Karlsruhe Institute of Technology

Algorithmic and Structural Properties of Subgraphs of Strong Grids

Bachelor's Thesis of

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(Jan Diester)
Karlsruhe, September 30, 2025

Abstract

Graph products have gained attention in recent years. It has been discovered that multiple notable classes of graphs are subgraphs of the product of simpler graphs. In this thesis, we focus on the Cartesian and strong product of two path graphs, referred to as *grids* and *strong grids*. We investigate the computational complexity of several well-known $\mathcal{NP}$-complete problems when constrained to arbitrary and induced subgraphs of grids and strong grids. We show that MINIMUM VERTEX COVER, MAXIMUM CUT and 3-COLORABILITY remain $\mathcal{NP}$-complete on induced subgraphs of strong grids, and that FEEDBACK VERTEX SET is $\mathcal{NP}$-complete even on induced subgraphs of grids. We provide proof that 3-EDGE-COLORABILITY is $\mathcal{NP}$-complete on subgraphs of strong grids, and that the problem 3-EDGE-COLORABILITY on planar graphs is polynomially equivalent to 3-EDGE-COLORABILITY on induced subgraphs of strong grids. It is a famous open problem, whether 3-EDGE-COLORABILITY on planar graphs is $\mathcal{NP}$-complete.

Furthermore, we investigate whether subgraphs G of strong grids with maximum degree $\Delta(G)$ are always Class I or can be Class II. We prove that for $\Delta(G) \in \{2, 3, 4\}$ there exist Class II subgraphs G of strong grids. We also show that for $\Delta(G_i) \in \{2, 3\}$ there exist Class II induced subgraphs G_i of strong grids. It is open whether there exist Class II subgraphs G of strong grids with $\Delta(G) \in \{5, 6, 7\}$, and Class II induced subgraphs of strong grids G_i with $\Delta(G_i) \in \{4, 5, 6, 7\}$. We present structural properties of subgraphs of strong grids, which are relevant to these open problems.

Deutsche Zusammenfassung

Produkte von Graphen haben in den letzten Jahren an Aufmerksamkeit gewonnen. Dazu beigetragen hat die Entdeckung, dass mehrere nennenswerte Graphklassen Teilgraphen vom Produkt von Graphen mit einfacherer Struktur sind. In dieser Arbeit liegt der Fokus auf dem Kartesischen Produkt und dem Strong Product zweier Pfade, auch Grids und Strong Grids genannt. Wir erkunden die Komplexität von einigen bekannten $\mathcal{NP}$-vollständigen Problemen, eingeschränkt auf Teilgraphen von Grids und Strong Grids. Wir zeigen, dass MINIMUM VERTEX COVER, MAXIMUM CUT und 3-COLORABILITY auf induzierten Teilgraphen von Strong Grids weiterhin $\mathcal{NP}$-vollständig sind, und dass FEEDBACK VERTEX SET auch auf induzierten Teilgraphen von Grids $\mathcal{NP}$-vollständig ist. Außerdem beweisen wir, dass 3-EDGE-COLORABILITY $\mathcal{NP}$-vollständig auf Teilgraphen von Strong Grids ist, und dass das Problem 3-EDGE-COLORABILITY von planaren Graphen polynomiell äquivalent ist zu 3-EDGE-COLORABILITY von induzierten Teilgraphen von Strong Grids. Es ist ein bekanntes offenes Problem, ob 3-EDGE-COLORABILITY von planaren Graphen $\mathcal{NP}$-vollständig ist.

Zusätzlich überprüfen wir, ob Teilgraphen G von Strong Grids mit Maximalgrad $\Delta(G)$ immer Class I sind oder Class II sein können. Wir zeigen, dass es für $\Delta(G) \in \{2, 3, 4\}$ Class II Teilgraphen G von Strong Grids gibt, jedoch nicht für $\Delta(G) \in \{1, 8\}$. Darüber hinaus zeigen wir für $\Delta(G_i) \in \{2, 3\}$, dass es Class II induzierte Teilgraphen G_i von Strong Grids gibt. Es ist offen ob es Class II Teilgraphen G von Strong Grids gibt, sodass $\Delta(G) \in \{5, 6, 7\}$, und ob es Class II induzierte Teilgraphen G_i von Strong Grids gibt, sodass $\Delta(G_i) \in \{4, 5, 6, 7\}$. Wir erläutern strukturelle Eigenschaften von Teilgraphen von Strong Grids, die von Bedeutung für diese offenen Probleme sind.

Contents

1. Introduction

In this thesis we examine the computational complexity of algorithmic problems on subgraphs of different products of path graphs. We also discuss some structural properties of these subgraphs, which are relevant to their chromatic index.

Recent results on product structure have shown that several notable graph classes are isomorphic to subgraphs of the strong product of simpler graphs [DMW23, UWY21, DHHW22, DJM$^+$20]. For instance, Dujmović et al. show that every planar graph is a subgraph of the strong product of a path and a graph with bounded treewidth [DJM$^+$20].

These discoveries raise the question whether product structure can be exploited to construct more efficient algorithms for some important graph problems.

This thesis focuses on subgraphs of Cartesian and strong products of two path graphs. We refer to the Cartesian product of paths as *grids*, and to the strong product as *strong grids* (also known as King's graphs). We consider both arbitrary and induced subgraphs of grids and strong grids. We refer to arbitrary subgraphs as *partial (strong) grids* and to induced subgraphs as *(strong) grid graphs*.

Because grids and their subgraphs are bipartite and planar, most of the problems investigated in this thesis are known to be solvable in polynomial time on these graphs. In the chapters on problems where that is the case, we exclusively consider subgraphs of strong grids.

We investigate the problems FEEDBACK VERTEX SET, MINIMUM VERTEX COVER, MAXIMUM CUT, 3-COLORABILITY and 3-EDGE-COLORABILITY. Other than 3-EDGE-COLORABILITY, these are all part of Karp's well-known set of 21 $\mathcal{NP}$-complete problems [Kar72].

All of the chosen problems have a high degree of practical relevance. The computation of minimum feedback vertex sets has been applied to multiple fields, including regulatory networks in biology [MFKS13], or Bayesian inference [BYGNR98]. The minimum vertex cover problem has several applications, for example in wireless sensor networks [KUD14]. Applications of the computation of maximum cuts in graphs include VLSI chip design [CZS$^+$19] and spin glass models in physics [BGJR88]. Graph coloring problems have been applied to use cases such as machine register allocation [CAC$^+$81], scheduling of courses in a timetable [GR17], and frequency assignment in signal transmission [Hal80].

In this thesis we show that FEEDBACK VERTEX SET, the only one of the problems which is not already known to be in $\mathcal{P}$ on subgraphs of grids, is $\mathcal{NP}$-complete on grid graphs. We

Problem	Grid graphs	Partial grids	Strong grid graphs	Partial strong grids
Feedback Vertex Set	NPC (*)	NPC (*)	NPC (*)	NPC (*)
Minimum Vertex Cover	P (bipartite)	P (bipartite)	NPC (*)	NPC (*)
Maximum Cut	P (planar)	P (planar)	NPC (*)	NPC (*)
3-Vertex-Colorability	P (bipartite)	P (bipartite)	NPC (*)	NPC (*)
3-Edge-Colorablity	P (bipartite)	P (bipartite)	Open	NPC (*)

Table 1.1: Summary of computational complexity results

$\Delta(G)$	Strong grid graphs	Partial strong grids
1	Class I	Class I
2	$\exists$ Class II example	$\exists$ Class II example
3	$\exists$ Class II example	$\exists$ Class II example
4	Open	$\exists$ Class II example
5	Open	Open
6	Open, has Δ-Matching	Open
7	Open, has Δ-Matching	Open, has Δ-Matching
8	Class I	Class I

Table 1.2: Summary of Class I / Class II results

provide proof that FEEDBACK VERTEX SET, MINIMUM VERTEX COVER, MAXIMUM CUT, 3-COLORABILITY and 3-EDGE-COLORABILITY are all $\mathcal{NP}$-complete on partial strong grids, and all but 3-EDGE-COLORABILITY are $\mathcal{NP}$-complete on strong grid graphs. Additionally, we show that the problem 3-EDGE-COLORABILITY on planar graphs is polynomially equivalent to 3-EDGE-COLORABILITY on strong grid graphs. Whether 3-EDGE-COLORABILITY on planar graphs is $\mathcal{NP}$-complete is a famous open problem.

Finally, we attempt to classify subgraphs of strong grids in relation to Vizing's Theorem, based on their maximum degree. Vizing's Theorem states that, for a graph G with maximum degree $\Delta(G)$, the chromatic index $\chi'(G)$ is in $\{\Delta(G), \Delta(G)+1\}$ [Viz64]. If $\chi'(G) = \Delta(G)$ holds, G is Class I, else G is Class II. We investigate whether there exist Class II partial strong grids and strong grid graphs, for every possible maximum degree ($\{1, \ldots, 8\}$). In some cases where no conclusion could be drawn, we discuss relevant structural properties.

Table 1.1 provides a summary of the computational complexity of the studied problems on subgraphs of (strong) grids. The contributions of this thesis are marked with '*'.

Table 1.2 summarizes our findings on the existence of Class II partial strong grids and strong grid graphs.

1.1 Related Work

We investigate the computational complexity of a selection of well-known algorithmic problems on subgraphs of (strong) grids. A lot of research has been carried out on the computational complexity of the problems we chose to investigate.

A finding that is of great relevance for the strong and Cartesian products of paths, where at least one of the paths has a constant length, is Courcelle's Theorem. Courcelle's Theorem states that all graph properties that can be expressed in monadic second order logic are decidable in linear time on graphs with bounded treewidth [Cou90]. Subgraphs of both the strong and Cartesian product of two paths have bounded treewidth, if one of the paths has bounded length. Since all of the problems discussed in this thesis can be expressed in monadic second order logic, Courcelle's Theorem implies they can be decided in linear time on this class of graphs. We investigate a more general case, by considering subgraphs of the Cartesian and strong product of two path graphs with unbounded length.

While the chosen problems have long been known to be $\mathcal{NP}$-complete on general graphs, a lot of research has been done on their computational complexity on restricted graph classes.

Polynomial-time algorithms on superclasses of subgraphs of grids, such as planar or bipartite graphs, are known for all of the chosen problems other than FEEDBACK VERTEX SET. Takaoka, Tayo and Ueno prove the $\mathcal{NP}$-hardness of FEEDBACK VERTEX SET on grid intersection graphs, which implies $\mathcal{NP}$-hardness on bipartite graphs [TTU12]. Garey, Johnson and Stockmeyer have shown that MINIMUM VERTEX COVER and 3-COLORABILITY are $\mathcal{NP}$-complete on planar graphs of maximum degree 3 [GJS76]. FEEDBACK VERTEX SET has been shown to be $\mathcal{NP}$-complete on planar graphs of maximum degree 4 [Spe83]. Other research finds that MAXIMUM CUT and 3-EDGE-COLORABILITY are $\mathcal{NP}$-complete on cubic graphs [BK99, Hol81].

The classification of graphs with respect to Vizing's theorem has also been covered extensively in literature, especially regarding planar graphs. Vizing proved that all planar graphs with a maximum degree $\Delta \geq 8$ are Class I, and conjectured that the same holds for planar graphs with $\Delta \geq 6$ [Viz64]. The $\Delta = 7$ case has since been proven true by Sanders and Zhao [SZ01]. Class II examples are known for planar graphs with $2 \leq \Delta \leq 5$. It remains open whether there exist Class II planar graphs with $\Delta = 6$. In this thesis we attempt to perform a similar classification, on arbitrary and induced subgraphs of strong grids.

Another problem that is related to this thesis is the embedding of graphs in grids. An embedding of a graph G into a graph H is an injective function, that maps the vertices of G to the vertices of H and the edges of G to the edges of H. Efficient embedding algorithms could be applied to identify whether an arbitrary graph is a subgraph of a (strong) grid. This is relevant to the application of algorithms whose runtime relies on input graphs being subgraphs of grids or strong grids. However, it has been shown that it is $\mathcal{NP}$-complete to embed tree graphs, both into grids and strong grids [Gre89, BEU23].

