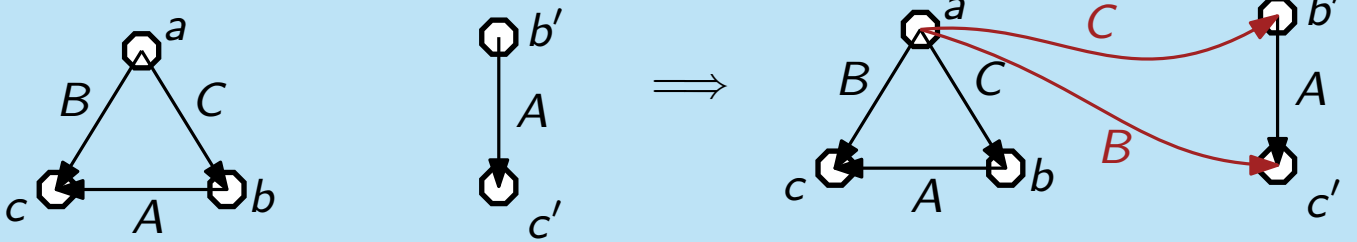
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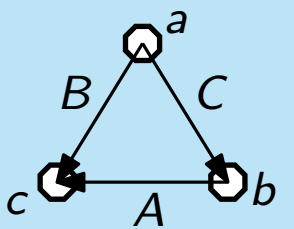


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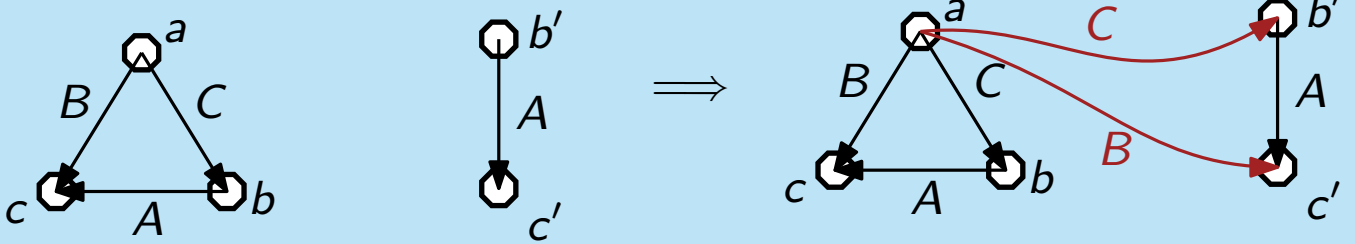
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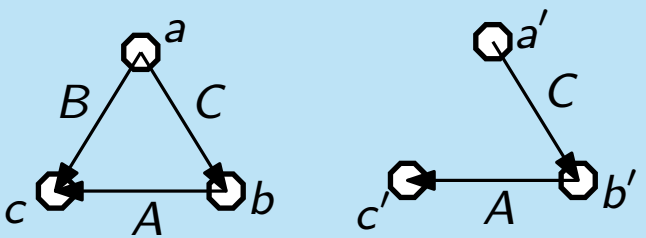
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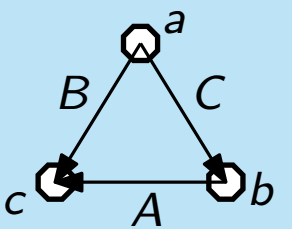
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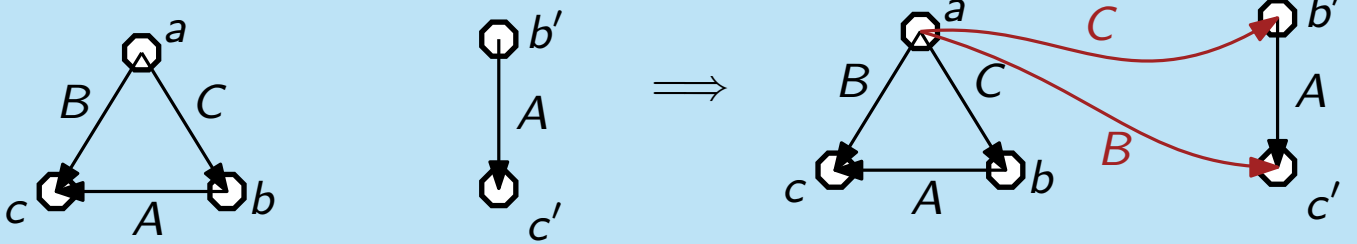
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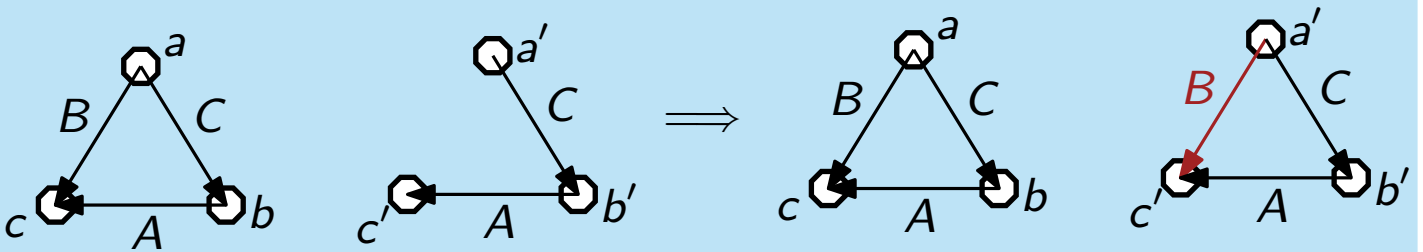
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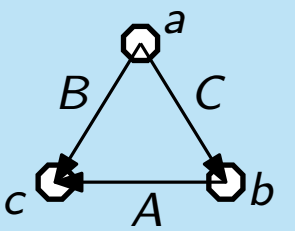
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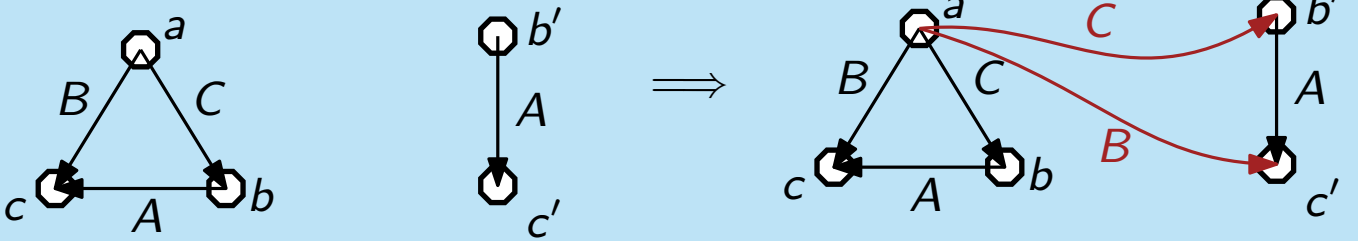
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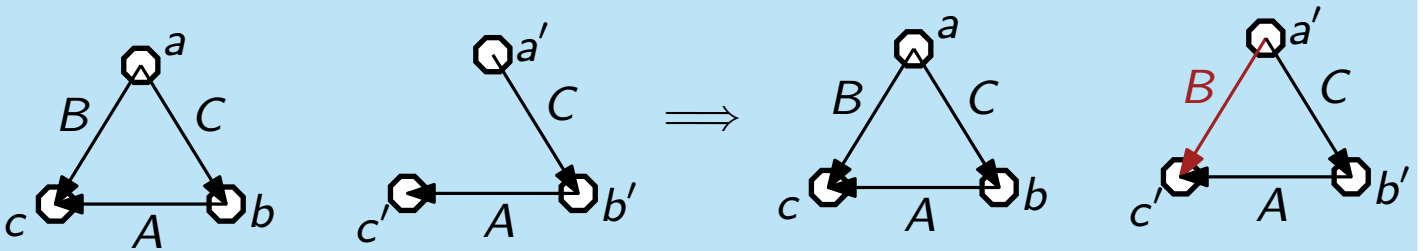
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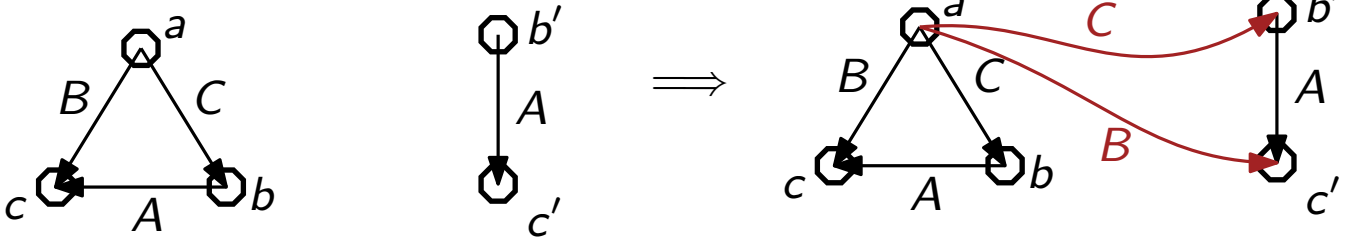


important:

- $A = C, B^{-1} = C, \ldots$ is possible!
- $a' = b, b' = c, \ldots$ is possible!

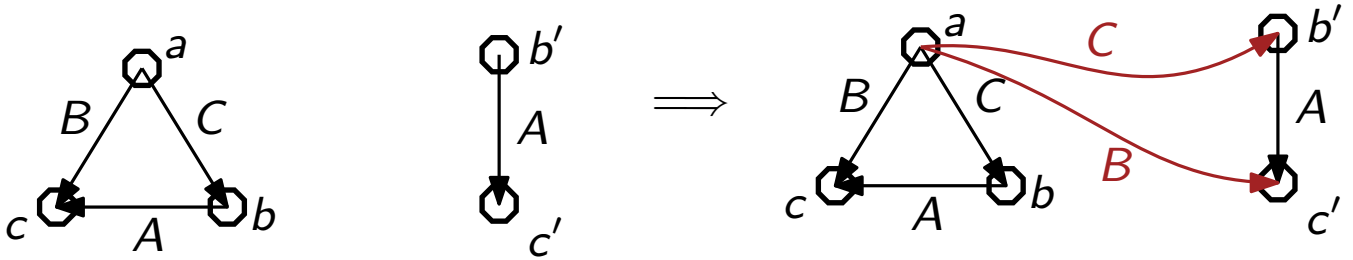
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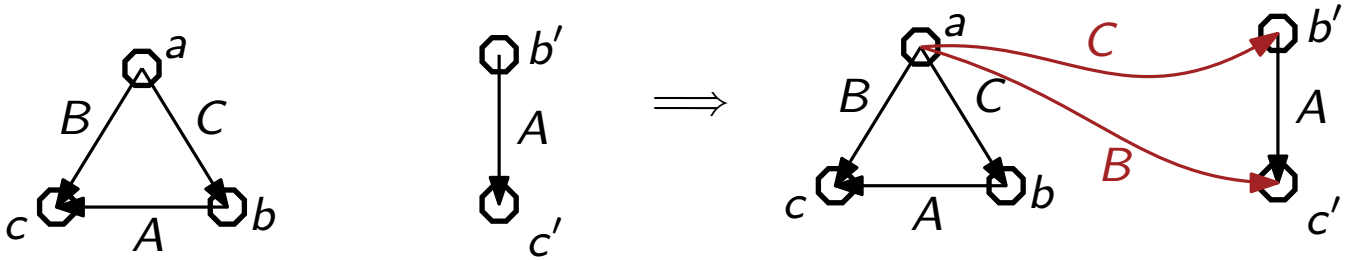
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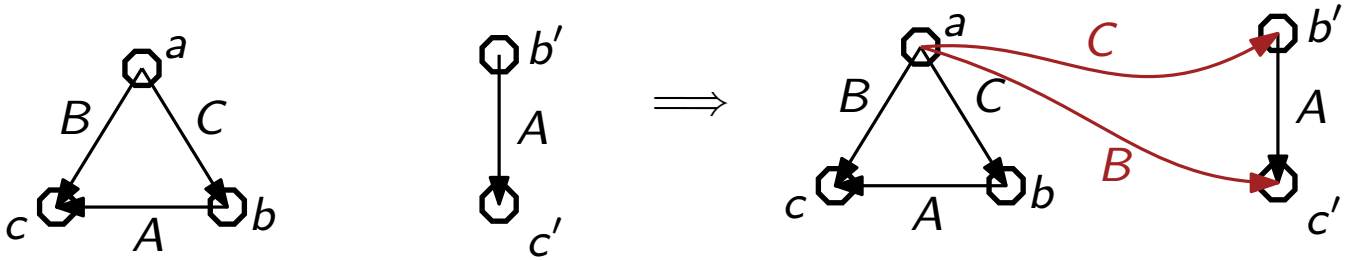
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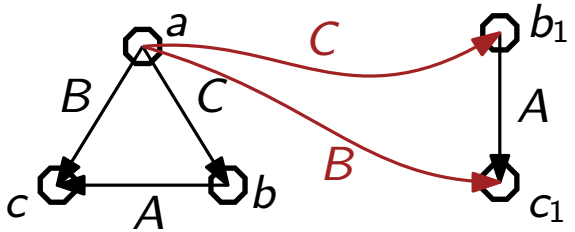
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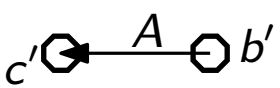
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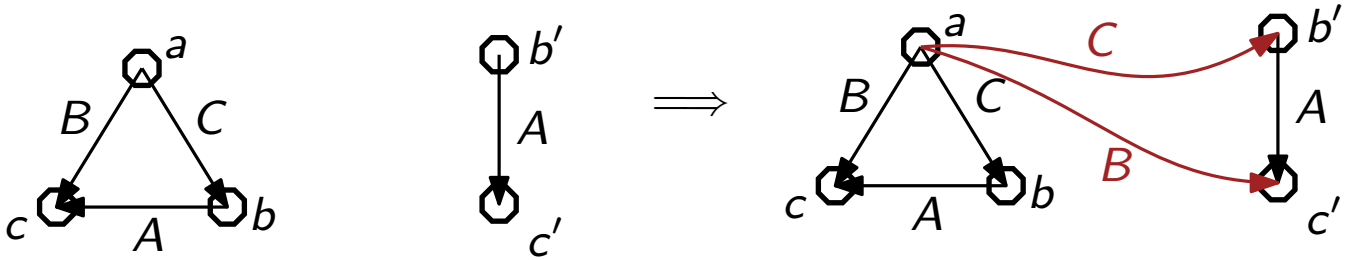
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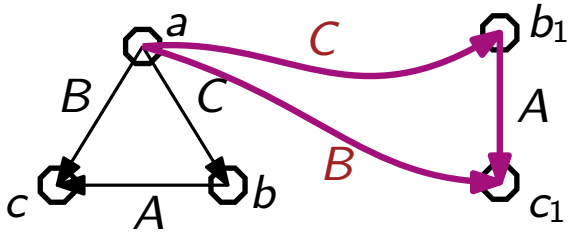


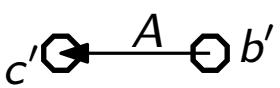
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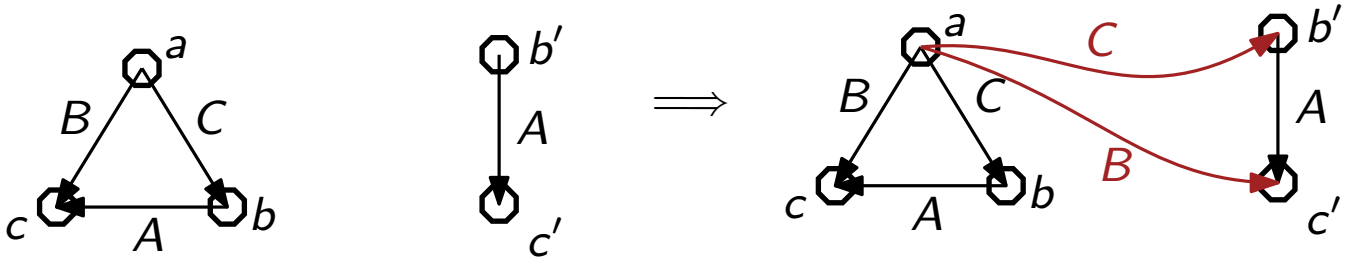
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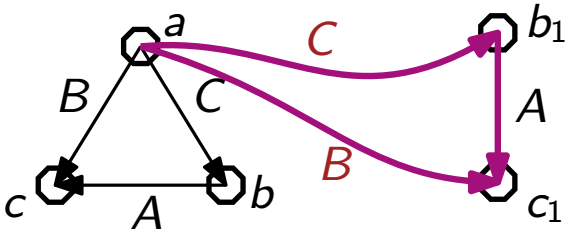


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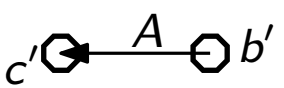
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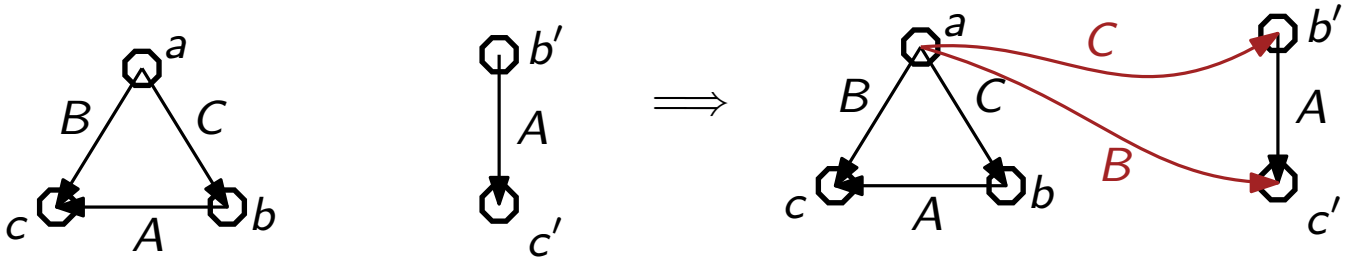


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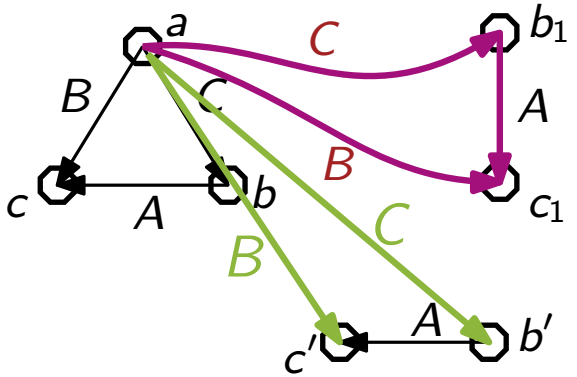


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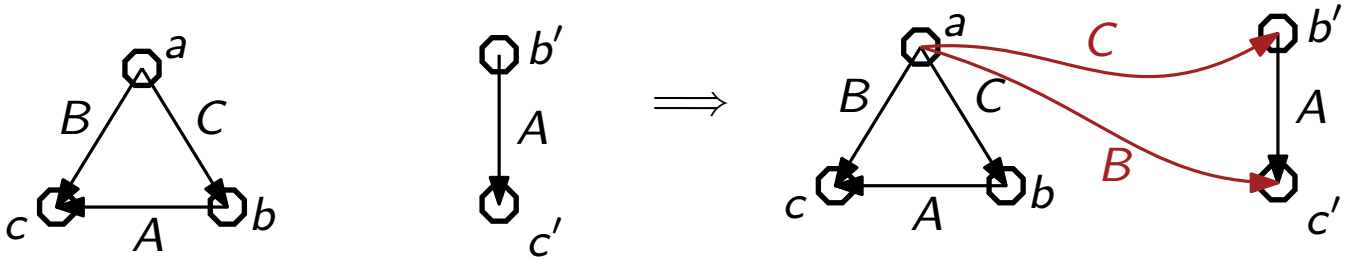
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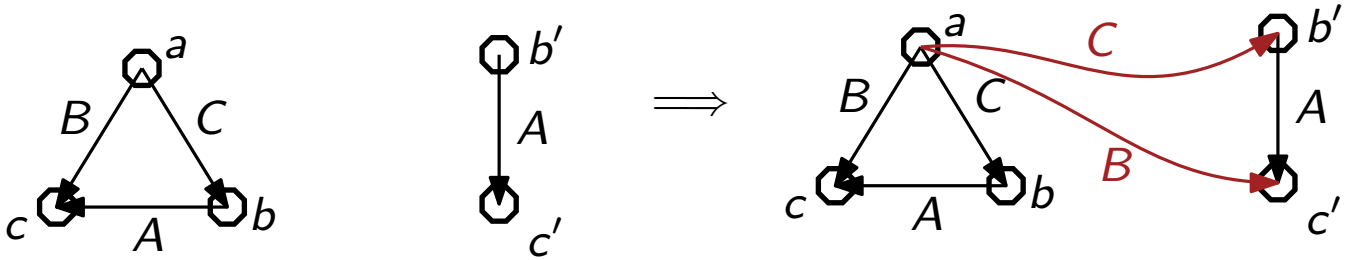


wlog only consider one step in Γ-chain $bc\Gamma^* b'c'$
two cases:

- $b = b'$ and $cc' \notin E(G)$
- $c = c'$ and $bb' \notin E(G)$

Triangle Lemma - Proof

Proof i)

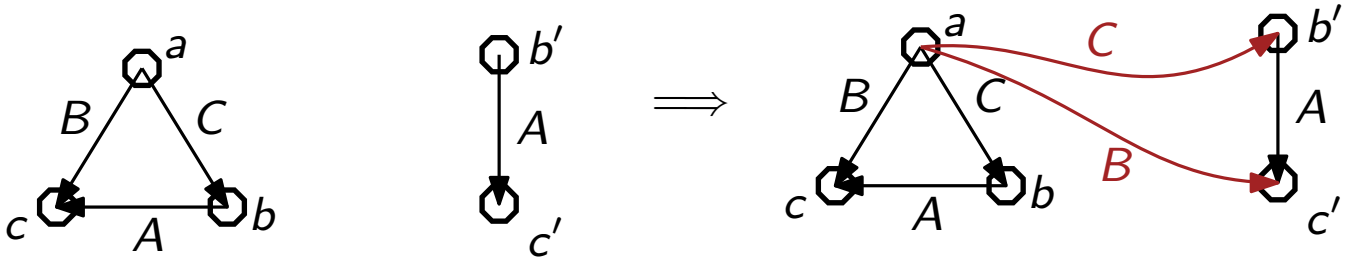


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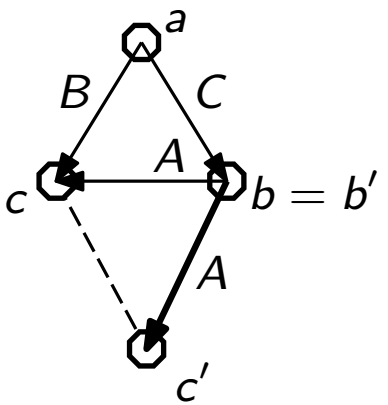
Triangle Lemma - Proof

Proof i)



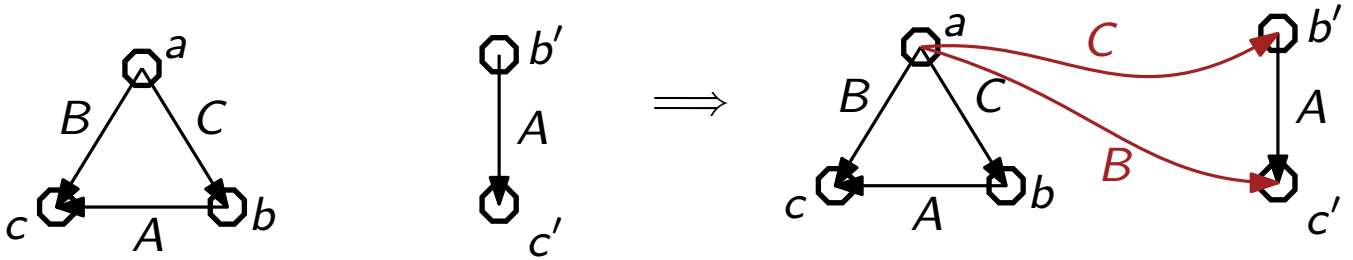
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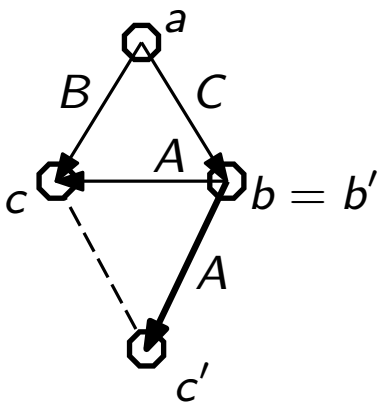
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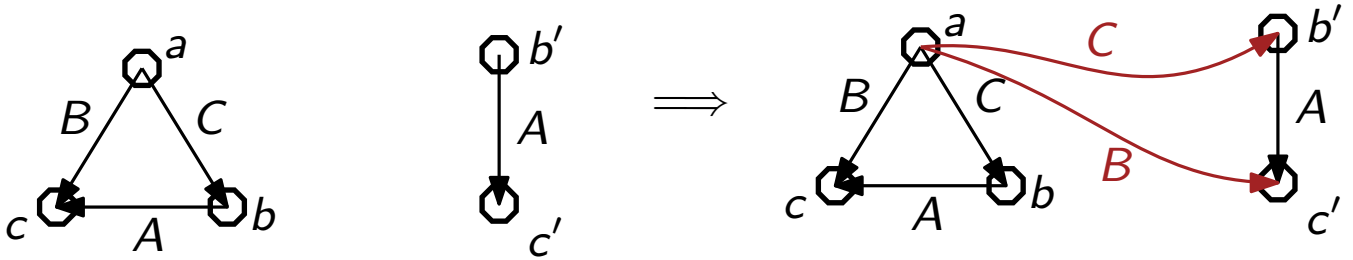
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observe: $c' \neq a$ otherwise $C^{-1} = A$



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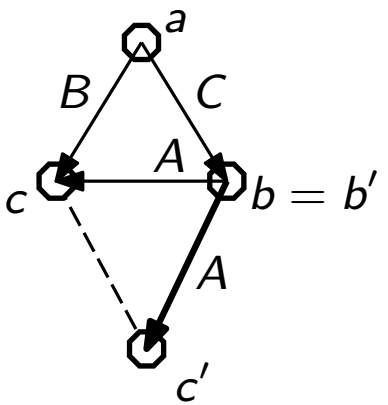
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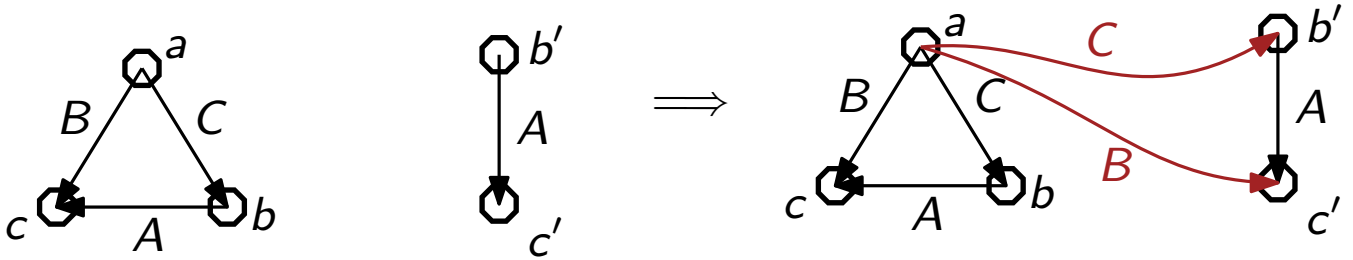
case $b = b'$ and $cc' \notin E(G)$:

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assume $ac' \notin E(G)$



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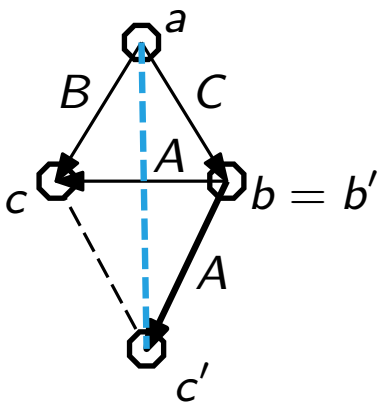
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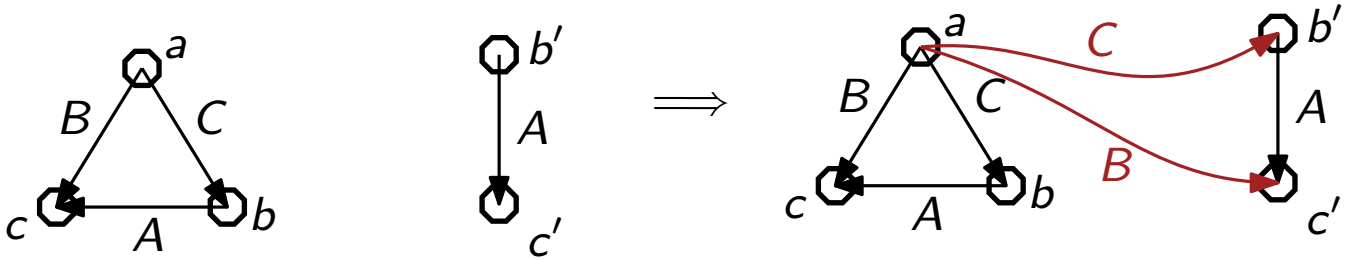
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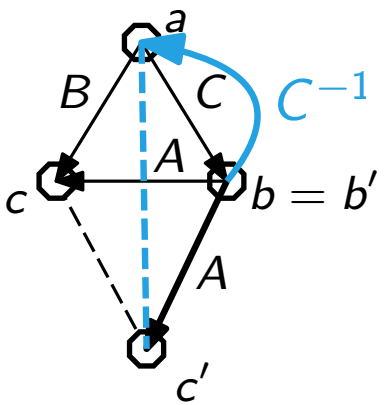
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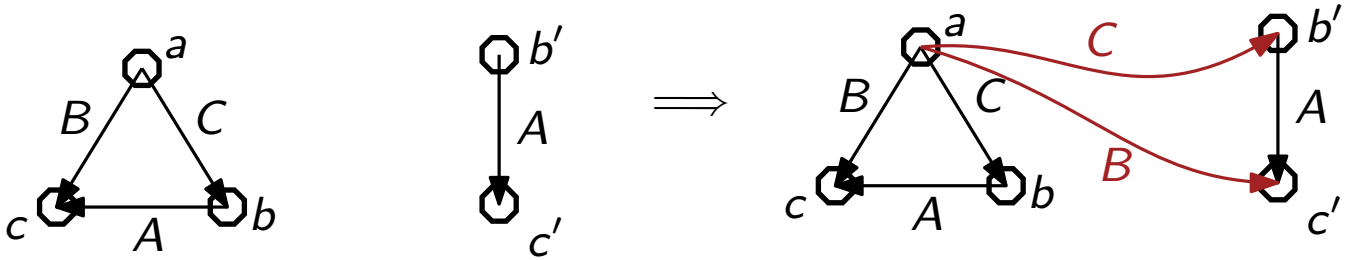
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then $ba\Gamma bc'$



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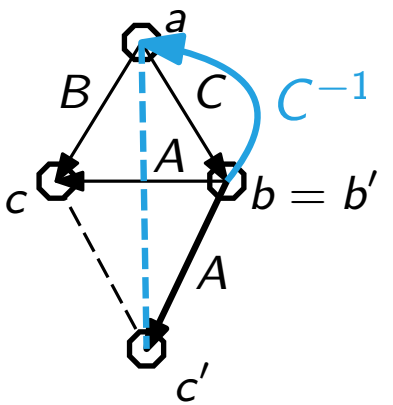
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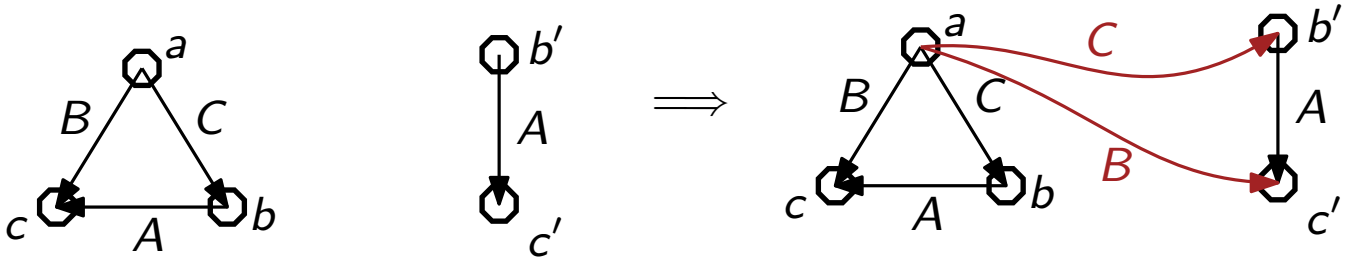
case $b = b'$ and $cc' \notin E(G)$:

observe: $c' \neq a$ otherwise $C^{-1} = A$
assume $ac' \notin E(G)$
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Triangle Lemma - Proof

Proof i)



wlog only consider one step in Γ-chain $bc\Gamma^* b'c'$

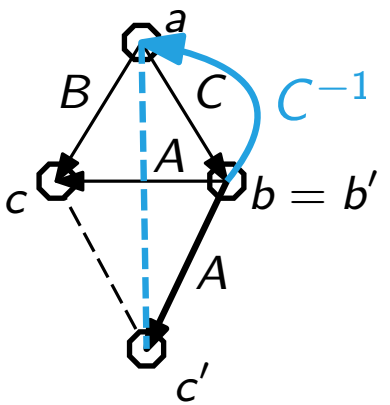
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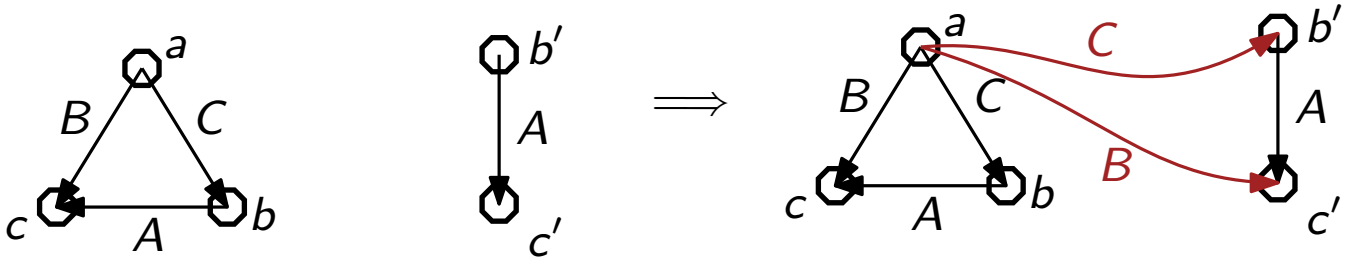
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Triangle Lemma - Proof

Proof i)

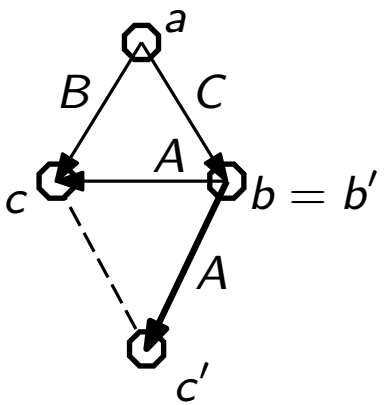


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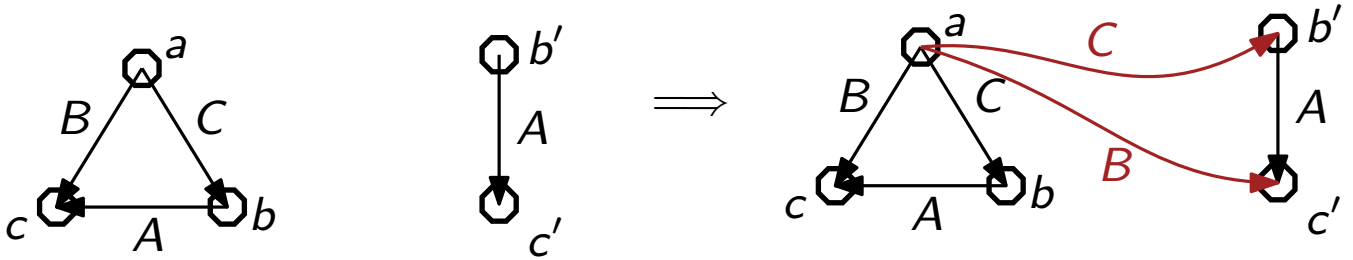
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Triangle Lemma - Proof

Proof i)

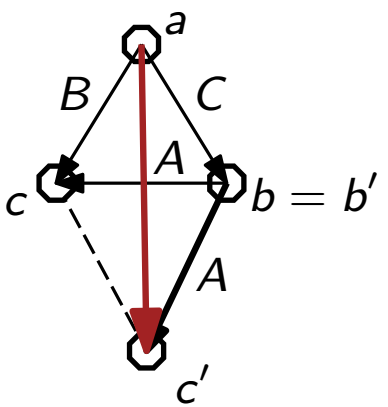


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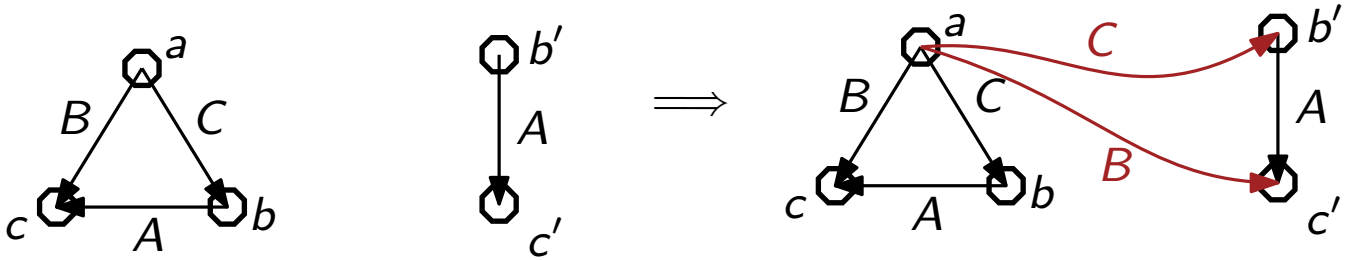
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Triangle Lemma - Proof

Proof i)



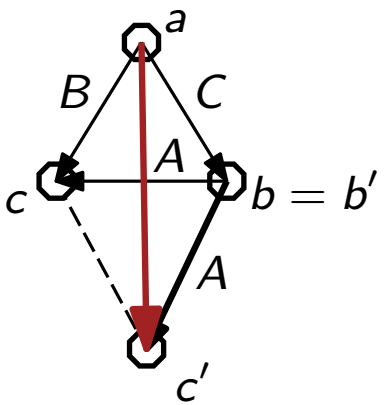
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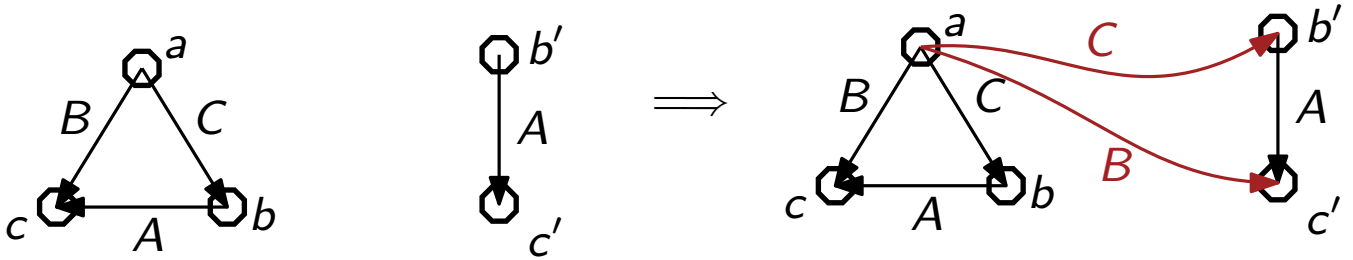
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Triangle Lemma - Proof

Proof i)



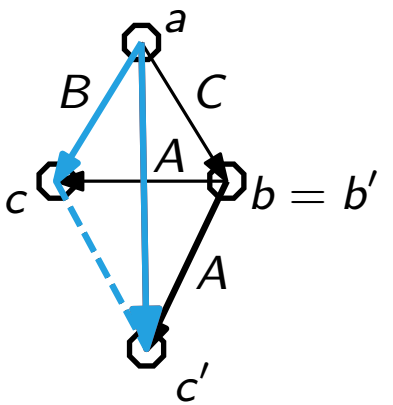
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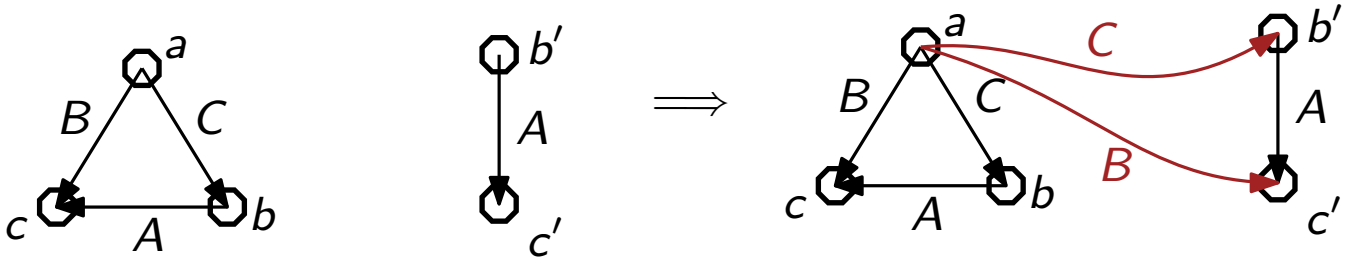
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Triangle Lemma - Proof

Proof i)



wlog only consider one step in Γ-chain $bc\Gamma^* b'c'$

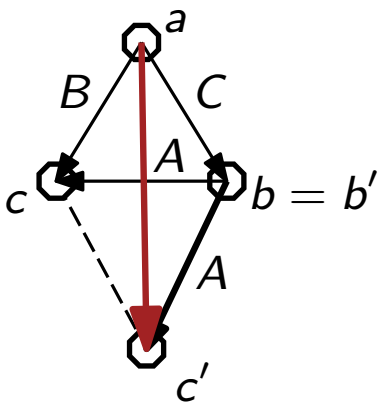
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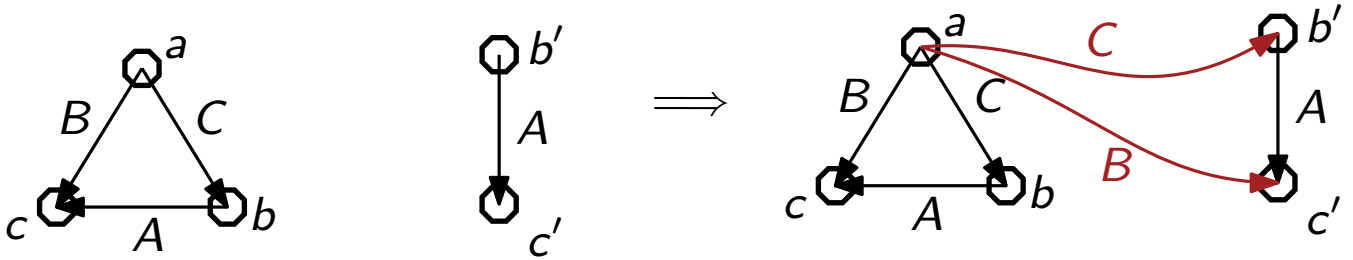
since $cc' \notin E(G)$ and $a = a$ we get $ac\Gamma ac'$

$ac' \in B$



Triangle Lemma - Proof

Proof i)



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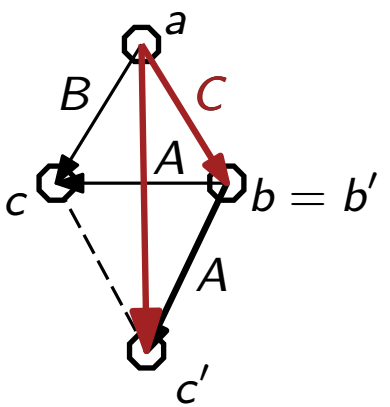
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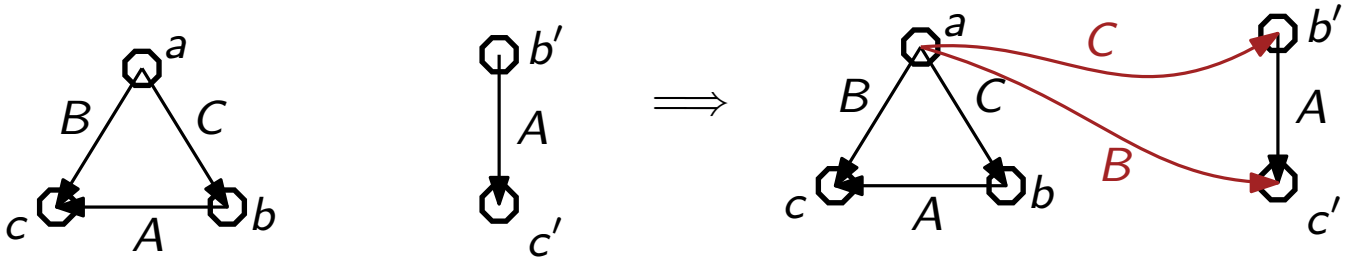
$ac' \in B$

$C \ni ab = ab'$



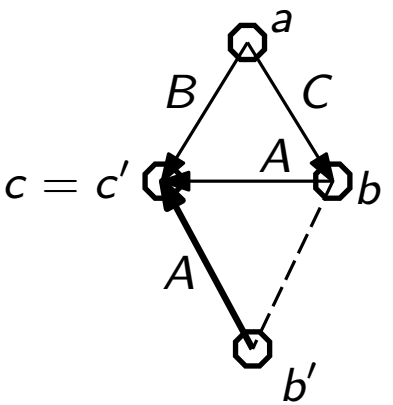
Triangle Lemma - Proof

Proof i)



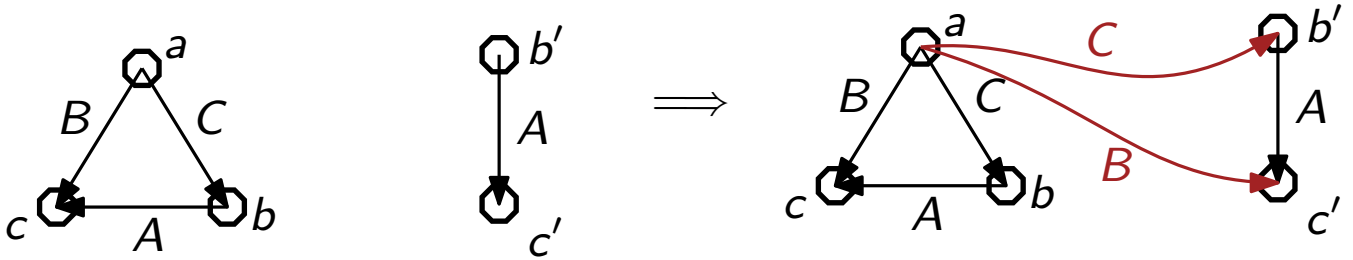
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Triangle Lemma - Proof

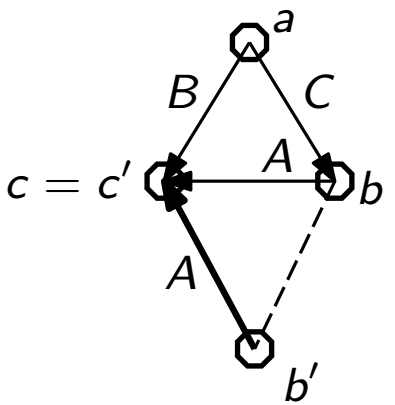
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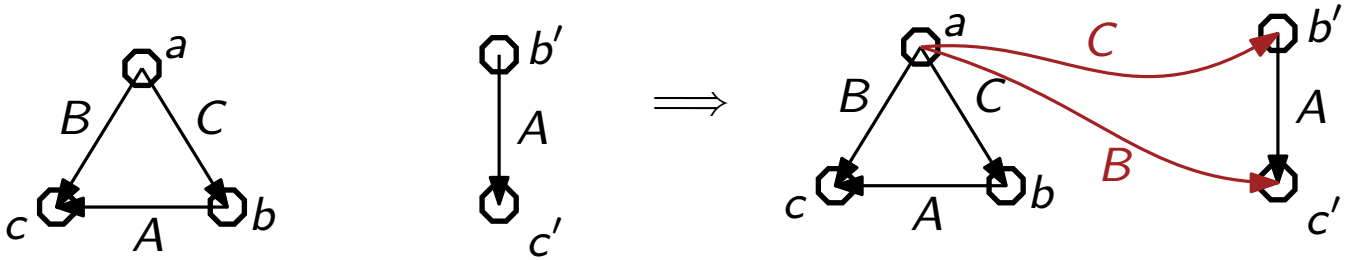
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Triangle Lemma - Proof

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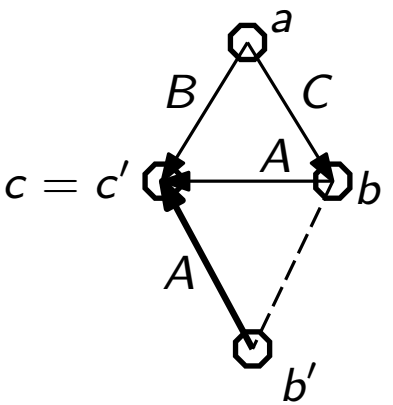


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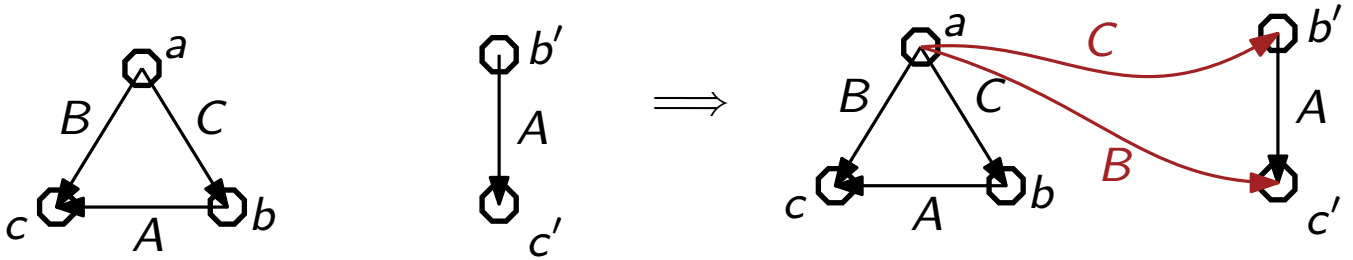
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Triangle Lemma - Proof

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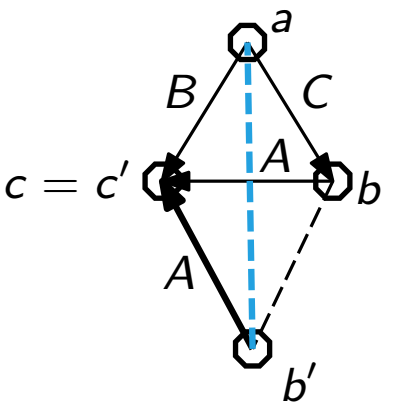


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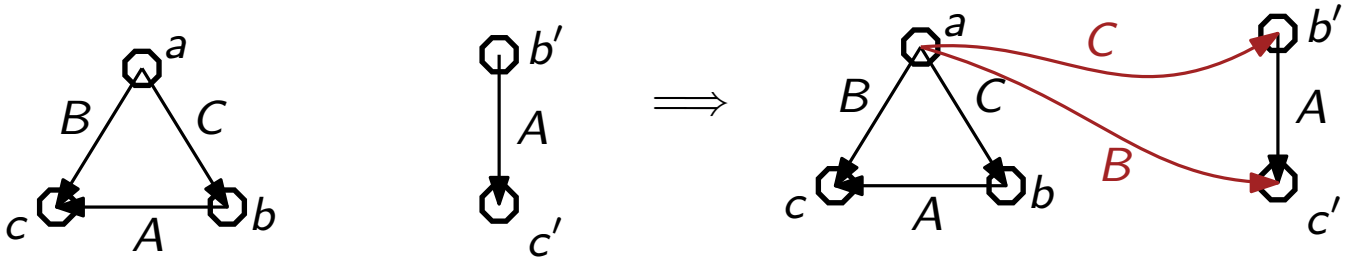
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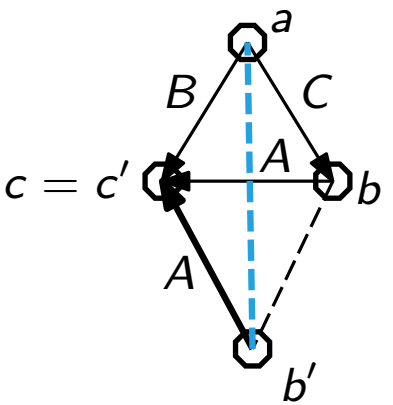
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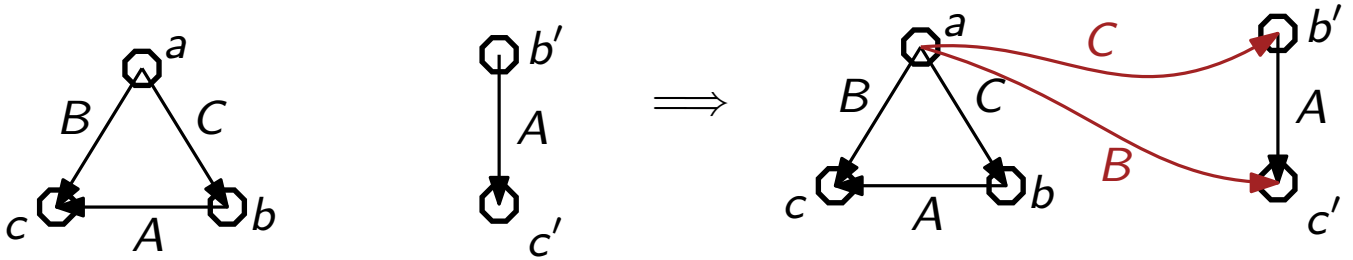
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Triangle Lemma - Proof

Proof i)



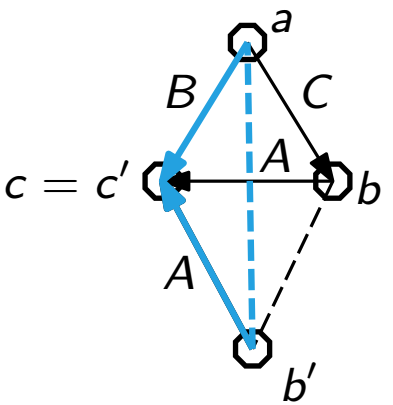
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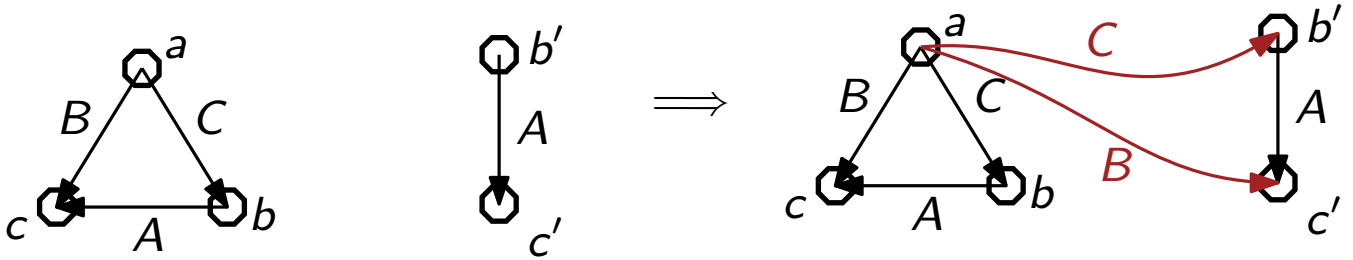
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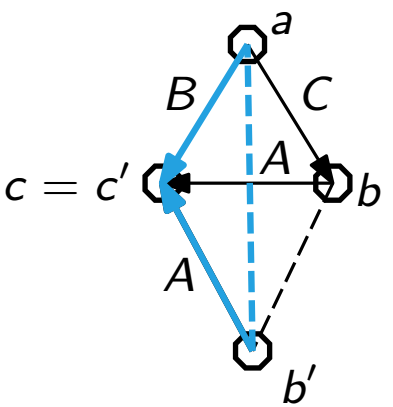
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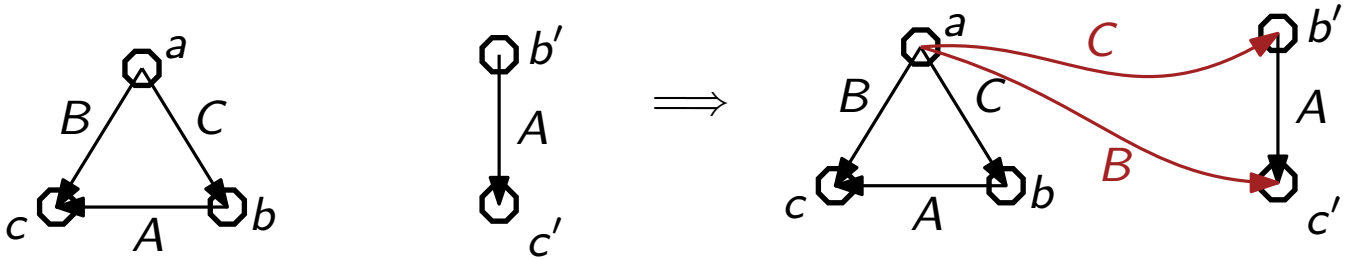
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Triangle Lemma - Proof

Proof i)

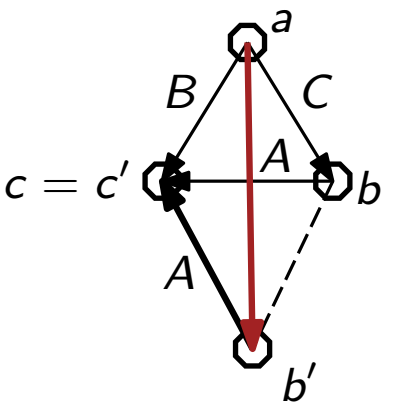


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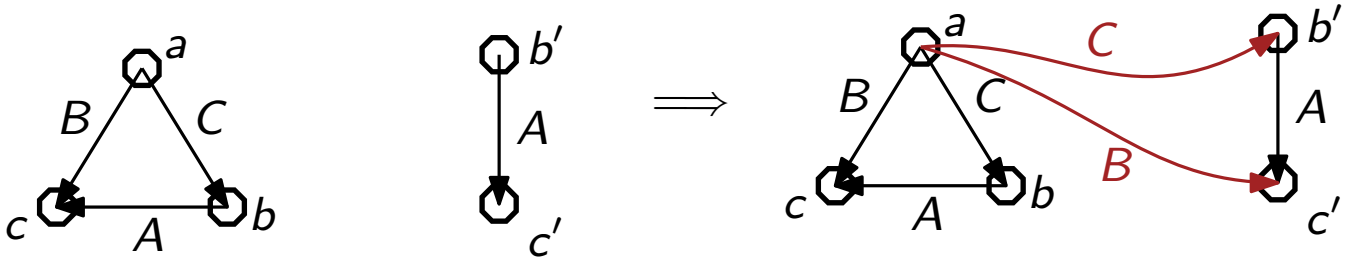
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Triangle Lemma - Proof

Proof i)



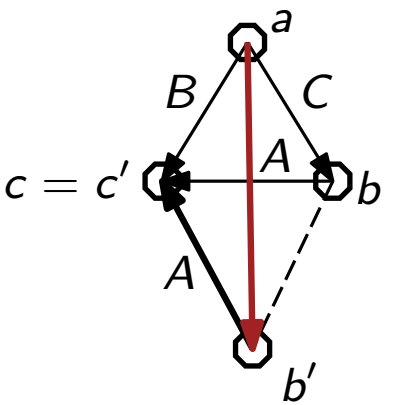
wlog only consider one step in Γ-chain $bc\Gamma^* b'c'$

case $c = c'$ and $bb' \notin E(G)$:

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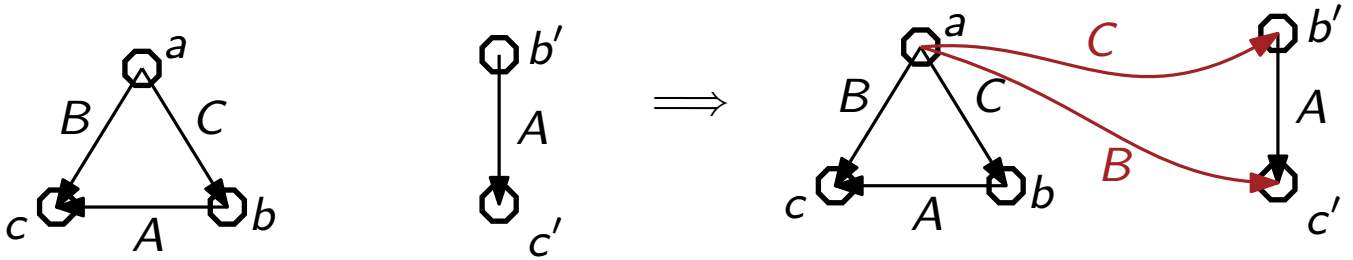
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since $bb' \notin E(G)$ and $a = a$ we get $ab\Gamma ab'$



Triangle Lemma - Proof

Proof i)



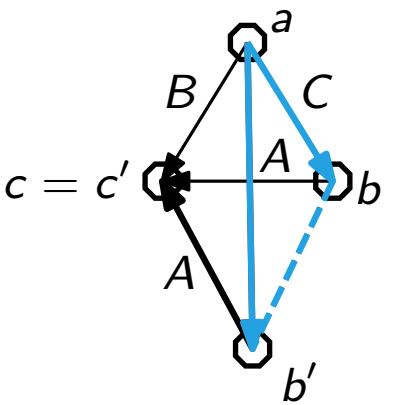
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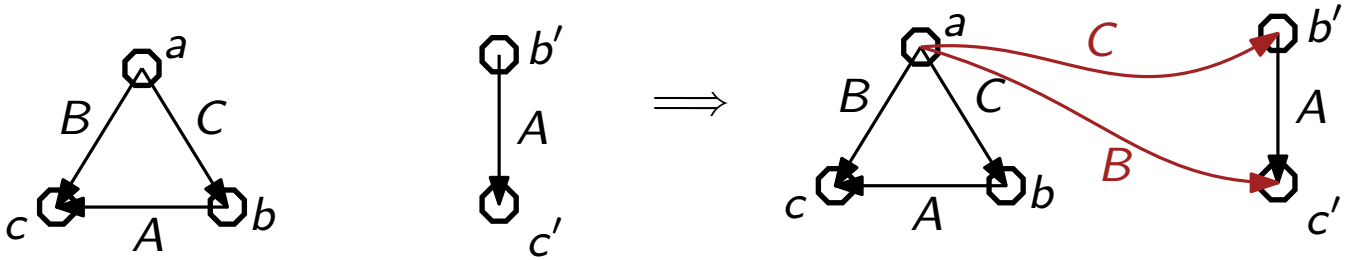
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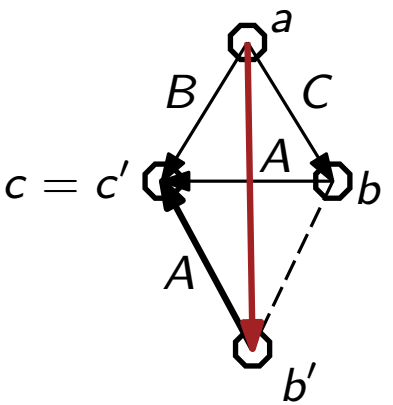
case $c = c'$ and $bb' \notin E(G)$:

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so $ab' \in E(G)$

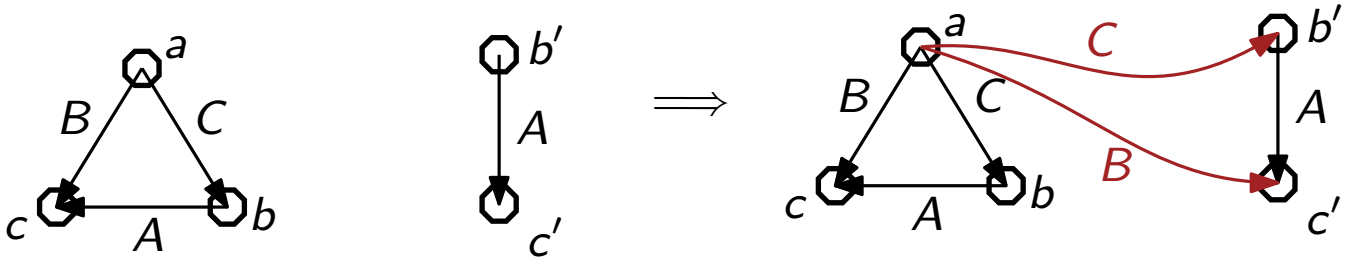
since $bb' \notin E(G)$ and $a = a$ we get $ab\Gamma ab'$

$ab' \in C$



Triangle Lemma - Proof

Proof i)



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case $c = c'$ and $bb' \notin E(G)$:

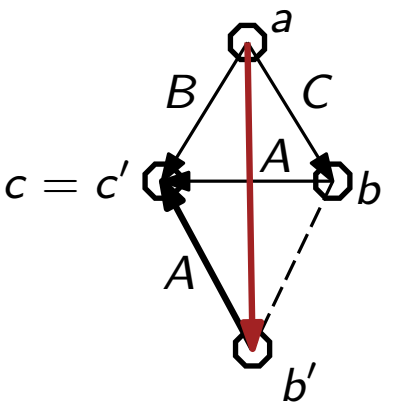
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so $ab' \in E(G)$

since $bb' \notin E(G)$ and $a = a$ we get $ab\Gamma ab'$

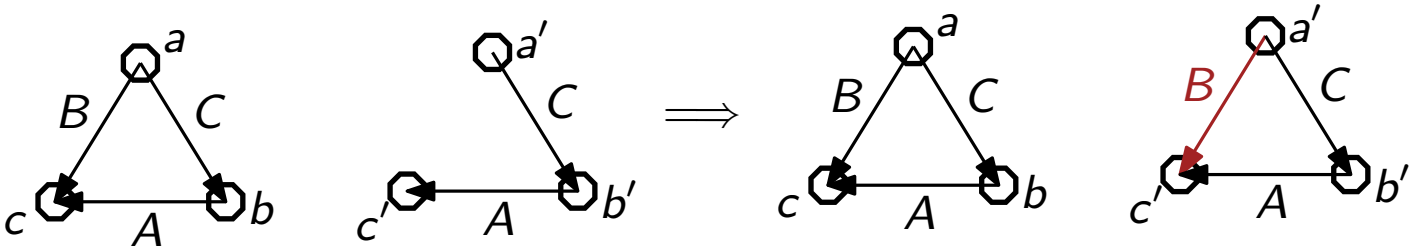
$ab' \in C$

$B \ni ac = ac'$



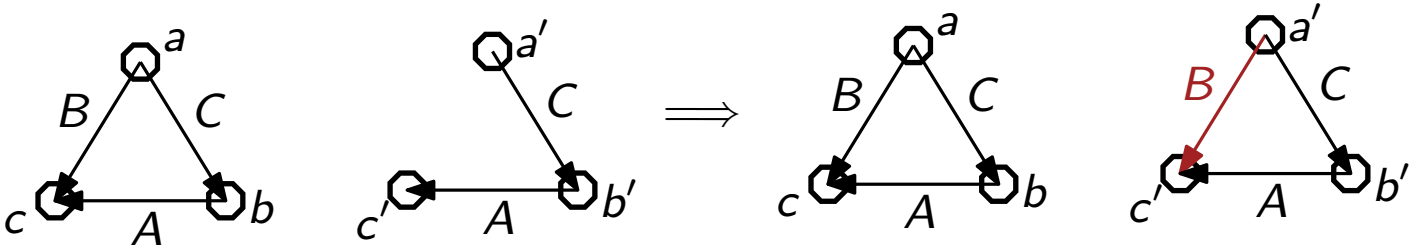
Triangle Lemma - Proof

Proof ii)



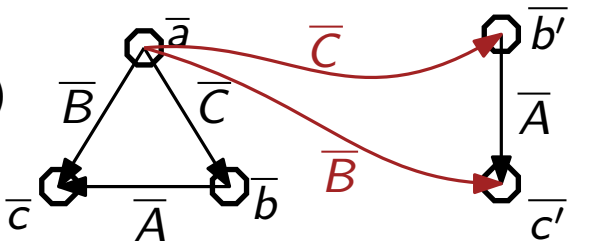
Triangle Lemma - Proof

Proof ii)



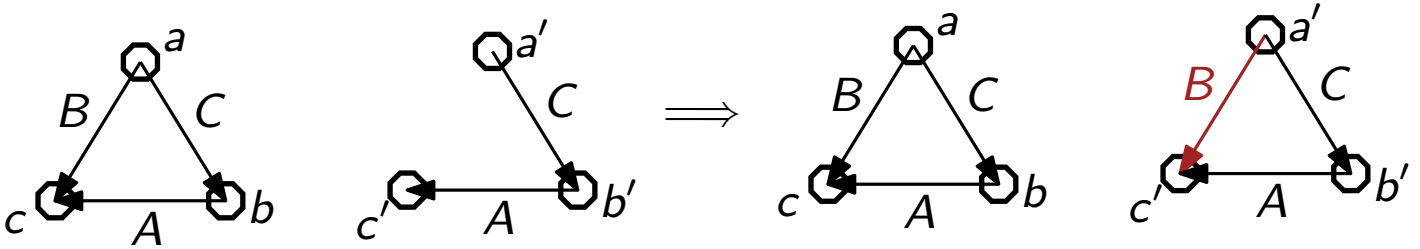
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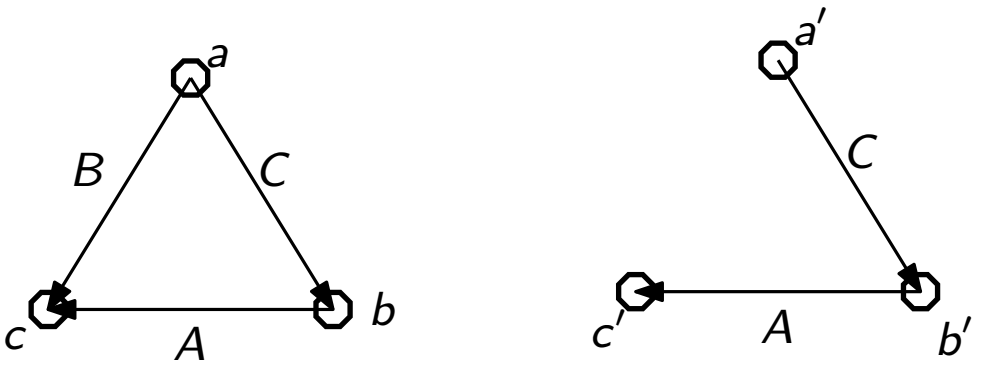


