Reconfiguration of Graphs under Edge Additions and Deletions

Bachelor's Thesis of

Lennart Blumenthal

At the KIT Department of Informatics
Institute of Theoretical Informatics

First examiner:	PD Dr. Torsten Ueckerdt
Second examiner:	Prof. Dr.-Ing. Marvin Künnemann
First advisor:	PD Dr. Torsten Ueckerdt

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Karlsruher Institut für Technologie
Fakultät für Informatik
Postfach 6980
76128 Karlsruhe

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I declare that I have developed and written the enclosed thesis completely by myself. I have not used any other than the aids that I have mentioned. I have marked all parts of the thesis that I have included from referenced literature, either in their original wording or paraphrasing their contents. I have followed the by-laws to implement scientific integrity at KIT.

Karlsruhe, 25. March 2026

L. Blumenthal

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(Lennart Blumenthal)

Abstract

Combinatorial reconfiguration examines if and how we can get from one configuration to another configuration using only a set of allowed moves. The field of combinatorial reconfiguration has only been around for the last two decades. As a result of this, many seemingly natural questions have not even been asked yet. This thesis introduces a new type of reconfiguration problem, which, to the best of our knowledge, has not been considered by any prior research.

We study the reconfiguration of graphs, where in one move we are allowed to either add or delete exactly one edge. We fix some graph class $\mathcal{G}$ and only allow graphs G which are in the graph class $\mathcal{G}$ as configurations. From this reconfiguration problem we define the associated reconfiguration graph $\mathcal{R}(\mathcal{G})$. We examine graph properties of $\mathcal{R}(\mathcal{G})$ like its chromatic number, distances, radius, and diameter for some families of graph classes $\mathcal{G}$. Furthermore, we calculate the exact values of these graph properties of $\mathcal{R}(\mathcal{G})$ for specific graph classes $\mathcal{G}$. Amongst them are planar graphs, graphs of bounded chromatic number, and chordal graphs.

Zusammenfassung

Das Gebiet der kombinatorischen Rekonfiguration untersucht, ob und wie wir, gegeben eine Menge an erlaubten Operationen, verschiedene Konfigurationen ineinander überführen können. Da die Forschung in diesem Gebiet erst in diesem Jahrhundert begonnen hat, gibt es noch viele unbetrachtete Fragen. Diese Arbeit führt ein neues Rekonfigurationsproblem ein, welches, soweit wir es wissen, noch nicht zuvor betrachtet wurde.

Wir beschäftigen uns mit der Rekonfiguration von Graphen. In einer Operation darf eine Kante des Graphen entfernt oder eine Neue hinzugefügt werden. Außerdem fixieren wir eine Graphklasse $\mathcal{G}$ und erlauben nur Graphen G aus $\mathcal{G}$ als Konfigurationen. Wir definieren den Rekonfigurationsgraphen $\mathcal{R}(\mathcal{G})$ zu diesem Problem und untersuchen einige seiner Eigenschaften, darunter seine chromatische Zahl, Distanzen sowie den Radius und Durchmesser. Zudem berechnen wir für einige Graphklassen $\mathcal{G}$ wie planare Graphen, Graphen mit beschränkter chromatischer Zahl sowie chordale Graphen die exakten Werte der Grapheigenschaften von $\mathcal{R}(\mathcal{G})$.

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1 Introduction

Most combinatorial puzzles can be modeled by their reconfiguration graph. A vertex of the reconfiguration graph represents one of the states of the puzzle, while the edges of the reconfiguration graph represent possible moves in the puzzle. Note that for any given puzzle there may exist multiple ways of modeling it as a reconfiguration graph. Consider a Rubik's Cube, for example. It can be modeled by a reconfiguration graph $\mathcal{R}_c$, where each vertex is a configuration of the cube. For now, let's say that a configuration is any arrangement of the cube that can be obtained by taking it apart and reassembling it in any way. An edge in $\mathcal{R}_c$ exists between two configurations if it is possible to get from one of them to the other by rotating one outer layer by 90 degrees. A small section of $\mathcal{R}_c$can be seen in Figure 1.1.

Once a reconfiguration graph for a given problem is defined, we can start to investigate its graph properties. Characterizations of the graph can then be used to deduce properties of the original problem. In our example of the Rubik's Cube, showing that $\mathcal{R}_c$ is connected would imply that the Rubik's Cube can be solved from any configuration. This is where our definition of a configuration becomes relevant. When defining the vertex set of $\mathcal{R}_c$ as the configuration reachable by taking apart and reassembling the cube, $\mathcal{R}_c$ is not connected. It follows from the results in [12] that the graph consists of 12 connected components, which are pairwise isomorphic. The configurations reachable by scrambling the cube without disassembling it are exactly those which lie in the same connected component as the solved state. So let us define a scramble as those configurations of a Rubik's Cube reached from the

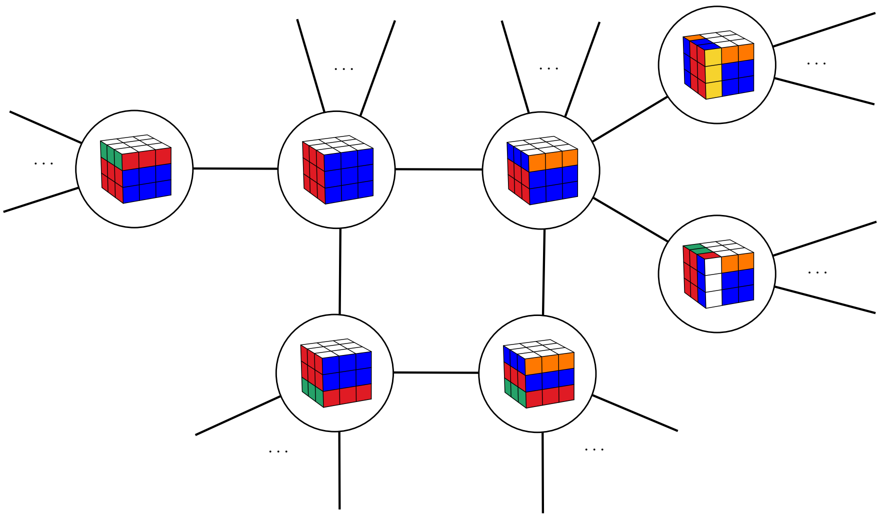


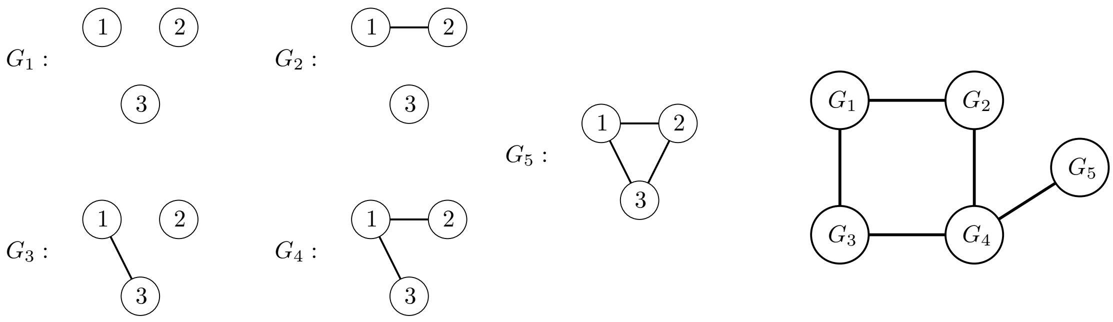
Figure 1.1: Section of $\mathcal{R}_c$

solved state by rotating the faces of the cube. Taking the scrambles of the cube as vertices gives us a different reconfiguration graph $\widetilde{\mathcal{R}}_c$, isomorphic to a connected component of $\mathcal{R}_c$. Thus $\widetilde{\mathcal{R}}_c$ is connected, so any scramble can be reached from any other by repeatedly rotating an outer layer of the cube by 90 degrees. One may now be interested in finding the minimum number of moves required to solve any scramble of a Rubik's Cube. This minimum number of moves corresponds to the eccentricity of the node representing the solved state of the cube in $\widetilde{\mathcal{R}}_c$. It can furthermore be shown that this is also equal to the radius of $\widetilde{\mathcal{R}}_c$.

Even though the idea of building a reconfiguration graph to model a combinatorial problem may seem quite natural, research in the field of reconfiguration problems and reconfiguration graphs only really took off in the mid 2000s. Thus, there are still many opportunities for new findings.

One can also define reconfiguration graphs on a graph class $\mathcal{G}$. That is, each vertex of the reconfiguration graph represents a graph G from the graph class $\mathcal{G}$. This is the type of reconfiguration graph considered in this thesis. Most previous research dealing with reconfiguration graphs of graph classes has considered edge flips of some kind, that is, the edges of the reconfiguration graph represent the operation of deleting one set of edges, while adding another set of edges of the same size. An example of this can be seen in [8] where the authors fix a degree sequence $S = (d_1, d_2, \dots, d_n)$ and consider the graph class $\mathcal{G} = \mathcal{G}(S)$ containing all graphs G realizing the degree sequence S. The edges of the reconfiguration graph represent a flip of exactly two edges, that is, removing two edges and adding two new ones. Amongst many other results, they show that this reconfiguration graph is connected for every degree sequence S. This information can then be used to guide the development of an efficient algorithm for enumerating all graphs G realizing a given degree sequence S.

For this thesis we also define a reconfiguration graph on a graph class. However, with a new way of defining the edges of the reconfiguration graph, we distinguish ourselves from all previous research. Let us fix a set of allowed graphs $\mathcal{G} = \{G_1, \dots, G_n\}$ which we call a graph class. Furthermore, we start with some graph $G \in \mathcal{G}$ from the graph class. We are allowed to either remove some edge from G or introduce a new edge which was not previously present, as long as the resulting graph G' is in the graph class $\mathcal{G}$ as well. Thus, the vertex set of the resulting reconfiguration graph $\mathcal{R}(\mathcal{G})$ is the graph class $\mathcal{G}$ itself. Edges of $\mathcal{R}(\mathcal{G})$ correspond to edge additions or removals in the underlying graphs. The formal definition can be found in Definition 1.6. For a small example, consider Figure 1.2. In this thesis we show a multitude of structural results about $\mathcal{R}(\mathcal{G})$ for different graph classes $\mathcal{G}$. We consider forests, planar graphs, graphs of bounded chromatic and many more. We hope that this can eventually lead to a better understanding of the graph classes $\mathcal{G}$ and give new ideas for improvements to algorithms on these graph classes.



(a) $\mathcal{G} = \{G_1, G_2, G_3, G_4, G_5\}$ **(b)** $\mathcal{R}(\mathcal{G})$

Figure 1.2: Example of a reconfiguration graph

1.1 Related Work

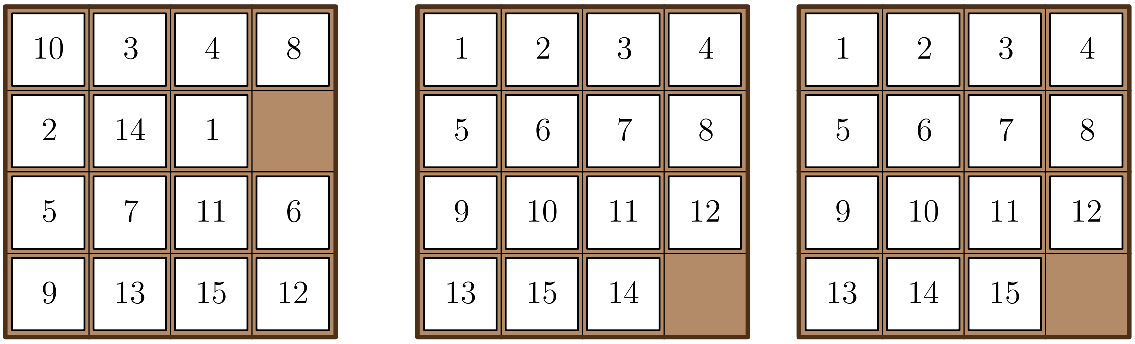
The exact type of reconfiguration graph that we take a look at in this thesis has not been considered by previous research. Thus, there is no related work which this thesis is directly based on and tries to extend. Instead, we use this section to give a rough overview of the developments in the research area of combinatorial reconfiguration. It consists of information acquired from research over the time of writing this thesis, as well as the list of papers in the field from [13], and the survey paper [22].

1.1.1 Early Work

While research in the field of reconfiguration only really picked up over the past 20 years, one can find related work which was published more than a century ago. One of the most prominent examples is the first structural consideration of the 15-puzzle from the year 1879 [20]. The 15-puzzle is a combinatorial puzzle which surfaced around that time, and is still well-known today. It consists of a square 4 by 4 grid, with 15 tiles numbered from 1 to 15 placed on it. The 16-th square is left empty. Thus, in one move a player can take any tile adjacent to the empty space and move it into the empty spot. One can scramble the puzzle into any start state. The goal then becomes to get back to the unique solved state, where the bottom right corner is left empty and all the 15 numbered tiles are sorted according to their numbers. An example of two different scrambles and the solved state can be seen in Figure 1.3. In 1879 it was shown that the 15-puzzle has scrambles that are not solvable [20]. Translating this into today's language of reconfiguration graphs, this means that the reconfiguration graph of the 15-puzzle is disconnected.

1.1.2 From Independent Work to Common Terminology

After the consideration of the 15-puzzle, some other combinatorial puzzles were analyzed in a similar manner over the next 100 years. But reconfiguration graphs do not only arise



(a) Solvable scramble **(b)** Unsolvable scramble **(c)** Solved state

Figure 1.3: The 15 puzzle

from reconfiguration problems. They can also be defined on solution spaces of problems. First examples of this can be found in the late 90s of the last century. One example are *Matching Graphs*, first appearing in 1998 [14]. Given a graph G, the matching graph $M(G)$ has all maximum matchings in G as vertices. Two vertices of $M(G)$, call them M_1 and M_2, which are maximum matchings in G, have an edge between them in $M(G)$, if and only if $|M_1 \setminus M_2| = 1$. In other words, matchings M_1 and M_2 are connected if one can remove one edge from M_1 and add a new edge to replace it, such that the resulting M_2 is once again a matching. This is also one of the first instances of a reconfiguration graph being explicitly defined to model a reconfiguration problem.

Another early application of reconfiguration graphs was to the solution space of the boolean satisfiability problem (SAT) first appearing at a conference in 2006 [17] and published as an article in 2009 [18]. In their work, they fix some SAT instance and define a reconfiguration graph, where each vertex corresponds to a variable assignment satisfying the instance. An edge exists between two assignments if they differ only in the truth value assigned to one variable. It is easy to imagine that properties of this reconfiguration graph describe the solution space of the SAT instance. These characterizations can then be used to better understand the runtime of SAT solving algorithms like DPLL and local search and guide the development of improvements to their runtime.

It should however be noted that up until this point, no common term for reconfiguration graphs was introduced. Instead, each paper chose some application-specific name for their construction. This also changed over the second half of the 2000s with the terms *reconfiguration* and *reconfiguration graph* becoming established and adopted by most authors working in the field. One of the earlier examples of these terms being used is from a conference paper which appeared in 2006 [9] which was published as an article two years later [10].

At the same time as terminology for reconfiguration research was standardized, research also aimed to unify many of the most common types of reconfiguration graphs into one framework. This can also be seen in [9]. However, they still use the term *chips* for what would later on be commonly named *tokens* [16]. The idea behind tokens can be described

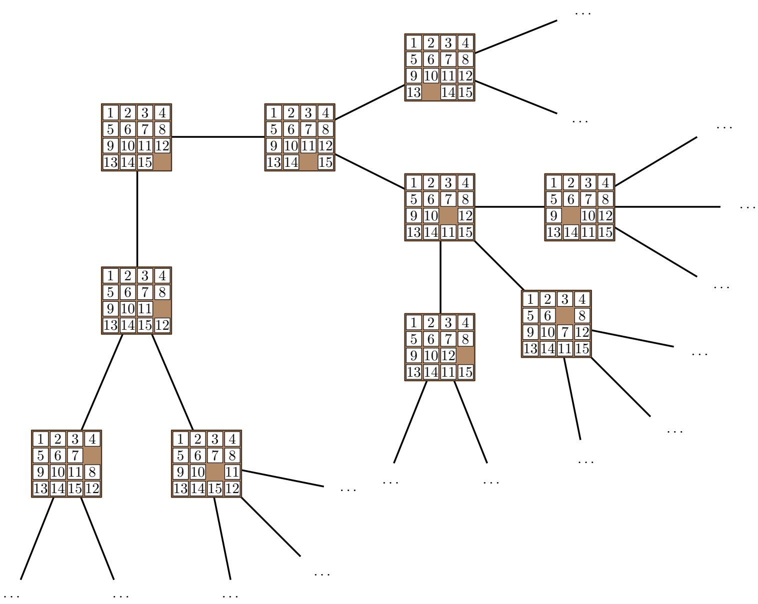


Figure 1.4: Section of the reconfiguration graph of the 15-puzzle

as follows. Fix some graph G and an integer k. We define a reconfiguration graph $\mathcal{R}_k^T(G)$. The vertices of $\mathcal{R}_k^T(G)$ are all configurations where k tokens are placed on the vertices of G. Edges are now commonly defined in one of two ways. Either we say that two token configurations are adjacent if we can get from one to the other by taking one token and moving it to an adjacent vertex. As we can imagine this process as sliding the token over an edge, this idea is called token sliding. On the other hand, one can also lift the restriction that the token must be moved to a neighboring vertex, which is then called token jumping. To model some problems, further restrictions on allowed token configurations can be defined. Furthermore, depending on the context, both token sliding and token jumping may be considered with either labeled or indistinguishable tokens. Also related to this, the model of token addition and removal was introduced, where the edges of our reconfiguration graph $\mathcal{R}^T(G)$ represent the removal of a token from a configuration.

It is easy to see that the aforementioned 15-puzzle can be modeled as a token sliding problem. Where the graph we consider is a grid graph on 16 vertices representing the game board and we have $k = 15$ labeled tokens. A section of the reconfiguration graph can be seen in Figure 1.4. But this framework was also used to model reconfiguration problems based on independent sets [21], vertex cover [19], or dominating sets [1], to name only some examples.

1.1.3 Recent Work

With this unified terminology the field of reconfiguration research community grew quickly. Before 2006 only few and unrelated papers dealing with reconfiguration problems were published. This number grew to a handful per year until the end of the 2000s and has since shown few signs of slowing down. Towards the second half of the 2010s more than fifty papers dealing with reconfiguration problems were published each year and in the 2020s this number has grown to around one hundred per year. Thus, it is not feasible to talk about all research that has been published over the last few years. So we use this opportunity to highlight some areas with recurring research over the past decade.

To start out, the previously mentioned token sliding and jumping reconfiguration problems have seen a lot of research activity. With many problems fitting in this framework considered over time, the largest research community has formed around independent set-based reconfiguration problems.

Another area of research with plentiful publications deals with coloring reconfiguration problems. That is, fixing a graph G and an integer $k \in \mathbb{N}$ and considering two different k-colorings C_1 and C_2. The question then becomes whether one can transform C_1 to C_2 using a set of allowed operations. Large parts of this research either allow the recoloring of a single vertex in one operation [11] or the swapping of the colors of two adjacent vertices [23]. But many further variants, like list coloring versions of these problems, exist.

There has also been quite a bit of work dealing with problems with a geometric component. For example, different notions of transforming non-crossing edge sets in the plane, like triangulations [2] or spanning trees [6], have been looked at.

1.2 Outline

But now let us get to our work. We fix a graph class $\mathcal{G}$ and define a reconfiguration graph $\mathcal{R}(\mathcal{G})$ on it, as can be seen in Figure 1.2. To obtain results, this thesis considers two types of graph classes.

