Karlsruhe Institute of Technology

Algorithmic Graph Theory

Problem Class 2 | 7 May 2025

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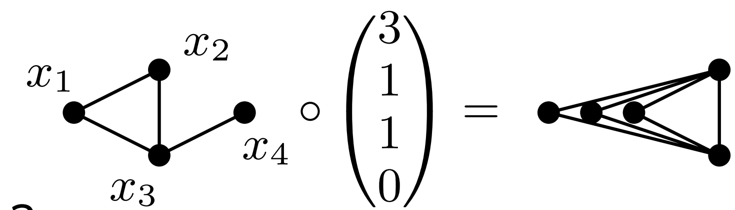
Update from the faculty

- exams may be taken in English or German

Graph G on vertices $x_1, \dots, x_n$, tuple $h = (h_1, \dots, h_n)$ of nonnegative integers

Recall $H = G \circ h$: $\quad V(H) = \{x_i^1, \dots, x_i^{h_i} \mid i \in [n]\}$, $\ E(H) = \{x_i^a x_j^b \mid x_i x_j \in E(G), a \in [h_i], b \in [h_j]\}$

New definition: $H = G \ominus h$ with
$V(H) = \{x_i^1, \dots, x_i^{h_i} \mid i \in [n]\}$,
$E(H) = \{x_i^a x_j^1 \mid x_i x_j \in E(G), a \in [h_i], h_j > 0\}$



(1) What is the difference between $\circ$ and $\ominus$?

(2) Can $\circ$, resp. $\ominus$, be realized by elementary operations?
I.e., is there a sequence $h^1, h^2, \dots$ of tuples, each with all entries 1 except for one 0 or 2, such that $G * h = G * h^1 * h^2 * \dots$ for each $* \in \{\circ, \ominus\}$? If so, does the order matter?

(3) Find a largest cycle, a largest induced cycle, ω, χ, α, and κ of $K_n \ominus (2, \dots, 2)$.

(4) Prove or disprove: If G is perfect, then ...

- $\omega(G \ominus h) = \chi(G \ominus h)$ (follow a proof of the lecture)
- $G \ominus h$ is perfect

Let $G_0 = K_2$ and $H_i = G_{i-1} \ominus (2, \dots, 2)$, $G_i = H_i + u + \{uv \mid v \in V(H_i) - V(G_{i-1})\}$, where u is a new vertex, $i \geq 1$

(5) Prove that $G_1, G_2, \dots$ are not perfect.

(6) Prove that $G_0, G_1, \dots$ are "far from perfect". For this, find $\omega(G_i)$ and $\chi(G_i)$, $i \geq 0$.

What is the difference?

Graph G on vertices $x_1, \dots, x_n$, tuple $h = (h_1, \dots, h_n)$ of nonnegative integers
Recall: $H = G \circ h$ defined as
$V(H) = \{x_i^1, \dots, x_i^{h_i} \mid i \in [n]\}$
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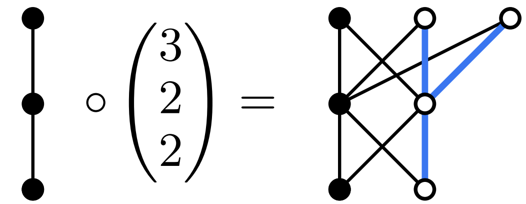

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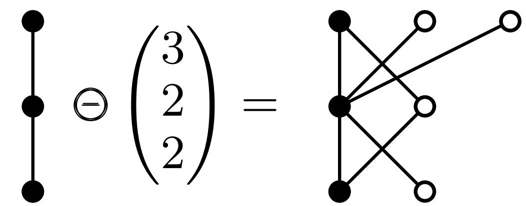
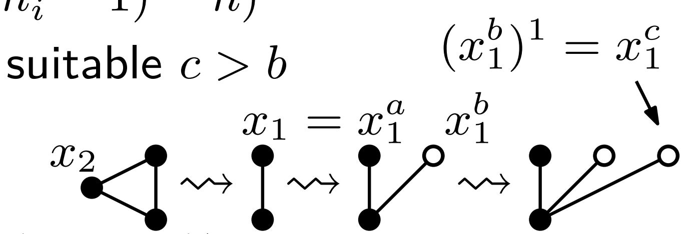


Claim: $(G \circ h) \circ h' = G \circ (h + h' - (1, \dots, 1))$ (padding h' with 1's as necessary)
so $G \circ (3, 0, 1) = G \circ (2, 0, 1) \circ (2, 1) = G \circ (1, 0, 1) \circ (2, 1) \circ (2, 1, 1)$