Triangle Lemma - Proof

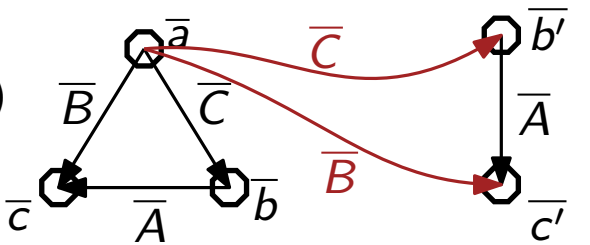
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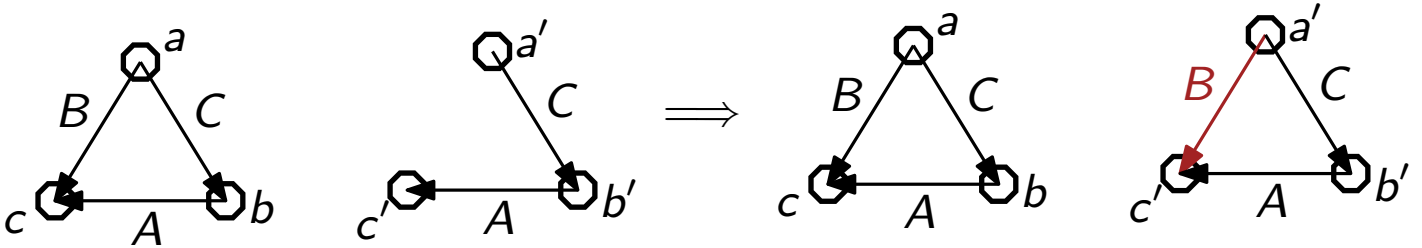


part i)



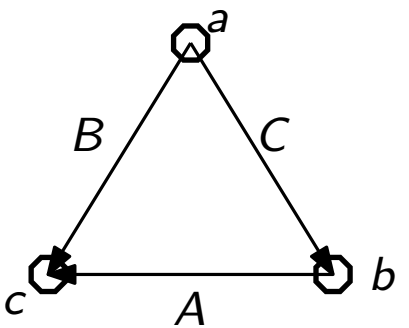
Triangle Lemma - Proof

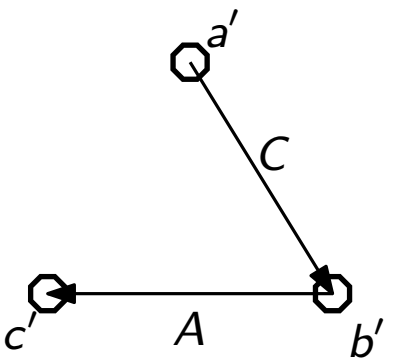
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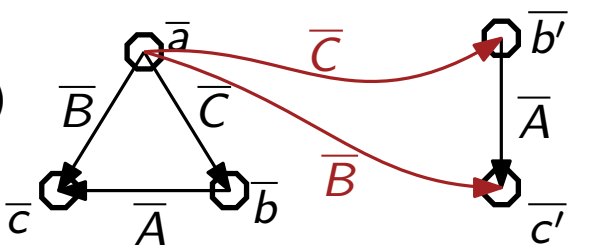
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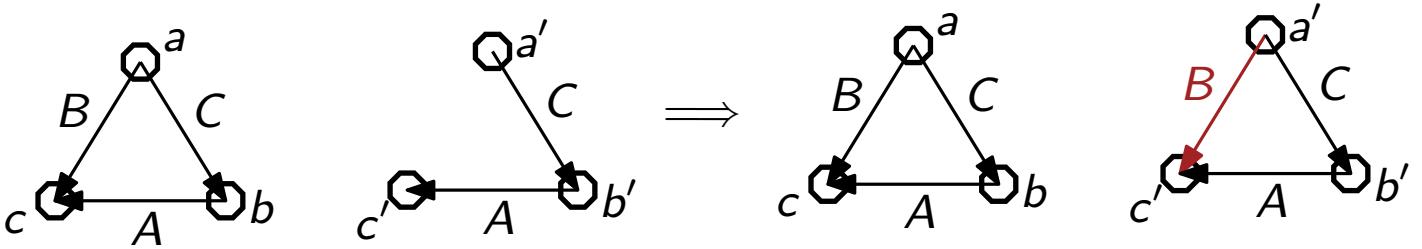


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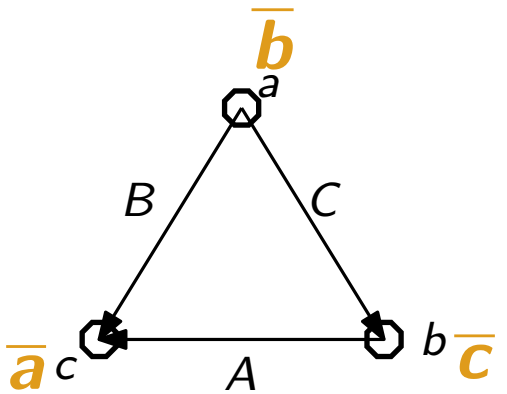
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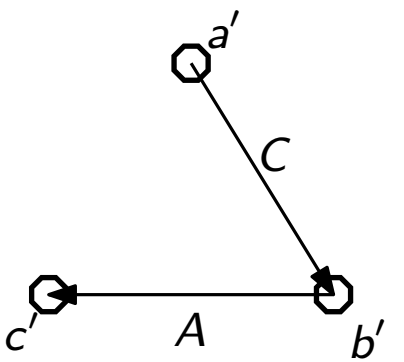
Proof ii)



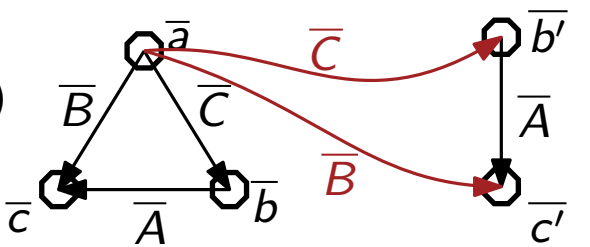
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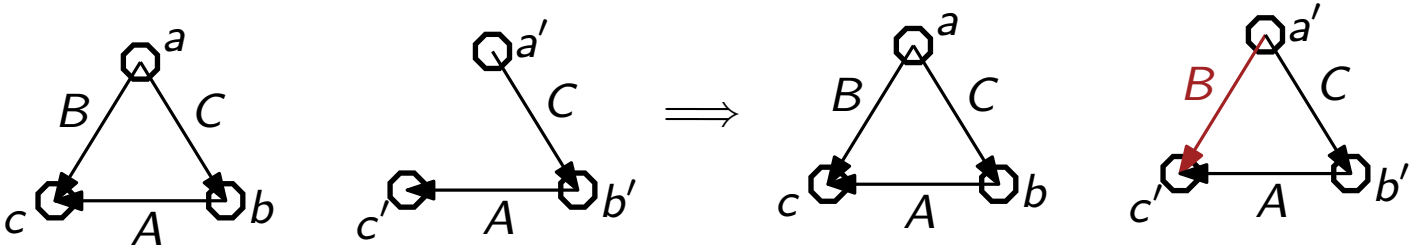


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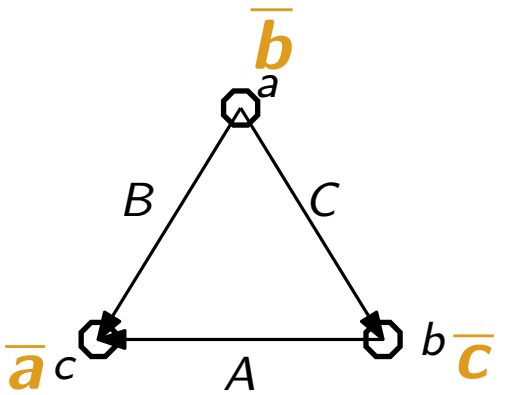
Triangle Lemma - Proof

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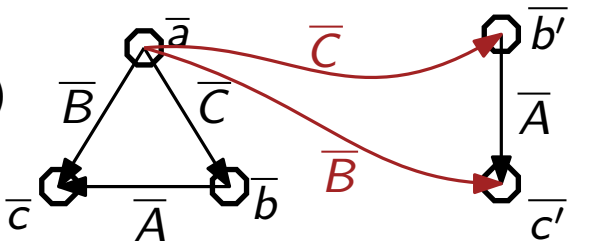


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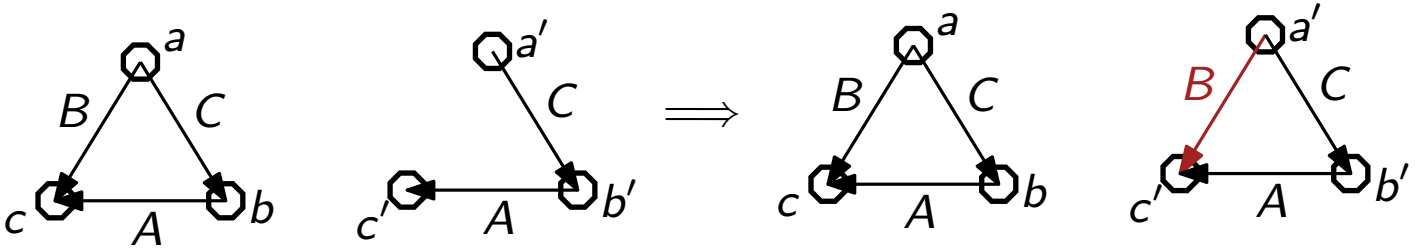


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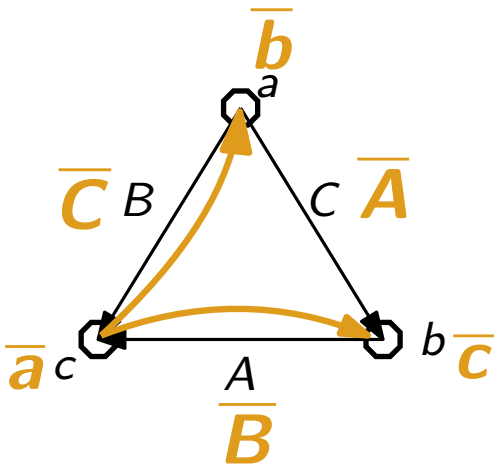
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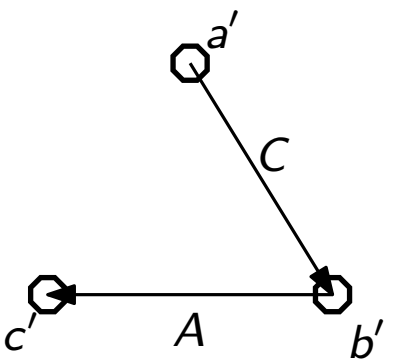
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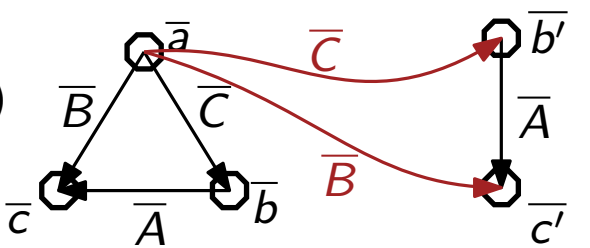
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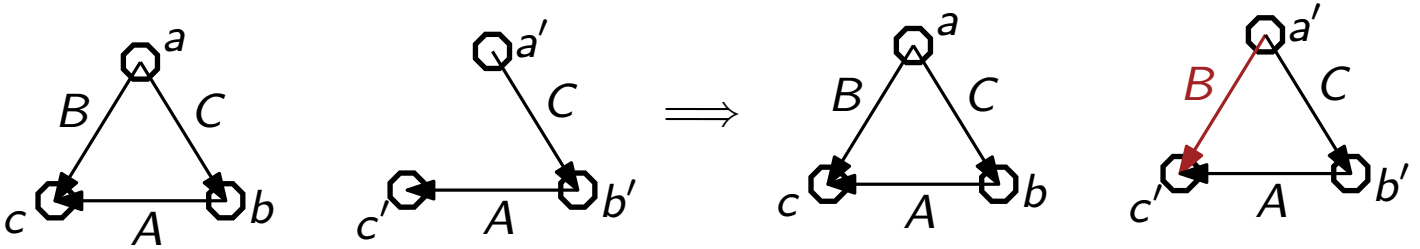


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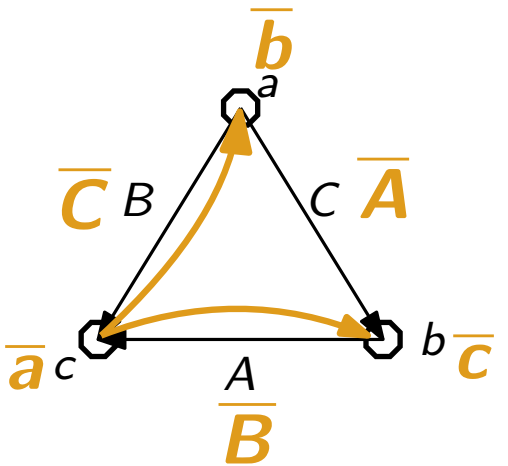
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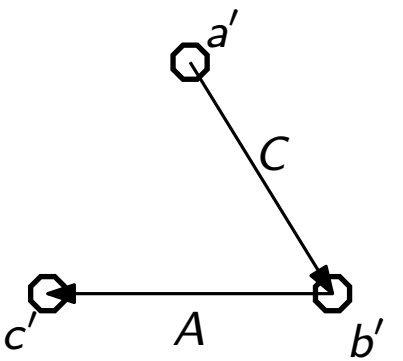
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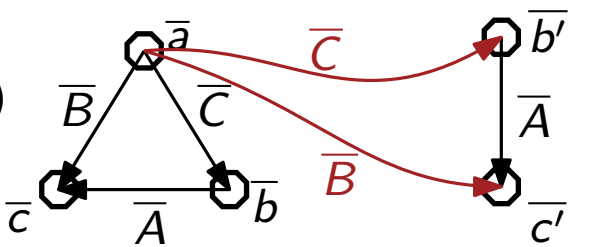
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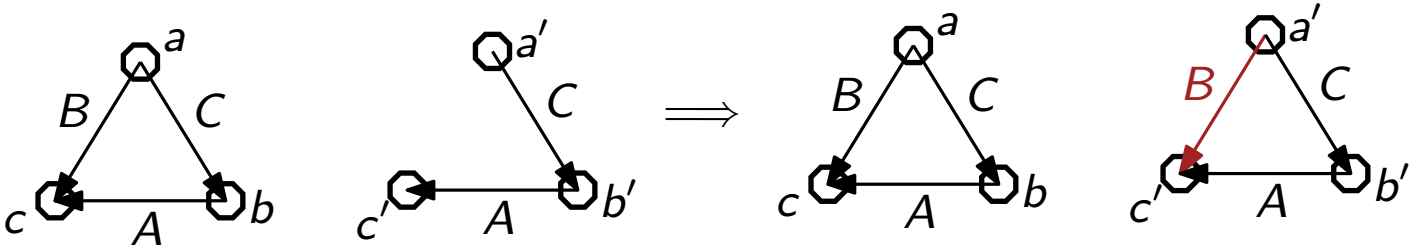


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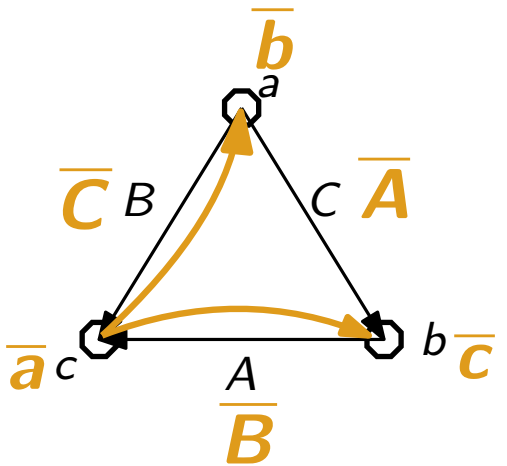
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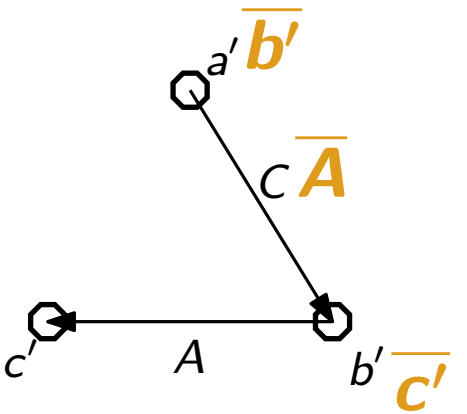
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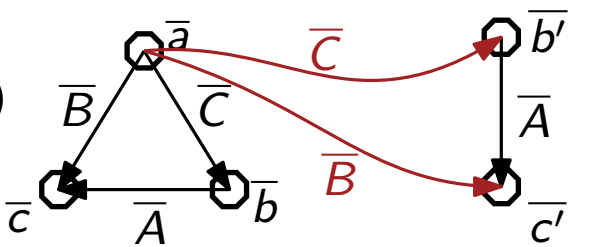
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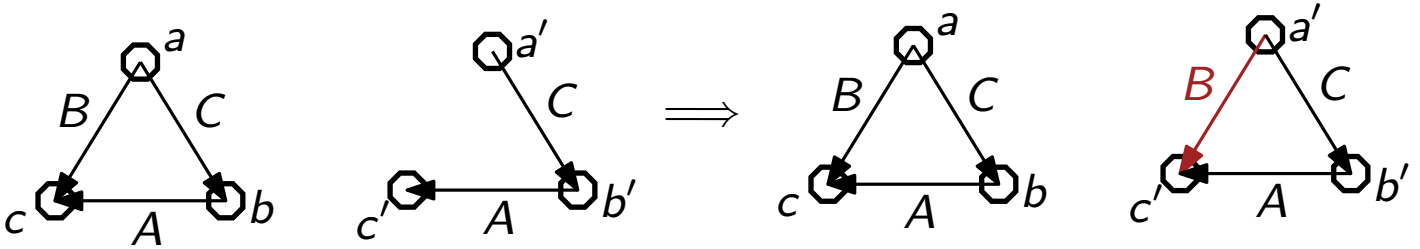


part i)



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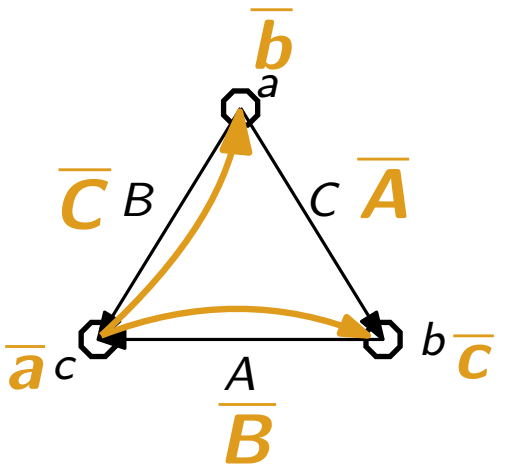
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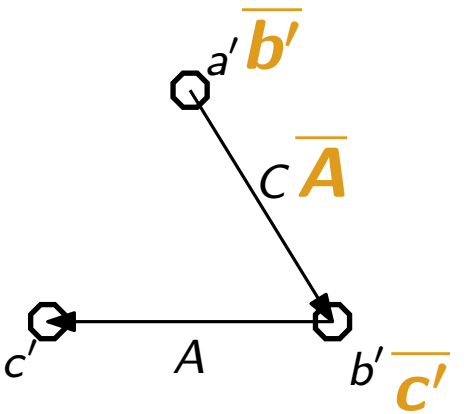


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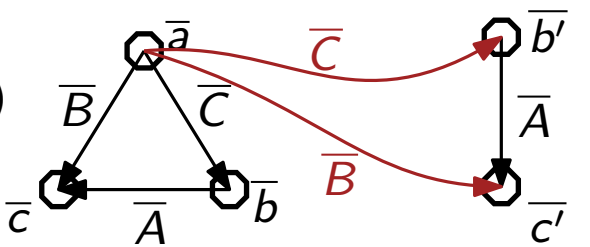
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$$\overline{A} \overset{?}{\neq} \overline{B}$$



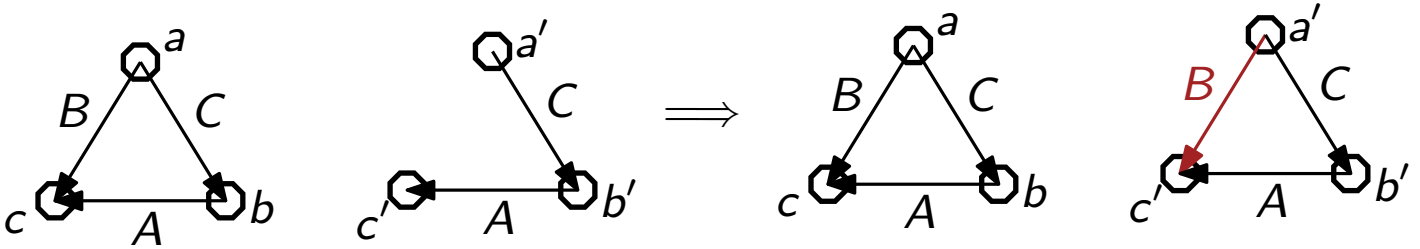


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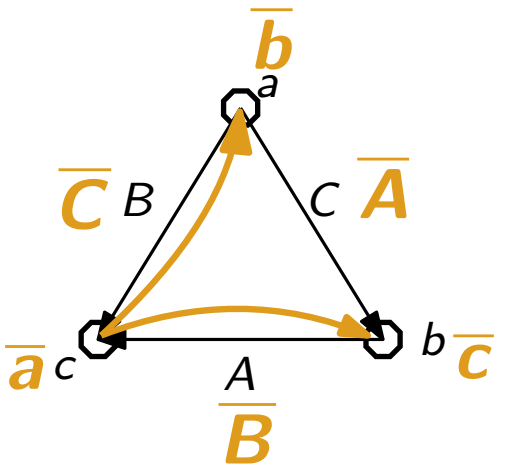
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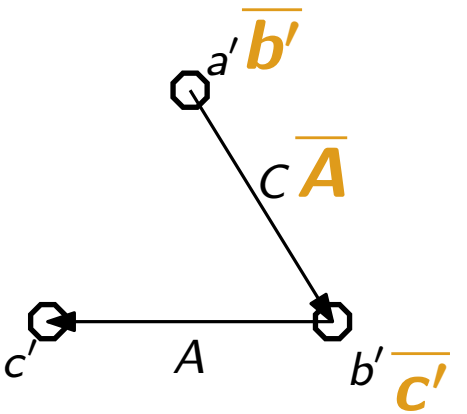


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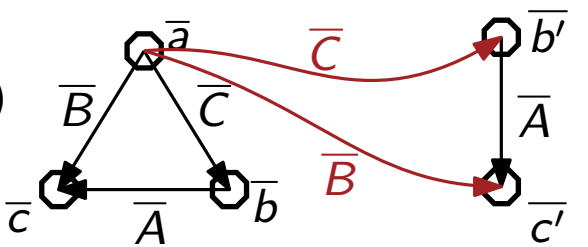
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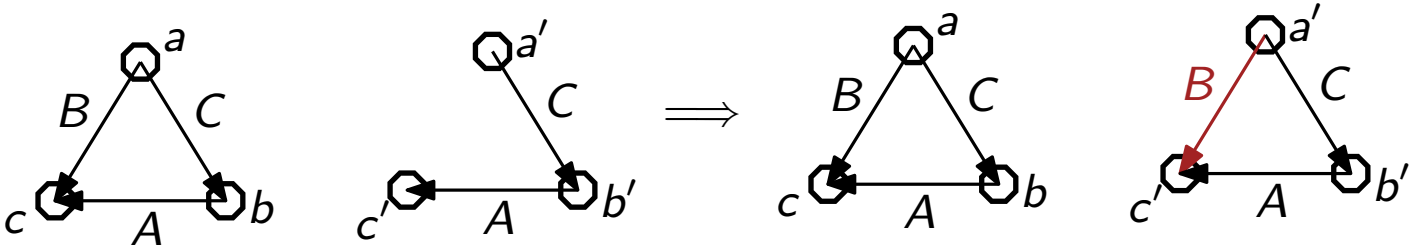


part i)



Triangle Lemma - Proof

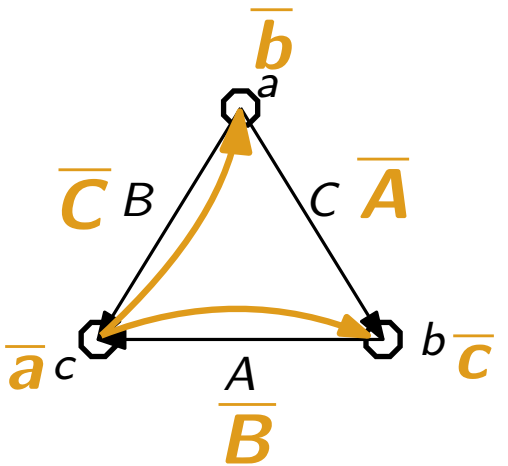
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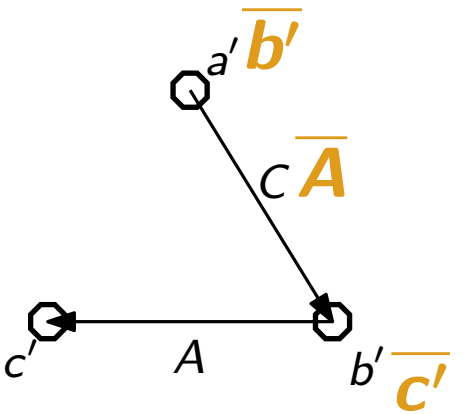


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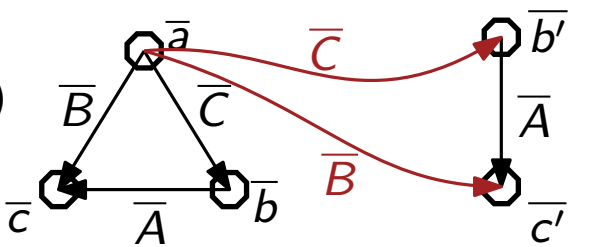
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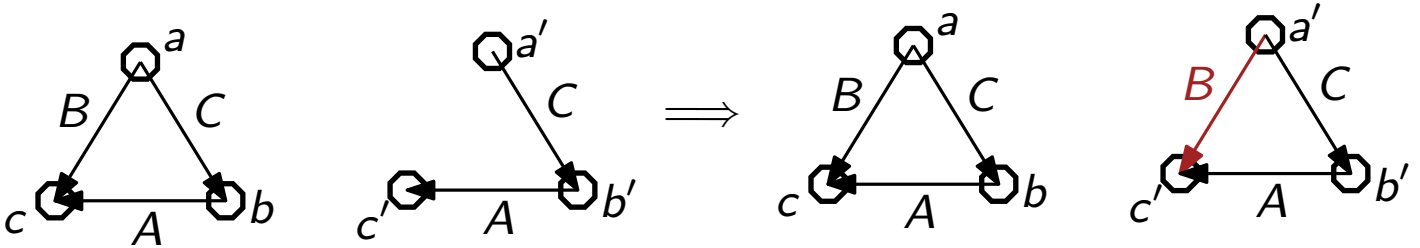


part i)



Triangle Lemma - Proof

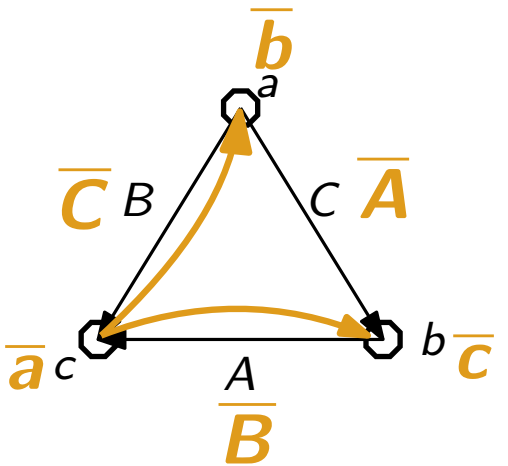
Proof ii)

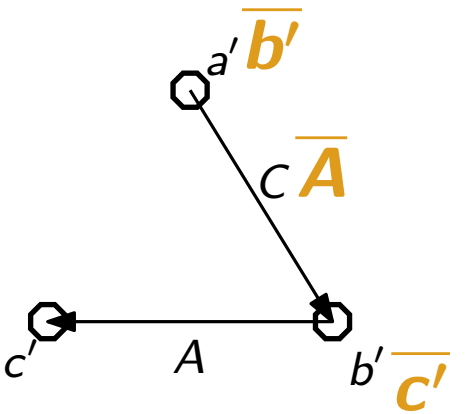


Goal: apply part i)

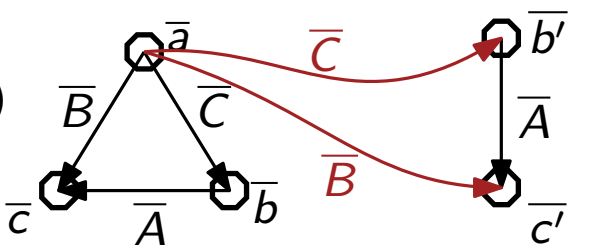
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- $a' = \overline{b'}, b' = \overline{c'}$

$$C = \overline{A} \neq \overline{B} = A^{-1}$$



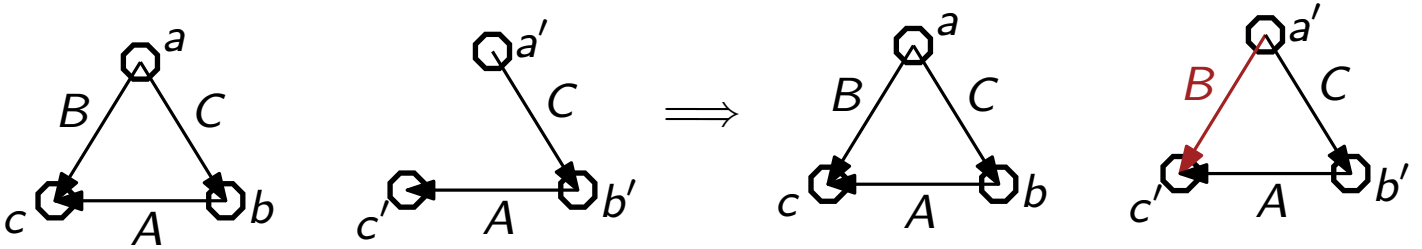


part i)



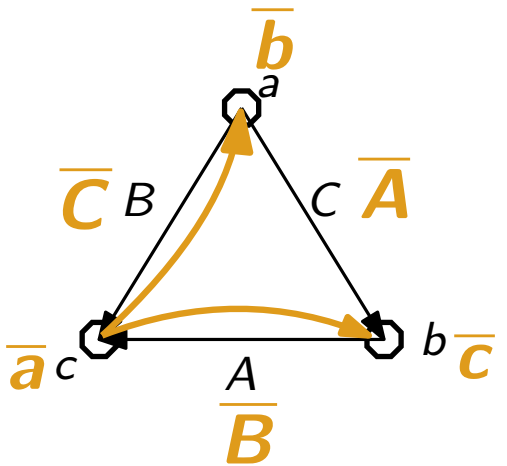
Triangle Lemma - Proof

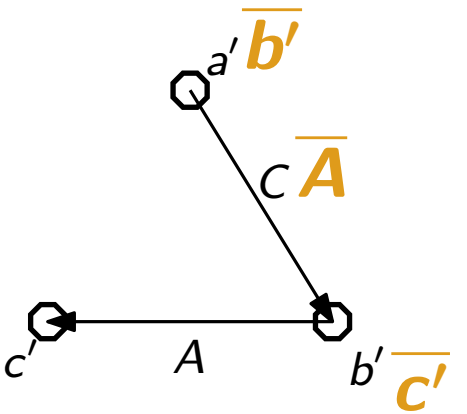
Proof ii)



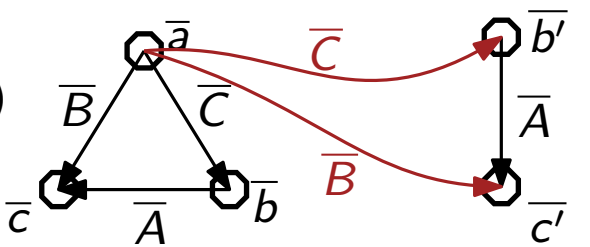
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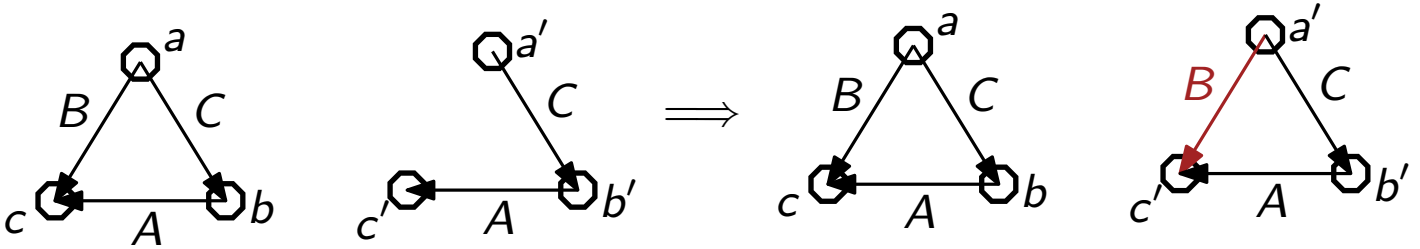


part i)



Triangle Lemma - Proof

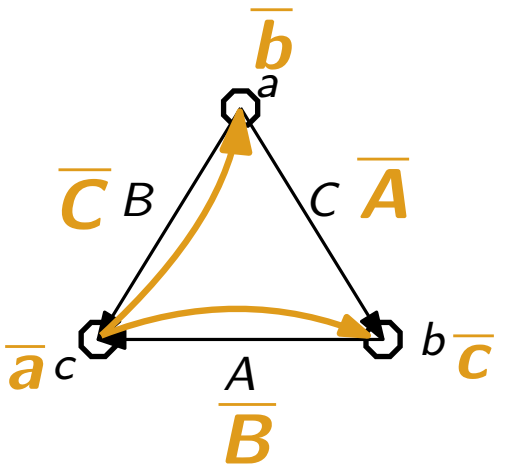
Proof ii)

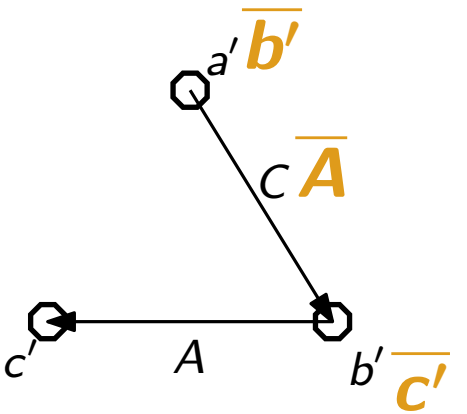


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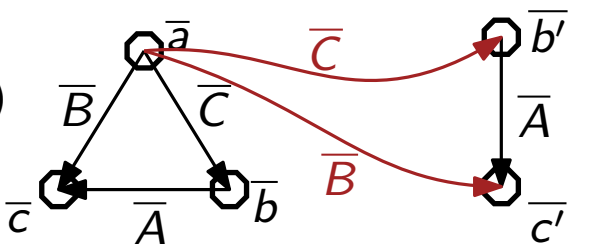
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$$C = \overline{A} \overset{?}{\neq} \overline{C}^{-1} = B$$



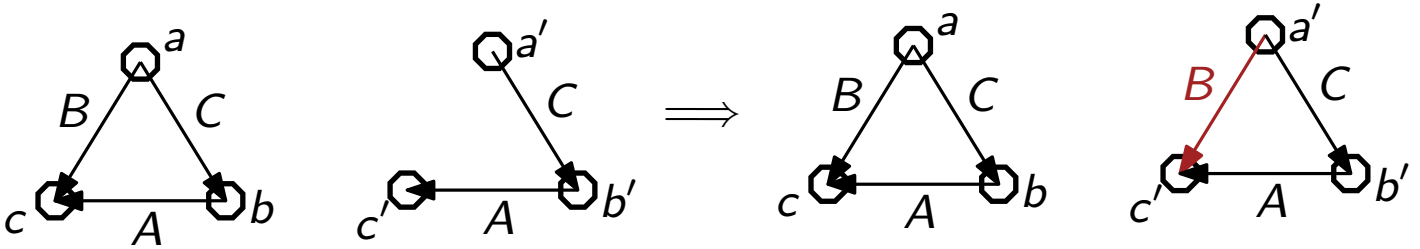


part i)



Triangle Lemma - Proof

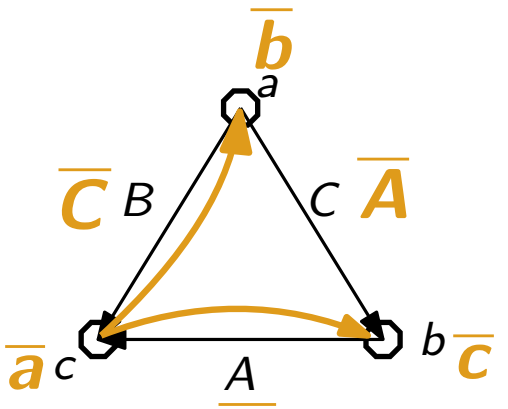
Proof ii)

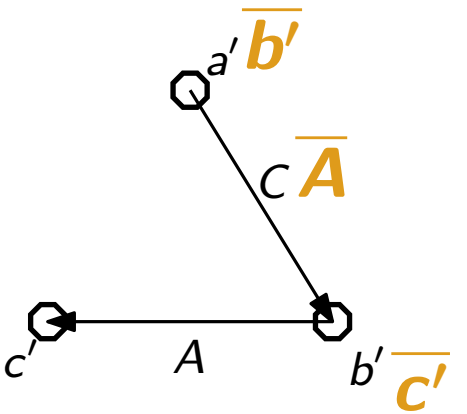


Goal: apply part i)

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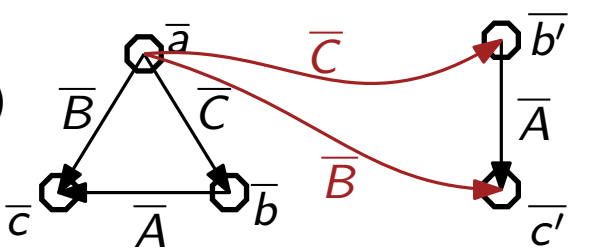
$$C = \overline{A} \overset{?}{\neq} \overline{C}^{-1} = B$$





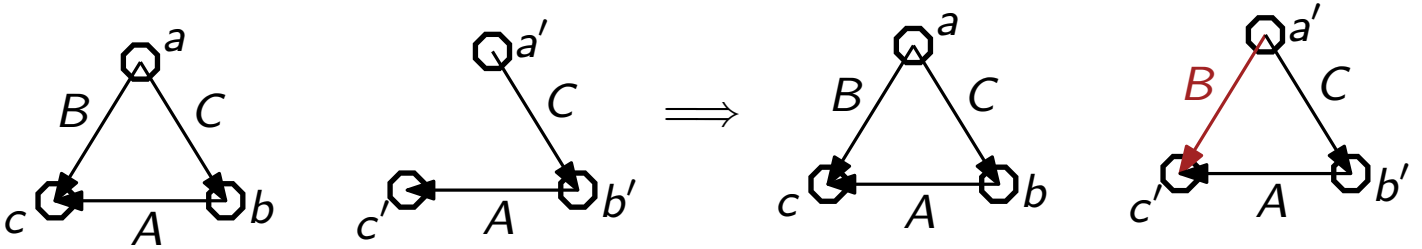
since we don't know whether $B \neq C$ or $B = C$
we have to do a case distinction

part i)



Triangle Lemma - Proof

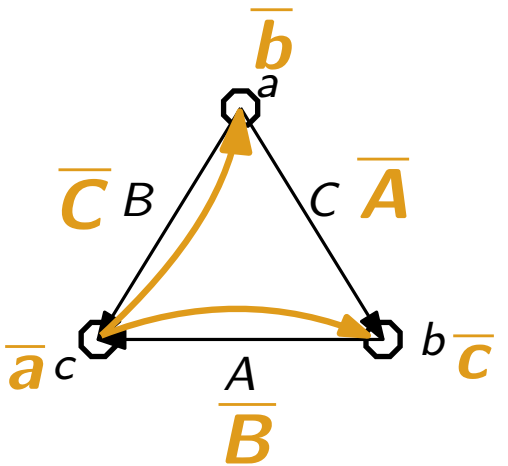
Proof ii)

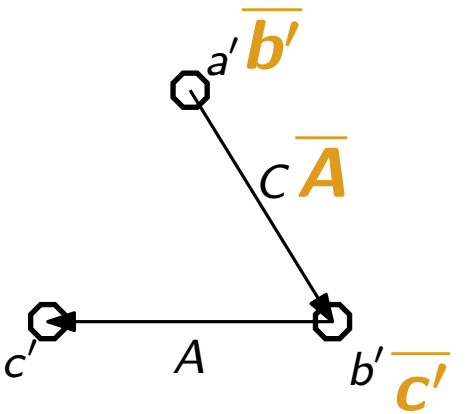


Goal: apply part i)

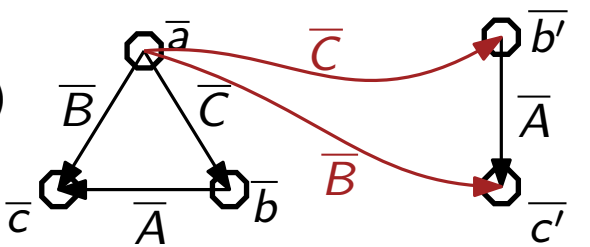
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case $B \neq C$



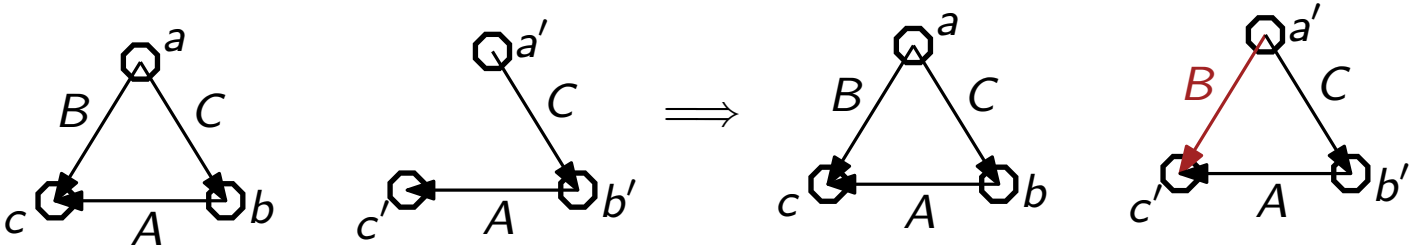


part i)



Triangle Lemma - Proof

Proof ii)

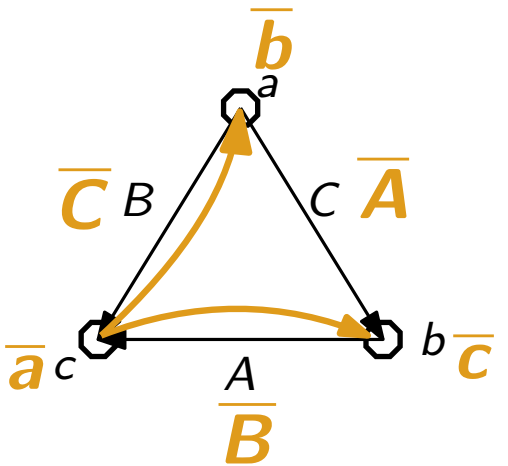


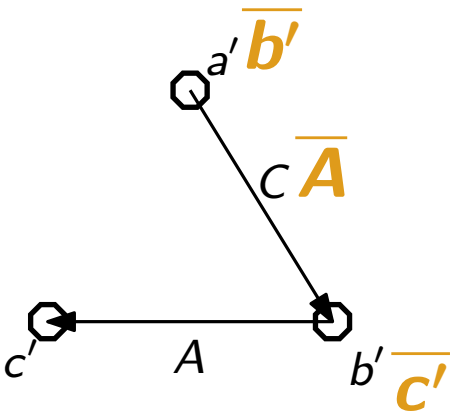
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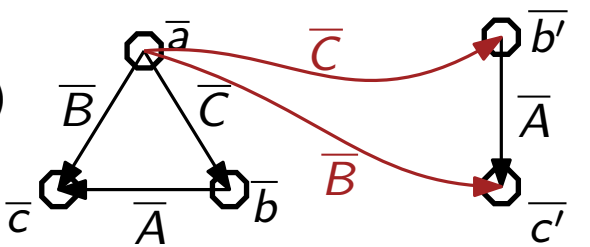
case $B \neq C$

we are allowed to apply part i)



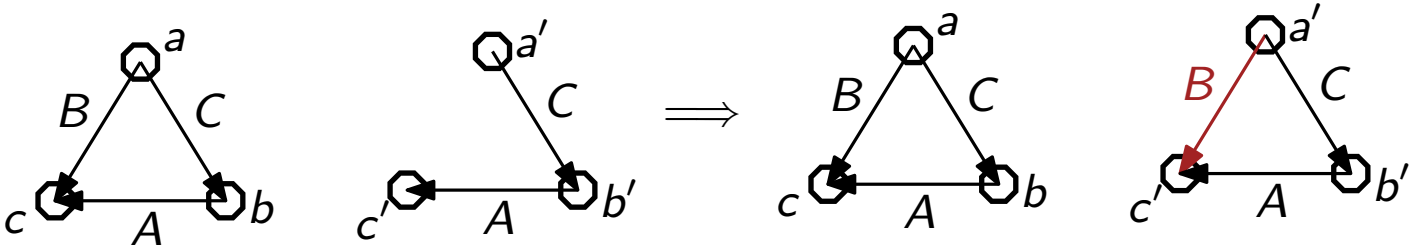


part i)



Triangle Lemma - Proof

Proof ii)



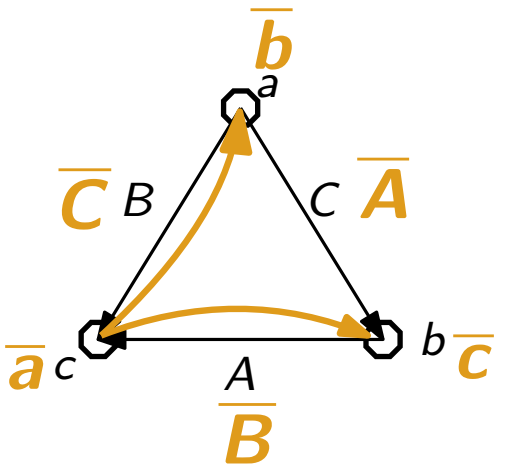
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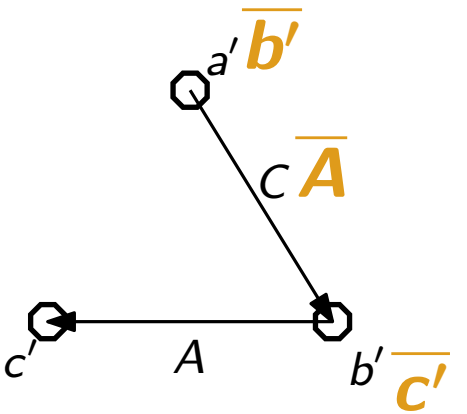
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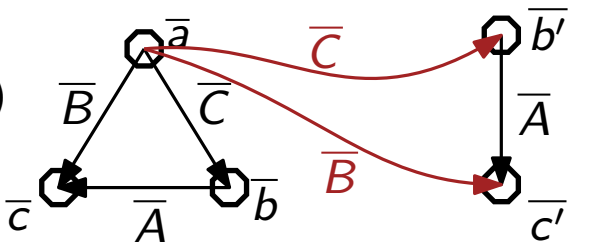
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obtain $\overline{a}\overline{b'} = ca' \in \overline{C} = B^{-1}$



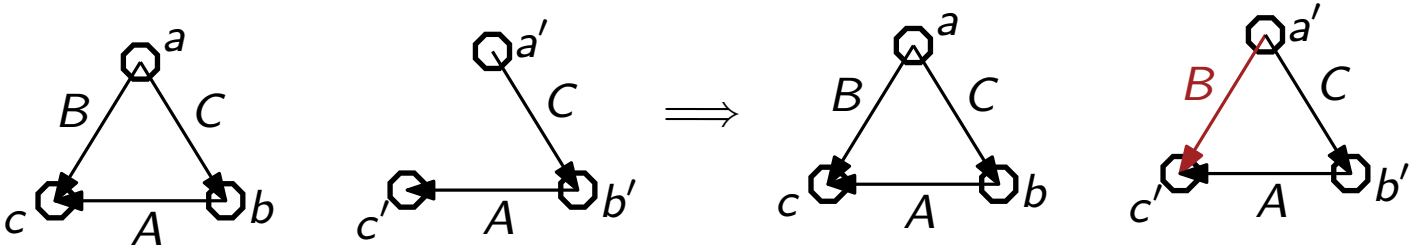


part i)



Triangle Lemma - Proof

Proof ii)



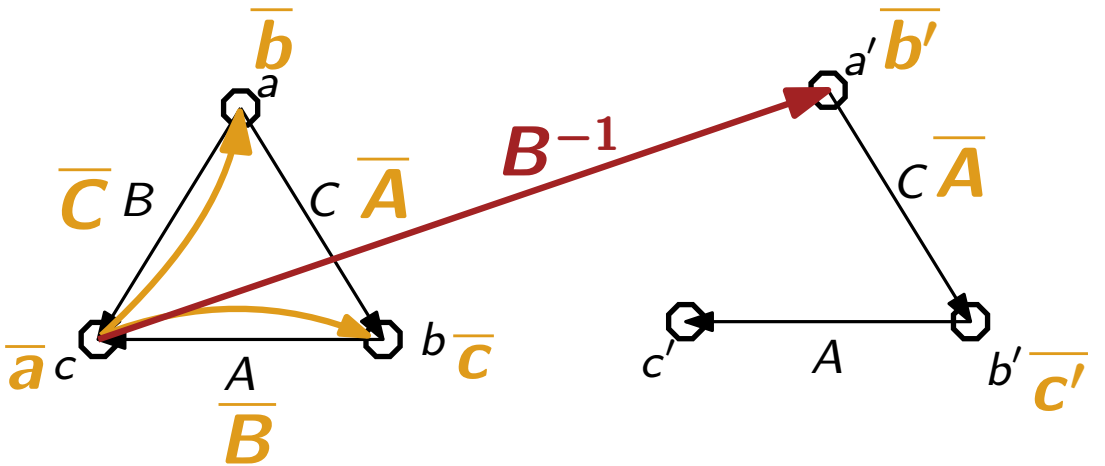
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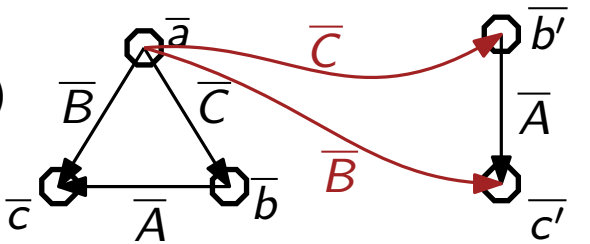
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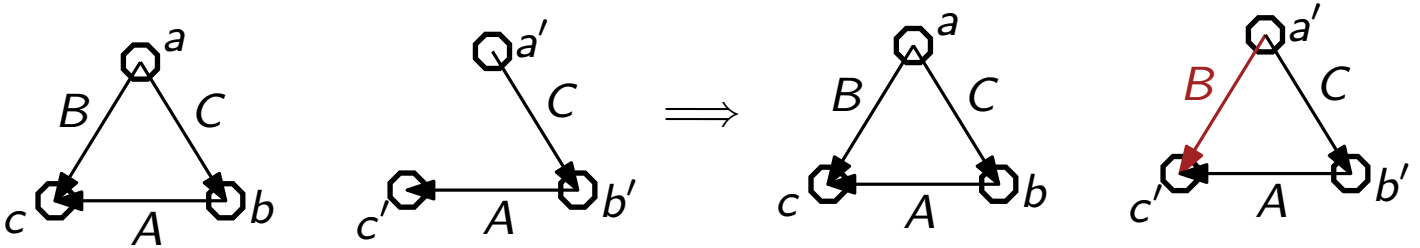


part i)



Triangle Lemma - Proof

Proof ii)



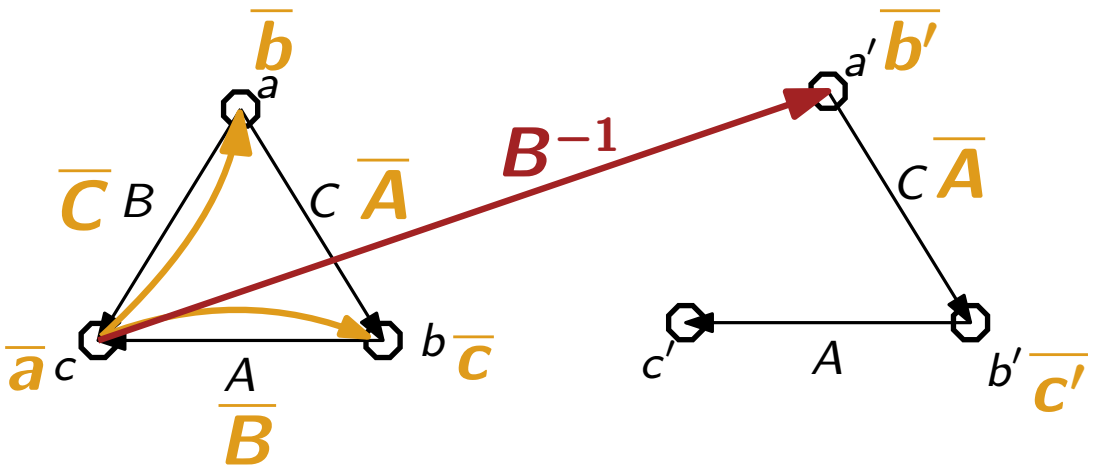
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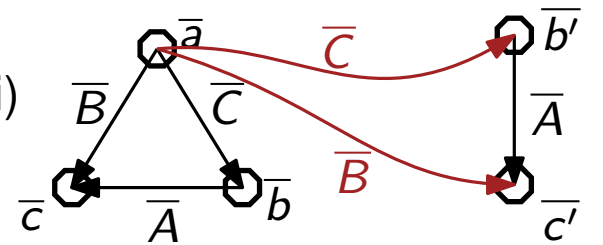
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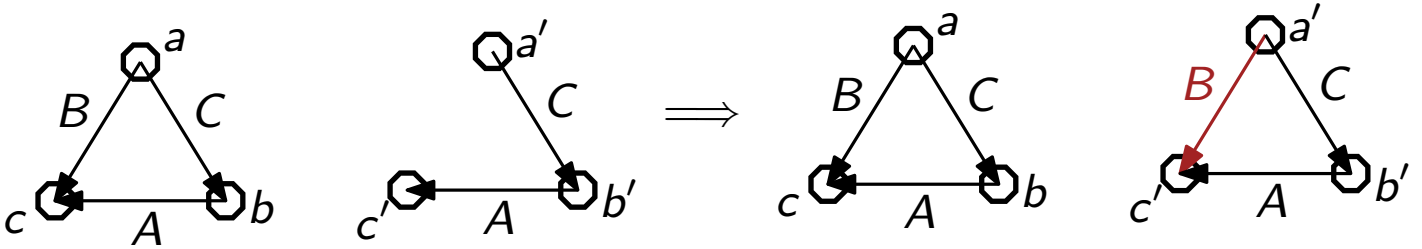


part i)



Triangle Lemma - Proof

Proof ii)



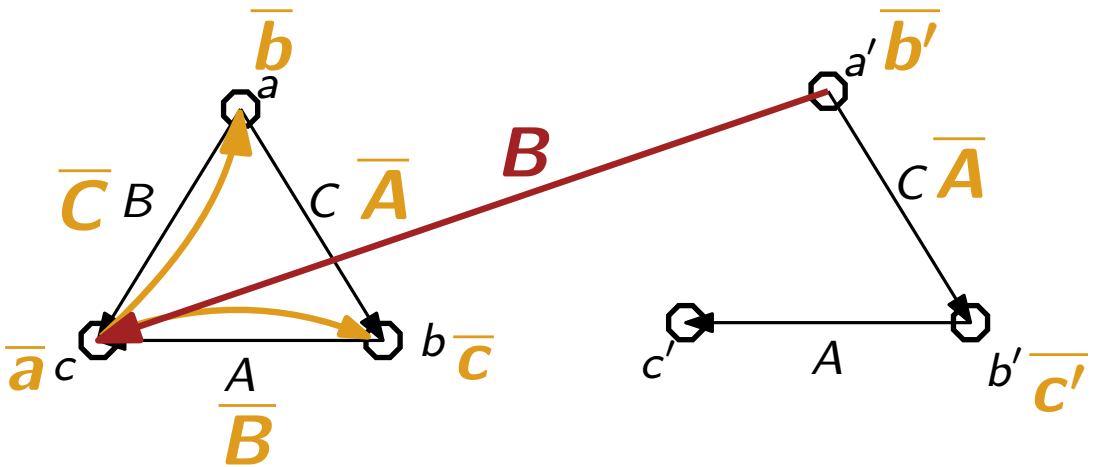
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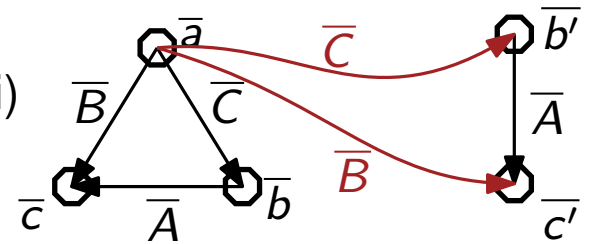
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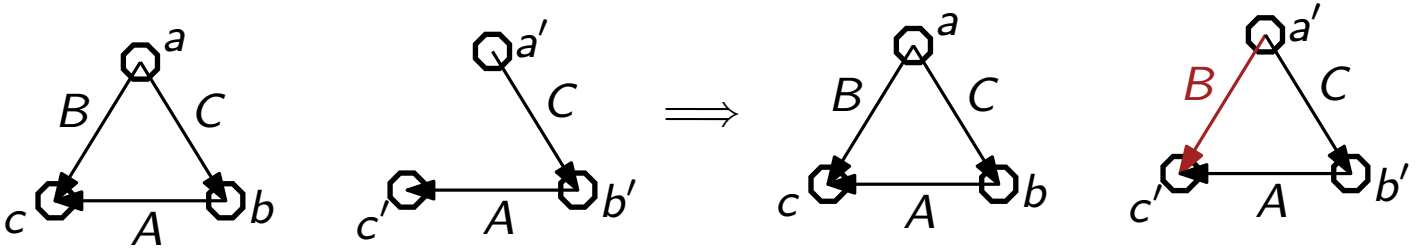


part i)



Triangle Lemma - Proof

Proof ii)

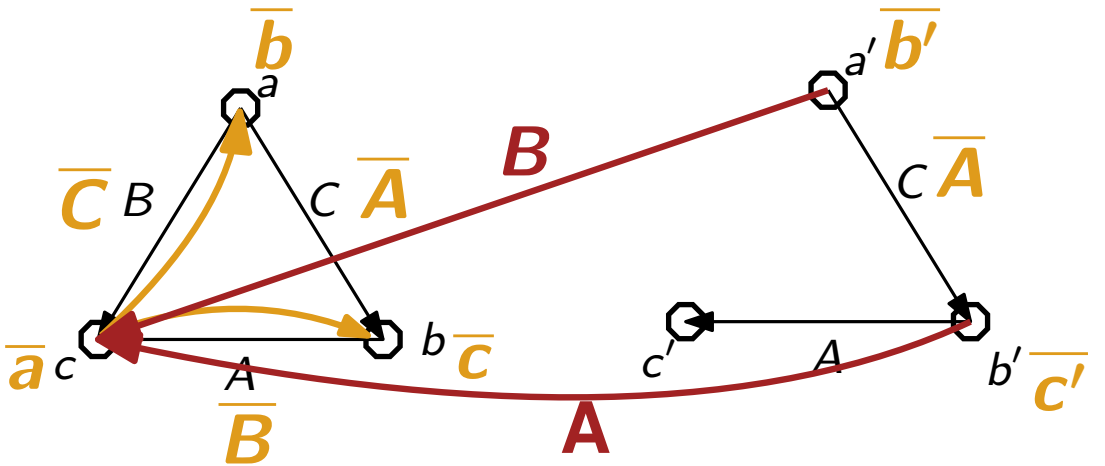


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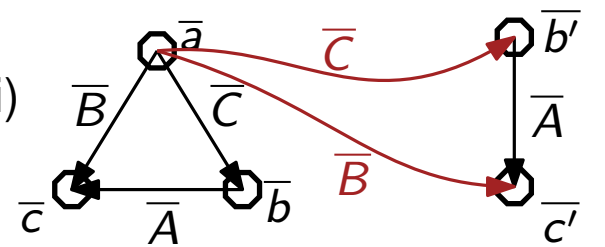
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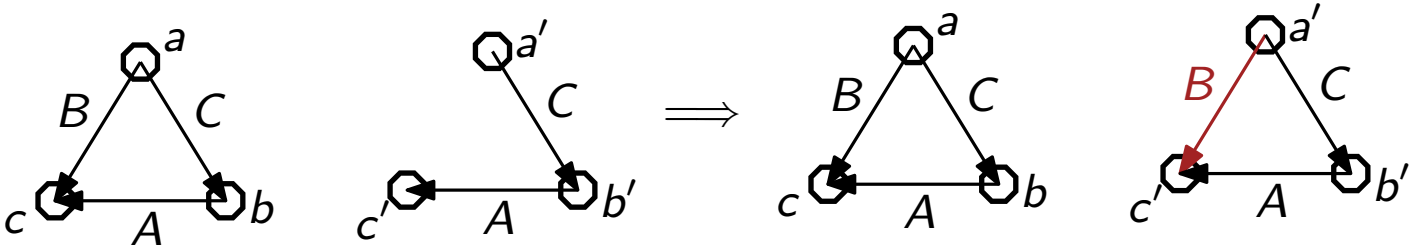
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part i)



Triangle Lemma - Proof

Proof ii)

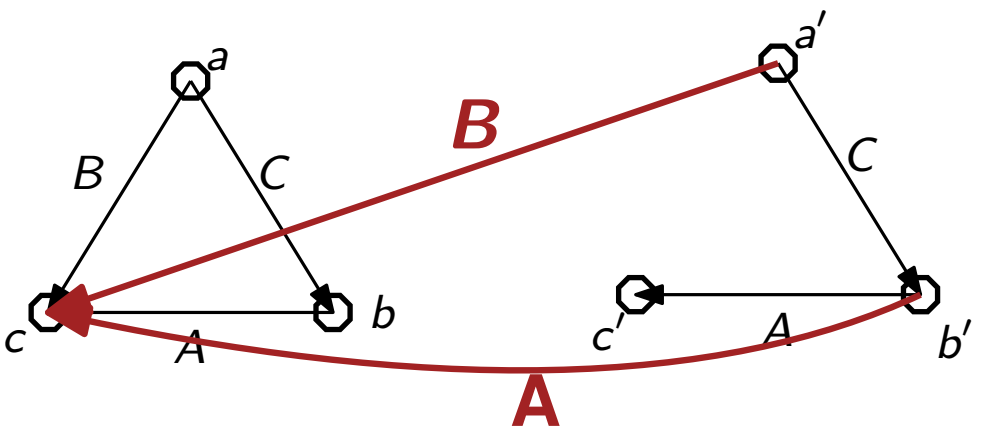


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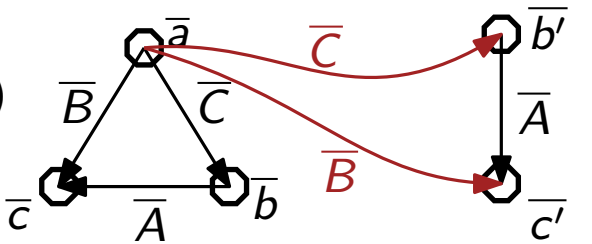
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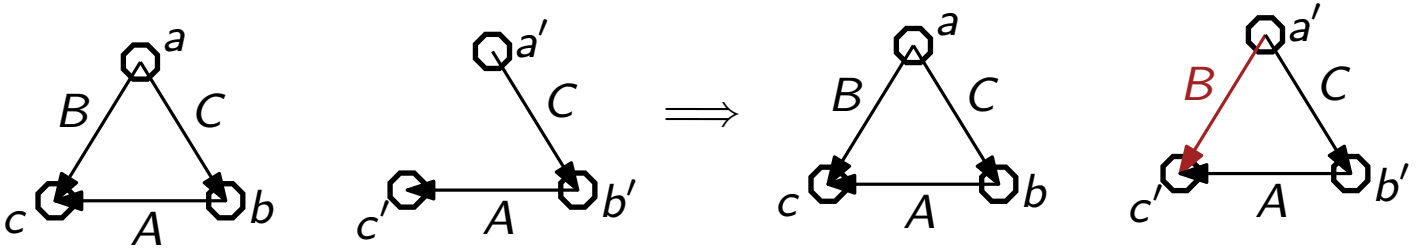
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Triangle Lemma - Proof

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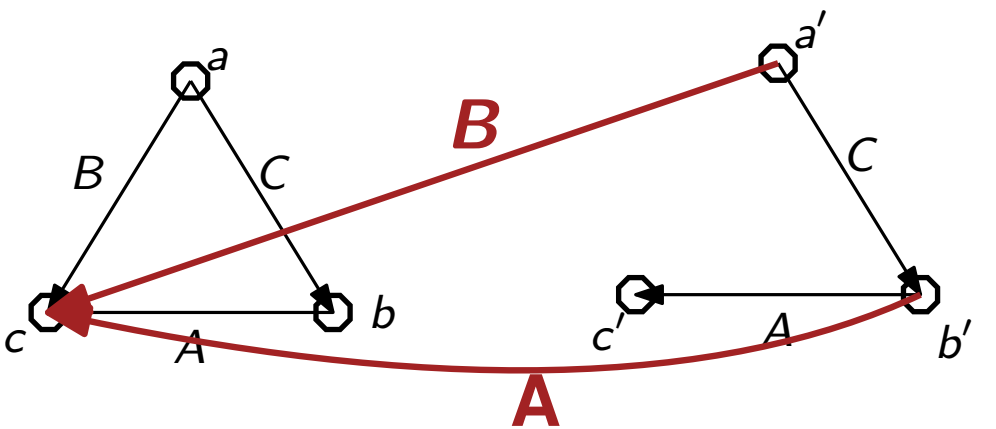


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case $B \neq C$

we are allowed to apply part i)

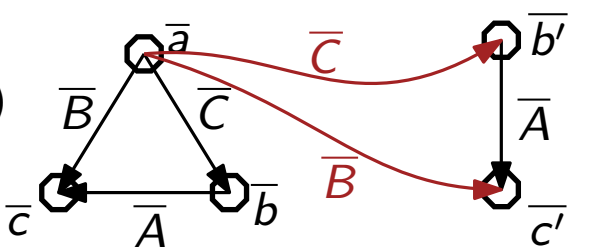


obtain $\overline{a}\overline{b'} = ca' \in \overline{C} = B^{-1} \iff a'c \in B$

obtain $\overline{a}\overline{c'} = cb' \in \overline{B} = A^{-1} \iff b'c \in A$

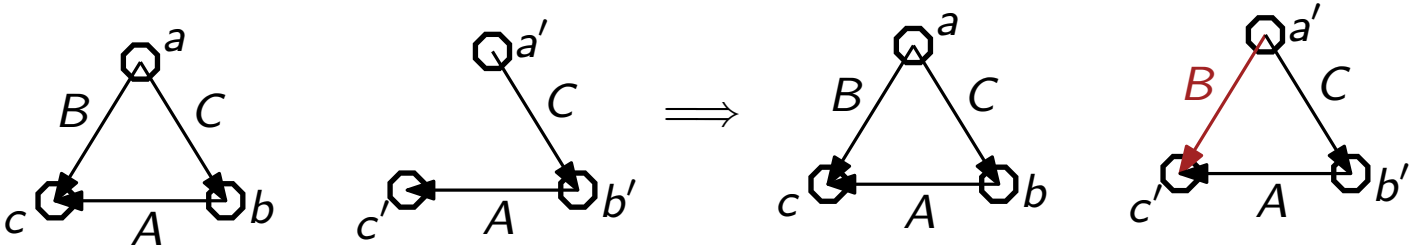
apply part i) to $a'b'c$ and $b'c'$

part i)



Triangle Lemma - Proof

Proof ii)

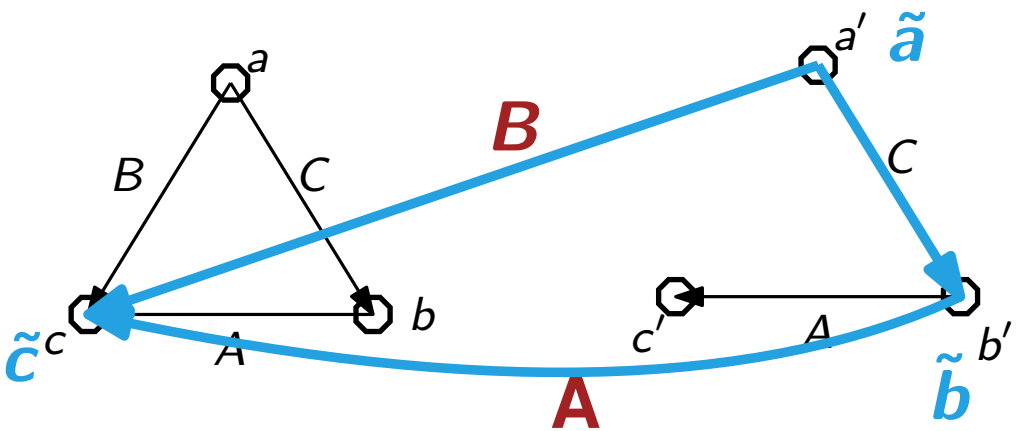


Goal: apply part i)

- $a = \overline{b}, b = \overline{c}, c = \overline{a}$
- $C = \overline{A}, A^{-1} = \overline{B}, B^{-1} = \overline{C}$
- $a' = \overline{b'}, b' = \overline{c'}$

case $B \neq C$

we are allowed to apply part i)

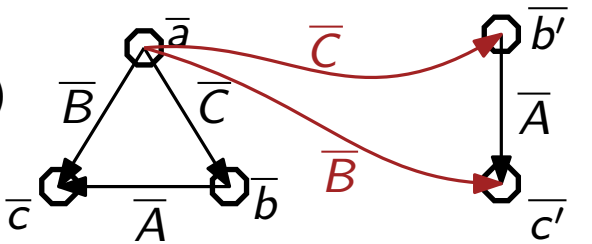


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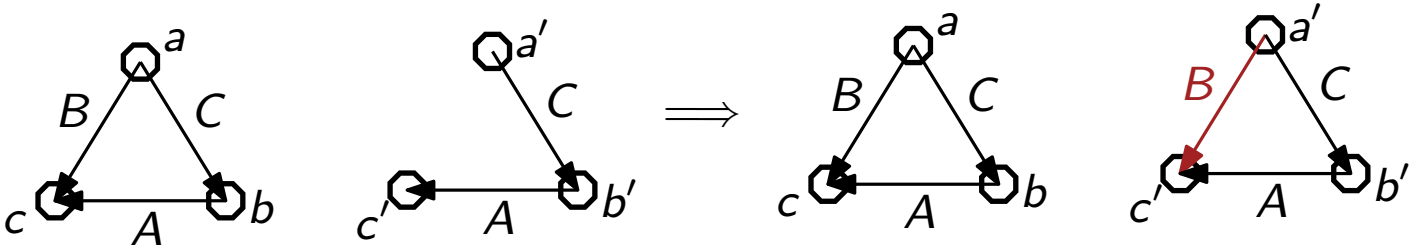
apply part i) to $a'b'c$ and $b'c'$

part i)



Triangle Lemma - Proof

Proof ii)