Firstly, we consider monotone graph classes $\mathcal{G}$. That is, for each graph $G \in \mathcal{G}$ every subgraph $H \subseteq G$ is also in the graph class $H \in \mathcal{G}$. We obtain formulas for calculating several graph parameters of $\mathcal{R}(\mathcal{G})$ in this case. Afterward, some time is spent on evaluating these formulas for specific graph classes like forests, planar graphs, and graphs with bounded chromatic number.

After this, we try to obtain similar results for graph classes $\mathcal{G}$ which satisfy a weaker version of monotonicity. It turns out that for most graph parameters of $\mathcal{R}(\mathcal{G})$ there is little hope for an easy formula. Instead, we find some upper and lower bounds for these parameters, which we show to be tight. Calculation of these bounds turns out to be simple in most cases, so we do not evaluate them explicitly for many graph classes.

1.3 Definitions

To avoid confusion, some less standardized or rarely used mathematical notation is given.

Definition 1.1. *We denote*

- *the set of natural numbers up to some* $n \in \mathbb{N}$ *by* $[n] := \{1, \ldots, n\}$.
- *the symmetric difference of two sets* S_1, S_2 *by* $S_1 \triangle S_2 := (S_1 \cup S_2) \setminus (S_1 \cap S_2)$.
- *the cartesian product of two sets* S_1, S_2 *by* $S_1 \times S_2 := \{(s_1, s_2) \mid s_1 \in S_1, s_2 \in S_2\}$.
- *for* $k \in \mathbb{N}$ *all* k *element subsets of a ground set* S *by* $\binom{S}{k} = \{A \subseteq S \mid |A| = k\}$.
- *for some set* S *the set of all subsets of* S *by* $2^S = \{A \mid A \subseteq S\}$.

As the following work takes a look at the structure of reconfiguration graphs, it is necessary to be familiar with the definition of a graph. As we only consider *finite*, *undirected*, *unweighted*, *simple* graphs, the following definition suffices.

Definition 1.2. *A graph* $G = (V, E)$ *consists of a finite* vertex set V *and an* edge set $E \subseteq \binom{V}{2}$.

Note that this definition directly guarantees that a graph has no self-loops or multi-edges, hence it is simple. It is also obvious that as V is finite, the edge set E is also finite and it holds that $|E| \leq 2^{\binom{|V|}{2}}$. When drawing graphs, we represent the vertex set by circles on a plane and edges by curves connecting the two vertices making up the edge.

With this definition in mind, we introduce a number of common graph classes, which are considered throughout this thesis.

Definition 1.3. *Let* $n, m \in \mathbb{N}$. *We denote*

- *the* empty graph *on* n *vertices by* $E_n = ([n], \emptyset)$.
- *the* clique *on* n *vertices by* $K_n = ([n], \binom{[n]}{2})$.
- *the* complete bipartite graph *with sides of size* n *and* m *by* $K_{n,m} = ([n+m], \{uv \mid u \in [n], v \in [m+n] \setminus [n]\})$
- *the* complete r-partite graph *with parts of size* $n_1, \ldots, n_r$ *by* $K_{n_1,\ldots,n_r}$. *That is, the graph made up of independent sets* $E_{n_1}, \ldots, E_{n_r}$ *where all other edges exist.*
- *the* path *on* n *vertices by* $P_n = ([n], \{i(i+1) \mid i \in [n-1]\})$.
- *the* cycle *on* n *vertices by* $C_n = ([n], \{i(i+1 \mod n) \mid i \in [n]\})$ *for* $n \geq 3$.

Furthermore, we need a set of operations on graphs. Notation for some of the most important ones is given in the following definition.

Definition 1.4. *Let $G = (V, E)$ and H be graphs. We denote*

- *the set of all vertices of G by $V = V(G)$ and the number of vertices by $|G| = |V(G)|$.*
- *the set of all edges of G by $E = E(G)$ and the number of edges by $\|G\| = |E(G)|$.*
- *the graph obtained from G by removing an edge $e \in E(G)$ by $G - e = (V, E \setminus \{e\})$.*
- *the subgraph induced by some subset of the vertices $\widetilde{V} \subseteq V$ as $G[\widetilde{V}] = \left(\widetilde{V}, E \cap \binom{\widetilde{V}}{2}\right)$.*
- *$H \subseteq_{ind} G$ if H is a subgraph of G induced by some subset of V.*
- *the complement of G by $\overline{G} = (V, \binom{V}{2} \setminus E)$.*
- *that G and H are isomorphic by $G \cong H$. That is, there exists a relabeling of the vertices of G such that it becomes equal to H.*

One should also be familiar with common graph parameters, as in the following, we analyze the structure of reconfiguration graphs using them. The most important parameters are the following.

Definition 1.5. *Let $G = (V, E)$ be a graph. We denote*

- *the degree of a vertex $v \in V(G)$ by $\deg_G(v) = |\{u \in V(G) \mid uv \in E(G)\}|$.*
  If the underlying graph is clear we may instead use $\deg(v)$.
- *the maximum degree of G by $\Delta(G) = \max_{v \in V(G)} \deg_G(v)$.*
- *the minimum degree of G by $\delta(G) = \min_{v \in V(G)} \deg_G(v)$.*
- *the distance between two vertices $u, v \in V$ in the graph G by $\mathrm{dist}_G(u, v)$.*
  If the underlying graph is clear, we may instead use $\mathrm{dist}(u, v)$.
- *the diameter of G by $\mathrm{diam}(G) = \max_{u,v \in V} \mathrm{dist}_G(u, v)$.*
- *the eccentricity of a vertex $v \in V(G)$ by $\epsilon(v) = \max_{v \in V} \mathrm{dist}_G(u, v)$.*
- *the radius of G by $\mathrm{rad}(G) = \min_{u \in V} \epsilon(u) = \min_{u \in V} \max_{v \in V} \mathrm{dist}_G(u, v)$.*
- *the center of G by $\mathrm{center}(G) = \{v \in V \mid \epsilon(v) = \mathrm{rad}(G)\}$.*
- *the size of a largest clique in G by $\omega(G) = \max\{n \in \mathbb{N} \mid K_n \subseteq G\}$.*
- *the size of a largest independent set in G by $\alpha(G) = \max\{|A| \mid A \subseteq V(G), \|G[A]\| = 0\}$.*
- *the chromatic number of G by $\chi(G) = \min\{n \in \mathbb{N} \mid G \text{ admits a proper } n\text{-coloring}\}$.*

This thesis deals with the reconfiguration problem of adding and deleting edges, while staying in a given graph class. Note that, as we only consider edge edits, the vertex set always stays the same. Thus, we only consider graph classes where each member graph has the same vertex set.

To formalize this reconfiguration problem, we define the reconfiguration graph of a graph class $\mathcal{G}$. It has the graphs in $\mathcal{G}$ as vertices and connects two graphs $H, G \in \mathcal{G}$ if one can obtain G from H by either adding or deleting one edge. This property is equivalent to saying that the symmetric difference of the edge sets of G and H has to contain exactly one edge e. Thus, we have $E(G) \triangle E(H) = \{e\}$ or in other words $|E(G) \triangle E(H)| = 1$.

Definition 1.6. *Let $G_1, G_2, \dots$ graphs on the same vertex set V. Then $\mathcal{G} = \{G_1, G_2, \dots\}$ is a* graph class. *The* reconfiguration graph of $\mathcal{G}$ *is the graph is given by*

$$\mathcal{R}(\mathcal{G}) := (\mathcal{G}, \{GH | G, H \in \mathcal{G}, |E(G) \triangle E(H)| = 1\}).$$

We call $|V|$ the dimension of $\mathcal{G}$*, as well as the dimension of $\mathcal{R}(\mathcal{G})$.*

Note that the vertices of the reconfiguration graph are now graphs themselves. Thus, it is important to differentiate between properties of graphs $G \in \mathcal{G}$ which are from the graph class and thus vertices in the reconfiguration graph and properties of the reconfiguration graph itself.

From now on, we assume, unless stated otherwise, that $\mathcal{G} \neq \emptyset$ and $V = [n]$ for some fixed $n \in \mathbb{N}$. Especially, this means that V and thus also $\mathcal{G}$ are finite. Therefore, $\mathcal{R}(\mathcal{G})$ is a finite graph.

Given two graphs $G, H \in \mathcal{G}$, we are interested in the distance between G and H in $\mathcal{R}(\mathcal{G})$, that is, the length of a shortest path connecting G and H in $\mathcal{R}(\mathcal{G})$. For brevity, we denote $\mathrm{dist}_{\mathcal{G}}(G, H) := \mathrm{dist}_{\mathcal{R}(\mathcal{G})}(G, H)$. If the underlying graph class is clear, we may also simply use $\mathrm{dist}(G, H)$.

Another fundamental theorem which should be mentioned here is the triangle inequality. It is best known as a property of a triangle, stating that for the side lengths $a, b, c \in \mathbb{R}$ it holds that $a + b \geq c$. However, it is a more general property of the metric of any metric space. Most commonly, we make use of the triangle inequality for the distance function on a graph. As this is well-known, we do not prove it here.

Lemma 1.7. *Let G be a graph. Then dist_G satisfies the triangle inequality. In other words, for arbitrary $u, v, x \in V(G)$ it holds that*

$$\mathrm{dist}_G(u, v) \leq \mathrm{dist}_G(u, x) + \mathrm{dist}_G(x, v).$$

It should also be mentioned that we use the triangle inequality for other distance metrics, without explicitly showing that it holds. One such example is the triangle inequality for the symmetric difference, or to be exact, the cardinality of the symmetric difference. This gives us the statement that for arbitrary sets A, B, C it holds that

$$|A \triangle B| \leq |A \triangle C| + |C \triangle B|.$$

2 Basics

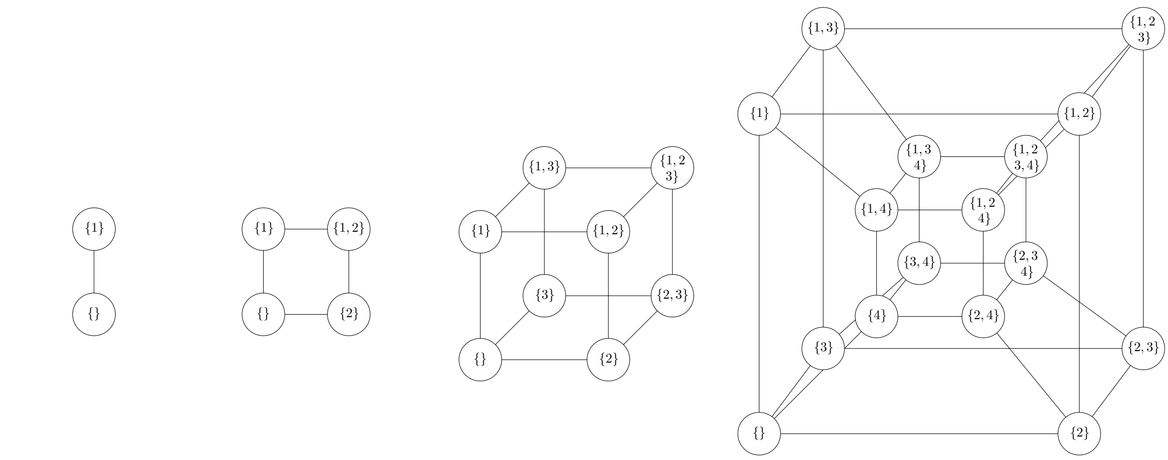
To get a better understanding of reconfiguration graphs, we start out by obtaining general results independent of the considered graph class. One of the most commonly examined graph parameters is the chromatic number of a graph. As the chromatic number is lower-bounded by the clique number of the graph, it is potent to take a look at both of them at once.

To tackle this, we show that for an arbitrary graph class $\mathcal{G}$ its reconfiguration graph $\mathcal{G}$ is a subgraph of a high dimensional hypercube. A hypercube, as the name suggests, is the graph representation of a higher dimensional cube. For $n = 1$ it represents a line segment, for $n = 2$ a square, and for $n = 3$ a cube. Illustrations for low dimensions can be seen in Figure 2.1. The formal definition for an arbitrary dimension is given below.

Definition 2.1. *The* hypercube of dimension $n \in \mathbb{N}$ *is the graph*

$$Q_n = (2^{[n]}, \{S_1S_2 | S_1, S_2 \subseteq [n], |S_1 \triangle S_2| = 1\}).$$

We consider an arbitrary graph class $\mathcal{G}$ of dimension n. We embed its reconfiguration graph $\mathcal{R}(\mathcal{G})$ in a hypercube of dimension $\binom{n}{2}$. This allows us to then use the well-studied properties of hypercubes to characterize the structure of the reconfiguration graph.



(a) dimension 1 **(b)** dimension 2 **(c)** dimension 3 **(d)** dimension 4

Figure 2.1: Hypercubes of different dimensions

Theorem 2.2. *Let $\mathcal{G}$ be a graph class of dimension n. Then $\mathcal{G}$ is isomorphic to a subgraph of $Q_{\binom{n}{2}}$.*

Proof. Let $\{e_1, e_2, \dots, e_{\binom{n}{2}}\} = \binom{[n]}{2}$ be a numbering of all possible edges in a graph on the vertex set $[n]$. Consider the mapping

$$\begin{aligned} \mathrm{f}\colon \mathcal{G} &\to 2^{[\binom{n}{2}]} \\ G &\mapsto \{i \mid e_i \in G\}. \end{aligned}$$

Using the fact that $|\,\mathrm{f}(G_1) \triangle \mathrm{f}(G_2)| = |E(G_1) \triangle E(G_2)|$, it is easy to verify that

$$Q_{\binom{n}{2}}[\{\mathrm{f}(G) \mid G \in \mathcal{G}\}] \cong \mathcal{R}(\mathcal{G}).$$

□

Hypercubes are bipartite, as one can 2-color them by assigning each vertex the color depending on the parity of the size of the set which labels the vertex. Furthermore, a hypercube of dimension n is n-regular, meaning that each vertex has degree n. As the chromatic number of a graph and vertex degrees of subgraphs are bounded by the respective value for the super graph we directly get the following results from Theorem 2.2.

Corollary 2.3. *For an arbitrary graph class $\mathcal{G}$, the reconfiguration graph $\mathcal{R}(\mathcal{G})$ is bipartite. Thus, it holds that*

$$\chi(\mathcal{R}(\mathcal{G})) \leq 2$$

as well as

$$\omega(\mathcal{R}(\mathcal{G})) \leq 2.$$

Corollary 2.4. *For an arbitrary graph class $\mathcal{G}$ of dimension n, it holds that*

$$\Delta(\mathcal{G}) \leq \binom{n}{2}.$$

Many other important graph parameters, like connectivity, diameter, and radius depend on distances between vertices. Thus, a lower bound for distances between vertices of $\mathcal{R}(\mathcal{G})$ proves to be useful. Let us consider two graphs $G, H \in \mathcal{G}$. Finding a shortest path in $\mathcal{R}(\mathcal{G})$ between G and H corresponds to finding a shortest sequence of edge additions and removals to transform G to H. However, the graph obtained after each operation has to be in $\mathcal{G}$. With this interpretation of distance in $\mathcal{R}(\mathcal{G})$, it is easy to see the following bound.

Lemma 2.5. *For a graph class $\mathcal{G}$ and two graphs $G, H \in \mathcal{G}$ it holds that*

$$\mathrm{dist}_{\mathcal{G}}(G, H) \geq |E(G) \triangle E(H)|.$$

Proof. Consider any shortest path $P = (G = G_0, G_1, \ldots, G_k = H)$ in $\mathcal{R}(\mathcal{G})$. By definition of $\mathcal{R}(\mathcal{G})$ we know that $|E(G_{i-1}) \triangle E(G_i)| = 1$ for all $1 \leq i \leq k$. Thus, applying the triangle inequality for the cardinality of the symmetric difference, we get that

$$|E(G) \triangle E(H)| \leq \sum_{i=1}^{k} |E(G_{i-1}) \triangle E(G_i)| = k = \operatorname{dist}_{\mathcal{R}(\mathcal{G})}(G, H).$$

□

In the remainder of this thesis, we consider graph classes with specific properties in order to obtain bounds on diameter, radius, and other graph parameters of reconfiguration graphs. In the following section we introduce a property which most considered graph classes satisfy.

2.1 Graph Classes Closed Under Isomorphism

In this section we take a quick look at graph classes which are closed under isomorphism, as these often arise naturally. Thus, nearly every well-studied graph class has this property.

Definition 2.6. *Let S be a finite set. We call a bijective mapping $p\colon S \to S$ a* permutation *of the set and denote the set of all permutations of S by* $\operatorname{Sym}(S)$.

For a graph G and a permutation of its vertex set $p\colon V(G) \to V(G)$ we denote the relabeling of its vertex set according to p by

$$p(G) := (V(G), \{p(u)p(v) \mid uv \in E(G)\}).$$

Definition 2.7. *A graph class $\mathcal{G}$ is called* isomorphism-closed *or* closed under isomorphism *if it holds that*

$$\forall G \in \mathcal{G}\ \forall p \in \operatorname{Sym}(V(G)) :\ p(G) \in \mathcal{G}.$$

In other words, taking any arbitrary graph G from the graph class $\mathcal{G}$ and rearranging the vertex labels of G, the newly obtained graph $p(G)$ is once again a member of $\mathcal{G}$. This should also give some intuition on why most graph classes commonly considered satisfy this property. They are usually defined by a certain property that each member graph has. While the exact form of these properties varies wildly, they are usually independent of vertex labels.

For example, consider the class of planar graphs. Recall that a graph G is planar if there exists some drawing of G on the plane, for which no two edges cross. This characterization is independent of the vertex labels themselves. Thus, one can follow that taking a planar graph G and a permutation p of its vertex set, $p(G)$ is planar as well. Therefore, the class of planar graphs is closed under isomorphism. This does not change when restricting the vertex set. Specifically, if we fix some $n \in \mathbb{N}$ and consider the graph class $\mathcal{G}$ of the planar graphs on vertex set $[n]$, then $\mathcal{G}$ is still closed under isomorphism.

The same goes for the graph class of connected graphs, acyclic graphs, chordal graphs, and comparability graphs to only give some examples. All of them are closed under isomorphism, no matter whether we restricted them to graphs with vertex set $[n]$ for some fixed $n \in \mathbb{N}$ or not.

It should however be mentioned that not all graph classes are closed under isomorphism. To give an example, fix some $n \in \mathbb{N}$ and consider the graph class

$$\mathcal{G} = \{G \text{ graph} \mid V(G) = [n], \text{vertex 1 has the highest degree}\}.$$

This graph class is not closed under isomorphism for $n \geq 3$, as rearranging vertex labels changes which vertex has the highest degree. Another example would be the class of chordal graphs on the vertex set $[n]$ for some fixed $n \geq 3$, for which the sequence $1, 2, \ldots, n$ is a perfect elimination scheme.

However, it still holds that the majority of popular graph classes are closed under isomorphism, as defining graph properties depending on the vertex labels tends to be arbitrary.

In the following chapters, we show results for graph classes with certain properties. As these properties are independent of isomorphism-closure, we shall at many points take a quick detour to consider isomorphism-closed graph classes and obtain stronger results for them.

3 Monotone Graph Classes

There are many graph properties that are monotone under edge addition. For example, the chromatic number of a graph χ cannot decrease under edge addition and is thus monotone increasing under edge addition. The same goes for the maximum degree Δ, the minimum degree δ, or clique number ω. Other graph properties like the independence number α of a graph, are monotone decreasing. We call such properties monotone.

Let's now fix a property p, which is monotone decreasing and some values $n, k \in \mathbb{N}$ and consider the graph class containing all graphs on the vertex set $[n]$ for which the property is at most k

$$\mathcal{G} := \{G \text{ graph} \mid V(G) = [n], p(G) \leq k\}.$$

Note that if we know that some graph $G \in \mathcal{G}$ is in the graph class, it directly follows that any subgraph $H \subseteq G$ is also in $\mathcal{G}$, as $\mathrm{p}(H) \leq \mathrm{p}(G) \leq k$. A graph class with the property that all subgraphs of member graphs are contained is called monotone decreasing.