- same number of vertices: $n + \sum(h_i - 1) + \sum(h'_i - 1) = n + (\sum(h_i + h'_i - 1) - n)$
- Identify vertices: Denote twins (due to h') of $x_i^b \in V(G \circ h)$ by x_i^c with suitable $c > b$
- edges with both endpoints in $G \circ h$: definitions coincide
- $x_i^a x_i^c \in E(G \circ h \circ h')$ with x_j^c twin of some x_j^b in $G \circ h$
  $\implies x_i^a x_j^b \in E(G \circ h) \implies x_i x_j \in E(G) \implies x_i^a x_j^c \in E(G \circ h + h' - (1, \dots, 1))$
- other direction analoguous



$\implies$ elementary operations $G \circ x_i$ and $G - x_i$ suffice (in any order)

What is the difference?

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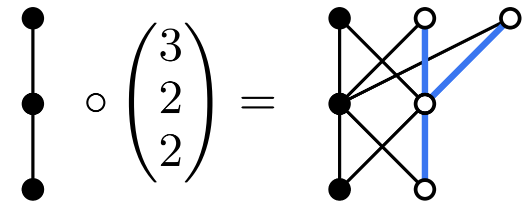

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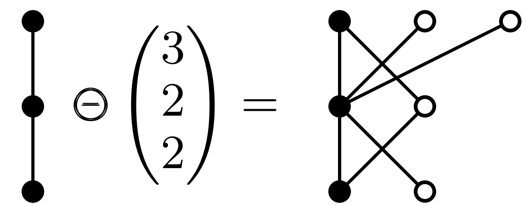
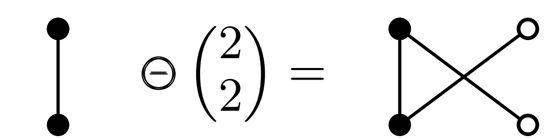
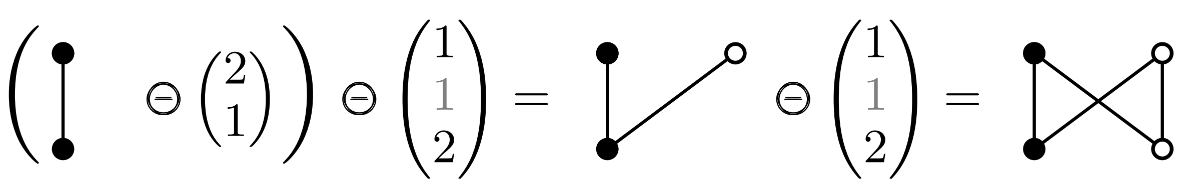


Does the same work for $\ominus$? $\rightarrow$ **No!**
What about other sequences?

What in the proof for $\circ$ fails for $\ominus$?





Claim: The graphs obtained form K_2 by elementary $\ominus$-replications are exactly the complete bipartite graphs. $\rightarrow$ no P_4

Therefore: single operation for all new vertices necessary for new vertices with $\ominus$

What is the difference?

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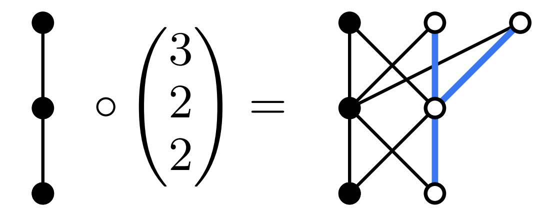

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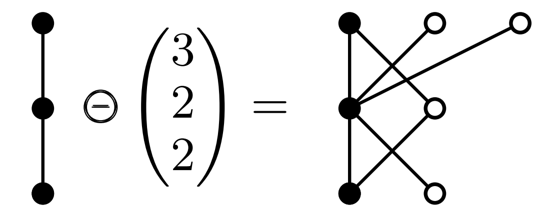

Does the same work for $\ominus$?
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What in the proof for $\circ$ fails for $\ominus$?