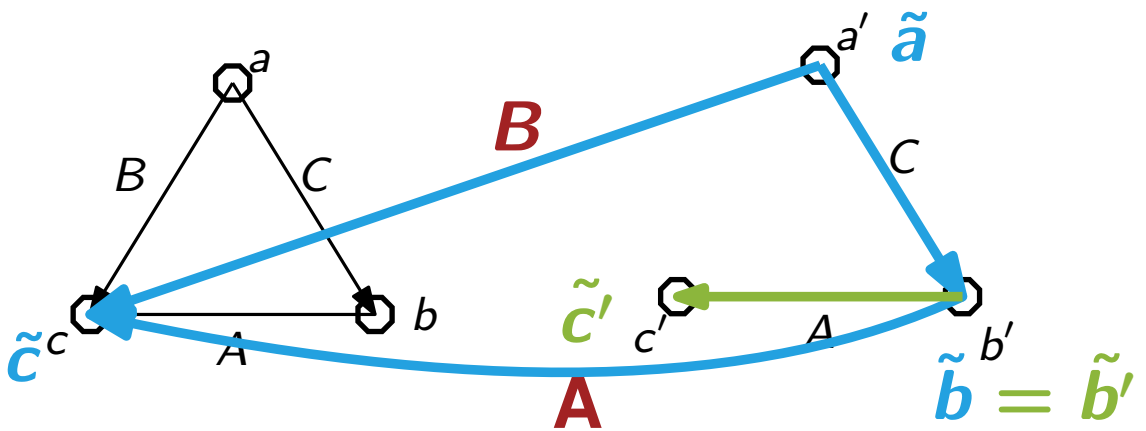


Goal: apply part i)

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case $B \neq C$

we are allowed to apply part i)

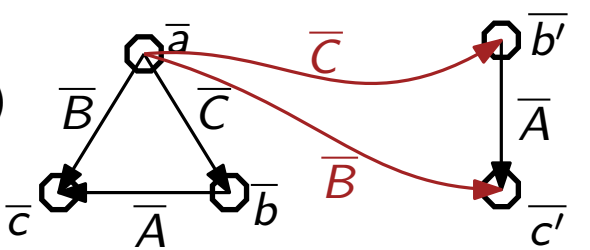


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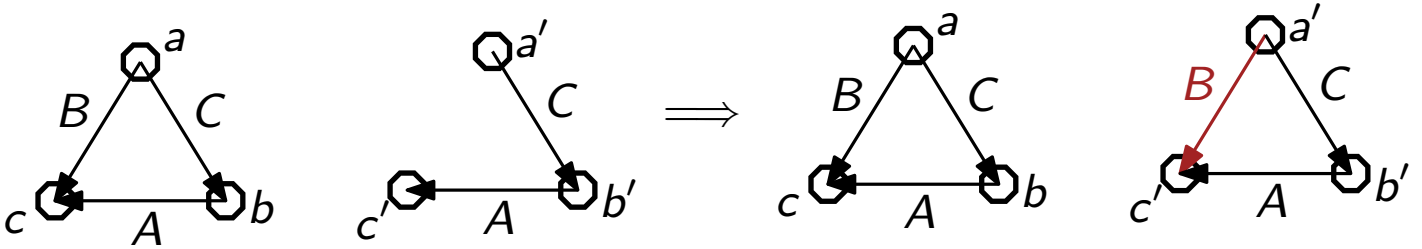
apply part i) to $a'b'c$ and $b'c'$

part i)



Triangle Lemma - Proof

Proof ii)

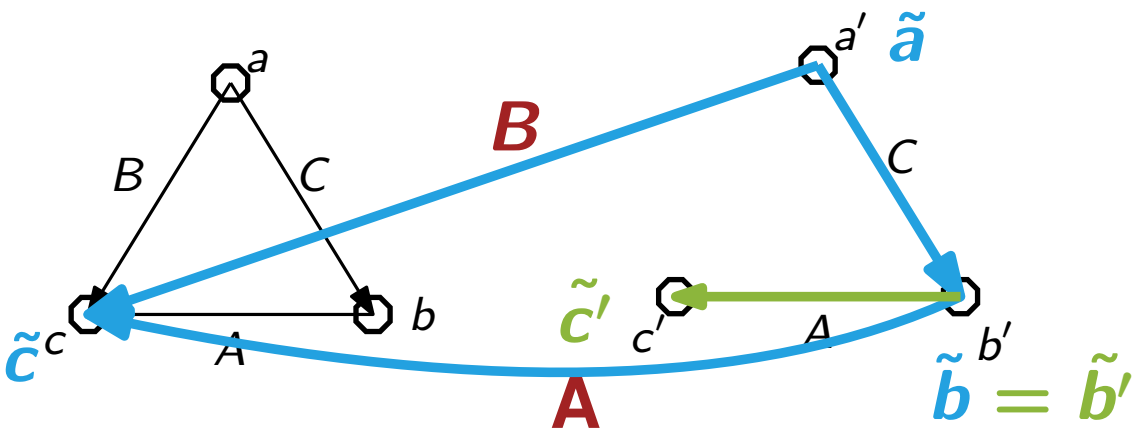


Goal: apply part i)

- $a = \overline{b}, b = \overline{c}, c = \overline{a}$
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case $B \neq C$

we are allowed to apply part i)



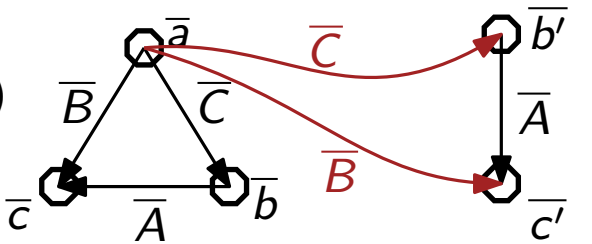
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apply part i) to $a'b'c$ and $b'c'$

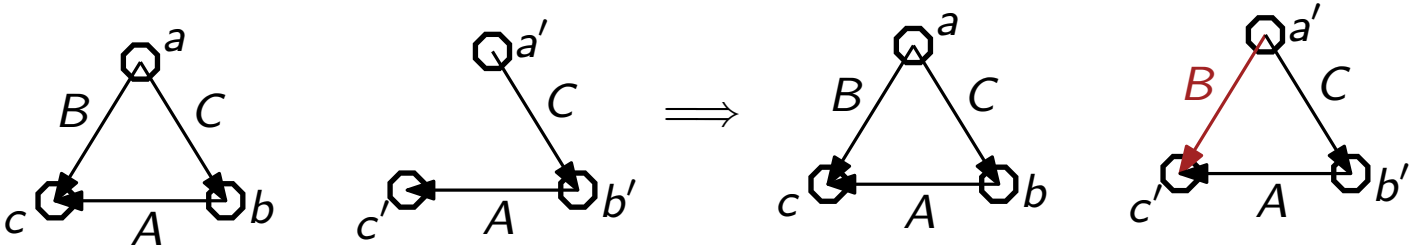
obtain $\tilde{a}\tilde{c'} = a'c' \in B$

part i)



Triangle Lemma - Proof

Proof ii)

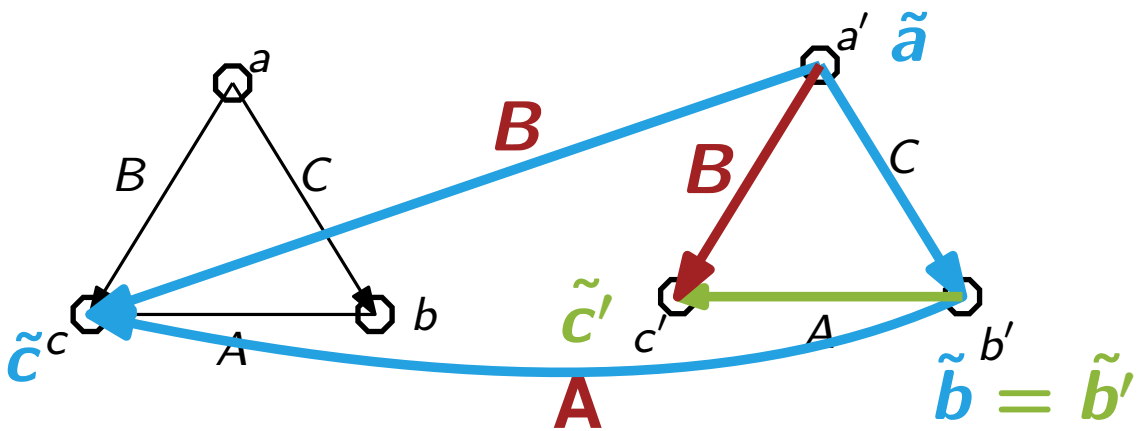


Goal: apply part i)

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- $a' = \overline{b'}, b' = \overline{c'}$

case $B \neq C$

we are allowed to apply part i)



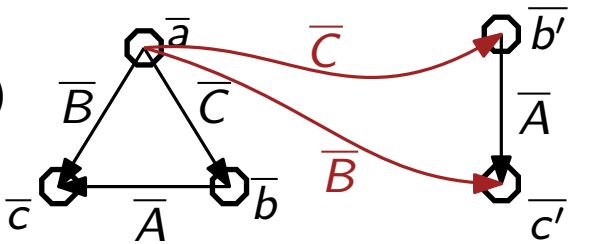
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apply part i) to $a'b'c$ and $b'c'$

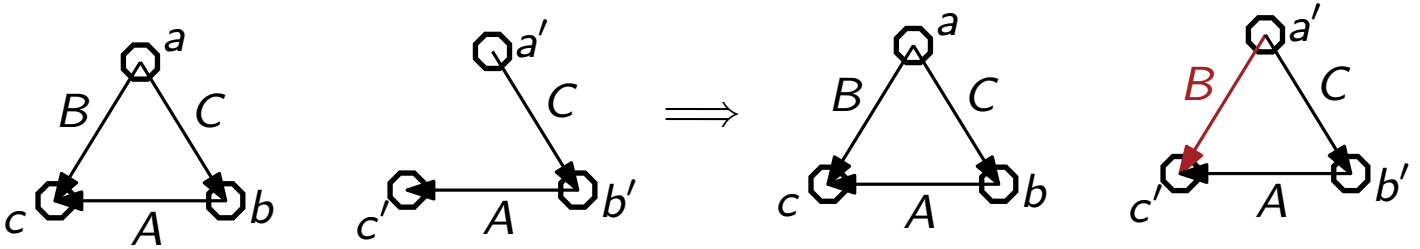
obtain $\tilde{a}\tilde{c'} = a'c' \in B$

part i)



Triangle Lemma - Proof

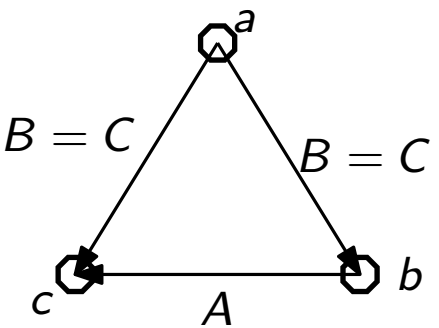
Proof ii)

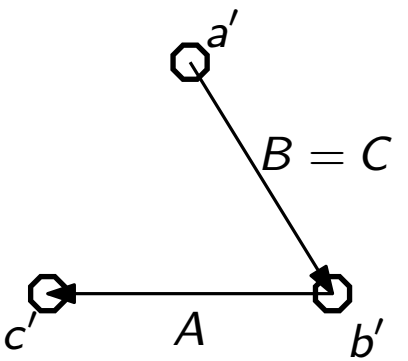


Goal: apply part i)

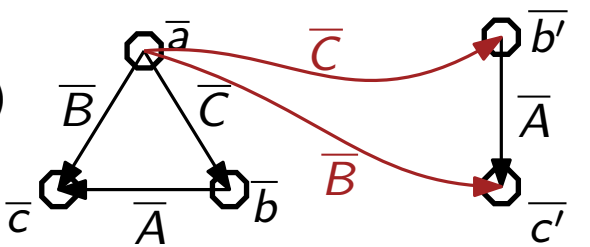
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- $a' = \overline{b'}, b' = \overline{c'}$

case $B = C$



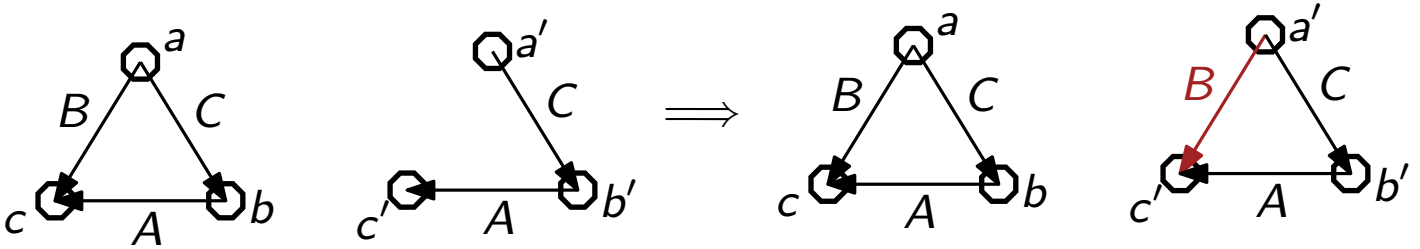


part i)



Triangle Lemma - Proof

Proof ii)

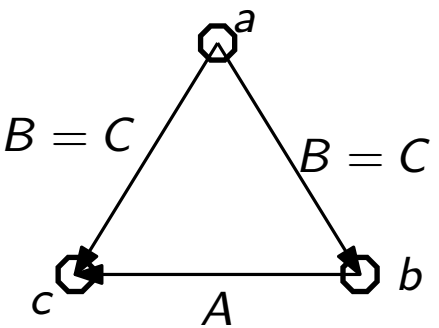


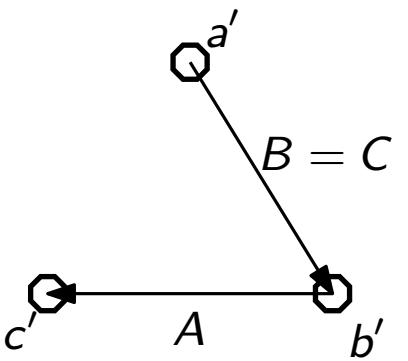
Goal: apply part i)

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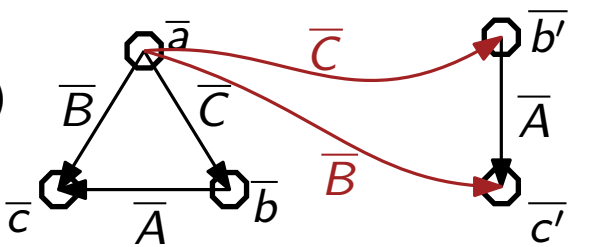
case $B = C$

can't use previous renaming



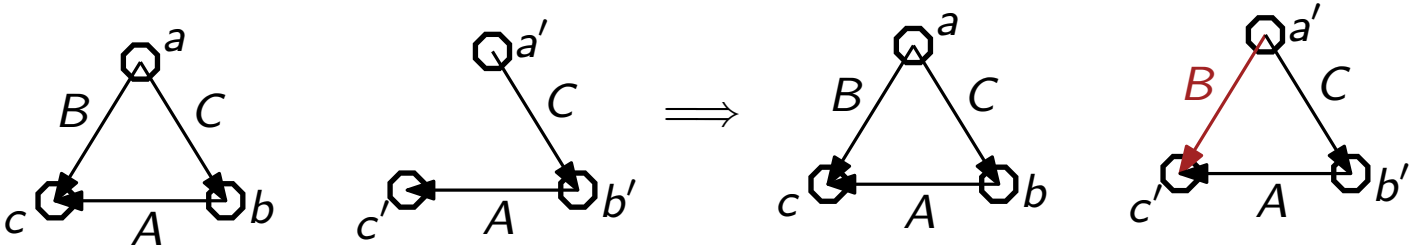


part i)

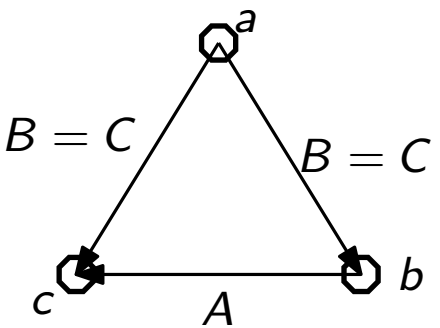


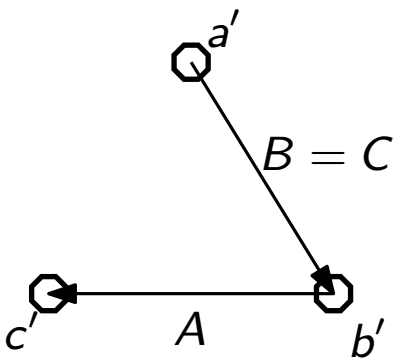
Triangle Lemma - Proof

Proof ii)

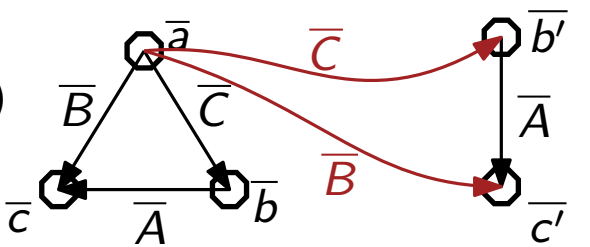


Goal: apply part i)
case $B = C$



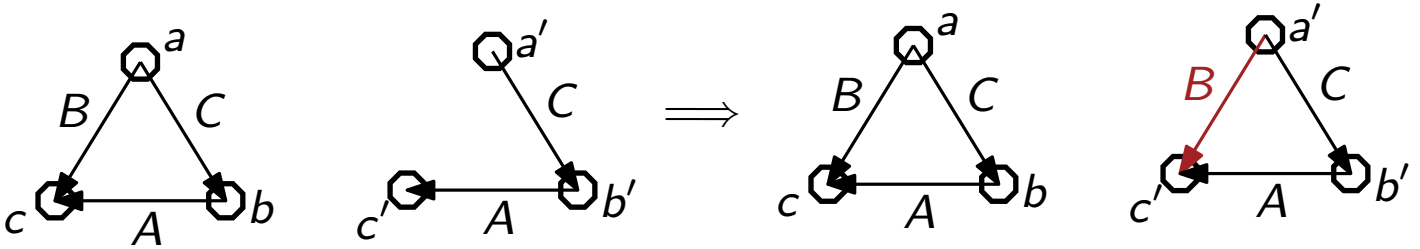


part i)

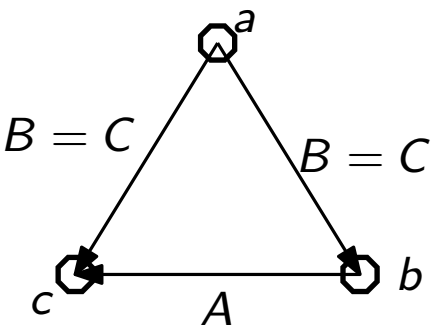


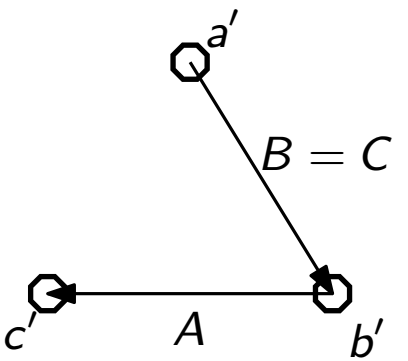
Triangle Lemma - Proof

Proof ii)

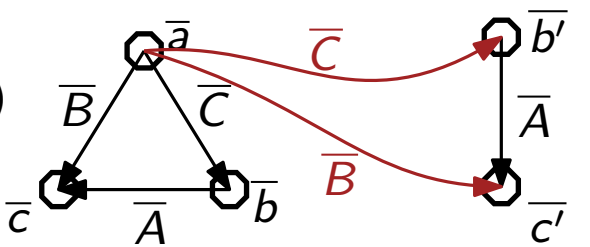


Goal: apply part i)
case $B = C$
directly apply part i) to abc and $b'c'$



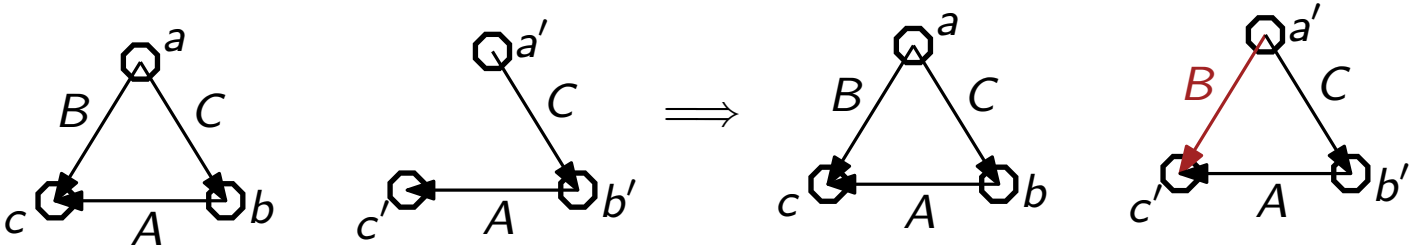


part i)

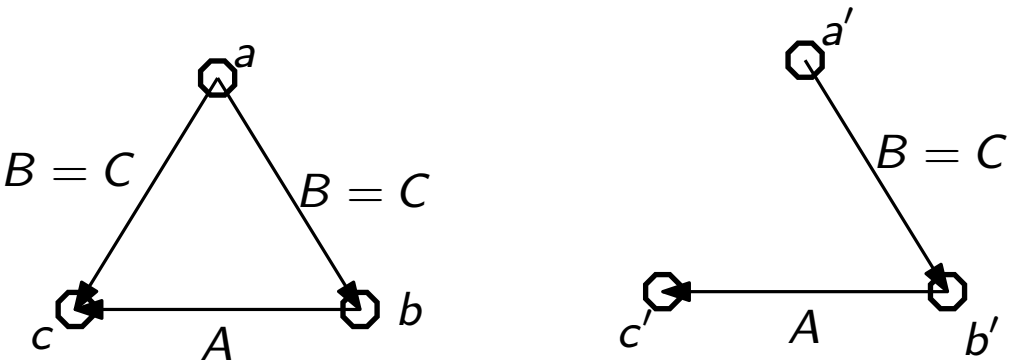


Triangle Lemma - Proof

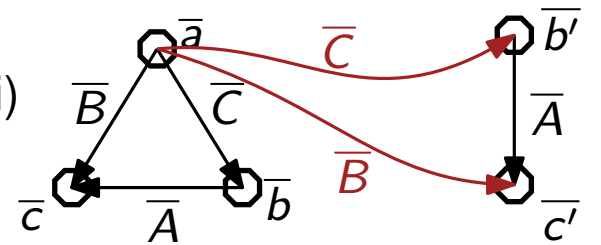
Proof ii)



Goal: apply part i)
case $B = C$
directly apply part i) to abc and $b'c'$
obtain $ab' \in C = B$ and $ac' \in B = C$

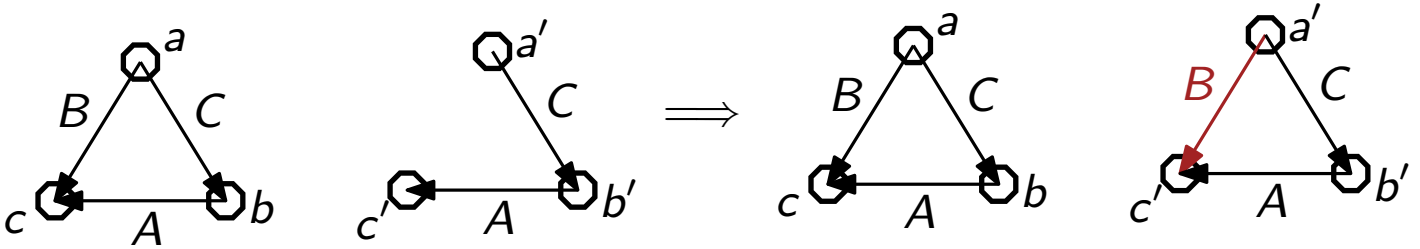


part i)

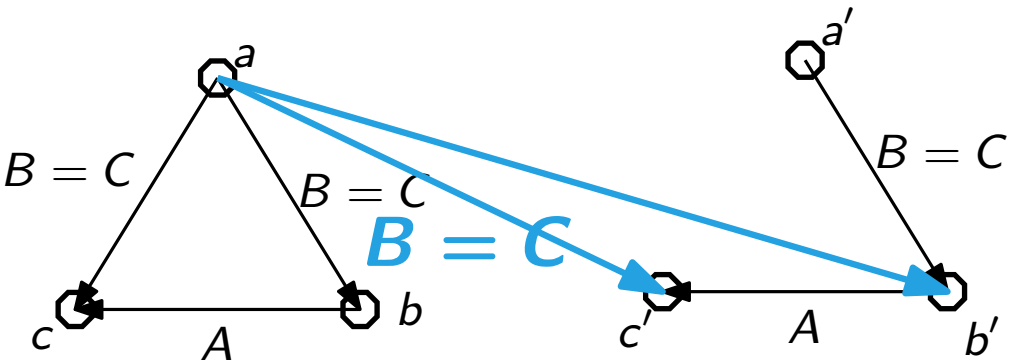


Triangle Lemma - Proof

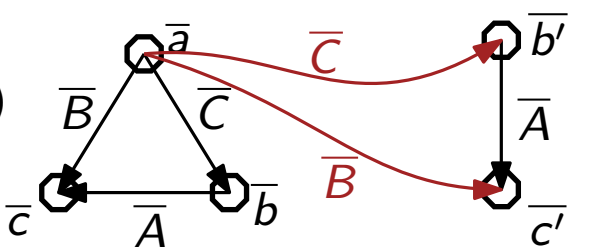
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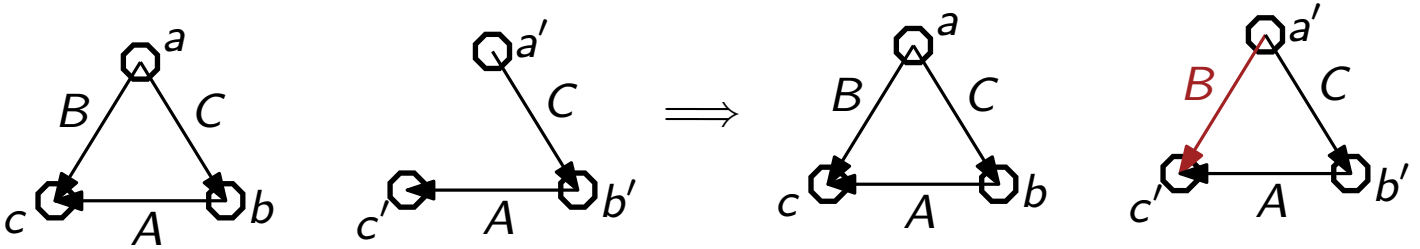


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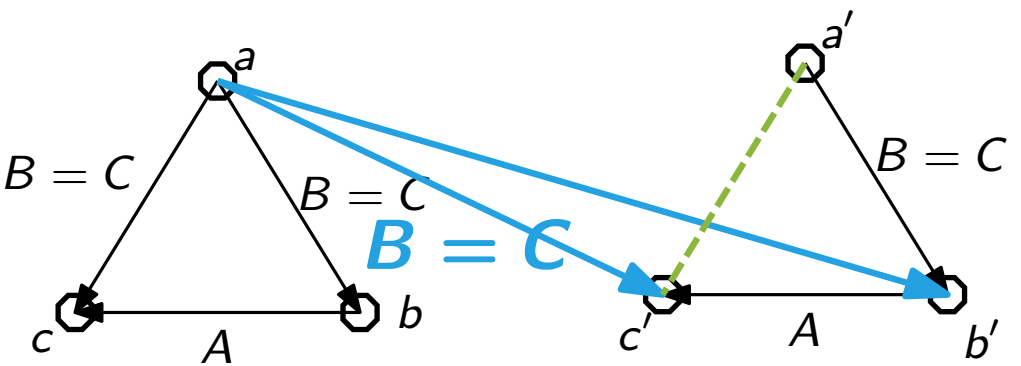


Triangle Lemma - Proof

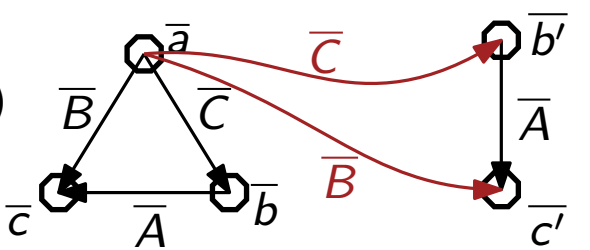
Proof ii)



Goal: apply part i)
case $B = C$
directly apply part i) to abc and $b'c'$
obtain $ab' \in C = B$ and $ac' \in B = C$
assume $a'c' \notin E(G)$

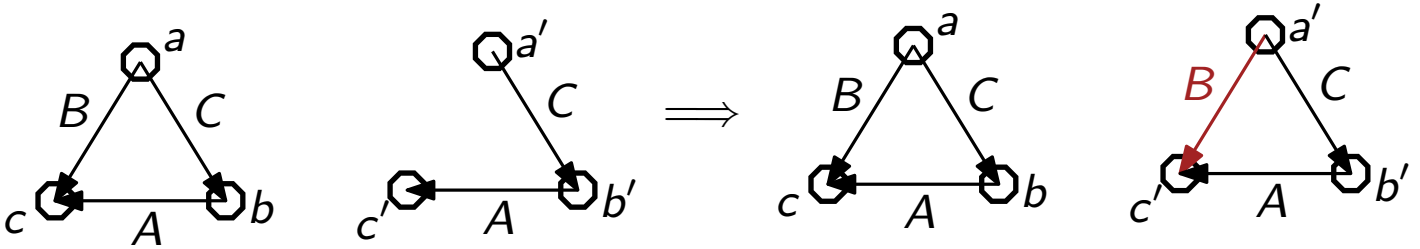


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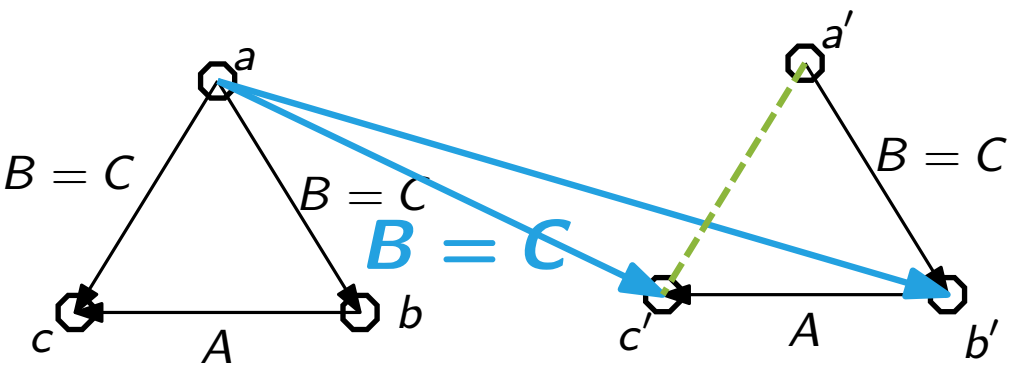


Triangle Lemma - Proof

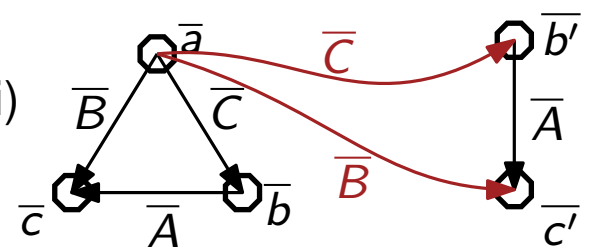
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Goal: apply part i)
case $B = C$
directly apply part i) to abc and $b'c'$
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assume $a'c' \notin E(G)$
then $b'a'\Gamma b'c'$

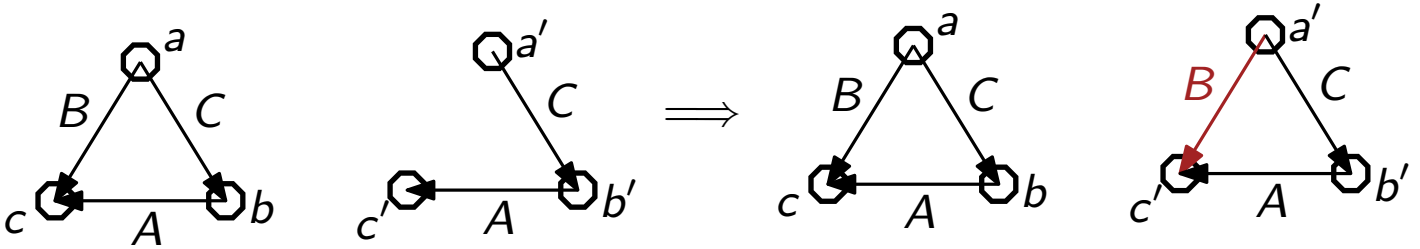


part i)

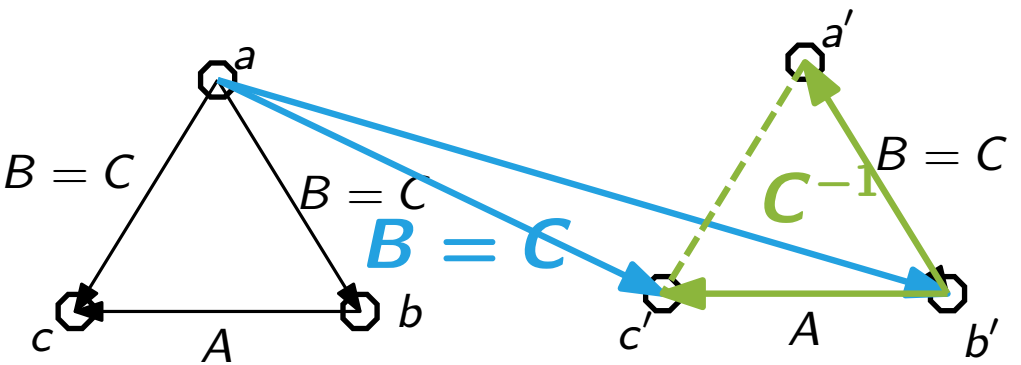


Triangle Lemma - Proof

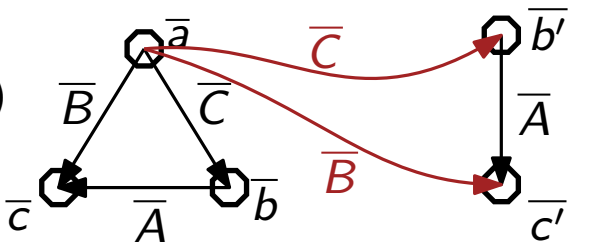
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Goal: apply part i)
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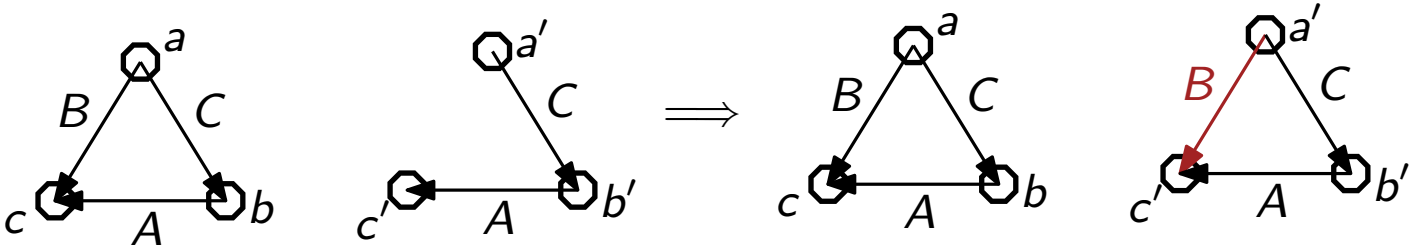


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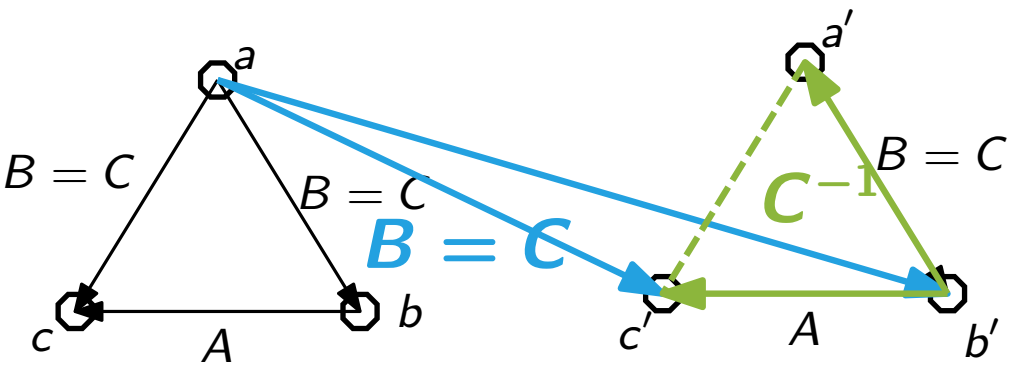


Triangle Lemma - Proof

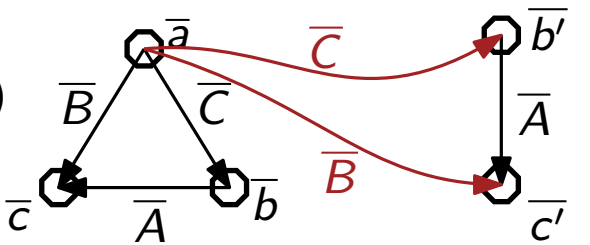
Proof ii)



Goal: apply part i)
case $B = C$
directly apply part i) to abc and $b'c'$
obtain $ab' \in C = B$ and $ac' \in B = C$
assume $a'c' \notin E(G)$
then $b'a'\Gamma b'c'$
so $C^{-1} = A$

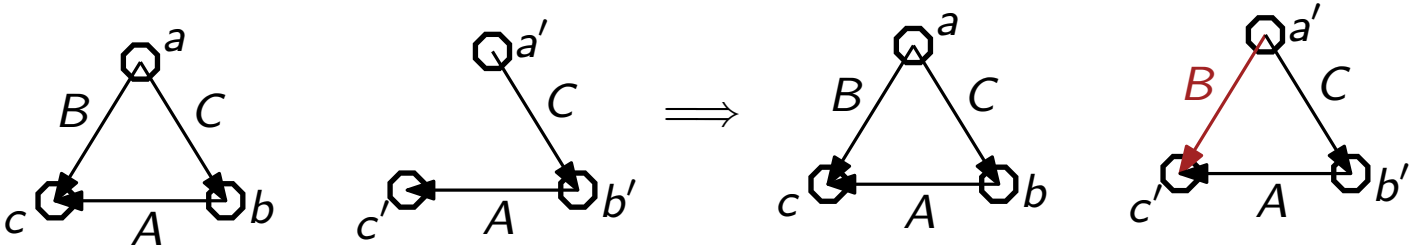


part i)

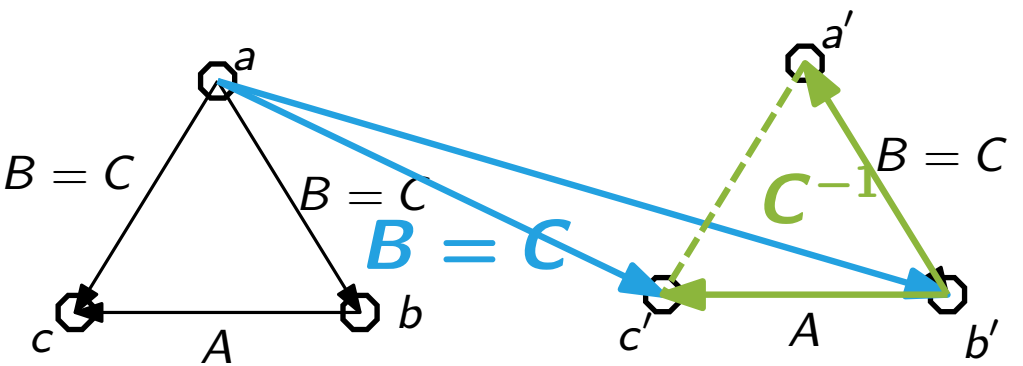


Triangle Lemma - Proof

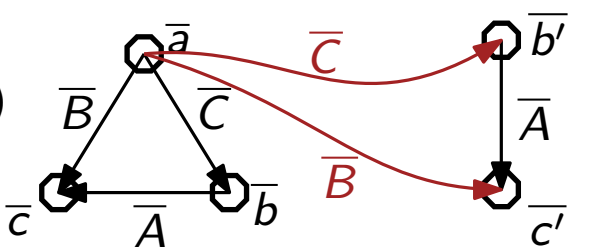
Proof ii)



Goal: apply part i)
case $B = C$
directly apply part i) to abc and $b'c'$
obtain $ab' \in C = B$ and $ac' \in B = C$
assume $a'c' \notin E(G)$
then $b'a'\Gamma b'c'$
so $C^{-1} = A$

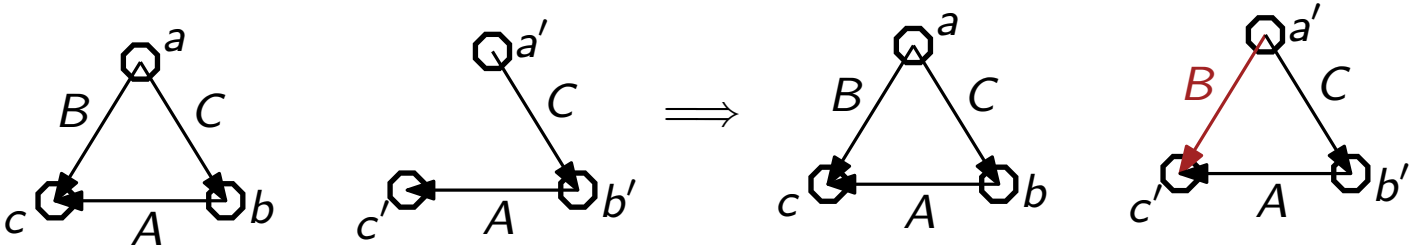


part i)

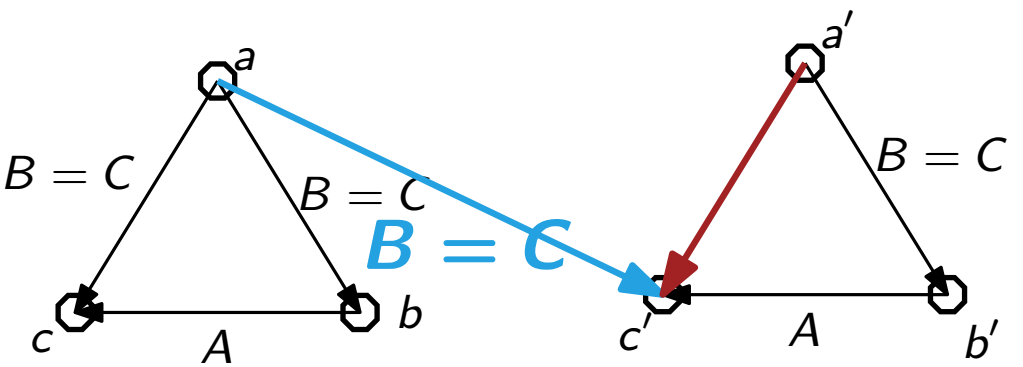


Triangle Lemma - Proof

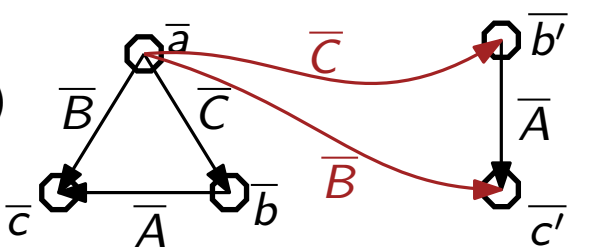
Proof ii)



Goal: apply part i)
case $B = C$
directly apply part i) to abc and $b'c'$
obtain $ab' \in C = B$ and $ac' \in B = C$
so $a'c' \in E(G)$

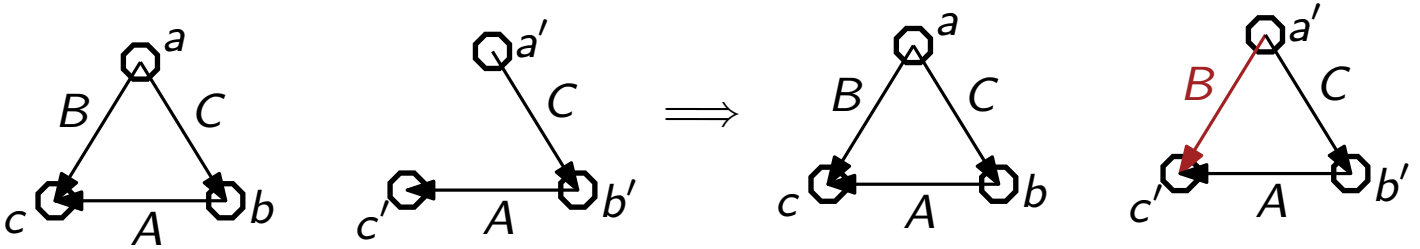


part i)

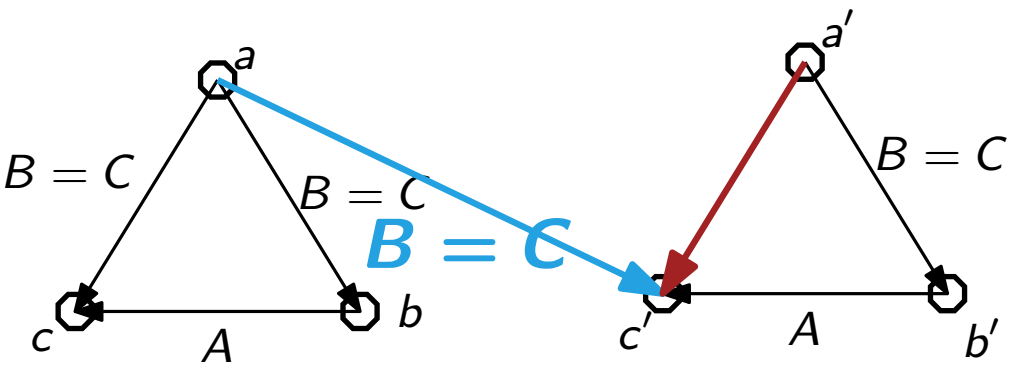


Triangle Lemma - Proof

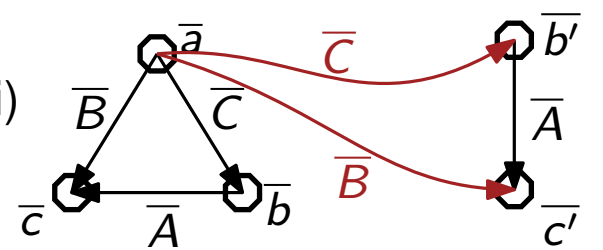
Proof ii)



Goal: apply part i)
case $B = C$
directly apply part i) to abc and $b'c'$
obtain $ab' \in C = B$ and $ac' \in B = C$
so $a'c' \in E(G)$
let $a'c' \in D \in \mathcal{I}(G)$

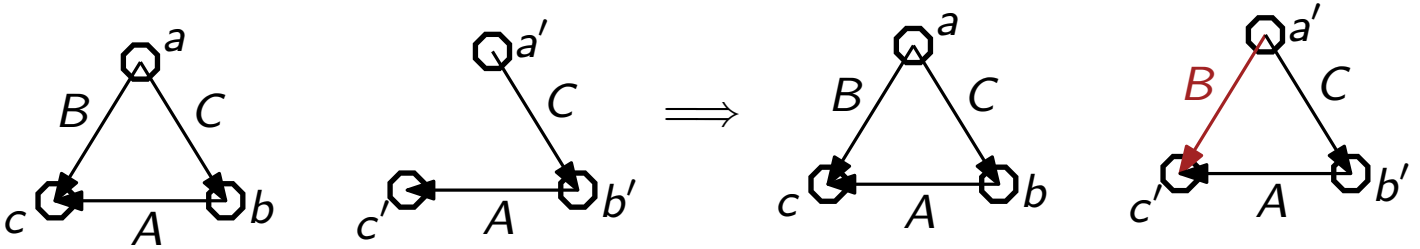


part i)

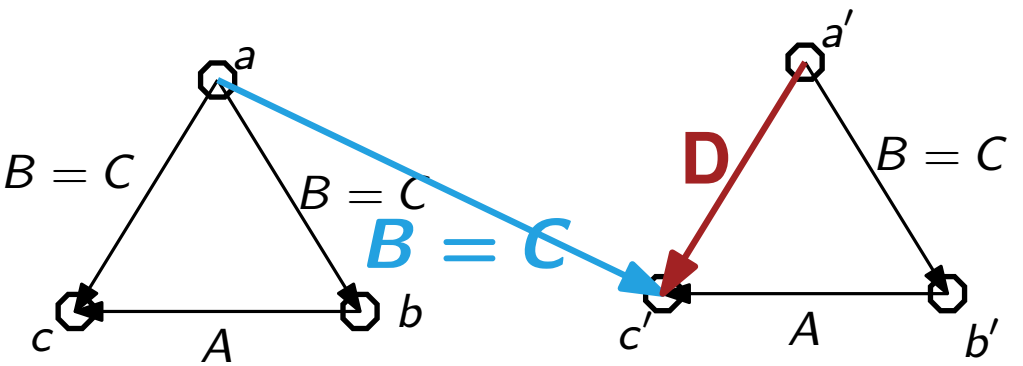


Triangle Lemma - Proof

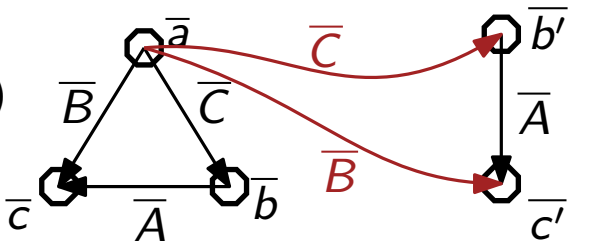
Proof ii)



Goal: apply part i)
case $B = C$
directly apply part i) to abc and $b'c'$
obtain $ab' \in C = B$ and $ac' \in B = C$
so $a'c' \in E(G)$
let $a'c' \in D \in \mathcal{I}(G)$

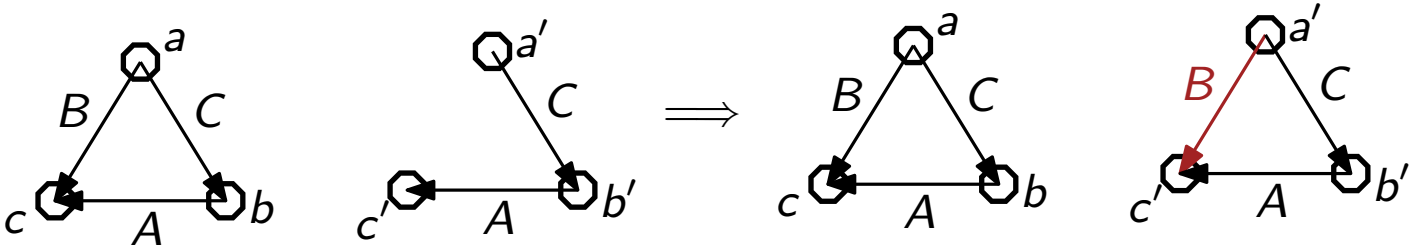


part i)



Triangle Lemma - Proof

Proof ii)



Goal: apply part i)

case $B = C$

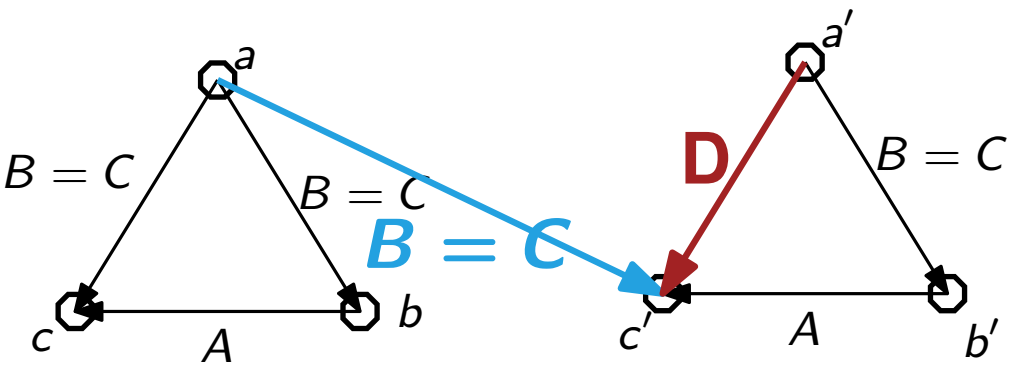
directly apply part i) to abc and $b'c'$

obtain $ab' \in C = B$ and $ac' \in B = C$

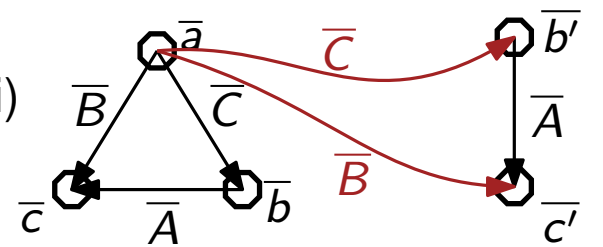
so $a'c' \in E(G)$

let $a'c' \in D \in \mathcal{I}(G)$

assume $D \neq B$ (otherwise we're done)

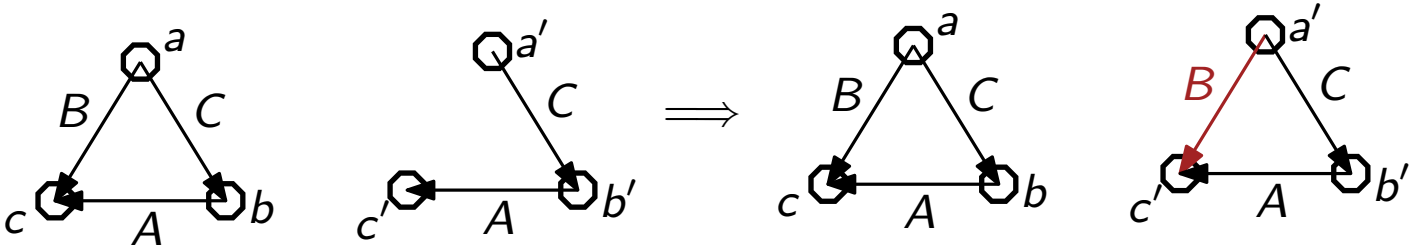


part i)



Triangle Lemma - Proof

Proof ii)



Goal: apply part i)

case $B = C$

directly apply part i) to abc and $b'c'$

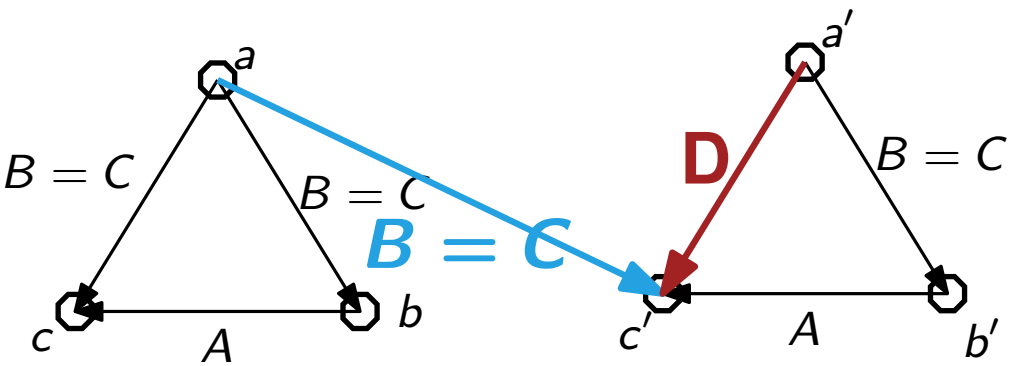
obtain $ab' \in C = B$ and $ac' \in B = C$

so $a'c' \in E(G)$

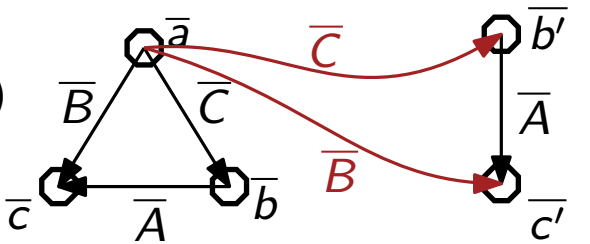
let $a'c' \in D \in \mathcal{I}(G)$

assume $D \neq B$ (otherwise we're done)

apply part i) to $c'b'a'$ and ba

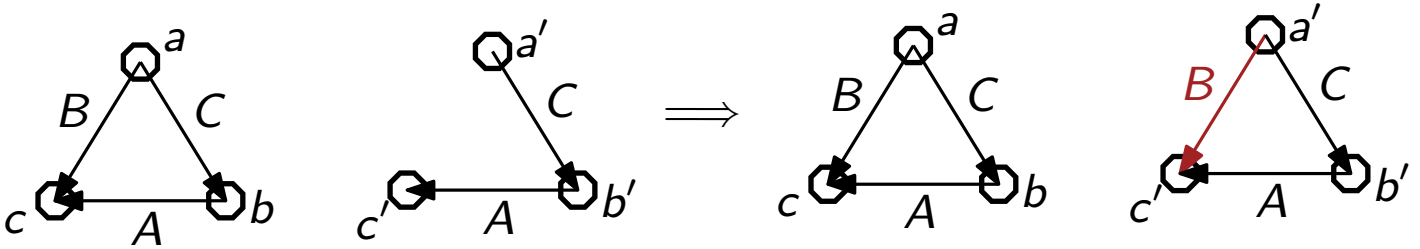


part i)

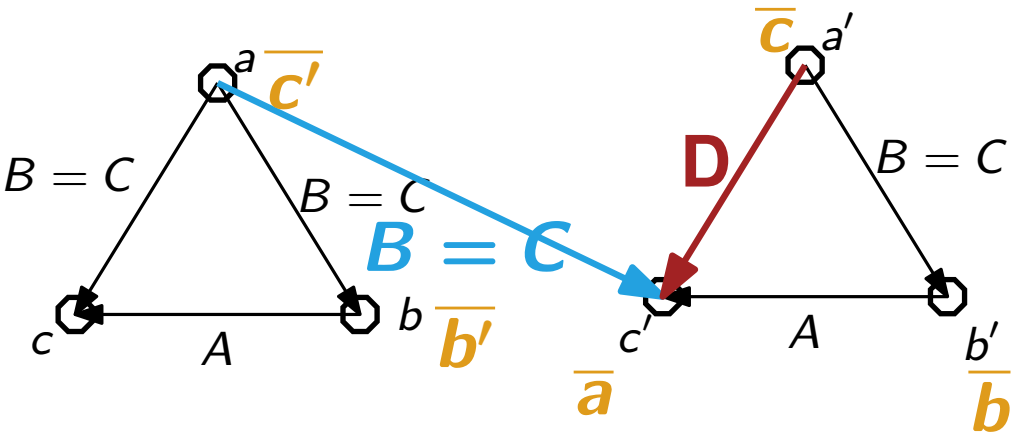


Triangle Lemma - Proof

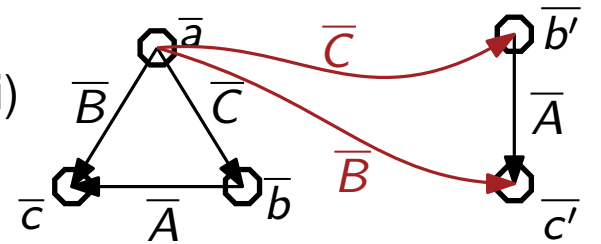
Proof ii)



Goal: apply part i)
case $B = C$
directly apply part i) to abc and $b'c'$
obtain $ab' \in C = B$ and $ac' \in B = C$
so $a'c' \in E(G)$
let $a'c' \in D \in \mathcal{I}(G)$
assume $D \neq B$ (otherwise we're done)
apply part i) to $c'b'a'$ and ba

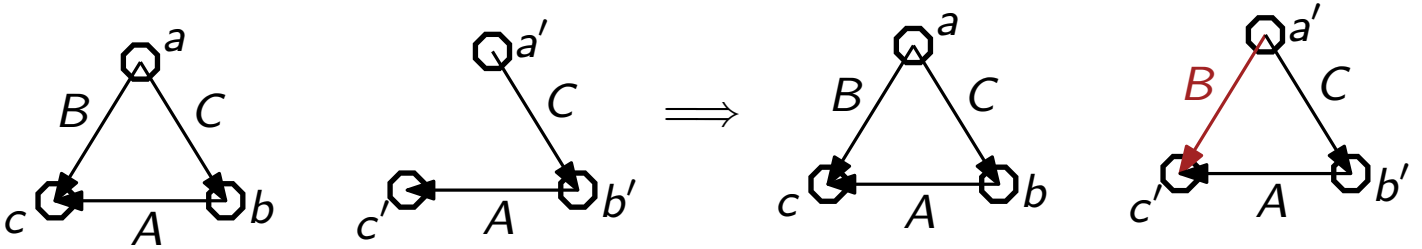


part i)



Triangle Lemma - Proof

Proof ii)



Goal: apply part i)

case $B = C$

directly apply part i) to abc and $b'c'$

obtain $ab' \in C = B$ and $ac' \in B = C$

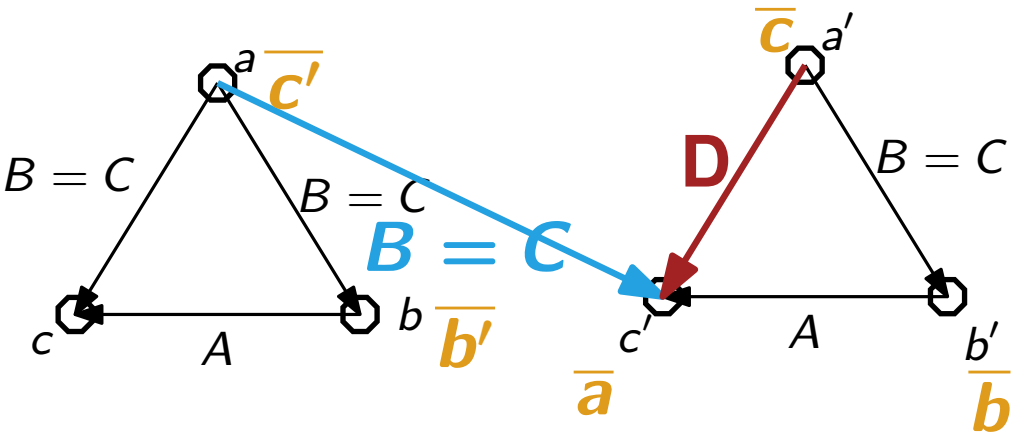
so $a'c' \in E(G)$

let $a'c' \in D \in \mathcal{I}(G)$

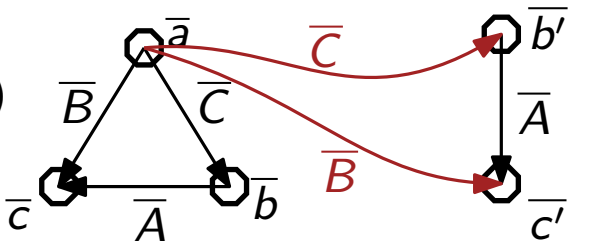
assume $D \neq B$ (otherwise we're done)

apply part i) to $c'b'a'$ and ba

$\overline{A} = B^{-1} = C^{-1}, \overline{C} = A^{-1}, \overline{B} = D^{-1}$

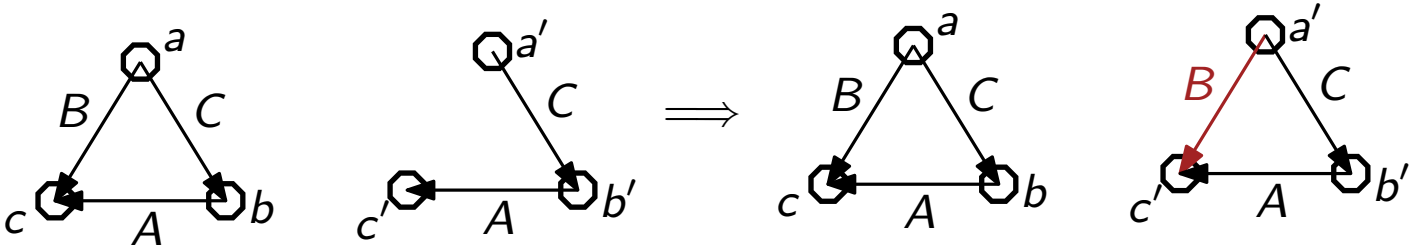


part i)



Triangle Lemma - Proof

Proof ii)



Goal: apply part i)

case $B = C$

directly apply part i) to abc and $b'c'$

obtain $ab' \in C = B$ and $ac' \in B = C$

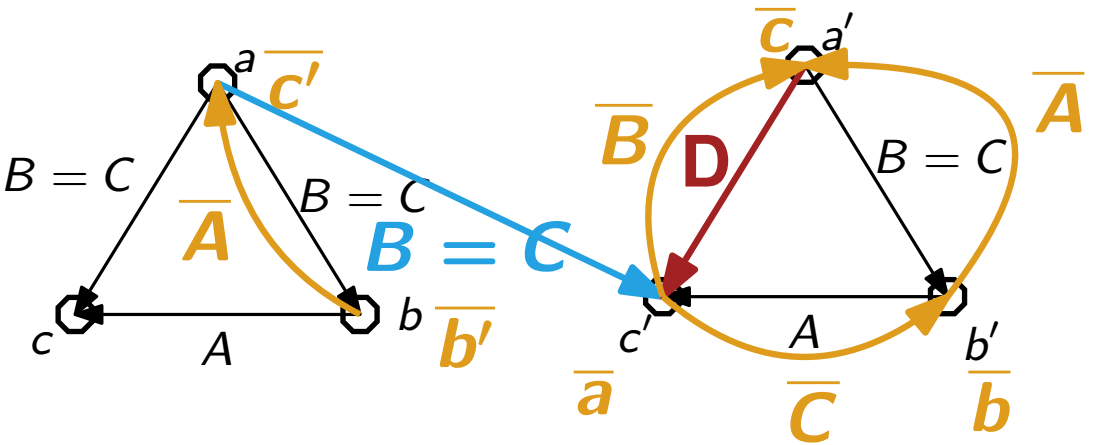
so $a'c' \in E(G)$

let $a'c' \in D \in \mathcal{I}(G)$

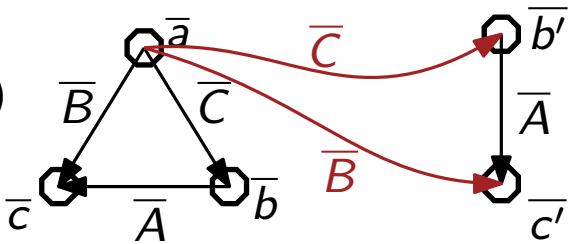
assume $D \neq B$ (otherwise we're done)

apply part i) to $c'b'a'$ and ba

$\overline{A} = B^{-1} = C^{-1}, \overline{C} = A^{-1}, \overline{B} = D^{-1}$

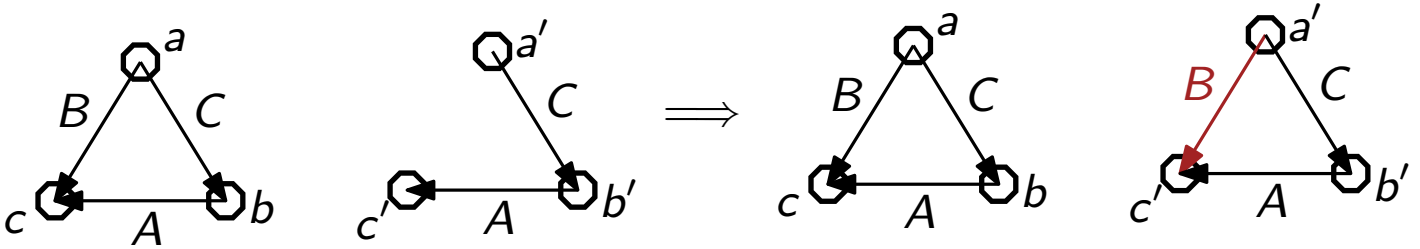


part i)



Triangle Lemma - Proof

Proof ii)



Goal: apply part i)

case $B = C$

directly apply part i) to abc and $b'c'$

obtain $ab' \in C = B$ and $ac' \in B = C$

so $a'c' \in E(G)$

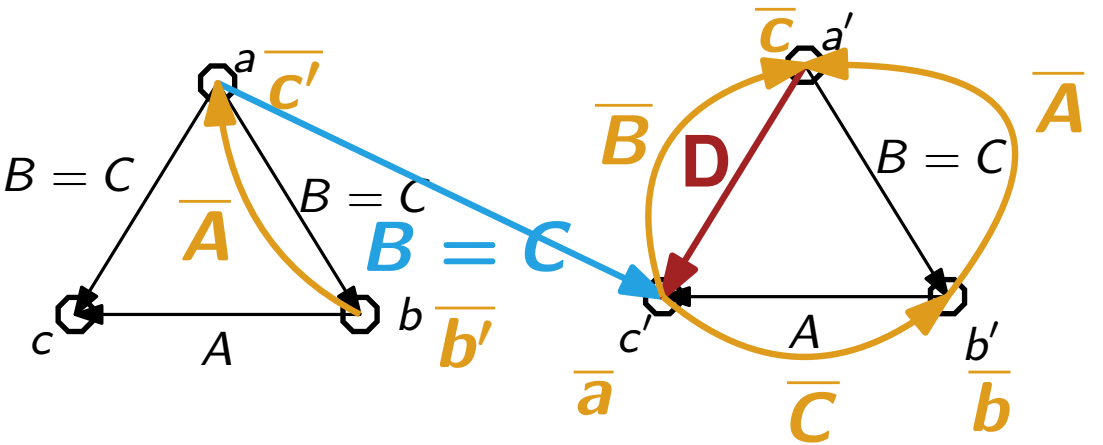
let $a'c' \in D \in \mathcal{I}(G)$

assume $D \neq B$ (otherwise we're done)

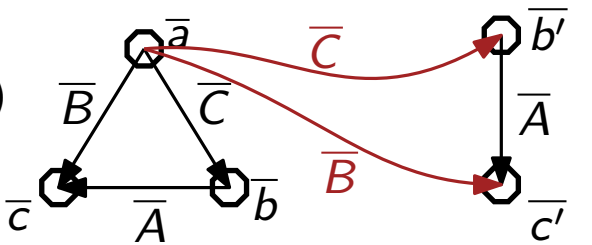
apply part i) to $c'b'a'$ and ba

$\overline{A} = B^{-1} = C^{-1}, \overline{C} = A^{-1}, \overline{B} = D^{-1}$

$\overline{A} = B^{-1} \neq D^{-1} = \overline{B}$

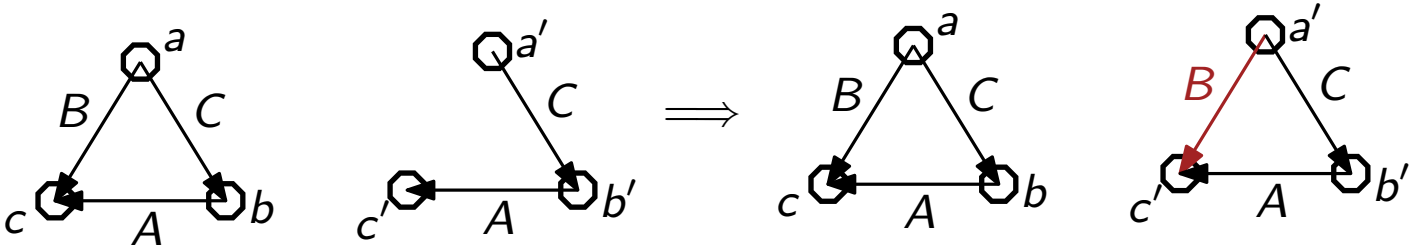


part i)