Such graph classes often arise naturally. One well-known example is the set of bipartite graphs on the vertex set $[n]$ for some $n \in \mathbb{N}$. In this case the monotone graph property would be the chromatic number χ and the upper bound would be $k = 2$.

Also note that we could have just as well considered a monotone increasing graph property or lower bounded the graph property instead of upper bounding it. In all of these cases we obtain a graph class satisfying either one of the following properties.

Definition 3.1. *A graph class $\mathcal{G}$ is monotone (decreasing) if*

$$\forall G \in \mathcal{G}\ \forall H \subseteq G\colon\ H \in \mathcal{G}$$

and monotone increasing if

$$\forall G \in \mathcal{G}\ \forall H \supseteq G\colon\ H \in \mathcal{G}.$$

To start out, it should be easy to see that for any monotone graph class $\mathcal{G}$ its reconfiguration graph $\mathcal{R}(\mathcal{G})$ is connected. If you struggle to see this, take a look at Lemma 3.5, which proves a stronger result. Thus, it is potent to study distances in $\mathcal{R}(\mathcal{G})$. Furthermore, we take a look at the radius and diameter of $\mathcal{R}(\mathcal{G})$, as these help us understand the global structure of distances in the graph.

In this chapter, unless mentioned otherwise, $\mathcal{G}$ is a monotone graph class. The main results are Theorem 3.2 and Theorem 3.3, giving us a way to calculate the diameter and radius of the reconfiguration graph $\mathcal{R}(\mathcal{G})$. Furthermore, we obtain strong results about the center of $\mathcal{R}(\mathcal{G})$ in Corollary 3.8 and Theorem 3.9.

Theorem 3.2. *Let $\mathcal{G}$ be a monotone graph class. Then it holds that*

$$\operatorname{diam}(\mathcal{R}(\mathcal{G})) = \max_{G,H\in\mathcal{G}} \|G \cup H\|.$$

Theorem 3.3. *Let $\mathcal{G}$ be a monotone graph class. Then it holds that*

$$\operatorname{rad}(\mathcal{R}(\mathcal{G})) = \max_{G\in\mathcal{G}} \|G\|.$$

3.1 Distances

As already mentioned, the reconfiguration graph $\mathcal{R}(\mathcal{G})$ for a monotone graph class $\mathcal{G}$ of dimension n is connected. This can be seen, as for $\mathcal{G} \neq \emptyset$ it holds that $E_n \in \mathcal{G}$, and fixing some graph from the graph class $G \in \mathcal{G}$, we can construct a path to E_n in $\mathcal{R}(\mathcal{G})$ by repeatedly deleting edges. This idea can also be applied to obtain a tight bound on the distances in $\mathcal{R}(\mathcal{G})$.

Lemma 3.4. *Let $\mathcal{G}$ be a monotone graph class and $G \in \mathcal{G}, H \subseteq G$. Then it holds that*

$$\operatorname{dist}(G, H) = \|G\| - \|H\|.$$

Proof. Note that by monotonicity of $\mathcal{G}$ it follows that $H \in \mathcal{G}$. We prove the claimed equality by showing "$\leq$" and "$\geq$" separately.

At first, note that as $H \subseteq G$ it holds that $E(G)\triangle E(H) = E(G) \setminus E(H)$. Applying Lemma 2.5 we get a lower bound on the distance $\operatorname{dist}(G, H) \geq |E(G)\triangle E(H)| = \|G\| - \|H\|$.

To find an upper bound of $\operatorname{dist}(G, H)$, let $k = \|G\| - \|H\|$ and denote the edges which are in G but not in H as $E(G) \setminus E(H) = \{e_1, \ldots, e_k\}$. We shall construct a path in $\mathcal{R}(\mathcal{G})$ from G to H, by starting from G and iteratively removing the edges $\{e_1, \ldots, e_k\}$.

Consider the path $P = (G = G_0, G_1, \ldots, G_k = H)$ where $G_i = G_{i-1} - e_i$. It has length k and is a path in $\mathcal{R}(\mathcal{G})$. Firstly, all G_i are subgraphs of G and thus by monotonicity of $\mathcal{G}$ combined with $G \in \mathcal{G}$ it follows that $G_i \in \mathcal{G}$. Furthermore, $|G_{i-1}\triangle G_i| = |\{e_i\}| = 1$ for $1 \leq i \leq k$ and thus by definition $G_{i-1}G_i \in E(\mathcal{R}(\mathcal{G}))$. As P is a path of length k we know that $\operatorname{dist}(G, H) \leq \|P\| = k$. □

We can now extend this to show a tight bound for the distance between any pair $G, H \in \mathcal{G}$. We do so by applying the previous lemma to $\operatorname{dist}(G, G \cap H)$ and $\operatorname{dist}(G \cap H, H)$ and using the triangle inequality for distances in a graph.

Lemma 3.5. *Let $G, H \in \mathcal{G}$. Then it holds that*

$$\operatorname{dist}(G, H) = |E(G)\triangle E(H)|.$$

Proof. The lower bound for the distance $\operatorname{dist}(G,H) \geq |E(G) \triangle E(H)|$ is given by Lemma 2.5.

Note that $G \cap H \subseteq G \in \mathcal{G}$ and thus by monotonicity of $\mathcal{G}$ we have $G \cap H \in \mathcal{G}$. With this, we can get the upper bound by applying the triangle inequality for the distance function and Lemma 3.4. Doing so, we get

$$\begin{aligned}\operatorname{dist}(G,H) &\leq \operatorname{dist}(G, G \cap H) + \operatorname{dist}(G \cap H, H) \\ &= \|G\| - \|G \cap H\| + \|H\| - \|G \cap H\| \\ &= |E(G) \triangle E(H)|.\end{aligned}$$

□

3.2 Diameter

This section consists of only one short proof, showing that

$$\operatorname{diam}(\mathcal{R}(\mathcal{G})) = \max_{G,H \in \mathcal{G}} \|G \cup H\|.$$

To follow along, it is crucial to keep the definition of the diameter of a graph G in mind

$$\operatorname{diam}(G) = \max_{u,v \in V} \operatorname{dist}_G(u,v).$$

The following proof shows a lower and upper bound on $\operatorname{diam}(\mathcal{R}(\mathcal{G}))$ separately. For the upper bound, we prove that for any arbitrary $G \in \mathcal{G}$ all other graphs of $\mathcal{G}$ are sufficiently close to it. For the lower bound, we construct a pair of graphs $G, H \in \mathcal{G}$, such that $\operatorname{dist}(G,H)$ is sufficiently large.

Proof of Theorem 3.2. Let n be the dimension of $\mathcal{G}$ and let us denote the maximal number of edges in the union of two graphs of the graph class $\mathcal{G}$ by $M = \max_{G,H \in \mathcal{G}} \|G \cup H\|$.

Let $G, H \in \mathcal{G}$ be arbitrary. Then, applying Lemma 3.5, we get an upper bound to the distance

$$\operatorname{dist}(G,H) = |E(G) \triangle E(H)| = \|G \cup H\| - \|G \cap H\| \leq \|G \cup H\| \leq M.$$

For the lower bound, let $G, H \in \mathcal{G}$ such that $\|G \cup H\|$ is maximal. We set $H' = ([n], E(H) \setminus E(G))$ and see that $H' \subseteq H \in \mathcal{G}$, giving us that $H' \in \mathcal{G}$. With Lemma 3.5 we now observe that

$$\operatorname{dist}(G,H') = |E(G) \triangle E(H')| = \|G \cup H'\| = \|G \cup H\| = M.$$

Thus, we have found a pair of vertices of $\mathcal{R}(\mathcal{G})$, with distance M.

Putting the two parts together, we get the claim

$$M \leq \max_{G,H \in \mathcal{G}} \operatorname{dist}(G,H) = \operatorname{diam}(\mathcal{R}(\mathcal{G})) = \max_{G,H \in \mathcal{G}} \operatorname{dist}(G,H) \leq M.$$

□

It should be said that while this result gives us an exact value for the diameter of the reconfiguration graph, there are still monotone graph classes for which calculation of the radius of their reconfiguration graph is non-trivial. An example of how this calculation may look can be seen in Section 3.6.2. There, we consider the class of planar graphs on the vertex set $[n]$ for different $n \in \mathbb{N}$ and calculate the diameter of their reconfiguration graphs.

3.3 Radius and Center

Let $\mathcal{G}$ be a monotone graph class. We also obtain a nice closed form for the radius of $\mathcal{R}(\mathcal{G})$

$$\mathrm{rad}(\mathcal{R}(\mathcal{G})) = \max_{G \in \mathcal{G}} \|G\|.$$

To follow along, one should have the definition of the graph radius in mind. It is given by

$$\mathrm{rad}(G) = \min_{u \in V} \epsilon(u) = \min_{u \in V} \max_{v \in V} \mathrm{dist}_G(u, v).$$

The following proof shows the equality in the claim of Theorem 3.3 by proving "≤" and "≥" separately. For the upper bound of the radius we show that the empty graph has sufficiently small eccentricity in $\mathcal{R}(\mathcal{G})$. For the lower bound, we consider an arbitrary graph $H \in \mathcal{G}$ and construct a graph sufficiently far away from it in $\mathcal{R}(\mathcal{G})$.

Proof of Theorem 3.3. Let n be the dimension of $\mathcal{G}$. Recall Lemma 3.5, which gives us $\mathrm{dist}(G, H) = |E(G) \triangle E(H)|$ for any two graphs $G, H \in \mathcal{G}$. This gets applied multiple times in the following argument.

By monotonicity of $\mathcal{G}$ and $\mathcal{G} \neq \emptyset$ we know that $E_n \in \mathcal{G}$. Choosing E_n as our central vertex in $\mathcal{R}(\mathcal{G})$ we get

$$\mathrm{rad}(\mathcal{R}(\mathcal{G})) \leq \epsilon(E_n) = \max_{G \in \mathcal{G}} \mathrm{dist}(E_n, G) = \max_{G \in \mathcal{G}} |\emptyset \triangle E(G)| = \max_{G \in \mathcal{G}} \|G\|.$$

In order to show the other inequality, we shall for arbitrary $H \in \mathcal{G}$ find $H_2 \in \mathcal{G}$ which has distance at least $\max_{G \in \mathcal{G}} \|G\|$ to H. It then follows that $\epsilon(H) \geq \max_{G \in \mathcal{G}} \|G\|$, giving us the claim.

Fix some $G_{max} \in \mathcal{G}$ with the maximum number of edges, and consider an arbitrary $H \in \mathcal{G}$. Let $H_2 = (V(H), E(G_{max}) \setminus E(H))$ and recognize that $H_2 \in \mathcal{G}$ as $H_2 \subseteq G_{max}$. We observe that

$$\mathrm{dist}(H, H_2) = |E(H) \triangle (E(G_{max}) \setminus E(H))| = |E(G_{max}) \cup (E(H) \setminus E(G_{max}))| \geq |E(G_{max})|.$$

With this, we can finally conclude that

$$\mathrm{rad}(\mathcal{R}(\mathcal{G})) = \min_{H \in \mathcal{G}} \epsilon(H) \geq \min_{H \in \mathcal{G}} \mathrm{dist}(H, H_2) \geq \max_{G \in \mathcal{G}} \|G\|.$$

□

This theorem gives us a nice way to calculate the radius of the reconfiguration graph for monotone graph classes. It should be noted that compared to the formula we found for the diameter, this one tends to be easier to calculate and is already well studied for most common graph classes.

We use the remainder of this section to characterize the center of the reconfiguration graph. Remember that the center of a graph is the set of vertices with minimal eccentricity. From the previous proof, one can directly see that for a monotone graph class $\mathcal{G}$ of dimension n, the empty graph $E_n \in \mathcal{G}$ is always in the center, but we find some stronger results.

Theorem 3.6. *Let $\mathcal{G}$ be a monotone graph class and $\mathcal{G}_{max} \subseteq \mathcal{G}$ be the set of all graphs in $\mathcal{G}$ with the maximum number of edges. For every graph in the center $H \in \mathrm{center}(\mathcal{R}(\mathcal{G}))$ it holds that*

$$H \subseteq \bigcap_{G \in \mathcal{G}_{max}} G.$$

Proof. Instead of proving it directly, we instead negate the claim and show that for $H \not\subseteq \bigcap_{G \in \mathcal{G}_{max}} G$ it follows that $H \notin \mathrm{center}(\mathcal{R}(\mathcal{G}))$.

Let $H \in \mathcal{G}$ such that $H \not\subseteq \bigcap_{G \in \mathcal{G}_{max}} G$. Thus, there is some $G_{max} \in \mathcal{G}_{max}$ such that $H \not\subseteq G_{max}$. Hence H has at least one edge not in G_{max}, in other words $|E(H) \setminus E(G_{max})| \geq 1$. Let $H_2 = ([n], E(G_{max}) \setminus E(H))$ and consider:

$$\begin{aligned}
\mathrm{dist}(H, H_2) &= |E(H) \triangle (E(G_{max}) \setminus E(H))| \\
&= |E(G_{max}) \cup (E(H) \setminus E(G_{max}))| \\
&= |E(G_{max})| + |E(H) \setminus E(G_{max})| \\
&> |E(G_{max})| = \mathrm{rad}(\mathcal{R}(\mathcal{G}))
\end{aligned}$$

Where the last equality stems from Theorem 3.3. It follows that H cannot be a central vertex. □

While the previous theorem used the set of maximum graphs $\mathcal{G}_{max}$ we can get a stronger statement, characterizing $\mathrm{center}(\mathcal{G})$ considering all maximal graphs $\mathcal{G}_{maximal}$. Recall that the difference between maximum and maximal graphs is a graph $G \in \mathcal{G}$ is maximal in the graph class, if no strict supergraph $H \subset G$ is contained in $\mathcal{G}$. Thus, there may be maximal graphs, which do not have the maximum number of edges and are thus not maximum.

Theorem 3.7. *Let $\mathcal{G}$ be a monotone graph class and k denote the maximum number of edges of any $G \in \mathcal{G}$ and $\mathcal{G}_{maximal} := \{G \in \mathcal{G} | H \in \mathcal{G}, G \subseteq H \Rightarrow G = H\}$ denote the set of maximal graphs in $\mathcal{G}$. Then for every $H \in \mathcal{G}$ it holds that*

$$H \in \mathrm{center}(\mathcal{R}(\mathcal{G})) \Leftrightarrow \forall G \in \mathcal{G}_{maximal} : |E(H) \setminus E(G)| \leq k - \|G\|.$$

The proof for this theorem is omitted, as it contains no new ideas, and is simply an extension of the proof of Theorem 3.3.

From the characterization of the graphs in the center of $\mathcal{R}(\mathcal{G})$ given by Theorem 3.7 and noticing that $|E(H) \setminus E(G)|$ can only get smaller if an edge is removed from H, the following can be seen.

Corollary 3.8. *Let $\mathcal{G}$ be a monotone graph class. Then it holds that* $\mathrm{center}(\mathcal{R}(\mathcal{G}))$ *is monotone.*

3.4 Monotone Graph Classes Closed Under Isomorphism

In this section we take a quick look at graph classes closed under isomorphism. While our claims about the diameter and radius of the reconfiguration graph $\mathcal{R}(\mathcal{G})$ are already tight, we obtain a rather loose characterization of the graph center. This can be improved if we know that $\mathcal{G}$ is closed under isomorphism. In fact, we obtain a full taxonomy.

Theorem 3.9. *Let $\mathcal{G}$ be a monotone graph class of dimension n that is closed under isomorphism. It holds that*

$$\mathrm{center}(\mathcal{R}(\mathcal{G})) = \begin{cases} V(\mathcal{R}(\mathcal{G})) = \mathcal{G} & \text{if } K_n \in \mathcal{G} \\ \{E_n\} & \text{otherwise} \end{cases}$$

Proof. First, consider the case where $K_n \in \mathcal{G}$. By monotonicity, $\mathcal{G}$ contains all graphs on the vertex set $[n]$.

By Theorem 3.3 we know that $\mathrm{rad}(\mathcal{R}(\mathcal{G})) = \max_{G \in \mathcal{G}} \|G\| = \|K_n\| = \binom{n}{2}$. Consider an arbitrary $H \in \mathcal{G}$ and observe that by Lemma 3.5 for any $G \in \mathcal{G}$ it holds that $\mathrm{dist}(H, G) = |E(G) \triangle E(H)| \leq \binom{n}{2}$. It directly follows that $\epsilon(H) \leq \binom{n}{2} = \mathrm{rad}(\mathcal{R}(\mathcal{G}))$ and thus $H \in \mathrm{center}(\mathcal{R}(\mathcal{G}))$. As $H \in \mathcal{G}$ was arbitrary we can conclude that $\mathrm{center}(\mathcal{R}(\mathcal{G})) = V(\mathcal{R}(\mathcal{G})) = \mathcal{G}$.

Now assume that $K_n \notin \mathcal{G}$. To prove the claim, we consider some graph G in $\mathcal{G}$ with the maximum number of edges. Using the fact that $\mathcal{G}$ is closed under isomorphism, we find a set of graphs with the maximal number of edges, such that no edge is contained in all of them. Applying Theorem 3.6 we get the desired result.

We have $K_n \notin \mathcal{G}$ by assumption. Especially we have $\max_{G \in \mathcal{G}} \|G\| < \binom{n}{2}$. Fix some graph $G \in \mathcal{G}$ with the maximum number of edges. As $\|G\| < \binom{n}{2}$, it has at least one non-edge $xy = e \in \binom{[n]}{2} \setminus E(G)$. If we now consider some arbitrary permutation p of the vertex set, we find that by definition $p(x)p(y) \notin E(p(\mathcal{G}))$. Also note that $p(G) \in \mathcal{G}$, as $\mathcal{G}$ is closed under isomorphism. Furthermore, $p(G)$ is another graph with the maximal number of edges as $\|G\| = \|p(G)\|$. Note that for each edge $uv \in \binom{[n]}{2}$ we can find a permutation of the vertex set p such that $u = p(x)$ and $v = p(y)$. Combining this with Theorem 3.6 we obtain, that for any $H \in \mathrm{center}(\mathcal{G})$ it holds that

$$H \subseteq \bigcap_{G \in \mathcal{G}_{max}} G \subseteq \bigcap_{p \in \mathrm{Sym}(V(G))} p(G) = E_n.$$

This directly gives us center$(\mathcal{G}) \subseteq \{E_n\}$. As the center of any graph is non-empty, we conclude that center$(\mathcal{G}) = \{E_n\}$. □

At this point it should once again be mentioned that nearly all commonly considered graph classes are isomorphism-closed. Thus, Theorem 3.9 gets applied multiple times in the upcoming Section 3.6 to find the center of $\mathcal{R}(\mathcal{G})$ for different graph classes $\mathcal{G}$.

3.5 Monotone Increasing Graph Classes

Up to this point we have only considered monotone decreasing graph classes. So one may wonder if similar results can be obtained for monotone increasing graph classes. It turns out this is possible. To see this, we first consider two possible ways of taking a complement of a graph class.

Definition 3.10. *Let $\mathcal{G}$ be a graph class of dimension n. Then the* complement *of $\mathcal{G}$ is*

$$\mathcal{G}^{\mathrm{c}} = \{G \text{ graph} \mid V(G) = [n], G \notin \mathcal{G}\}.$$

In other words, the complement of $\mathcal{G}$ contains exactly those graphs on the same vertex set, which are not in $\mathcal{G}$. A second definition of a complement of a graph class can be obtained by taking complements of each member graph $G \in \mathcal{G}$.