- same number of vertices: $n + \sum(h_i - 1) + \sum(h_i' - 1) = n + (\sum(h_i + h_i' - 1) - n)$
- Identify vertices: Denote twins (due to h') of $x_i^b \in V(G \circ h)$ by x_i^c with suitable $c > b$
- edges with both endpoints in $G \circ h$: definitions coincide
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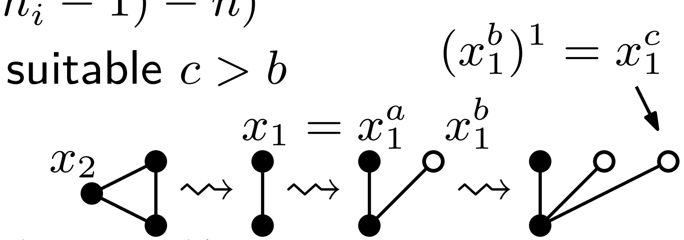
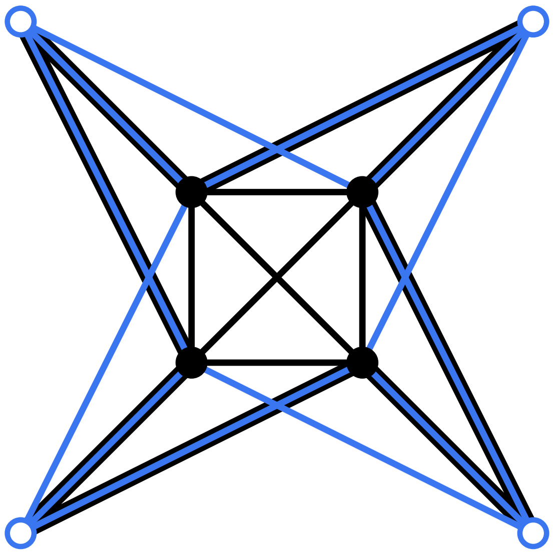
  $\implies x_i^a x_j^b \in E(G \circ h)$ $\boxed{\not\Leftarrow}$ $x_i x_j \in E(G) \implies x_i^a x_j^c \in E(G \circ h + h' - (1, \dots, 1))$
- other direction analoguous

$\implies$ elementary operations $G \circ x_i$ and $G - x_i$ suffice (in any order)



$K_n \ominus (2, \ldots, 2)$



- Hamilton cycle (length $2n$)
- largest induced cycle: triangle
  - larger induced cycle contains ≤ 2 black vertices (clique)
  - all black vertices are consecutive (clique)
  - blue vertices not adjacent $\implies$ only one blue vertex
- $\omega = \chi = \alpha = \kappa = n$

Perfect?

Lemma (lecture): If G is perfect, then $G \circ h$ is perfect.

Goal: adapt proof for $\ominus$

Observation: If h is a 0-1-tuple, then $G \ominus h = G \circ h \subseteq_{\text{ind}} G$

Let h' be such that $h'_i = \begin{cases} 1 & \text{if } h_i > 0 \\ 0 & \text{if } h_i = 0 \end{cases}$ and $G' = G \ominus h' \subseteq_{\text{ind}} G$

Since G is perfect, we have $\omega(G') = \chi(G')$.

- new vertices in $G \ominus h$ form an independent set $\implies \leq 1$ new vertex in every clique
  If new vertex x_i^a in largest clique C, then $C - x_i^a + x_i^1 \subseteq G'$ is clique of same size.
  $\implies \omega(G \ominus h) = \omega(G')$
- Observe: $N(x_i^a) \subseteq N(x_i)$ for each twin x_i^a of x_i
  Let $c' : V(G') \to [\chi(G')]$ be a proper coloring.
  Now $c(x_i^a) = c'(x_i)$ is a proper coloring of $G \ominus h$.
  $\implies \chi(G \ominus h) = \chi(G')$

So, $\omega(G \ominus h) = \omega(G') = \chi(G') = \chi(G \ominus h)$

Perfect?

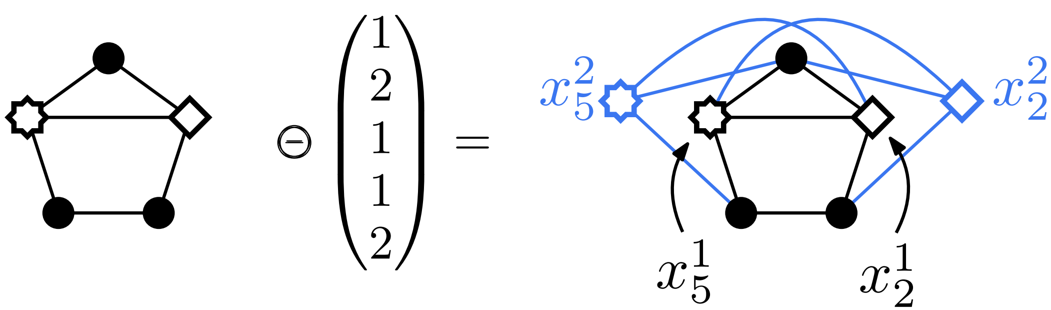
Lemma (lecture): If G is perfect, then $G \circ h$ is perfect.