Triangle Lemma - Proof

Proof ii)



Goal: apply part i)

case $B = C$

directly apply part i) to abc and $b'c'$

obtain $ab' \in C = B$ and $ac' \in B = C$

so $a'c' \in E(G)$

let $a'c' \in D \in \mathcal{I}(G)$

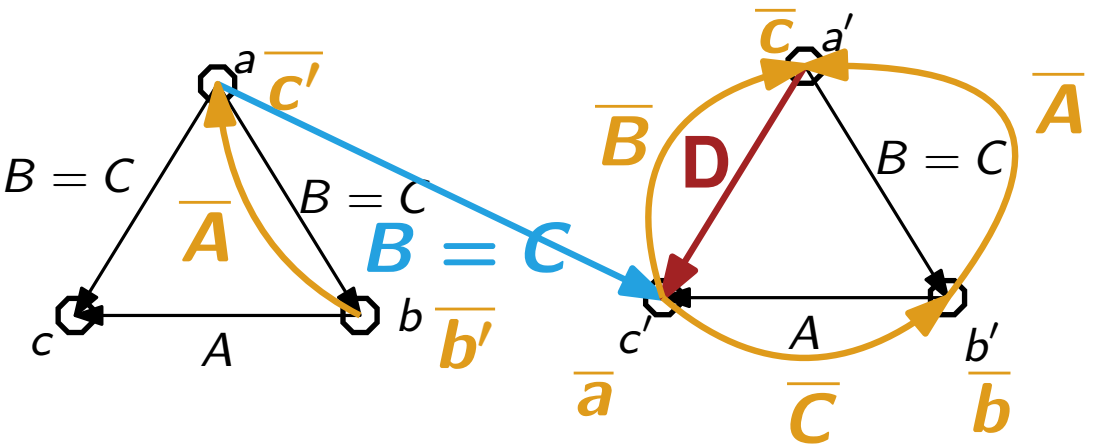
assume $D \neq B$ (otherwise we're done)

apply part i) to $c'b'a'$ and ba

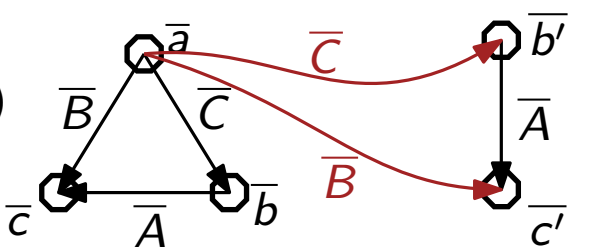
$\overline{A} = B^{-1} = C^{-1}, \overline{C} = A^{-1}, \overline{B} = D^{-1}$

$\overline{A} = B^{-1} \neq D^{-1} = \overline{B}$

$\overline{A} = C^{-1} \neq A = \overline{C}^{-1}$

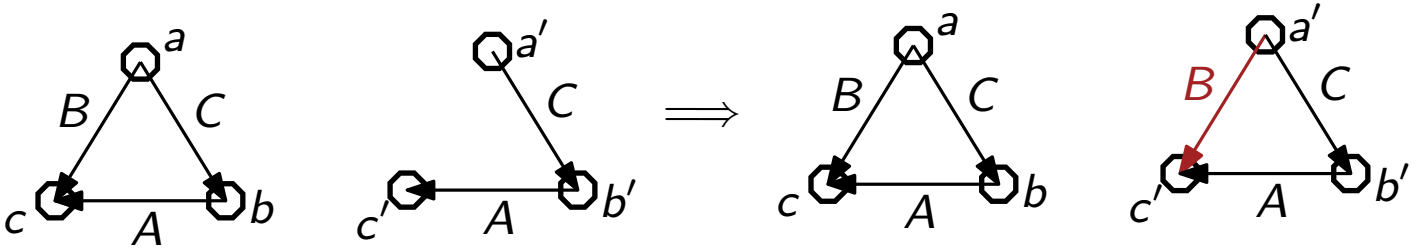


part i)



Triangle Lemma - Proof

Proof ii)



Goal: apply part i)

case $B = C$

directly apply part i) to abc and $b'c'$

obtain $ab' \in C = B$ and $ac' \in B = C$

so $a'c' \in E(G)$

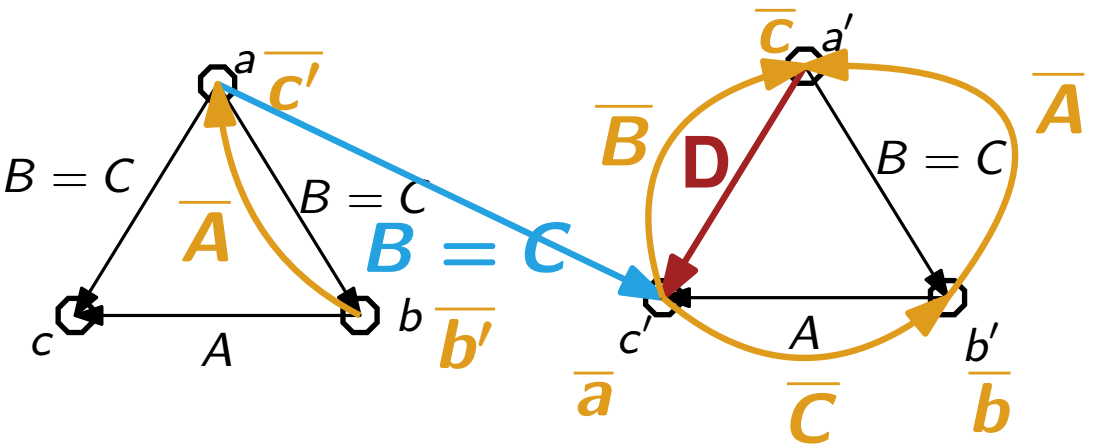
let $a'c' \in D \in \mathcal{I}(G)$

assume $D \neq B$ (otherwise we're done)

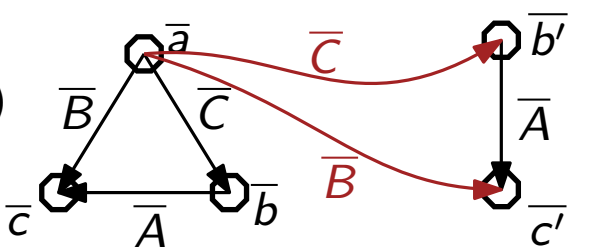
apply part i) to $c'b'a'$ and ba

$\overline{A} = B^{-1} = C^{-1}, \overline{C} = A^{-1}, \overline{B} = D^{-1}$

obtain $\overline{a}\overline{c'} = c'a \in \overline{B} = D^{-1} \iff ac' \in D$

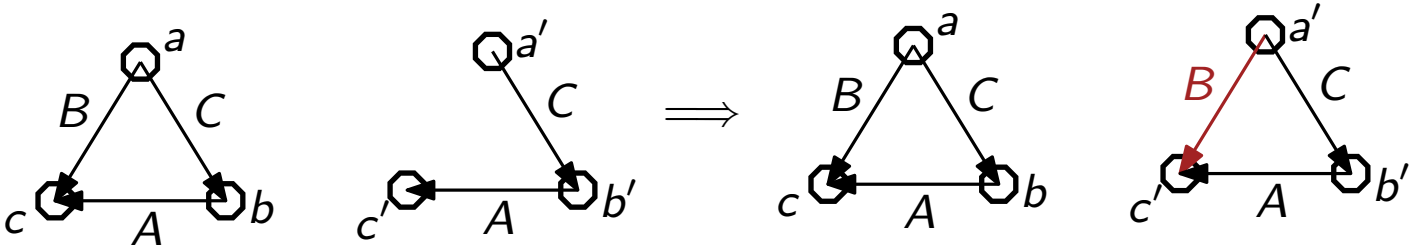


part i)



Triangle Lemma - Proof

Proof ii)



Goal: apply part i)

case $B = C$

directly apply part i) to abc and $b'c'$

obtain $ab' \in C = B$ and $ac' \in B = C$

so $a'c' \in E(G)$

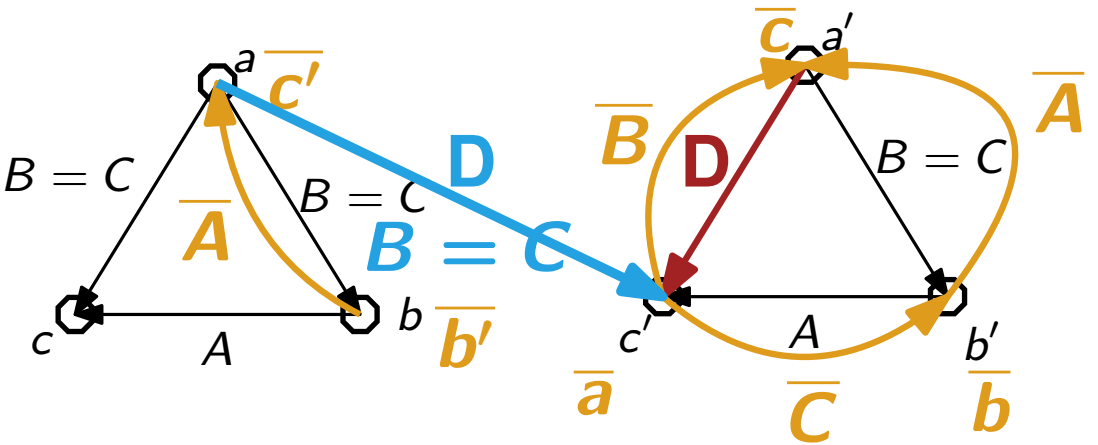
let $a'c' \in D \in \mathcal{I}(G)$

assume $D \neq B$ (otherwise we're done)

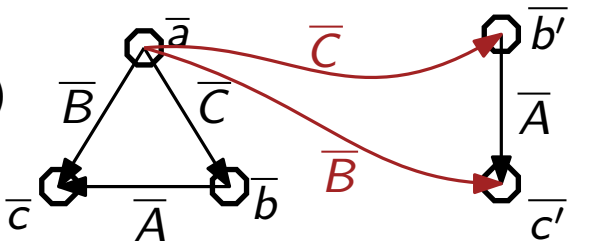
apply part i) to $c'b'a'$ and ba

$\overline{A} = B^{-1} = C^{-1}, \overline{C} = A^{-1}, \overline{B} = D^{-1}$

obtain $\overline{a}\overline{c'} = c'a \in \overline{B} = D^{-1} \iff ac' \in D$

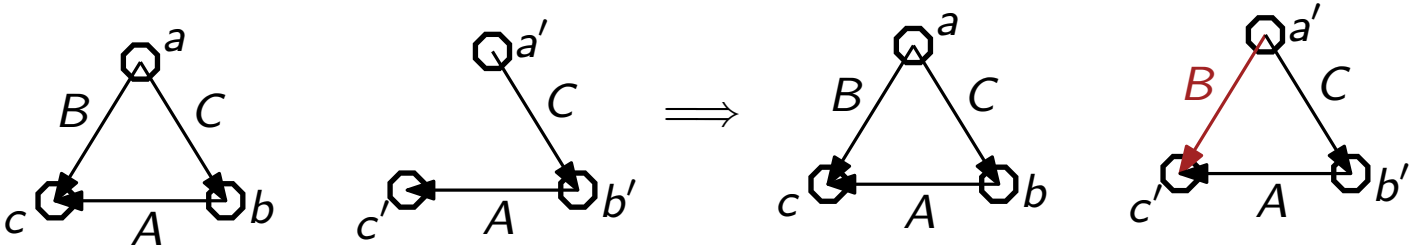


part i)



Triangle Lemma - Proof

Proof ii)



Goal: apply part i)

case $B = C$

directly apply part i) to abc and $b'c'$

obtain $ab' \in C = B$ and $ac' \in B = C$

so $a'c' \in E(G)$

let $a'c' \in D \in \mathcal{I}(G)$

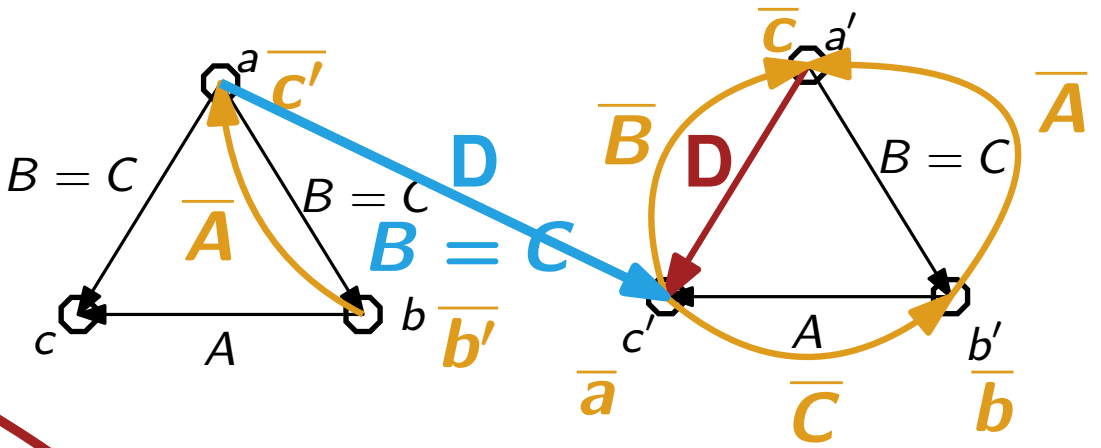
assume $D \neq B$ (otherwise we're done)

apply part i) to $c'b'a'$ and ba

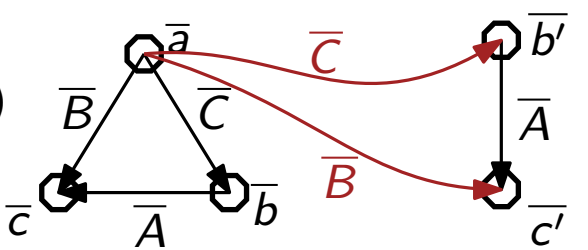
$\overline{A} = B^{-1} = C^{-1}, \overline{C} = A^{-1}, \overline{B} = D^{-1}$

obtain $\overline{a}\overline{c'} = c'a \in \overline{B} = D^{-1} \iff ac' \in D$

but previously we obtained $ac' \in B$ so $B = D$

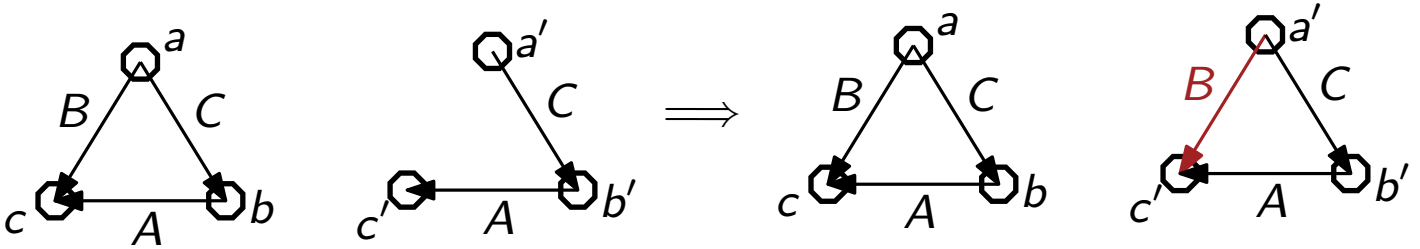


part i)



Triangle Lemma - Proof

Proof ii)



Goal: apply part i)

case $B = C$

directly apply part i) to abc and $b'c'$

obtain $ab' \in C = B$ and $ac' \in B = C$

so $a'c' \in E(G)$

let $a'c' \in D \in \mathcal{I}(G)$

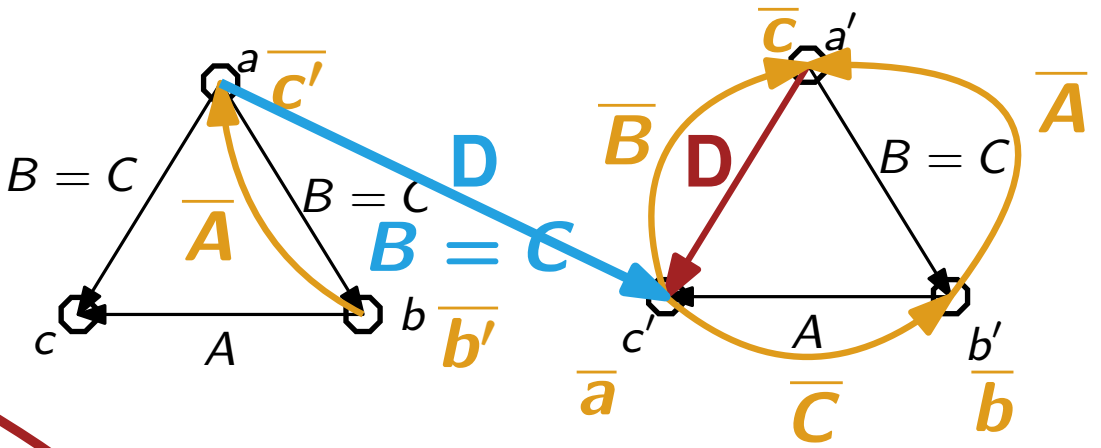
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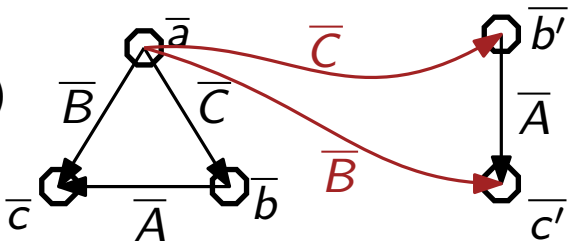
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part i)



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$$A \in \mathcal{I}(G) \implies A = A^{-1} \text{ or } \left[\begin{array}{l} A \cap A^{-1} = \varnothing \textbf{ and} \\ A, A^{-1} \text{ transitive} \end{array} \right]$$

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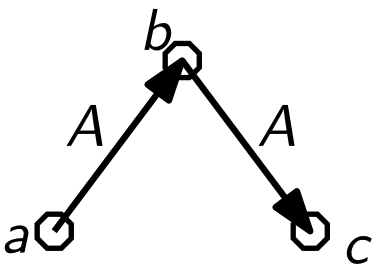
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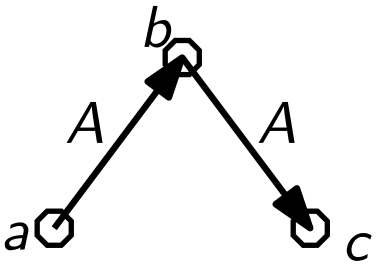
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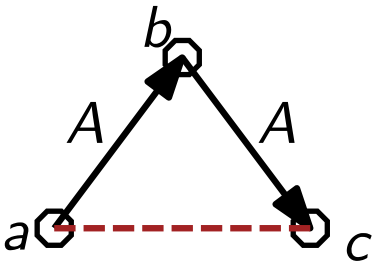
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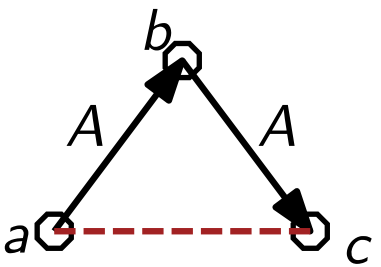
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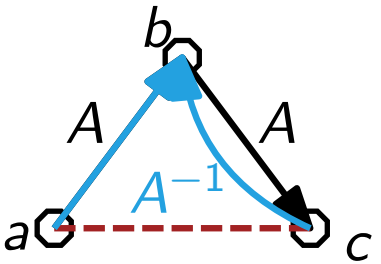
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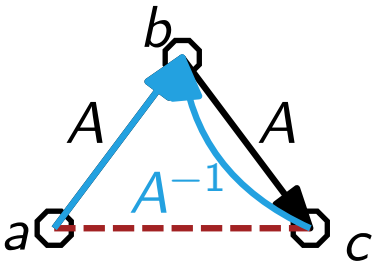
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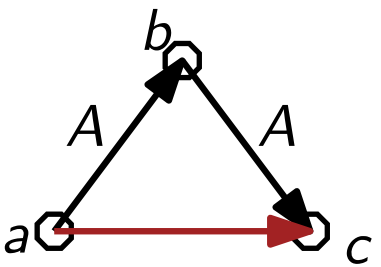
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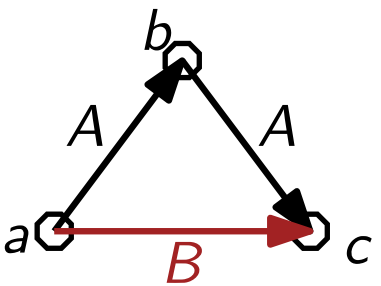
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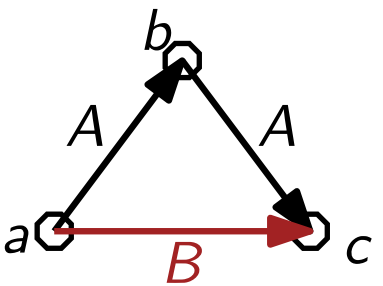
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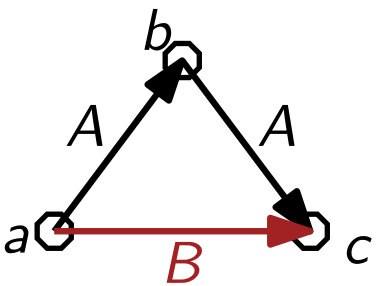
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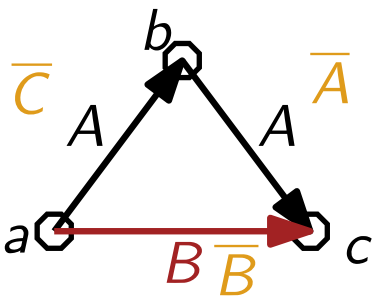
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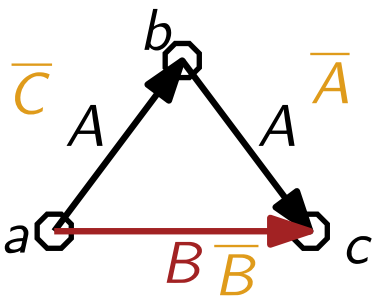
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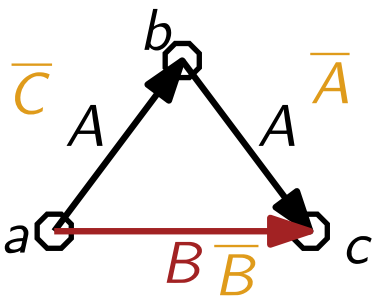
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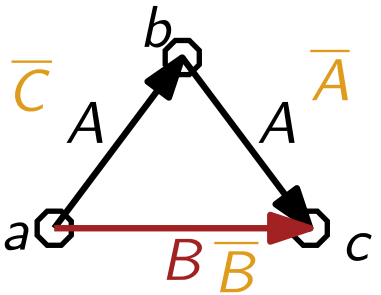
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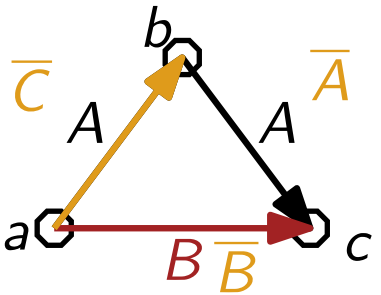
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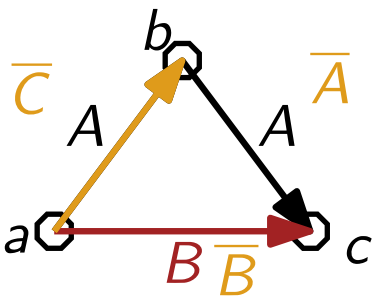
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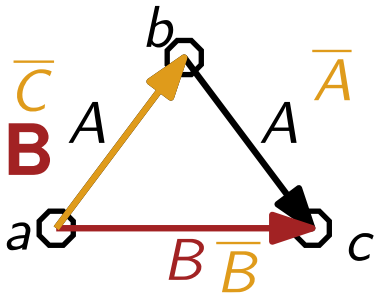
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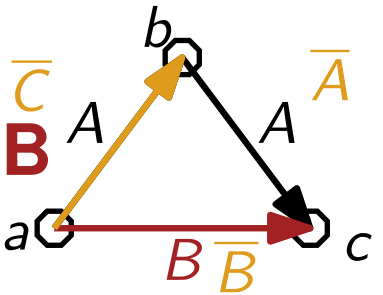
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use $ab = b'c'$ since $\overline{C} = A = \overline{A}$ to apply part i) of triangle lemma

obtain $ac' = ab \in \overline{B} = B$

so $B = A$

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previous gamma relations remain but new ones can alsobe created
implication classes can merge
$\implies$ $D \in \mathcal{I}(G)$ then $D =$ disjoint union of some previous implication classes

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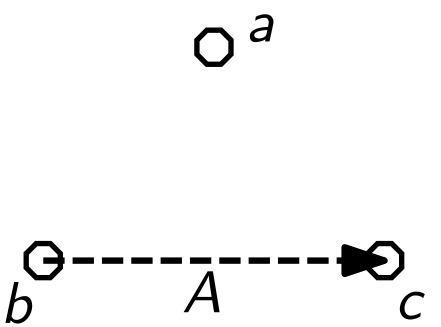
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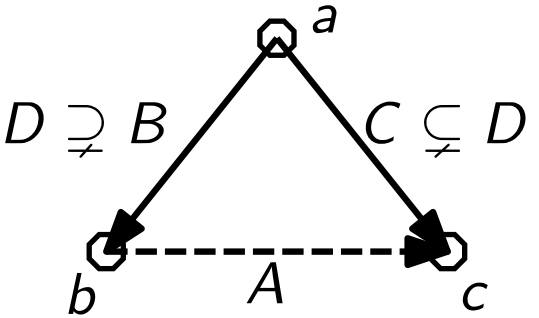
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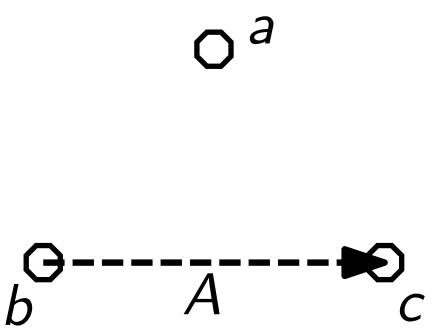
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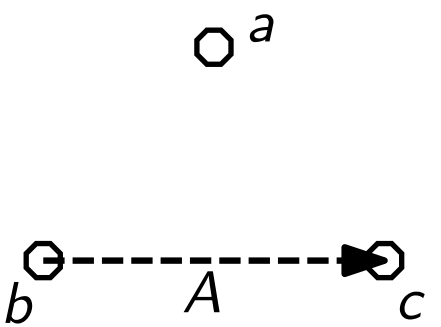
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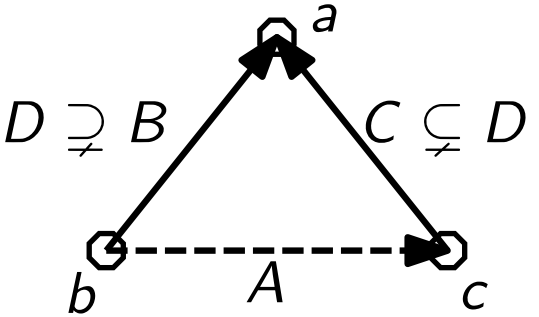
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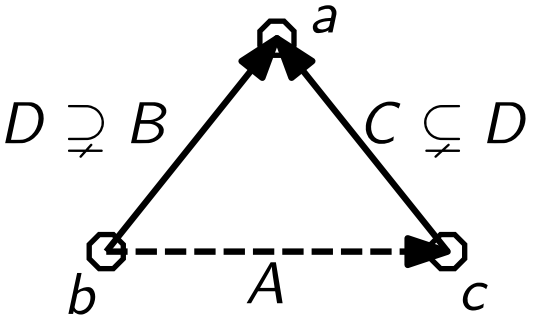
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wlog only first case (otherwise consider D^{-1})



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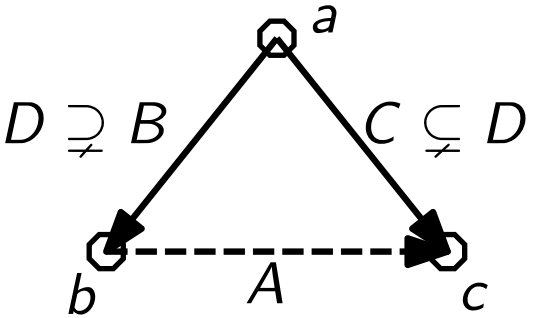
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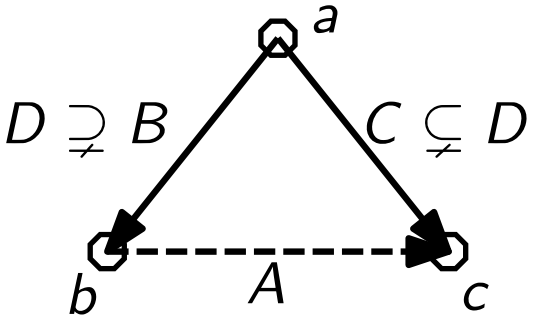
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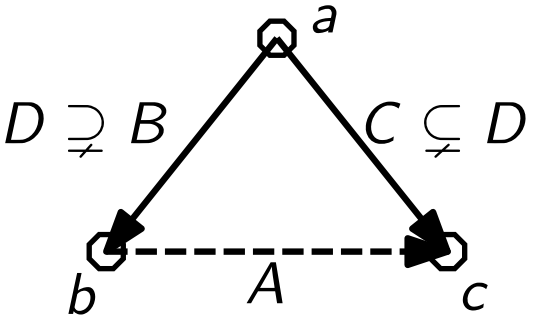
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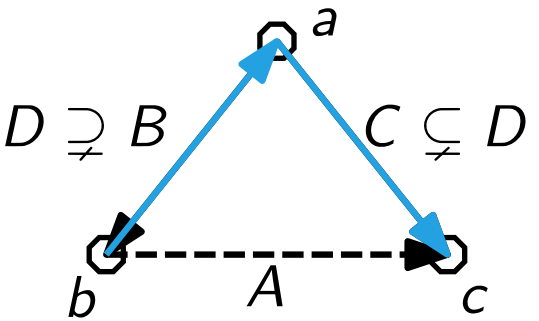
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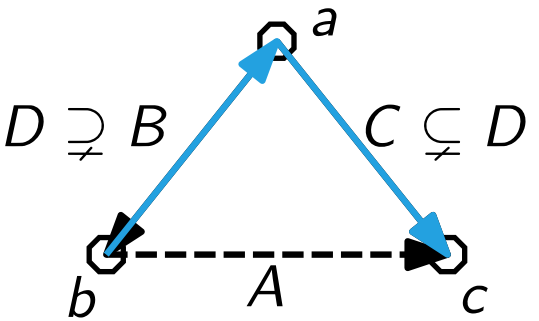
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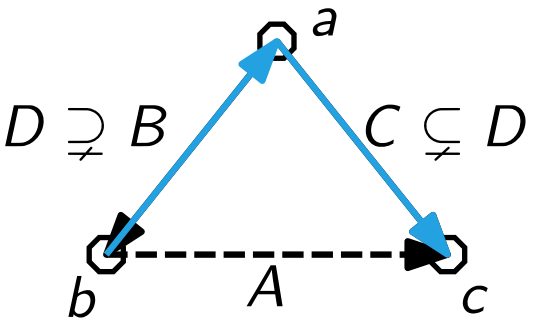
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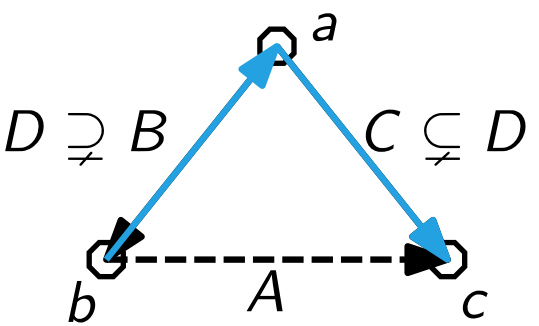
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by Thm 4.4 B is either transitive or $B = B^{-1}$

since $ac \in A \neq B$ we get $B = B^{-1} = C$ $\lightning$ B, C disjoint

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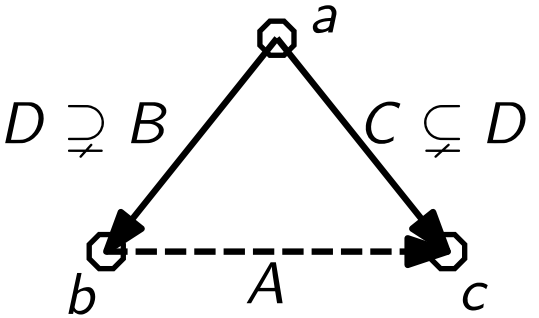
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so $B \neq C^{-1}$ and $\hat{B} \neq C$. Since $\hat{A} \neq \hat{B}, \hat{C}$ we have rainbow triangle



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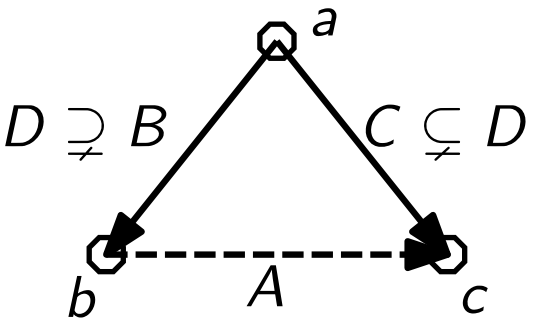
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Let X be a different term in the disjoint union of D



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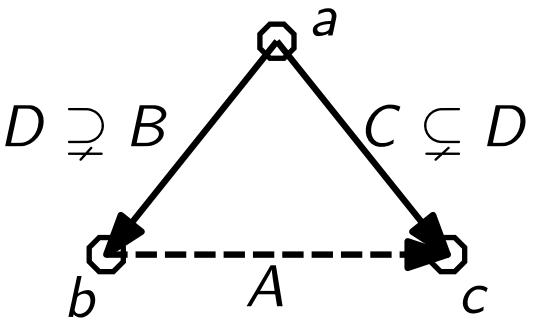
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B must have also merged with X (or is equal to X)



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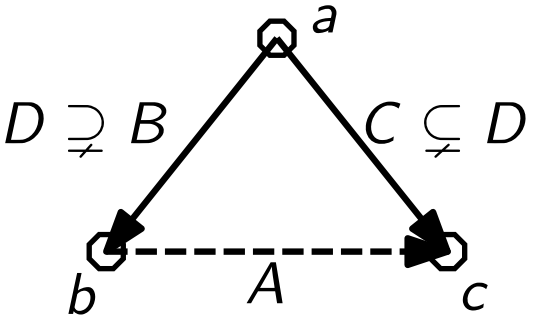
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Let X be a different term in the disjoint union of D

B must have also merged with X (or is equal to X)

like before triangle $a'b'c'$



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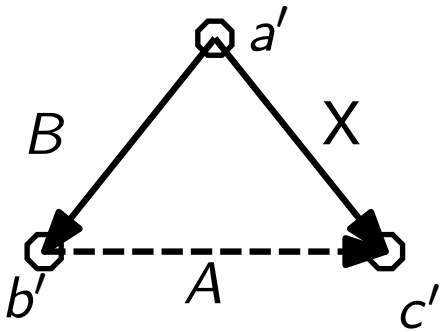
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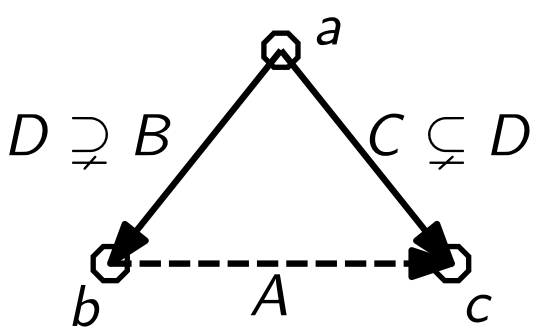
case 1: $D = B + C + \ldots$ with $B, C, \ldots \in \mathcal{I}(G)$

Let X be a different term in the disjoint union of D

B must have also merged with X (or is equal to X)

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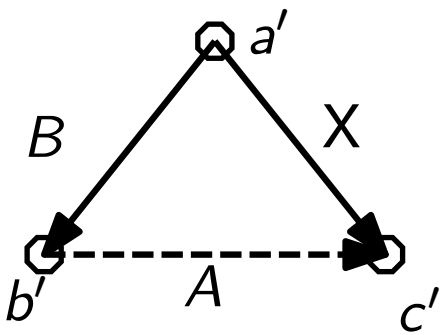
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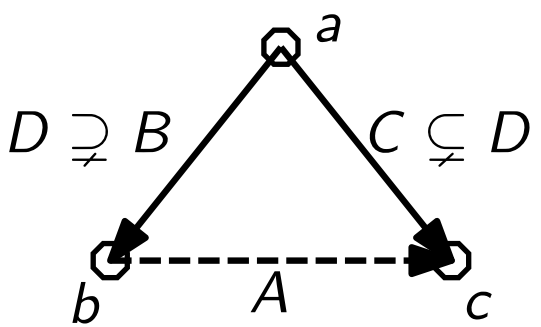
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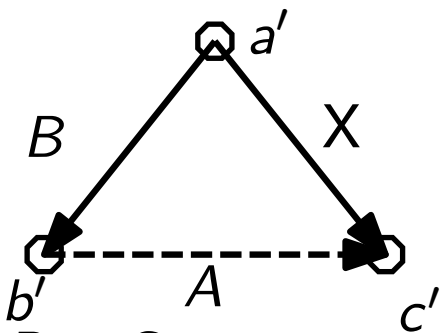
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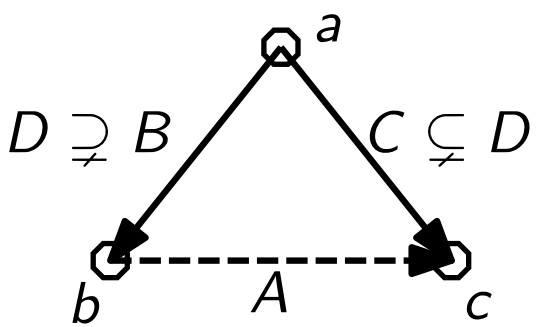
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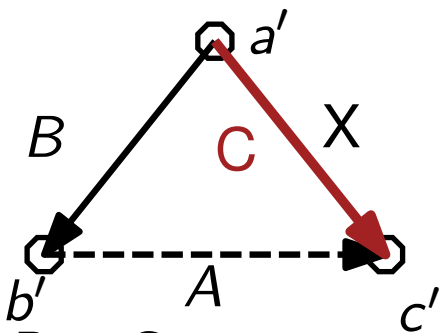
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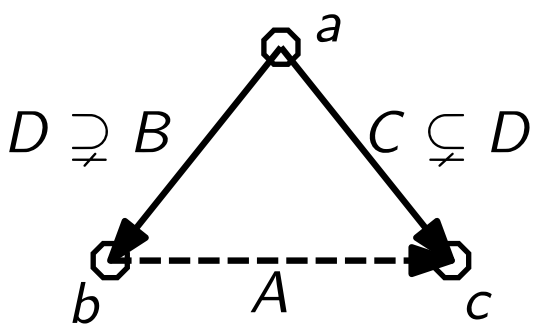
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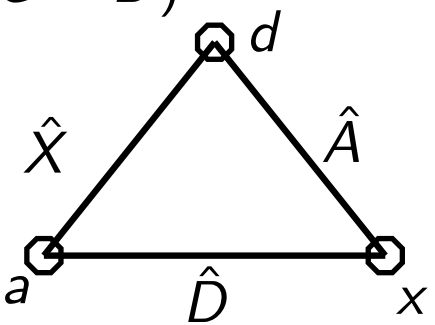
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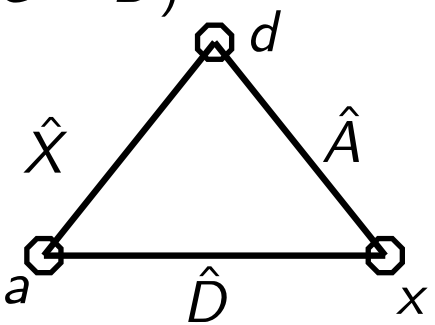
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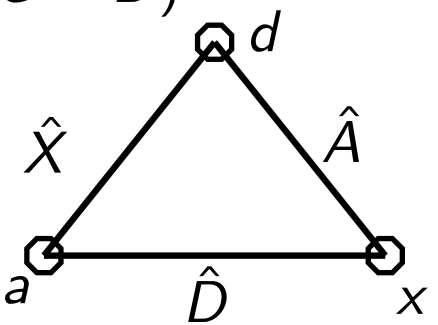
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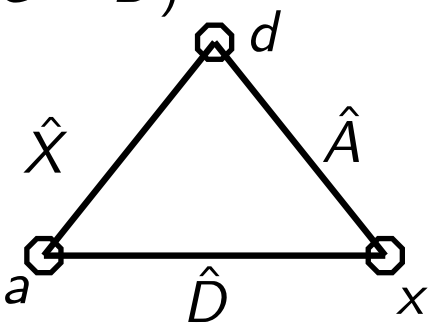
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For every graph G, the following are equivalent:

- G is a comparability graph
- $A \cap A^{-1} = \varnothing$ for all $A \in \mathcal{I}(G)$
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”i) $\Longrightarrow$ ii)” follows directly from corollary of Theorem 4.1

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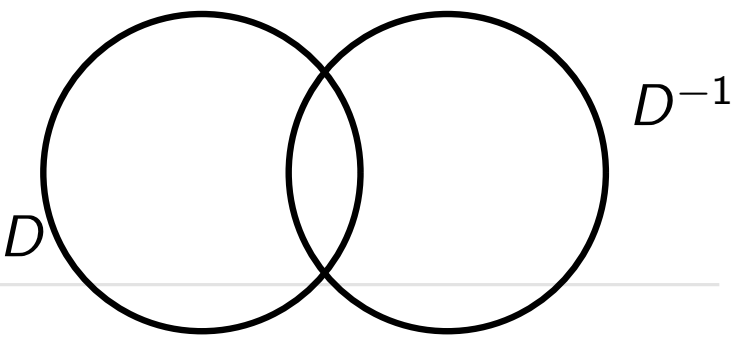
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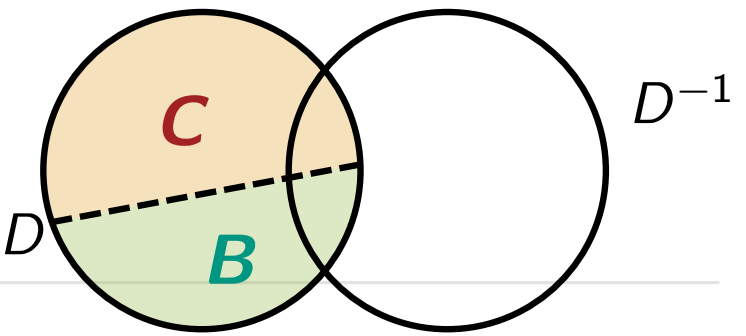
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to apply induction we have to verify $D \cap D^{-1} = \varnothing$ for all $D \in \mathcal{I}(G - \hat{B}_1)$

Thm 4.6: case $D = B + C$ for $B, C \in \mathcal{I}(G)$



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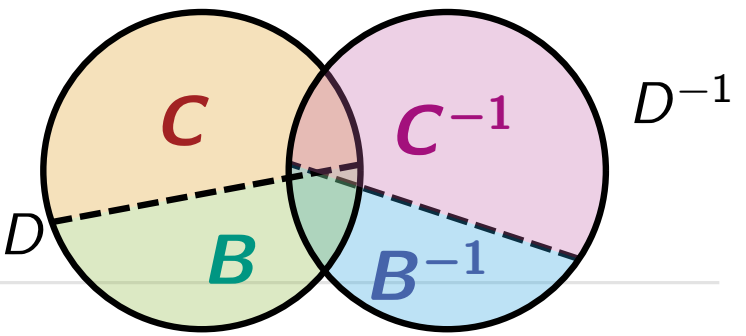
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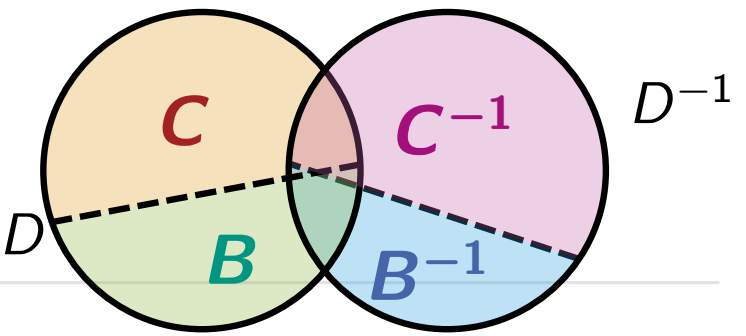
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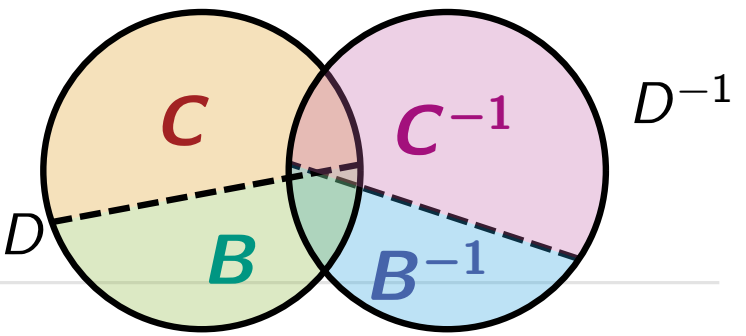
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$$D \cap D^{-1} = (B+C) \cap (B+C)^{-1} = (B+C) \cap (B^{-1}+C^{-1})$$



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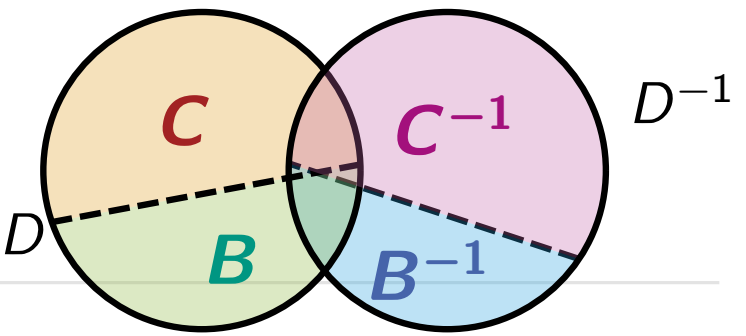
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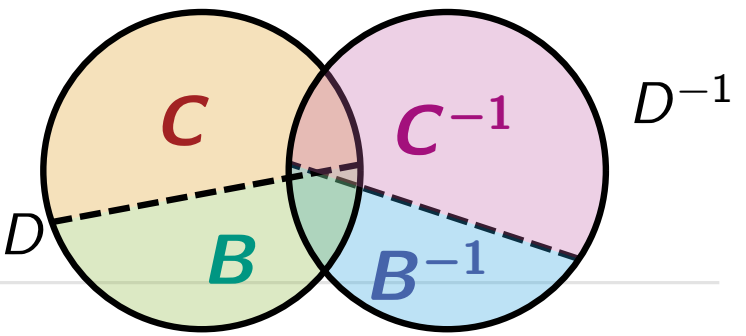
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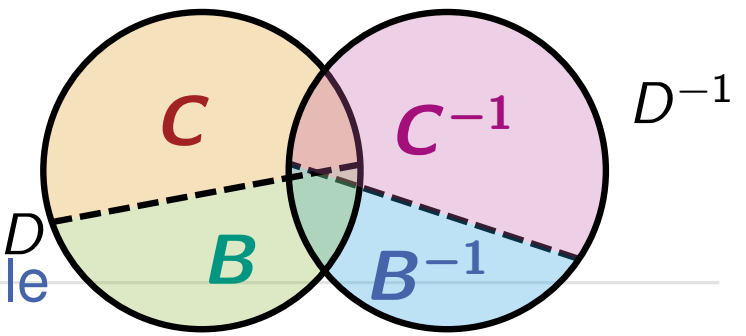
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$$D \cap D^{-1} = (B+C) \cap (B+C)^{-1} = (B+C) \cap (B^{-1}+C^{-1})$$
$$= (B \cap B^{-1}) \cup (B \cap C^{-1}) \cup (C \cap B^{-1}) \cup (C \cap C^{-1})$$

by ii) $B \neq C^{-1}$ and $C \neq B^{-1}$ since $\hat{A}, \hat{B}, \hat{C}$ rainbowtriangle



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ii) holds for $G - \hat{B}_1$

by induction $B_i \cap B_i^{-1} = \varnothing$ for all $i \geq 2$

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"iii) $\Longrightarrow$ i)"

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"iii) $\Longrightarrow$ i)" $[B_1, ..., B_k]$ G-decomposition. Induction on k:

$k = 1$: $B_1 \cap B_1^{-1} = \varnothing$

Thm 4.4 says B_1 is transitive orientation (since $B_1 = G$)

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fullfills iii) for $G - B_1$ since all $G - \hat{B}_1$ decompositions $[C_1, ..., C_{k'}]$ can be made to G-decomposition $[B_1, C_1, ..., C_{k'}]$ so by iii) $C_i \cap C_i = \varnothing$

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by induction $G - \hat{B}_1$ is a comp. graph with trans. or. $T = B_2 + ... + B_k$

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orientation: B_1, B_1^{-1} disjoint and T, B_1 disjoint

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orientation: B_1, B_1^{-1} disjoint and T, B_1 disjoint

so $xy \in T + B_1 \iff yx \notin T$ and $yx \notin B_1 \iff yx \notin T + B_1$

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transitive: let $ab, bc \in T + B_1$

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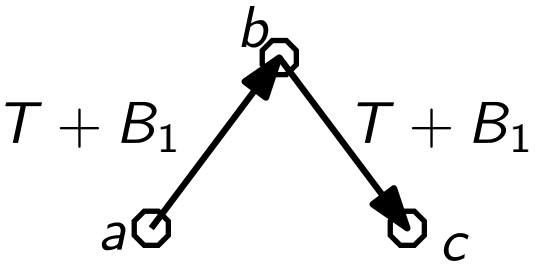
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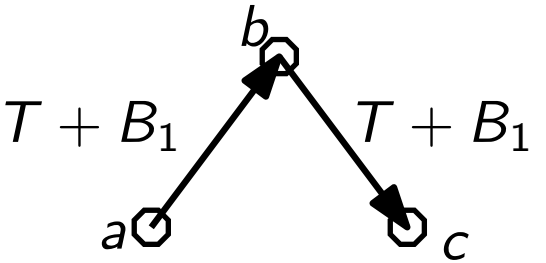
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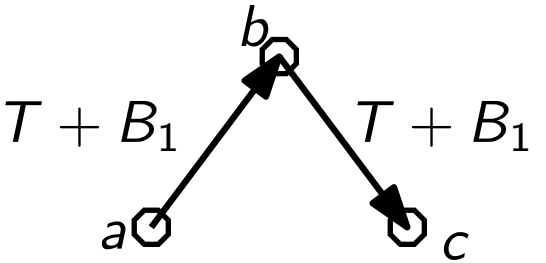
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transitive: let $ab, bc \in T + B_1$

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If both ab and bc are in T or in B_1 then $ac \in T + B_1$ by their respective transitivity



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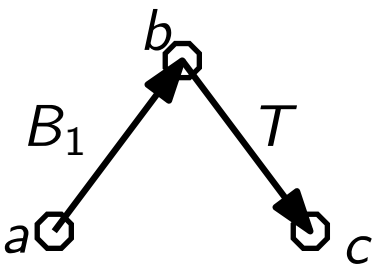
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transitive: let $ab \in B_1, bc \in T$ wlog



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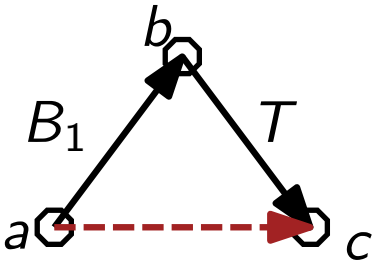
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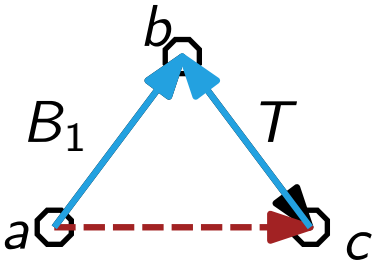
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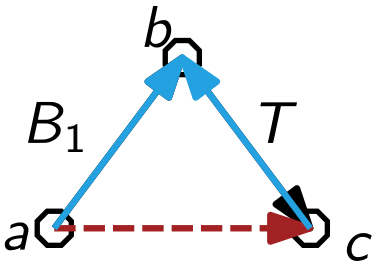
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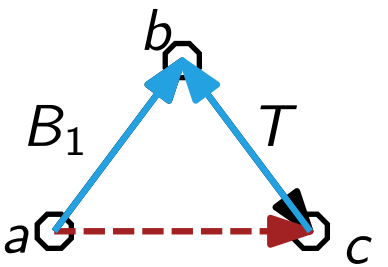
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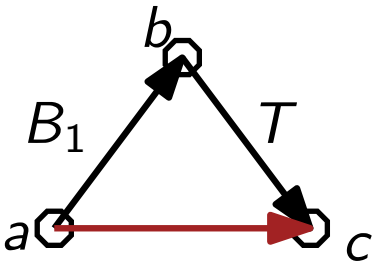
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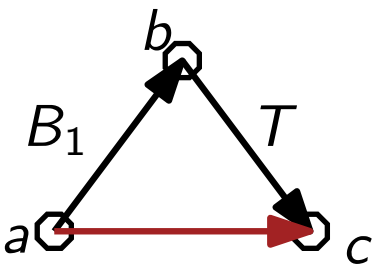
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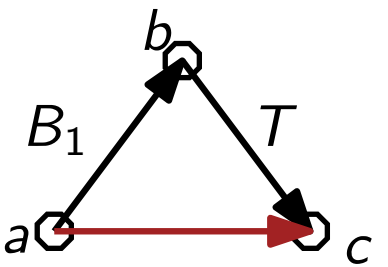
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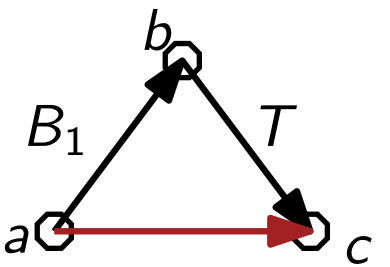
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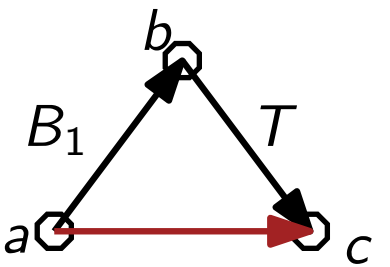
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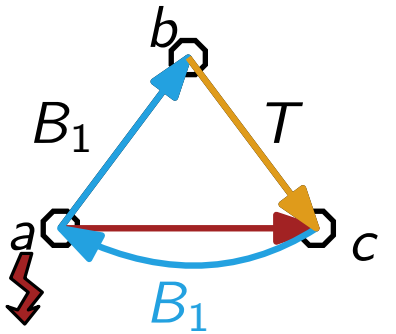
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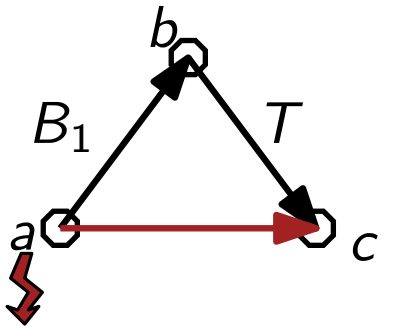
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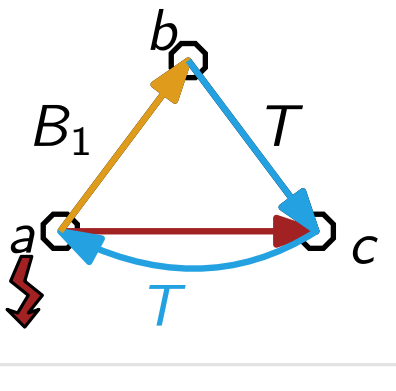
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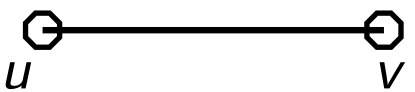
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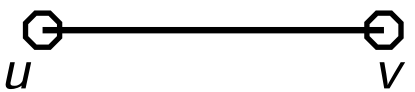
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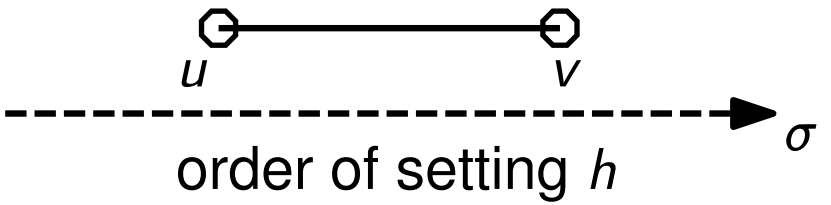
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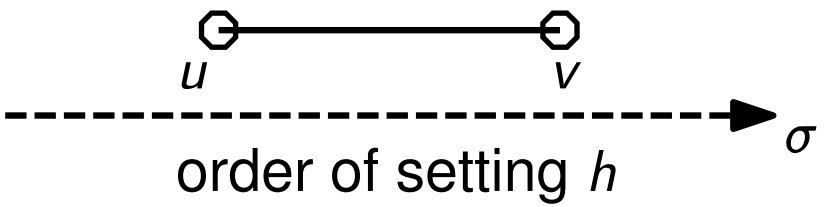
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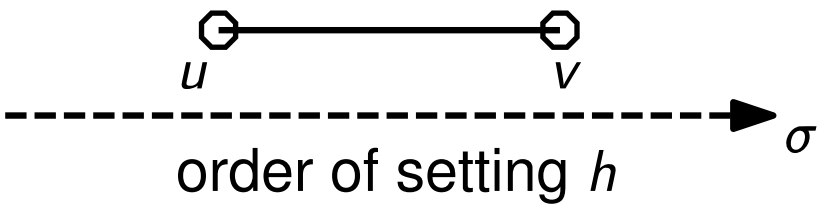
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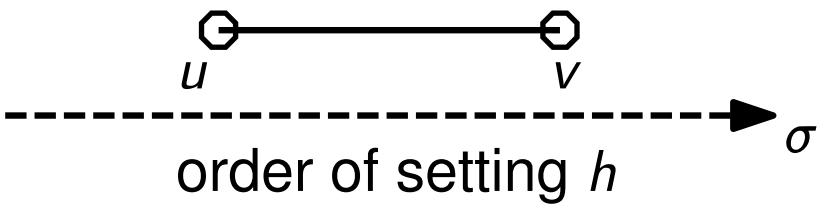
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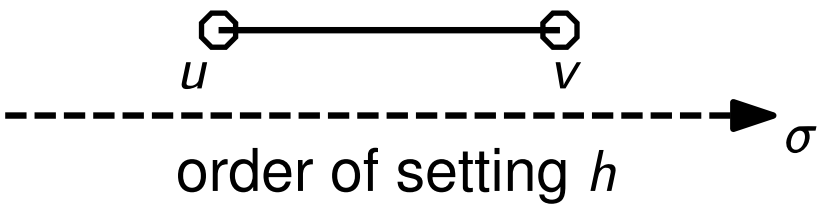
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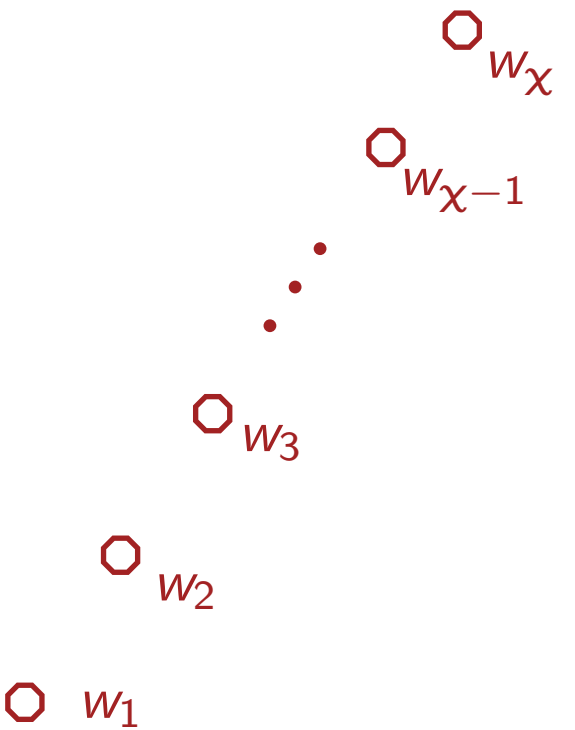
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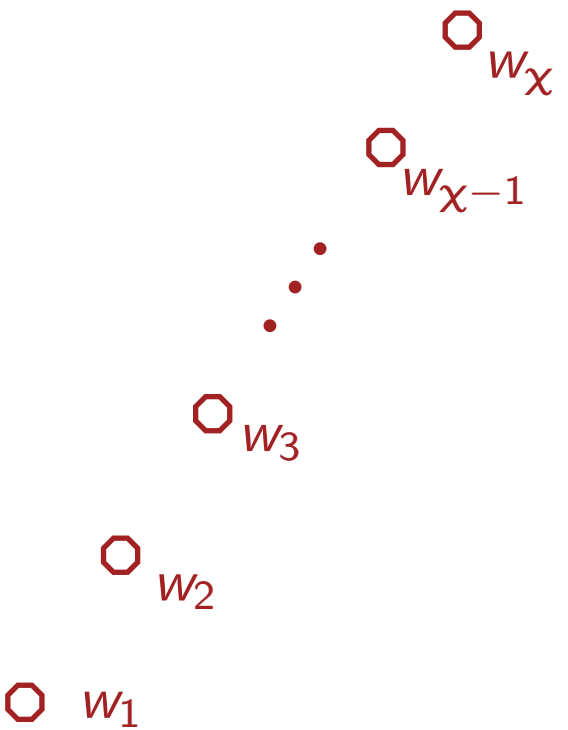
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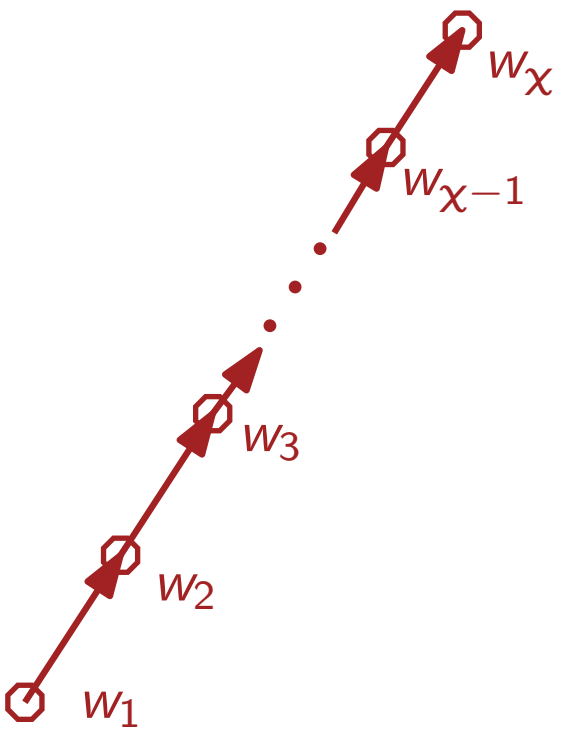
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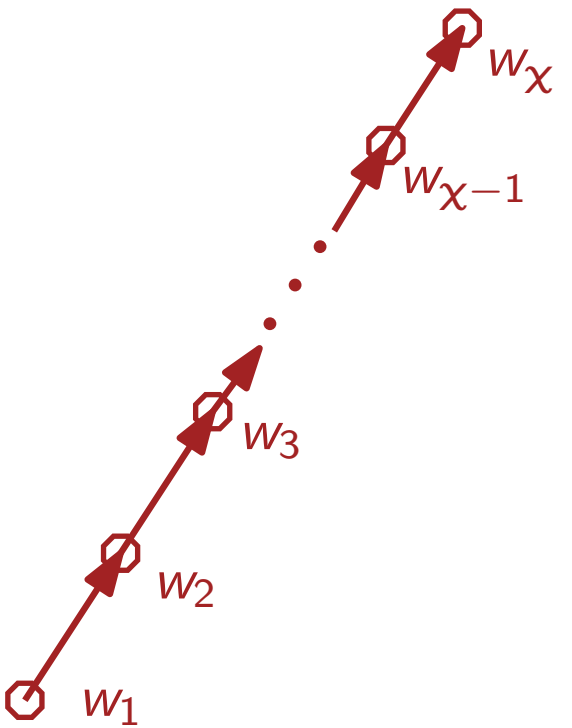
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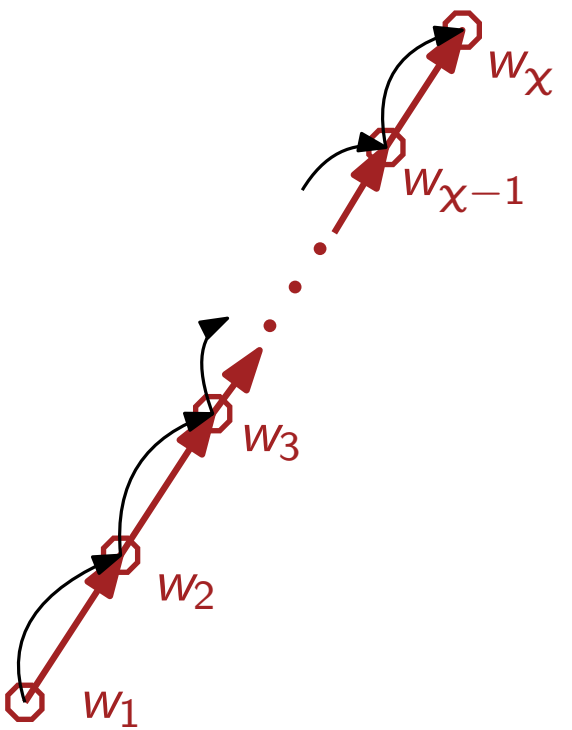
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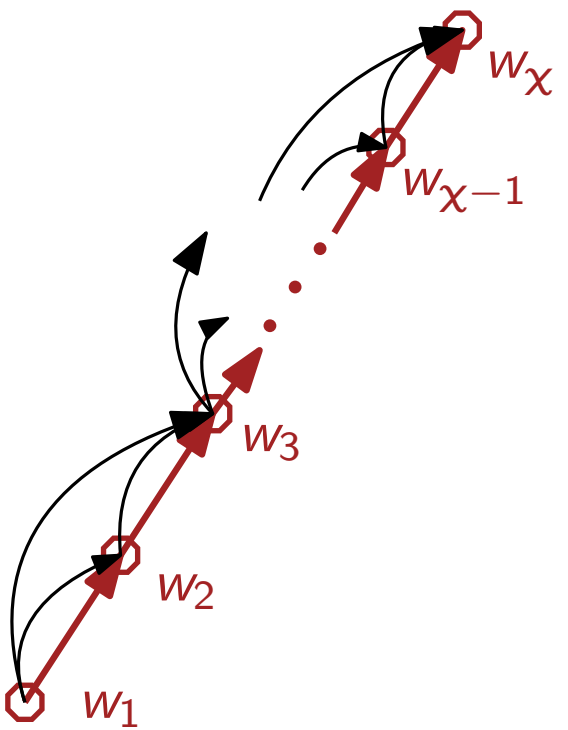
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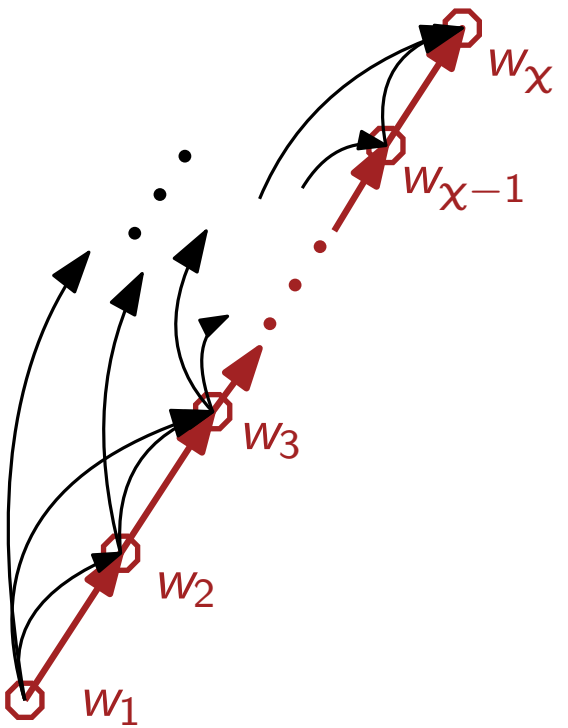
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Let $C = \{w_\chi, \ldots, w_1\}$