Definition 3.11. *Let $\mathcal{G}$ be a graph class of dimension n. Then the* element-wise complement *of $\mathcal{G}$ is*

$$\mathcal{G}_{compl} = \{\overline{G} \mid G \in \mathcal{G}\}.$$

It is easy to see that the following correlation between the monotonicity of $\mathcal{G}$, $\mathcal{G}^{\mathrm{c}}$, and $\mathcal{G}_{compl}$ holds.

Observation 3.12. *Let $\mathcal{G}$ be a graph class.*

- *If $\mathcal{G}$ is monotone decreasing, then $\mathcal{G}^{\mathrm{c}}$ and $\mathcal{G}_{compl}$ are monotone increasing.*
- *If $\mathcal{G}$ is monotone increasing, then $\mathcal{G}^{\mathrm{c}}$ and $\mathcal{G}_{compl}$ are monotone decreasing.*

Let us only discuss the second item of Observation 3.12, as the first item of Observation 3.12 can be shown analogously. Consider some monotone increasing graph class $\mathcal{G}$.

For $\mathcal{G}_{compl}$, monotonicity is easiest to see. Consider some $G \in \mathcal{G}_{compl}$ and $H \subseteq G$. We need to show that $H \in \mathcal{G}_{compl}$. At first, note that as $H \subseteq G$ it holds that $\overline{H} \supseteq \overline{G}$. Plugging in the definition of $\mathcal{G}_{compl}$ and keeping in mind that $\mathcal{G}$ is monotone increasing we get

$$G \in \mathcal{G}_{compl} \Rightarrow \overline{G} \in \mathcal{G} \Rightarrow \overline{H} \in \mathcal{G} \Rightarrow H \in \mathcal{G}_{compl}.$$

For $\mathcal{G}^c$ monotonicity may not be as intuitive. For the sake of contradiction assume we have $G \in \mathcal{G}^c$ and $H \subseteq G$ with $H \notin \mathcal{G}^c$. By definition of $\mathcal{G}^c$ this means that $H \in \mathcal{G}$. As $\mathcal{G}$ is monotone increasing by assumption and $H \subseteq G$, we find that $G \in \mathcal{G}$, giving us a contradiction to $G \in \mathcal{G}^c$. Thus, for $G \in \mathcal{G}^c$ and $H \subseteq G$ it holds that $H \in \mathcal{G}^c$.

If we have a monotone increasing graph class $\mathcal{G}$, one can now apply our results for monotone graph classes to $\mathcal{G}^c$ or $\mathcal{G}_{compl}$ to better understand $\mathcal{R}(\mathcal{G}^c)$ and $\mathcal{R}(\mathcal{G}_{compl})$. As we want to obtain results for $\mathcal{R}(\mathcal{G})$, we need one more insight.

Lemma 3.13. *For a graph class $\mathcal{G}$, it holds that $\mathcal{R}(\mathcal{G}) \cong \mathcal{R}(\mathcal{G}_{compl})$.*

Proof. This can be seen by considering the map

$$\begin{aligned} f : \mathcal{G} &\to \mathcal{G}_{compl} \\ G &\mapsto \overline{G}. \end{aligned}$$

It is easy to verify that f is an isomorphism between $\mathcal{R}(\mathcal{G})$ and $\mathcal{R}(\mathcal{G}_{compl})$. □

Keeping in mind that all previously considered graph properties, like degree conditions, graph diameter, and radius, are invariant under graph isomorphism, we can combine Observation 3.12 and Lemma 3.13 to obtain the following results.

Theorem 3.14. *Let $\mathcal{G}$ be a monotone increasing graph class. Then it holds that*

$$\operatorname{diam}(\mathcal{R}(\mathcal{G})) = \binom{n}{2} - \min_{G,H \in \mathcal{G}} \|G \cap H\|.$$

Theorem 3.15. *Let $\mathcal{G}$ be a monotone increasing graph class. Then it holds that*

$$\operatorname{rad}(\mathcal{R}(\mathcal{G})) = \binom{n}{2} - \min_{G \in \mathcal{G}} \|G\|.$$

Proofs of Theorems 3.14 and 3.15 are omitted, as the claims are directly obtained from Theorem 3.2 and Theorem 3.3, respectively.

In the same way, one can also obtain twins of Corollary 3.8 and Theorem 3.9 for monotone increasing graph classes, but this is not presented in detail. One may also consider the other definition of the complement of a graph class $\mathcal{G}^c$ in order to characterize $\mathcal{R}(\mathcal{G}^c)$. However, as $\mathcal{R}(\mathcal{G}^c)$ cannot be related to $\mathcal{R}(\mathcal{G})$ easily, this gets more involved and is not carried out here.

One may also be tempted to consider graph classes that are monotone increasing as well as monotone decreasing. However, the following shall make it obvious why we do not do so.

Theorem 3.16. *Let $\mathcal{G}$ be a graph class of dimension n, which is monotone increasing as well as monotone decreasing. Then it holds that $\mathcal{G} = \{G \mid G$ graph on vertex set $[n]\}$*

Proof. Remember that we only consider non-empty graph classes, otherwise $\mathcal{G} = \emptyset$ would be an exception. Since $\mathcal{G} \neq \emptyset$, we find some $G \in \mathcal{G}$. As $\mathcal{G}$ is monotone increasing and $K_n \supseteq G$, we have that $K_n \in \mathcal{G}$. Monotonicity of $\mathcal{G}$ then gives us that for any arbitrary graph H on vertex set $[n]$ we can use $H \subseteq K_n$ to follow $H \in \mathcal{G}$. □

With this, we hope that it is clear why monotone increasing graph classes are not considered further. In the following, we may instead take a look at some well-known graph classes, like planar graphs or bipartite graphs and apply the previously found statements to them, giving us structural characterizations of their respective reconfiguration graphs.

3.6 Application of Results to Graph Classes

This section applies the results we have for monotone graph classes $\mathcal{G}$ to some well-known examples of monotone graph classes. Especially, we calculate the diameter and radius of the reconfiguration graph $\mathcal{R}(\mathcal{G})$ with the use of Theorem 3.2 and Theorem 3.3. Moreover, we show that each of them is closed under isomorphism and therefore apply Theorem 3.9 to obtain the structure of the center of $\mathcal{R}(\mathcal{G})$.

3.6.1 Acyclic Graphs

We start out with a graph class for which calculation of the radius and diameter is easy. Consider the family of graph classes

$$\mathcal{F}_n = \{G \text{ graph on vertex set } [n] \mid G \text{ is acyclic}\}.$$

In other words, $\mathcal{F}_n$ contains exactly the forests on n vertices. It should be easy to see that $\mathcal{F}_n$ is both monotone and closed under isomorphism for arbitrary $n \in \mathbb{N}$.

Lemma 3.17. *Let $n \in \mathbb{N}$ and $\mathcal{F}_n = \{G$ graph on vertex set $[n] \mid G$ is acyclic$\}$. It holds that*

- *$\mathcal{F}_n$ is monotone.*
- *$\mathcal{F}_n$ is closed under isomorphism.*

Proof. We tackle monotonicity first. Consider some $G \in \mathcal{F}$. As G does not contain a cycle, it is obvious that any $H \subseteq G$ cannot contain any cycle either, as H is obtained from G by several edge deletions. It follows that $H \in \mathcal{F}_n$.

Isomorphism-closure of $\mathcal{F}_n$ is easy to see as well. Consider any $G \in \mathcal{F}$ and for the sake of contradiction assume there is a permutation $p \in \mathrm{Sym}([n])$ such that $p(G) \notin \mathcal{F}$. Thus, there exists a cycle of length $k \geq 3$ on the vertices $v_1, \ldots, v_k$ in $p(G)$. With this it is not hard to see that $p^{-1}(v_1), \ldots, p^{-1}(v_k)$ is a cycle in G. This is a contradiction to $G \in \mathcal{F}$, letting us conclude that the assumption was false. Isomorphism closure follows directly. □

With this result, we can apply Theorem 3.3 and Theorem 3.9 to obtain an exact formula for $\mathrm{rad}(\mathcal{R}(\mathcal{F}))$ as well as a taxonomy of $\mathrm{rad}(\mathcal{R}(\mathcal{F}))$.

Theorem 3.18. *Let $n \in \mathbb{N}$ and $\mathcal{F}_n = \{G$ graph on vertex set $[n] \mid G$ is acyclic$\}$. It holds that*

- $\mathrm{rad}(\mathcal{R}(\mathcal{F}_n)) = n - 1$.
- $\mathrm{center}(\mathcal{R}(\mathcal{F}_n)) = \begin{cases} \mathcal{F}_n & \text{if } n \leq 2 \\ \{E_n\} & \text{if } n \geq 3 \end{cases}$

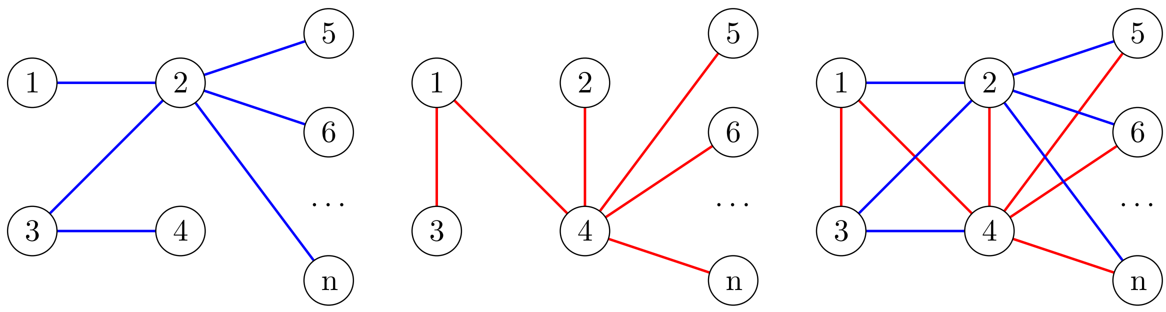
Proof. By Lemma 3.17 $\mathcal{F}_n$ is monotone. Thus, we can apply Theorem 3.3 to get that $\mathrm{rad}(\mathcal{R}(\mathcal{F}_n)) = max_{G \in \mathcal{F}_n} \|G\|$. It is well known that the edge maximal acyclic graphs are the trees. Furthermore, any tree on n vertices has $n - 1$ edges, giving us the claimed $\mathrm{rad}(\mathcal{R}(\mathcal{F}_n)) = n - 1$.

Note that for $n = 1, 2$ the clique K_n is acyclic and thus $K_n \in \mathcal{F}_n$. On the other hand, for any other $n \geq 3$ the clique K_n contains a triangle and therefore is acyclic. Therefore, $K_n \notin \mathcal{F}_n$ for $n \geq 3$. As $\mathcal{F}_n$ is closed under isomorphism and monotone by Lemma 3.17 we can use Theorem 3.9. Together with the previous observations, we get the claimed result. □

Now that we have found a formula for the radius of $\mathcal{R}(\mathcal{F}_n)$ and characterized the center, we turn to calculation of the diameter. For this, we use monotonicity of $\mathcal{F}_n$ to apply Theorem 3.2. Some basic observations about acyclic graphs then give us the final result.

Theorem 3.19. *Let $n \in \mathbb{N}$ and $\mathcal{F}_n = \{G$ graph on vertex set $[n] \mid G$ is acyclic$\}$. It holds that*

$$\mathrm{diam}(\mathcal{R}(\mathcal{F}_n)) = \min\{\binom{n}{2}, 2n - 2\}.$$



(a) T_1 **(b)** T_2 **(c)** $T_1 \cup T_2$

Figure 3.1: Two edge disjoint trees on the vertex set $[n]$ for $n \geq 4$

Proof. By Lemma 3.17 we know that $\mathcal{F}_n$ is monotone, letting us apply Theorem 3.2. We thus have

$$\operatorname{diam}(\mathcal{R}(\mathcal{F}_n)) = \max_{G,H \in \mathcal{F}_n} \|G \cup H\|.$$

It is easy to see that for $n \leq 3$ we can split the edges of K_n into two forests, giving us that $\operatorname{diam}(\mathcal{R}(\mathcal{F}_n)) = \binom{n}{2}$. For larger $n \geq 4$ we can always find two edge-disjoint trees. In the following we use v_i to denote vertex i. One construction of such trees is

$$T_1 = ([n], \{v_1v_2, v_2v_3, v_3v_4\} \cup \{v_2v_i \mid 5 \leq i \leq n\})$$

$$T_2 = ([n], \{v_2v_4, v_4v_1, v_1v_3\} \cup \{v_4v_i \mid 5 \leq i \leq n\})$$

See Figure 3.1 for a visualization. It is easy to see that T_1, T_2 are acyclic graphs on the vertex set $[n]$ and thus $T_1, T_2 \in \mathcal{F}_n$. Furthermore, they are edge disjoint $E(T_1) \cap E(T_2) = \emptyset$ and satisfy $\|T_1\| = \|T_2\| = n-1$. Thus, we have $\|T_1 \cup T_2\| = 2n-2$. As any acyclic graph can have at most $n-1$ edges, this is optimal, letting us conclude that $\operatorname{rad}(\mathcal{R}(\mathcal{F})) = 2n-2$ for $n \geq 4$. Together with the previous result of $\operatorname{rad}(\mathcal{R}(\mathcal{F})) = \binom{n}{2}$ for $n \leq 3$ we get the claim. $\square$

This finishes up the section about acyclic graphs. We hope that it has become clear how we can use our results for monotone graph classes to calculate graph parameters $\operatorname{rad}(\mathcal{R}(\mathcal{G}))$ and $\operatorname{diam}(\mathcal{R}(\mathcal{G}))$ explicitly. The same is done for other monotone graph classes in the upcoming sections. It should be noted that, while the basic structure of these sections is the same, proofs become more involved. Especially the calculation of the diameter turns out to require some effort in both cases.

3.6.2 Planar Graphs

For this section we consider the graph classes

$$\mathcal{P}_n = \{G \text{ planar graph on the vertex set } [n]\}$$

for $n \in \mathbb{N}$. We start out by showing that $\mathcal{P}_n$ is monotone as well as closed under isomorphism, and then apply the results from this chapter to get a better understanding of the reconfiguration graph $\mathcal{R}(\mathcal{P}_n)$.

Lemma 3.20. *Let $n \in \mathbb{N}$ and $\mathcal{P}_n = \{G$ planar graph on the vertex set $[n]\}$. It holds that $\mathcal{P}_n$ is*

- *monotone.*
- *closed under isomorphism.*

Proof. We first tackle monotonicity. Consider any arbitrary $G \in \mathcal{P}$. By definition G is a planar graph on the vertex set $[n]$. Consider some $H \subseteq G$, which is still a graph on the vertex set $[n]$. Furthermore, it has to be planar, as fixing some planar embedding of G and removing the edges from $E(G) \setminus E(H)$, we obtain a planar embedding of H.

The graph class is also closed under isomorphism, as taking any $G \in \mathcal{P}$ and fixing some planar embedding of it, we can relabel its vertices arbitrarily to obtain another graph on the vertex set $[n]$ with the same planar embedding. □

Now that this is shown, we can apply Theorem 3.3 and Theorem 3.9. To get the final result, we need some well-known facts about planar graphs. Firstly K_4 is planar. Thus, K_1, K_2, K_3 are planar as well. On the other hand, for any larger $n \geq 5$ the clique K_n is not planar. Furthermore, it is well known that for each $n \geq 5$ the maximum number of edges a planar graph on n vertices can have is $3n-6$. For each such n there also exists a planar graph with $3n-6$ edges. With this, we can get the final results.

Theorem 3.21. *Let $n \in \mathbb{N}$ and $\mathcal{P}_n = \{G$ planar graph on the vertex set $[n]\}$. It holds that*

- $\text{rad}(\mathcal{R}(\mathcal{P}_n)) = \begin{cases} \binom{n}{2} & \text{if } n \leq 4 \\ 3n-6 & \text{if } n \geq 5 \end{cases}$
- $\text{center}(\mathcal{R}(\mathcal{P}_n)) = \begin{cases} \mathcal{P}_n & \text{if } n \leq 4 \\ \{E_n\} & \text{if } n \geq 5 \end{cases}$

Proof. We start out with the radius of the reconfiguration graph. By Theorem 3.3 we know that $\text{rad}(\mathcal{R}(\mathcal{P}_n)) = max_{G \in \mathcal{P}_n} \|G\|$. Recall that for $n = 1, 2, 3, 4$ the complete graph K_n is planar, while for larger n there exists some maximal planar graph on $3n-6$ edges and no planar graph can have more than $3n-6$ edges. This gives us the claimed formula for the radius.

For the graph center, we once again use the fact that for $n = 1, 2, 3, 4$ the complete graph K_n is planar and thus $K_n \in \mathcal{P}_n$. On the other hand, for any $n \geq 5$ the complete graph K_n is not planar, which gives us that $K_n \notin \mathcal{P}_n$. In both cases we can apply Theorem 3.9 to get the claim. □

We also calculate the diameter of the reconfiguration graph $\text{diam}(\mathcal{R}(\mathcal{P}_n))$. Doing so requires some more work and more advanced claims about planar graphs. Especially, we need results from [5] and [7], which allow us to find answers for $n \leq 8$ and $n \geq 12$ respectively. The gap $n = 9, 10, 11$ is then closed by computer search. With this, we can finally get the exact values for all $n \in \mathbb{N}$.

Theorem 3.22. *Let $n \in \mathbb{N}$ and $\mathcal{P}_n = \{G$ planar graph on the vertex set $[n]\}$. The diameter of $\mathcal{R}(\mathcal{P}_n)$ has the following value.*

$$\operatorname{diam}(\mathcal{R}(\mathcal{P}_n)) = \begin{cases} \binom{n}{2} & \text{if } n \leq 8 \\ 35 = \binom{n}{2} - 1 & \text{if } n = 9 \\ 43 = \binom{n}{2} - 2 & \text{if } n = 10 \\ 51 = \binom{n}{2} - 4 & \text{if } n = 11 \\ 6n - 12 & \text{if } n \geq 12 \end{cases}$$

Proof. We start out by considering the result for $n \leq 8$. This is directly given by [5], where it is shown that for $n \leq 8$, K_n is the union of two planar graphs G, H. Thus, we can apply Theorem 3.3 to get $\operatorname{diam}(\mathcal{R}(\mathcal{P}_n)) = \max_{G,H \in \mathcal{P}_n} \|G \cup H\| = \|K_n\| = \binom{n}{2}$.

We now take a look at the case $n \geq 12$. As a planar graph on n vertices has at most $3n - 6$ edges, it is easy to see that for two planar graphs G, H on the vertex set $[n]$, it holds that $\|G \cup H\| \leq 2 \cdot (3n - 6) = 6n - 12$. To get equality with this upper bound, we need to find two maximum planar graphs G and H with disjoint edge sets for each $n \geq 12$. The existence of such G and H is given in [7]. With this, we get the claim.