2. Preliminaries

In this thesis we only consider undirected graphs without loops, and without multiple edges between the same vertices.

2.1 Definitions

2.1.1 Induced Subgraphs

Let $G = (V, E)$ be a graph, and $S \subseteq V$. We call the graph $G[S] := (S, \{e = (u, v) \in E \mid u, v \in S\}$ the subgraph of G *induced* by S. Furthermore, we say that $G[S]$ is an *induced subgraph* of G.

2.1.2 Intersection, Union and Difference of Graphs

Let $G_1 = (V_1, E_1), G_2 = (V_2, E_2)$ be two not necessarily disjoint graphs. We define the graph $G_1 \cap G_2$ as $(V_1 \cap V_2, E_1 \cap E_2)$. Analogously $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$.

We define the graph $G_1 \setminus G_2 := G_1[V_1 \setminus V_2]$.

2.1.3 Products of graphs

We use two different definitions of graph products: the *Cartesian product* and the *strong product*. We only consider products of two graphs.

Let $G_1 = (V_1, E_1), G_2 = (V_2, E_2)$ be two graphs. We denote the Cartesian product of G_1 and G_2 as $G_1 \Box G_2$, and the strong product as $G_1 \boxtimes G_2$.

The vertex set of $G_1 \Box G_2$ is the Cartesian product of the vertex sets of G_1 and G_2, $V_1 \times V_2$.

To avoid confusion, we denote a tuple of two vertices u, v as $\langle u, v\rangle$, and an edge between u, v as (u, v). If $\langle v_1, v_2\rangle \in V_1 \times V_2$, then $v_1 \in V_1$ and $v_2 \in V_2$. For our purposes, the order of vertices does not matter when defining edges: $(u, v) = (v, u)$.

The edge set $E_\Box$ of $G_1 \Box G_2$ is given by $\{(\langle u, v\rangle, \langle u', v'\rangle) \mid (u = u' \wedge \exists e = (v, v') \in E_2) \vee (v = v' \wedge \exists e' = (u, u') \in E_1)\}$.

The vertex set of the strong product $G_1 \boxtimes G_2$ is also $V_1 \times V_2$. The edge set $E_\boxtimes$ of $G_1 \boxtimes G_2$ is given by $\{(\langle u, v\rangle, \langle u', v'\rangle) \mid (u = u' \wedge \exists e = (v, v') \in E_2) \vee (v = v' \wedge \exists e' = (u, u') \in E_1) \vee (\exists e_1 = (u, u') \in E_1 \wedge e_2 = (v, v') \in E_2)\}$.

$G_1 \Box G_2$ is a subgraph of $G_1 \boxtimes G_2$. A simple example of the Cartesian and strong product of two graphs is shown in Figure 2.1.

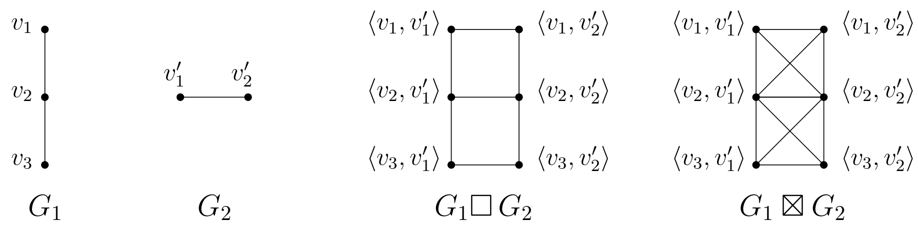


Figure 2.1: Cartesian and strong product example

2.1.4 Grids and their subgraphs

The Cartesian product of two paths is known as a *grid*. We refer to the strong product of two paths as a *strong grid*. Given two paths P_n, P_m, with lengths n and m respectively, we call their Cartesian product $P_n \square P_m$ an $n \times m$ grid. Analogously, $P_n \boxtimes P_m$ is an $n \times m$ strong grid.

We refer to subgraphs of grids as *partial grids*, and induced subgraphs as *grid graphs*. Similarly, we call subgraphs of strong grids *partial strong grids*, and induced subgraphs *strong grid graphs*.

In this thesis, we always consider grids that are embedded in the plane.

2.1.5 Grid Coordinates

When a vertex v is placed in a grid, it receives a unique integer coordinate (v_x, v_y) in the grid. For two vertices u, v with grid coordinates (u_x, u_y) and (v_x, v_y), v is to the right of u if $v_x > u_x$. If $v_y > u_y$ v is above u. Vertices u, v are referred to as *diagonal* to each other if $|v_x - u_x| = |v_y - u_y|$. An edge $e = (u, v)$ is referred to as *diagonal* if its endpoints u, v are diagonal to each other.

2.1.6 Maximum Degree

We define the *maximum degree* of a graph $G = (V, E)$ as the greatest degree among all vertices in V. $\Delta(G)$ denotes the maximum degree of G.

2.1.7 Neighbor Vertices

Let $G = (V, E)$ be a graph, and let $v \in V$ be a vertex. We refer to the vertices in V, which are connected to v via an edge in E as *neighbors* of v. $N(G, v)$ denotes the set of all neighbors of v in G.

Let $V_s \subseteq V$. We define $N(G, V_s) := \bigcup_{v_s \in V_s} N(G, v_s) \setminus V_s$.

2.1.8 Subdivision of a graph

Let $G = (V, E)$ be a graph, and $e = (u, v) \in E$ an edge. An edge-subdivision of e is carried out as follows: e is removed from E. A new vertex v' is added to V. Two new edges (u, v') and (v', v) are added to E.

A graph $G' = (V \cup V_s, E')$ is called a subdivision of G, if it can be obtained by carrying out a series of edge-subdivisions on G. The vertex set V of G remains part of the vertex set of G', as none of the vertices from V are removed when carrying out edge-subdivisions. Vertices $v \in V$ are referred to as *original vertices*, $v' \in V_s$ are referred to as *subdivision vertices*. For edges $e \in E$, $s(G', e)$ denotes the set of vertices from V_s, which subdivide e in G'.

2.1.9 Even Subdivision of a graph

Let $G = (V, E)$ be a graph. A graph $G' = (V', E')$ is called an *even subdivision* of G, if G' is a subdivision of G, such that for all edges $e \in E$ $|s(G', e)|$ is even.

2.1.10 Matchings

Let $G = (V, E)$ be a graph. We refer to $M \subseteq E$ as a *matching*, if no edges in M have an endpoint in common. We say M *covers* a vertex $v \in V$, if there exists an edge in M that has v as an endpoint.

A matching that covers all vertices in V is referred to as a *perfect matching* of G.

2.1.11 Vertex Ordering

We define 4 quasiorders with which we can compare vertices based on their grid coordinates. A *left-ordering* orders along the x-axis in descending order, a *right-ordering* in ascending order. A *bottom-ordering* orders along the y-axis in descending order, a *top-ordering* in ascending order.

We define 8 total orderings for vertices in a grid: *top-right-*, *right-top-*, *top-left-*, *left-top-*, *bottom-right-*, *right-bottom-*, *bottom-left-*, and *left-bottom-orderings.*

A top-right-ordering first orders vertices using a top-ordering. It then orders the vertices, which have the same y-coordinates, using a right-ordering. In contrast, a right-top-ordering first orders vertices using a right-ordering, then uses a top-ordering to sort vertices with the same x-coordinates.

The other orderings are defined analogously.

Let $G = (V, E)$ be a partial strong grid, and $S \subseteq V$.

We can order the vertices in S using any of the 8 total orderings defined above. We define the *top-right-*, *right-top-*, *top-left-*, *left-top-*, *bottom-right-*, *right-bottom-*, *bottom-left-*, and *left-bottom vertex* of S, as the vertex in S that is greatest in the corresponding total ordering.

For example, the top-right vertex of S is the greatest vertex in a top-right-ordering, so the vertex with the greatest x-coordinate among the vertices with the greatest y-coordinate in S.

Analogously, we define the *top-right-*, *right-top-*, *top-left-*, *left-top-*, *bottom-right-*, *right-bottom-*, *bottom-left-*, and *left-bottom neighbor* of S, as the vertex in $N(G, S)$ that is greatest in the corresponding total ordering.

2.1.12 Orthogonal Drawings

An orthogonal drawing of a graph $G = (V, E)$ is an embedding in the plane where all edges are drawn as alternating sequences of vertical and horizontal line segments.

Let $e = (u, v) \in E$ be an edge, drawn using k line segments $\{\ell_1, ..., \ell_k\}$ in an orthogonal drawing of G. If $k = 1$, e is drawn as a straight line from u to v. Else, e starts with line segment ℓ_1 with an endpoint at u, and ends with line segment ℓ_k with an endpoint at v. Each line segment ℓ_i with $i \notin \{1, k\}$ shares one endpoint with ℓ_{i-1} and ℓ_{i+1}. A line segment ℓ_i $(i \neq 1)$ always has a different orientation (vertical or horizontal) than the previous line segment ℓ_{i-1}. The point where a horizontal and vertical line segment meet is referred to as a *bend*. An example of an orthogonal drawing can be seen in Figure 2.2.

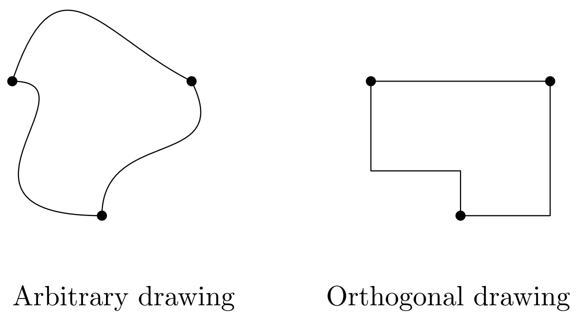


Figure 2.2: Example of an orthogonal drawing

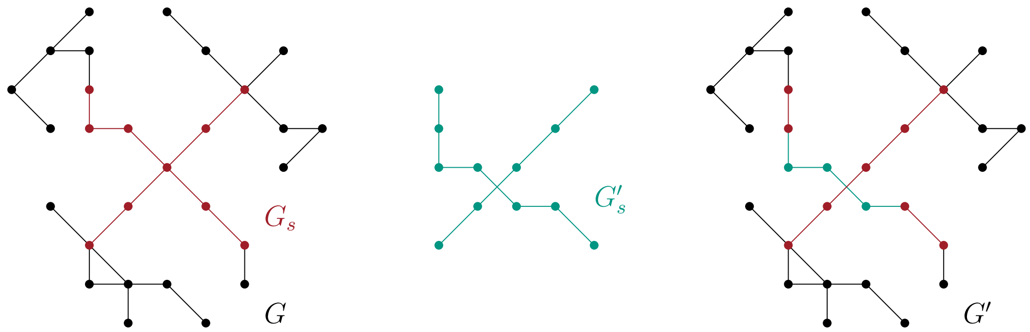


Figure 2.3: Example of a subgraph replacement

2.1.13 Subgraph replacement in partial strong grids

Let $G = (V, E)$ be a partial strong grid. Let $G_s = (V_s, E_s)$ be a subgraph of G. Let $x_{\text{min}}, x_{\text{max}}, y_{\text{min}}, y_{\text{max}}$ be the lowest and highest x- and y-coordinates in G_s.

Let $G'_s = (V'_s, E'_s)$ be a different partial strong grid, such that the lowest and highest x- and y-coordinates in G'_s are the same as in G_s. We define $V_\cap \subseteq V_s$ as follows: $V_\cap := \{v \in V_s \mid \exists v' \in V'_s : v'_x = v_x \wedge v'_y = v_y\}$. $V_\cap$ contains the vertices in V_s, for which a vertex with the same coordinate exists in V'_s. For a vertex $v \in V_\cap$, $c(v)$ denotes the vertex with the same coordinate in V'_s. $V'_\cap \subseteq V'_s$ denotes the set of vertices in G'_s, for which there exists a vertex with the same coordinate in V_s.