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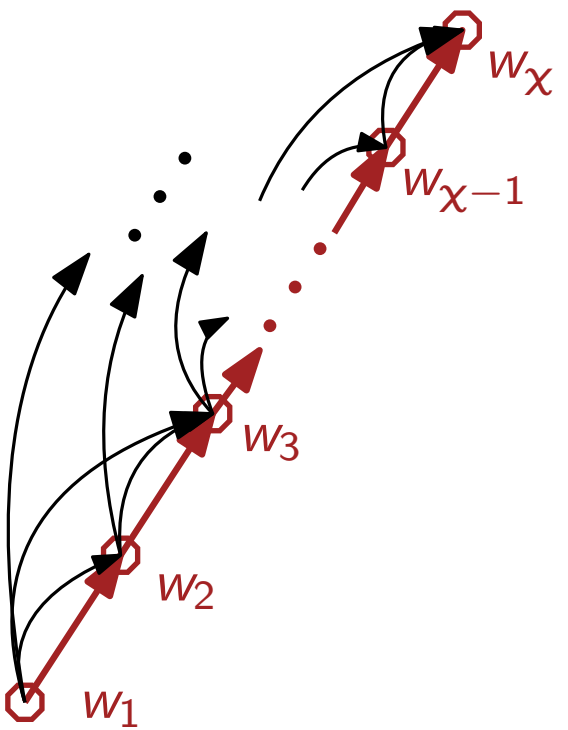
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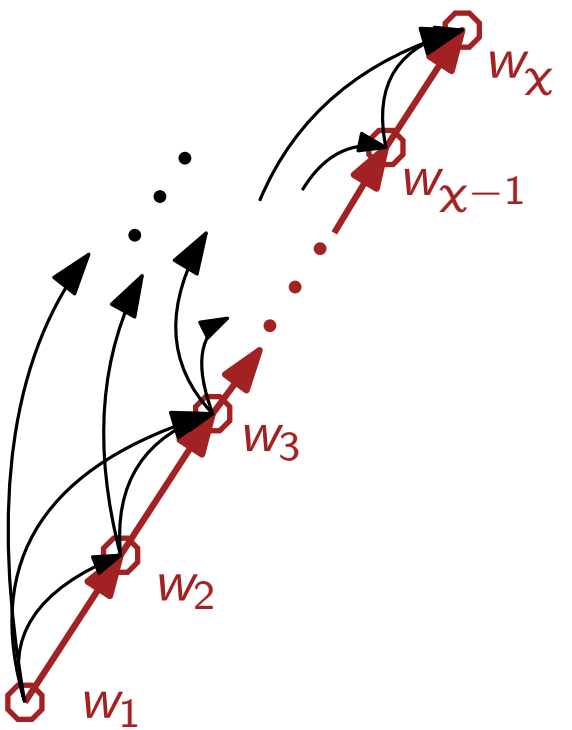
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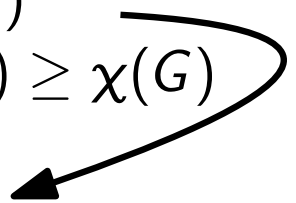
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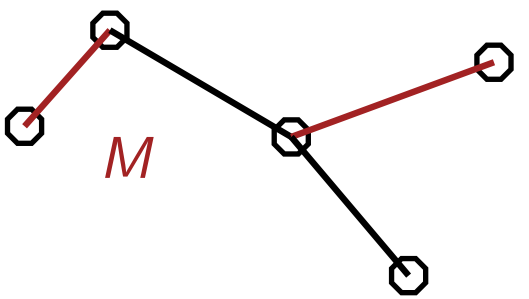
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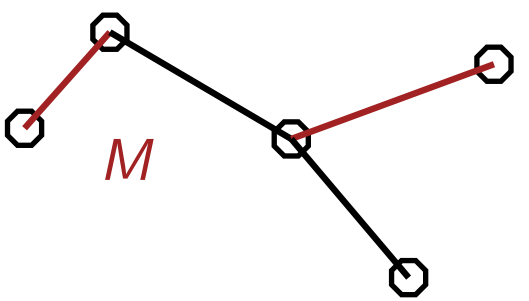
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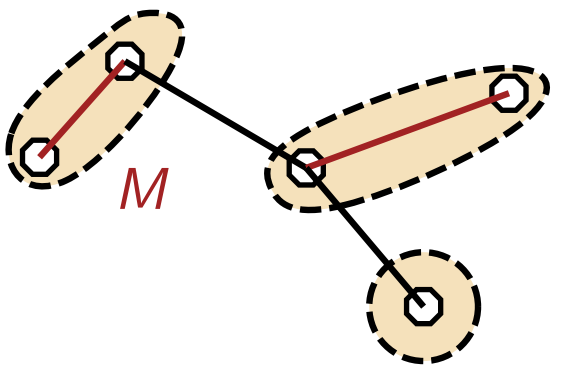
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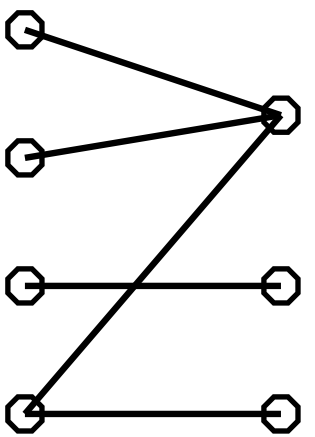
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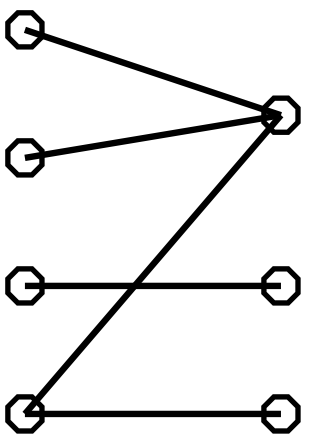
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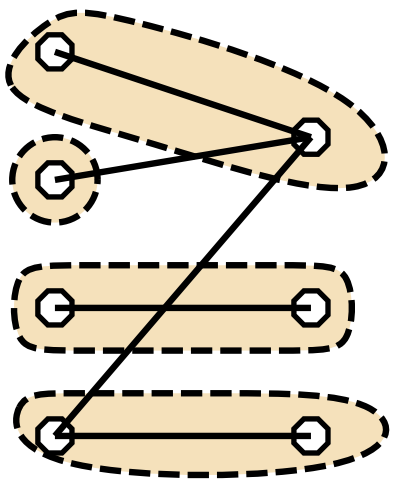
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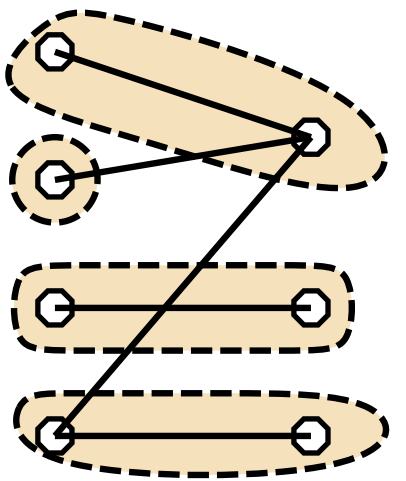
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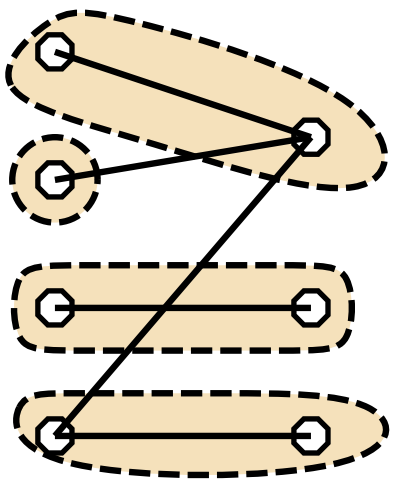
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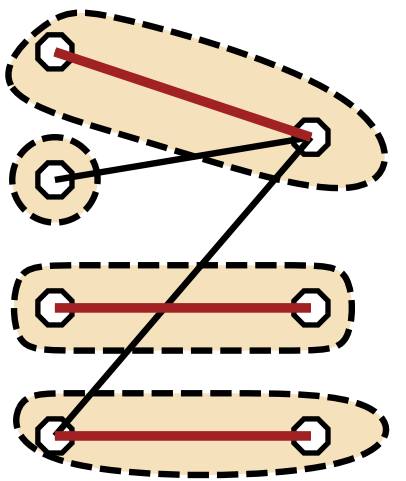
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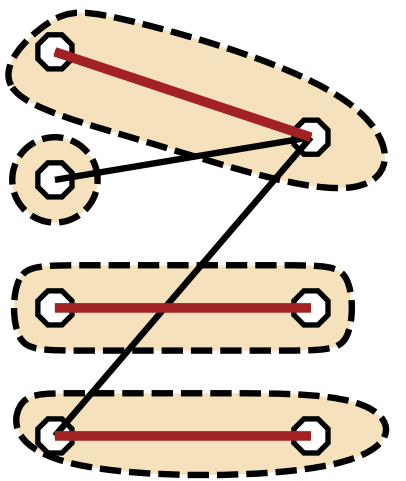
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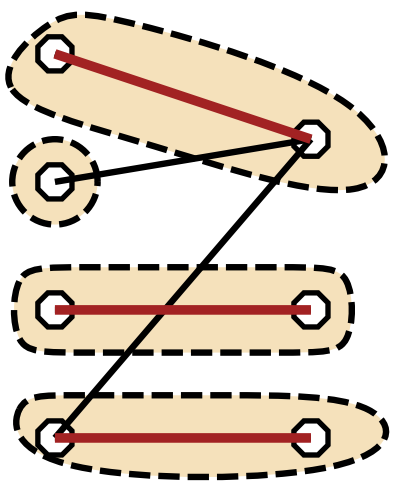
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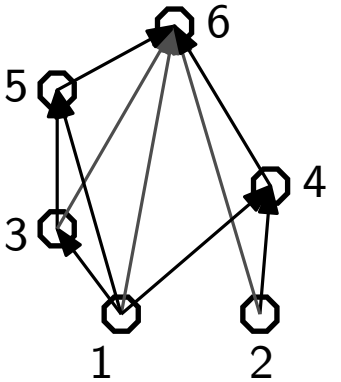
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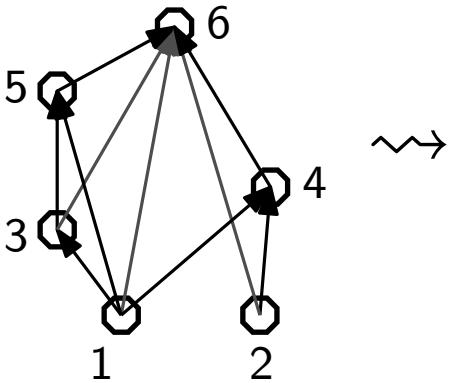
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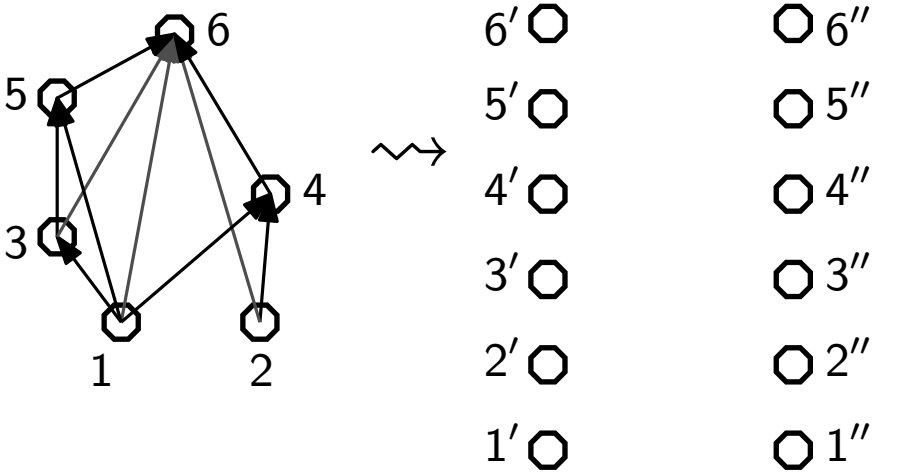
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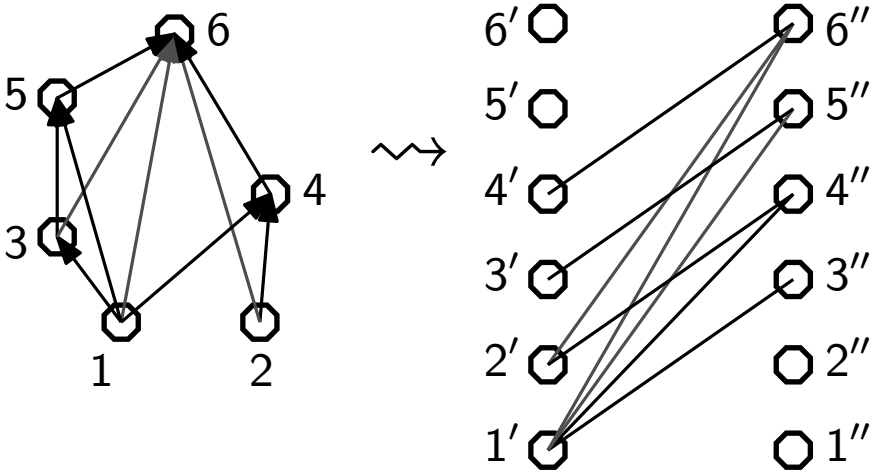
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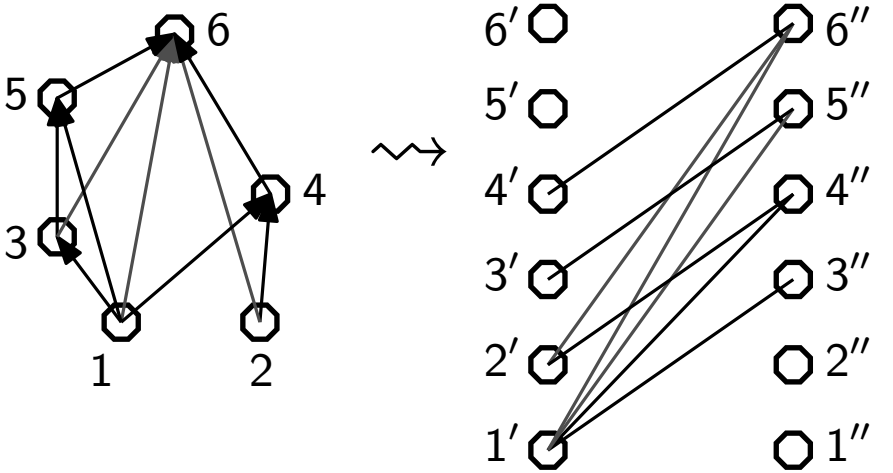
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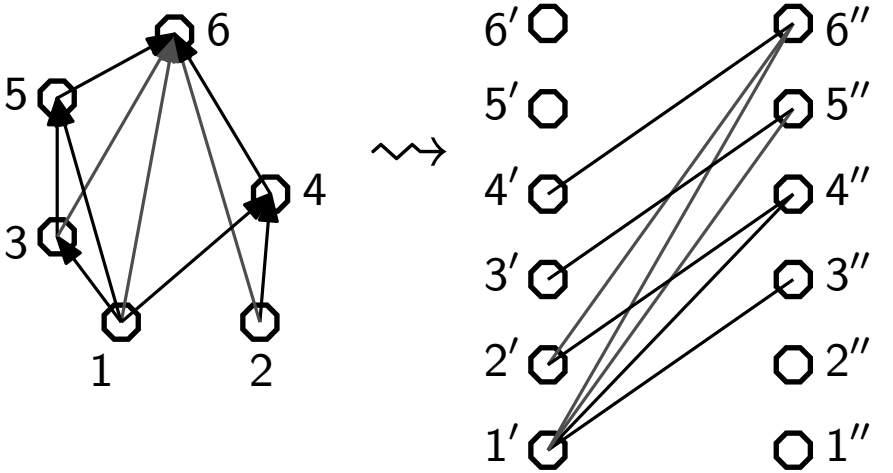
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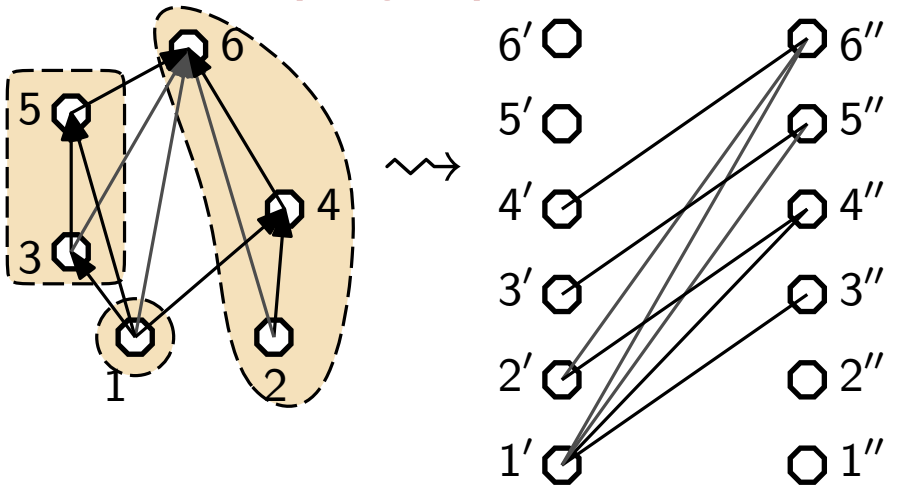
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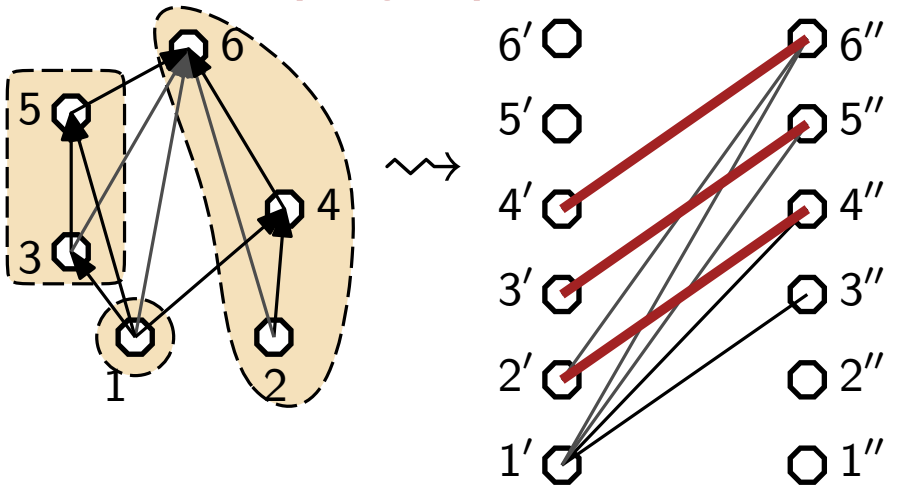
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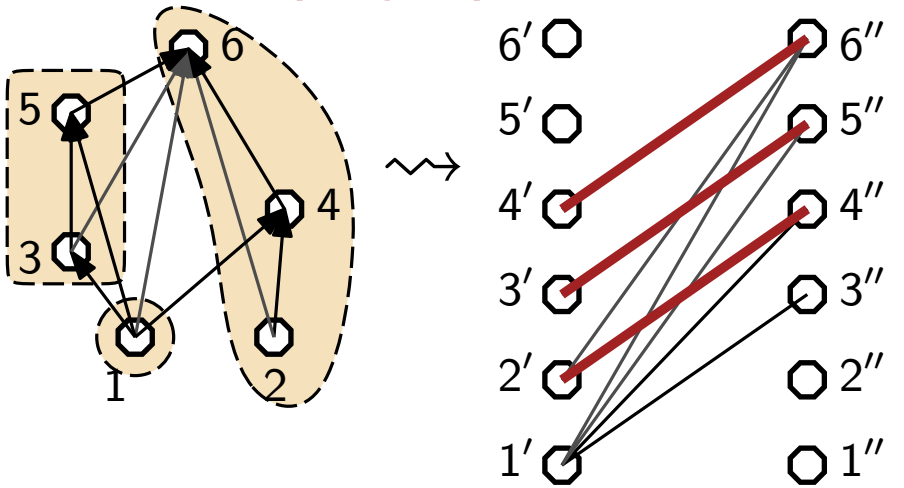
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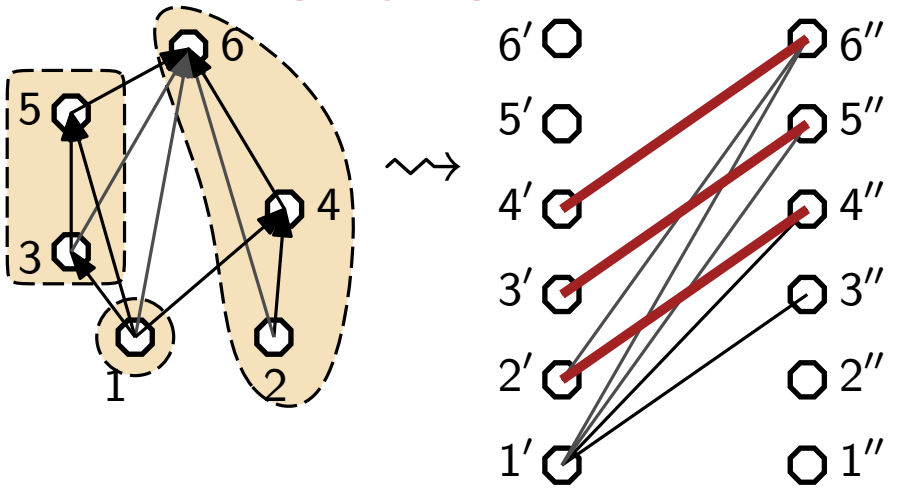
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Goal / Proof sketch of Algorithm 9

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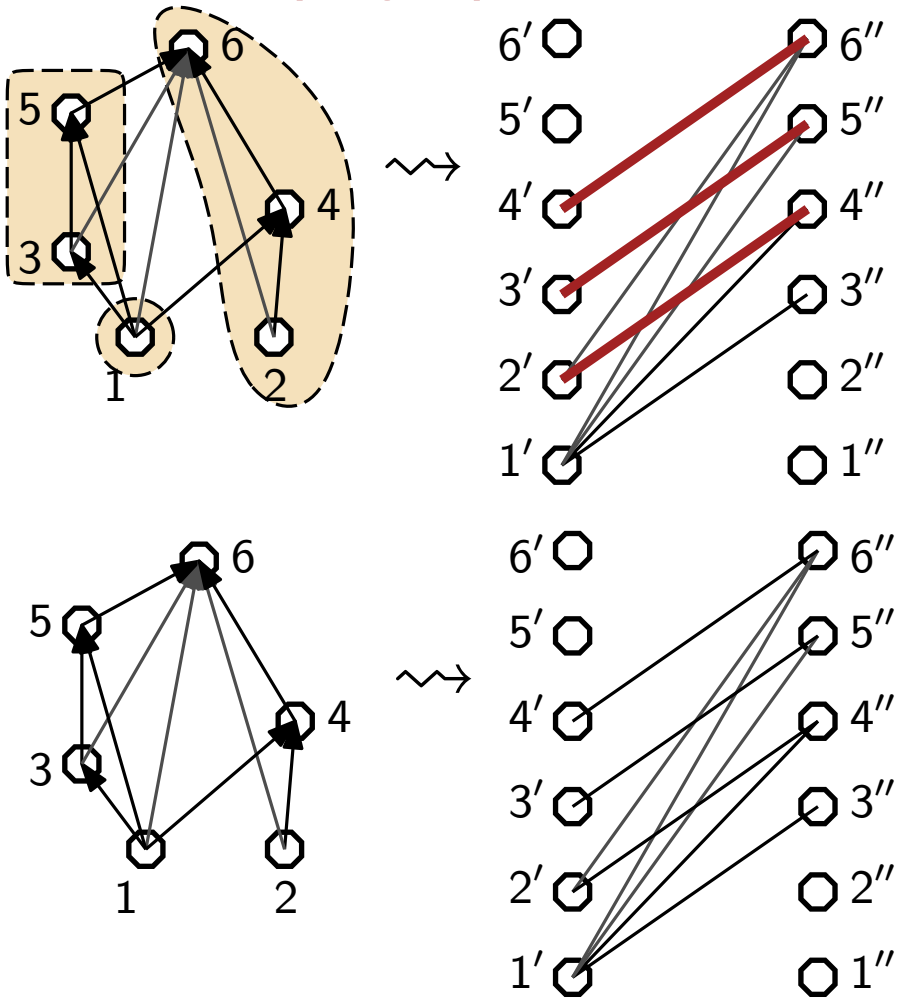
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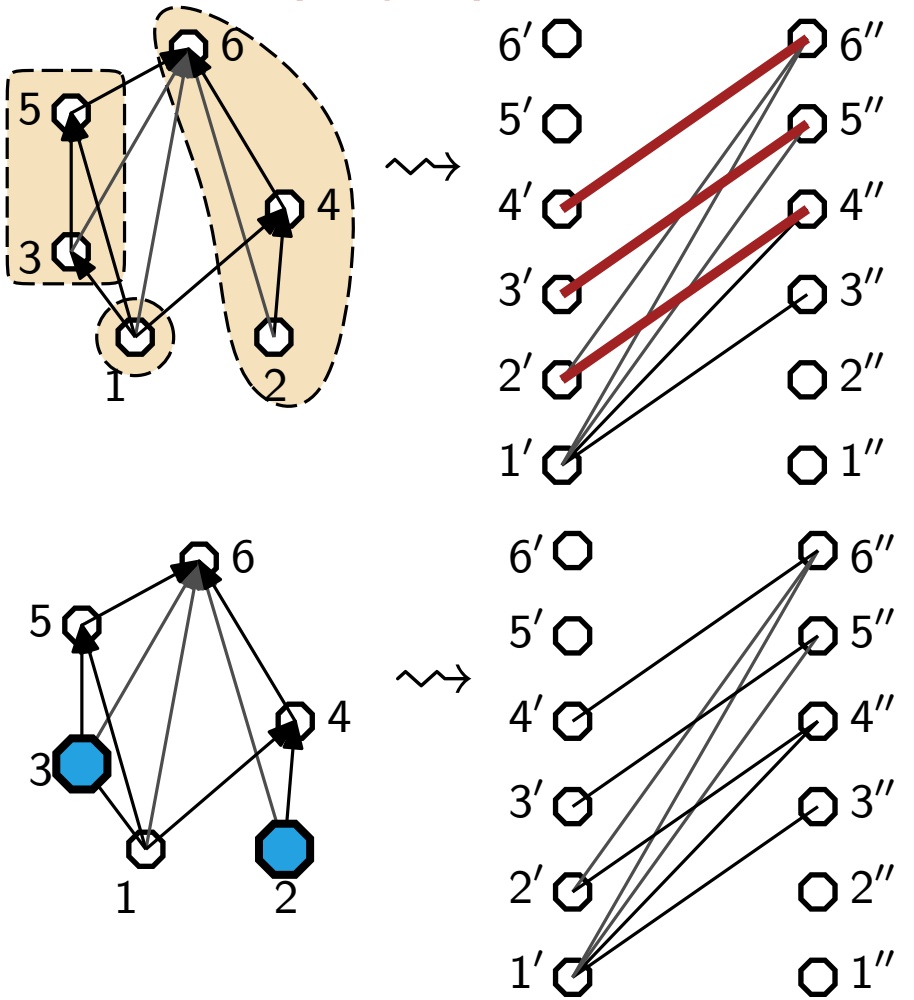
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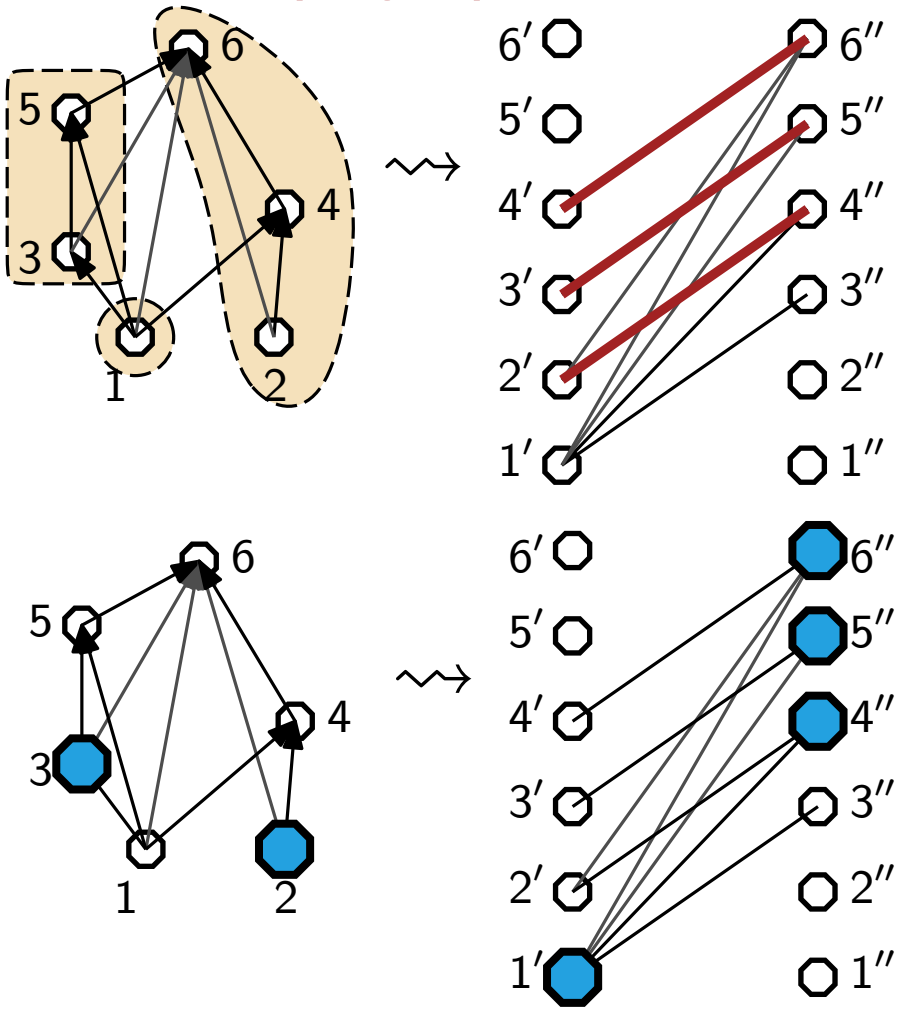
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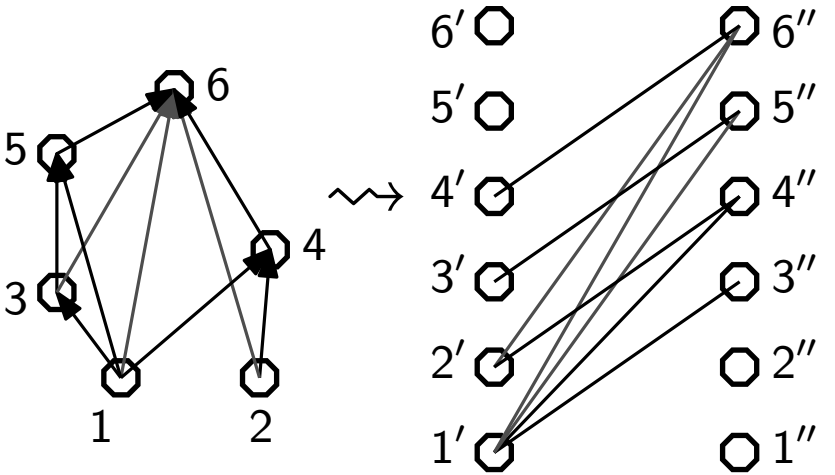
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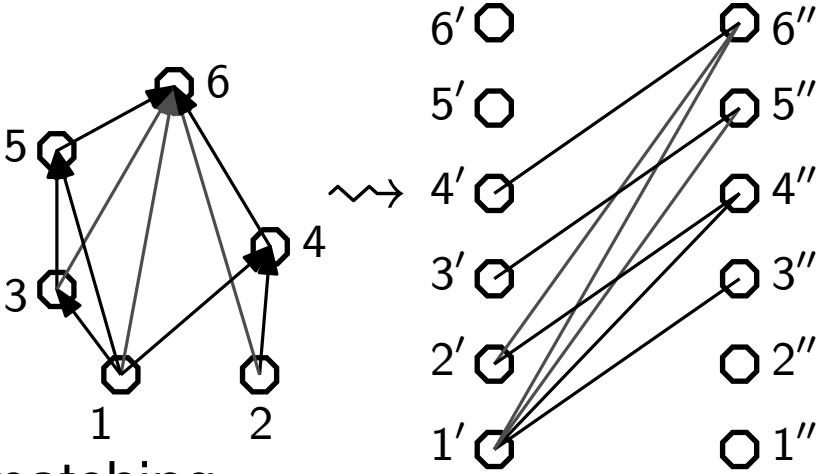
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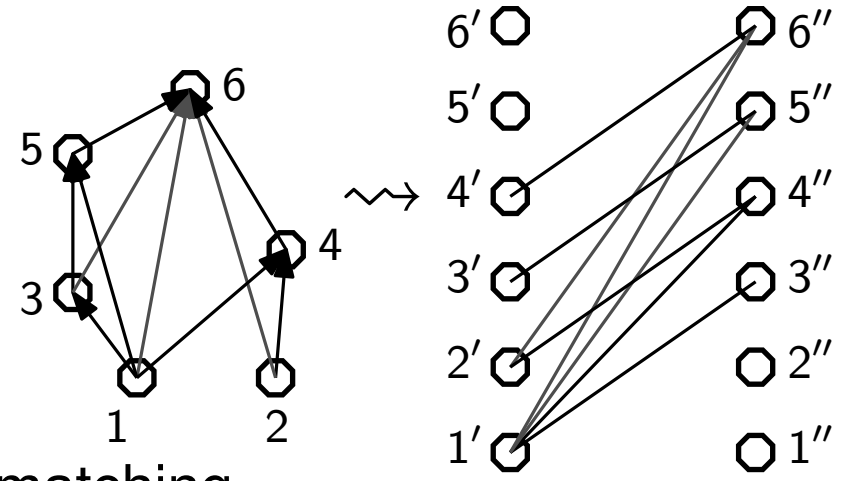
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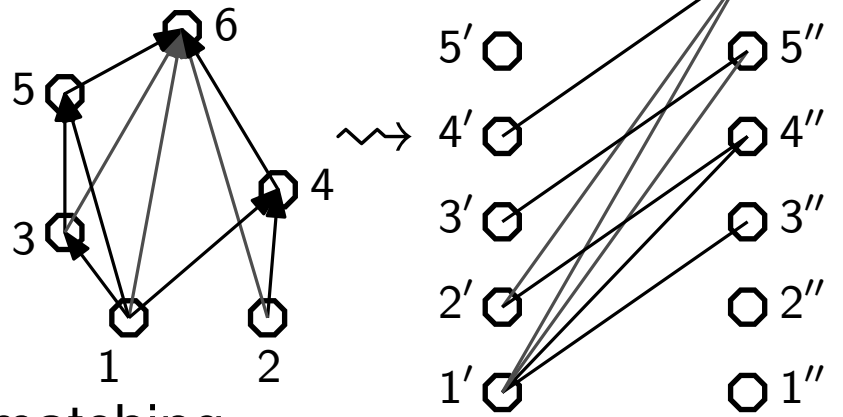
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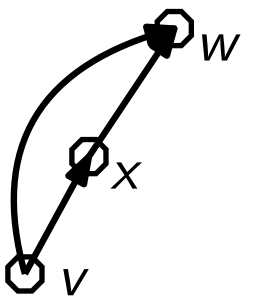
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v, w consecutive



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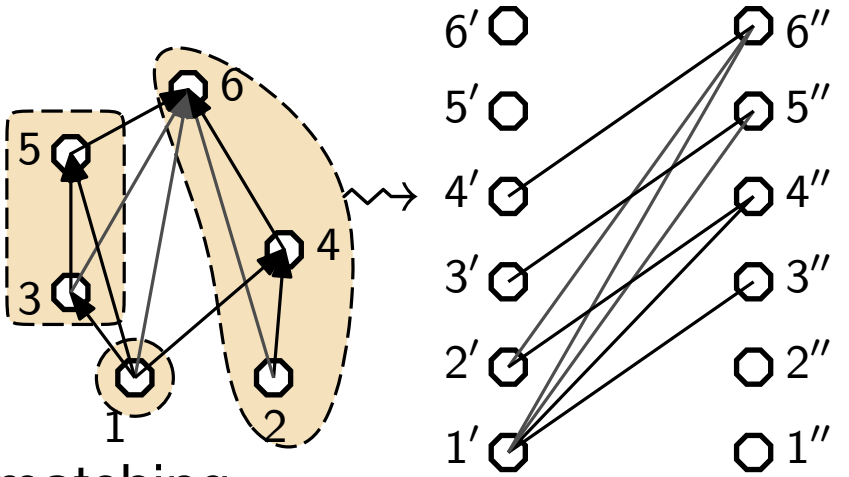
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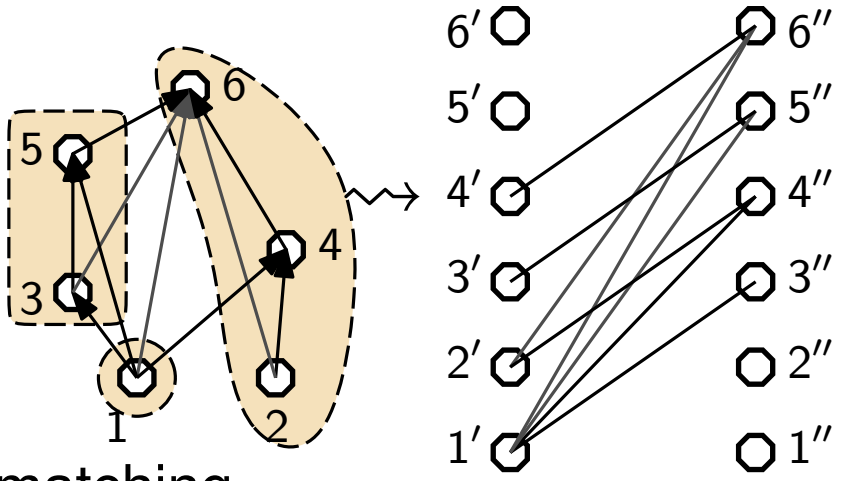
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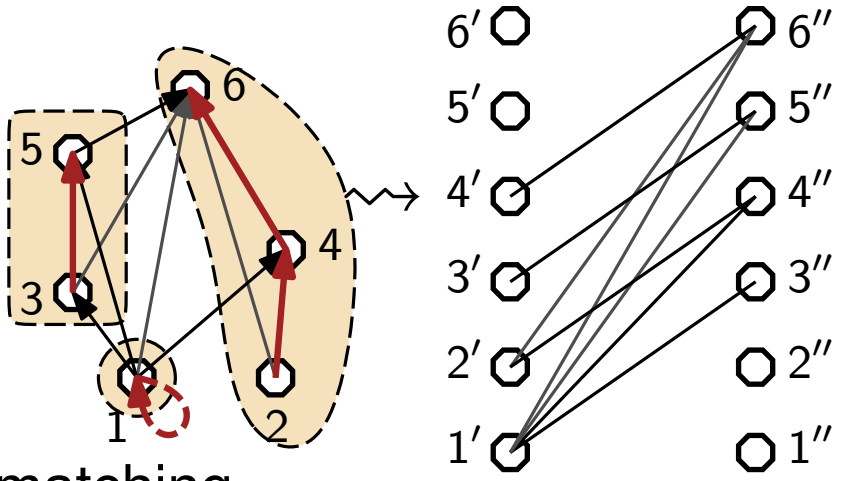
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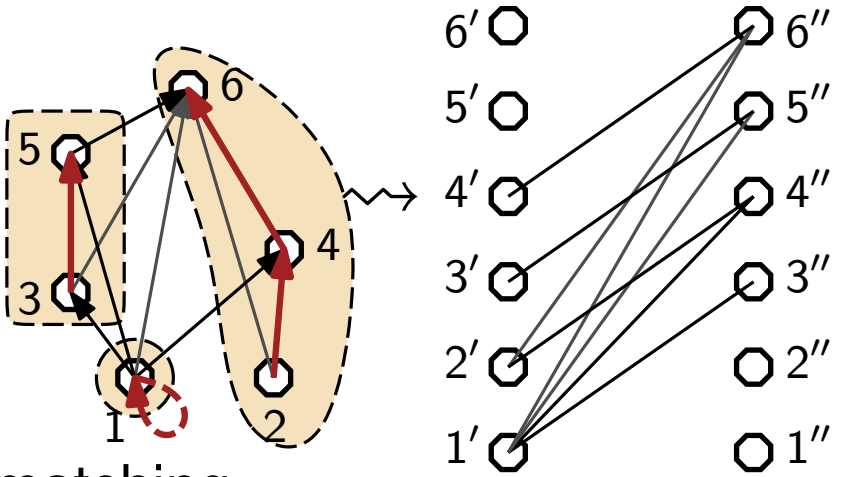
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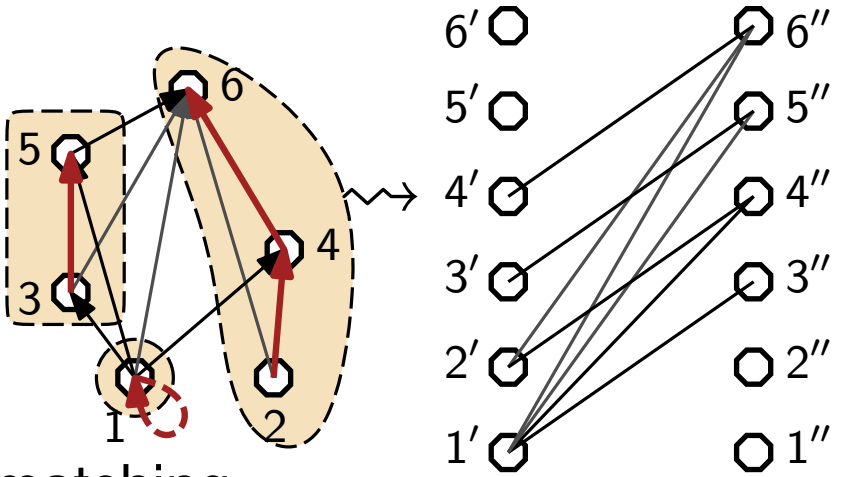
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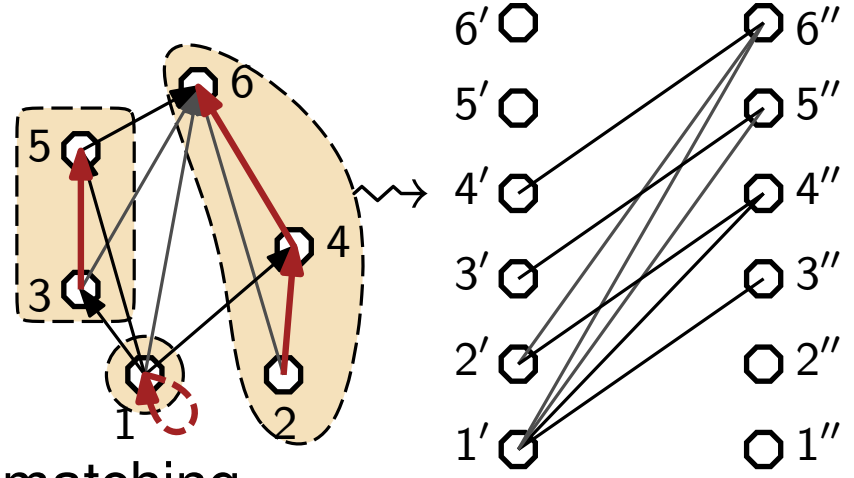
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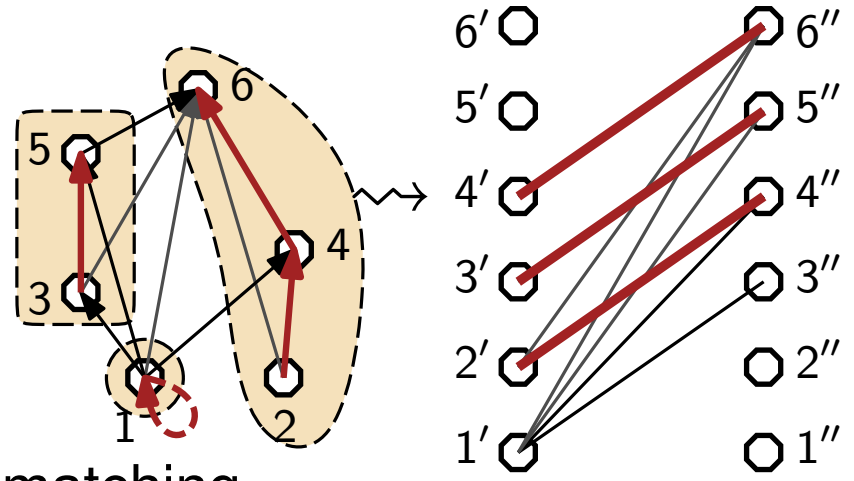
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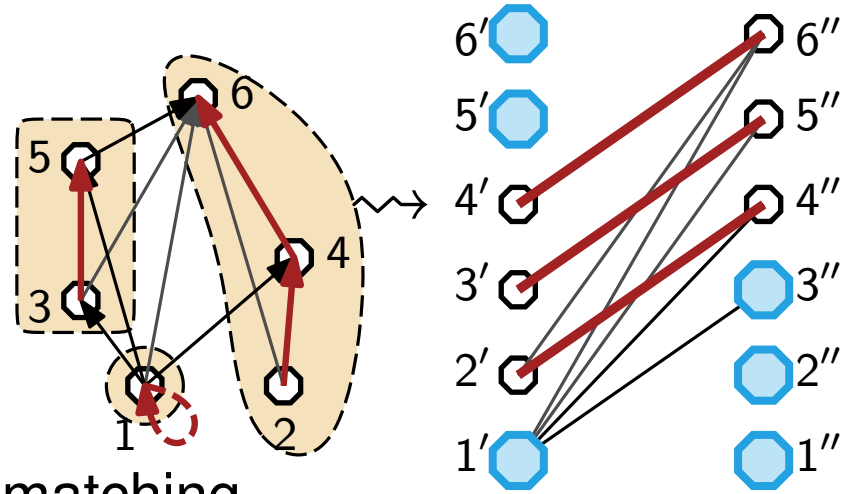
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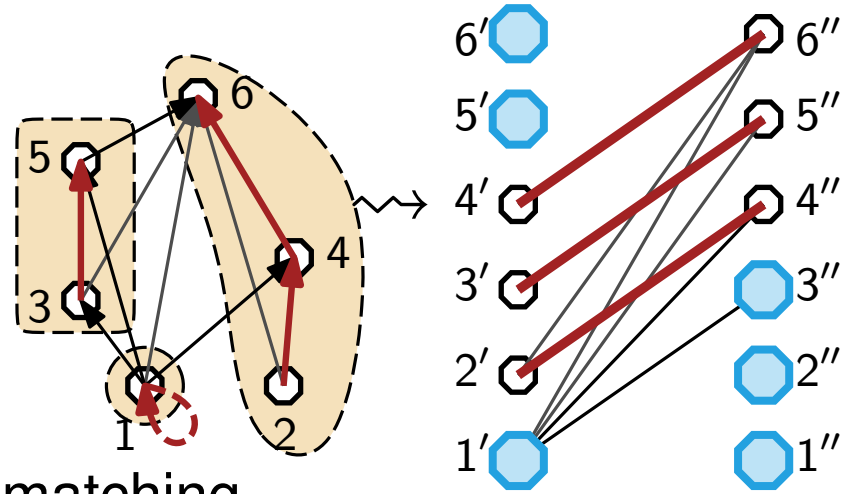
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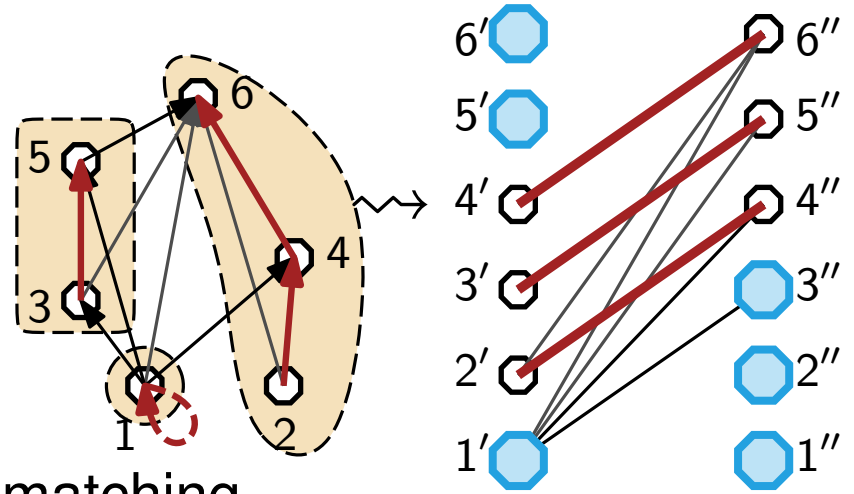
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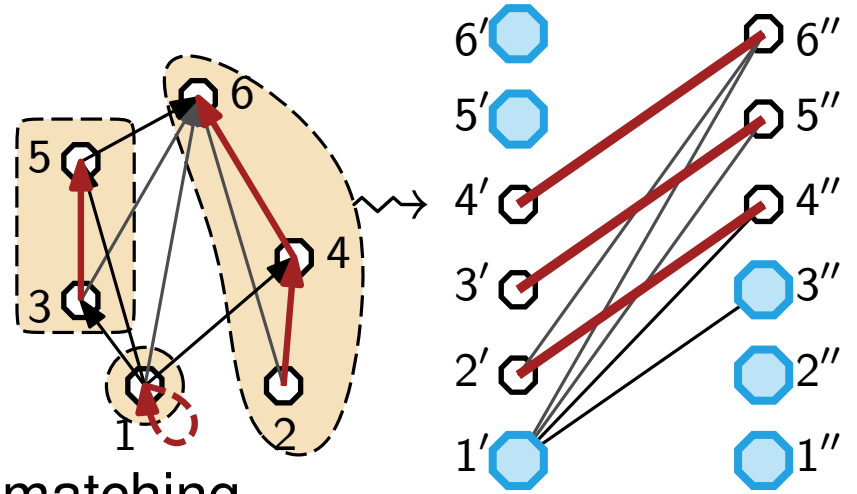
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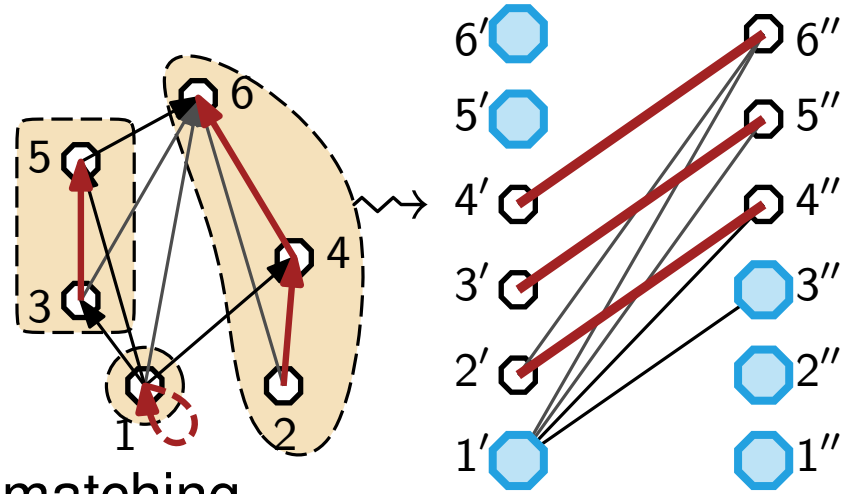
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this also allows reconstructing the path i.e. the "$\Leftarrow$"

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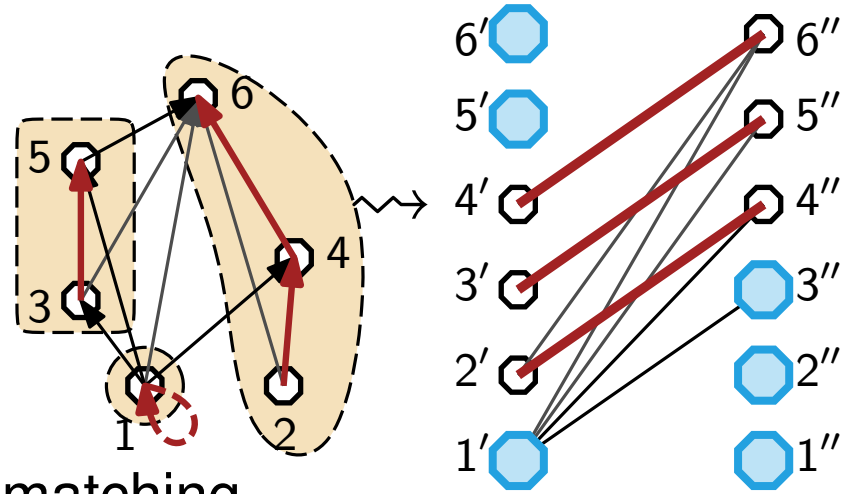
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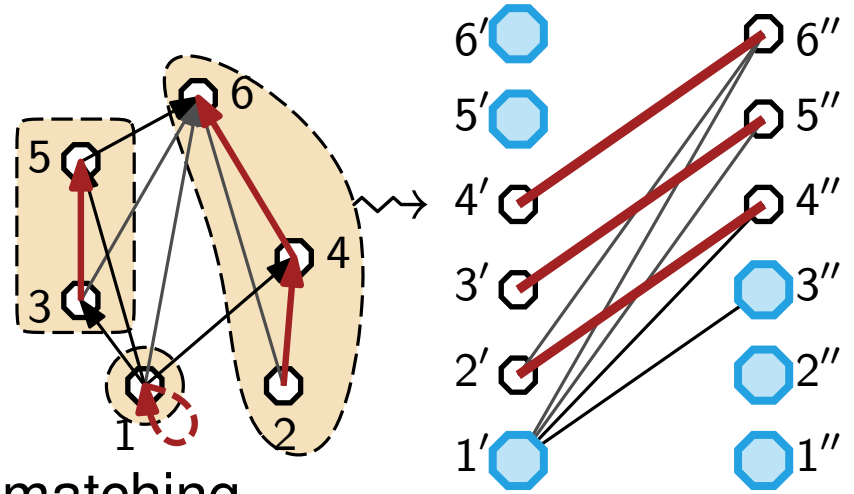
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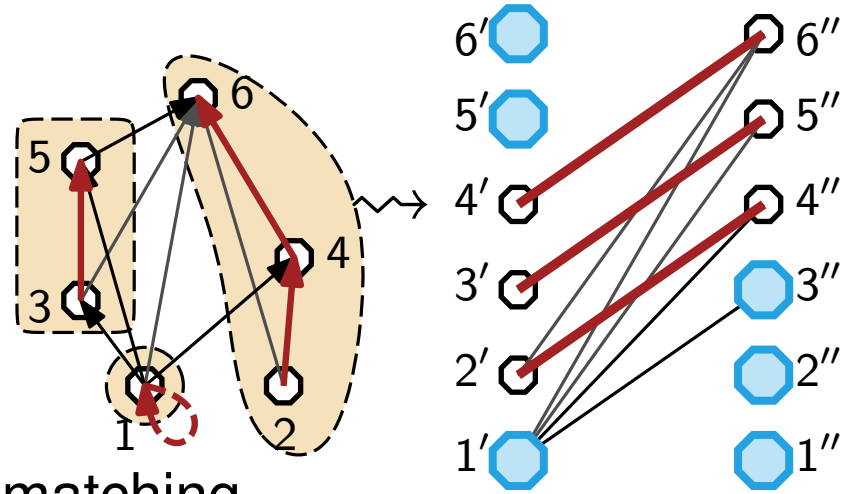
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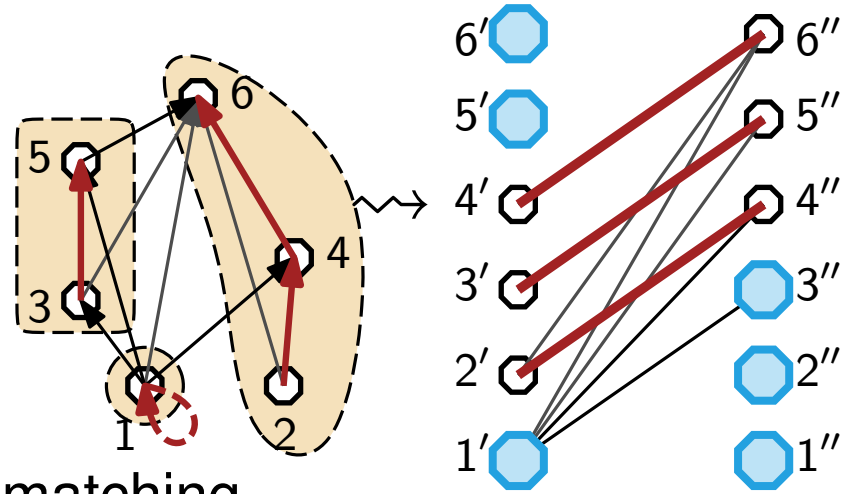
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- $V' = \{v' \mid v \in V\}$
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we will prove this rule to be correct:

v, w consecutive in clique cover $\overset{\text{rule}}{\iff}$ $v'w'' \in M$ matching

$\exists V_1 + \ldots + V_k$ clique cover of G
$\iff$ k disjoint cliques in G
$\iff$ k directed disjoint paths in G, F
$\iff$ $2k$ starts of disjoint paths in G, F
(count vertices on path of length 0 twice)
$\iff$ $2k$ vertices in B unmatched
$\iff$ $2 \cdot |V| - 2k$ vertices in B matched
$\iff$ there exists matching M with $|M| = |V| - k$

vertex v unmatched if $\forall e \in M : v \notin e$

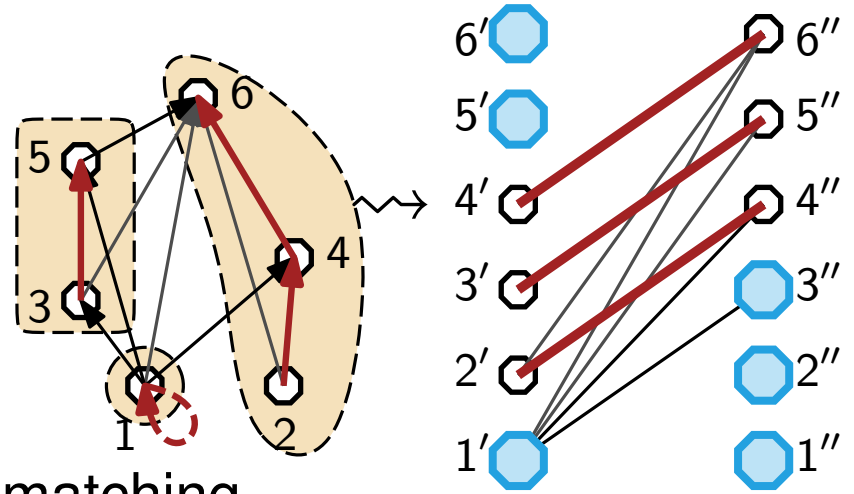
vertex v matched if $\exists e \in M : v \notin e$

divide by 2 for edges since v in at most one edge in M
M is matching since we only pick consecutive edges i.e. at most one per vertex

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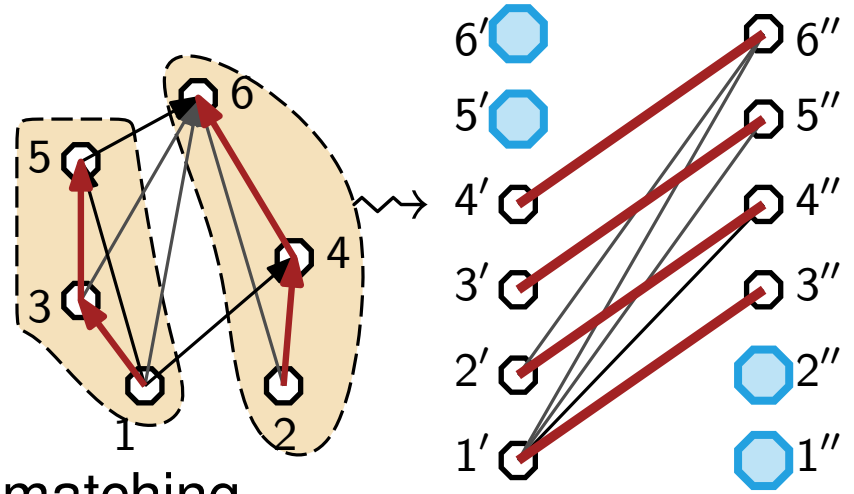
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(count vertices on path of length 0 twice)
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$\iff$ minimal matching M of B has $|M| = |V| - \kappa(G)$

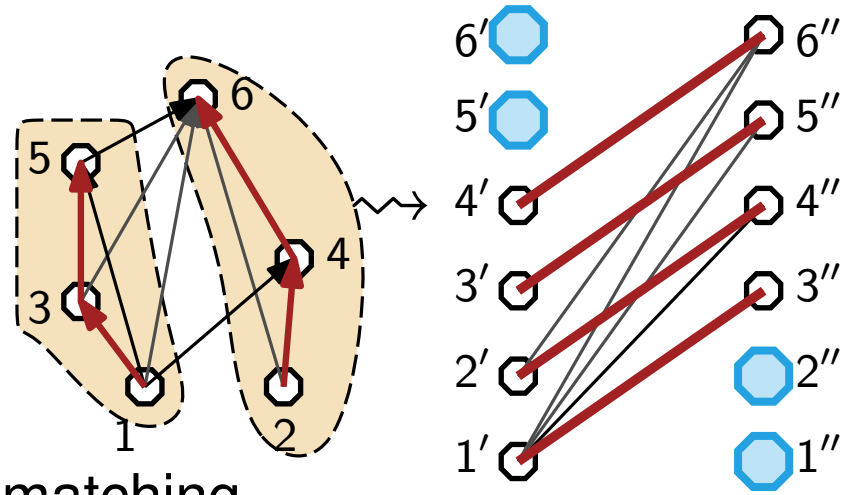
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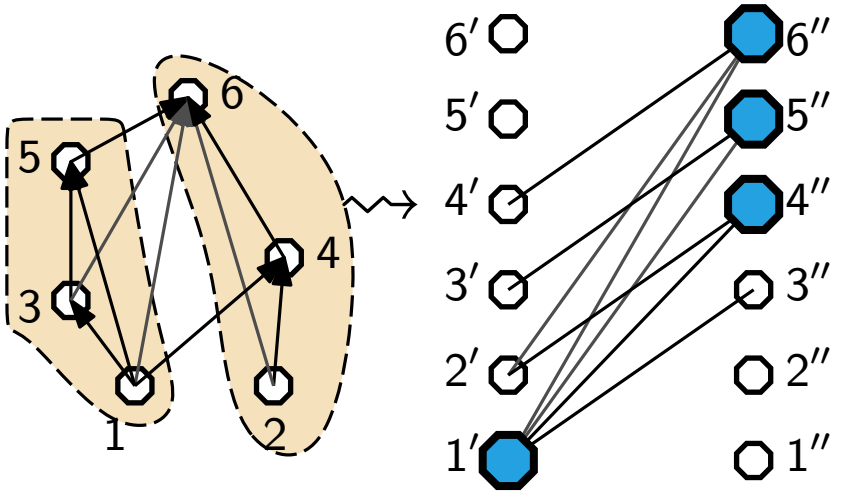
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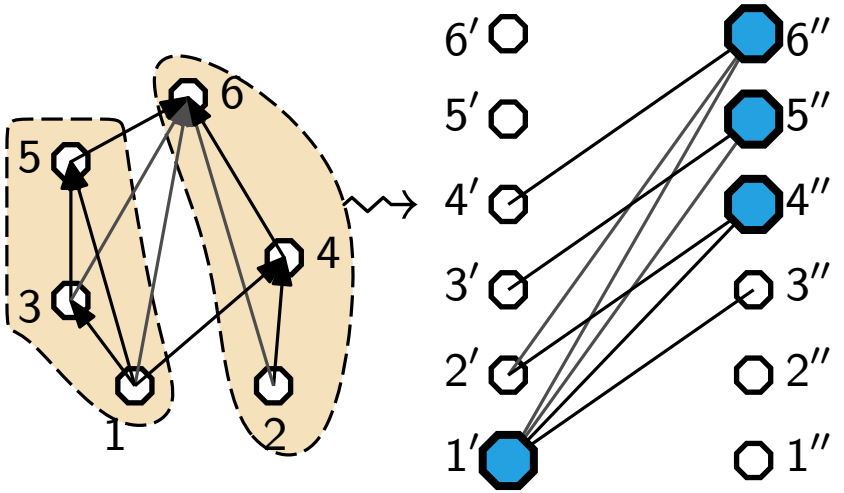
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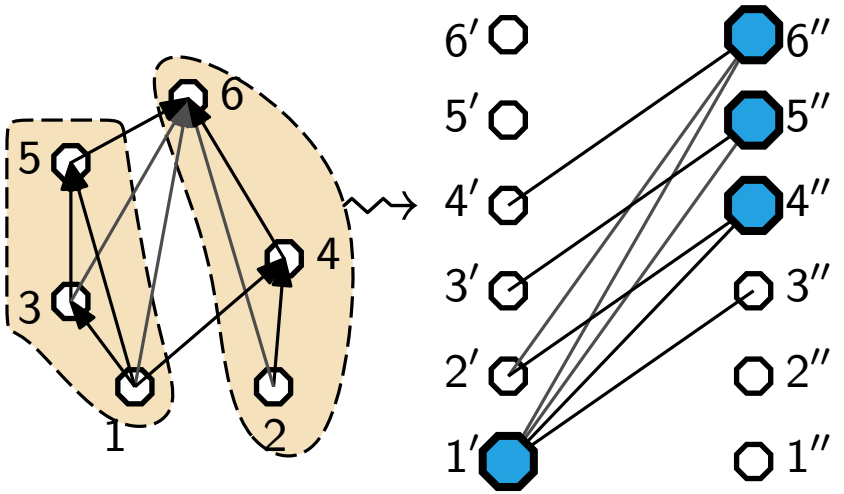
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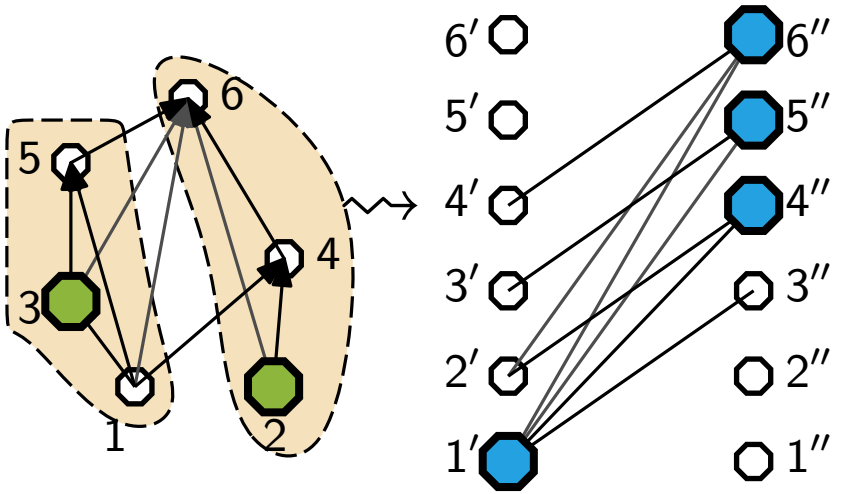
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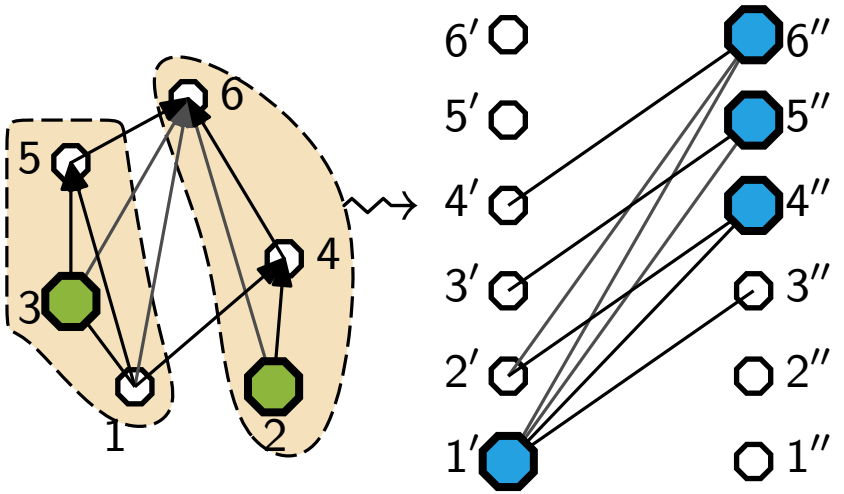
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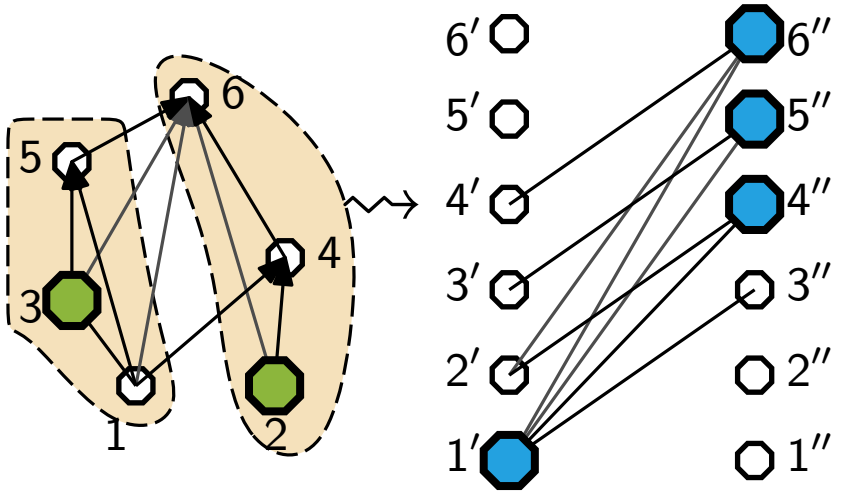
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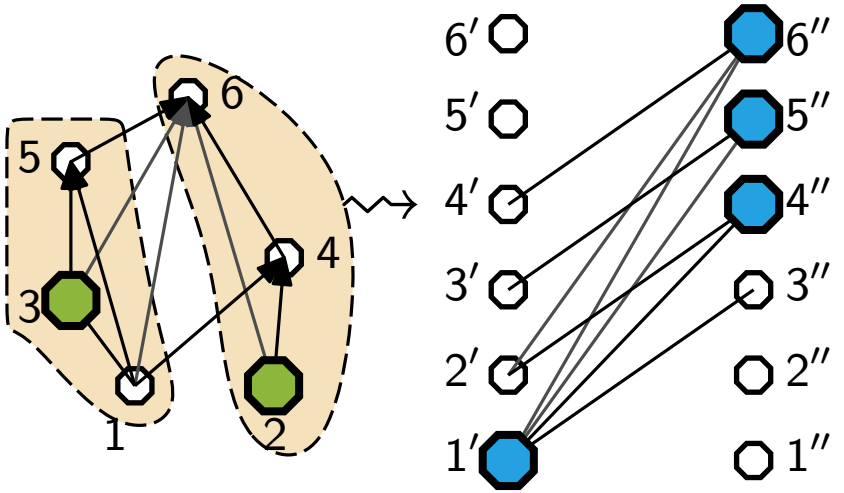
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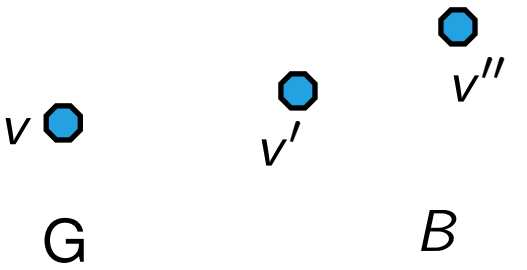
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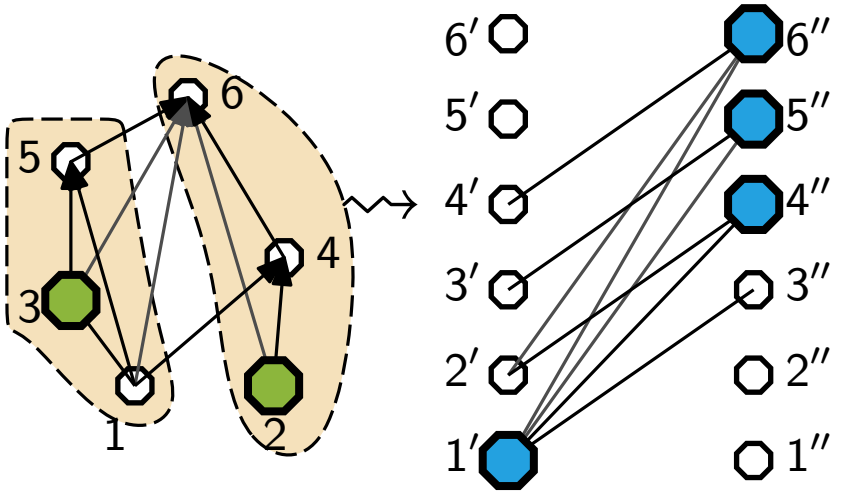
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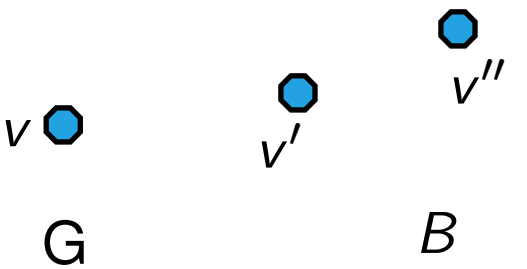
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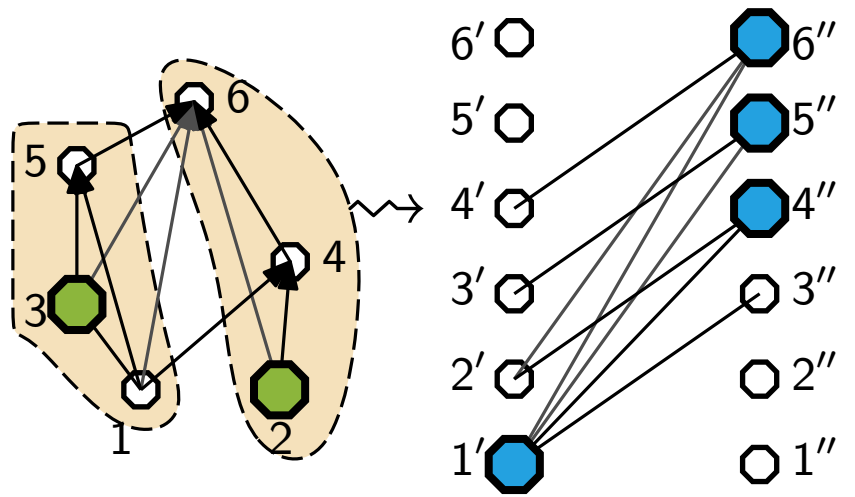
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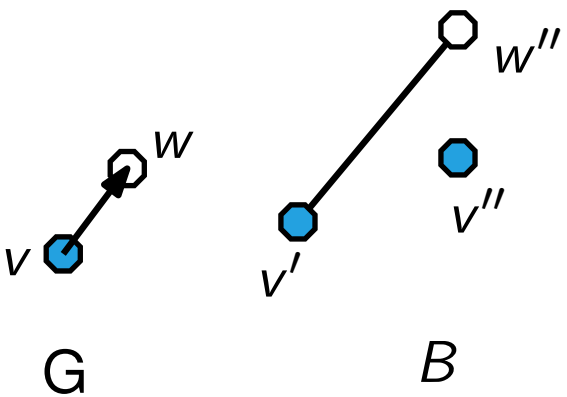
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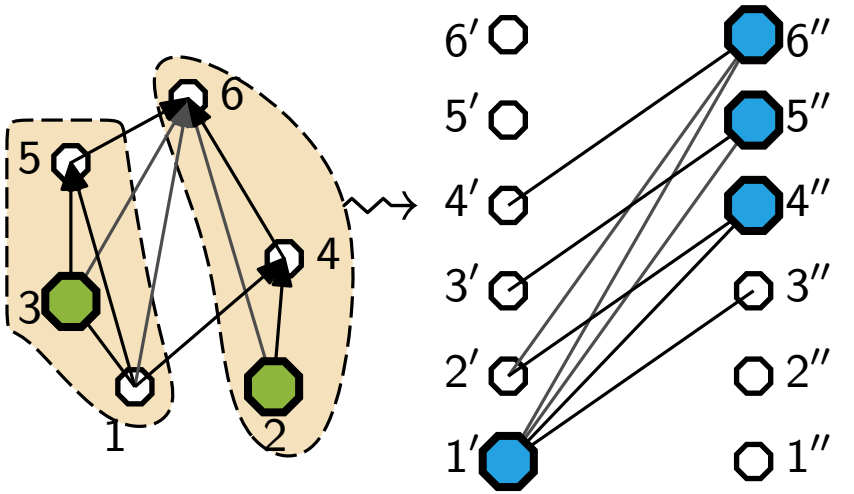