Finally, the values of $\operatorname{diam}(\mathcal{R}(\mathcal{P}_n))$ for $9 \leq n \leq 12$ are found by computer search. Some details for optimizing this are discussed in the following. The resulting planar graphs can be seen in Figure 3.2 □

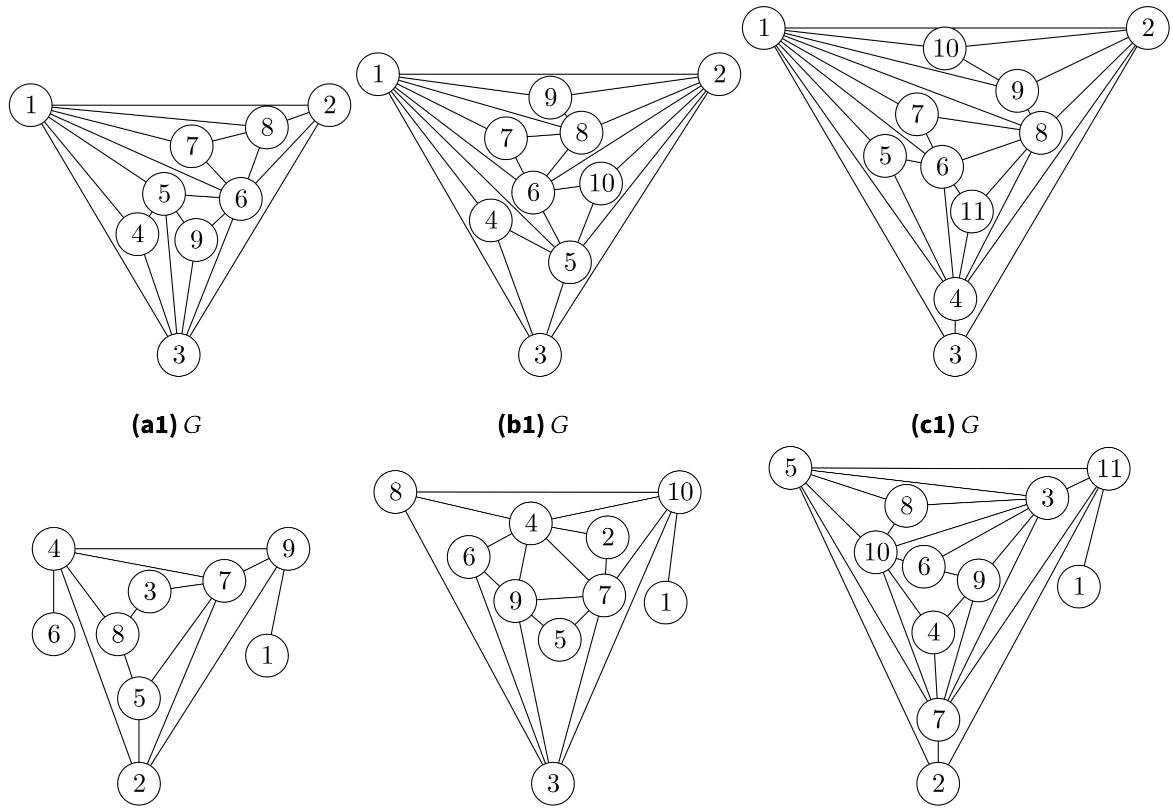
In the following, we discuss efficient ways to calculate $\max_{G,H \in \mathcal{P}_n} \|G \cup H\|$. Likely the most obvious ways to brute force these values would be to iterate over all pairs of planar graphs G, H and calculating $\|G \cup H\|$. However, this turns out to be too slow.

Alternatively, one can iterate over all graphs $G \subseteq K_n$ and for each check if it can be represented as the union of two planar graphs. This approach has the benefit that we can iterate G in order of decreasing $\|G\|$. Once we find one, which can be represented as the union of two planar graphs, we know that $\|G\|$ is the answer and can stop the search. Especially as the answers to $\operatorname{diam}(\mathcal{R}(\mathcal{P}))$ are quite close to $\binom{n}{2}$, we do not have to iterate over many G. With this, we only need an efficient way to check whether a fixed G is the union of two planar graphs. For this, we use the following observation.

Observation 3.23. *Let $n \in \mathbb{N}$ and H_1, H_2 be planar graphs on the vertex set $[n]$ and $G := H_1 \cup H_2$. Then there exists a maximal planar graph M such that $G - M$ is planar.*

This is easy to see, as $G - H_1 \subseteq H_2$ is already planar. Thus, we can add edges to H_1 until it is maximal planar and call the resulting graph M. Realizing that $G - M \subseteq G - H_1 \subseteq H_2$ is planar then gives us the claim.

Using this, we can check if some graph G can be expressed as the union of two planar graphs. We iterate over all maximal planar graphs H on the vertex set $[n]$ and then run a planarity check on $G - H$. Furthermore, it can be noticed that as we iterate over all graphs G



(a1) G **(b1)** G **(c1)** G

(a2) H **(b2)** H **(c2)** H

(a) $n = 9$ **(b)** $n = 10$ **(c)** $n = 11$

Figure 3.2: Planar graphs $G, H \in \mathcal{P}_n$ with maximal $\|G \cup H\|$ for $n = 9, 10, 11$

without considering isomorphism, it suffices to only iterate over all maximal planar graphs H up to isomorphism.

Algorithm 1

procedure CheckBiplanar(G)
  for M maximal planar graph on vertex set $V(G)$ up to isomorphism **do**
    if $G - M$ is planar **then**
      return true
  return false

procedure BruteForce(n)
  $S \leftarrow \binom{[n]}{2}$
  for $m \leftarrow \binom{n}{2}, \dots, 0$ **do**
    for $E \in \binom{S}{m}$ **do**
      $G \leftarrow ([n], E)$
      if CheckBiplanar(G) **then return** m
  return -1

Putting the described ideas together and using some implementation of a planarity check, we get Algorithm 1. It should be noted that planarity can be checked in linear time, and most graph libraries provide an efficient implementation of such an algorithm.

Algorithm 1 is quick enough for $9 \leq n \leq 11$, as even for $n = 11$ there are only 1249 maximal planar graphs up to isomorphism. Our implementation of this technique only requires on the order of 10^7 planarity checks until a solution is found. The set of maximal planar graphs up to isomorphism required for the loop in line 2 can be generated by a variety of graph libraries.

3.6.3 Graphs With Bounded Chromatic Number

To start out, we consider the class of bipartite graphs on n vertices for some $n \in \mathbb{N}$

$$\mathcal{B}_n = \{G \text{ graph on vertex set } [n] \mid \chi(G) \leq 2\}.$$

We start out by showing that $\mathcal{B}_n$ is monotone as well as closed under isomorphism. Then we apply previous results from this chapter to get a better understanding of the reconfiguration graph $\mathcal{R}(\mathcal{B}_n)$.

Lemma 3.24. *Let $n \in \mathbb{N}$ and $\mathcal{B}_n = \{G \text{ graph on vertex set } [n] \mid \chi(G) \leq 2\}$. The graph class $\mathcal{B}_n$ is monotone as well as closed under isomorphism.*

Proof. It is well known that χ is monotone. In other words, for a graph G and a subgraph $H \subseteq G$ it holds that $\chi(H) \leq \chi(G)$. This directly gives us monotonicity of $\mathcal{B}_n$.

Isomorphism closure is also easy to see, as the definition of χ is independent of vertex labels. □

Using Lemma 3.24, we can now apply the formulas for calculating radius and diameter. Doing so, we obtain the following result.

Theorem 3.25. *Let $n \in \mathbb{N}$ and $\mathcal{B}_n = \{G \text{ graph on vertex set } [n] \mid \chi(G) \leq 2\}$. It holds that*

- $\mathrm{rad}(\mathcal{R}(\mathcal{B}_n)) = \lfloor \frac{n}{2} \rfloor \cdot \lceil \frac{n}{2} \rceil$.
- $\mathrm{diam}(\mathcal{R}(\mathcal{B}_n)) = \binom{n}{2} - \binom{\lfloor \frac{n}{4} \rfloor}{2} - \binom{\lfloor \frac{n+1}{4} \rfloor}{2} - \binom{\lfloor \frac{n+2}{4} \rfloor}{2} - \binom{\lfloor \frac{n+3}{4} \rfloor}{2}$.

Proof. As this is needed for the later proof, we start out by characterizing edge maximal bipartite graphs. Fix some maximal bipartite graph $G \in \mathcal{B}_n$, and consider some proper 2-coloring of its vertices $c\colon V(G) \to \{1, 2\}$. Let $a = |c^{-1}(\{1\})|$ and $b = |c^{-1}(\{2\})|$ the number of vertices colored with colors 1 and 2 respectively. For any pair of vertices $u, v \in V(G)$ such that $c(u) = 1$ and $c(v) = 2$ it holds that $uv \in E(G)$. Otherwise, we could add the edge uv to G, obtaining a bipartite supergraph of G, which is a contradiction to the maximality of G. Furthermore, for any pair of vertices $u, v \in V(G)$ colored in the same color $c(u) = c(v)$, there can be no edge uv in G. Therefore, it holds that G is the complete bipartite graph $G \cong K_{a,b}$, giving us that $\|G\| = a \cdot b$. Note that as each maximal bipartite graph has the form $K_{a,b}$ for some $a, b \in \mathbb{N}_0$ satisfying $a + b = n$, any maximum bipartite graph also has to have this form.

Using Lemma 3.24 we can apply Theorem 3.3 and get $\operatorname{rad}(\mathcal{R}(\mathcal{B}_n)) = \max_{G \in \mathcal{B}_n} \|G\|$. With the previous characterization of maximum bipartite graphs, we get that $\max_{G \in \mathcal{B}_n} \|G\| = \max_{0 \leq a \leq n} \|K_{a,n-a}\| = \max_{0 \leq a \leq n} a \cdot (n - a)$. It turns out that this maximum is achieved for $a = \left\lfloor \frac{n}{2} \right\rfloor$ giving us that

$$\operatorname{rad}(\mathcal{R}(\mathcal{B}_n)) = \max_{G \in \mathcal{B}_n} \|G\| = \max_{0 \leq a \leq n} a \cdot (n - a) = \left\lfloor \frac{n}{2} \right\rfloor \cdot \left\lceil \frac{n}{2} \right\rceil$$

For the diameter we use Lemma 3.24 to apply Theorem 3.2, giving us that $\operatorname{diam}(\mathcal{R}(\mathcal{B}_n)) = \max_{G,H \in \mathcal{B}_n} \|G \cup H\|$. Note that adding edges to G or H respectively can only increase $\|G \cup H\|$. Thus, it suffices to consider pairs of maximal bipartite graphs G, H. With the previous characterization of maximal bipartite graphs, we know that $G = K_{a,n-a}$ and $H = K_{b,n-b}$ for some $0 \leq a, b \leq n$. Let A, A_2 be the vertex sets of the two sides of G and let B, B_2 be the sides of H. We thus know that for the number of edges in their union it holds that $\|G \cup H\| = \binom{n}{2} - \binom{|A \cap B|}{2} - \binom{|A_2 \cap B|}{2} - \binom{|A \cap B_2|}{2} - \binom{|A_2 \cap B_2|}{2}$. With this, it is easy to see that

$$\operatorname{diam}(\mathcal{R}(\mathcal{B}_n)) = \max_{G,H \in \mathcal{B}_n} \|G \cup H\| = \max_{\substack{0 \leq a,b,c,d \leq n \\ a+b+c+d=n}} \binom{n}{2} - \left(\binom{a}{2} + \binom{b}{2} + \binom{c}{2} + \binom{d}{2} \right)$$

It can be verified, that this term is maximal for $a = \left\lfloor \frac{n}{4} \right\rfloor$, $b = \left\lfloor \frac{n+1}{4} \right\rfloor$, $c = \left\lfloor \frac{n+2}{4} \right\rfloor$, $d = \left\lfloor \frac{n+3}{4} \right\rfloor$ giving us the claim. □

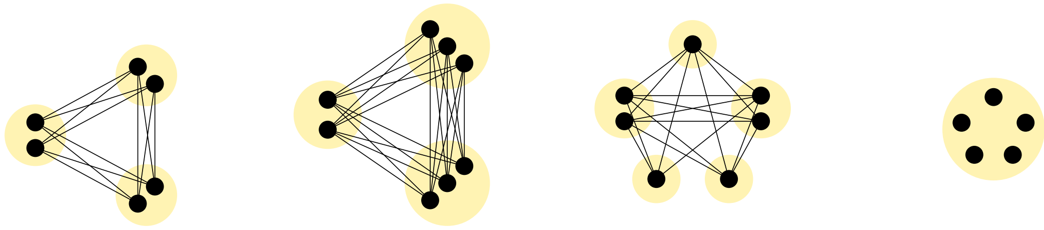
We generalize the results seen for bipartite graphs to graph classes of bounded chromatic number. For this we fix $n, r \in \mathbb{N}$ consider all graphs on vertex set $[n]$ which can be r-colored

$$\mathcal{G}_n^{\chi \leq r} = \{G \text{ graph on vertex set } [n] \mid \chi(G) \leq r\}.$$

Note that for $r = 2$ it holds that $\mathcal{G}_n^{\chi \leq 2} = \mathcal{B}_n$, for which we already have the results. Furthermore, for $r = 1$ it holds that $\mathcal{G}_n^{\chi \leq 1} = \{E_n\}$ as each graph containing an edge needs at least two colors in a proper coloring. Thus, it may seem pointless to consider $r = 1, 2$. However, we do not exclude them, as the following results and proofs work for them as well.

Once again, we use monotonicity of χ and invariance of χ under vertex relabeling, to obtain monotonicity and isomorphism-closure of $\mathcal{G}_n^{\chi \leq r}$.

Lemma 3.26. *Let $n, r \in \mathbb{N}$ and $\mathcal{G}_n^{\chi \leq r} = \{G$ graph on vertex set $[n] \mid \chi(G) \leq r\}$. The graph class $\mathcal{G}_n^{\chi \leq r}$ is monotone as well as closed under isomorphism.*



(a) $T(6,3)$ **(b)** $T(8,3)$ **(c)** $T(7,5)$ **(d)** $T(5,1)$

Figure 3.3: Some Turán graphs

Before calculating the diameter and radius for $\mathcal{R}(\mathcal{G}_n^{\chi \leq r})$ we show some auxiliary lemmas that are needed for the calculation.

Similarly to the characterization for maximum bipartite graphs, we need some insight into the maximum number of edges in a graph $G \in \mathcal{G}_n^{\chi \leq r}$. For this purpose, we introduce the well-known Turán graph.

Definition 3.27 (Turán graph). *Let $n, r \in \mathbb{N}$. Write $n = q \cdot r + s$ such that $s, q \in \mathbb{N}_0$ and $s < r$. Then the* Turán graph *$T(n,r)$ is the graph made up of s copies of E_{q+1} and $r - s$ copies of E_q, where each edge between vertices of different independent sets exists. In other words*

$$T(n,r) = K_{q,q,\ldots,q+1,q+1}$$

where the term q appears $r - s$ times and the term $q + 1$ appears s times.

The number of edges in a Turán graph is called the Turán number. *We denote it by $t(n,r) = \|T(n,r)\|$.*

Examples of some Turán graphs can be seen in Figure 3.3. Turán graphs are of interest to us because of Turán's theorem. We do not prove Turán's theorem here, as it is well known and several proofs can be found in literature.

Theorem 3.28 (Turán's theorem). *For every $n, r \in \mathbb{N}$, the Turán graph $T(n,r)$ has the maximum number of edges among all graphs on n vertices which do not contain K_{r+1} as a subgraph.*

We can now use this to obtain a result on the maximum number of edges in any graph with chromatic number at most r.

Corollary 3.29. *For every $n, r \in \mathbb{N}$, the Turán graph $T(n,r)$ has the maximum number of edges among all graphs G on n vertices which satisfy $\chi(G) \leq r$. In other words, it holds that*

$$\max_{G \in \mathcal{G}_n^{\chi \leq r}} \|G\| = t(n,r).$$

Proof. To start out, one should note that $\chi(T(n,r)) \leq r$, as we can color each of the r independent set in its own color. Thus, it holds that

$$\max_{G \in \mathcal{G}_n^{\chi \leq r}} \|G\| \geq \|T(n,r)\| = t(n,r).$$

To get the second direction, we apply Turán's theorem. Note that any graph containing a K_{r+1} has chromatic number at least $r+1$. As Turán's theorem tells us that any graph G on n vertices with more than $t(n,r)$ edges contains a K_{r+1}, we can conclude that $G \notin \mathcal{G}_n^{\chi \leq r}$. This gives us

$$\max_{G \in \mathcal{G}_n^{\chi \leq r}} \|G\| \leq \|T(n,r)\| = t(n,r)$$

which completes the proof. □

Let us now take some time to calculate $t(n,r)$ explicitly for cases where n is an integer multiple of r, as this is needed later on.

Lemma 3.30. *Let $r, k \in \mathbb{N}$ and $n = r \cdot k$. Then it holds that*

$$T(n,r) = \binom{r}{2} \cdot \left(\frac{n}{r}\right)^2 = \frac{n^2 \cdot (r-1)}{2r}$$

Proof. Let $V_1, \ldots, V_r$ denote the vertex sets of the independent sets from the construction of $T(n,r)$. We know that $|V_i| = \frac{n}{r} = k$ for $i \in [r]$. By definition the edges of $T(n,r)$ are exactly the pairs $u \in V_i$, $v \in V_j$ where $i \neq j$. Thus, we have

$$T(n,r) = \sum_{\substack{i,j \in [r] \\ i<j}} |V_i| \cdot |V_j| = \sum_{\substack{i,j \in [r] \\ i<j}} \left(\frac{n}{r}\right)^2 = \binom{r}{2} \cdot \left(\frac{n}{r}\right)^2.$$

□

Before we start to prove the main theorem, we still need an observation about the chromatic number of the union of graphs.

Lemma 3.31. *Let $n \in \mathbb{N}$ and G, H be graphs on the vertex set $[n]$. It holds that*

$$\chi(G \cup H) \leq \chi(G) \cdot \chi(H).$$

Proof. Fix a proper $\chi(G)$-coloring of G and call it $c_G \colon [n] \to [\chi(G)]$. Do the same for H to obtain c_H. We use these to construct $c \colon [n] \to [\chi(G)] \times [\chi(H)]$ a proper coloring of $G \cup H$. As $|[\chi(G)] \times [\chi(H)]| = \chi(G) \cdot \chi(H)$ this then directly gives us the claim.

For any vertex $v \in [n]$ color it according to $c(v) = (c_G(v), c_H(v))$. We now need to show that c is a proper coloring of $G \cup H$. To do so, consider some edge $uv = e \in E(G \cup H)$.

Case $e \in E(G)$: As c_G is a proper coloring of G we know that $c_G(u) \neq c_G(v)$. Therefore, we have $c(u) = (c_G(u), c_H(u)) \neq (c_G(v), c_H(v)) = c(v)$.

Case $e \notin E(G)$: As the edge e appears in the union of G and H but not in G we know that $e \in E(H)$. Thus, we can use c_H to apply the same argument as in the first case to get $c(u) \neq c(v)$.

Note that in both cases we have $c(u) \neq c(v)$. Thus, as e was arbitrary, we know that c is a proper coloring, concluding the proof. □

We can now turn our attention to proving the main result of this section. The proof applies the previous lemmas to obtain the radius and diameter of $\mathcal{R}(\mathcal{G}_n^{\chi \leq r})$.

Theorem 3.32. *Let $n, r \in \mathbb{N}$ and $\mathcal{G}_n^{\chi \leq r} = \{G \text{ graph on vertex set } [n] \mid \chi(G) \leq r\}$. It holds that*

- $\mathrm{rad}(\mathcal{R}(\mathcal{G}_n^{\chi \leq r})) = t(n, r)$.
- $\mathrm{diam}(\mathcal{R}(\mathcal{G}_n^{\chi \leq r})) = t(n, r^2)$.

Proof. Remember that $\mathcal{G}_n^{\chi \leq r}$ is monotone by Lemma 3.26.

To show the first claim of the theorem we apply Theorem 3.3 to get $\mathrm{rad}(\mathcal{R}(\mathcal{G}_n^{\chi \leq r})) = \max_{G \in \mathcal{G}_n^{\chi \leq r}} \|G\|$. Corollary 3.29 then gives us $\max_{G \in \mathcal{G}_n^{\chi \leq r}} \|G\| = t(n, r)$. The claim directly follows.

The proof of the second part of the theorem is a bit more involved. We start by showing that $t(n, r^2)$ is an upper bound to the diameter. Afterwords, we construct graphs $G, H \in \mathcal{G}_n^{\chi \leq r}$ which satisfy $\|G \cup H\| = t(n, r^2)$. The claim then follows.