We define a new graph $G' = (V', E')$, which is the result of replacing the subgraph G_s with the graph G'_s in G. Only vertices not in $V_\cap$ are replaced, so $V' := (V \setminus (V_s \setminus V_\cap)) \cup (V'_s \setminus V'_\cap)$. All edges between vertices in $V \setminus (V_s \setminus V_\cap)$ in G are included in E'. The same holds for edges between vertices in $V'_s \setminus V'_\cap$ in G'_s. There is an edge between $v^1_\cap, v^2_\cap \in V' \cap V_\cap$ in G', if and only if $c(v^1_\cap), c(v^2_\cap)$ were adjacent in G'_s. There is an edge between $v_\cap \in V' \cap V_\cap$ and $v' \in V' \cap V'_s$, if and only if $c(v_\cap)$ and v' were adjacent in G'_s.

Essentially, we replace the subgraph G_s with the graph G'_s in G, to obtain G'. However, if a new vertex in G' would be placed at coordinates where there already was a vertex in G, we keep the vertex from G in G' instead of replacing it.

An example of a subgraph replacement is shown in Figure 2.3.

2.2 Embedding graphs in a grid

Many of the polynomial-time reductions in this thesis embed a graph $G = (V, E)$ in a (strong) grid, so that the embedded graph is a subdivision of G. Only connected graphs need to be transformed; all the problems studied in this thesis can be solved for non-connected graphs by solving them separately on the connected components. The transformations discussed here are limited to graphs with a maximum degree of 4.

Construction 2.1. Let $G = (V, E)$ be a planar graph of maximum degree 4.

We compute an orthogonal drawing of G. This is done using the linear time algorithm presented in [BK98], which can compute an orthogonal drawing of connected graphs with a maximum degree of 4, in an $n \times n$ grid. The drawing has at most $2n + 4$ bends. The algorithm places all vertices and bends at integer coordinates in the generated drawing, which means they align with the locations of points in a grid. When given a planar graph with a planar embedding, the drawing computed by the algorithm is also planar.

The orthogonal drawing is placed on an $n \times n$ grid, so that its vertices align with the grid points. Then the edges are subdivided at all grid points they cross. The resulting graph, referred to as $G_{grid} = (V_{grid}, E_{grid})$, is both a subdivision of G and a partial grid.

The construction of G_{grid} has polynomial runtime in n, and since G_{grid} is within an $n \times n$ grid, $|V_{grid}| \leq n^2$.

Construction 2.2. Let $G = (V, E)$ be a (not necessarily planar) graph with maximum degree 4.

When G is non-planar, any two edges $e_1 = (u_1, v_1), e_2 = (u_2, v_2) \in E$ that cross each other meet at integer coordinates in an orthogonal drawing of G, which is obtained using the algorithm presented in [BK98].

Let (c_x, c_y) be the coordinates of the crossing of two edges $e_1, e_2 \in E$ in the orthogonal drawing of G. We place a vertex v_c at (c_x, c_y). We delete the edges e_1, e_2, and replace them with edges $e_1^1 = (u_1, v_c), e_1^2 = (v_c, v_1), e_2^1 = (u_2, v_c), e_2^2 = (v_c, v_2)$. We perform this modification for every crossing in G, to obtain the graph $G' = (V', E')$.

G' is a planar graph. We apply Construction 2.1 to G', resulting in the output partial grid G_{grid}.

Since the number of crossings in G is polynomial in n, $|V'|$ is also polynomial in n. $|V_{grid}|$ is polynomial in $|V'|$ (see Construction 2.1).The polynomial runtime of the construction of G_{grid} follows from the polynomial runtime of 2.1.

2.2.1 Constructing induced subgraphs of grids

When constructing a (strong) grid graph G_{grid_graph} which is a subdivision of another graph $G = (V, E)$, we need to ensure no unwanted edges are induced in G_{grid_graph}.

Using Construction 2.1, a partial grid G_{grid} which is a subdivision of G can be obtained. However, vertices can be placed too close to each other in G_{grid}, inducing unwanted edges. Also, some of the upcoming constructions replace subgraphs $G_{\subseteq}$ of G_{grid} with other graphs G_{new}, where G_{new} takes up more space than $G_{\subseteq}$.

To address these issues, an operation to increase the space between vertices in partial grids is needed. This operation should ensure that, when applied to G_{grid}, the resulting graph G'_{grid} remains a subdivision of G and a partial (strong) grid.

Observation 2.3. *Given a partial strong grid $G_{grid} = (V_{grid}, E_{grid})$, a partial strong grid $G'_{grid} = (V'_{grid}, E'_{grid})$ can be constructed, so that G'_{grid} is a subdivision of G_{grid}, and the distance within the grid between any two vertices $u, v \in E \cap E'$ is twice as much in G'_{grid} as in G_{grid}. If G_{grid} is a partial grid, G'_{grid} is also a partial grid.*

This can be done in polynomial time in $n = |V|$.

The transformation from G_{grid} to G'_{grid} will be referred to as a resolution doubling operation.

(a) Before resolution doubling (b) After resolution doubling

Figure 2.4: Example resolution doubling

Proof. Given a partial strong grid $G = (V, E)$ on an $n \times n$ grid, we construct a new graph $G''_{grid} = (V''_{grid}, E''_{grid})$ in a $2n \times 2n$ grid. Each vertex $v \in V$ with coordinates (x, y) in the $n \times n$ grid is placed at coordinates $(2x, 2y)$ in G''_{grid}, doubling the distances between all vertices. All edges $e = (u, v) \in E$ are drawn as a straight line between the same vertices in G''_{grid}.

We subdivide all edges $e'' \in E''_{grid}$, at every grid point they cross, where there is not already a vertex, to obtain the graph G'_{grid}.

G'_{grid} is a subdivision of G_{grid} in a $2n \times 2n$ grid, and the time required to compute it is naturally polynomial in n. If G_{grid} contains no diagonal edges, neither does G'_{grid}.

An example is depicted in Figure 2.4. □

Construction 2.4. Let $G = (V, E)$ be a planar graph of maximum degree 4.

When transforming $G = (V, E)$ using Construction 2.1 to obtain G_{grid}, the subdivision vertices of two different edges, or two original vertices, may be only one unit apart in G_{grid}. In a (strong) grid graph, this would induce an edge connecting two vertices that were not connected in G, or an edge between the subdivisions of two different edges.

To avoid this, we use the resolution doubling operation from Observation 2.3. When such an operation is carried out on G_{grid} to obtain the graph $G'_{grid} = (V'_{grid}, E'_{grid})$, it results in an additional unit of space between vertices that would otherwise be one unit apart, preventing these two cases of unwanted induced edges.

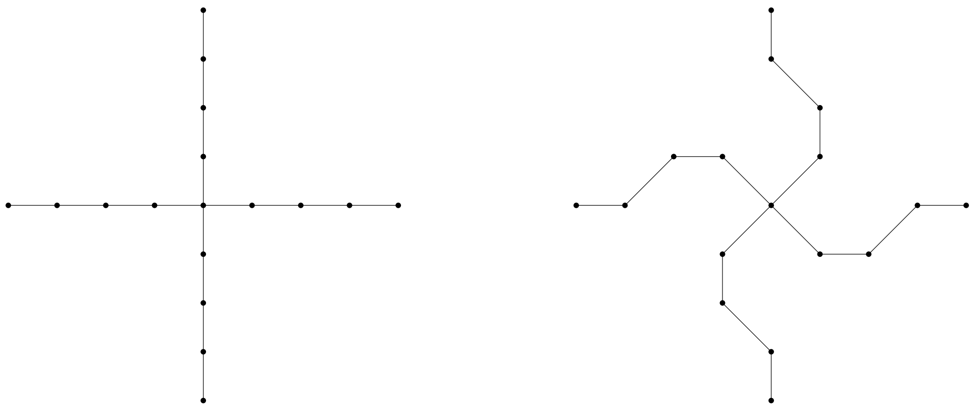
G'_{grid} is a subdivision of G and a grid graph. The resolution doubling operation can be performed in polynomial time in n, and since G'_{grid} is within a $2n \times 2n$ grid, $|V'_{grid}| \leq 4n^2$.

Construction 2.5. Let $G = (V, E)$ be a planar graph of maximum degree 4.

We perform Construction 2.4 on G to obtain the graph G^1_{grid}. G^1_{grid} is a subdivision of G and a grid graph, but not necessarily a strong grid graph.

At bends in G^1_{grid} along the subdivision of edges, an unwanted diagonal edge is induced in strong grid graphs. This can be prevented by removing the vertex on the bend, including all adjacent edges. The diagonal edge between the two vertices adjacent to the removed vertex remains. By performing this modification at all bends in G^1_{grid}, we obtain the graph G^2_{grid}.

Another issue is that horizontal and vertical edges adjacent to an original vertex $v \in V$ induce edges between each other in G^2_{grid} (and G^1_{grid}). To avoid this, we first perform two resolution doubling operations on G^2_{grid}, to create space for a subgraph replacement, resulting in the graph G^3_{grid}. In G^3_{grid}, for each edge adjacent to an original vertex $v \in V$, there exists a path of length 4 that contains the adjacent edge and ends in v. We replace



(a) After two resolution doubling operations (b) After replacement

Figure 2.5: Transformation of degree 4 vertex for strong grid graphs

all such path subgraphs in G^3_{grid} with modified graphs, as shown in Figure 2.5, to obtain $G_{strgrid} = (V_{strgrid}, E_{strgrid})$. Since the edges adjacent to v are all diagonal after the replacement, no unwanted edges are induced between them.

$G_{strgrid}$ is both a strong grid graph and a subdivision of G.

As stated in Construction 2.1, G'_{grid} can be computed in polynomial time. To obtain $G_{strgrid}$, 2 resolution doubling operations are carried out on G'_{grid}. Hence $G_{strgrid}$ is within a $8n \times 8n$ grid, and $|V_{strgrid}| \leq 64n^2$. The runtime to carry out the resolution doubling operations is polynomial in n. Replacement of a subgraph is possible in constant time, and is carried out for all bends and original vertices, of which there can only be $\mathcal{O}(n)$.

2.2.2 Transforming non-planar graphs

Non-planar graphs and their subdivisions cannot be embedded in a partial grid, since grids are planar. In strong grids, diagonal edges can cross each other. Since this would induce a K_4 in strong grid graphs, the upcoming constructions cannot produce strong grid graphs which are subdivisions of the original graph.

Construction 2.6. Let $G = (V, E)$ be a (potentially non-planar) graph $G = (V, E)$, with maximum degree 4.

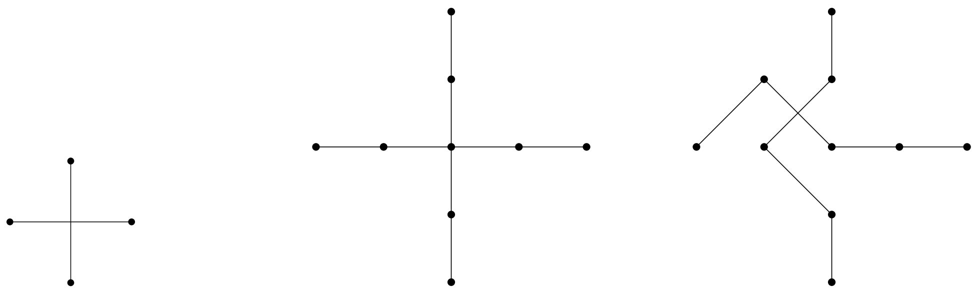
Construction 2.2 accepts non-planar input graphs G_{np}, but does not produce subdivisions of G_{np}.

Let G''_{grid} be the result of performing Construction 2.2 on G. To create space for a subgraph replacement, we perform a resolution doubling operation (Observation 2.3) on G''_{grid}, to obtain the new graph G'_{grid}. We replace all crossing subgraphs in G'_{grid} as illustrated in Figure 2.6, to obtain G_{grid}. G_{grid} is a partial strong grid, and a subdivision of G.

Since the number of crossings is polynomial in n, and they can each be replaced in constant time, the runtime of this transformation is polynomial in n. G_{grid} is within a $2n \times 2n$ grid, hence $|V| \leq 4n^2$.

Construction 2.7. Let $G = (V, E)$ be a graph with maximum degree 4.