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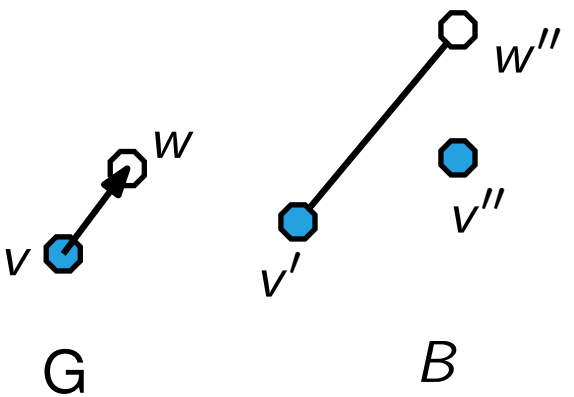
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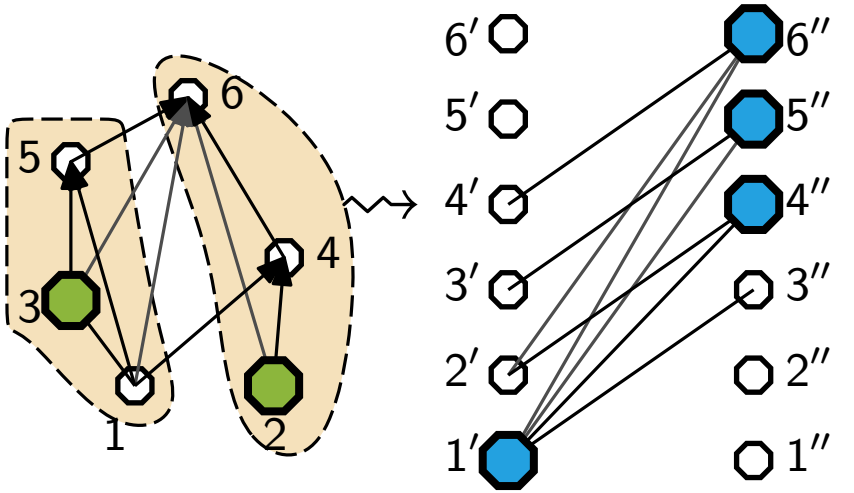
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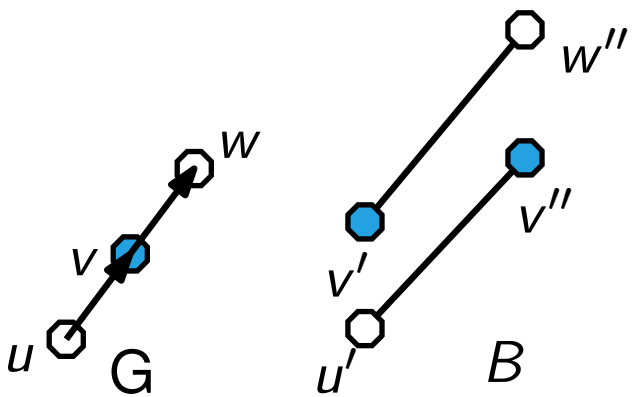
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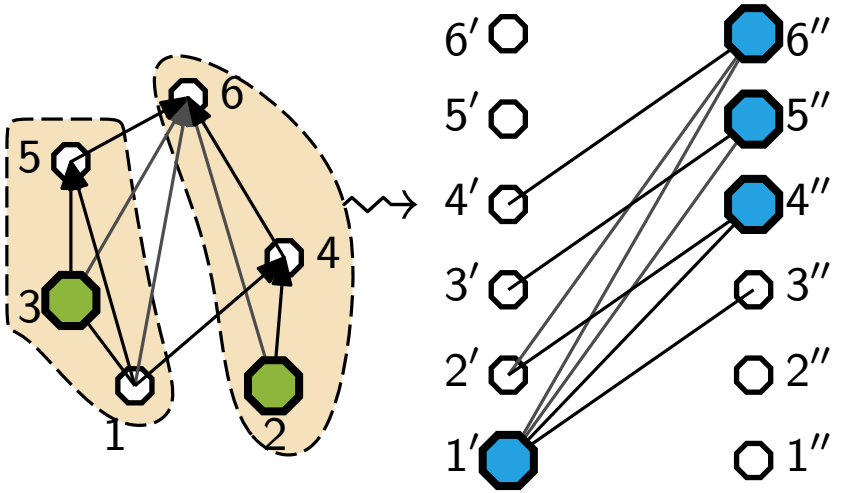


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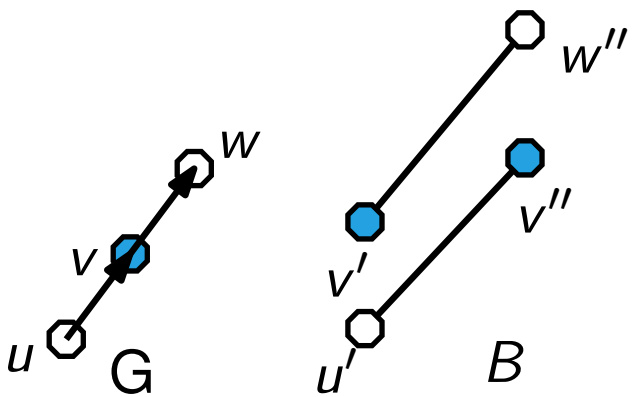
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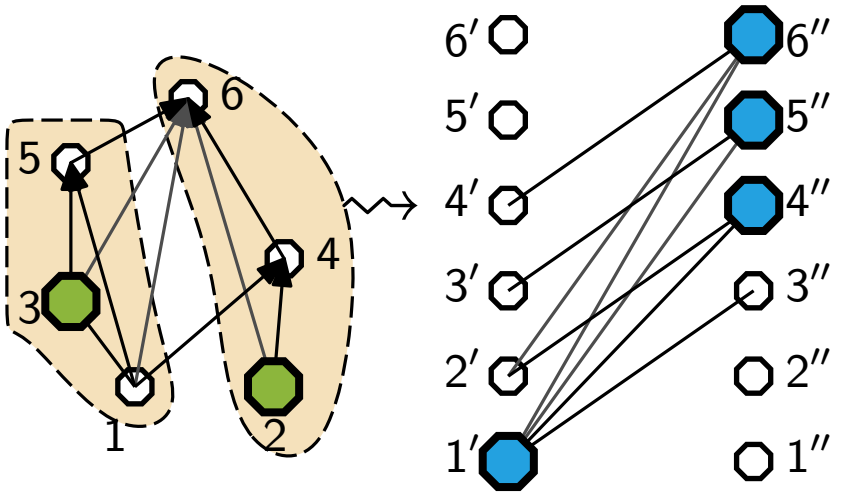
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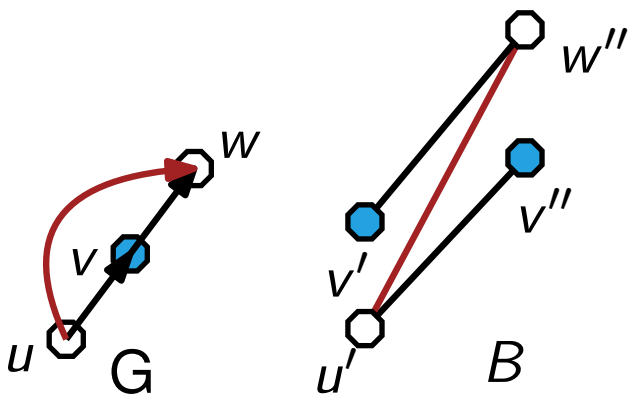
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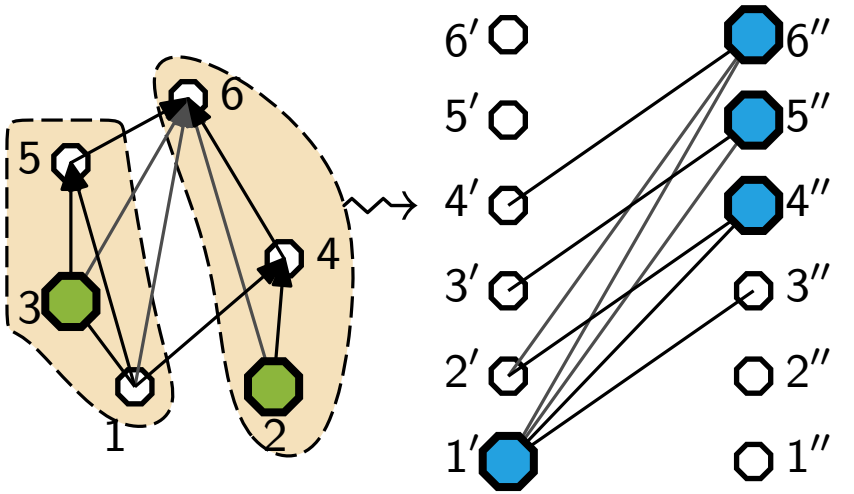
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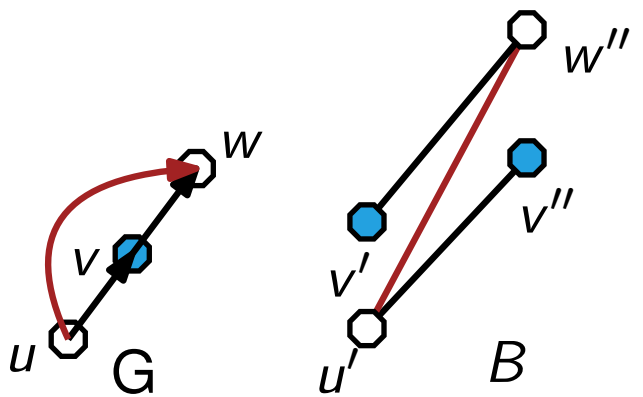
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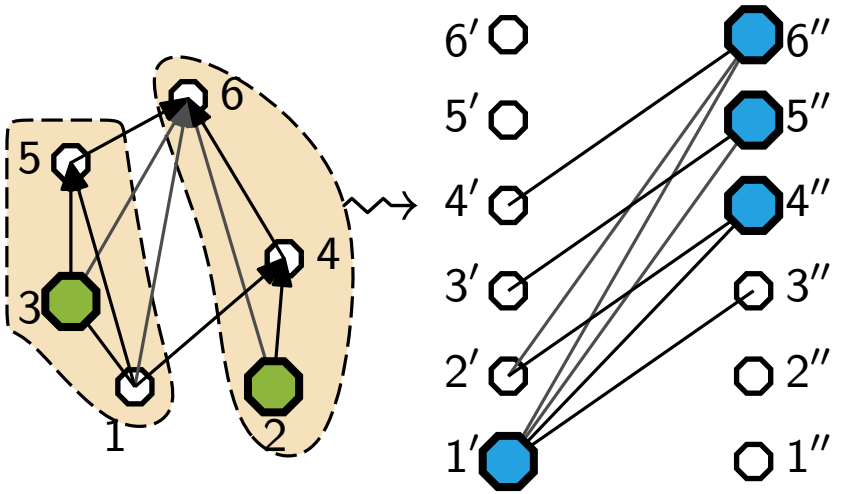
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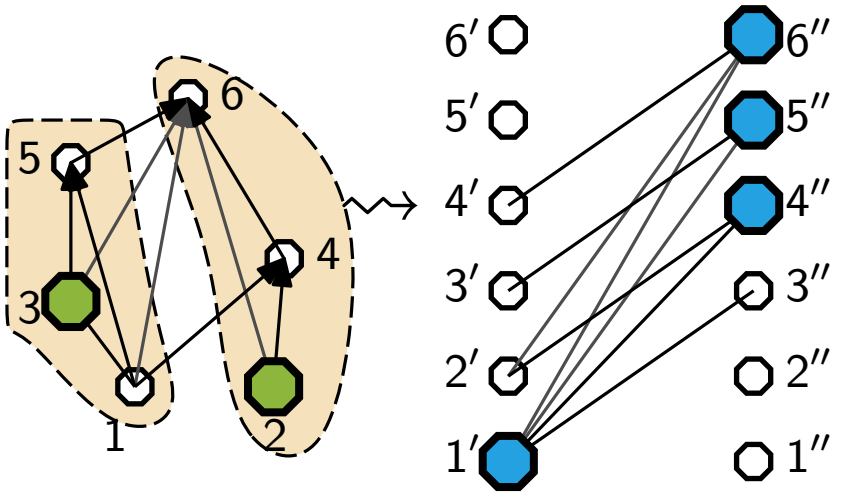


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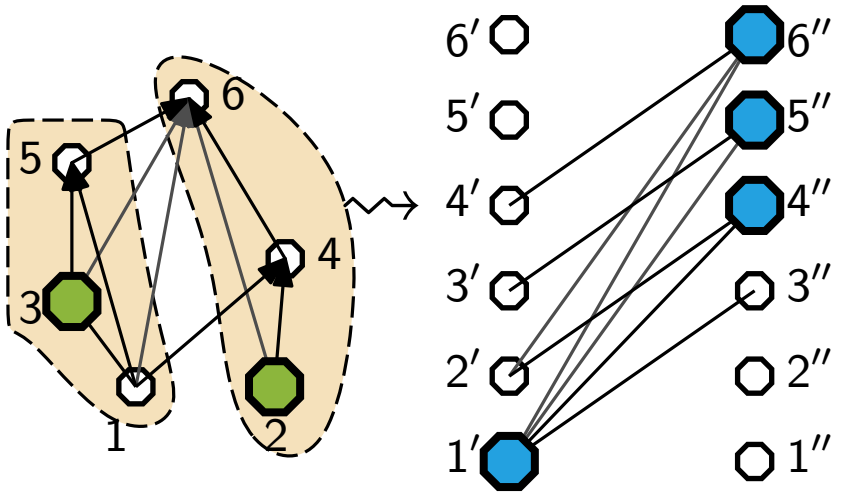
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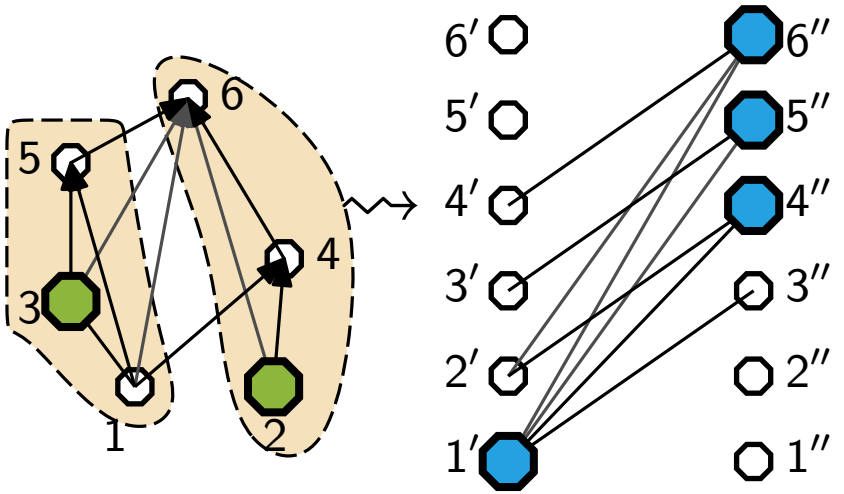
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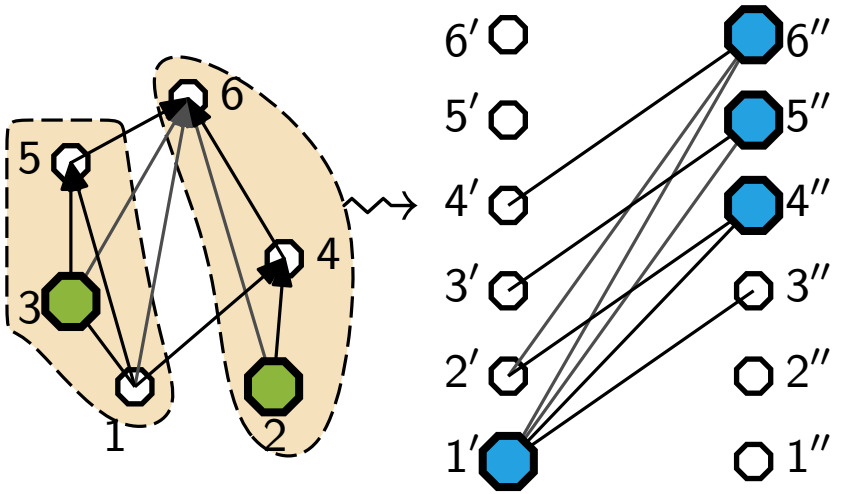
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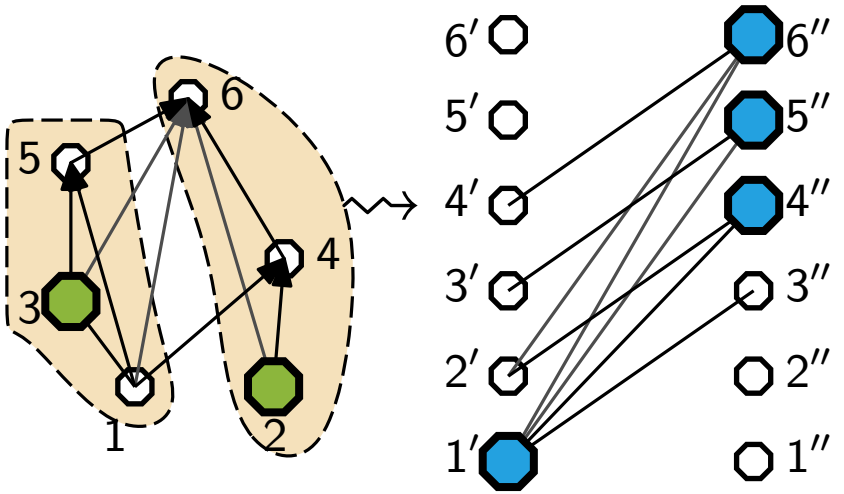
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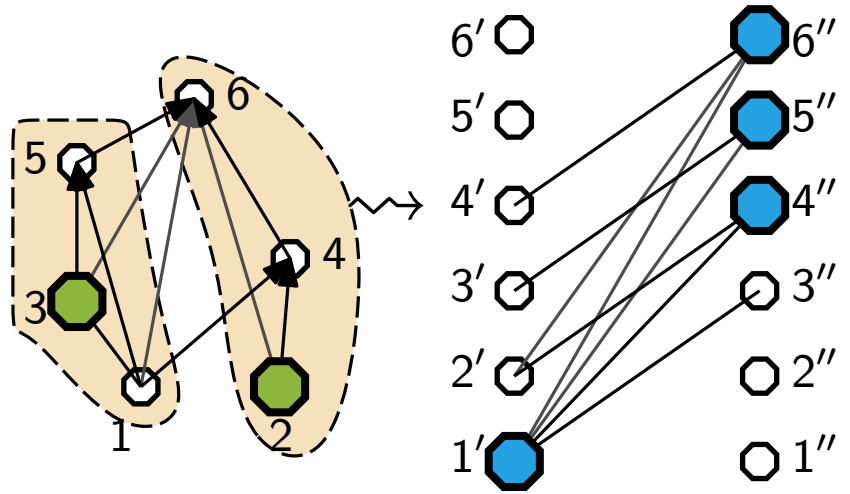
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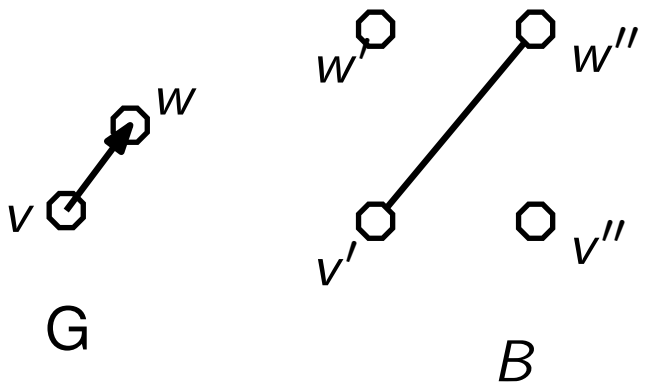
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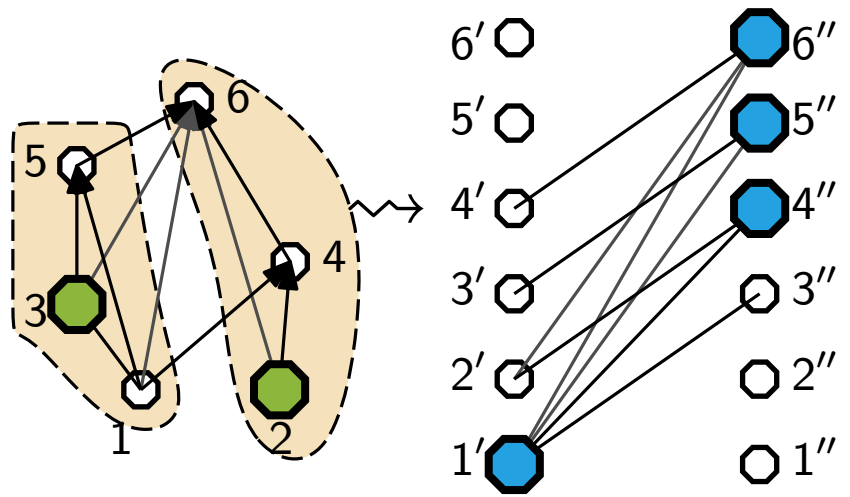
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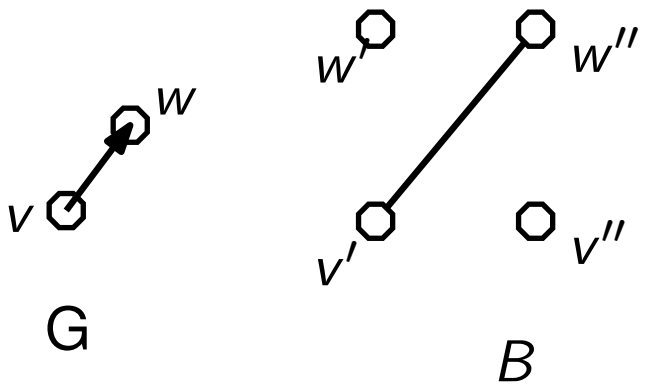
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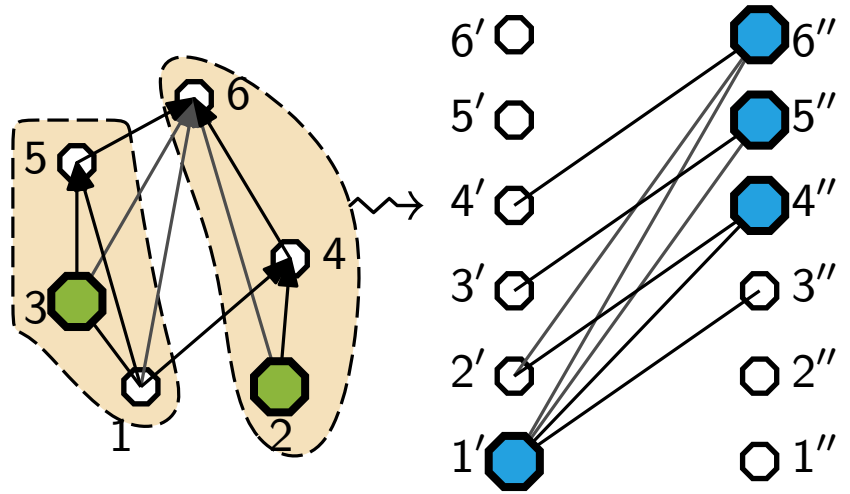
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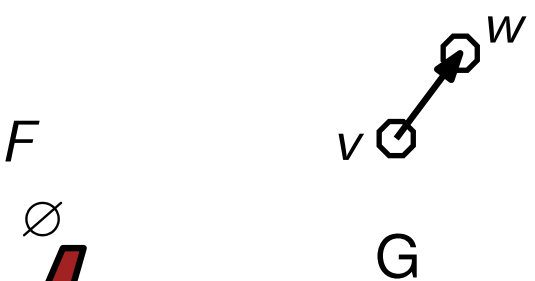
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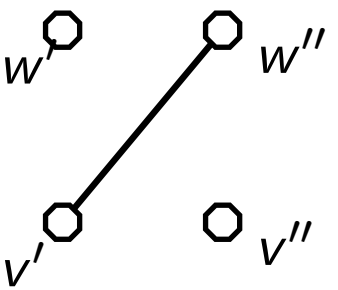
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G



B

S vertex cover

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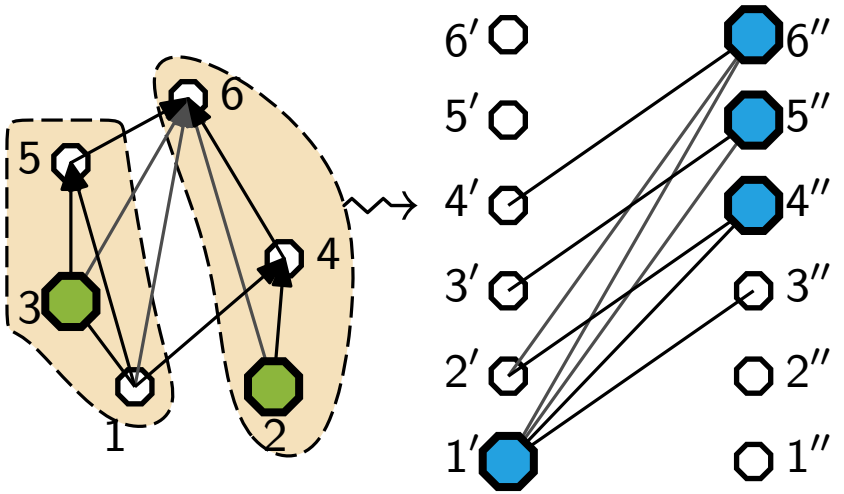
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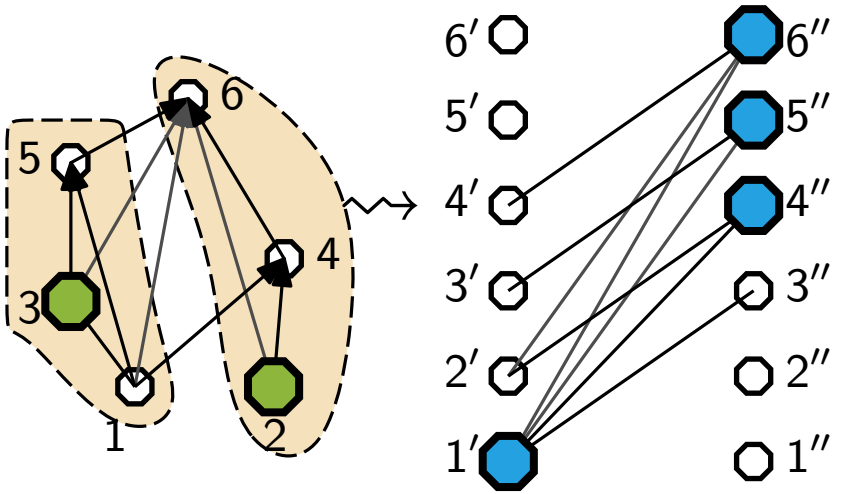
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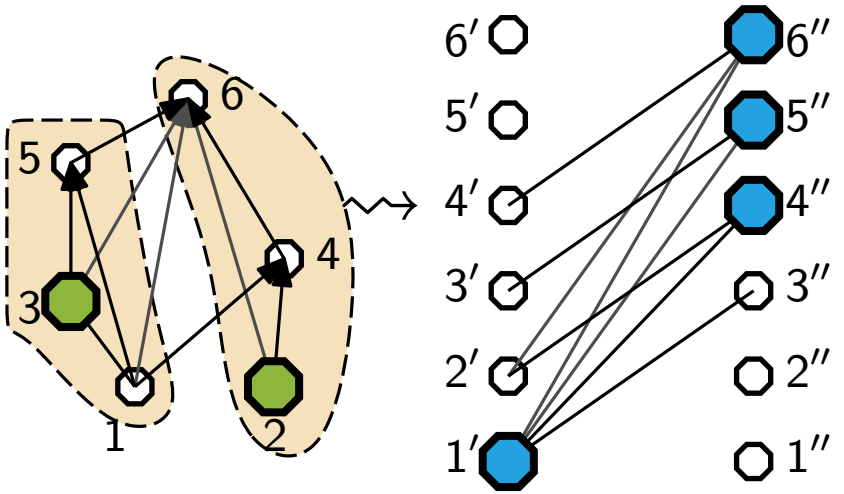
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↳ by Obs. 1 for all graphs



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- G is chordal and $\overline{G}$ is chordal i.e. G split graph
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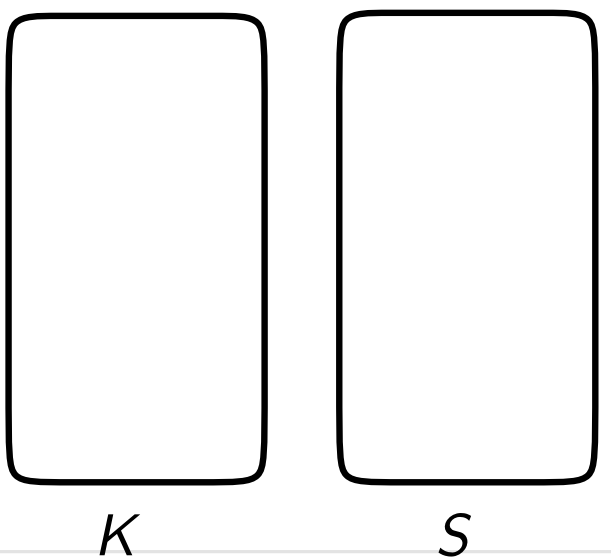
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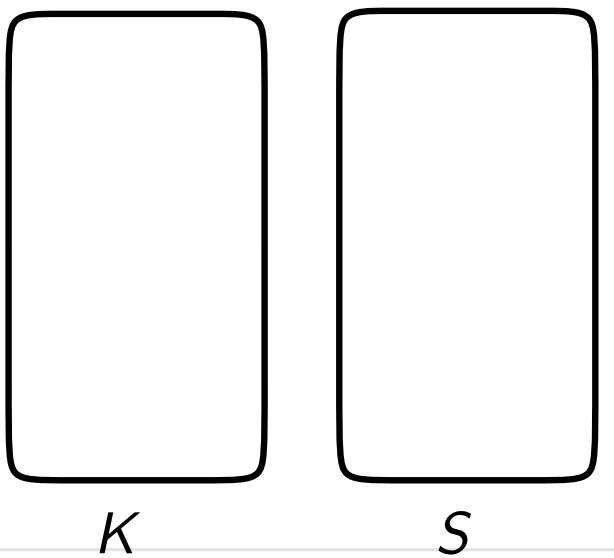
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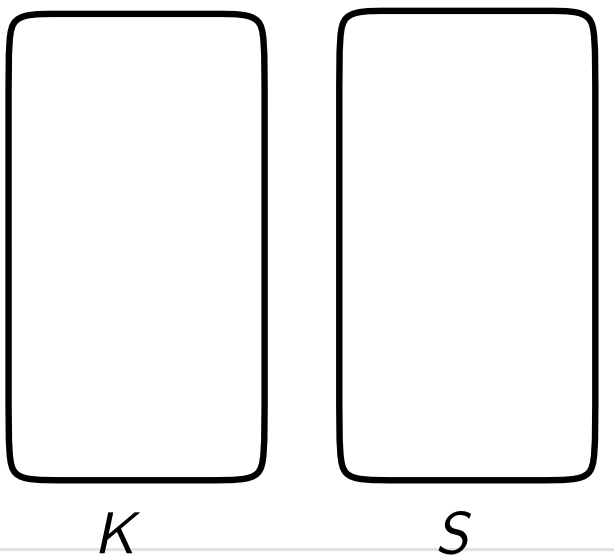
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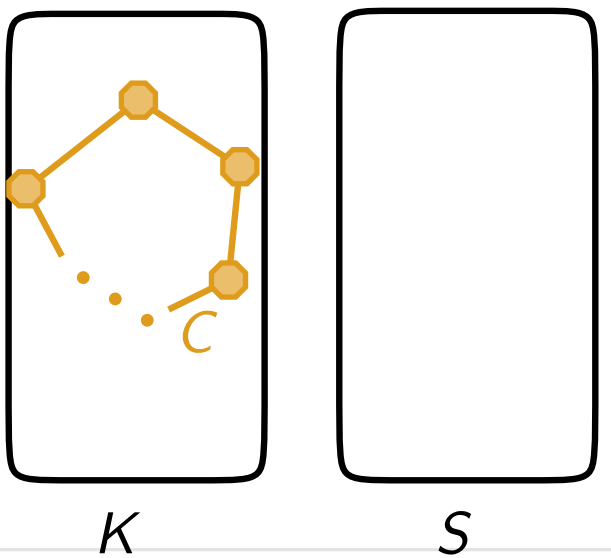
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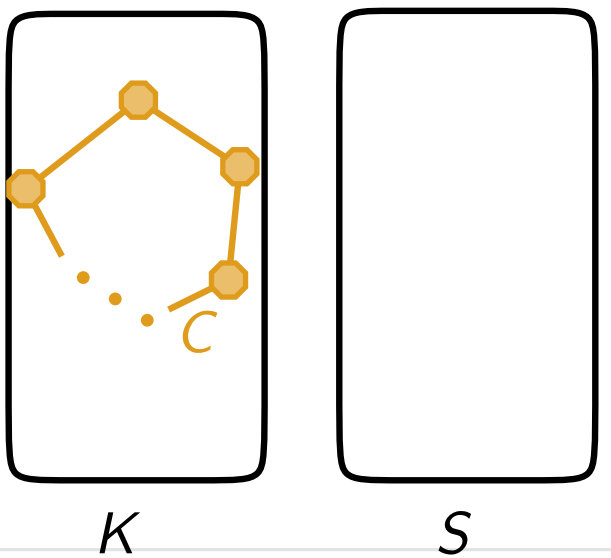
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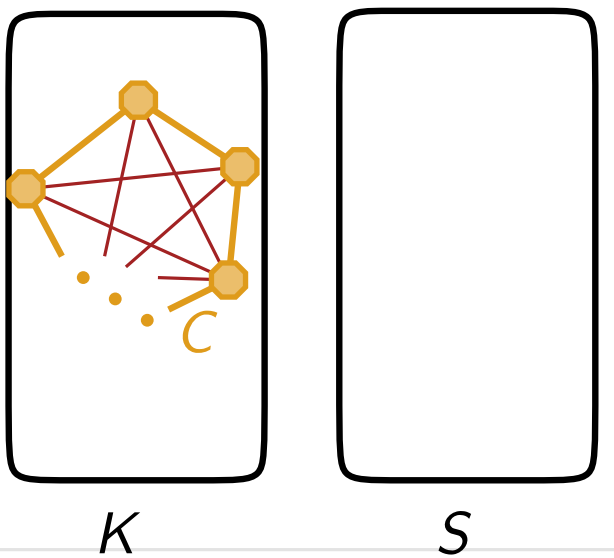
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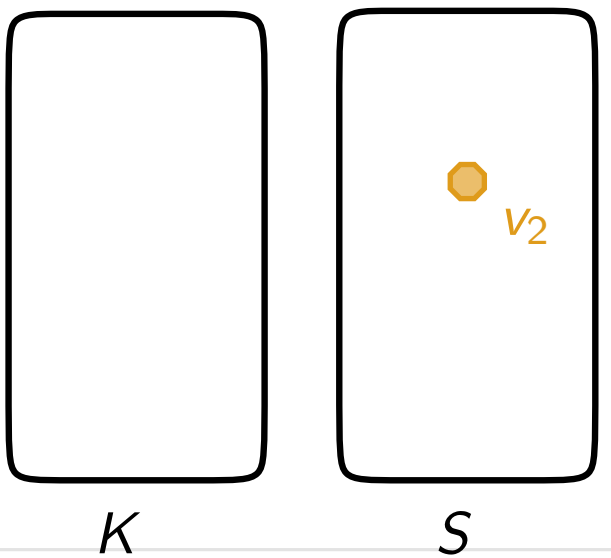
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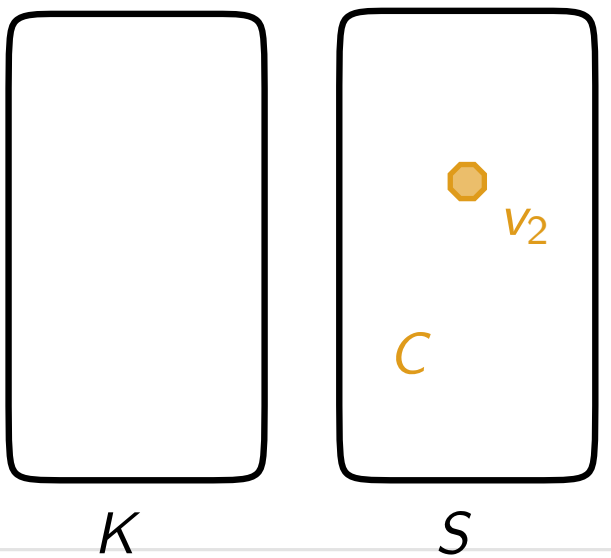
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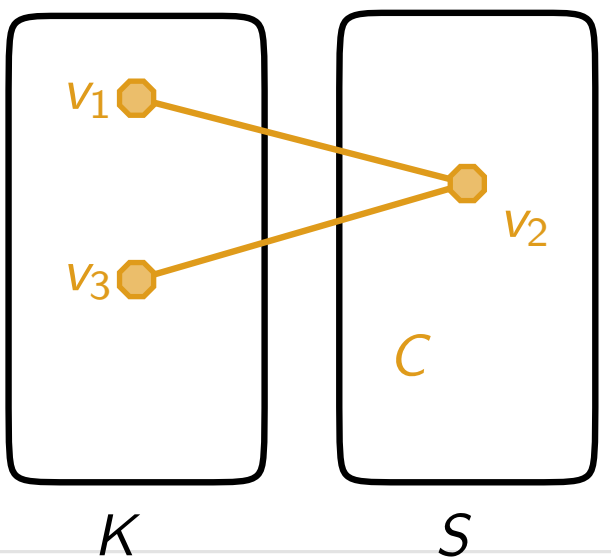
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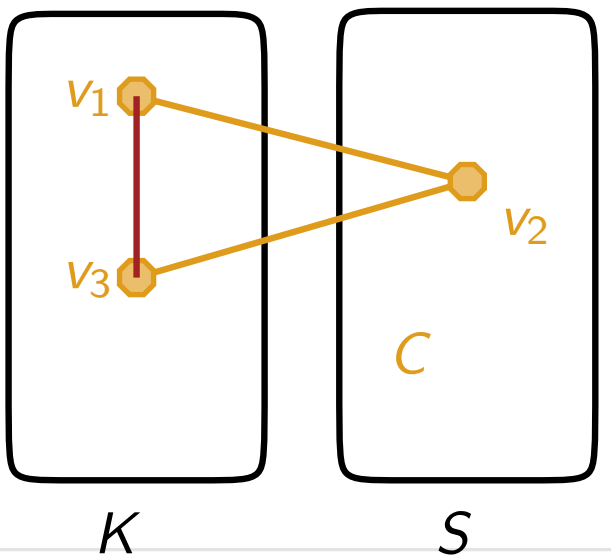
We have $V = K + S$ where K is a clique and S an i-set

Let C be a cycle of length ≥ 4

case $C = [v_1, v_2, v_3, v_4, \ldots]$ with $v_2 \in S$

then $v_1, v_3 \in K$ since S is i-set

chord $v_1 v_3 \in E$ since K is clique



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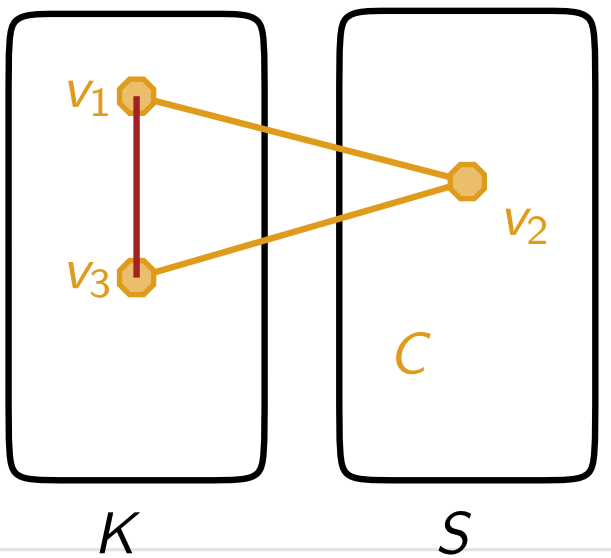
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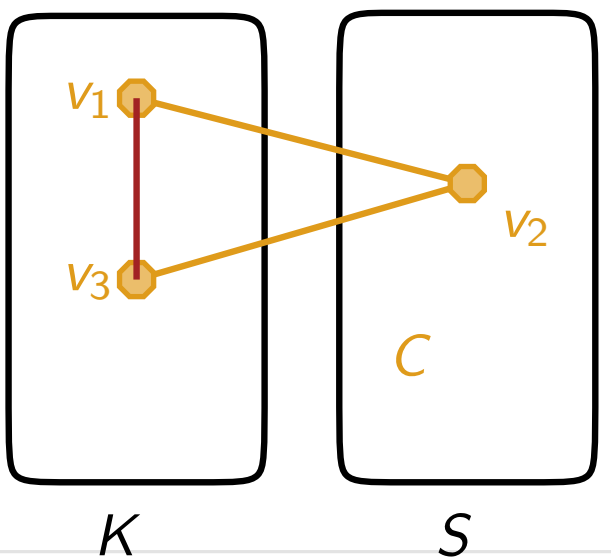
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since K and S just swap in $\overline{G}$ we get $\overline{G}$ chordal by the same argument



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the interesting part is, that this condition is sufficient here as well

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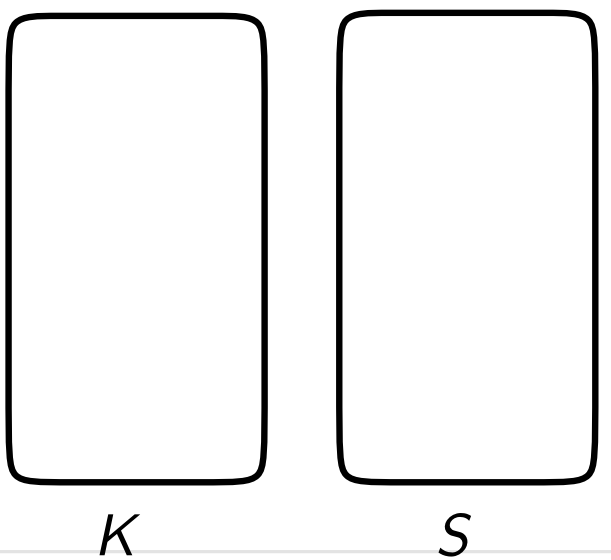
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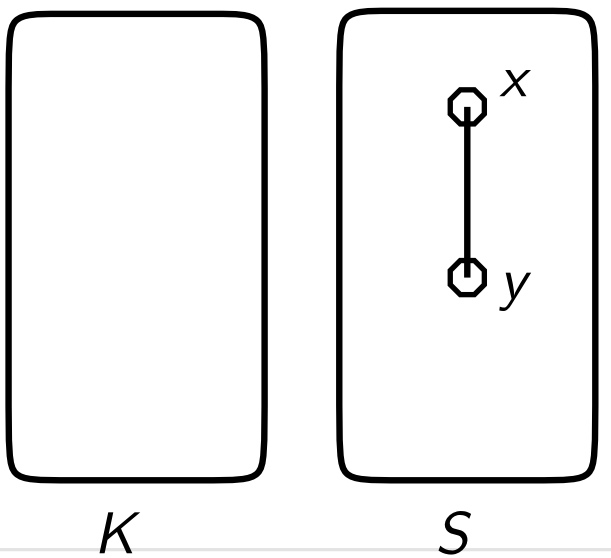
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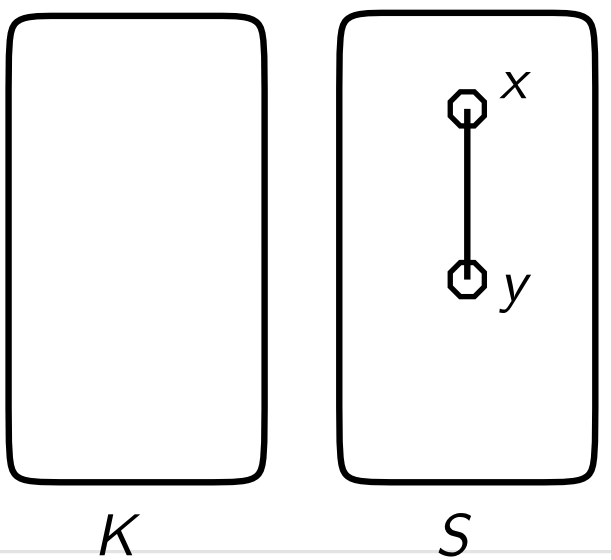
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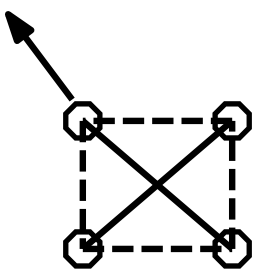
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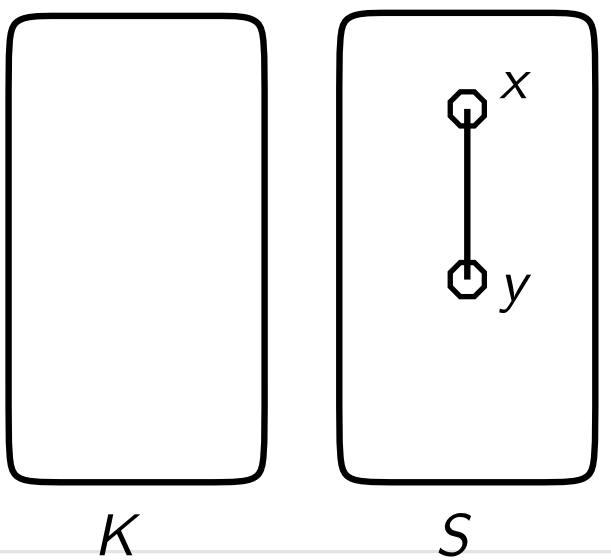
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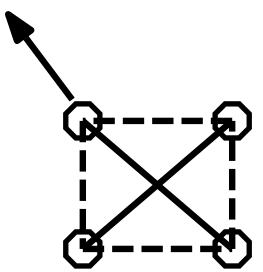
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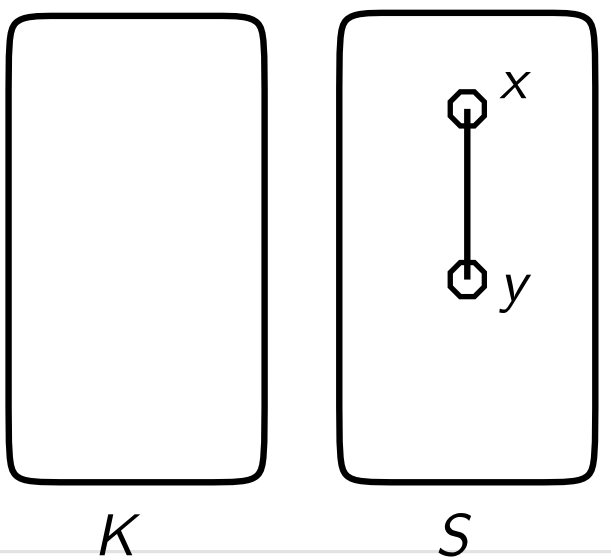
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$2K_2 \subseteq_{\text{ind}} G$