As $\mathcal{G}_n^{\chi \leq r}$ is monotone we can apply Theorem 3.2 to get $\mathrm{diam}(\mathcal{R}(\mathcal{G}_n^{\chi \leq r})) = \max_{G, H \in \mathcal{G}_n^{\chi \leq r}} \|G \cup H\|$. Fix some pair $G, H \in \mathcal{G}_n^{\chi \leq r}$. By definition, we know that $\chi(G), \chi(H) \leq r$. Applying Lemma 3.31 we get that $\chi(G \cup H) \leq r^2$. With this, we can use the Corollary 3.29 of Turán's theorem, giving us that $\|G \cup H\| \leq t(n, r^2)$. As G and H are arbitrary we know that

$$\mathrm{diam}(\mathcal{R}(\mathcal{G}_n^{\chi \leq r})) = \max_{G, H \in \mathcal{G}_n^{\chi \leq r}} \|G \cup H\| \leq t(n, r^2).$$

Let us now construct a pair of graphs $G, H \in \mathcal{G}_n^{\chi \leq r}$, such that their union has $t(n, r^2)$ edges. Start out by setting $G = T(n, r) \in \mathcal{G}_n^{\chi \leq r}$. Denote the vertices of the r independent sets from the construction of $T(n, r)$ by $V_1, \ldots, V_r$. We set H as the disjoint union of Turán graphs $T(|V_1|, r), \ldots, T(|V_r|, r)$ on the vertex sets $V_1, \ldots, V_r$ respectively. There are no edges between vertices $u \in V_i, v \in V_j$ for $i \neq j$. For a visualization of how this may look consider Figure 3.4. It is easy to see that $\chi(H) \leq r$, as we can color each component $H[V_i]$ independently in r colors. Thus, we have $H \in \mathcal{G}_n^{\chi \leq r}$.

It remains to be shown that $\|G \cup H\| = t(n, r^2)$. As this is not very difficult but extremely tedious, we show the calculation only for the case where n is a multiple of r^2. Let $k \in \mathbb{N}$

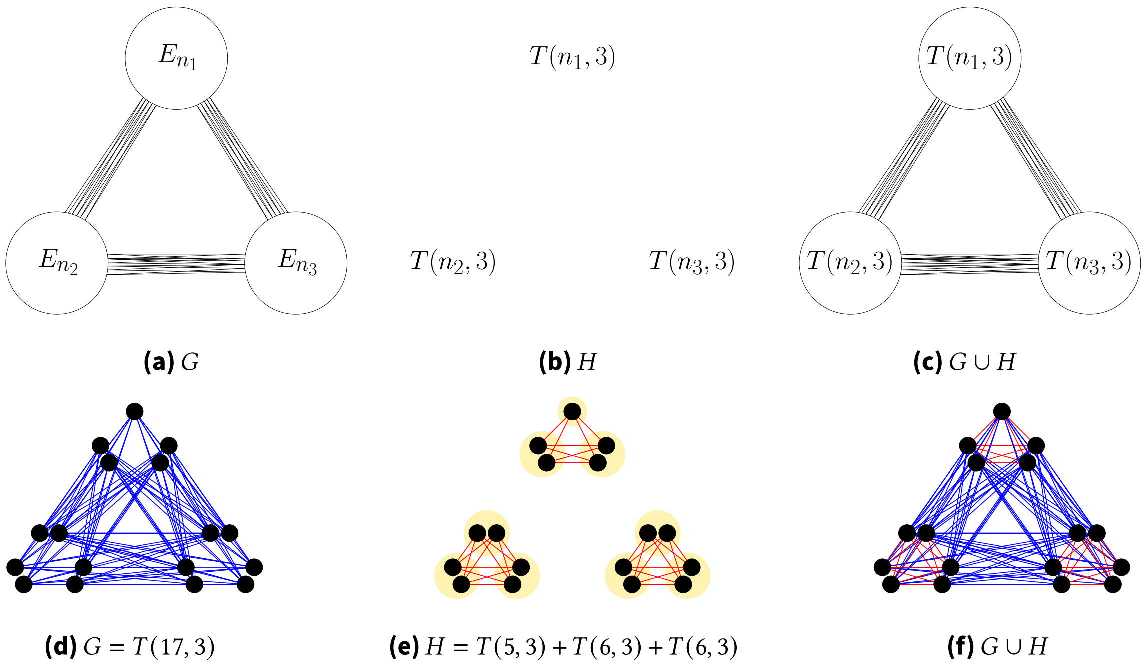


Figure 3.4: Edge maximal union of graphs with chromatic number at most $k = 3$

such that $n = k \cdot r^2$. Note that the independent sets $V_1, \ldots, V_r$ in G all have size $\frac{n}{r} = k \cdot r$. Thus, H is the disjoint union of r copies of $T(k \cdot r, k)$. By construction G and H have no common edges, as G only has edges between $u \in V_i, v \in V_j$ for $i \neq j$ while H only has edges where $i = j$. With this insight, we can calculate the number of edges $\|G \cup H\| = \|G\| + \|H\| - \|G \cap H\| = \|G\| + \|H\|$ by calculating $\|G\|$ and $\|H\|$ separately. For G we get this directly from Lemma 3.30, stating that

$$\|G\| = \frac{n^2 \cdot (r-1)}{2r} = \frac{k^2 \cdot r^3 \cdot (r-1)}{2}.$$

We can also use Lemma 3.30 to calculate the number of edges of each of the r copies of $T(k \cdot r, r)$, giving us that

$$\|H\| = r \cdot T(k \cdot r, r) = r \cdot \frac{(k \cdot r)^2 \cdot (r-1)}{2r} = \frac{k^2 \cdot r^2 \cdot (r-1)}{2}$$

Putting this together and using Lemma 3.30 for one last time, we get

$$\begin{aligned} \|G \cup H\| = \|G\| + \|H\| &= \frac{k^2 \cdot r^3 \cdot (r-1) + k^2 \cdot r^2 \cdot (r-1)}{2} \\ &= \frac{k^2 \cdot r^2 \cdot (r+1) \cdot (r-1)}{2} = \frac{n^2 \cdot (r^2-1)}{2r^2} = t(n, r^2) \end{aligned}$$

□

This concludes the section dealing with graphs of bounded chromatic number. In the following, we consider graphs with bounded clique number. As for any graph G it holds that $\omega(G) \leq \chi(G)$, one may expect the reconfiguration graph to examine similar graph properties to the ones proven in this section.

3.6.4 Graphs With Bounded Clique Number

We also consider sets of graphs with bounded clique number. More specifically for fixed $n, k \in \mathbb{N}$ we consider

$$\mathcal{G}_n^{\omega \leq k} = \{G \text{ graph on vertex set } [n] \mid \omega(G) \leq k\}.$$

We once again find that $\mathcal{G}_n^{\omega \leq k}$ is monotone and closed under isomorphism. The proof of these facts shall not be exercised here, as it is not hard, and many similar proofs can be found in this chapter.

Lemma 3.33. *Let $n, k \in \mathbb{N}$ and $\mathcal{G}_n^{\omega \leq k} = \{G \text{ graph on vertex set } [n] \mid \omega(G) \leq k\}$. It holds that $\mathcal{G}_n^{\omega \leq k}$ is monotone as well as closed under isomorphism.*

As chromatic number and clique number are closely related, it is quite intuitive to expect that for graphs with bounded clique number the reconfiguration graph $\mathcal{R}(\mathcal{G}_n^{\omega \leq k})$ exhibits similar graph parameters to the reconfiguration graph $\mathcal{R}(\mathcal{G}_n^{\chi \leq r})$ for graphs with bounded chromatic number. Interestingly enough, it turns out that we get the same formula for the radius of the reconfiguration graph, but we have no such luck with the diameter.

Theorem 3.34. *Let $n, k \in \mathbb{N}$ and $\mathcal{G}_n^{\omega \leq k} = \{G \text{ graph on vertex set } [n] \mid \omega(G) \leq k\}$. It holds that*

- $\mathrm{rad}(\mathcal{R}(\mathcal{G}_n^{\omega \leq k})) = t(n, k)$.
- $\mathrm{diam}(\mathcal{R}(\mathcal{G}_n^{\omega \leq k})) \geq t(n, k^2)$.

Proof. Remember that by Lemma 3.33 we know that $\mathcal{G}_n^{\omega \leq k}$ is monotone. Therefore, we can apply Theorem 3.3 to get $\mathrm{rad}(\mathcal{R}(\mathcal{G}_n^{\omega \leq k})) = \max_{G \in \mathcal{G}_n^{\omega \leq k}} \|G\|$. Theorem 3.28 then gives us $\max_{G \in \mathcal{G}_n^{\omega \leq k}} \|G\| = t(n, k)$. The claim follows.

For the second part of the theorem, we consider the results we know for graphs of bounded chromatic number. Observe that for any graph G it holds that $\omega(G) \leq \chi(G)$. It directly follows that $\mathcal{G}_n^{\chi \leq k} \subseteq \mathcal{G}_n^{\omega \leq k}$. Therefore, we can use the construction given in the proof of Theorem 3.32 to get $\mathrm{diam}(\mathcal{R}(\mathcal{G}_n^{\omega \leq k})) \geq t(n, k^2)$. □

Note that the reason why we do not get an upper bound to $\mathrm{diam}(\mathcal{R}(\mathcal{G}_n^{\omega \leq k}))$ is that we do not have an upper bound for the clique number of the union of two graphs, similar to the one we saw in Lemma 3.31 for chromatic numbers. For a better understanding of the diameter of $\mathcal{R}(\mathcal{G}_n^{\omega \leq k})$ we consider the following.

Theorem 3.35. *Let $n \in \mathbb{N}$ and $k \geq 2 \cdot \log_2(n)$. Then it holds that*

$$\mathrm{diam}(\mathcal{R}(\mathcal{G}_n^{\omega \leq k})) = \binom{n}{2}.$$

Proof. The following proof is a standard application of the probabilistic method. We prove the result by considering a random graph G on the vertex set $[n]$ and showing that with positive probability the graph G satisfies $\omega(G), \omega(G^c) \leq k$. Thus, we can follow that there exists some such G. By the definition of the graph class we therefore have $G \in \mathcal{G}_n^{\omega \leq k}$ as well as $G^c \in \mathcal{G}_n^{\omega \leq k}$. With this, we can apply Theorem 3.2 to get

$$\operatorname{diam}(\mathcal{R}(\mathcal{G}_n^{\omega \leq k})) = \max_{H_1, H_2 \in \mathcal{G}_n^{\omega \leq k}} \|H_1 \cup H_2\| \geq \|G \cup G^c\| = \binom{n}{2}.$$

Furthermore, we have $\max_{H_1, H_2 \in \mathcal{G}_n^{\omega \leq k}} \|H_1 \cup H_2\| \leq \|K_n\| = \binom{n}{2}$ giving us the claim.

Without loss of generality let $n \geq 2$ as the statement trivially holds for $n = 1$.

Fix $p = \frac{1}{2}$ and consider an Erdős–Rényi graph $G = G(n, p)$. That is, the graph on the vertex set $[n]$, which has each edge independently with probability $p = \frac{1}{2}$. We start out by calculating the probability that G has a clique number larger than k.

$$\begin{aligned}
\mathbb{P}[\omega(G) > k] &= \mathbb{P}[\omega(G) \geq k+1] = \mathbb{P}[\exists A \subseteq [n], |A| = k+1 : G[A] \cong K_{k+1}] \\
&\leq \sum_{A \in \binom{[n]}{k+1}} \mathbb{P}[G[A] \cong K_{k+1}] = \binom{n}{k+1} \cdot p^{\binom{k}{2}} \\
&\leq \frac{n^{k+1}}{(k+1)! \cdot 2^{\frac{k \cdot (k+1)}{2}}} = \frac{1}{(k+1)!} \left(\frac{n}{2^{\frac{k}{2}}}\right)^{k+1} \\
&\leq \frac{1}{(k+1)!} \left(\frac{n}{2^{\log_2(n)}}\right)^{k+1} = \frac{1}{(k+1)!} \left(\frac{n}{n}\right)^{k+1} = \frac{1}{(k+1)!} \\
&\leq \frac{1}{(2 \cdot \log_2(n) + 1)!} \leq \frac{1}{6} < \frac{1}{2}
\end{aligned}$$

The first inequality is obtained from the union bound and the second is obtained by plugging the definition of a binomial coefficient for $\binom{n}{k+1}$ and using $\frac{n!}{(n-k-1)!} \leq n^{k+1}$. The remaining inequalities are simple applications of $k \geq 2 \cdot \log_2(n)$ and $n \geq 2$.

Furthermore, fix some graph H on the vertex set $[n]$. It is easy to see that as we have $p = \frac{1}{2}$ it holds that $\mathbb{P}[G^c = H] = 2^{-\binom{n}{2}} = \mathbb{P}[G = H]$. Let E denote the event "$\omega(G) \leq k$" and E_c denote the event "$\omega(G^c) \leq k$". We use $\overline{E}$ to denote the complement of E, which is "$\omega(G) > k$" and $\overline{E}_c$ as the complement of E_c.

By the previous results, we know that $\mathbb{P}[\overline{E}] = 1 - \mathbb{P}[E] < 1 - \frac{1}{2} = \frac{1}{2}$ and $\mathbb{P}[\overline{E}_c] = \mathbb{P}[\overline{E}]$. Putting this together, we have

$$\begin{aligned}
\mathbb{P}[\omega(G) \leq k \text{ and } \omega(G^c) \leq k] &= \mathbb{P}[E \cap E_c] = 1 - (\mathbb{P}[\overline{E} \cap E_c] + \mathbb{P}[E \cap \overline{E}_c] + \mathbb{P}[\overline{E} \cap \overline{E}_c]) \\
&= 1 - (\mathbb{P}[\overline{E} \cap E_c] + \mathbb{P}[\overline{E}_c]) \geq 1 - (\mathbb{P}[\overline{E}] + \mathbb{P}[\overline{E}_c]) \\
&= 1 - 2 \cdot \mathbb{P}[\overline{E}] > 1 - 2 \cdot \frac{1}{2} = 0.
\end{aligned}$$

As our random graph satisfies the property with positive probability, we know that there exists some graph G with $\omega(G), \omega(G^c) \leq k$, concluding the proof. □

Note that this also directly gives us a case where the lower bound on $\operatorname{diam}(\mathcal{G}_n^{\omega\leq k})$ given in Theorem 3.34 is not tight. More precisely, combining Thorem 3.35 and Theorem 3.30 gives us that for $k = 2 \cdot \lceil log(n) \rceil$ it holds that

$$\frac{t(n,k^2)}{\operatorname{diam}(\mathcal{R}(\mathcal{G}_n^{\omega\leq k}))} \approx 1 - \frac{1}{\log(n)^2}.$$

Let us consider the case $\operatorname{diam}(\mathcal{R}(\mathcal{G}_n^{\omega\leq k})) = \binom{n}{2}$ from another view point. For this, one needs to be familiar with the definition of the Ramsey numbers.

Definition 3.36. *For $k \in \mathbb{N}$ the (diagonal) Ramsey number is*

$$R(k,k) = \min\{n \in \mathbb{N} \mid \textit{every } 2\textit{-coloring of the edges of } K_n \textit{ contains a monochromatic } K_k\}.$$

Theorem 3.37. *Let $n, k \in \mathbb{N}$. Then it holds that*

$$\operatorname{diam}(\mathcal{R}(\mathcal{G}_n^{\omega\leq k})) = \binom{n}{2} \Leftrightarrow n \leq R(k+1,k+1) - 1.$$

Proof. By Theorem 3.2 we know that $\operatorname{diam}(\mathcal{R}(\mathcal{G}_n^{\omega\leq k})) = \max_{G,H\in\mathcal{G}_n^{\omega\leq k}} \|G \cup H\|$.

We start by showing the implication '$\Rightarrow$'. So $\binom{n}{2} = \operatorname{diam}(\mathcal{R}(\mathcal{G}_n^{\omega\leq k})) = \max_{G,H\in\mathcal{G}_n^{\omega\leq k}} \|G \cup H\|$ holds. Fix $G, H \in \mathcal{G}_n^{\omega\leq k}$ with $\|G \cup H\| = \binom{n}{2}$. We construct a 2-edge-coloring $c\colon \binom{n}{2} \to [2]$ of K_n, which contains no monochromatic clique of size $k + 1$.

$$c(e) = \begin{cases} 1 & \text{if } e \in E(G) \\ 2 & \text{otherwise} \end{cases}$$

Assume that K_n contains a monochromatic clique of size $k+1$ under the coloring c. Call the vertex set of the monochromatic clique $A \in \binom{[n]}{k+1}$.

Case the clique has color 1: Then we know that for each $u, v \in A$ it holds that $c(uv) = 1$ and thus $uv \in E(G)$. Thus, $G[A] \cong K_{k+1}$ and therefore $\omega(G) \geq k+1$ which is a contradiction to $G \in \mathcal{G}_n^{\omega\leq k}$.

Case the clique has color 2: Then we know that for each $u, v \in A$ it holds that $c(uv) = 2$ and thus $uv \notin E(G)$. As $|E(G) \cup E(H)| = \binom{n}{2}$ we know that $E(G) \cup E(H) = \binom{[n]}{2}$. With this, we directly have that $uv \in E(H)$. Thus, $H[A] \cong K_{k+1}$ and $\omega(H) \geq k+1$ which is a contradiction to $H \in \mathcal{G}_n^{\omega\leq k}$.

As we get a contradiction in both cases, we know that the coloring c of K_n does not contain a monochromatic clique of size $k + 1$ and thus $n < R(k+1, k+1)$. As n and $R(k+1, k+1)$ are integers we have $n \leq R(k+1, k+1) - 1$.

The other implication '$\Leftarrow$' can be shown in a similar way. Starting out with $n \leq R(k+1, k+1) - 1$ we can fix a 2-edge-coloring c of K_n which contains no monochromatic K_{k+1}. It is now easy to see that $G = ([n], c^{-1}(\{1\})) \in \mathcal{G}_n^{\omega\leq k}$ and $H = ([n], c^{-1}(\{2\})) \in \mathcal{G}_n^{\omega\leq k}$. Furthermore, we have $\|G \cup H\| = \binom{n}{2}$, giving us the claim. □

Using Theorem 3.37 and combining it with the known bound on the diagonal Ramsey number $R(k,k) \leq 4^k$, one could obtain a proof of Theorem 3.35. In the same way it is possible to combine better upper bounds of $R(k,k)$ with Theorem 3.37 to get stronger versions of Theorem 3.35, but this shall not be exercised here.

Instead, let us take a moment to consider the implications this has on the hardness of calculating $\operatorname{diam}(\mathcal{R}(\mathcal{G}_n^{\omega\leq k}))$. By Theorem 3.37 deciding whether $\operatorname{diam}(\mathcal{R}(\mathcal{G}_n^{\omega\leq k})) = \binom{n}{2}$ is equivalent to determining if $n < R(k+1,k+1)$ holds. Therefore, calculation of $\operatorname{diam}(\mathcal{R}(\mathcal{G}_n^{\omega\leq k}))$ is at least as hard as calculating $R(k+1,k+1)$. While no explicit hardness results for the calculation of Ramsey numbers exist, there is no known way to efficiently compute them. Up until this day, $R(5,5)$ is not known exactly. Instead, $43 \leq R(5,5) \leq 46$ are the best known bounds shown in [15] and [3]. Therefore, there is little hope of easy calculation of $\operatorname{diam}(\mathcal{R}(\mathcal{G}_{45}^{\omega\leq 4}))$. Calculation of $R(6,6)$ is commonly believed to not be possible in the foreseeable future. For larger $k > 6$ calculating $R(k,k)$ is even harder. Thus, calculating $\operatorname{diam}(\mathcal{R}(\mathcal{G}_n^{\omega\leq k}))$ for $k \geq 6$ is hard for at least some $n \in \mathbb{N}$.