We obtain $G^0_{grid} = (V_{grid}, E_{grid})$ by applying Construction 2.2 to G. Then, we obtain $G^1_{grid} = (V_{grid}, E_{grid})$ by "rotating" G^0_{grid} by 45° clockwise. More precisely, it is embedded in a diagonal grid of size $n \times n$, which is an induced subgraph of a larger strong grid. This is illustrated in figure 2.7. G^1_{grid} and G^0_{grid} are isomorphic.



(a) Original (b) After resolution doubling (c) After replacement

Figure 2.6: Transformation of a crossing

(a) Before rotation (b) After rotation

Figure 2.7: *Rotation* of a graph

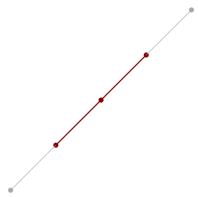
If G is non-planar, the crossings need to be modified. In G^1_{grid}, a crossing subgraph is a subdivision vertex with exactly 4 adjacent vertices, one on each diagonal. A resolution doubling operation (Observation 2.3) is performed on G^1_{grid} to create space for the replacement of these subgraphs, resulting in the graph G^2_{grid}. All crossing subgraphs in G^2_{grid} are replaced, as shown in Figure 2.8, resulting in the graph G^3_{grid}.

We transform G^3_{grid} to G^4_{grid} by performing two resolution doubling operations, which creates enough space to guarantee that the upcoming subgraph replacement does not induce unwanted edges.

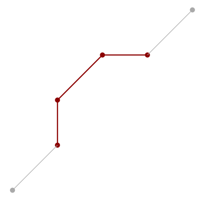
Then, the subdivisions of all original edges $e = (u, v) \in E$, which have an odd number of subdivision vertices in G^4_{grid}, are altered. The subdivisions in G^4_{grid} of edges from E, with an odd number of subdivision vertices, have at least 7 subdivision vertices. When counting the subdivision vertices starting with those closest to u, the subgraph induced by the third, fourth and fifth vertex is a diagonal path with three vertices. This subgraph is replaced as shown in Figure 2.9, on every subdivision of an edge from E in G^4_{grid} which has an odd number of subdivision vertices. This results in the graph $G'_{grid} = (V'_{grid}, E'_{grid})$.

(a) After resolution doubling (b) After replacement

Figure 2.8: Crossing subgraph for even subdivision



(a) Before replacement



(b) After replacement

Figure 2.9: Replacement for even subdivision

G'_{grid} is a subdivision of G, where each original edge is subdivided by an even number of vertices. If G was planar, G'_{grid} is even a strong grid graph: the "rotation" by 45° makes the modification of bends and original vertices, performed in Construction 2.5, superfluous. Both the size of V'_{grid} and the runtime needed to compute G'_{grid} are polynomial in n.

3. Feedback Vertex Set

For a graph $G = (V, E)$, a feedback vertex set of G is a set $X \subseteq V$ whose removal leaves G without cycles. More precisely, X is a feedback vertex set of G if and only if the graph $G' := G[V \setminus X]$ does not contain cycles.

The FEEDBACK VERTEX SET problem is defined as follows: Given an instance (G, k), with a graph G and $k \in \mathbb{N}$, it should be decided whether (G, k) is a YES- or a NO-instance. (G, k) is a YES-instance if there exists a feedback vertex set X of G, with $|X| \leq k$. Else, (G, k) is a NO-instance.

FEEDBACK VERTEX SET is known to be $\mathcal{NP}$-complete on planar graphs of maximum degree 4 [Spe88]. It is also $\mathcal{NP}$-complete on bipartite graphs [TTU12].

Lemma 3.1. *Let $G = (V, E)$ be a graph, and $G' = (V', E')$ a subdivision of G. Let $X \subseteq V$ be a feedback vertex set of G. Then X is also a feedback vertex set of G'.*

Proof. Let $C = (V_c, E_c)$ be a cycle subgraph in G. There exists a vertex $v_c \in V_c \cap X$. Since G' is a subdivision of G, any cycle in G' is the subdivision of exactly one cycle in G. $C' = G'[(\bigcup_{e \in E_c} s(G', e)) \cup V_c]$ is the subdivision of the cycle C in G'. So the removal of $v_c \in V_c$ also breaks the cycle C'.

Since the removal of X breaks all cycles in G, it follows that its removal also breaks all cycles in G'. □

Lemma 3.2. *Let $G = (V, E)$ be a graph, and $G' = (V', E')$ a subdivision of G. Let $X' \subseteq V'$ be a feedback vertex set of G', with $|X'| = k$. Then there exists a feedback vertex set $X \subseteq V$ of G with $|X| \leq k$.*

Proof. We construct a new set $X \subseteq V$ from X'. We add all vertices from $X' \cap V$ to X. For each $v' \in X' \setminus V$, let the vertex $v \in V$, with shortest distance to v' in G', be in X. We claim X is a feedback vertex set of G.

Consider a cycle subgraph $C = (V_c, E_c)$ in G, and its subdivision $C' = (V'_c, E'_c)$ in G'. Naturally, if V_c contains a vertex from $(X' \cap V) \subseteq X$, the removal of that vertex will result in the cycle C being broken.

If that is not the case, V'_c contains a vertex v'_s from $X' \setminus V$, and there exists an edge $e_c = (u_c, v_c) \in E_c$ in C such that $v'_s \in s(G', e_c)$. Then one of u_c, v_c is the original vertex with the shortest distance to v'_s in C'. The removal of a vertex in $\{u_c, v_c\}$, at least one of which is included in X, will break the cycle C. □

Theorem 3.3. *FEEDBACK VERTEX SET is $\mathcal{NP}$-complete on grid graphs.*

Proof. Since FEEDBACK VERTEX SET on planar graphs of maximum degree 4 is $\mathcal{NP}$-complete, it is clear that FEEDBACK VERTEX SET on grid graphs is in $\mathcal{NP}$.

To prove $\mathcal{NP}$-hardness, we perform a polynomial-time reduction from FEEDBACK VERTEX SET on planar graphs of maximum degree 4, to FEEDBACK VERTEX SET on grid graphs.

Let (G, k) be an instance of FEEDBACK VERTEX SET on planar graphs of maximum degree 4. We obtain a subdivision G_{grid} of G using Construction 2.4. Then G_{grid} is a grid graph, and (G_{grid}, k) is an instance of FEEDBACK VERTEX SET on grid graphs.

If (G, k) is a YES-instance, G has a feedback vertex set X with $|X| \leq k$. Due to Lemma 3.1, X is also a feedback vertex set of G_{grid}, and (G_{grid}, k) is a YES-instance.

If (G_{grid}, k) is a YES-instance, G_{grid} has a feedback vertex set X' with $|X'| \leq k$. Due to Lemma 3.2, X' is also a feedback vertex set of G, so (G, k) is a YES-instance.

Consequently (G, k) is a YES-instance if and only if (G_{grid}, k) is a YES-instance. □

4. Minimum Vertex Cover

A vertex cover of a graph $G = (V, E)$ is a set of vertices $C \subseteq V$, such that $\forall e = (u, v) \in E : u \in C \vee v \in C$. A set of vertices $C_{min} \subseteq V$ is a minimum vertex cover if it is a vertex cover of G with the lowest number of vertices.

The MINIMUM VERTEX COVER problem is defined as follows: Given an instance (G, k), with a graph G and a number $k \in \mathbb{N}$, it should be decided whether (G, k) is a YES- or a NO-instance. If G has a minimum vertex cover of size at most k, (G, k) is a YES-instance, else it is a NO-instance.

MINIMUM VERTEX COVER is known to be solvable in polynomial time on bipartite graphs [Whi84], of which grid graphs and partial grids are subclasses. On planar graphs of maximum degree 4, MINIMUM VERTEX COVER is $\mathcal{NP}$-complete [GJS76].

Lemma 4.1. *Let $P = (V, E)$ be a path of odd length $2k + 1, k \in \mathbb{N}$, between vertices u and v. Then any vertex cover C of P contains at least $k + 1$ vertices, of which at least k are in $V \setminus \{u, v\}$.*

Proof. There are $2k + 1$ edges along the path from u to v, of which at least $2k - 1$ are not covered by u or v. On P, a vertex $v' \in C$ can only cover up to two edges. To cover the $2k - 1$ edges not covered by u or v, at least $\lceil (2k - 1)/2 \rceil = k$ vertices from $V \setminus \{u, v\}$ are included in C. Analogously, to cover the $2k + 1$ edges in P, at least $\lceil (2k + 1)/2 \rceil = k + 1$ from V are included in C. □

Lemma 4.2. *Let $G = (V, E)$ be a graph, and $G' = (V', E')$ an even subdivision of G. Let $C' \subseteq V'$ be a vertex cover of G'. Then a vertex cover C'' of G' with $|C''| \leq |C'|$ exists, so that $C'' \cap V$ is a vertex cover in G. The time required to compute C'' from C' is polynomial in $|V'|$.*

Proof. If $C' \cap V$ is already a vertex cover in G, the claim is fulfilled by simply selecting $C'' = C'$.

We assume $C' \cap V$ is not a vertex cover in G. Let $E_u \subseteq E$ be the set of edges in G not covered by $C' \cap V$.

We construct a vertex cover C'' of G' as follows: Let $V'_u = \bigcup_{e_u \in E_u} s(G', e_u)$ be the set of subdivision vertices of the edges in E_u. We add all vertices in $C' \setminus V'_u$ to C''.

Let $e_u = (v_u^1, v_u^2) \in E_u$ be an edge, with $k := |s(G', e_u)|$. Since G' is an even subdivision, k is even. Then, due to Lemma 4.1, $|C' \cap s(G', e_u)| \geq k/2 + 1$. Since e_u was not covered by $C' \cap V$, neither of v_u^1, v_u^2 was in $C' \setminus V_u'$. We add v_u^1, and the $k/2$ vertices in $s(G', e_u)$ with even distance to v_u^1, to C''.

As a result, for every edge $e_u = (v_u^1, v_u^2) \in E_u$, $|C'' \cap (s(G', e_u) \cup \{v_u^1, v_u^2\})| \leq |C' \cap (s(G', e_u) \cup \{v_u^1, v_u^2\})|$. Also, C'' covers all edges in G' with both endpoints in $s(G', e_u) \cup \{v_u^1, v_u^2\}$. Since C'' contains the same vertices as C' from $V \setminus (\bigcup_{e_u=(v_u^1,v_u^2)\in E_u} s(G', e_u) \cup \{v_u^1, v_u^2\})$, it follows that $|C''| \leq |C'|$, and that C'' is a vertex cover of G'. The runtime of both the computation of E_u and the construction of C'' is polynomial in $|V'|$. □

Lemma 4.3. *Let $G = (V, E)$ be a graph, and $G' = (V', E')$ an even subdivision of G. Let $C \subseteq V$ be a vertex cover of G. Then there exists a vertex cover C' of G', with $|C'| = |C| + \sum_{e\in E} |s(G', e)|/2$.*

Proof. We construct a new set $C' \subseteq V'$. We add all vertices from C to C'. For every edge $e = (u, v) \in E$, we select an endpoint $v_c \in \{u, v\}$, which is in C. We include all vertices in $s(G_{grid}, e)$ with even distance to v_c in C'. The size of C' is $|C'| = |C| + \sum_{e\in E} |s(G', e)|/2$.

C' is a vertex cover of G': for every edge $e \in E$, C' covers all edges in the subgraph of G' induced by $s(G', e) \cup \{u, v\}$. The union of all these subgraphs, $\bigcup_{e=(u,v)\in E} s(G', e) \cup \{u, v\}$, contains all edges in E'. □

Lemma 4.4. *Let $G = (V, E)$ be a graph, and $G' = (V', E')$ an even subdivision of G. Let C' be a minimum vertex cover of G', such that $C = C' \cap V$ is a vertex cover in G.*

Then C is a minimum vertex cover of G, and $|C'| = |C| + \sum_{e\in E} |s(G', e)|/2$.

Proof. For the sake of contradiction, assume there exists a vertex cover C_2 of G, with $|C_2| < |C|$.

Due to Lemma 4.3, there exists a vertex cover C_2' of G', with $|C_2'| = |C_2| + \sum_{e\in E} |s(G', e)|/2$.