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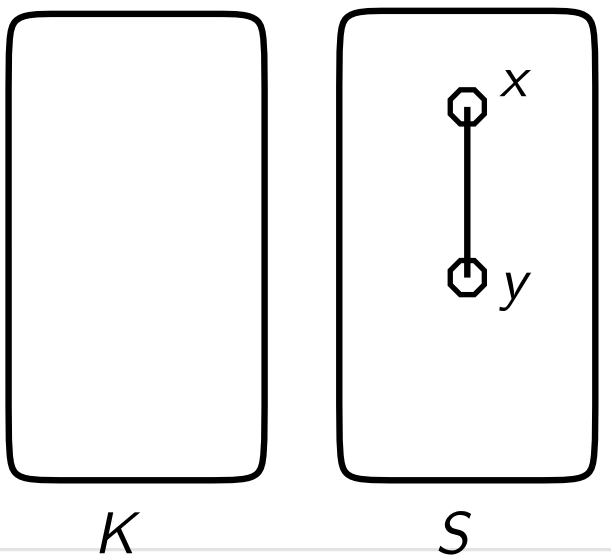
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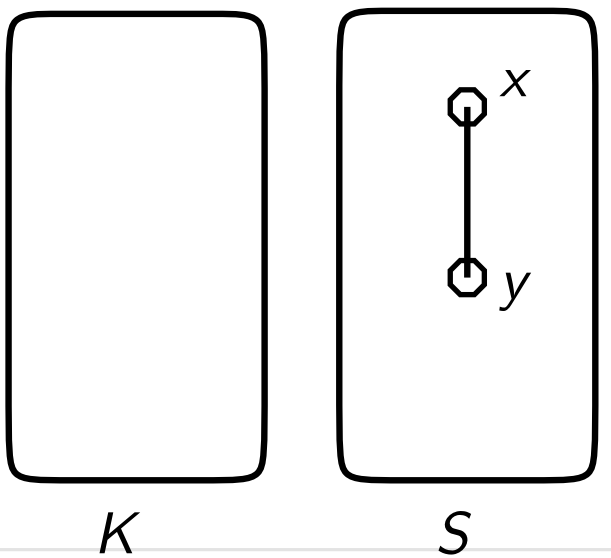
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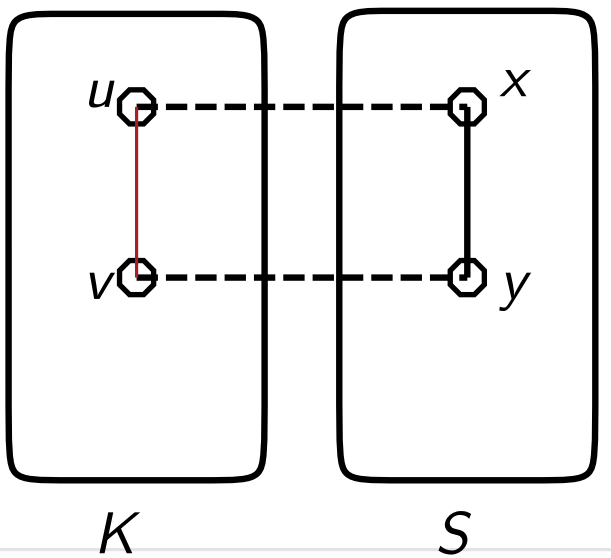
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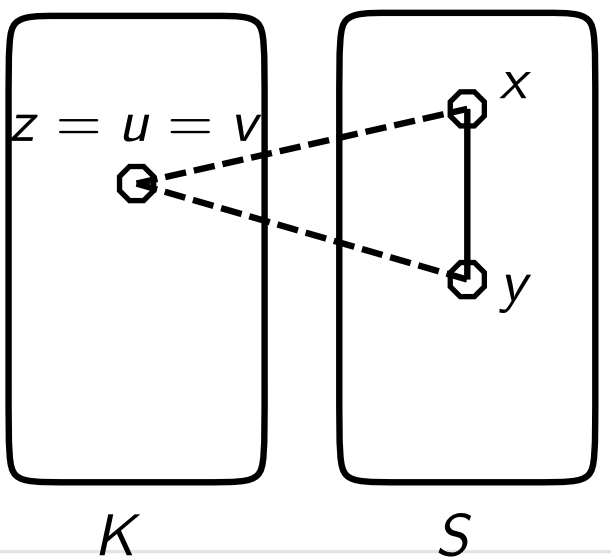
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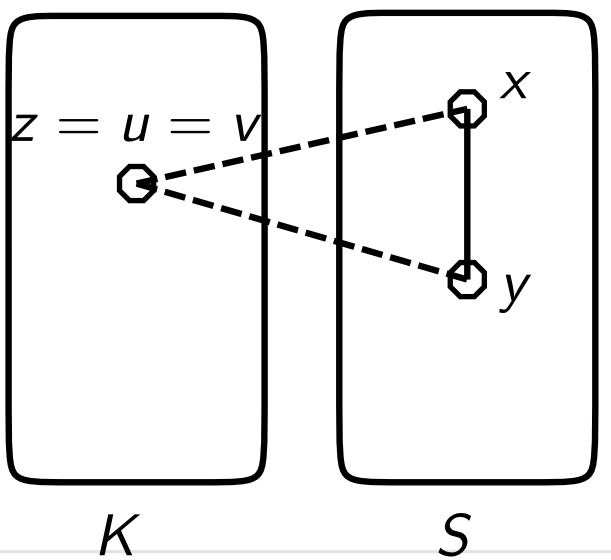
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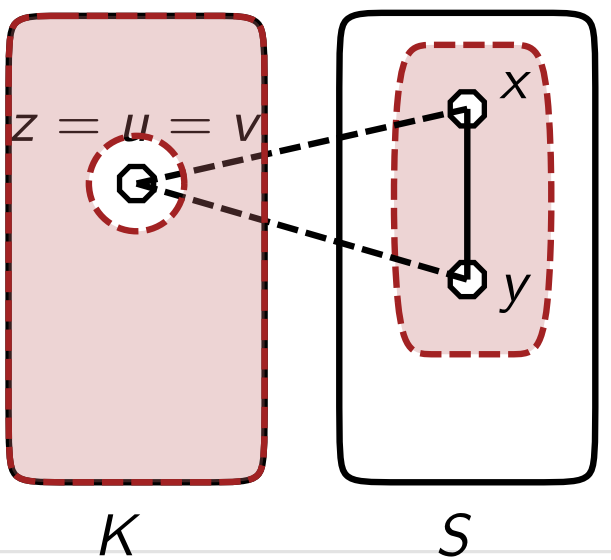
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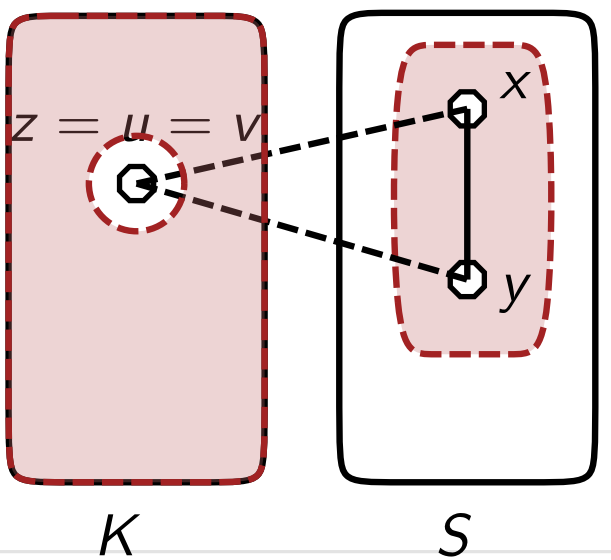
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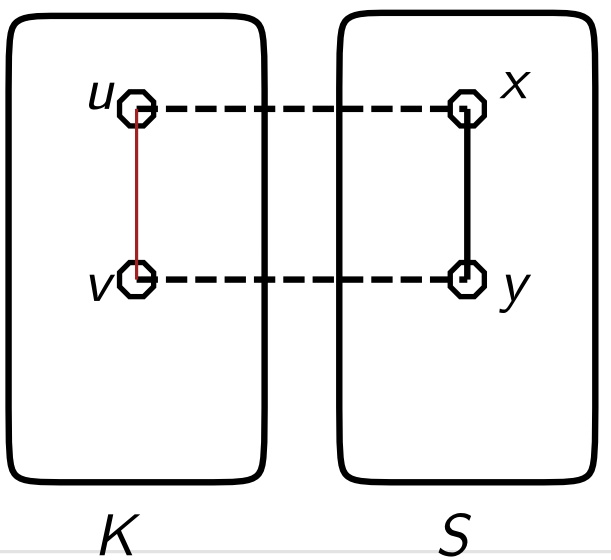
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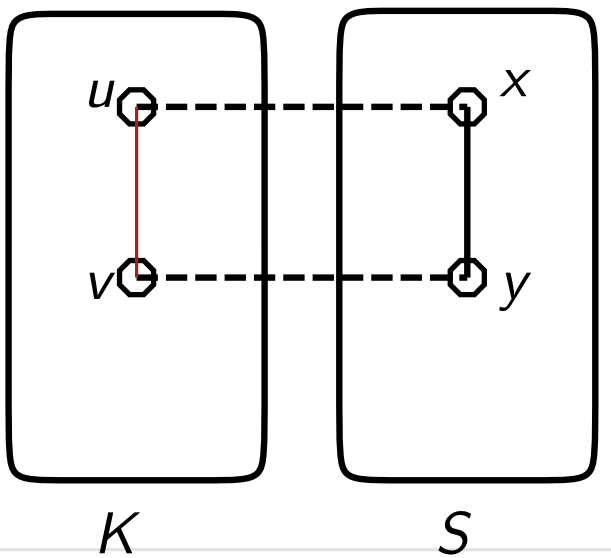
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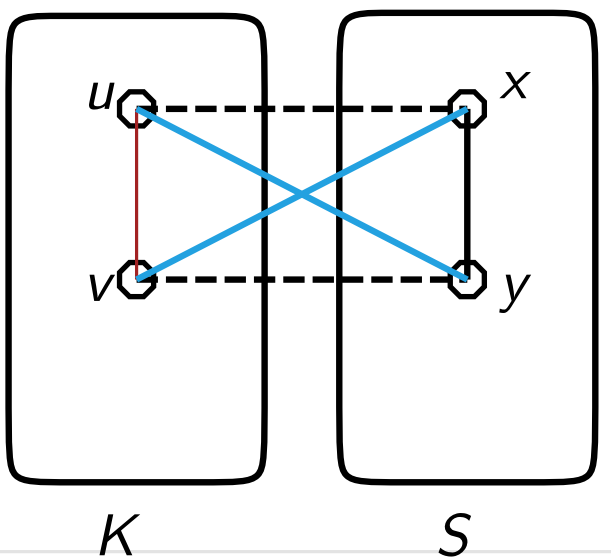
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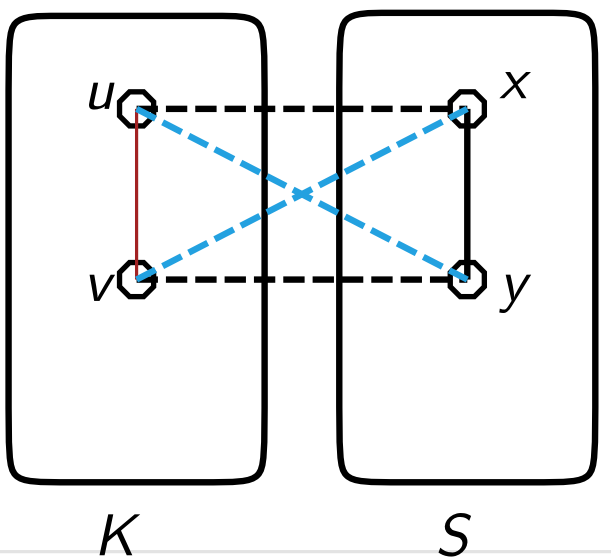
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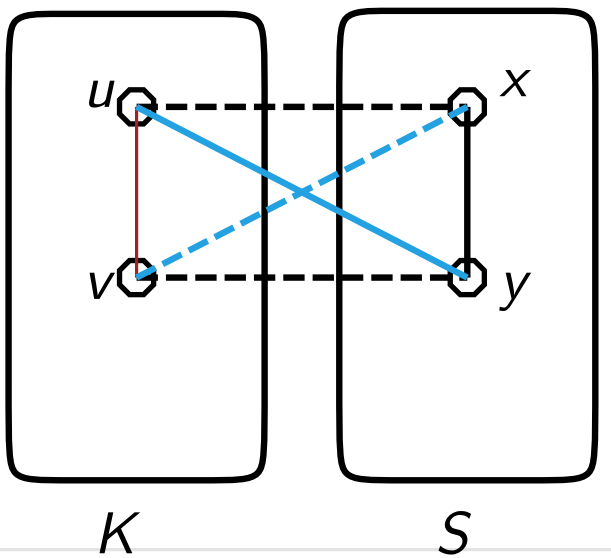
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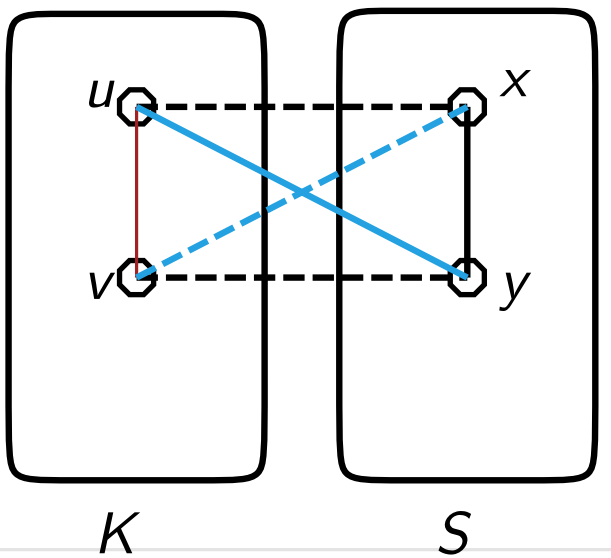
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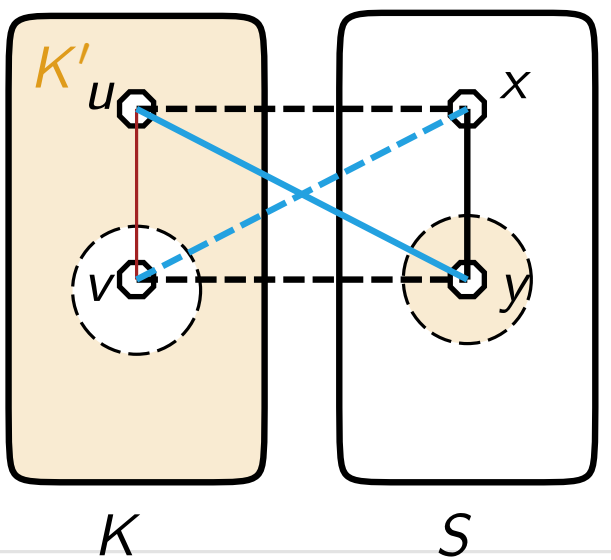
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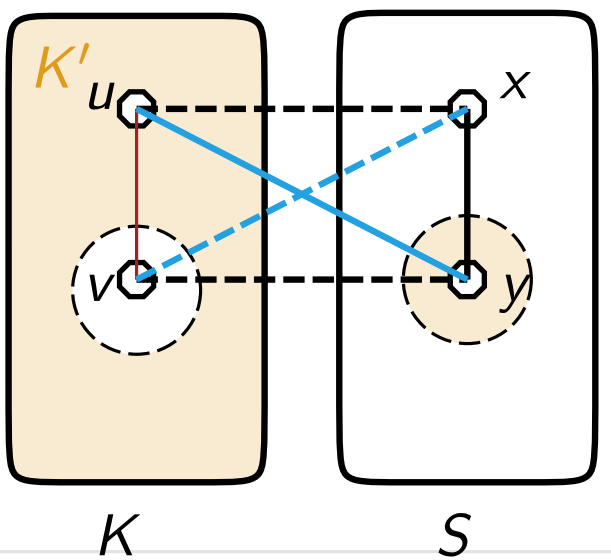
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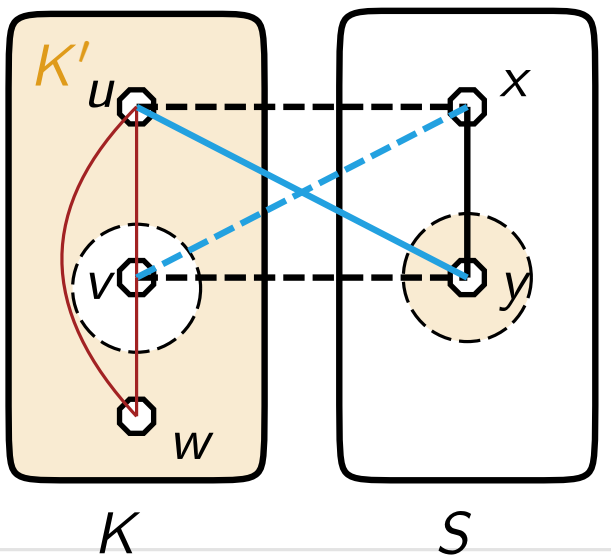
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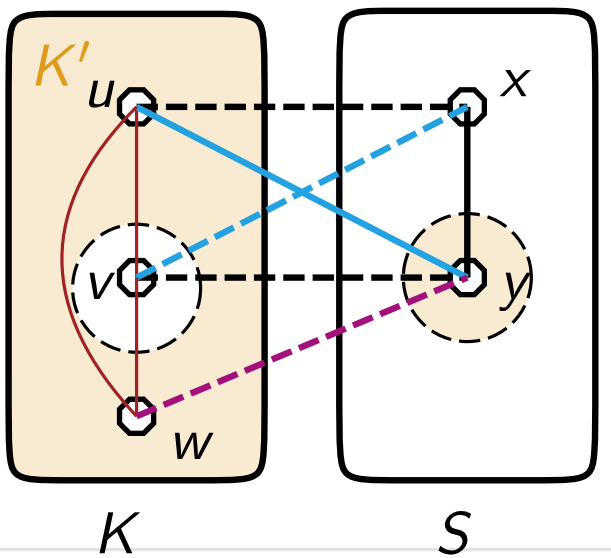
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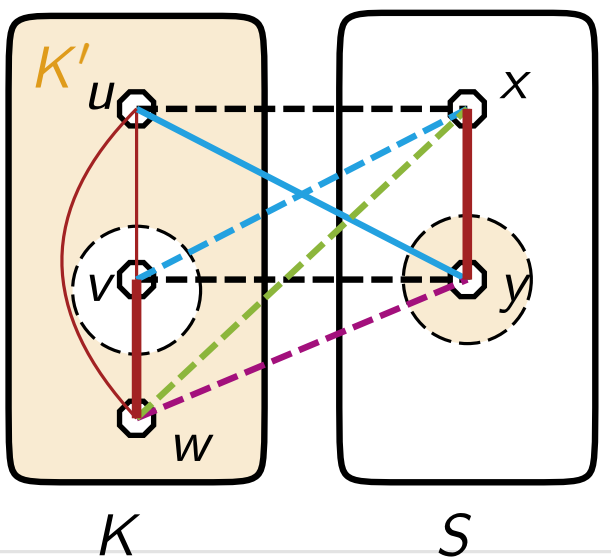
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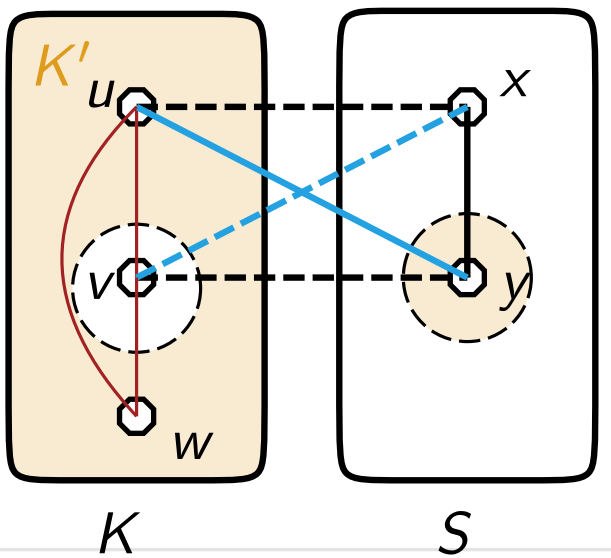
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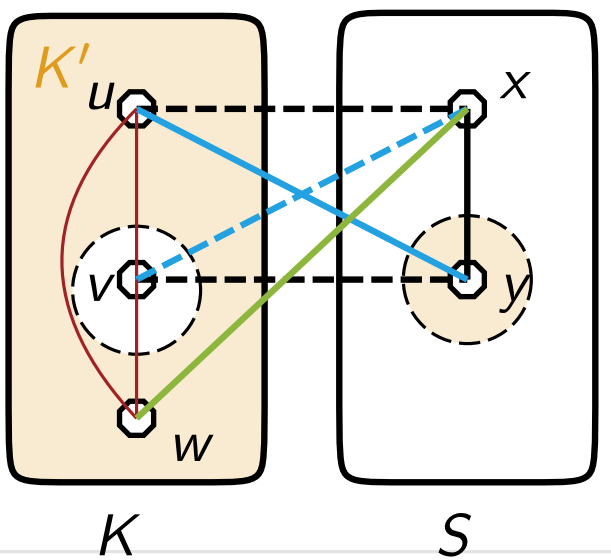
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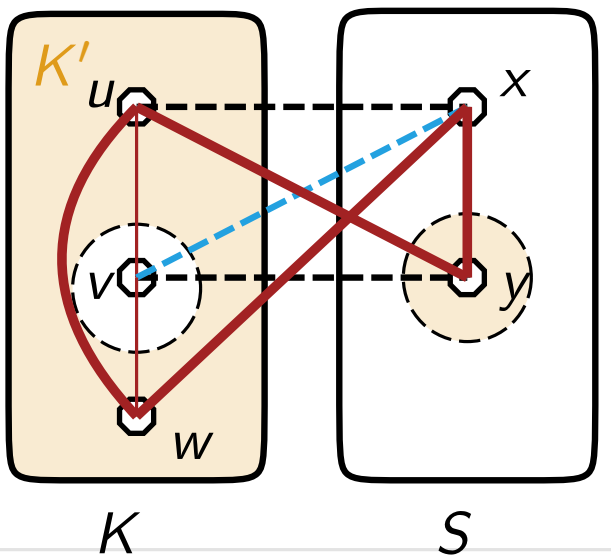
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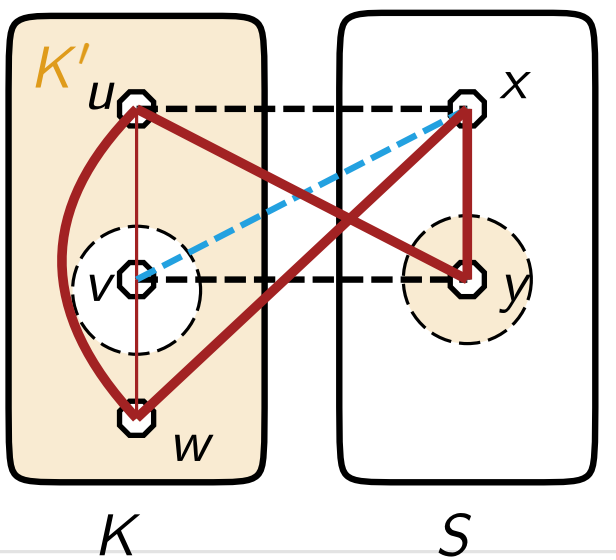
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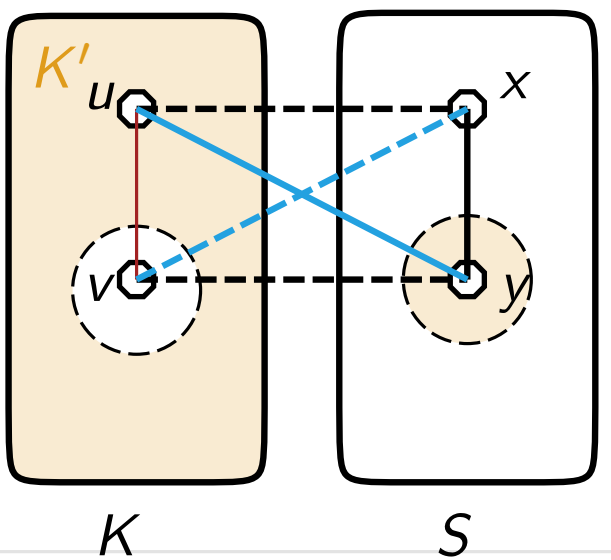
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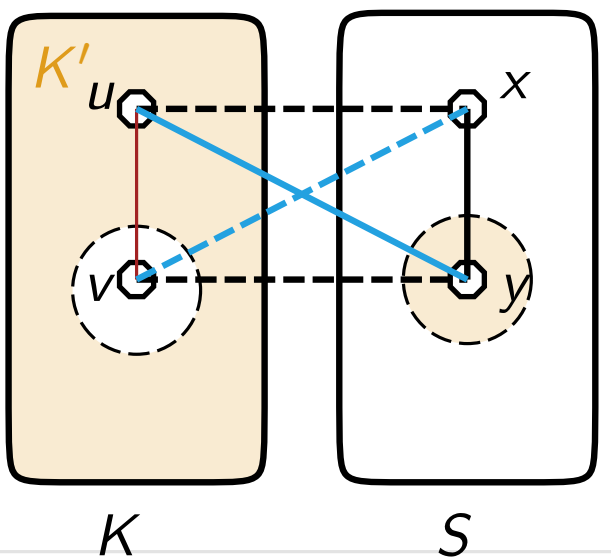
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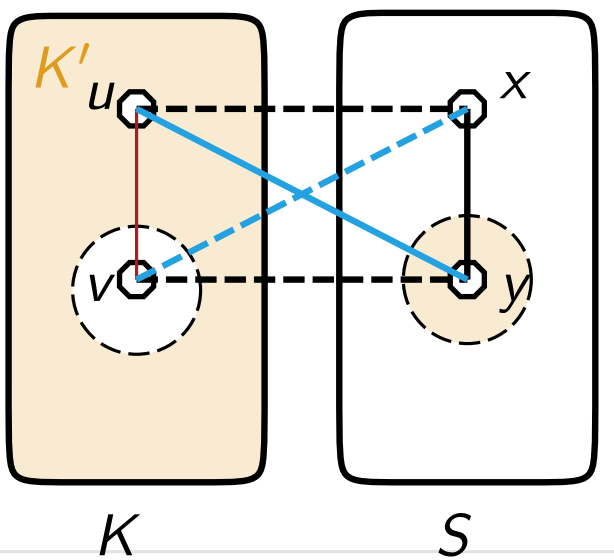
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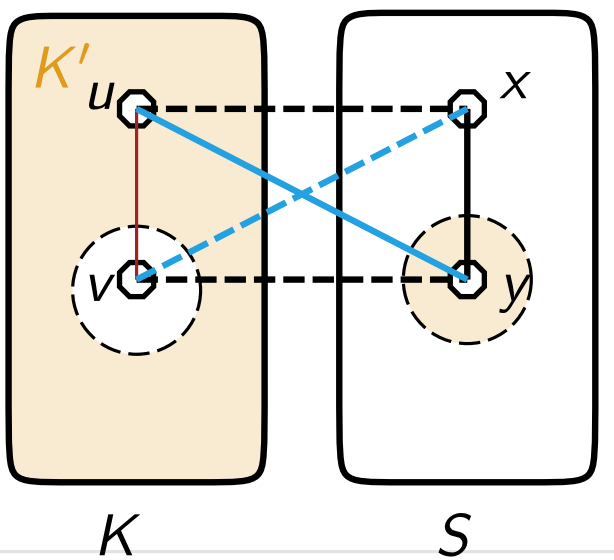
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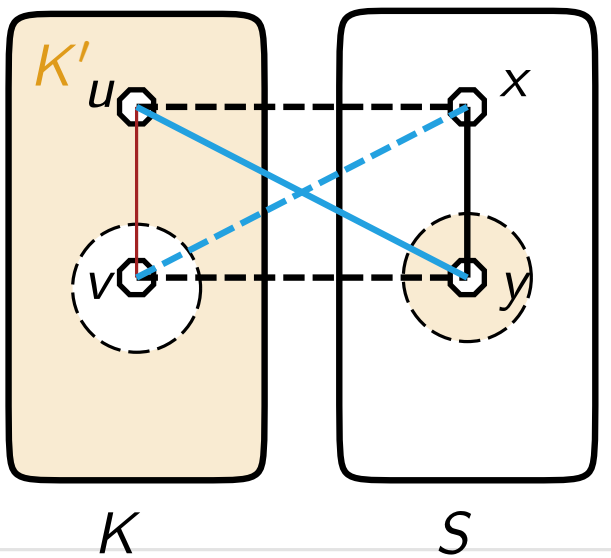
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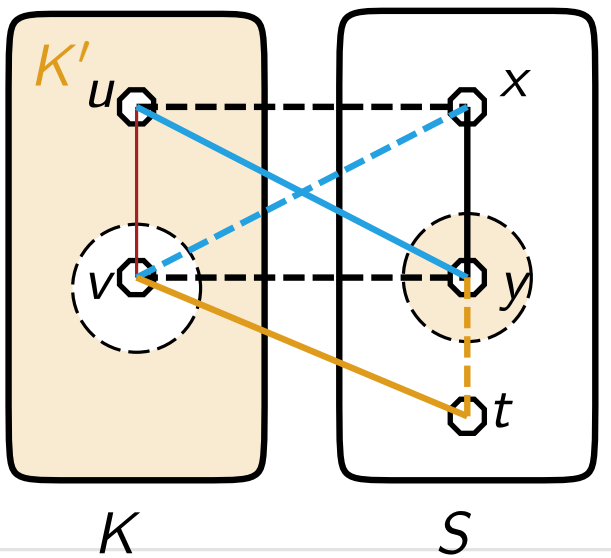
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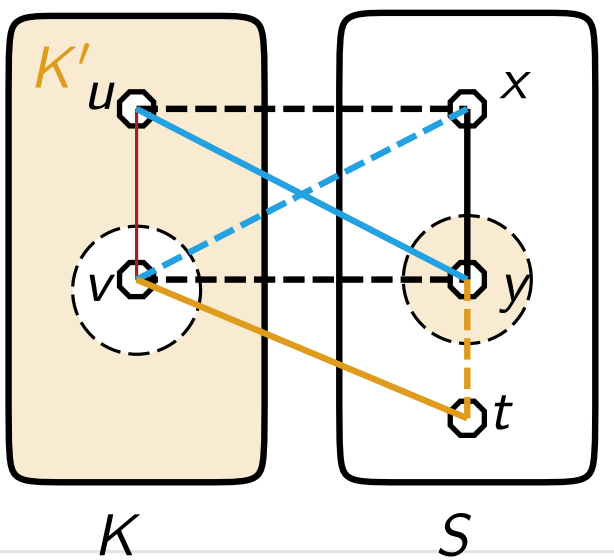
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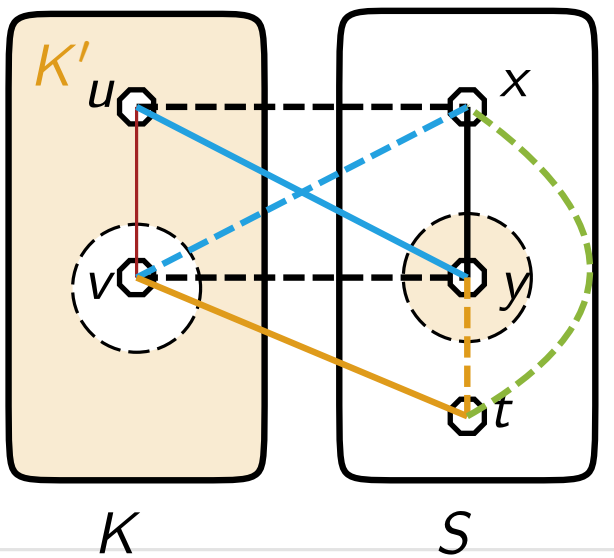
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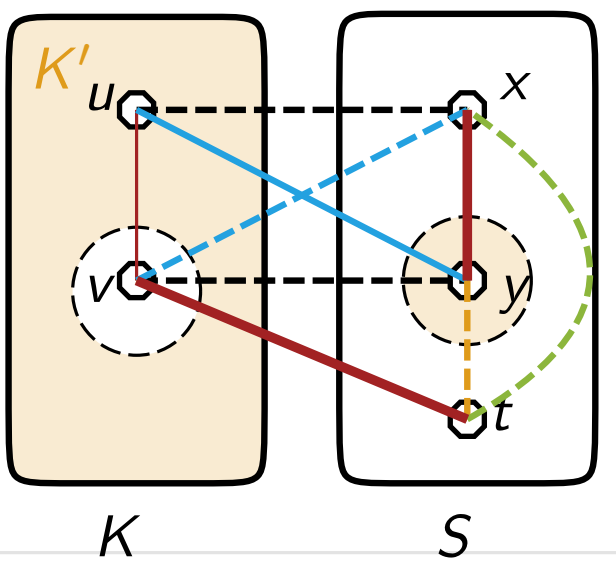
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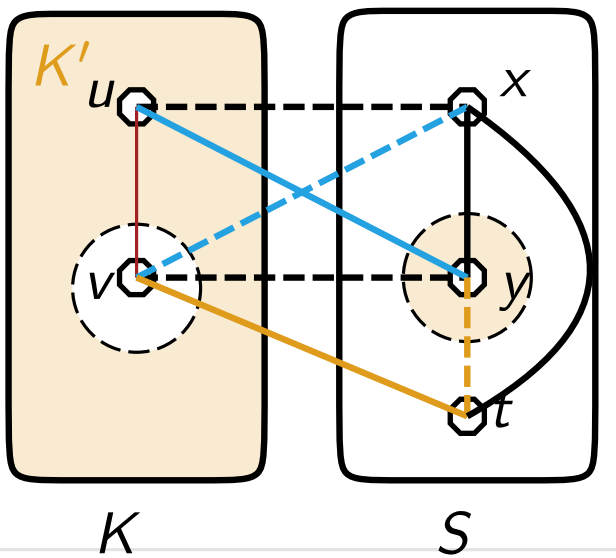
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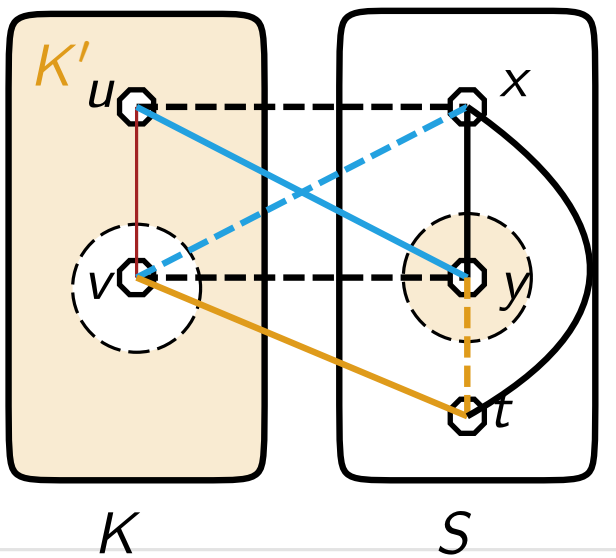
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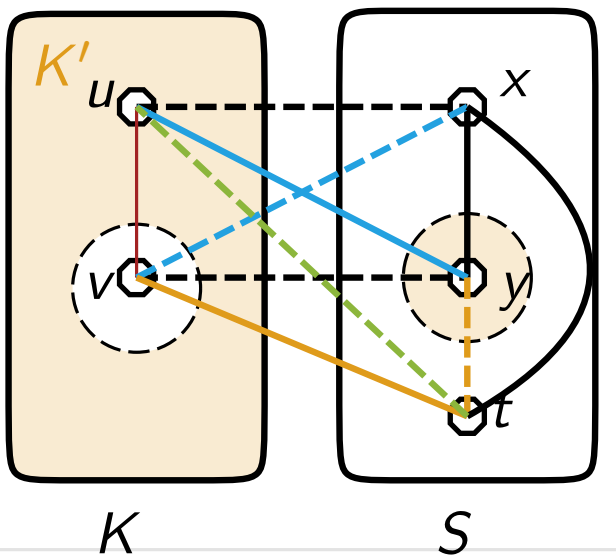
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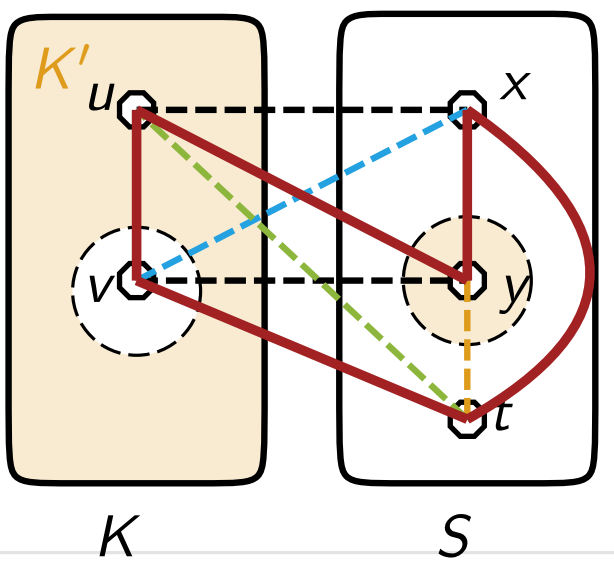
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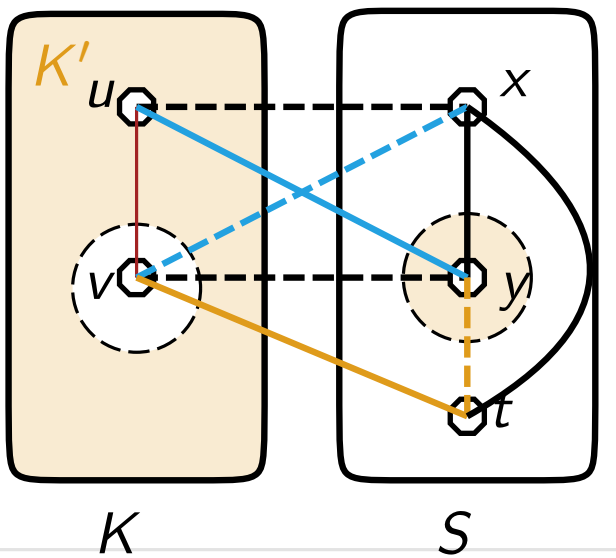
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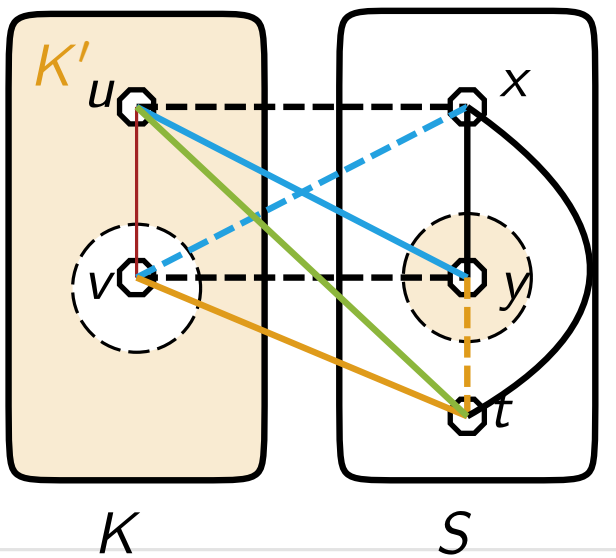
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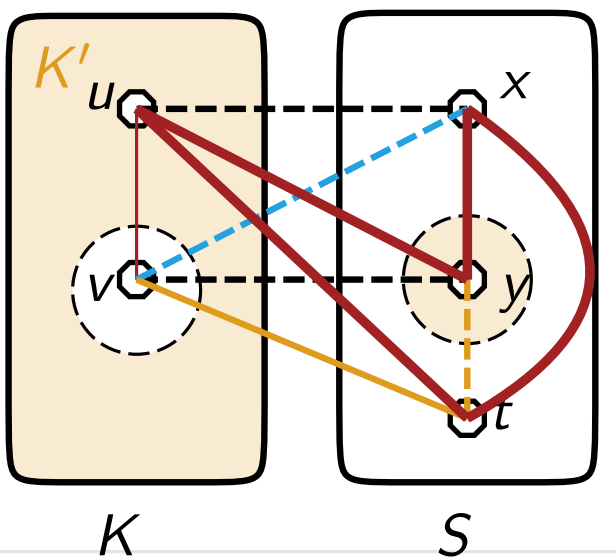
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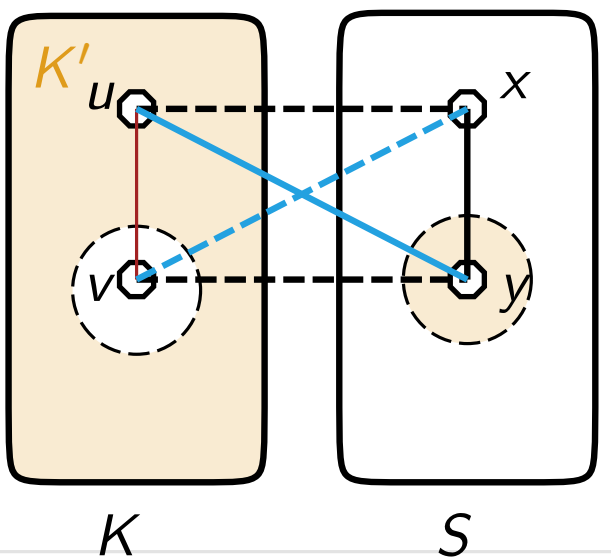
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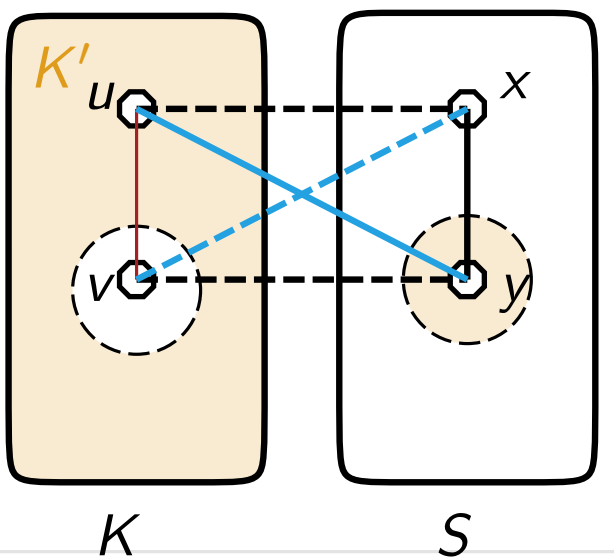
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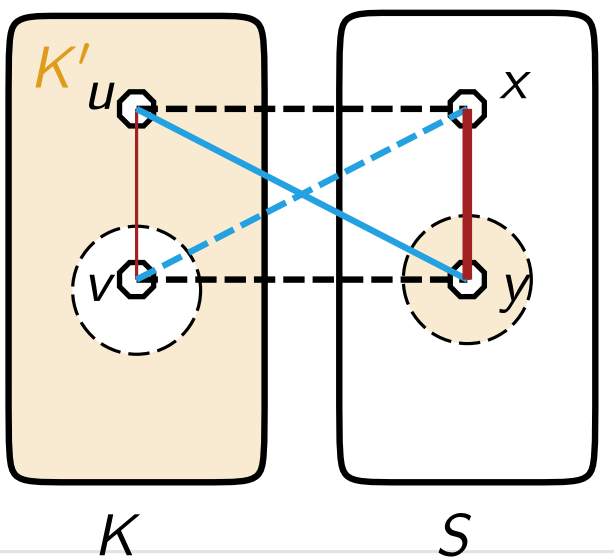
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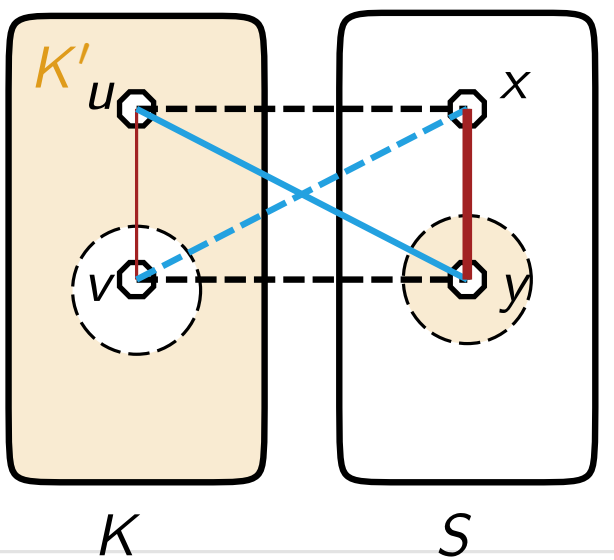
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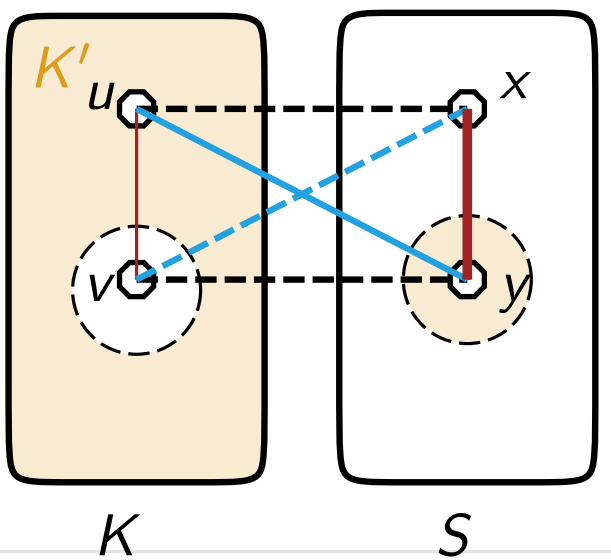
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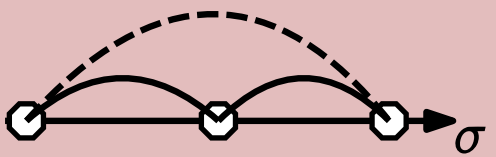
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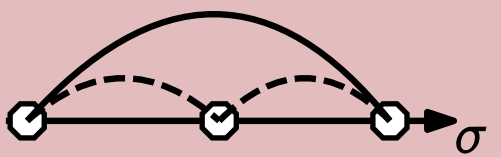
Characterization of permution graphs : Statement

Theorem

For every graph $G = (V, E)$ the following are equivalent:

- G and $\overline{G}$ are comparability graphs i.e. G permutation graph
- there exists a vertex ordering σ of G

without 

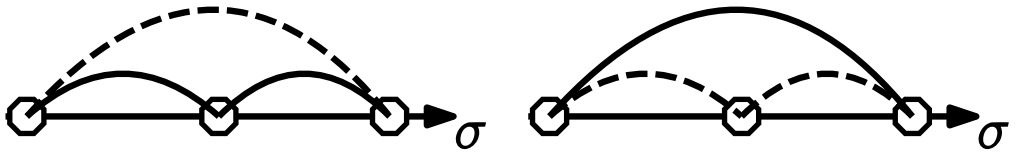
and without 

- there exists an embedding $V \to \mathbb{R}^2$ s.t.

$$uv \in E \iff [u_x < v_x \iff u_y < v_y]$$

Characterization of permutation graphs : Proof

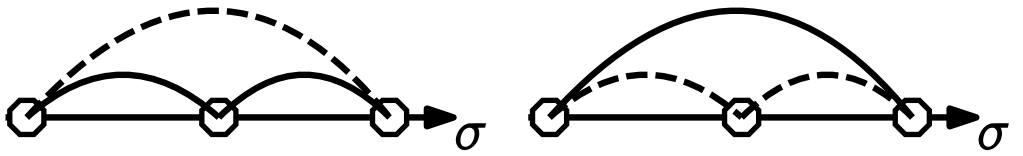
”(i) $\Longrightarrow$ (ii)” construct vtx ordering w/o



Characterization of permutation graphs : Proof

”(i) $\Longrightarrow$ (ii)” construct vtx ordering w/o

G has trans. orientation (V, F_1)

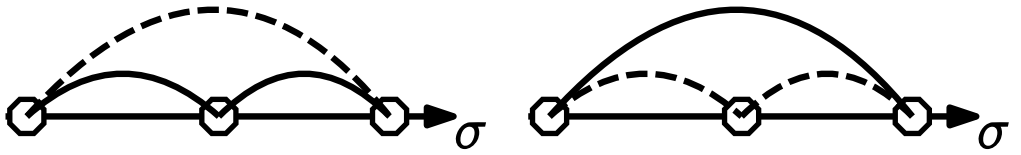


Characterization of permutation graphs : Proof

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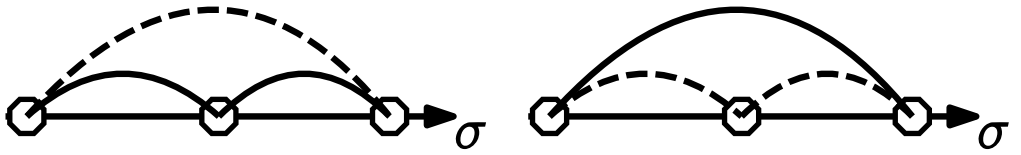
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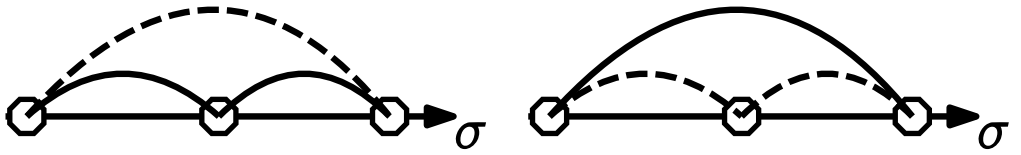


$F = F_1 + F_2$ orientation of complete graph on V

Characterization of permutation graphs : Proof

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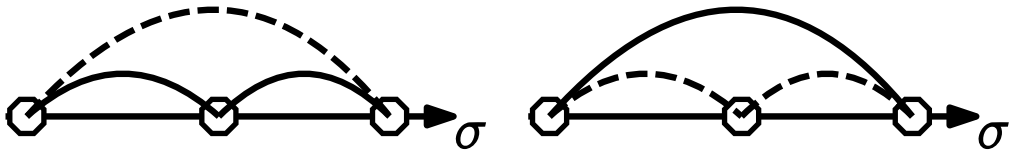


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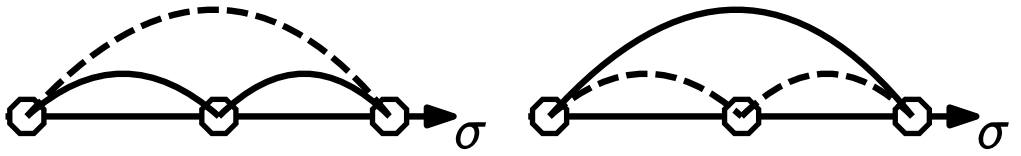
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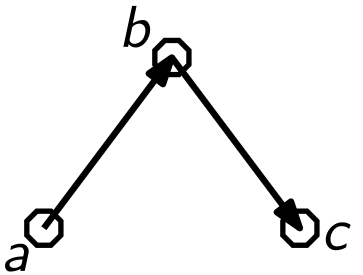


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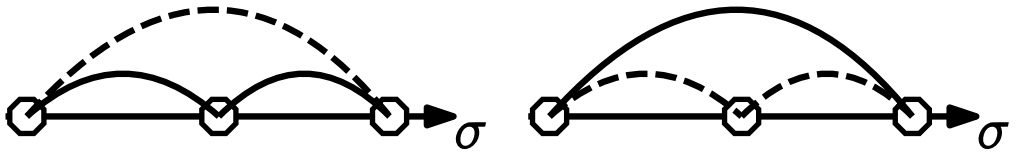
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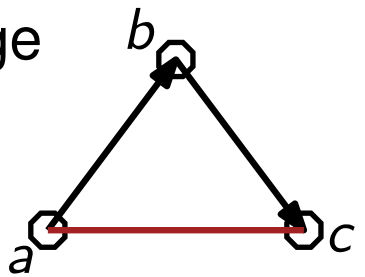
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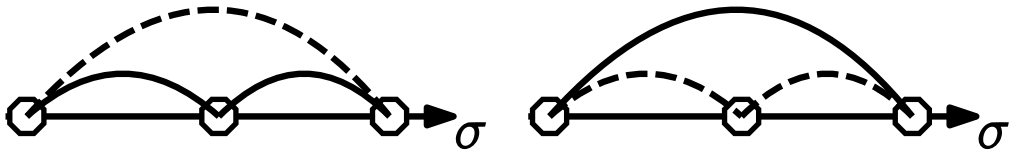
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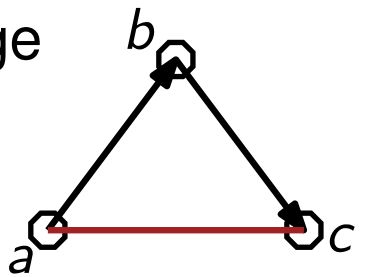
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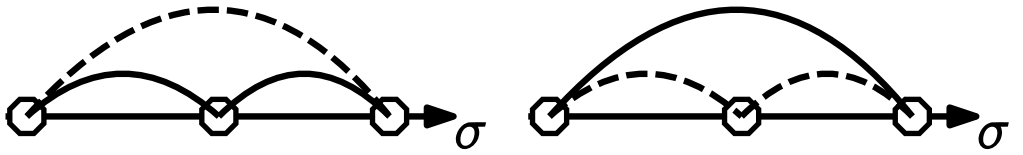
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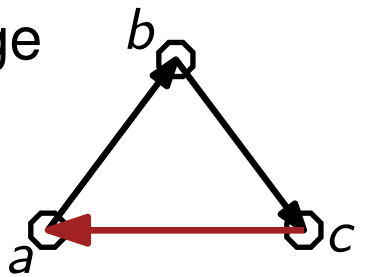
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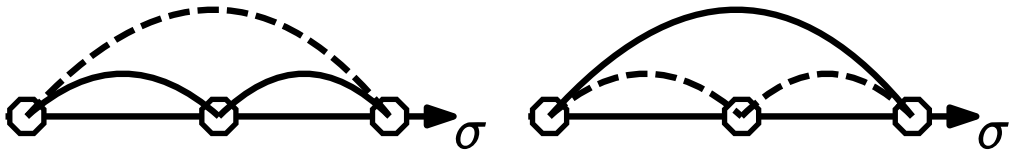
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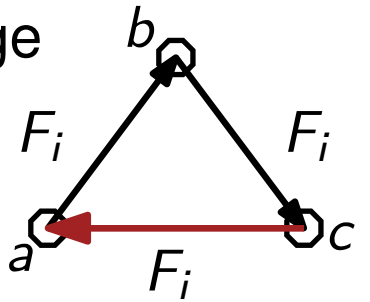
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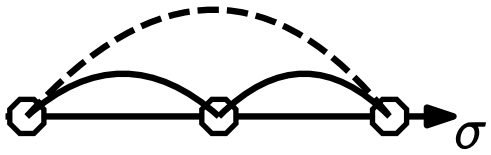
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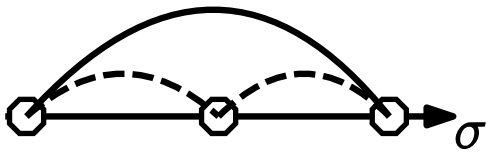
case $ab, bc, ca \in F_i$ for an $i \in \{1, 2\}$



Characterization of permutation graphs : Proof

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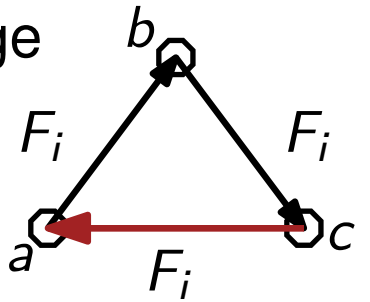
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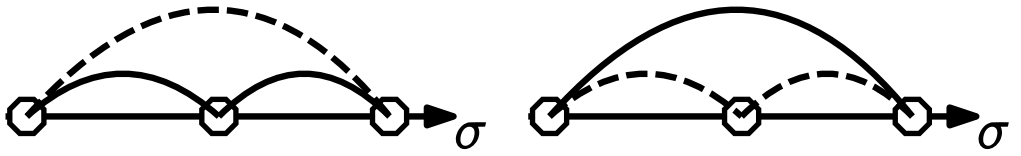
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Characterization of permutation graphs : Proof

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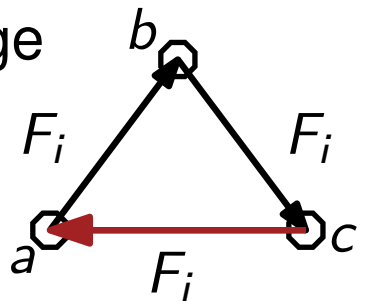
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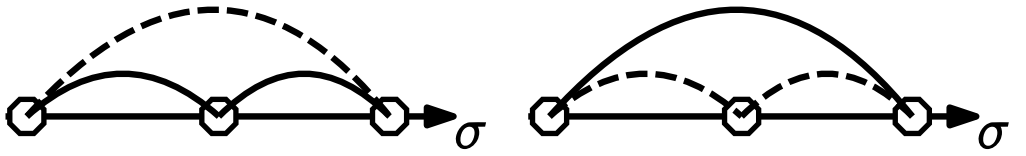
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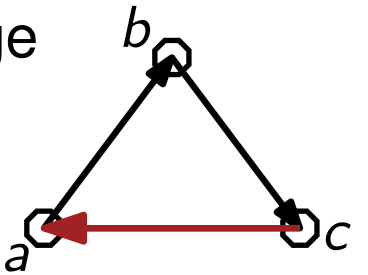
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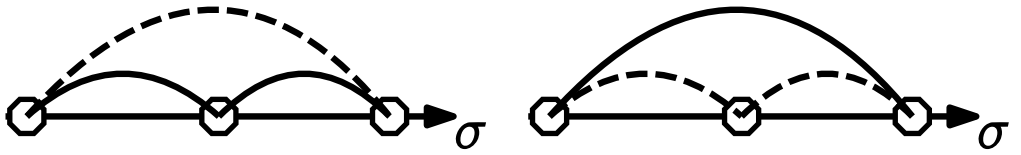
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in the other case two consecutive edges are in F_i and the other edge is in $F_j \neq F_i$

Characterization of permutation graphs : Proof

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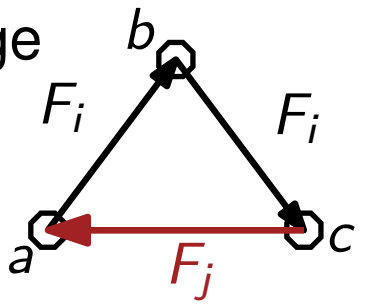
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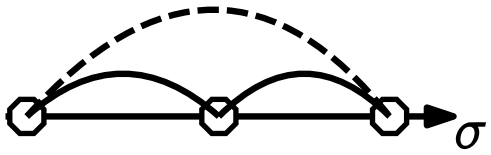
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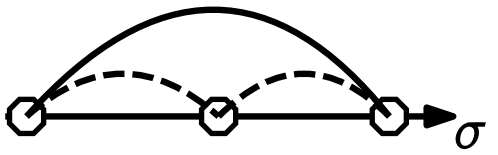
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wlog (using symmetry) $ab, bc \in F_i$ and $ca \in F_j$

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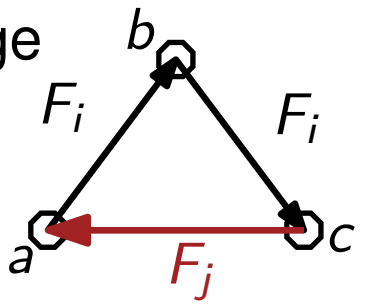
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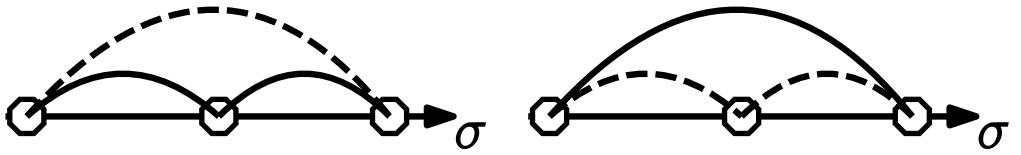
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but $ac \notin F_i$ contradicts trans. of F_i ↯

Characterization of permutation graphs : Proof

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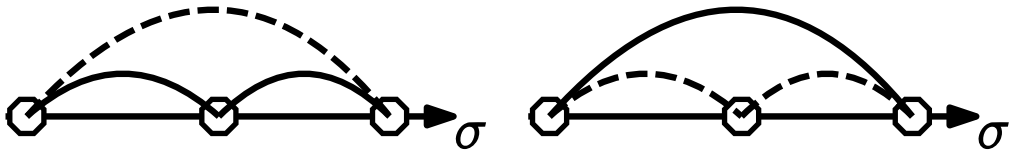


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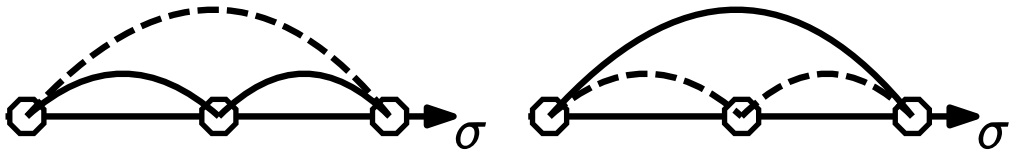
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So F is transitive
set σ to be the topological ordering of F

Characterization of permutation graphs : Proof

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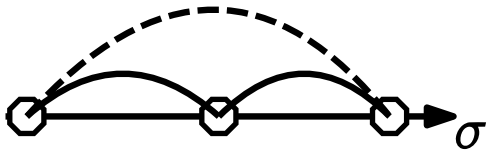
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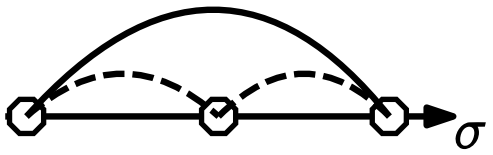
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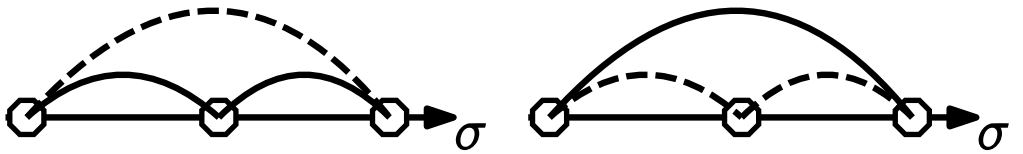


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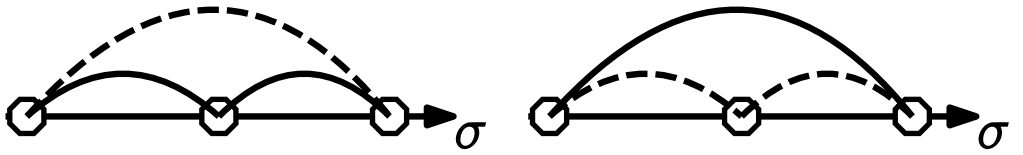
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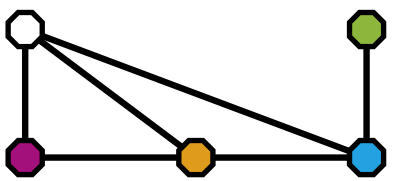
”(ii) $\Longrightarrow$ (iii)” given vtx ordering σ w/o

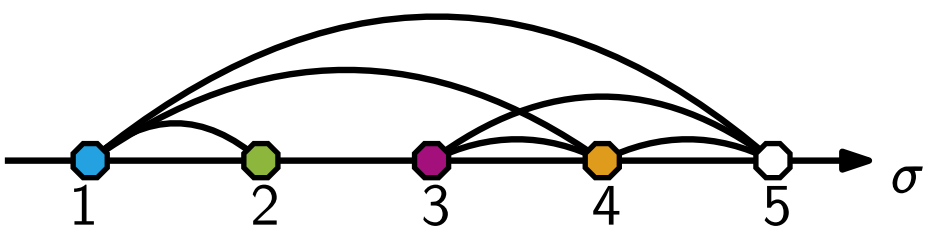


Characterization of permutation graphs : Proof

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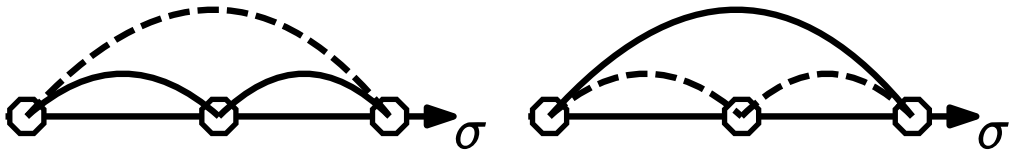




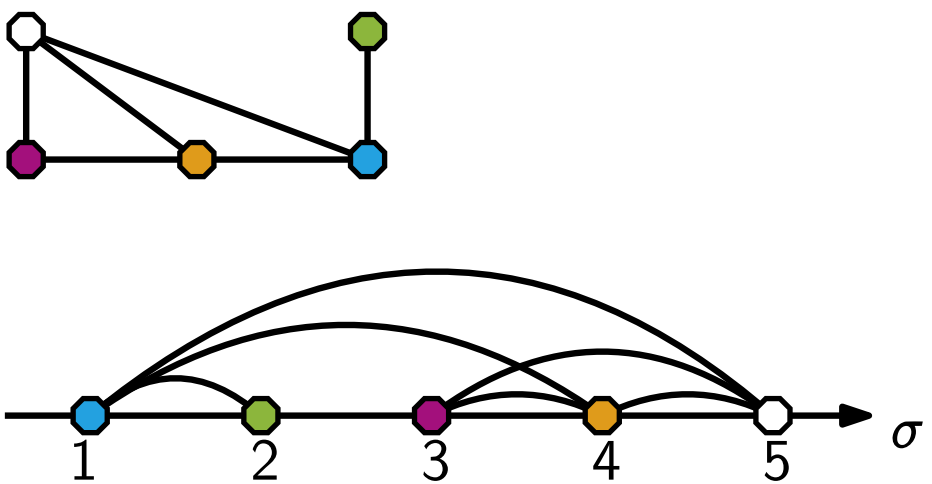


Characterization of permutation graphs : Proof

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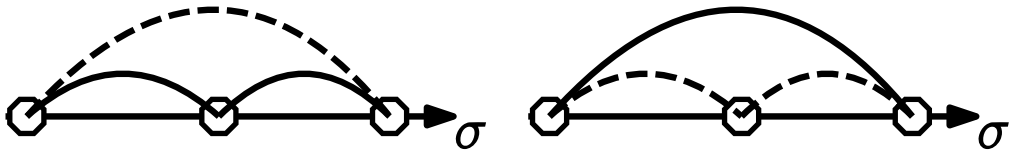


want to find embedding $V \to \mathbb{R}^2$ with $uv \in E \iff [u_x < v_x \iff u_y < v_y]$



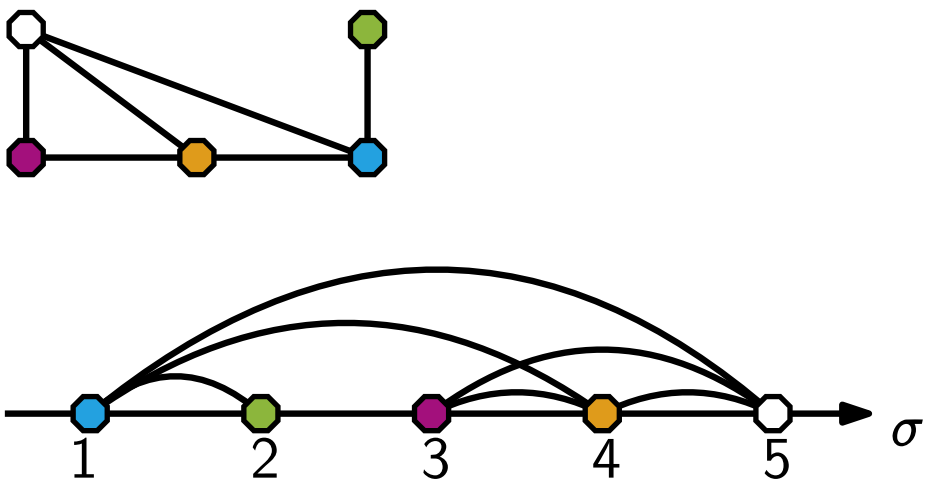
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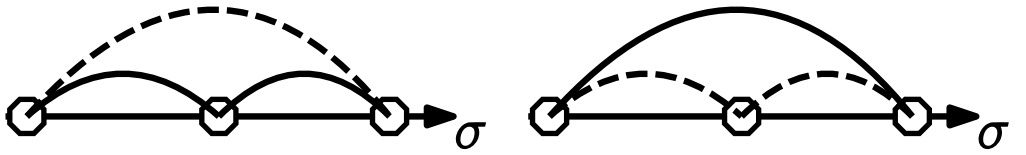
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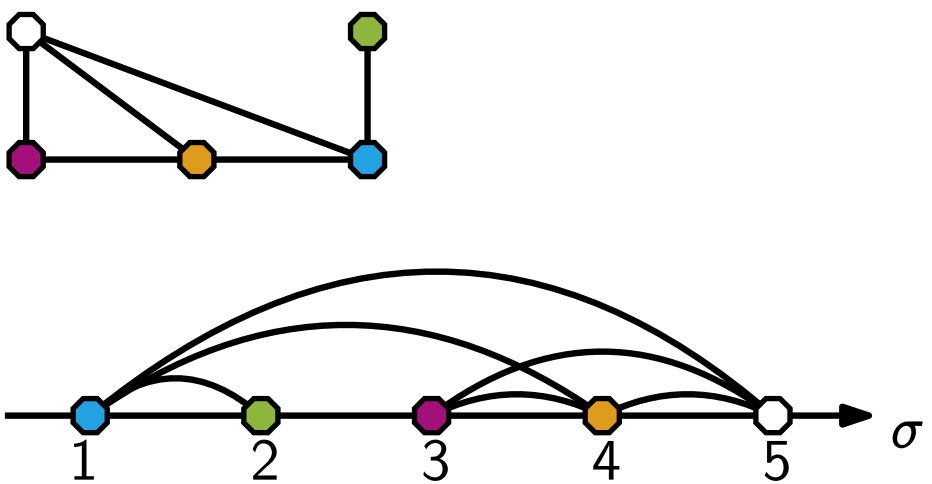
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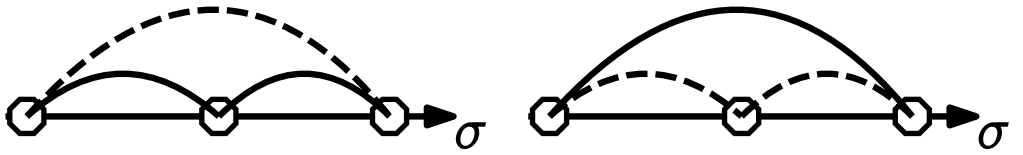
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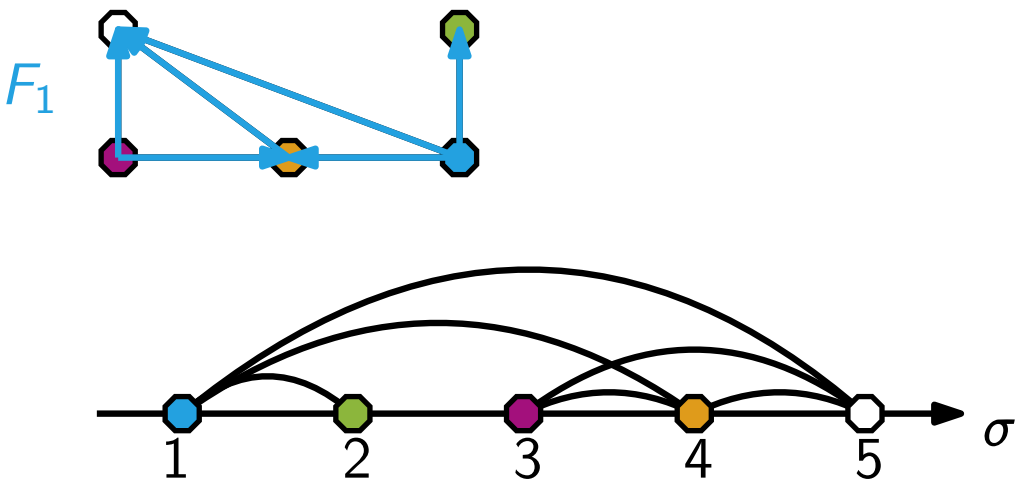
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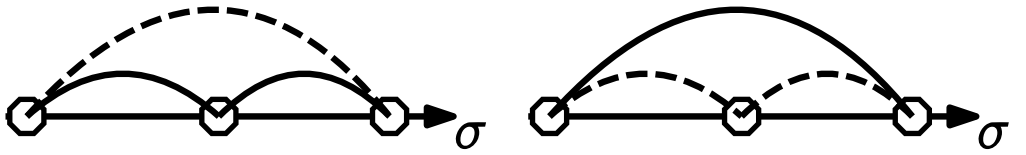
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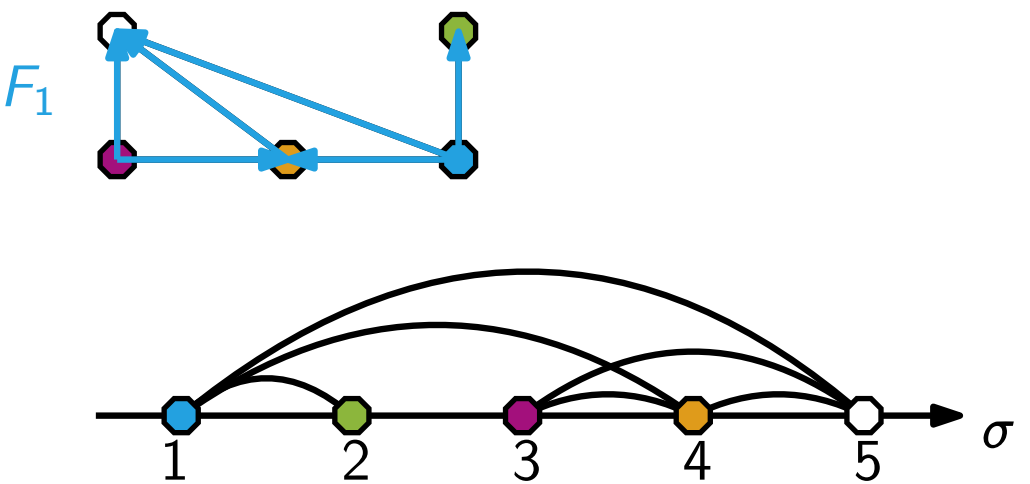


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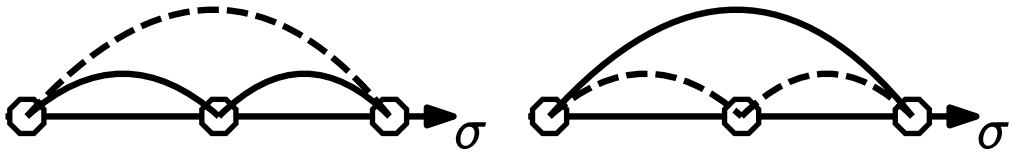
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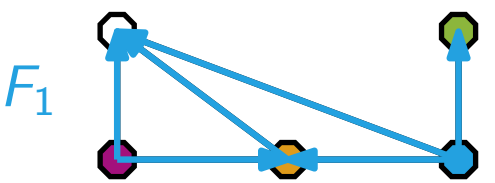


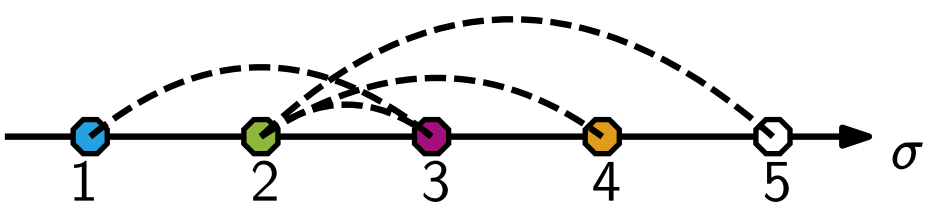
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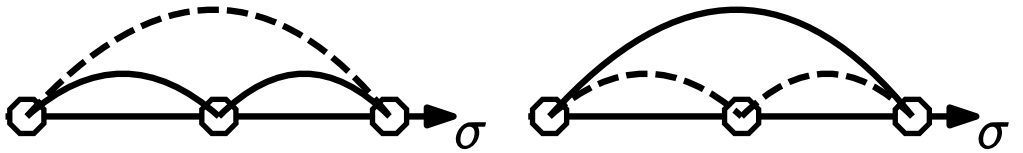
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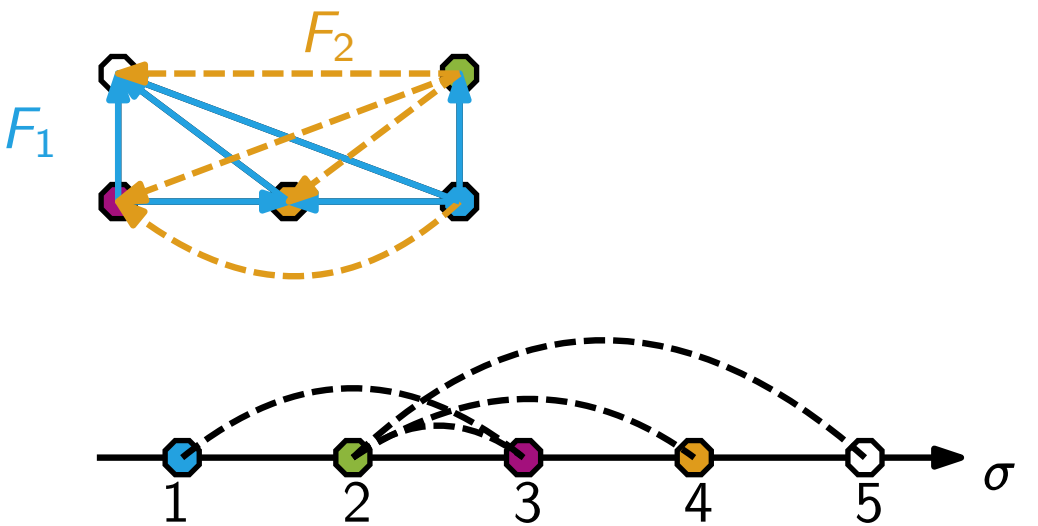


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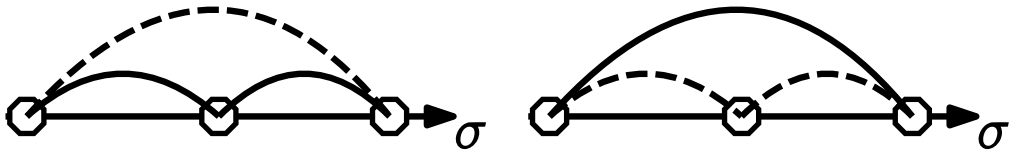
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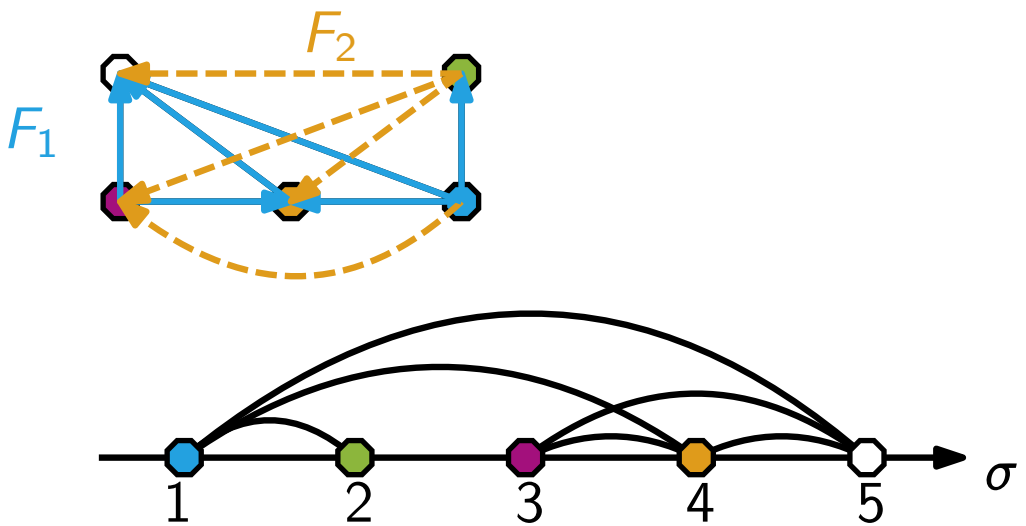
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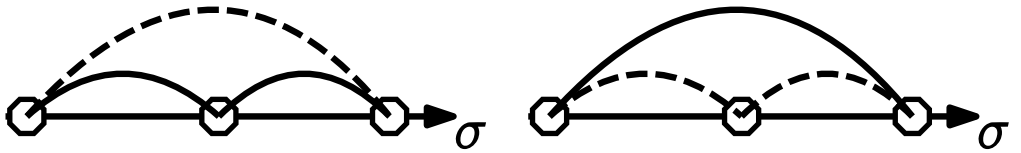
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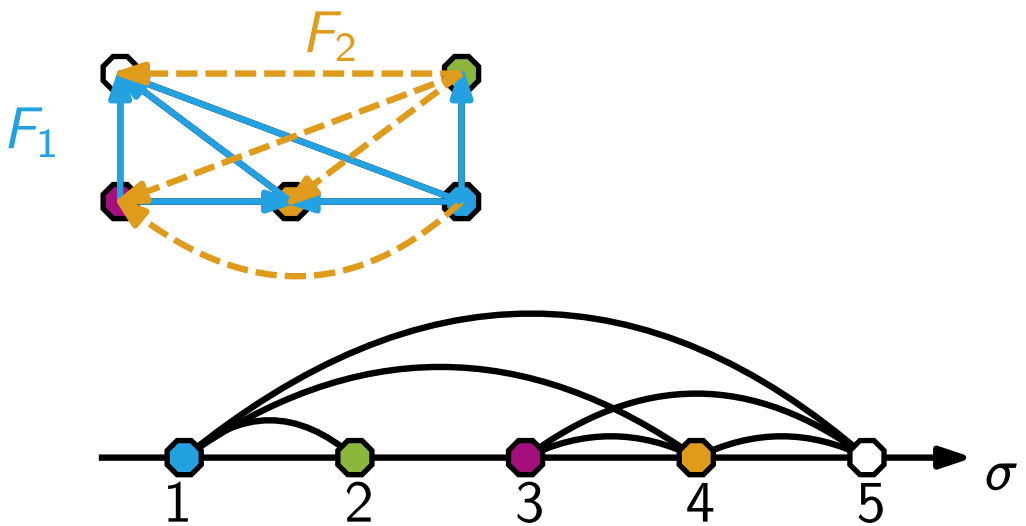
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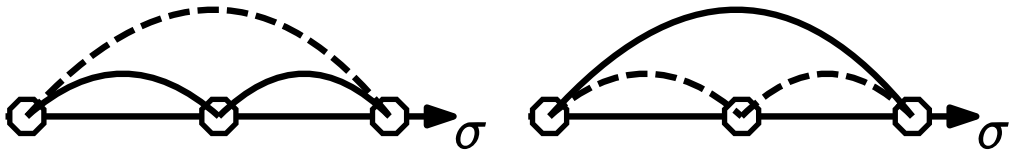
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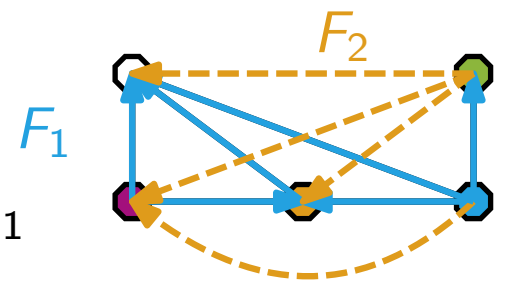
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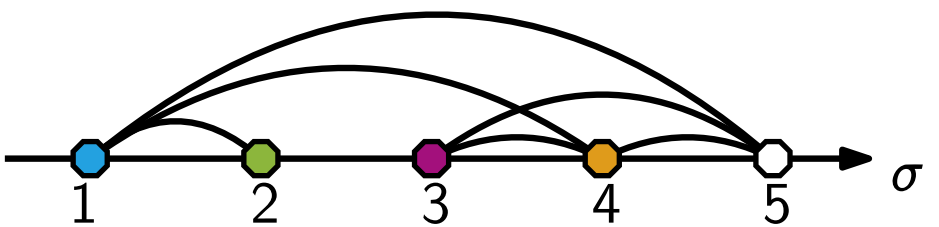
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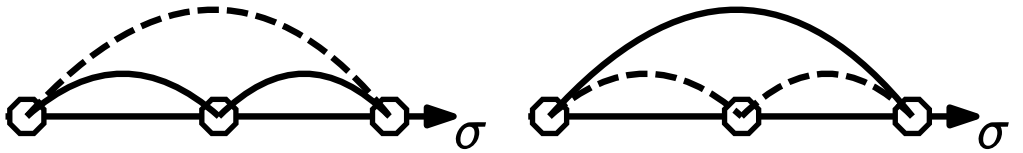
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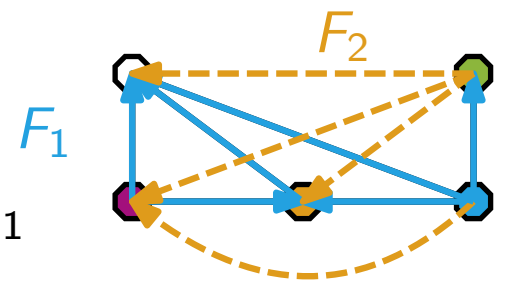
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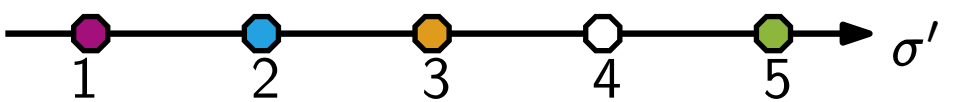
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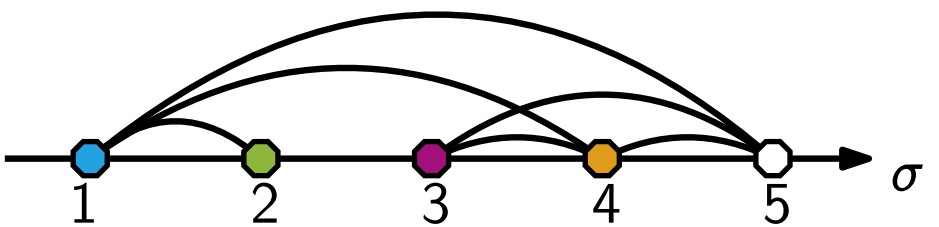
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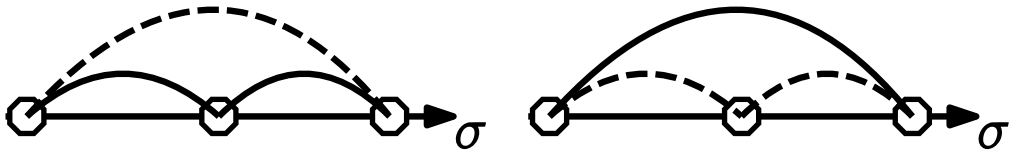






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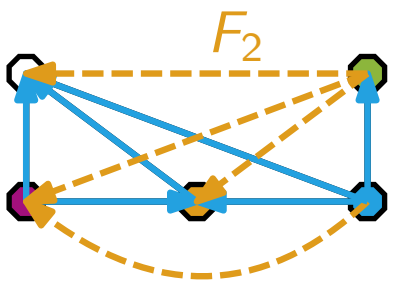
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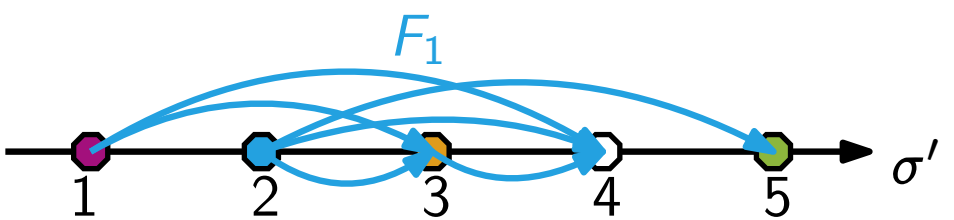
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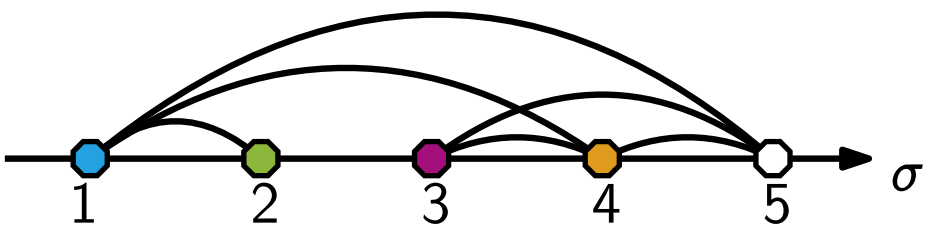
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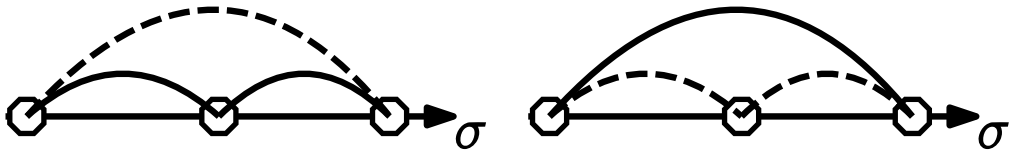






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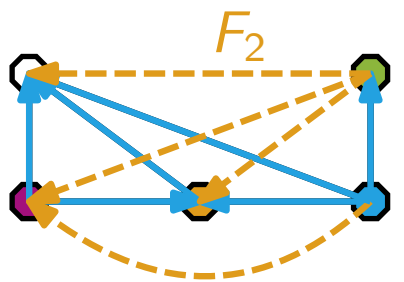
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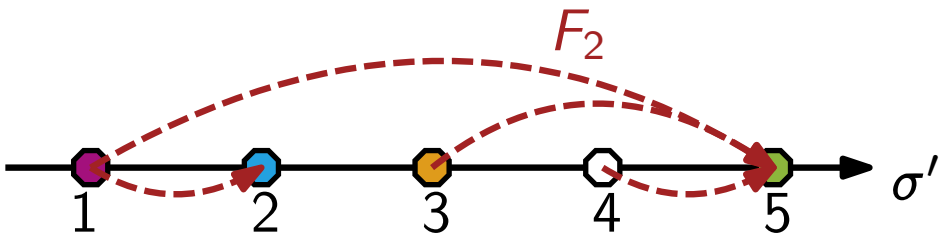
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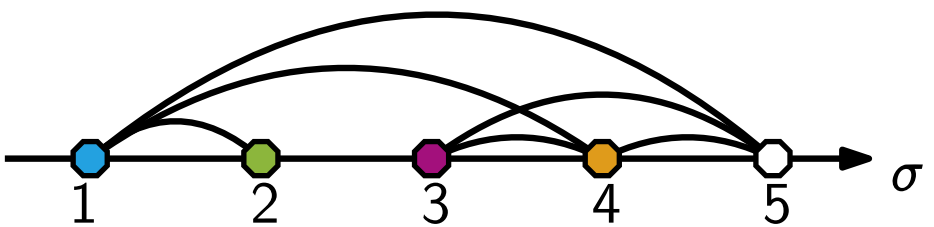
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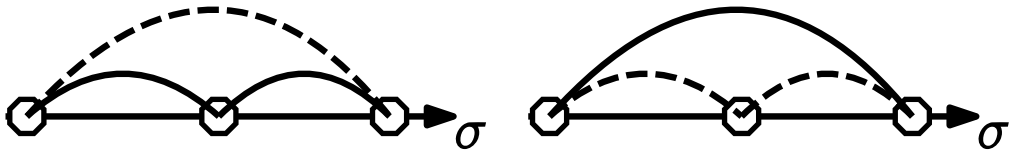