3.6.5 Other Monotone Graph Classes

One can consider further monotone graph classes. For example, the class of outer planar graphs or the class of triangle-free graphs are monotone but not considered here. One can also construct further monotone graph classes by bounding other monotone graph properties similar to what we did for chromatic number and clique number in Section 3.6.3 and Section 3.6.4. Some examples that come to mind are independence number, min degree, max degree, the number of connected components, treewidth, and others. Note that for the independence number, one has to lower-bound it, while the other mentioned properties should be upper-bounded, to obtain a monotone graph class. For time reasons we do not exercise this here.

4 Weakly Monotone Graph Classes

In this chapter we consider a property of graph classes, which is weaker than monotonicity. Recall that for a graph class $\mathcal{G}$ to be monotone, it needs to hold that for any graph of the graph class $G \in \mathcal{G}$ all subgraphs $H \subseteq G$ are also in the graph class $H \in \mathcal{G}$. This is equivalent to the property that for any graph of the class $G \in \mathcal{G}$ we can remove any edge $e \in E(G)$ and the resulting graph $G - e$ is still in the graph class $\mathcal{G}$. This is the part of the definition of monotonicity that we weaken. Namely, we do not require that $G - e \in \mathcal{G}$ for every edge $e \in E(G)$, but instead there only has to exist some edge $e \in E(G)$ with this property.

Definition 4.1. *A graph class $\mathcal{G}$ of dimension n is weakly monotone (decreasing) if*

$$\forall G \in \mathcal{G} \text{ with } \|G\| \geq 1\ \exists H \subseteq G\colon\ \|H\| + 1 = \|G\|,\ H \in \mathcal{G}$$

and weakly monotone increasing if

$$\forall G \in \mathcal{G} \text{ with } \|G\| \leq \binom{n}{2} - 1\ \exists H \supseteq G\colon\ \|H\| = \|G\| + 1, H \in \mathcal{G}.$$

While this property may seem arbitrary at first, we later on find many graph classes which are not monotone but satisfy the weak version of monotonicity defined in 4.1. Among them are chordal graphs, comparability graphs, interval graphs and many more.

In this chapter $\mathcal{G}$ is a weakly monotone graph class. The main results in this section are Theorem 4.2 and Theorem 4.3. Note that, as weak monotonicity is a weaker property than monotonicity, these results are also weaker than what we saw in the corresponding result Theorem 3.2 and Theorem 3.3 in the previous chapter. But it turns out we can still obtain a 2-aproximation of the diameter and radius respectively.

Theorem 4.2. *Let $\mathcal{G}$ be a weakly monotone graph class. It holds that*

$$\max_{G \in \mathcal{G}} \|G\| \leq \operatorname{diam}(\mathcal{R}(\mathcal{G})) \leq 2 \cdot \max_{G \in \mathcal{G}} \|G\|.$$

Furthermore, these bounds are tight.

Theorem 4.3. *Let $\mathcal{G}$ be a weakly monotone graph class. It holds that*

$$\frac{\max_{G \in \mathcal{G}} \|G\|}{2} \leq \operatorname{rad}(\mathcal{R}(\mathcal{G})) \leq \max_{G \in \mathcal{G}} \|G\|.$$

Furthermore, these bounds are tight.

4.1 Distances

To start out, it is easy to see that for any weakly monotone graph class $\mathcal{G}$ the reconfiguration graph $\mathcal{R}(\mathcal{G})$ is connected. This can be seen by constructing a path from an arbitrary $G \in \mathcal{G}$ to the empty graph, by repeatedly deleting an edge. By the definition of weak monotonicity, we can in each step choose an edge to delete, such that the graph obtained from edge removal is still in $\mathcal{G}$. As the reconfiguration is connected, it is potent to consider distances in $\mathcal{R}(\mathcal{G})$. We find certain bounds on distances which are used to prove the theorems about the radius and diameter of $\mathcal{R}(\mathcal{G})$.

We apply the same idea described in the paragraph above to an arbitrary $G \in \mathcal{G}$ to obtain an upper bound on $\text{dist}_{\mathcal{R}(\mathcal{G})}(E_n, G)$ where n is the dimension of $\mathcal{G}$. Combining this with the lower given by Lemma 2.5, we find that the distance is given by $\|G\|$.

Lemma 4.4. *Let $\mathcal{G}$ be a weakly monotone graph class of dimension n and $G \in \mathcal{G}$ a member graph. Then it holds that:*

$$\text{dist}(E_n, G) = \|G\|$$

Proof. Lemma 2.5 directly gives us that

$$\text{dist}(E_n, G) \geq \|G\|.$$

It remains to be shown that $\|G\|$ is also an upper bound for the distance. We prove this by constructing a path between G and E_n of the desired length by iteratively removing edges from G until we arrive at E_n.

Construct a graph sequence starting with $G = G_0$ and for $1 \leq i \leq \|G\|$ iteratively choosing $G_i \subseteq G_{i-1}$ with $\|G_i\| + 1 = \|G_{i-1}\|$ which is also in the graph class, so $G_i \in \mathcal{G}$ holds. The existence of such a graph is given by weak monotonicity of $\mathcal{G}$.

Note that $(G = G_0, G_1, \ldots, G_{\|G\|} = E_n)$ is a path in $\mathcal{R}(\mathcal{G})$ as $|G_i \triangle G_{i-1}| = 1$ and thus $G_i G_{i-1} \in E(\mathcal{R}(\mathcal{G}))$. Therefore, we have a path between G and E_n of length $\|G\|$ giving us that

$$\text{dist}(E_n, G) \leq \|G\|.$$

□

With this, we can now prove bounds on the distance between an arbitrary pair of graphs. However, it turns out that in contrast to previous distance formulas, we do not get some exact expression, but instead a separate lower and upper bound. Following the proof of this corollary, we also quickly give examples where either bound is tight.

Corollary 4.5. *Let $\mathcal{G}$ be a weakly monotone graph class and $G, H \in \mathcal{G}$. It holds that*

$$|E(G) \triangle E(H)| \leq \text{dist}(G, H) \leq \|G\| + \|H\|.$$

Proof. The first inequality is given by Lemma 2.5.

To show the second inequality, we start out by noticing that for any weakly monotone graph class $\mathcal{G}$ of dimension n it holds that $E_n \in \mathcal{G}$. For the sake of contradiction assume that $\mathcal{G} \neq \emptyset$ weakly monotone with $E_n \notin \mathcal{G}$. Consider some $G \in \mathcal{G}$ with the minimum number of edges. As $G \neq E_n$ it holds that $\|G\| \geq 1$. By the definition of weak monotonicity we find some $H \in \mathcal{G}$ with $H \subseteq G$ and $\|H\| + 1 = \|G\|$. This implies that $\|H\| < \|G\|$, giving us a contradiction to the minimality of G.

With this, we can apply the triangle inequality for the distance function and Lemma 4.4 to obtain

$$\mathrm{dist}(G, H) \leq \mathrm{dist}(G, E_n) + \mathrm{dist}(H, E_n) = \|G\| + \|H\|.$$

□

We now take some time to convince ourselves that the bounds given in Lemma 4.5 are tight. We have already seen a lot of examples where $|E(G) \triangle E(H)| \leq \mathrm{dist}(G, H)$ is tight. To start out in Lemma 4.4 we saw that when setting $H = E_n$ and letting $G \in \mathcal{G}$ arbitrary in any weakly monotone graph class $\mathcal{G}$ of dimension n then it follows that $\mathrm{dist}(G, H) = \|G\| = |E(G) \triangle E(H)|$. On the other hand, in the previous chapter we saw that if $\mathcal{G}$ is monotone, then $|E(G) \triangle E(H)| = \mathrm{dist}(G, H)$ for arbitrary $G, H \in \mathcal{G}$. As every monotone graph class $\mathcal{G}$ is also weakly monotone, the bound $|E(G) \triangle E(H)| \leq \mathrm{dist}(G, H)$ also achieves equality for all graph classes seen in Chapter 3.

The second bound $\mathrm{dist}(G, H) \leq \|G\| + \|H\|$ shown in Lemma 4.5 is also tight. To start out, we find that if $G, H \in \mathcal{G}$ are edge disjoint $E(G) \cap E(H) = \emptyset$ then the upper and lower bounds coincide $|E(G) \triangle E(H)| = \|G\| + \|H\|$ and thus the upper bound is tight. On the other hand, we can also construct weakly monotone graph classes $\mathcal{G}_k$ for arbitrary $k \in \mathbb{N}$ such that there exist graphs $G, H \in \mathcal{G}_k$ with

$$|E(G) \triangle E(H)| + k = \mathrm{dist}(G, H) = \|G\| + \|H\|.$$

For an explicit construction of such $\mathcal{G}_k$ take a look at Example 4.6.

As we do not get exact formulas for distances in reconfiguration graphs of weakly monotone graph classes, it is no surprise that we do not get exact formulas for the radius and diameter of the reconfiguration graph either.

4.2 Diameter

This section proves Theorem 4.2, giving us bounds on the diameter of $\mathcal{R}(\mathcal{G})$ for weakly monotone graph classes $\mathcal{G}$. While the proof of the bounds themselves is short, we spend some time giving intuition on why no stronger bounds can be found.

Proof of Theorem 4.2. For the first inequality consider some $H \in \mathcal{G}$ with the maximum number of edges. We then get

$$\operatorname{diam}(\mathcal{R}(\mathcal{G})) \geq \operatorname{dist}(E_n, H) = \|H\| = \max_{G \in \mathcal{G}} \|G\|.$$

For the second inequality we apply Lemma 4.4 to obtain

$$\operatorname{diam}(\mathcal{R}(\mathcal{G})) = \max_{G,H \in \mathcal{G}} \operatorname{dist}(G, H) \leq \max_{G,H \in \mathcal{G}} \|G\| + \|H\| = 2 \cdot \max_{G \in \mathcal{G}} \|G\|.$$

To show that these bounds are tight we consider some examples.

Let G be an arbitrary graph with $m = \|G\|$ and number its edges $E(G) = \{e_1, \dots, e_m\}$. Consider the weakly monotone graph class $\mathcal{G} = \{G, G - \{e_1\}, G - \{e_1, e_2\}, \dots, G - \{e_1, \dots, e_m\}\}$. It is easy to see that $\mathcal{R}(\mathcal{G}) \cong P_m$ and thus it holds that

$$\operatorname{diam}(\mathcal{R}(\mathcal{G})) = m = \max\{\|G\| \mid G \in \mathcal{G}\}.$$

On the other hand, let two graphs G, H on the same vertex set with disjoint edge-sets $E(G) \cap E(H) = \emptyset$ of size $m = \|G\| = \|H\|$. Denoting their edges $E(G) = \{e_1, \dots, e_m\}$ and $E(G) = \widetilde{e}_1, \dots, \widetilde{e}_m$ we can define the weakly monotone graph class $\mathcal{G} = \{G, G - \{e_1\}, \dots, G - \{e_1, \dots, e_m\}, H, H - \widetilde{e}_1, \dots, H - \{\widetilde{e}_1, \dots, \widetilde{e}_m\}\}$. For this graph class it holds that:

$$\operatorname{diam}(\mathcal{R}(\mathcal{G})) = 2 \cdot m = 2 \cdot \max\{\|G\| \mid G \in \mathcal{G}\}$$

□

It should also be noted that while the two examples provided in the previous proof still satisfy $\operatorname{diam}(\mathcal{R}(\mathcal{G})) = \max_{G,H \in \mathcal{G}} \|G \cup H\|$, which we know to be true for any monotone graph class $\mathcal{G}$. But this does not hold in general for weakly monotone classes. To see this consider the following examples.

Example 4.6. *Fix some $k \in \mathbb{N}$ and two graphs G, H on the same vertex set, with edge sets $E(G) = \{e_1, \dots, e_k, f, g\}$ and $E(H) = \{e_1, \dots, e_k\}$. Consider the graph class*

$$\begin{aligned}\mathcal{G} = &\{G, G - \{e_1\}, \dots, G - \{e_1, \dots, e_k\}, G - \{e_1, \dots, e_k, f\}, E_n\} \\ &\cup \{H, H - \{e_1\}, \dots, H - \{e_1, \dots, e_k\}\}.\end{aligned}$$

It is obvious that $\mathcal{G}$ is weakly monotone. Furthermore, it can be verified that $\operatorname{diam}(\mathcal{R}(\mathcal{G})) = \operatorname{dist}(G, H) = 2 \cdot k + 2$ but all $F \in \mathcal{G}$ satisfy $F \subseteq G$ and thus $\max_{G,H \in \mathcal{G}} \|G \cup H\| = \|G\| = k + 2$. Especially this gives us $\operatorname{diam}(\mathcal{R}(\mathcal{G})) = \max_{G,H \in \mathcal{G}} \|G \cup H\| + k$.

On the other hand lets consider another construction.

Example 4.7. *Fix some $k \in \mathbb{N}$ and two graphs G, H on the same vertex set, with edge sets $E(G) = \{e_1, \ldots, e_k, f_1, \ldots, f_k\}$ and $E(H) = \{e_1, \ldots, e_k, g_1, \ldots g_k\}$. Consider the graph class*

$$\begin{aligned}\mathcal{G} =& \{G, G - \{f_1\}, \ldots, G - \{f_1, \ldots, f_k\}\} \\ & \cup \{H, H - \{g_1\}, \ldots, H - \{g_1, \ldots, g_k\}\} \\ & \cup \{G \cap H - \{e_1\}, \ldots, G \cap H - \{e_1, \ldots e_k\}\}.\end{aligned}$$

Note that the graph class $\mathcal{G}$ from Example 4.7 is weakly monotone. Furthermore, it holds that $\operatorname{diam}(\mathcal{R}(\mathcal{G})) = 2k$. On the other hand $\max_{G_1, G_2 \in \mathcal{G}_k} \|G_1 \cup G_2\| = \|G \cup H\| = 3k$. Thus, we have $\operatorname{diam}(\mathcal{R}(\mathcal{G})) + k = \max_{G,H \in \mathcal{G}} \|G \cup H\|$.

Note that one could for example use $\frac{\max_{G,H \in \mathcal{G}} \|G \cup H\|}{2}$ as a lower bound of $\operatorname{diam}(\mathcal{R}(\mathcal{G}))$, but this is weaker than $\max_{G \in \mathcal{G}} \|G\|$. The same goes for $2 \cdot \max_{G,H \in \mathcal{G}} \|G \cup H\|$ as an upper bound. With this, we can conclude that $\max_{G,H \in \mathcal{G}} \|G \cup H\|$ cannot be used to obtain a stronger upper or lower bound than the one given in Theorem 4.2.

4.3 Radius

In this section we prove the bounds on the radius of $\mathcal{R}(\mathcal{G})$ given in Theorem 4.3.

Proof of Theorem 4.3. Let n be the dimension of the graph class.

For the first inequality consider some $G \in \mathcal{G}$ with the maximum number of edges. Consider an arbitrary $H \in \mathcal{G}$ and note that

$$\epsilon(H) \geq \max(\operatorname{dist}(H, E_n), \operatorname{dist}(H, G)) \geq \max(\|H\|, \|G\| - \|H\|) \geq \frac{\|G\|}{2}$$

Thus giving us the desired result

$$\operatorname{rad}(\mathcal{R}(\mathcal{G})) = \min_{G \in \mathcal{G}} \epsilon(G) \geq \frac{\max_{G \in \mathcal{G}} \|G\|}{2}.$$

For the second inequality we consider E_n as a center of the reconfiguration graph and apply Lemma 4.4 to obtain:

$$\operatorname{rad}(\mathcal{R}(\mathcal{G})) \leq \epsilon(E_n) = \max_{G \in \mathcal{G}} \operatorname{dist}(G, E_n) = \max_{G \in \mathcal{G}} \|G\|$$

To get equality in the upper bound one can consider any monotone graph class. Another construction can be obtained from any graph G on $\|G\| = m$ edges labeled $E(G) = \{e_1, \ldots, e_m\}$, considering the graph class:

$$\mathcal{G} = \{G, G - \{e_1\}, \ldots, G - \{e_1, \ldots, e_{m-1}\}, E_n, G - \{e_m\}, \ldots, G - \{e_m, \ldots, e_2\}\}$$

This graph class is weakly monotone and it can be verified that $\mathcal{R}(\mathcal{G}) \cong C_{2\cdot m}$ and thus $\mathrm{rad}(\mathcal{R}(\mathcal{G})) = \mathrm{rad}(C_{2\cdot m}) = m = \max_{G\in\mathcal{G}} \|G\|$.

For equality in the lowerbound, let G be an arbitrary graph with $m = \|G\|$ edges labeled $E(G) = \{e_1, \dots, e_m\}$. Consider the weakly monotone graph class $\mathcal{G} = \{G, G - \{e_1\}, G - \{e_1, e_2\}, \dots, G - \{e_1, \dots, e_m\}\}$. It is easy to see that $\mathcal{R}(\mathcal{G}) \cong P_m$ and thus:

$$\mathrm{rad}(\mathcal{R}(\mathcal{G})) = \left\lceil \frac{m}{2} \right\rceil = \left\lceil \frac{\max_{G\in\mathcal{G}} \|G\|}{2} \right\rceil$$

For even m this achieves equality.

□

This closes out the proofs of theorems for graph properties of $\mathcal{R}(\mathcal{G})$ for an arbitrary weakly monotone graph class $\mathcal{G}$. In the following, we take a look at weakly monotone graph classes $\mathcal{G}$ with further properties.

4.4 Weakly Monotone Graph Classes Closed Under Isomorphism

In cases where the graph class is closed under isomorphism, one might still be hopeful to get a nice characterization of $\mathrm{center}(\mathcal{R}(\mathcal{G}))$, similar to what we saw for monotone graph classes in Theorem 3.9. But the following construction shows that there is little hope.

Example 4.8. *Let $n = 4$ and consider the graph class*

$$\mathcal{G} = \{G \mid V(G) = [4], \|G\| \leq 3 \text{ or } \|G\| = 4 \text{ and each vertex has degree } 2\}.$$

This graph class is weakly monotone and closed under isomorphism. However, the center of the reconfiguration graph is given by

$$\mathrm{center}(\mathcal{R}(\mathcal{G})) = \{E_n\} \cup \{([4], \{xy, yz\}) \mid x, y, z \in [4] \text{ pairwise distinct}\}.$$

Note that this is neither the whole graph class nor trivial. Furthermore, the subgraph of $\mathcal{R}(\mathcal{G})$ induced by the center is not even connected. Therefore, we do not pursue the characterization of the center further.

4.5 Weakly Monotone Increasing Graph Classes

Remember our discussion of monotone increasing graph classes in Section 3.5. There we had Lemma 3.13 stating that $\mathcal{R}(\mathcal{G}) \cong \mathcal{R}(\mathcal{G}_{compl})$ for an arbitrary graph class $\mathcal{G}$. This once again lets us transfer all results we had for a monotone increasing graph class to weakly monotone ones. Thus, we do not take a closer look at them.

But instead, we consider graph classes $\mathcal{G}$ which are both weakly monotone decreasing and weakly monotone increasing. While in the monotone case in section 3.5 this directly implies that $\mathcal{G}$ contains every graph on the vertex set $[n]$ for some $n \in \mathbb{N}$, this is not the case here. Instead, $\mathcal{G}$ still has non-trivial structure. In the upcoming Section 4.6 we find that many naturally occurring graph classes, which are weakly monotone decreasing are also weakly monotone increasing. For $\mathcal{G}$, which are weakly monotone decreasing and increasing, we obtain an exact value of the diameter of its reconfiguration graph.