Let $e \in E$ be an edge in G, with $|s(G', e)| = k$. Due to Lemma 4.1, at least $k/2$ of the subdivision vertices in $s(G', e)$ are included in every vertex cover of G. Hence $|C'| \geq |C| + \sum_{e\in E} |s(G', e)|/2$.

Since $|C_2| < |C|$, it follows that $|C_2'| < |C'|$, which is in contradiction to C' being a minimum vertex cover of G'. Consequently, $C = C' \cap V$ is a minimum vertex cover of G.

Let $e_2 = (u_2, v_2) \in E$ be an edge, without loss of generality assume $u_2 \in C'$. We claim C' contains exactly half of the subdivision vertices in $s(G', e_2)$. It does not contain fewer due to Lemma 4.1. Due to Lemma 4.3, there exists a vertex cover C''' of G', with $|C'''| = |C| + \sum_{e\in E} |s(G', e)|/2$. Since C' is a minimum vertex cover, $|C'| \leq |C| + \sum_{e\in E} |s(G', e)|/2$. Since $C = C' \cap V$, this is not possible if C' contains more than half of the vertices in $s(G', e_2)$ for any edge $e_2 \in E$.

Consequently, the number of subdivision vertices in C' is exactly $\sum_{e\in E} |s(G', e)|/2$, and $|C'| = |C' \cap V| + \sum_{e\in E} |s(G', e)|/2$. □

Theorem 4.5. *MINIMUM VERTEX COVER is $\mathcal{NP}$-complete on strong grid graphs and partial strong grids.*

Proof. Since MINIMUM VERTEX COVER on arbitrary graphs is $\mathcal{NP}$-complete, it is clear that MINIMUM VERTEX COVER on strong grid graphs is in $\mathcal{NP}$.

To prove $\mathcal{NP}$-hardness, we perform a polynomial-time reduction from MINIMUM VERTEX COVER on planar graphs of maximum degree 4, to MINIMUM VERTEX COVER on strong grid graphs.

Let (G, k) be an instance of MINIMUM VERTEX COVER on planar graphs of maximum degree 4, with a planar graph $G = (V, E)$ of maximum degree 4 and $k \in \mathbb{N}$. We define an instance (G_{grid}, k') of MINIMUM VERTEX COVER on strong grid graphs. Let G_{grid} be the result of applying Construction 2.7 to G. Since G is planar, G_{grid} is a strong grid graph. We set $k' = k + \sum_{e \in E} |s(G_{grid}, e)|/2$.

If (G_{grid}, k') is a YES-instance, G_{grid} has a minimum vertex cover C' with $|C'| \leq k'$. Due to Lemma 4.2, there also exists a minimum vertex cover C'' of G_{grid}, so that $|C''| \leq |C'|$ and $C := C'' \cap V$ is a vertex cover in G. Lemma 4.4 states that $|C''| = |C| + \sum_{e \in E} |s(G_{grid}, e)|/2$. As a result $|C| \leq k' - \sum_{e \in E} |s(G_{grid}, e)|/2 = k$, so (G, k) is also a YES-instance.

If (G, k) is a YES-instance, there exists a vertex cover C of G with $|C| \leq k$. Due to Lemma 4.3, there exists a vertex cover C' of G', with $|C'| = |C| + \sum_{e \in E} |s(G', e)|/2 = k'$. Hence, (G_{grid}, k') is also a YES-instance of MINIMUM VERTEX COVER on strong grid graphs.

As a result, (G, k) is a YES-instance if and only if (G_{grid}, k') is a YES-instance. □

5. Maximum-Cut

A cut $C = (S, T)$ is a partition of the vertices of a graph $G = (V, E)$ into two disjoint sets S and T. The cut-set of the cut C is defined as $\{e = (u, v) \in E \mid u \in S, v \in T\}$, and $|C|$ refers to the size of the cut-set of C. An edge will be referred to as *covered* by C if it is part of the cut-set of C. A cut in G is called a maximum cut if there is no cut of larger size in G.

The MAXIMUM CUT problem is defined as follows: Given an instance (G, k), with a graph G and $k \in \mathbb{N}$, it should be decided whether (G, k) is a YES- or a NO-instance. (G, k) is a YES-instance if G has a maximum cut with size at least k, else it is a NO-instance.

MAXIMUM CUT is known to be solvable in polynomial time on planar graphs [Had75], which grid graphs and partial grids are subclasses of. On cubic graphs the problem is $\mathcal{NP}$-complete [GP95].

5.1 Maximum cut on partial strong grids

Lemma 5.1. *Let $G = (V, E)$ be a graph, and $G' = (V', E')$ an even subdivision of G. Let $C' = (S', T')$ be a cut in G', and $C = (S, T)$ a cut in G, with $S \subseteq S'$, $T \subseteq T'$.*

Then, if an edge $e = (u, v) \in E$ is not covered by C, there exists a corresponding edge $e' = (u', v')$ which is not covered by C', with $u', v' \in \{u, v\} \cup s(G', e)$.

If an edge $e_2 = (u_2, v_2) \in E$ is covered by C, all edges with both endpoints in $\{u_2, v_2\} \cup s(G', e_2)$ are covered by C', if C' is a maximum cut of G'.

Proof. Let $e = (u, v) \in E$ be an edge that is not covered by C. Then, u, v are in the same set in C, without loss of generality assume $u, v \in S$. It follows that $u, v \in S'$ in C'.

For all edges between vertices in $\{u, v\} \cup s(G', e)$ to be covered by C', all vertices in $s(G', e)$ with odd distance to u or v would have to be in T', and all edges with even distance to u or v in S'. However, since $|s(G', e)|$ is even, any vertex in $s(G', e)$ with even distance to u has odd distance to v, and vice versa. Hence, at least one edge with both endpoints in $\{u, v\} \cup s(G', e)$ is not covered by C'.

Let $e_2 = (u_2, v_2) \in E$ be covered by C. This means u_2 and v_2 were partitioned into different sets S, T in G. Then, u_2 and v_2 are also in different sets S' and T' in G', without loss of generality assume $u_2 \in S'$, $v_2 \in T'$.

We now assume C' is a maximum cut of G', we claim C' covers all edges with both endpoints in $\{u_2, v_2\} \cup s(G', e_2)$. For the sake of contradiction, assume there exists such an edge which is not covered by C'.

Let $C'' = (S'', T'')$ be a cut of G', which is identical to C' outside of $s(G', e_2)$. Let the vertices in $s(G', e_2)$ with odd distance to u_2 be in T'', those with even distance to u_2 in S''. C'' covers all edges with both endpoints in $\{u, v\} \cup s(G', e_2)$, which is at least one edge more than C' covers among this subset of edges.

Since C' and C'' are identical outside $s(G', e_2)$, and vertices on $s(G', e_2)$ have no neighbors outside of $\{u, v\} \cup s(G', e_2)$, C'' covers more edges than C', in contradiction to C' being a maximum cut of G'. □

Lemma 5.2. *Let $G = (V, E)$ be a graph, and $G' = (V', E')$ an even subdivision of G. Given a cut $C = (S, T)$ of G, we can construct a cut $C' = (S', T')$ of G', such that the number of edges in E not covered by C is equal to the number of edges in E' not covered by C'.*

Proof. Let $S_2 \subseteq S_2'$ and $T_2 \subseteq T_2'$. For every edge $e = (u, v) \in E$, let all vertices in $s(G', e)$ with even distance to u be in the same set as u, and let the remaining vertices in $s(G', e)$ be in the other set.

If u, v are in different sets, C' covers all edges along the subgraph of G' induced by $\{u, v\} \cup s(G', e)$. Else, C' covers all but one edge along this induced subgraph. Hence, for every edge C does not cover, there is exactly one edge C' does not cover. □

Lemma 5.3. *Let $G = (V, E)$ be a graph, and $G' = (V', E')$ an even subdivision of G. Let $C' = (S', T')$ be a maximum cut in G'. Then $C = (S = S' \cap V, T = T' \cap V)$ is a maximum cut in G.*

Proof. For the sake of contradiction, assume there was a cut $C_2 = (S_2, T_2)$ in G, and $|C_2| > |C|$. Then the number of edges not covered by C is greater than the number of edges not covered by C_2. Since $S \subseteq S'$ and $T \subseteq T'$, due to Lemma 5.1, the number of edges not covered by C is equal to the number of edges not covered by C'.

We construct a cut $C_2' = (S_2', T_2')$ in G' from the cut C_2, as described in Lemma 5.2. The number of edges in E' not in the cut-set of C_2' is equal to the number of edges in E not in the cut-set of C_2. Since $|C_2| > |C|$, it follows that $|C_2'| > |C'|$, which is in contradiction to C' being a maximum cut of G'. □

Lemma 5.4. *Let $G = (V, E)$ be a graph, and $G' = (V', E')$ an even subdivision of G. Let $C = (S, T)$ be a maximum cut of G. Then there exists a maximum cut $C' = (S', T')$ of G', such that $S \subseteq S'$ and $T \subseteq T'$.*

Proof. We construct C' from C as described in Lemma 5.2. The number of edges in E' not covered by C' is equal to the number of edges in E not covered by C. We claim C' is a maximum cut of G'.

For the sake of contradiction assume there was a cut $C_2' = (S_2', T_2')$ of G', with $|C_2'| > |C'|$. Due to Lemma 5.1, the cut $C_2 = (S_2 = S_2' \cap V, T_2 = T_2' \cap V)$ leaves fewer edges in E uncovered than C. Hence $|C_2| > |C|$, in contradiction to C being a maximum cut of G. □

(a) With induced edges (b) Without induced edges

Figure 5.1: Crossing subgraphs

Theorem 5.5. *MAXIMUM CUT is $\mathcal{NP}$-complete on partial strong grids.*

Proof. Since MAXIMUM CUT on arbitrary graphs is $\mathcal{NP}$-complete, it is clear that MAXIMUM CUT on partial strong grids is in $\mathcal{NP}$.

To prove $\mathcal{NP}$-hardness, we perform a polynomial-time reduction from MAXIMUM CUT on cubic graphs to MAXIMUM CUT on partial strong grids.

Let (G, k) be an instance of MAXIMUM CUT on cubic graphs, with a cubic graph $G = (V, E)$ and $k \in \mathbb{N}$.

We construct an instance (G_{grid}, k') of MAXIMUM CUT on partial strong grids in polynomial time. G is transformed to a partial strong grid $G_{grid} = (V_{grid}, E_{grid})$, which is an even subdivision of G, using Construction 2.7. k' is computed as $k' = k + \sum_{e \in E} |s(G_{grid}, e)|$.

If (G, k) is a YES-instance, there exists a cut $C = (S, T)$ of G, such that $|C| \geq k$. Due to Lemma 5.4 there exists a cut $C' = (S', T')$ in G_{grid}, where the number of edges in E_{grid} not covered by C' is equal to the number of edges in E not covered by C.

So $|E_{grid}| - |C'| = |E| - |C|$. The size of E_{grid} can be rewritten as $|E_{grid}| = |E| + \sum_{e \in E} |s(G_{grid}, e)|$. It follows that $|C'| = |E_{grid}| - |E| + |C| = |C| + \sum_{e \in E} |s(G_{grid}, e)| \geq k + \sum_{e \in E} |s(G_{grid}, e)| = k'$. Therefore, (G_{grid}, k') is also a YES-instance.

If (G_{grid}, k') is a YES-instance, there exists a cut $C' = (S', T')$ of G_{grid}, such that $|C'| \geq k'$. Due to Lemma 5.3, there exists a cut $C = (S, T)$ in G, where the number of edges in E_{grid} not covered by C' is equal to the number of edges in E not covered by C.

Consequently $|C| = |E| - (|E_{grid}| - |C'|) = |C'| - \sum_{e \in E} |s(G_{grid}, e)| \geq k$, and (G, k) is a YES-instance.

(G, k) is a YES-instance if and only if (G_{grid}, k') is a YES-instance. □

5.2 Maximum cut on strong grid graphs

When Construction 2.7 is performed on a non-planar cubic graph G, the resulting graph G_{grid} is not a strong grid graph. G_{grid} can be transformed to a strong grid graph $G_{strgrid}$ by simply adding in all edges that should be induced. However, this results in a K_4 at every crossing, so that $G_{strgrid}$ is not a subdivision of G. We want to ensure that, given a maximum cut $C' = (S', T')$ of $G_{strgrid}$, the cut $C = (S' \cap V, T' \cap V)$ is a maximum cut of G.