Goal: adapt proof for $\ominus$

Observation: If h is a 0-1-tuple, then $G \ominus h = G \circ h \subseteq_{\text{ind}} G$

Have: $\omega(G \ominus h) = \chi(G \ominus h)$

But what about induced subgraphs? $\rightarrow \chi(C_5) \neq \omega(C_5)$

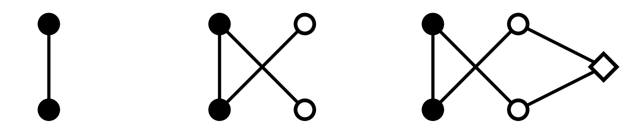


perfect

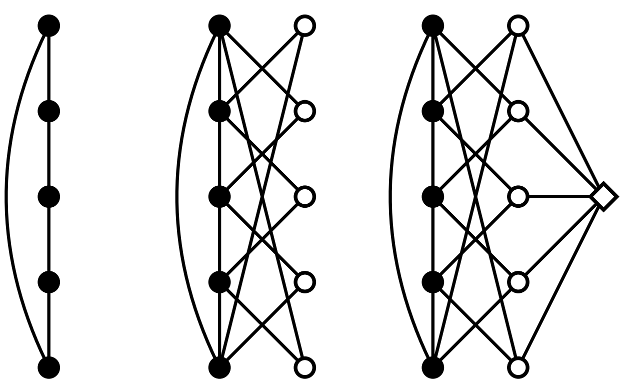
not perfect

Far from perfect (Mycielski 1955)

Let $G_0 = K_2$ and $H_i = G_{i-1} \ominus (2, \ldots, 2)$, $G_i = H_i + u + \{uv \mid v \in V(H_i) - V(G_{i-1})\}$, where u is a new vertex, $i \geq 1$

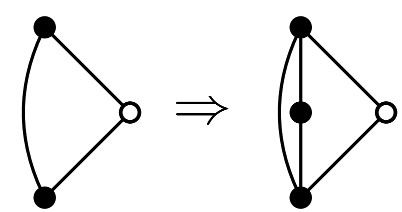


G_0 H_1 G_1



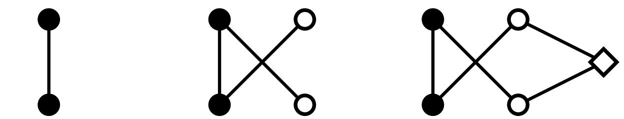
G_1 H_2 G_2

- $G_0 = K_2$ is perfect
- $G_i, i > 0$ contain induced $C_5 \implies$ not perfect
- $\omega(G_i) = 2$ for all i by induction:
  - triangle does not contain $\diamond$
  - only one $\circ$
  - so at least two $\bullet$, but they form triangle with twin of chosen $\circ$
  - contradiction: $\bullet$ do not form a triangle by induction

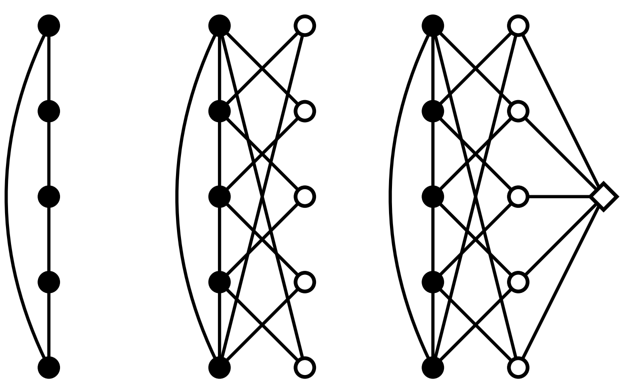


Far from perfect (Mycielski 1955)

Let $G_0 = K_2$ and $H_i = G_{i-1} \ominus (2, \ldots, 2)$, $G_i = H_i + u + \{uv \mid v \in V(H_i) - V(G_{i-1})\}$, where u is a new vertex, $i \geq 1$



G_0 H_1 G_1



G_1 H_2 G_2

- $\chi(G_i) \geq i + 2$ by induction:
  - let $k = i + 1$
  - let c be a k-coloring that minimizes the number of vertices colored in the same color as $\diamond$
  - w. l. o. g $c(\diamond) = k$
  - If there is a $\bullet$-vertex x_i colored with k, then its neighborhood contains all other colors (choice of c)
  - thus, the neighborhood of its twin x_i^2 contains all colors $1, \ldots, k-1$ and $c(x_i^2) = k$
  - not a proper coloring
  - so the $\bullet$-vertices admit a $(k-1)$-coloring, contradiction to induction
- $\chi(G_i) \leq i + 2$: copy colors for $\circ$ and use new color for $\diamond$