Theorem 4.9. *Let $\mathcal{G}$ be a weakly monotone increasing and decreasing graph class of dimension n. Then the diameter of the reconfiguration graph satisfies*

$$\operatorname{diam}(\mathcal{R}(\mathcal{G})) = \binom{n}{2}.$$

Proof. We show the equality $\operatorname{diam}(\mathcal{R}(\mathcal{G})) = \binom{n}{2}$ by showing '≤' and '≥' separately.

To start out note that as $\mathcal{G}$ is weakly monotone decreasing, we have $E_n \in \mathcal{G}$. On the other hand $\mathcal{G}$ is weakly monotone increasing, which gives us that $K_n \in \mathcal{G}$.

As $\mathcal{G}$ is weakly monotone we can apply Theorem 4.2 to get

$$\operatorname{diam}(\mathcal{R}(\mathcal{G})) \geq \max_{G \in \mathcal{G}} \|G\| \geq \|K_n\| = \binom{n}{2}.$$

Now it remains to be shown that $\operatorname{diam}(\mathcal{R}(\mathcal{G})) \leq \binom{n}{2}$. We do so by considering arbitrary $G, H \in \mathcal{G}$ and showing that they have distance at most $\binom{n}{2}$. It follows that $\operatorname{diam}(\mathcal{R}(\mathcal{G})) = \max_{G,H \in \mathcal{G}} \operatorname{dist}_{\mathcal{R}(\mathcal{G})}(G, H) \leq \binom{n}{2}$.

Fix $G, H \in \mathcal{G}$. We show $\operatorname{dist}(G, H) \leq \binom{n}{2}$ by considering two cases.

Case $\|G\| + \|H\| \leq \binom{n}{2}$: By Lemma 4.4 we know that $\operatorname{dist}(G, E_n) = \|G\|$ and $\operatorname{dist}(H, E_n) = \|H\|$. Therefore, using the triangle inequality for distances we get that $\operatorname{dist}(G, H) \leq \operatorname{dist}(G, E_n) + \operatorname{dist}(E_n, H) \leq \binom{n}{2}$.

Case $\|G\| + \|H\| > \binom{n}{2}$: We can do the same as in the previous case, using K_n instead of E_n. The proof that $\operatorname{dist}(G, K_n) = \binom{n}{2} - \|G\|$ is omitted, as it can directly be followed from the combination of Lemma 3.13 and Lemma 4.4. With this we get $\operatorname{dist}(G, H) \leq \operatorname{dist}(G, K_n) + \operatorname{dist}(K_n, H) = 2 \cdot \binom{n}{2} - (\|G\| + \|H\|) \leq \binom{n}{2}$. □

Even though we got an exact number for $\operatorname{diam}(\mathcal{R}(\mathcal{G}))$ for weakly monotone increasing and decreasing graph classes $\mathcal{G}$, we have no such luck for the radius. For arbitrary $n \in \mathbb{N}$ one can construct a graph class $\mathcal{G}$ which is weakly monotone increasing and decreasing, such that $\mathcal{R}(\mathcal{G}) \cong C_{2\cdot\binom{n}{2}}$ as well as one which gives us $\mathcal{R}(\mathcal{G}) \cong P_{\binom{n}{2}}$. Therefore, we cannot improve on the result of Theorem 4.3.

4.6 Application of Results to Graph Classes

In this section we apply the results from this chapter to some graph classes. We mainly focus on chordal and comparability graphs. Both of them turn out to be weakly monotone increasing and decreasing. Therefore, we get an exact diameter of the reconfiguration graph from Theorem 4.9. For the radius, we only get the bounds from Theorem 4.3. One can likely obtain better bounds or an exact formula for either one, but we do not do so here.

4.6.1 Comparability Graphs

We start out by considering comparability graphs, as the proofs in this section do not require any external results. To start out, remember the definition of a comparability graph.

Definition 4.10. *A graph G is called a* comparability graph*, if there exists an orientation of its edges that is transitive and does not contain a directed cycle. That is, if we have $u, v, w \in V(G)$ and an edge oriented from u to v as well as an edge from v to w, then we must also have an edge oriented from u to w. We call such an orientation of $E(G)$ a* transitive orientation.

We denote a transitive orientation O by a binary relation $<_O$ where for $x, y \in V(G)$ the pair (x, y) is in the relation $<_O$, if and only if $xy \in E(G)$ and the edge xy is oriented from x to y. In the following, $<_O$ is used in infix notation. So we may write $x <_O y$ if (x, y) is in the relation $<_O$.

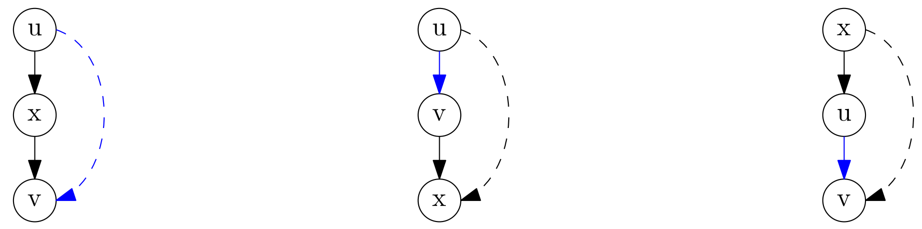
In the following, we use the terms minimal or maximal for vertices in a comparability graph with a fixed transitive ordering O. A vertex v is minimal if there is no $u \in V(G)$ with $u <_O v$. A vertex v is maximal if there is no $u \in V(G)$ with $v <_O u$.

We define the family of graph classes

$$\mathcal{G}_n^{cmp} = \{G \text{ comparability graph on vertex set } [n]\}.$$

To apply the results from this chapter we show that for $n \in \mathbb{N}$ the graph class $\mathcal{G}_n^{cmp}$ is weakly monotone decreasing and increasing.

Theorem 4.11. *Let $n \in \mathbb{N}$ and $\mathcal{G}_n^{cmp} = \{G \text{ comparability graph on vertex set } [n]\}$. It holds that $\mathcal{G}_n^{cmp}$ is weakly monotone increasing and weakly monotone decreasing.*



(a) Edge removal **(b)** Edge addition first case **(c)** Edge addition second case

Figure 4.1: Visualization of bad configurations when adding or removing an edge to a comparability graph. Dotted lines represent non-edges.

Proof. Remember the definition of weak monotonicity. To get weak decreasing monotonicity we need to show that for any $G \in \mathcal{G}_n^{cmp} \setminus \{E_n\}$ we can remove one edge $e \in E(G)$ from G such that $G - e$ is once again in the graph class $\mathcal{G}_n^{cmp}$. Fix some $G \in \mathcal{G}_n^{cmp} \setminus \{E_n\}$. As G is a comparability graph, we can fix some transitive orientation of its edges $\mathcal{O}$. We pick an edge $uv \in V(G)$, with $u <_\mathcal{O} v$ and minimal v. Let $e = uv$, we find that $G - e$ is once again a comparability graph. This can be verified by considering the edge orientation of $G - e$ orienting each edge according to $\mathcal{O}$ and showing that it is transitive and acyclic. This shall not be conducted here, as the details of the proof are simple. The intuition is that the only way to break transitivity is by deleting an edge $u <_\mathcal{O} v$ such that there exists some $x \in V(G)$ with $u <_\mathcal{O} x <_\mathcal{O} v$. See Figure 4.1a for a visualization. This cannot happen with our choice of e, as otherwise x could have been chosen instead of v, giving us a contradiction to the minimality of v. With this we have found an edge $e \in E(G)$ such that $G - e \in \mathcal{G}_n^{cmp}$. Therefore, $\mathcal{G}_n^{cmp}$ is weakly monotone decreasing.

Weak increasing monotonicity of $\mathcal{G}_n^{cmp}$ can be shown in a similar way. Fix some $G \in \mathcal{G}_n^{cmp} \setminus \{K_n\}$. We need to find some edge $e \in \binom{[n]}{2} \setminus E(G)$, which we can add to G to get a graph which is once again in the graph class $\mathcal{G}_n^{cmp}$. Fix a transitive ordering of the edges of G and denote it $\mathcal{O}$. Pick a vertex $v \in V(G)$ which is not connected to all other $V(G)$ and which is maximal with this property. The existence of such v is trivial as $G \neq K_n$. Pick some u such that $uv \notin E(G)$ and which is minimal with this property. If we set $e = uv$, the graph $G' = ([n], E(G) \cup \{e\})$ resulting from adding the edge e to G is a comparability graph once again. This can be seen by considering the ordering $\mathcal{O}'$, which orders each edge of G according to $\mathcal{O}$ and the new edge uv in the direction $v <_{\mathcal{O}'} u$. It can be shown that $\mathcal{O}'$ is a transitive ordering of $E(G')$. This proof is not shown here. The intuition behind this is that there are only two ways to obtain a non-transitive $\mathcal{O}'$. They are depicted in Figure 4.1b and Figure 4.1c. It is easy to verify that neither one of them can occur with our choice of $e = uv$. Thus, we have $G' \in \mathcal{G}_n^{cmp}$. As $G \in \mathcal{G}_n^{cmp} \setminus \{K_n\}$ was arbitrary, this gives us weak increasing monotonicity of $\mathcal{G}_n^{cmp}$. □

By Theorem 4.11 we know that $\mathcal{G}_n^{cmp}$ is weakly monotone increasing and decreasing. Thus, we can apply Theorem 4.9 to get an exact formula for the diameter of $\mathcal{R}(\mathcal{G}_n^{cmp})$. Furthermore, K_n is a comparability graph and thus it holds that $\max_{G \in \mathcal{G}_n^{cmp}} = \binom{n}{2}$. With this, we can apply Theorem 4.3 to obtain bounds on the radius of $\mathcal{R}(\mathcal{G}_n^{cmp})$. In total, we have the following result.

Theorem 4.12. *For $n \in \mathbb{N}$ it holds that*

- $\frac{\binom{n}{2}}{2} \leq \mathrm{rad}(\mathcal{R}(\mathcal{G}_n^{cmp})) \leq \binom{n}{2}$.
- $\mathrm{diam}(\mathcal{R}(\mathcal{G}_n^{cmp})) = \binom{n}{2}$.

One may be able to get tighter bounds on $\mathrm{rad}(\mathcal{R}(\mathcal{G}_n^{cmp}))$, but we did not take a look at this.

4.6.2 Chordal Graphs

In this section we take a look at chordal graphs. The proof of weak monotonicity heavily relies on the characterization of chordal graphs by perfect elimination orderings. To start out, remember the definition of a chordal graph.

Definition 4.13. *A graph G is chordal if and only if it contains no induced cycle of length k for all $k \geq 4$. In other words, every cycle in G that is not a triangle has a chord.*

To prove weak monotonicity of $\mathcal{G}^{chr}$ we need a well-known characterization of chordal graphs. It shall not be proven here. Proofs of Theorem 4.15 can be found in literature.

Definition 4.14. *For a graph G on $n \in \mathbb{N}$ vertices, a* perfect elimination ordering *is an ordering $v_1, \ldots, v_n \in V(G)$ of its vertices such that the right neighborhood of each vertex induces a clique. Formally, for any $1 \leq i \leq n$ the induced subgraph $G[\{v_j \mid j > i \text{ and } v_i v_j \in E(G)\}]$ is a clique.*

Theorem 4.15. *A graph G is chordal if and only if G has a perfect elimination ordering.*

In the following, we deal with the family of graph classes

$$\mathcal{G}_n^{chr} = \{G \text{ chordal graph on vertex set } [n]\}.$$

With Theorem 4.15 we have an easy proof for weak increasing and decreasing monotonicity of $\mathcal{G}_n^{chr}$, which is carried out in the following.

Theorem 4.16. *Let $n \in \mathbb{N}$ and $\mathcal{G}_n^{chr} = \{G \text{ chordal graph on vertex set } [n]\}$. It holds that $\mathcal{G}_n^{chr}$ is weakly monotone increasing and weakly monotone decreasing.*

Proof. To show that $\mathcal{G}_n^{chr}$ is weakly monotone decreasing, we need to show that for any $G \in \mathcal{G}_n^{chr}$ which is not the empty graph E_n, we can remove one edge $e \in E(G)$ such that $G - e$ is chordal. Such an edge can be found by considering a perfect elimination scheme $v_1, \ldots, v_n$ of G. Choose e as $v_i v_j \in E(G)$ with minimal i. If there are multiple, we can choose any. The proof of chordality of $G - e$ is straightforward. One can check that $v_1, \ldots, v_n$ is a perfect elimination scheme of $G - e$. As we have found $e \in E(G)$ such that $G - e \in \mathcal{G}_n^{chr}$ for arbitrary $G \in \mathcal{G}_n^{chr} \setminus \{E_n\}$, we can follow that $\mathcal{G}_n^{chr}$ is weakly monotone decreasing.

Weak increasing monotonicity can be shown in a similar way, by once again fixing $G \in \mathcal{G}_n^{chr}$ and some perfect elimination ordering $v_1, \ldots, v_n$. We can now pick $e = v_i v_j \in \binom{[n]}{2} \setminus E(G)$ with maximal j. It can once again be checked that $v_1, \ldots, v_n$ is a perfect elemintation ordering of $G' = ([n], E(G) \cup \{e\})$ giving us weak increasing monotonicity □

With Theorem 4.16 we can apply Theorem 4.3 and Theorem 4.9. Observing that K_n is chordal, we know that $\max_{G \in \mathcal{G}_n^{chr}} = \binom{n}{2}$, giving us the following result.

Theorem 4.17. *For $n \in \mathbb{N}$ it holds that*

- $\frac{\binom{n}{2}}{2} \leq \mathrm{rad}(\mathcal{R}(\mathcal{G}_n^{chr})) \leq \binom{n}{2}$.
- $\mathrm{diam}(\mathcal{R}(\mathcal{G}_n^{chr})) = \binom{n}{2}$.

Tighter bounds on $\mathrm{rad}(\mathcal{R}(\mathcal{G}_n^{chr}))$ may exist, but we did not try to find any.

4.6.3 Other Weakly Monotone Graph Classes

Many other weakly monotone graph classes exist. For example, interval graphs, split graphs, and permutation graphs are all weakly monotone. It actually turns out that all of them are weakly monotone increasing and decreasing.

But there also exist graph classes that are weakly monotone decreasing but not increasing. For example, all graph classes which are monotone decreasing but not monotone increasing, like those considered in Section 3.6. However, we were not able to find any well-known or truly natural graph classes which are weakly monotone decreasing but not increasing, without already being monotone. But one can still construct a graph class of this kind. For example, fix some $n \in \mathbb{N}$ and consider $\mathcal{G} = \{G \text{ graph on vertex set [n]} \mid G \text{ contains exactly one cycle or } G \text{ is a path}\}$.

However, we do not consider any of the aforementioned other graph classes in detail for time reasons. Instead, let us finishes up our work on weakly monotone graph classes. We shall wrap up this thesis with a conclusion.

5 Conclusion

In this thesis, we considered graph classes $\mathcal{G}$ where each graph $G \in \mathcal{G}$ has the vertex set $V(G) = [n]$ for some fixed natural number $n \in \mathbb{N}$. On this type of graph class, we defined a reconfiguration graph $\mathcal{R}(\mathcal{G})$, with the graph class as the vertex set $V(\mathcal{R}(\mathcal{G})) = \mathcal{G}$ and edges corresponding to edge additions and deletions in the underlying graphs. As this type of reconfiguration graph has not been considered in previous work, we started out by proving basic properties of this type of reconfiguration graph independent of the graph class. Most importantly we found that $\mathcal{R}(\mathcal{G})$ is always an induced subgraph of a hypercube and thus bipartite.

After having laid the groundwork, we took a deeper look at monotone graph classes $\mathcal{G}$. Using the structure of the graph class $\mathcal{G}$ we found the following formulas for the reconfiguration graph $\mathcal{R}(\mathcal{G})$ in Theorem 3.2 and Theorem 3.3 respectively.

$$\begin{aligned}\operatorname{diam}(\mathcal{R}(\mathcal{G})) &= \max_{G,H\in\mathcal{G}} \|G \cup H\| \\ \operatorname{rad}(\mathcal{R}(\mathcal{G})) &= \max_{G\in\mathcal{G}} \|G\|\end{aligned}$$

Furthermore, in Theorem 3.9 we saw that for a graph class $\mathcal{G}$ which is monotone as well as isomorphism-closed, the center of the reconfiguration graph $\operatorname{center}(\mathcal{R}(\mathcal{G}))$ consists either of only the empty graph or all graphs in $\mathcal{G}$.

Afterwords we applied these results to specific monotone graph classes $\mathcal{G}$, calculating exact values for $\operatorname{rad}(\mathcal{R}(\mathcal{G}))$ and $\operatorname{diam}(\mathcal{R}(\mathcal{G}))$. Amongst the considered graph classes were forests, planar graphs, and graphs with bounded chromatic number. For future work, it is possible to extend this to more graph classes. Most interestingly, we could show that for graphs of bounded clique number, the calculation of the diameter of the reconfiguration graph is at least as hard as calculating Ramsey numbers.

Furthermore, we also considered a weaker version of monotonicity. For weakly monotone graph classes $\mathcal{G}$ we obtained some upper and lower bounds on distances between vertices in $\mathcal{R}(\mathcal{G})$, instead of the exact formula we got for monotone graph classes. Thus, we do not get exact formulas for the diameter and radius of the reconfiguration graph. Instead, we got the following bounds in Thorem 4.2 and Theorem 4.3 respectively.

$$\begin{aligned}\max_{G\in\mathcal{G}} \|G\| \leq \operatorname{diam}(\mathcal{R}(\mathcal{G})) &\leq 2\max_{G\in\mathcal{G}} \|G\| \\ \frac{\max_{G\in\mathcal{G}} \|G\|}{2} \leq \operatorname{rad}(\mathcal{R}(\mathcal{G})) &\leq \max_{G\in\mathcal{G}} \|G\|\end{aligned}$$

This leaves two obvious ways to extend this work. One could continue and try to find formulas for other graph properties of $\mathcal{R}(\mathcal{G})$. In this thesis independence number, edge chromatic number, treewidth, or degree properties like the minimum or average degree were not considered. We believe that most of these depend too much on the local structure of the graph class $\mathcal{G}$ for there to exist closed formulas calculating them in a general setting. However, fixing a certain graph class, like graphs of bounded chromatic number or planar graphs, calculation seems feasible.

Another way to continue the presented work would be to consider other families of graph classes. This thesis only considered monotone and weakly monotone graph classes $\mathcal{G}$. In both cases, we could directly show connectivity of the reconfiguration graph $\mathcal{R}(\mathcal{G})$. However, this does not hold for general graph classes. One of the first graph classes we had in mind to continue work on is the class given by a forbidden induced subgraph H. Let us fix some graph H and natural number $n \in \mathbb{N}$ and consider $\mathcal{G} = \{G \in \text{Forb}_{\text{ind}}(H) \mid V(G) = [n]\}$. Now connectivity of $\mathcal{R}(\mathcal{G})$ becomes non-trivial and depends on the structure of H. It should be mentioned that some work related to this exists. The existence of an H-induced-saturated graph on n vertices is equivalent to the existence of an isolated vertex in $\mathcal{R}(\mathcal{G})$. Some work on the existence of these induced-saturated graphs has been done [4], but not much is known yet.

But many other graph classes $\mathcal{G}$ could be considered. For example, picking some graph property like chromatic number or treewidth alongside some constant $k \in \mathbb{N}$ and considering the class of graphs for which the chosen graph property is exactly k. In these cases answering questions like connectivity of $\mathcal{R}(\mathcal{G})$ seems feasible.

All in all, there are a lot of ways to continue research dealing with our definition of a reconfiguration graph. We are hoping that further work in this area will be done. The hope is that for certain graph classes $\mathcal{G}$ it will be possible to use properties of the reconfiguration graph $\mathcal{R}(\mathcal{G})$ to obtain a better understanding of the structure of the graph class $\mathcal{G}$ itself.

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