However, this is not necessarily the case: Figure 5.1 depicts Construction 2.7 applied to a crossing in a non-planar cubic graph. A cut (S, T) is shown. (S, T) is a maximum cut in the crossing subgraph with induced edges, but it is not a maximum cut in the subgraph without the induced edges.

Consequently, given a maximum cut (S', T') of $G_{strgrid}$, $C = (S' \cap V, T' \cap V)$ may not be a maximum cut of G.

We intend to replace the crossing subgraphs in G_{grid} with a modified crossing subgraph G'_c (Figure 5.2), which is a strong grid graph, resulting in a strong grid graph $G'_{strgrid}$.

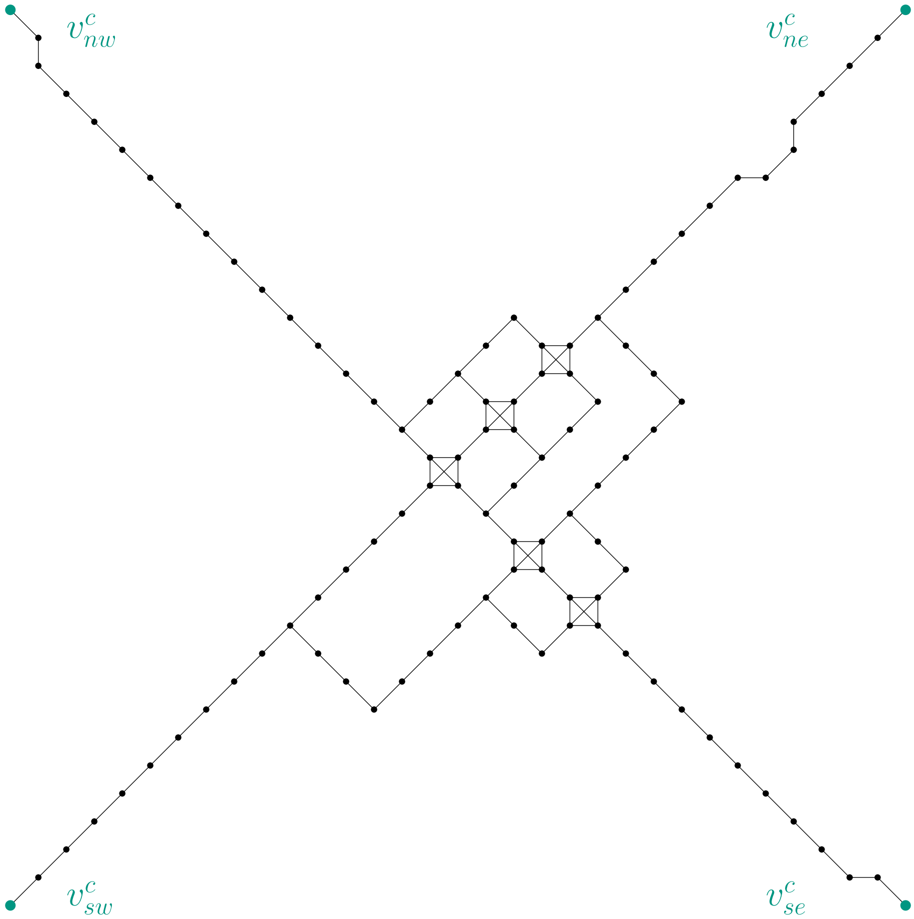


Figure 5.2: Modified crossing subgraph G'_c

Given a maximum cut $C'' = (S'', T'')$ of $G'_{strgrid}$, the cut $C = (S'' \cap V, T'' \cap V)$ should be a maximum cut of G.

In the following lemmas we show some relevant properties of G'_c and its subgraphs. Construction 5.6 describes the structure of some of the cuts which are used in these lemmas.

Construction 5.6. Let $\{v'_{ne}, v'_{nw}, v'_{se}, v'_{sw}\}$ be the top-right, top-left, bottom-right and bottom-left vertices of $G''_c = (V''_c, E''_c)$ (as shown in Figure 5.3). Given a partition of the vertices in $\{v'_{ne}, v'_{nw}, v'_{se}, v'_{sw}\}$ into the sets $S' \subseteq V''_c$ or $T' \subseteq V''_c$, we can construct a cut $C' = (S', T')$ of G'' by adding the remaining vertices to either S' or T'.

Consider the vertices diagonal to v'_{ne}. Those with an even distance to v'_{ne} are added to the same set (S'or T') as v'_{ne}, the other vertices diagonal to v'_{ne} are added to the other set. Analogous steps are repeated with v'_{nw}.

Then, there exists exactly one K_4 (in the center of G''_c), that has had a set assigned to all of its vertices v'_1, v'_2, v'_3, v'_4. Let v'_1, v'_2 be diagonal to each other. There exist two paths of odd length ≥ 3, which consist exclusively of diagonal edges, between v'_1 and v'_2. We add the vertices on these paths, which have an even distance to v'_1, to the same set as v'_1, and the other vertices on the paths to the other set.

We repeat analogous steps for the vertices along the two odd length (≥ 3) paths between v'_3, v'_4. We include vertices along the paths with even distance to v'_3 in the same set as v'_3, and the other vertices in the other set.

By performing this construction, every vertex in V''_c is included in exactly one set, S' or T'. This results in the cut $C' = (S', T')$.

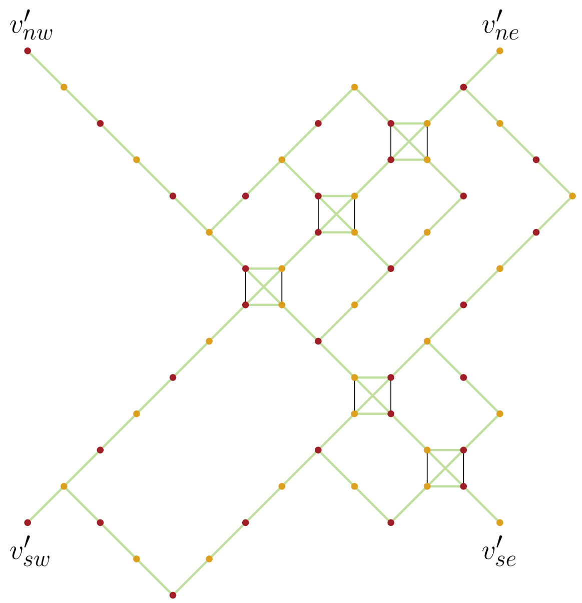


Figure 5.3: Relevant subgraph G''_c of G'_c, highlighted maximum cut

Lemma 5.7. *Let $C = (S, T)$ be a cut in an unmodified crossing subgraph G_c (Figure 5.1a), with $|C| = 2$, and let C_{max} be a maximum cut of G_c. Let $v_{ne}, v_{nw}, v_{se}, v_{sw}$ be the top-right, top-left, bottom-right and bottom-left vertices in G_c. Let $v'_{ne}, v'_{nw}, v'_{se}, v'_{sw}$ be the top-right, top-left, bottom-right and bottom-left vertices in the graph G''_c (Figure 5.3).*

Let $C' = (S', T')$ be a cut in G''_c such that $v'_i \in S' \iff v_i \in S$ ($i \in \{ne, nw, se, sw\}$), and let C'_{max} be a maximum cut of G''_c.

Then $0 = |C_{max}| - |C| \leq |C'_{max}| - |C'|$, and there exists a C' such that equality holds.

Proof. Due to $|C| = 2$, it follows that $v_{ne} \in S \iff v_{sw} \notin S$, $v_{nw} \in S \iff v_{se} \notin S$, and $|C_{max}| - |C| = 0$. Since $|C'_{max}| - |C'| \geq 0$ we only need to show that there exists a cut C' with $|C'| = |C'_{max}|$, which fulfills the specified conditions.

A cut within a K_4 can only cover 4 of its 6 edges. Therefore, any cut $C'' = (S'', T'')$ in G''_c that covers all non-K_4-edges, and 4 out of 6 edges within each K_4 subgraph, would be a maximum cut.

We partition $\{v'_{ne}, v'_{nw}, v'_{se}, v'_{sw}\}$ into S', T' such that $v'_i \in S' \iff v_i \in S$ ($i \in \{ne, nw, se, sw\}$). Let $C' = (S', T')$ be the cut we obtain, when the remaining vertices are assigned to S' or T' as described in Construction 5.6. Then C' covers all non-K_4-edges, and 6 edges within each K_4. Hence $|C'| = |C_{max}|$. □

Lemma 5.8. *Let $C'_1 = (S'_1, T'_1)$ be a cut of the graph $G'_1 = (V'_1, E'_1)$ (see Figure 5.4), such that there exist two vertices $v_1, v_2 \in V'_1$ within the same K_4 subgraph, that are diagonal to each other and part of the same set, S'_1 or T'_1. Let C'_{1max} be a maximum cut of G'_1.*

Then $|C'_{1max}| - |C'_1| \geq 1$.

Proof. A maximum cut C'_{1max} of G'_1 is highlighted in Figure 5.4. It covers all non-K_4-edges, and 4 K_4-edges within each K_4 subgraph of G'_1.

For the sake of contradiction, assume there exists a cut $C''_1 = (S''_1, T''_1)$ of G'_1, where to vertices $v_1, v_2 \in V'_1$, that are diagonal to each other and part of the same K_4 subgraph, with $|C''_1| = |C'_{1max}|$. Without loss of generality let $v_1, v_2 \in S''_1$.

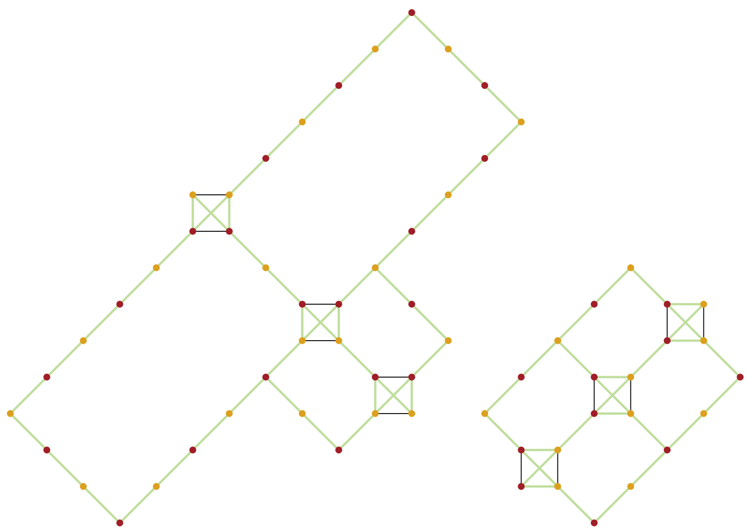


Figure 5.4: Relevant Subgraphs of G''_c, highlighted maximum cut

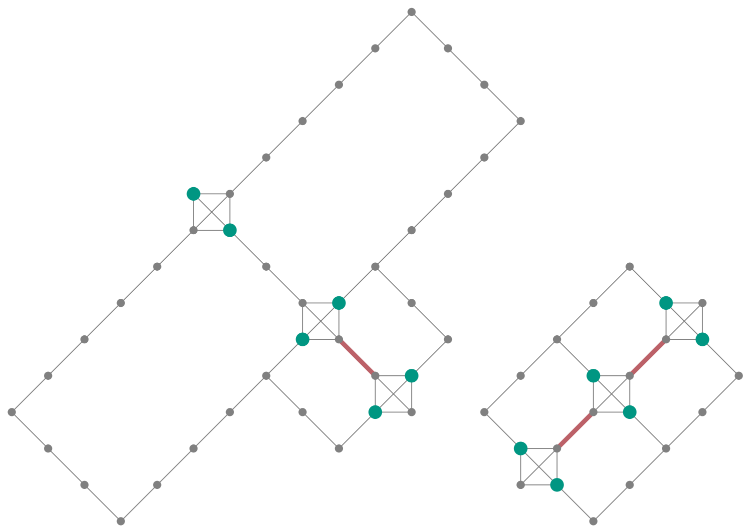


Figure 5.5: Missing edges in cut-sets of G'_1 and G'_2

Exactly two vertices within every K_4 subgraph are in S''_1: else C''_1 would cover fewer edges than C'_{1max}. Let $((v_1)_x, (v_1)_y)$ and $((v_2)_x, (v_2)_y)$ be the grid coordinates of v_1 and v_2, we assume without loss of generality $(v_1)_x < (v_2)_x$, $(v_1)_y < (v_2)_y$.