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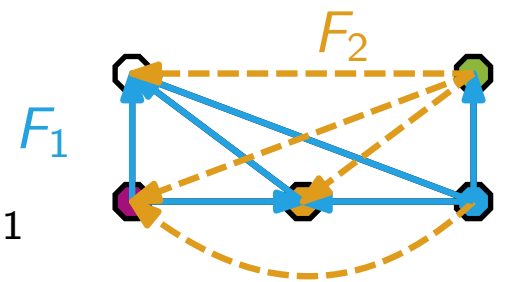
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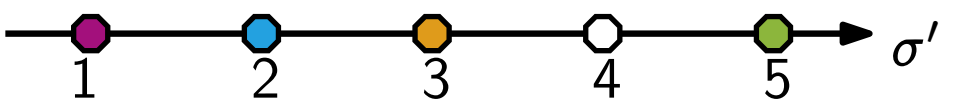
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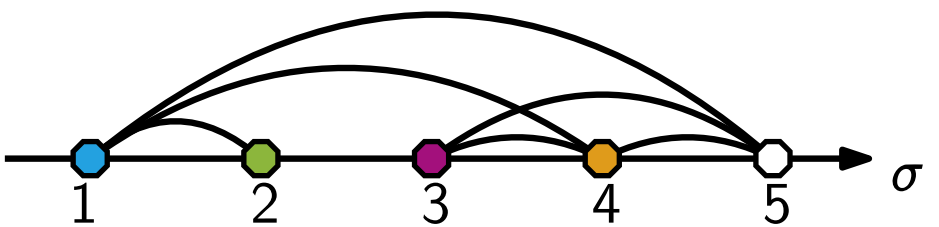
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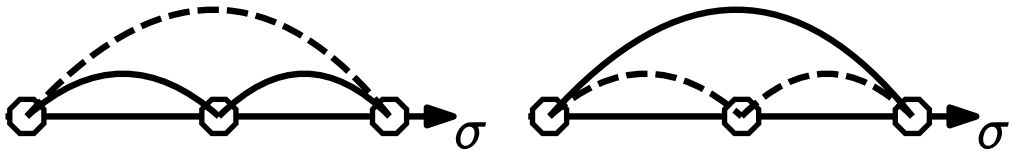






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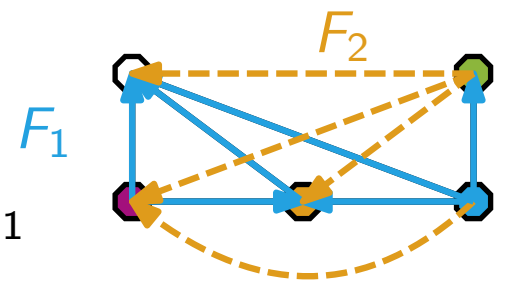
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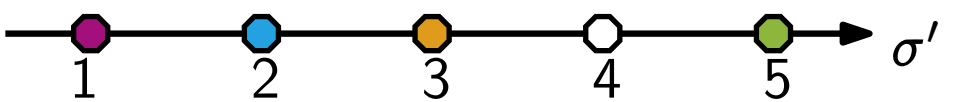
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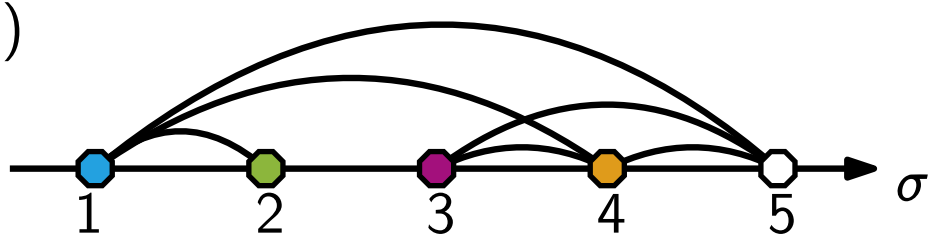
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define embedding $V \to \mathbb{R}^2, v \mapsto (\sigma(v), \sigma'(v))$

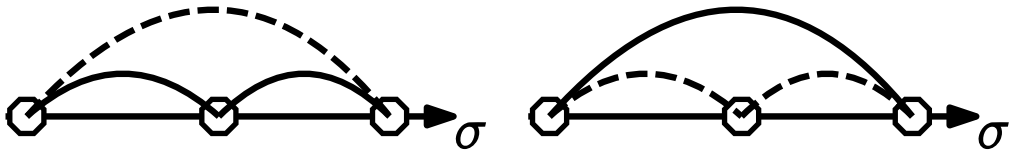






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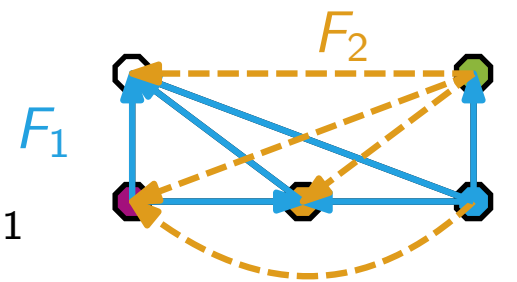
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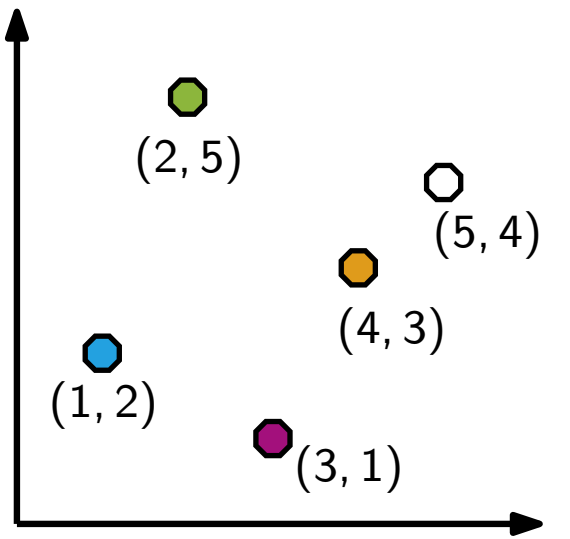
F_1 and F_2 are transitive since non-transitivity violates pattern

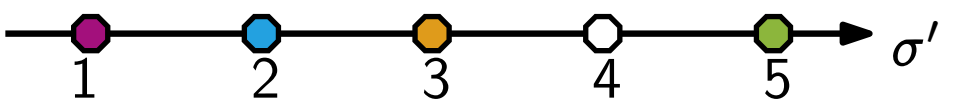
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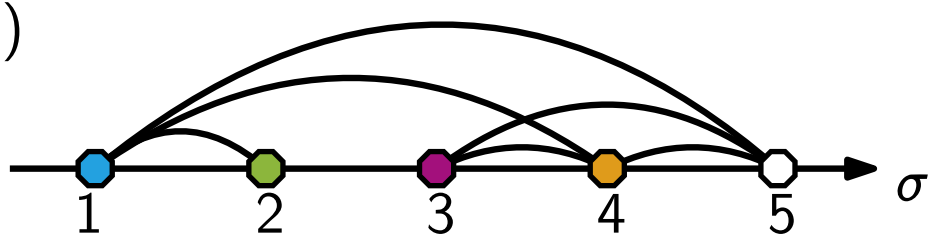
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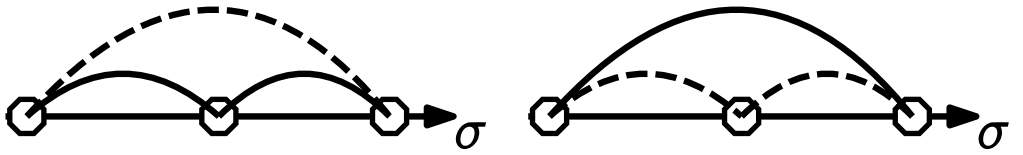






Characterization of permutation graphs : Proof

"(ii) $\Longrightarrow$ (iii)" given vtx ordering σ w/o



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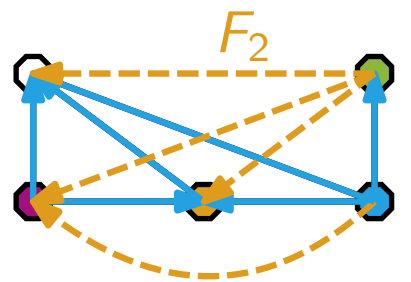
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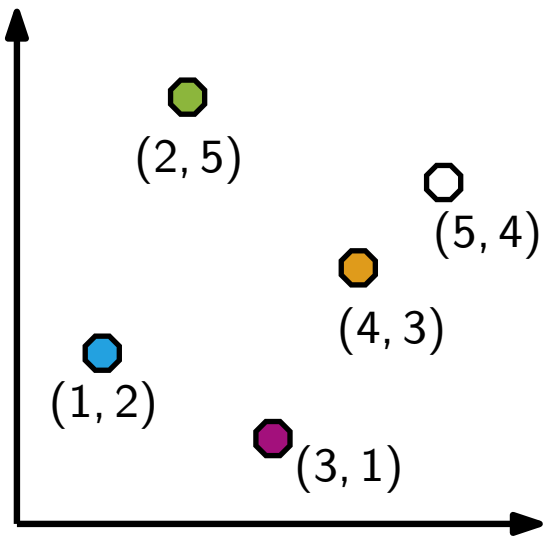
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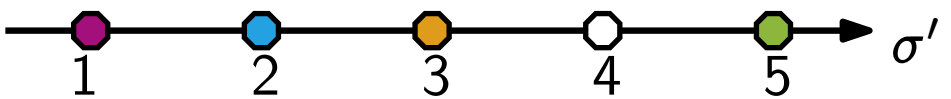
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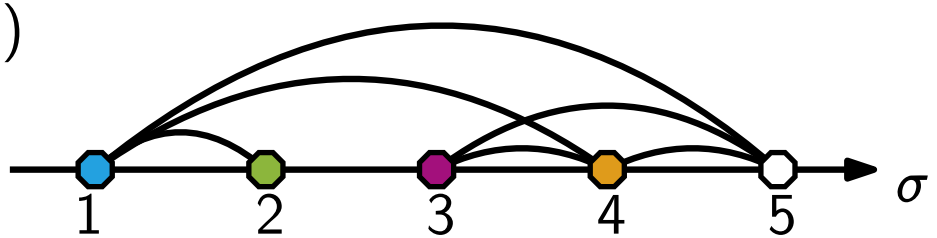
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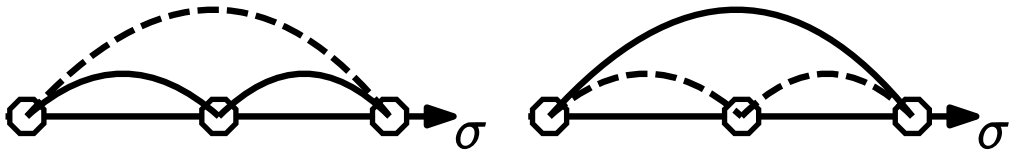






Characterization of permutation graphs : Proof

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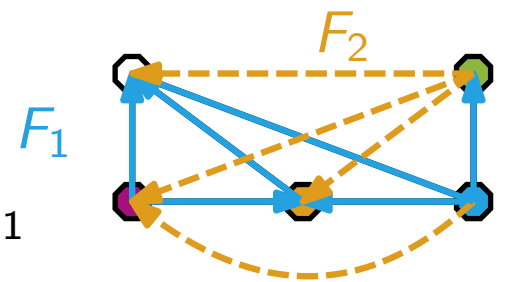
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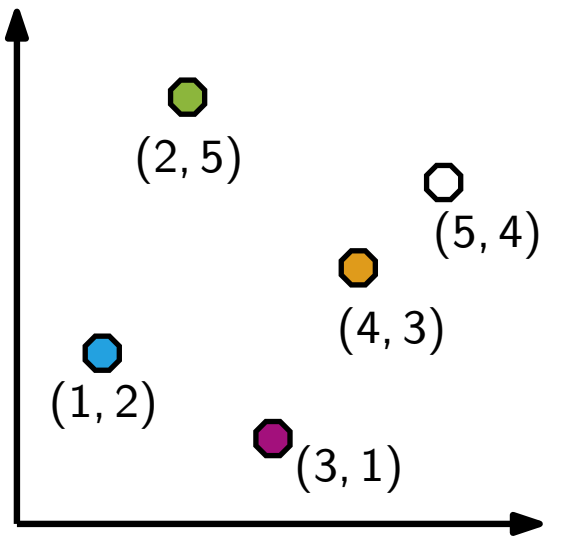
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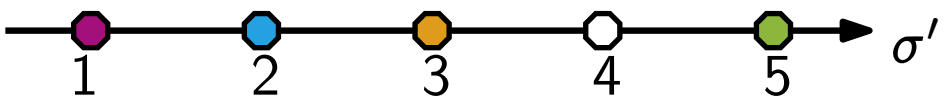
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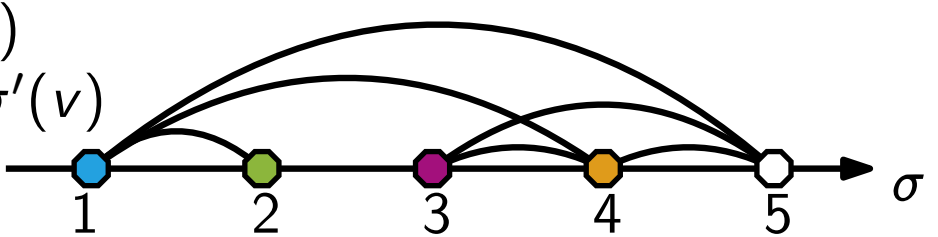
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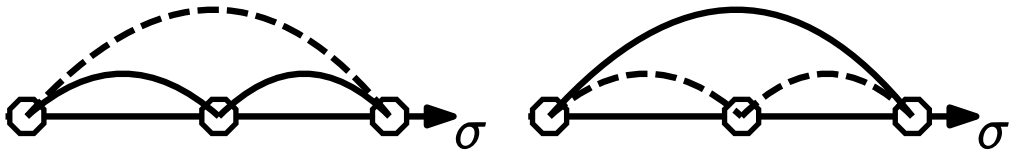






Characterization of permutation graphs : Proof

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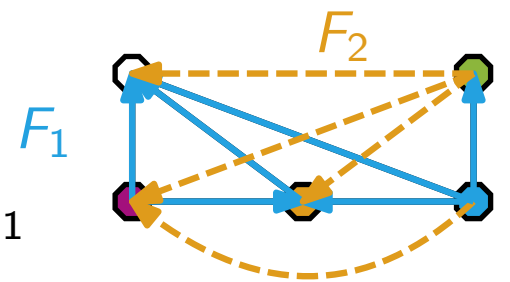
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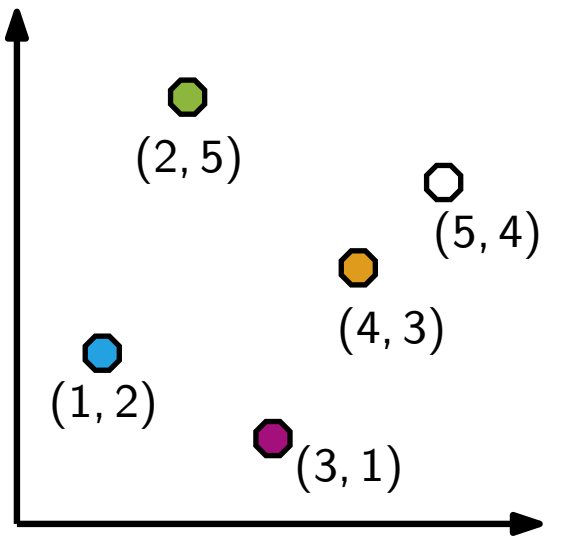
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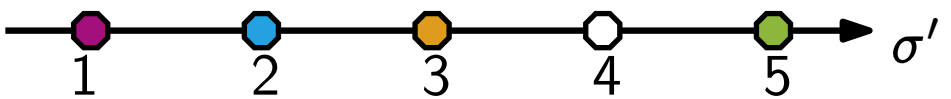
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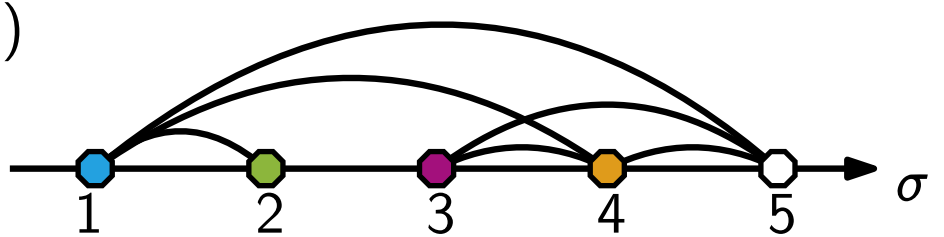
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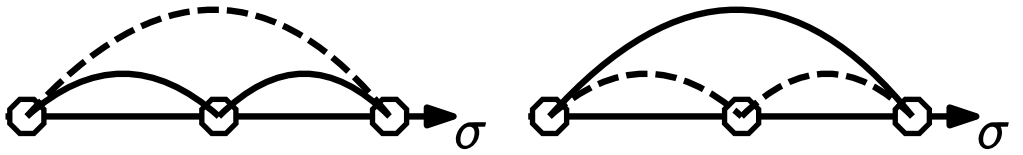






Characterization of permutation graphs : Proof

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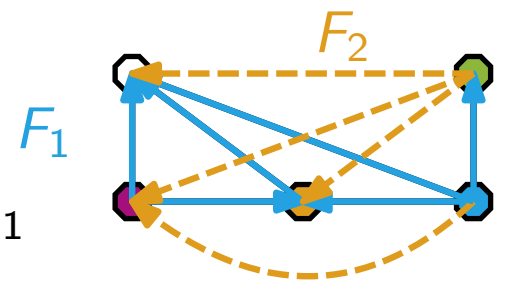
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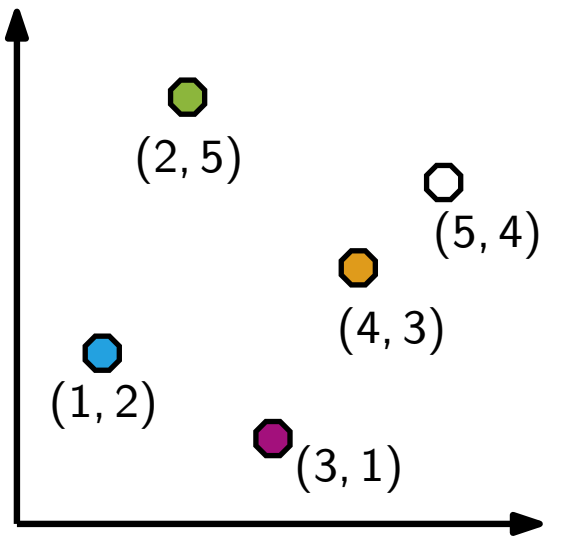
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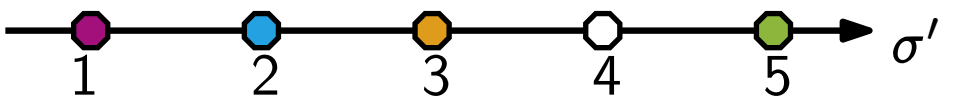
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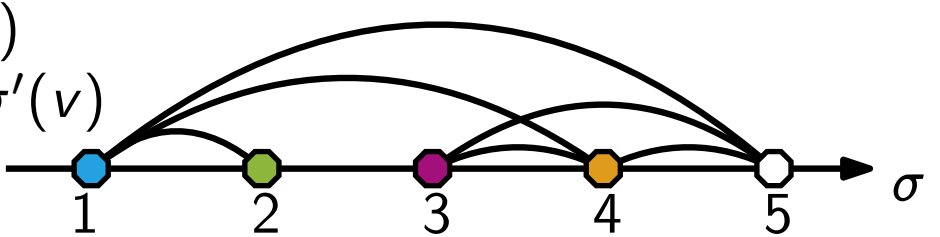
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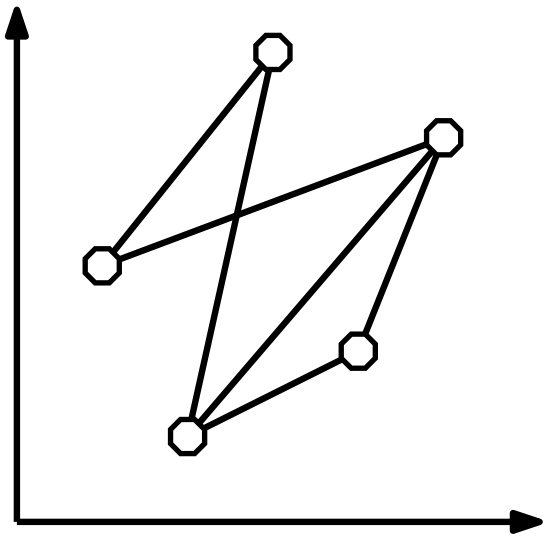






Characterization of permutation graphs : Proof

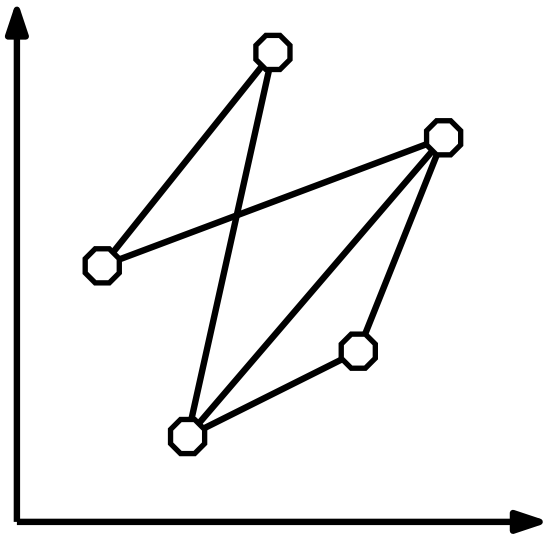
”(iii) $\Longrightarrow$ (i)” given embedding $V \to \mathbb{R}^2$ with $uv \in E \iff [u_x < v_x \iff u_y < v_y]$



Characterization of permutation graphs : Proof

”(iii) $\Longrightarrow$ (i)” given embedding $V \to \mathbb{R}^2$ with $uv \in E \iff [u_x < v_x \iff u_y < v_y]$

find trans. or. of G and $\overline{G}$

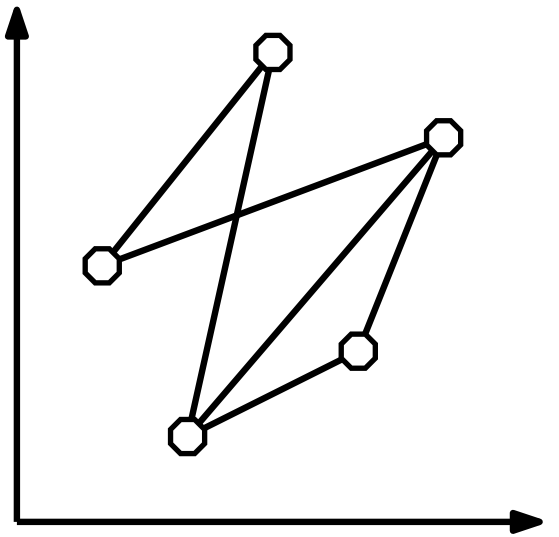


Characterization of permutation graphs : Proof

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orientate $uv \in E$ from u to v in F_1 : $\Longleftrightarrow$



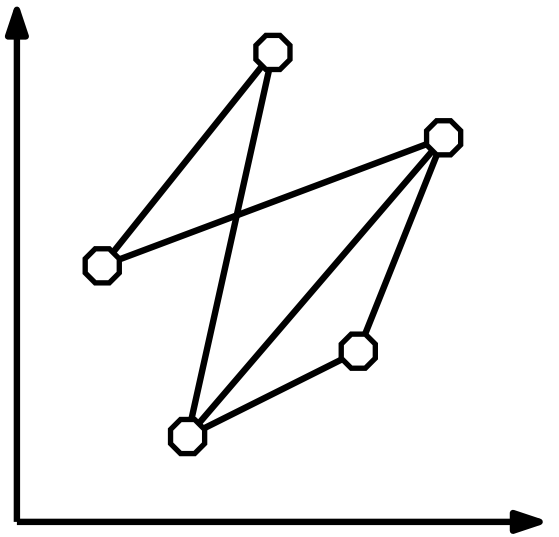
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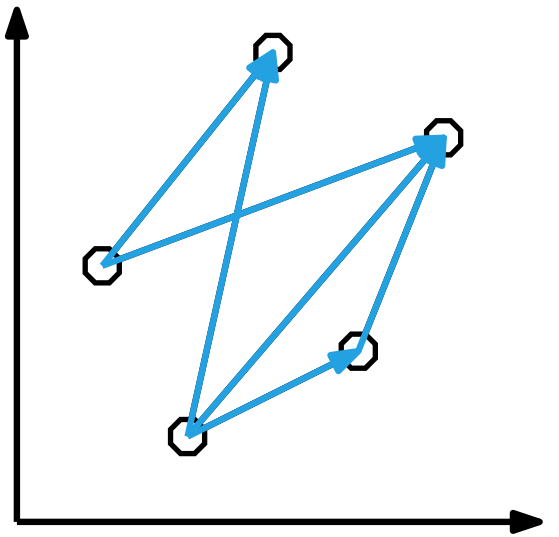
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Characterization of permutation graphs : Proof

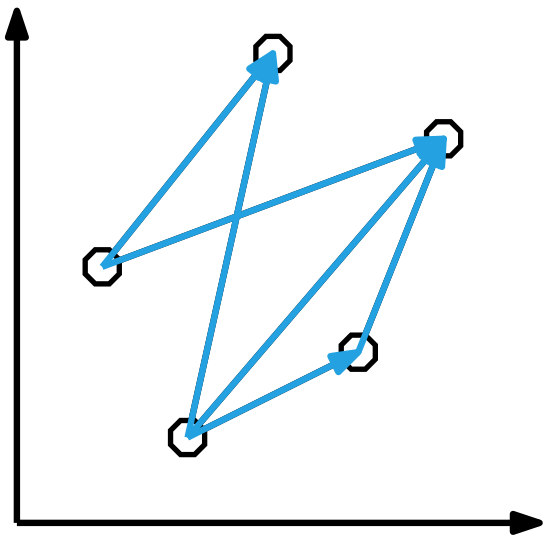
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bottom left relation is transitive $\Longrightarrow$ G comp. graph



Characterization of permutation graphs : Proof

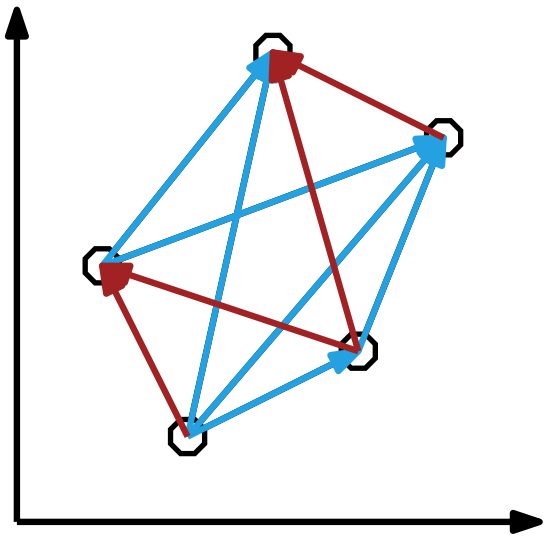
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orientate $uv \in E$ from u to v in F_1 : $\iff$ (F_2)

u is on the bottom left of v ($\iff$ right) $\iff u_x \leq v_x$ and $u_y \leq v_y$

bottom left relation is transitive ($\Longrightarrow$ right) G ($\overline{G}$) comp. graph



Characterization of intervall graphs: Statement

Theorem

For every graph $G = (V, E)$ the following are equivalent:

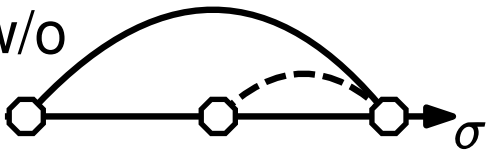
- G is an intervall graph
- there exists a vertex ordering σ of G without



- G is chordal and $\overline{G}$ is comparability graph
- $C_4 \not\subseteq_{\text{ind}} G$ and $\overline{G}$ is comparability graph
- There exists an ordering $A_1, \ldots, A_x$ of the inclusion-maximal cliques in G s.t. $\forall v \in V$ the numbers in $\{i \mid v \in A_i\}$ are consecutive in $\{1, \ldots, x\}$ i.e. an intervall of $\{1, \ldots, x\}$

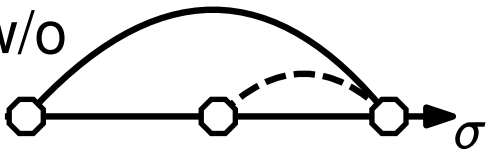
Characterization of intervall graphs: Proof

”(i) $\Longrightarrow$ (ii)” given G intervall graph, construct vtx ordering σ w/o



Characterization of intervall graphs: Proof

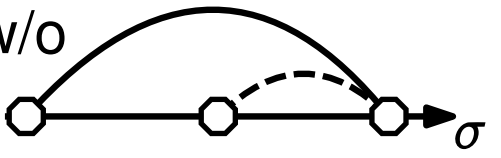
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obtain intervall representation $I_v = [v_l, v_r]$ for $v \in V$

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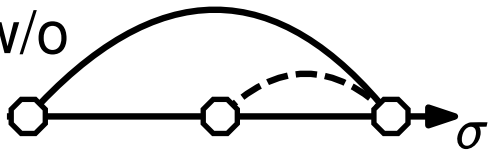
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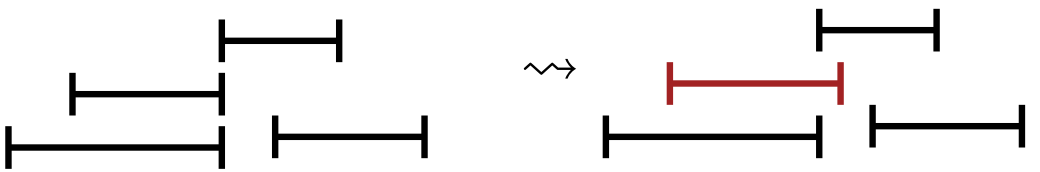
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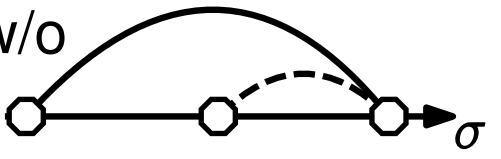
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we can ”nudge” intervall endpoints a bit to the right



Characterization of intervall graphs: Proof

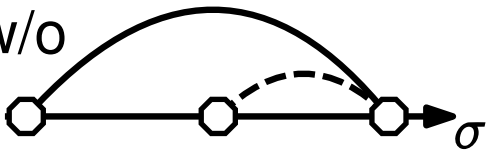
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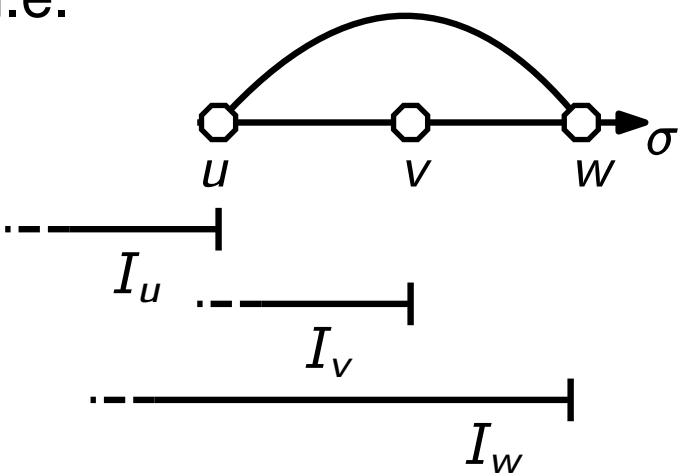
obtain intervall representation $I_v = [v_l, v_r]$ for $v \in V$
wlog all endpoints v_r are distinct
define σ as the left to right ordering of endpoints i.e.
$u <_\sigma v \iff u_r < v_r$

Characterization of intervall graphs: Proof

”(i) $\Longrightarrow$ (ii)” given G intervall graph, construct vtx ordering σ w/o

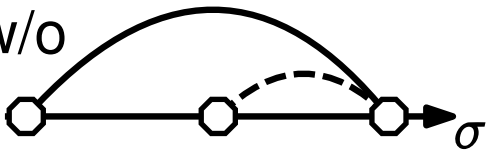


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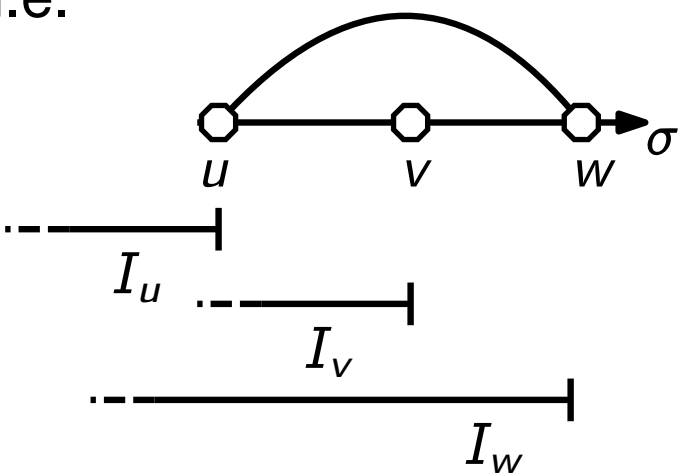


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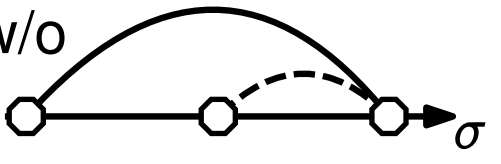


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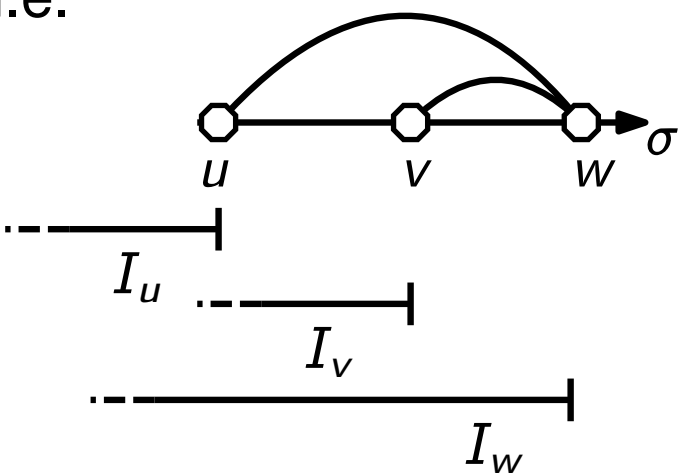


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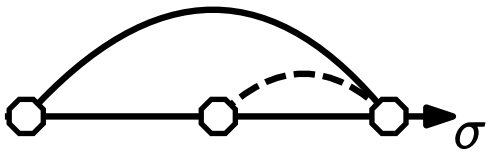


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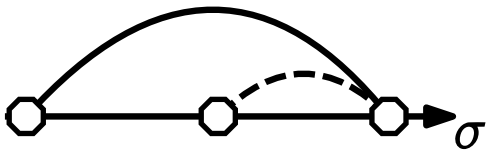
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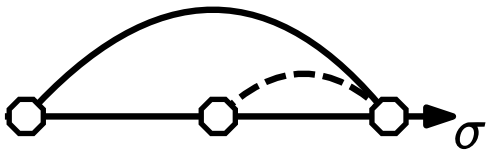
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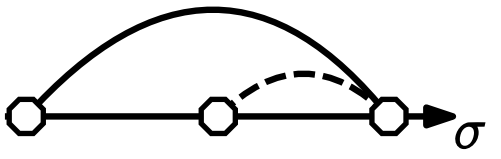


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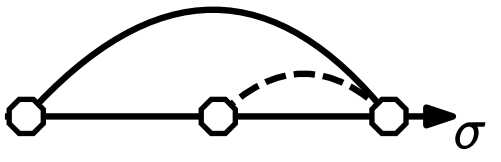


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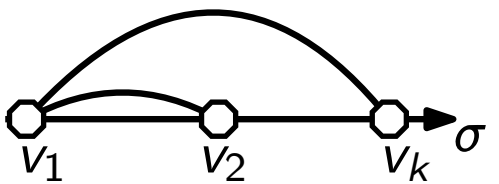
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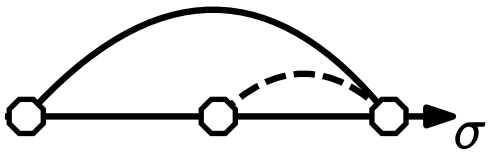
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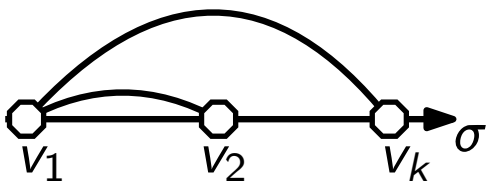
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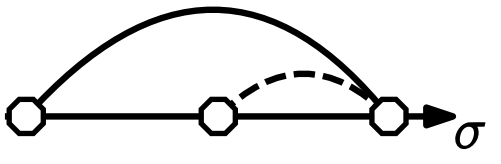
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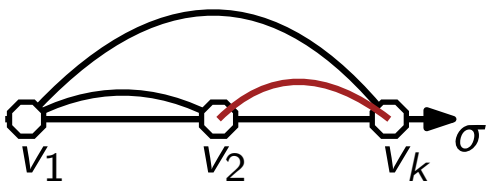
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by vertex ordering $v_2 v_k \in E$ i.e. a chord

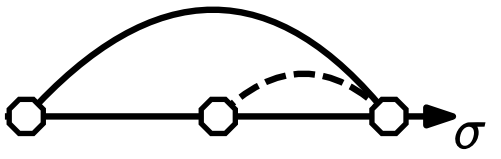


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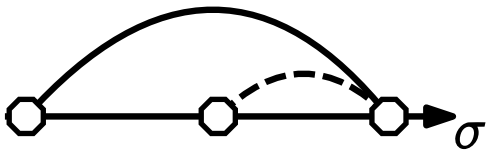
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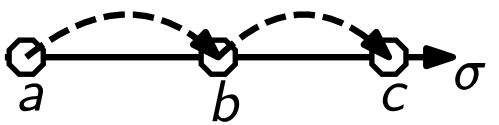
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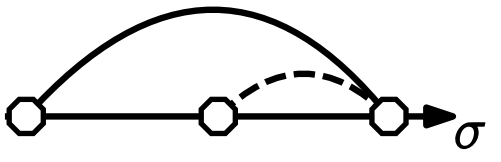
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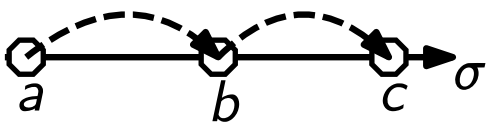


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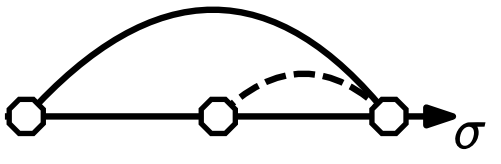
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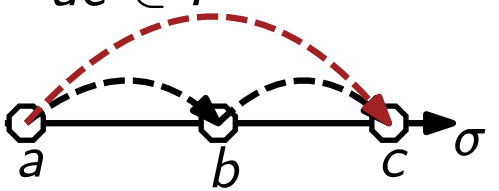


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Characterization of intervall graphs: Proof

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interesting because it is sufficient

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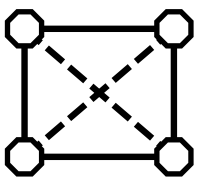
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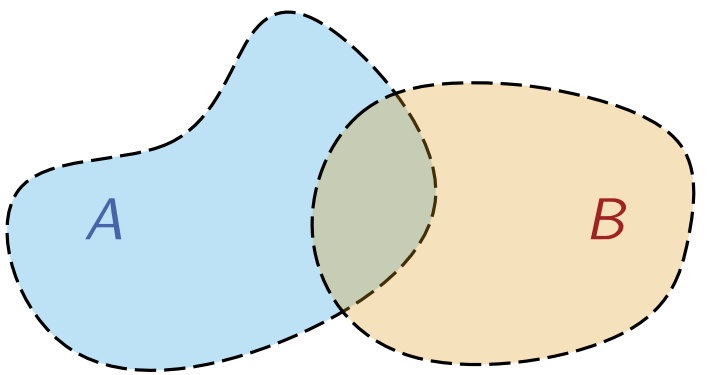
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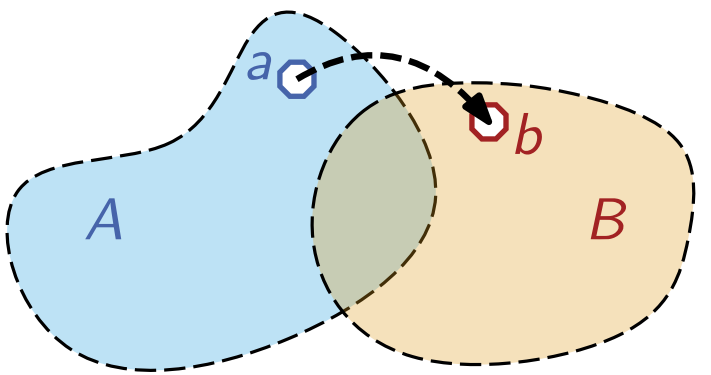
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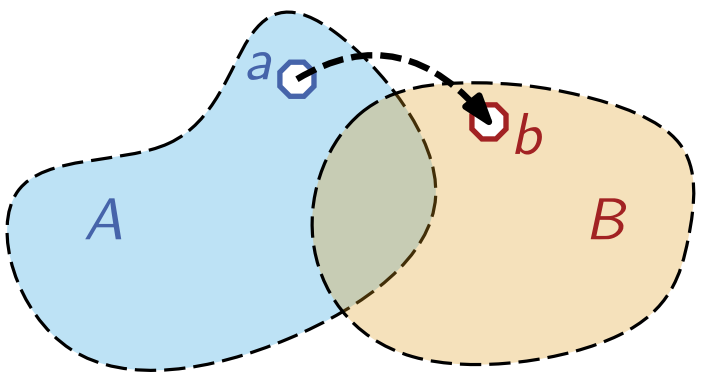
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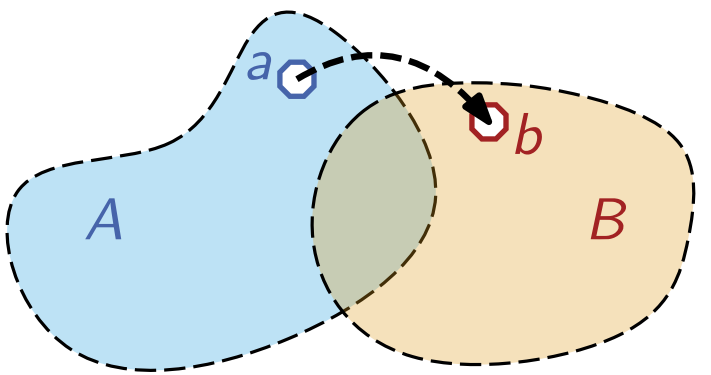
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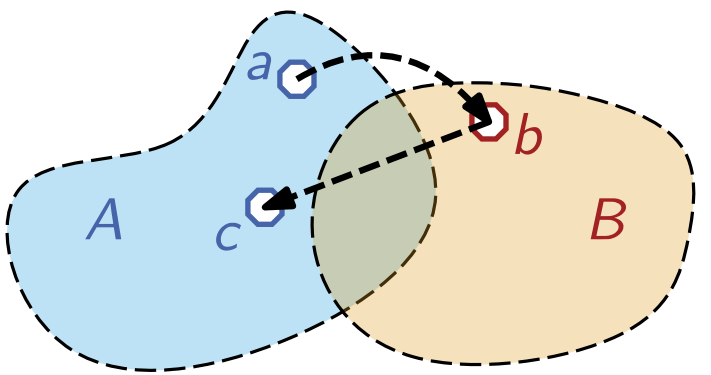
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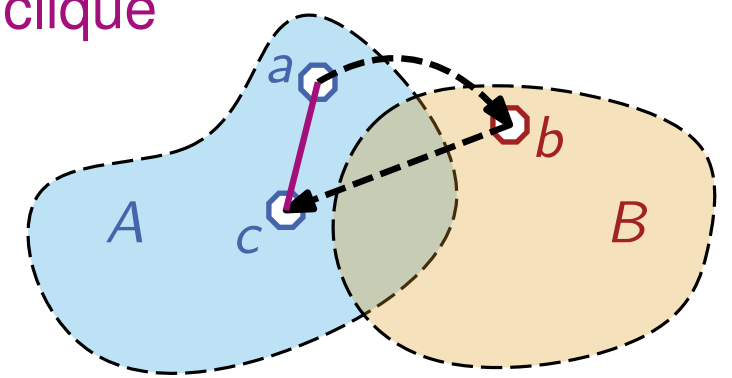
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by A clique

first case:



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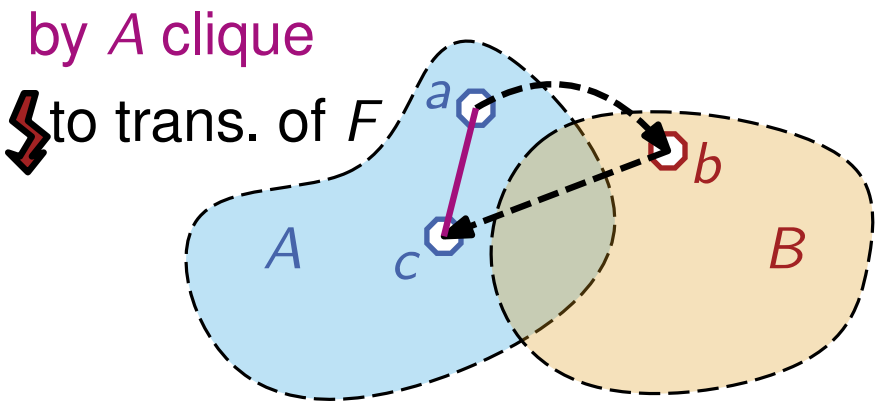
Let F be a trans. orientation of $\overline{G}$
we define order between incl.-maximal cliques $A \neq B$: $A < B \iff$
there is a non-edge $ab \in F$ between $A - B$ and $B - A$

$ab \notin E$ between $A - B$ and $B - A$ always exists since both are maximal

we will now show that $<$ is well-defined

assume $A < B$ and $B < A$. This amounts to 3 cases (see sketch)

first case:



Characterization of intervall graphs: Proof

"(iv) $\Longrightarrow$ (v)" given $C_4 \not\subseteq_{\text{ind}} G$ and $\overline{G}$ comp. graph
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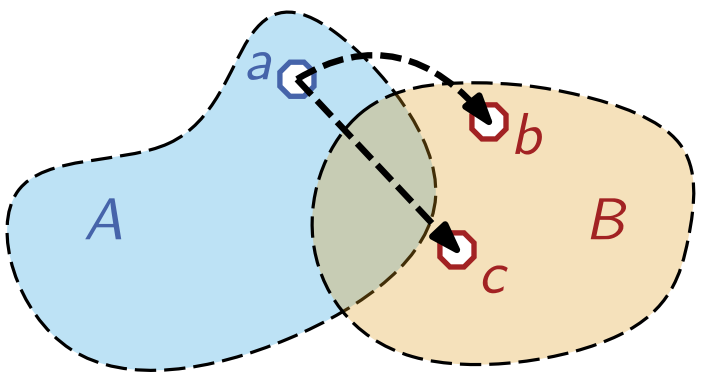
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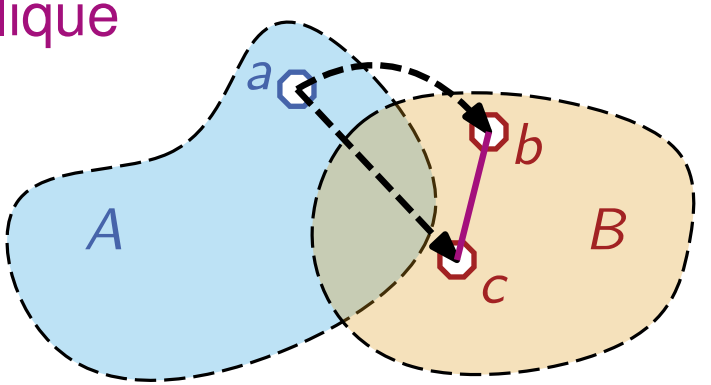
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we will now show that $<$ is well-defined

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by B clique

second case:



Characterization of intervall graphs: Proof

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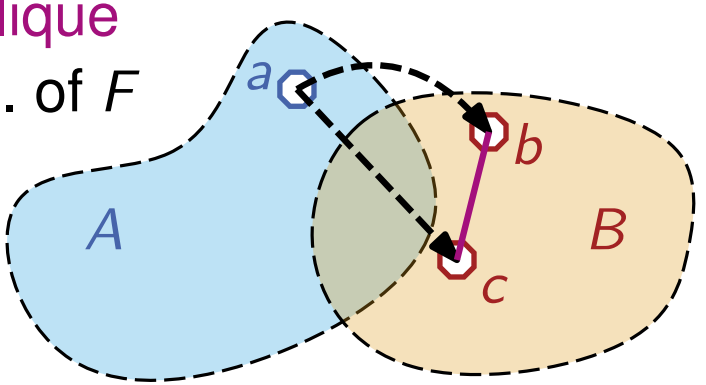
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by B clique
to trans. of F

second case:



Characterization of intervall graphs: Proof

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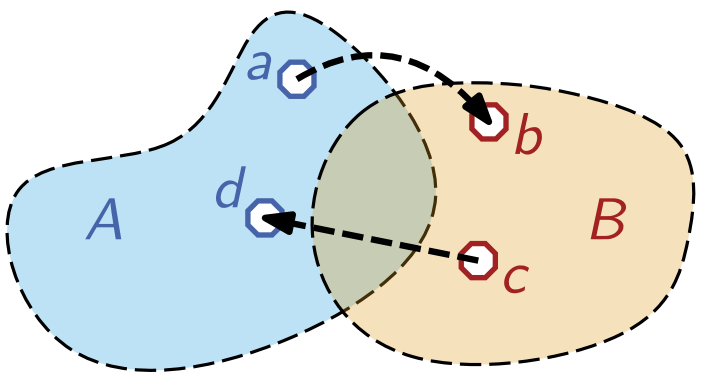
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third case:



Characterization of intervall graphs: Proof

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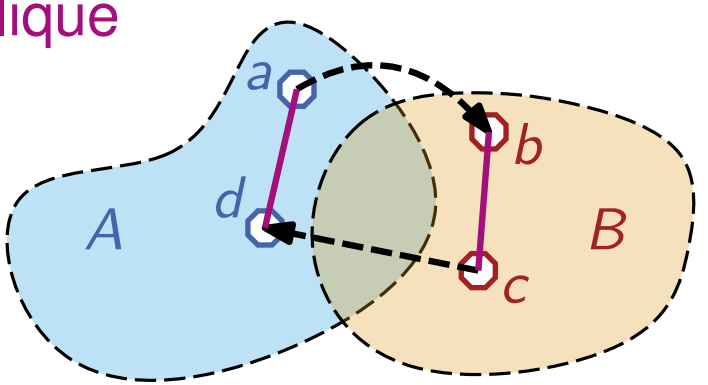
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by B clique

third case:



Characterization of intervall graphs: Proof

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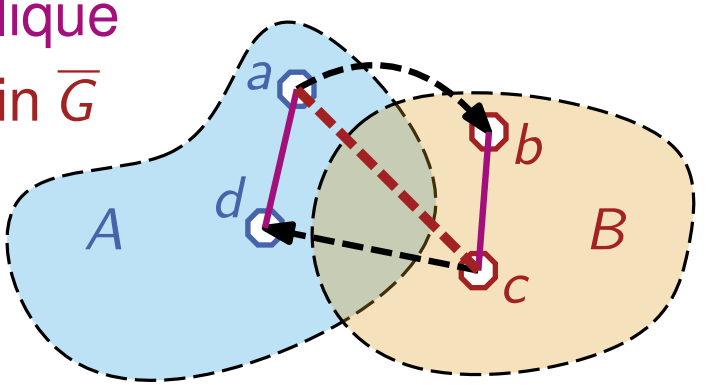
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by B clique
no K_2 in $\overline{G}$

third case:



Characterization of intervall graphs: Proof

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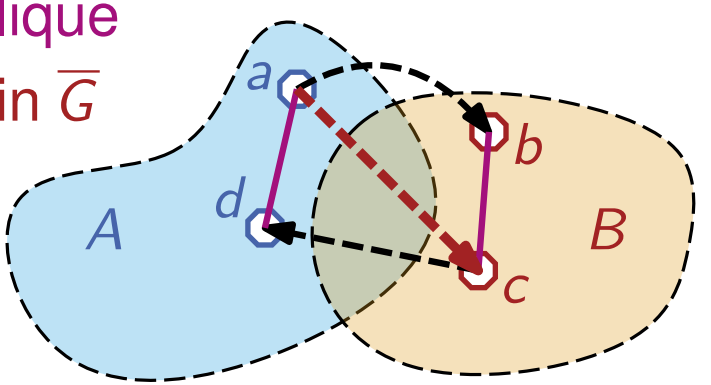
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by B clique
no K_2 in $\overline{G}$



third case: a)

Characterization of intervall graphs: Proof

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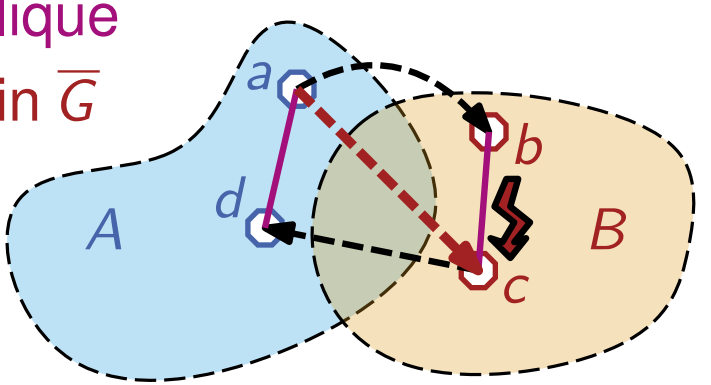
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we will now show that $<$ is well-defined

assume $A < B$ and $B < A$. This amounts to 3 cases (see sketch)

by B clique
no K_2 in $\overline{G}$



third case: a)

to trans. of F

Characterization of intervall graphs: Proof

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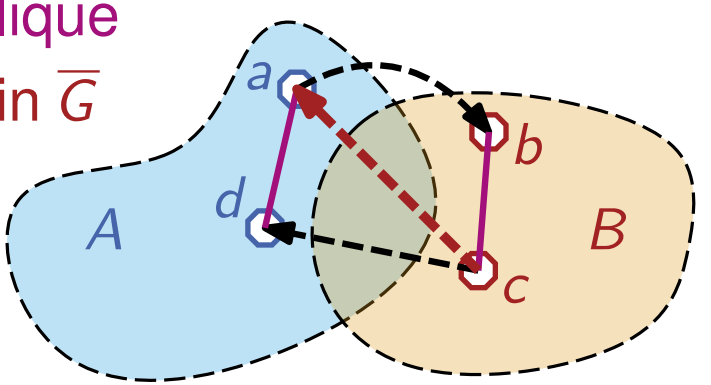
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we will now show that $<$ is well-defined

assume $A < B$ and $B < A$. This amounts to 3 cases (see sketch)

by B clique
no K_2 in $\overline{G}$

third case:b)



Characterization of intervall graphs: Proof

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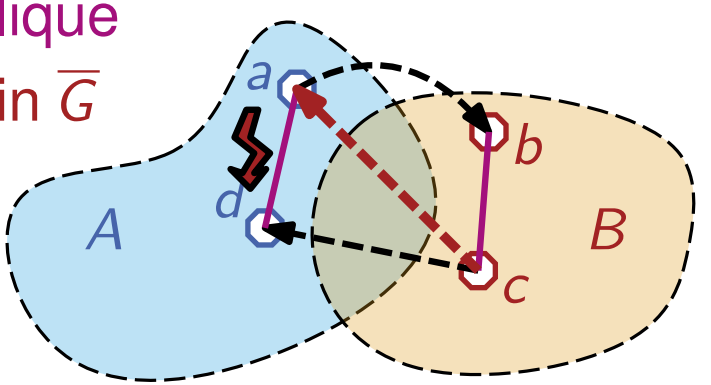
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we will now show that $<$ is well-defined

assume $A < B$ and $B < A$. This amounts to 3 cases (see sketch)

by B clique
no K_2 in $\overline{G}$



third case:b)

to trans. of F

Characterization of intervall graphs: Proof

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Goal: $<$ is transitive

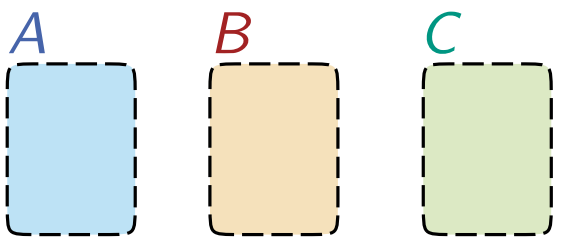
Characterization of intervall graphs: Proof

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Goal: $<$ is transitive
Let $A < B, B < C$



Characterization of intervall graphs: Proof

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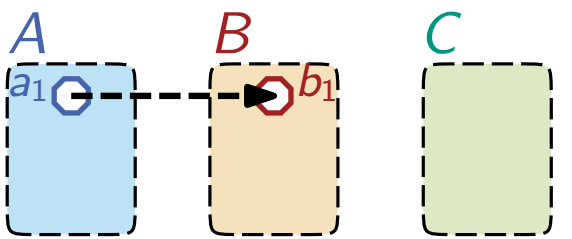
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Goal: $<$ is transitive

Let $A < B, B < C$

- $a_1 b_1 \in F$ with $a_1 \in A - B, b_1 \in B - A$



Characterization of intervall graphs: Proof

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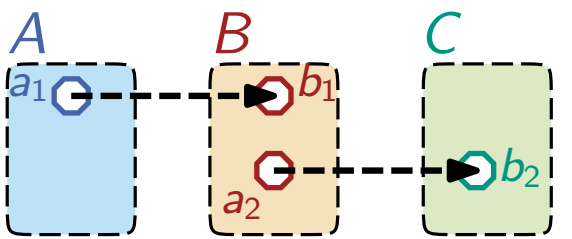
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Goal: $<$ is transitive

Let $A < B, B < C$

- $a_1 b_1 \in F$ with $a_1 \in A - B, b_1 \in B - A$
- $a_2 b_2 \in F$ with $a_2 \in B - C, b_2 \in C - B$



Characterization of intervall graphs: Proof

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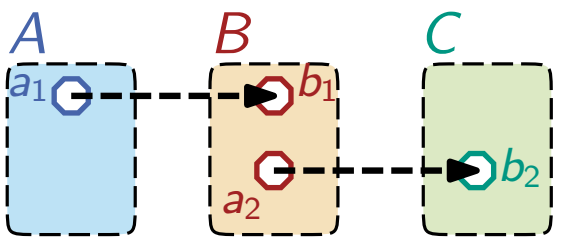
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Goal: $<$ is transitive

Let $A < B, B < C$

- $a_1 b_1 \in F$ with $a_1 \in A - B, b_1 \in B - A$
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Characterization of intervall graphs: Proof

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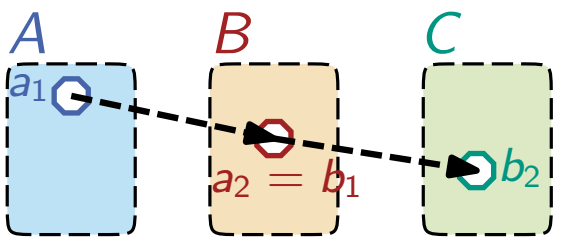
Goal: $<$ is transitive

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- $a_1 b_1 \in F$ with $a_1 \in A - B, b_1 \in B - A$
- $a_2 b_2 \in F$ with $a_2 \in B - C, b_2 \in C - B$

$b_2 \notin A$ since otherwise $B < A$

if $b_1 = a_2$ by transitivity of F also $a_1 b_2 \in F$ so $A < C$



Characterization of intervall graphs: Proof

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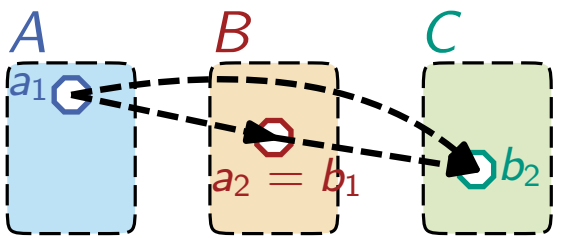
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if $b_1 = a_2$ by transitivity of F also $a_1 b_2 \in F$ so $A < C$



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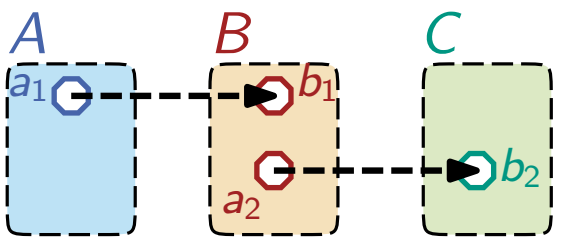
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$b_2 \notin A$ since otherwise $B < A$

$b_1 \neq a_2$:



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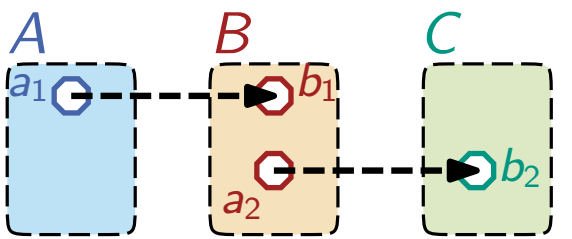
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$b_1 \neq a_2$:

if $a_1 a_2 \notin E$ or $b_1 b_2 \notin E$ we’re done by trans. of F

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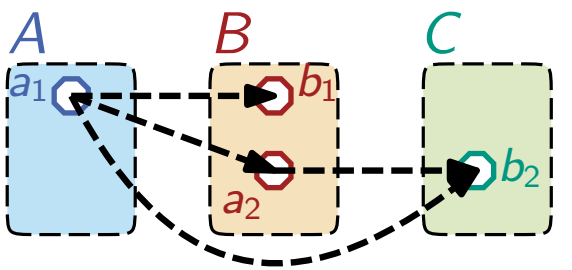
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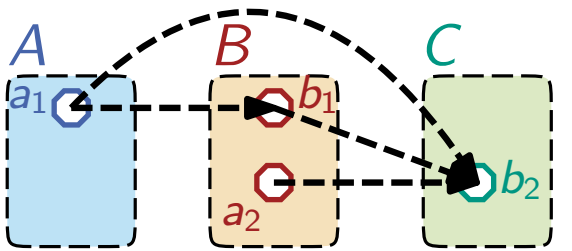
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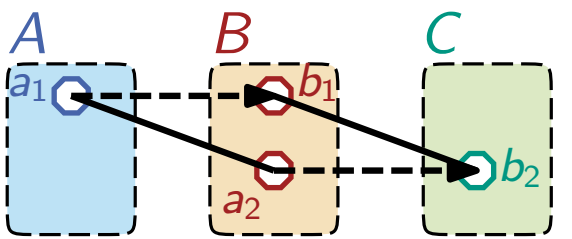
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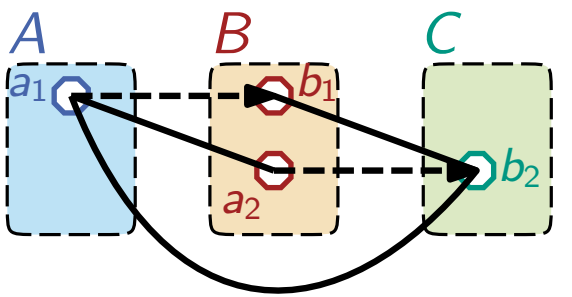
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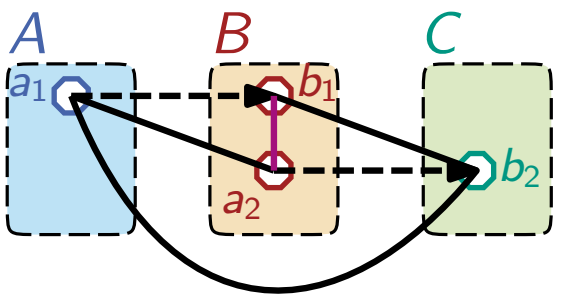
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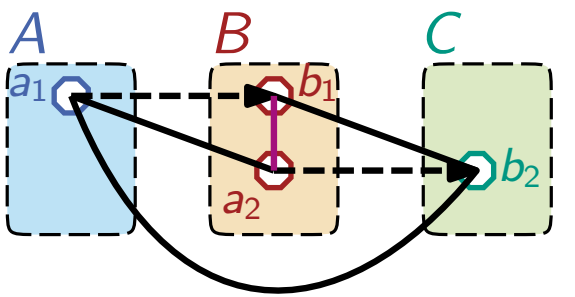
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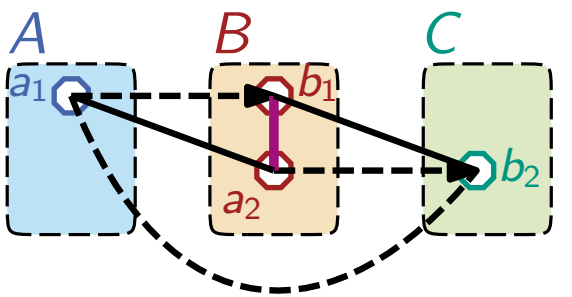
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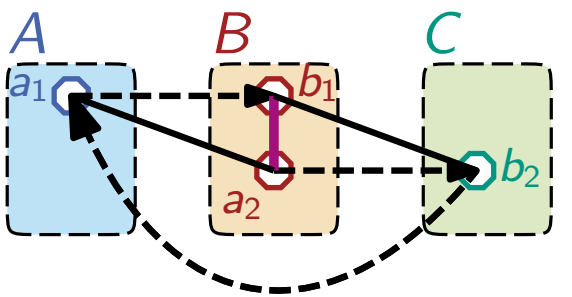
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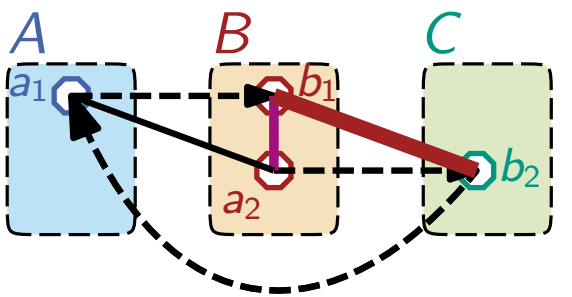
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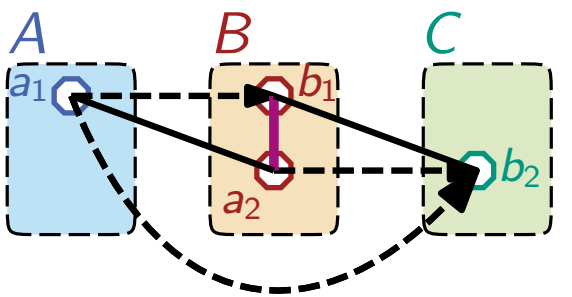
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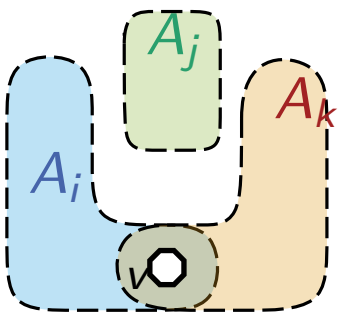
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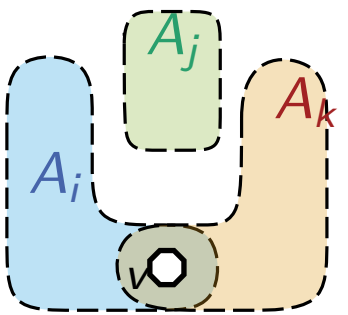
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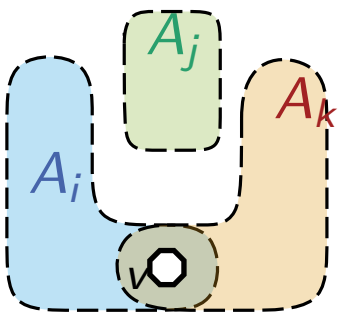
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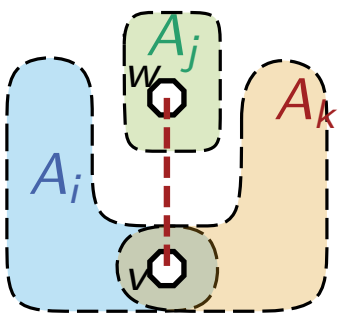
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since A_j maximal there must be a non-edge $vw \notin E$ between v and $w \in A_j$



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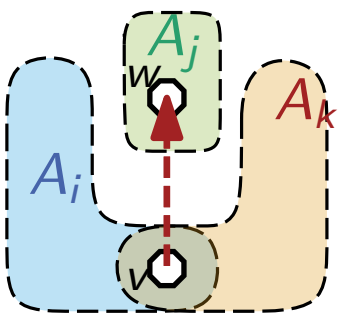
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if $vw \in F \implies A_k < A_j$



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we define order between incl.-maximal cliques $A \neq B$: $A < B \iff$
there is a non-edge $ab \in F$ between $A - B$ and $B - A$

We have total order $A_1 < ... < A_x$. Show ”consecutivity”

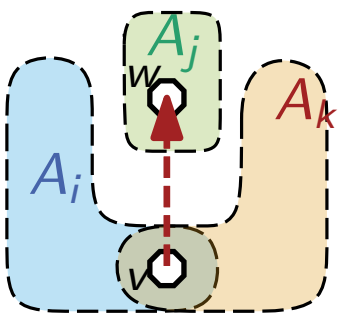
Let $v \in A_i \cap A_k$ with $i < j < k$

Goal $v \in A_j$

Assume $v \notin A_j$

since A_j maximal there must be a non-edge $vw \notin E$ between v and $w \in A_j$

if $vw \in F \implies A_k < A_j$ ↯



Characterization of intervall graphs: Proof

”(iv) $\Longrightarrow$ (v)” given $C_4 \not\subseteq_{\text{ind}} G$ and $\overline{G}$ comp. graph
construct ordering $A_1, \ldots, A_x$ of the inclusion-maximal cliques in G s.t.
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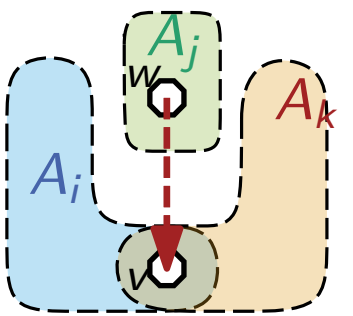
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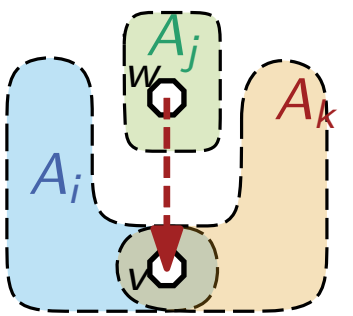
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Characterization of intervall graphs: Proof

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