Let K_4^1 be the K_4 subgraph that contains v_1, v_2, and let v_3, v_4 be the other vertices in K_4^1. Then $v_3, v_4 \in T''_1$.

Consider a path from v_1 to v_2, with odd length ≥ 3, which exclusively consists of diagonal edges. Two such paths exist within G'_1.

Let P_1 be such a path. P_1 contains exactly one edge (v'_1, v'_2) within a different K_4 (K_4^2, with vertices $\{v'_1, v'_2, v'_3, v'_4\}$). Let v'_1 be the vertex with the lower x-coordinate among v'_1, v'_2.

Out of $\{v'_1, v'_2\}$, v'_1 is closer to v_1 on P_1. For C''_1 to cover all non-K_4-vertices, if v'_1 has even distance to v_1, $v'_1 \in S''_1$ holds, else $v'_1 \in T''_1$. The same argument applies for the vertices v_2, v'_2.

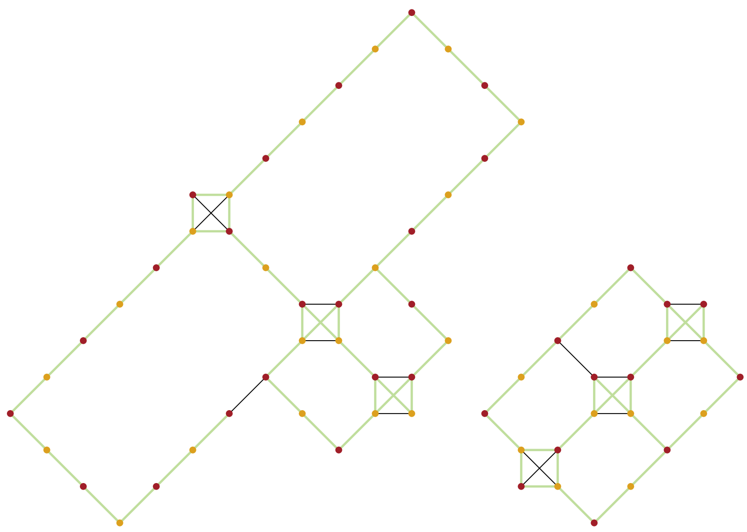


Figure 5.6: Cuts C^1_{-1} of G'_1 and C^2_{-1} of G'_2

Consequently, the vertices in G'_1, which are highlighted green in Figure 5.5, are all in the same set. The vertices, which are part of K_4 subgraphs and are not highlighted, are in the other set. Since two of the K_4 subgraphs are connected by an edge e_{nc} (colored red in Figure 5.5), e_{nc} is not part of the cut-set of C''_1. The edge e_{nc} is a non-K_4-edge. Since C'_{1max} covers all non-K_4-edges, and optimally covers all K_4 subgraphs, this is in contradiction to $|C''_1| = |C_{1max}|$.

The bound $|C'_{1max}| - |C'_1| \geq 1$ is tight. A cut $C^1_{-1} = (S^1_{-1}, T^1_{-1})$ with $|C^1_{-1}| = |C'_{1max}| - 1$, such that the vertices v_1, v_2 are part of the same set (S^1_{-1} or T^1_{-1}), is highlighted in Figure 5.6. □

Lemma 5.9. *Let $C'_2 = (S'_2, T'_2)$ be a cut of the graph $G'_2 = (V'_2, E'_2)$, such that there exist two vertices $u_1, u_2 \in V'_2$ within the same K_4 subgraph, that are diagonal to each other and part of the same set, S'_2 or T'_2. Let C'_{2max} be a maximum cut of G'_2.*

Then $|C'_{2max}| - |C'_2| \geq 1$.

Proof. This can be shown using analogous arguments to those in Lemma 5.8: We assume for the sake of contradiction there exists a cut C''_2 of G'_2, where two vertices u_1, u_2, which are diagonal to each other and part of the same K_4 subgraph, are assigned to the same set.

A maximum cut C'_{2max} of G'_2 is highlighted in Figure 5.4. The K_4-vertices highlighted green in Figure 5.5 are in the same set when $|C''_2| = |C'_{2max}|$ holds, due to analogous arguments to Lemma 5.8. However, when that is the case, an edge between two K_4 subgraphs is not covered by C''_2. Hence $|C''_2| < |C'_{2max}|$.

The bound $|C'_{2max}| - |C'_2| \geq 1$ is tight. A cut $C^2_{-1} = (S^2_{-1}, T^2_{-1})$ with $|C^2_{-1}| = |C'_{2max}| - 1$, such that the vertices u_1, u_2 are part of the same set (S^2_{-1} or T^2_{-1}), is highlighted in Figure 5.6. □

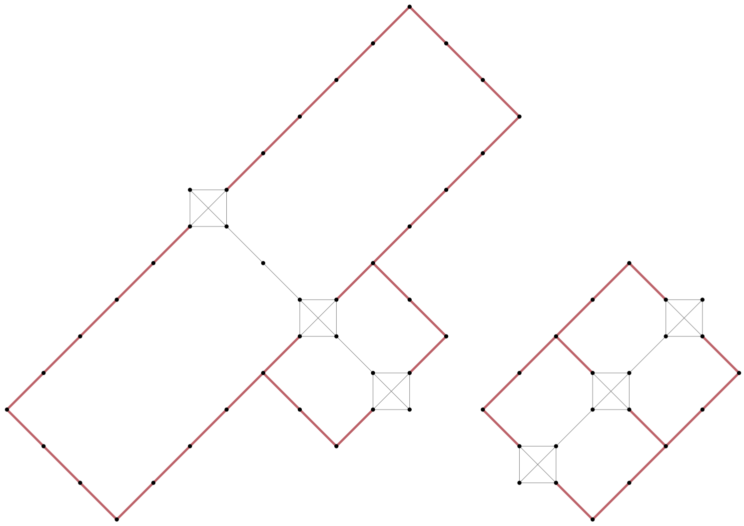


Figure 5.7: Edge subsets of G'_1 and G'_2

Lemma 5.10. *Let $C = (S, T)$ be a cut in an unmodified crossing subgraph G_c (Figure 5.1a), with $|C| = 1$, and let C_{max} be a maximum cut of G_c. Let $v_{ne}, v_{nw}, v_{se}, v_{sw}$ be the top-right, top-left, bottom-right and bottom-left vertices in G_c. Let $v'_{ne}, v'_{nw}, v'_{se}, v'_{sw}$ be the top-right, top-left, bottom-right and bottom-left vertices in the graph G''_c (Figure 5.3). Let $C' = (S', T')$ be a cut in G''_c such that $v'_i \in S' \iff v_i \in S$ ($i \in \{ne, nw, se, sw\}$), and let C'_{max} be a maximum cut of G''_c.*

Then $1 = |C_{max}| - |C| \leq |C'_{max}| - |C'|$, and there exists a C' such that equality holds.

Proof. A cut $C' = (S', T')$ with $v'_i \in S' \iff v_i \in S$ ($i \in \{ne, nw, se, sw\}$), and $|C'| = |C'_{max}| - 1$ can be constructed as described in Construction 5.6. (S', T') covers all but one non-K_4-edge, and 4 edges within each K_4 subgraph.

For the sake of contradiction, assume there exists a cut $C'' = (S'', T'')$ of G''_c, with $v'_i \in S'' \iff v_i \in S$ ($i \in \{ne, nw, se, sw\}$), and $|C''| > |C'|$.

Among $\{v'_{ne}, v'_{nw}, v'_{se}, v'_{sw}\}$, there exists a pair (v'_{ne}, v'_{sw}) or (v'_{nw}, v'_{se}), such that both vertices are in the same set (S'' or T''). Let this pair be (u, v), without loss of generality assume $u, v \in S''$. Consider the (unique) path P with odd length ≥ 3 between u and v, that only contains vertices diagonal to u and v. If a non-K_4-edge along P were not covered by C'', $|C''| < |C'_{max}|$, which would be in contradiction to $|C''| > |C'|$. Due to its odd length, and since C'' covers all non-K_4-edges along P, C'' leaves a K_4-edge along P uncovered. It follows that C'' includes two vertices v_1, v_2, which are diagonal to each other and part of the same K_4 subgraph, in the same set (S'' or T'').

C'_{max} covers the same amount of edges in G'_1 and G'_2 as their maximum cuts, C'_{1max} and C'_{2max}, do. Due to Lemmas 5.8, 5.9, including v_1, v_2 in the same set (S'' or T'') results in C'' covering fewer edges than C'_{max} in G'_1 or G'_2.

Hence $|C''| \leq |C'_{max}| - 1 = |C'|$. □

Lemma 5.11. *Let C'_{1max} be a maximum cut of G'_1. Let $E^1_\subset$ be the set of non-K_4-edges in G'_1, which are adjacent to an edge $e = (x, y)$ such that both x and y are non-K_4-vertices. The edges in $E^1_\subset$ are marked red in Figure 5.7.*

Let $C'_1 = (S'_1, T'_1)$ be a cut of G'_1 such that:

- *There exist two vertices u, v, that are diagonal to each other and part of the same K_4 subgraph. Also, $u_x < v_x$ and $u_y < v_y$ (see Definition 2.1.5), and u, v are assigned to the same set (S_1' or T_1').*
- *C_1' covers all edges in $E_{\subset}^1$.*

Then $|C_{1max}'| - |C_1'| \geq 2$.

Proof. We assume for the sake of contradiction, that there exists a cut C_1' that fulfills the conditions of this lemma, with $|C_{1max}'| - |C_1'| \leq 1$. Without loss of generality assume $u, v \in S_1'$. There exist two paths P_1, P_2 between u, v, with odd lengths ≥ 3, which consist exclusively of diagonal edges. On both P_1 and P_2, at least one edge cannot be covered by C_1'. These edges are K_4-edges, as all non-K_4-edges along P_1, P_2 are part of $E_{\subset}^1$. Since no K_4-edge is part of both paths, there is a unique uncovered K_4-edge on each path.

At least two of the K_4 subgraphs in G_1' are covered optimally by C_1', meaning exactly two of their vertices are in S_1' and T_1'.

Case 1: All K_4 subgraphs are covered optimally by C_1'.

In G_1', there is an edge $e = (u', v')$ between two K_4 subgraphs K_4^1 and K_4^2, and a path of length two between two K_4 subgraphs, K_4^2 and K_4^3.

Since all edges in $E_{\subset}^1$ are covered by C_1', the vertices highlighted green in Figure 5.5 are part of the same set, S_1' or T_1'. The remaining K_4 vertices in G_1' are all in the other set.

Consequently, u, v are both part of the same set, and u', v' are both part of the other set. Hence e is not covered by C_1'.

For the same reason, the top-right and bottom-left vertices of K_4^2 are in a different set than the top-right and bottom-left vertices of K_4^3. Since all K_4 subgraphs are covered optimally, the same holds for the top-left and bottom-right vertices of K_4^2 and K_4^3. Let $e_1 = (v_1, v_2), e_2 = (v_2, v_3)$ be the edges on the path of length 2 between K_4^2 and K_4^3, and let v_1 and v_3 be the K_4-vertices. Then v_1 and v_3 are in different sets.

Therefore, no matter which set v_2 is assigned to, either e_1 or e_2 is not covered by C_1'. Hence $|C_1'| \leq |C_{1max}'| - 2$, assuming all K_4 subgraphs are covered optimally by C_1'.

Case 2: Exactly two K_4 subgraphs, K_{4opt}^1 and K_{4opt}^2, are covered optimally by C_1'

Due to the same arguments, if exactly two K_4 subgraphs, K_{4opt}^1 and K_{4opt}^2, are covered optimally by C_1', and there is an edge between K_{4opt}^1 and K_{4opt}^2, that edge is not covered by C_1'. If there is a path of length 2 between K_{4opt}^1 and K_{4opt}^2, one of the edges along that path is uncovered. Since the third K_4 subgraph ($\neq K_4^1, K_4^2$) is not covered optimally, this also results in $|C_{1max}'| - |C_1'| \geq 2$ in both cases.

If there is neither an edge, nor a path of length 2 between K_{4opt}^1 and K_{4opt}^2, observe the third K_4 subgraph, $K_4^{3'}$, which is not covered optimally. Then $K_4^{3'}$ is connected to one of K_{4opt}^1, K_{4opt}^2 via an edge, and connected to the other via a path of length 2. Without loss of generality let there be an edge $e' = (v_3', v_1')$ between $K_4^{3'}$ and K_{4opt}^1. Let v_3' be part of $K_4^{3'}$.

If a K_4 has all of its vertices in the same set, S_1' or T_1', $|C_{1max}'| - |C_1'| > 2$. So $K_4^{3'}$ has exactly 3 vertices in the same set.

Recall that there exist two vertices u, v, that are diagonal to each other and part of the same K_4 subgraph, K_4^{uv} , with $u, v \in S_1'$. Also, $u_x < v_x$ and $u_y < v_y$. Consider the top-right and bottom-left vertex v_{ne}, v_{sw} of any other K_4 subgraph ($\neq K_4^{uv}$). In order for C_1' to cover all edges in $E_{\subset}^1$, either $v_{ne}, v_{sw} \in S_1'$, or $v_{ne}, v_{sw} \in T_1'$ holds.

This means that the top-right and bottom-left vertices v^3_{ne}, v^3_{sw} of $K^{3'}_4$, are in the same set as the top-right and bottom-left vertices of K^1_{4opt}. The other two vertices of K^1_{4opt}, which includes v'_1, are part of the other set.

Therefore, when the edge $e' = (v'_3, v'_1)$ between K^1_{4opt} and $K^{3'}_4$ is covered, v'_3 is in the same set as v^3_{ne}, v^3_{sw}. Then the remaining vertex, v''_3, of $K^{3'}_4$ is in the other set.

Finally, let $e''_1 = (v''_2, v''), e''_2 = (v'', v''_3)$ be the two edges on the path of length 2 between $K^{3'}_4$ and K^2_{4opt}, where v''_2 is part of K^2_{4opt}. The top-right and bottom-left vertices of K^2_{4opt} are not in the same set as v^3_{ne}, v^3_{sw}. The other two vertices of K^2_{4opt}, which includes v''_2, are in the same set as v^3_{ne}, v^3_{sw}.

So v''_3 and v''_2 are in different sets. No matter whether $v'' \in S'_1$ or $v'' \in T'_1$, either e''_1 or e''_2 is not covered by C'_1. Since $K^{3'}_4$ is also covered suboptimally, this results in $|C'_{1max}| - |C'_1| \geq 2$. □

Lemma 5.12. *Let C'_{2max} be a maximum cut of G'_2. Let $E^2_{\subset}$ be the set of non-K_4-edges in G'_2, which are also adjacent to at least one other non-K_4-edge. The edges in $E^2_{\subset}$ are marked red in Figure 5.7.*

Let $C'_2 = (S'_2, T'_2)$ be a cut of G'_2 such that:

- *There exist two vertices u, v, that are diagonal to each other and part of the same K_4 subgraph. Also, $u_x < v_x$ and $u_y > v_y$, and u, v are assigned to the same set (S'_2 or T'_2).*
- *C'_2 covers all edges in $E^2_{\subset}$.*

Then $|C'_{2max}| - |C'_2| \geq 2$.

Proof. We assume for the sake of contradiction, that there exists a cut C'_2 that fulfills the conditions of this lemma, with $|C'_{2max}| - |C'_2| \leq 1$. Without loss of generality assume $u, v \in S'_2$. There exist two paths P_1, P_2 between u, v, with odd length ≥ 3, which consist exclusively of diagonal edges. On both P_1 and P_2, at least one edge cannot be covered by C'_2. These edges are K_4-edges, as all non-K_4-edges along P_1, P_2 are part of $E^2_{\subset}$. Since no K_4-edge is part of both paths, there is a unique uncovered K_4-edge on each path.

At least two of the K_4 subgraphs in G'_2 are covered optimally by C'_2, which means exactly two of their vertices are in S'_2 and T'_2.

Case 1: All K_4 subgraphs of G'_2 are covered optimally by C'_2

Since $E^2_{\subset}$ is fully covered, the vertices highlighted green in Figure 5.5 are in the same set (S'_2 or T'_2). All K_4 subgraphs are covered optimally, so the remaining K_4-vertices in G'_2 are in the other set. Hence, the two non-K_4-edges between K_4 subgraphs are uncovered, resulting in $|C'_{2max}| - |C'_2| \geq 2$.

Case 2: Exactly two K_4 subgraphs in G'_2, K^1_4 and K^2_4, are covered optimally by C'_2

Since C'_2 covers all edges in $E^2_{\subset}$, the vertices highlighted green in Figure 5.5 are in the same set. K^1_4 and K^2_4 are covered optimally, so their remaining vertices are in the other set. Therefore, if there is an edge between K^1_4 and K^2_4, that edge would not be covered. Since the third K_4 subgraph, K^3_4, is not covered optimally, this would result in $|C'_{2max}| - |C'_2| = 2$.

If there are no edges adjacent to both K^1_4 and K^2_4, observe K^3_4, which is not covered optimally. If a K_4 has all of its vertices in the same set, S'_2 or T'_2, $|C'_{2max}| - |C'_2| > 2$. So K^3_4 has 3 vertices in the same set.

Since all edges in $E^2_\subset$ are covered by C'_2, the top-left and bottom-right vertices of K^3_4 are in the same set as u, v, which is S'_2. Naturally, if u, v themselves are part of K^3_4 this is also true.

One of the remaining vertices of K^3_4 is in S'_2, the other in T'_2. Since there is no edge between K^1_4, K^2_4, there is an edge between K^1_4 and K^3_4, and between K^2_4 and K^3_4. One of these edges has an endpoint at the top-right vertex of K^3_4, the other has an endpoint at the bottom-left vertex of K^3_4. The other endpoints of these edges, which are not part of K^3_4, are all in T'_2.

No matter whether the top-right or bottom-left vertex of K^3_4 is the one in T'_2, one of these edges is not covered by C'_2. Consequently, $|C'_{2max}| - |C'_2| \geq 2$. □

Lemma 5.13. *Let $C = (S, T)$ be a cut in an unmodified crossing subgraph G_c (Figure 5.1a), with $|C| = 0$, and let C_{max} be a maximum cut of G_c. Let $v_{ne}, v_{nw}, v_{se}, v_{sw}$ be the top-right, top-left, bottom-right and bottom-left vertices in G_c. Let $v'_{ne}, v'_{nw}, v'_{se}, v'_{sw}$ be the top-right, top-left, bottom-right and bottom-left vertices in the graph G''_c (Figure 5.3). Let $C' = (S', T')$ be a cut in G''_c such that $v'_i \in S' \iff v_i \in S$ ($i \in \{ne, nw, se, sw\}$), and let C'_{max} be a maximum cut of G''_c.*

Then $2 = |C_{max}| - |C| \leq |C'_{max}| - |C'|$, and there exists a C' such that equality holds.

Proof. A cut $C' = (S', T')$ with $v'_i \in S' \iff v_i \in S$ ($i \in \{ne, nw, se, sw\}$), and $|C'| = |C_{max}| - 2$ can be constructed as described in Construction 5.6. (S', T') then covers all but two non-K_4-edges, and 6 edges within each K_4 subgraph.

For the sake of contradiction, assume there exists a cut $C'' = (S'', T'')$ of G''_c with $v'_i \in S'' \iff v_i \in S$ ($i \in \{ne, nw, se, sw\}$), and $|C''| > |C'|$.

Consider the path P_1 between v'_{nw}, v'_{se}, that only contains vertices diagonal to v'_{nw}, v'_{se}, and the path P_2 between v'_{ne} and v'_{sw}, that only contains vertices diagonal to v'_{ne} and v'_{sw}. On at least one of these paths, all non-K_4-edges are covered by C''. P_1, P_2 have odd lengths.

Case 1: C'' covers all non-K_4-edges on both P_1 and P_2.

Then, there exist K_4-edges (u_1, v_1) on P_1, and (u_2, v_2) on P_2, that are not covered by C''. We can assume, without loss of generality, that $(u_1)_x < (v_1)_x$, $(u_1)_y > (v_1)_y$, $(u_2)_x < (v_2)_x$, and $(u_2)_y < (v_2)_y$. Due to Lemmas 5.8, 5.9, C'' covers fewer edges than C'_{max} in both G'_1 and G'_2.

Let $E^1_\subset$ be the set of non-K_4-edges in G'_1, which are adjacent to an edge $e = (x, y)$ such that both x and y are non-K_4-vertices. Let $E^2_\subset$ be the set of non-K_4-edges in G'_2, which are also adjacent to at least one other non-K_4-edge (see Figure 5.7). Due to Lemmas 5.11, 5.12, C'' leaves at least one edge in $E^1_\subset$, and one edge in $E^2_\subset$ uncovered. Since $E^1_\subset \cap E^2_\subset = \varnothing$, these two edges are distinct, and $|C'_{max}| - |C''| \geq 2$, in contradiction to $|C''| > |C'|$.

Case 2: C'' covers all non-K_4-edges on exactly one of P_1, P_2.

Let $P_i, i \in \{1, 2\}$ be the path on which all non-K_4-edges are covered, let P_j be the other path. C'' leaves a non-K_4-edge $e_j = (u_j, v_j)$ on P_j, and at least one K_4-edge $e_i = (u_i, v_i)$ on P_i uncovered.

C'' covers at least one edge less than C'_{max} on G'_i, due to Lemmas 5.8, 5.9. Due to Lemmas 5.11, 5.12, that uncovered edge $e_i = (u_i, v_i)$ is in $E^i_\subset$.

If $e_i \neq e_j$, then $|C'_{max}| - |C''| \geq 2$.

Let $K^{1\cap 2}_4$ be the K_4 subgraph in G''_c that is part of both G'_1 and G'_2. If the top-left and bottom-right vertices of $K^{1\cap 2}_4$ are in the same set, S'' or T'', and the other vertices in $K^{1\cap 2}_4$

are in the other set, then C'' leaves an edge in G'_1 and G'_2 uncovered, due to Lemmas 5.8, 5.9. Due to Lemmas 5.11, 5.12, and $E^1_{\subset} \cap E^2_{\subset} = \varnothing$, the two uncovered edges would be distinct, so $|C'_{max}| - |C''| \geq 2$.

If three vertices in $K_4^{1\cap 2}$ are in the same set, one edge is lost compared to C'_{max}, as $K_4^{1\cap 2}$ is not covered optimally. As previously established, there is also a non-K_4-edge e_j which is not covered by C'', which would result in $|C'_{max}| - |C''| \geq 2$.

Hence, no two vertices in $K_4^{1\cap 2}$ which are diagonal to each other can be in the same set in C'', and e_i is part of one of the other K_4 subgraphs.

All K_4 subgraphs $\neq K_4^{1\cap 2}$ have exactly two vertices in S'' and T''. Else, C'' covers fewer edges than C'_{max} in that K_4, in addition to not covering e_j, which results in $|C'_{max}|-|C''| \geq 2$.

Let K'_4 be the K_4 subgraph that contains e_i. Due to $K'_4 \neq K_4^{1\cap 2}$, there exists another K_4 subgraph, K''_4, such that there is an edge between K'_4 and K''_4. Let $G'_i, i \in \{1, 2\}$ be the subgraph from Figure 5.4, which K'_4, K''_4 are part of. Let u'_i, v'_i be the other two vertices in K'_4 which, as previously established, are in the other set than u_i, v_i.

Consider the odd length path P'_i between u'_i, v'_i in G'_i, that passes through K''_4 and consists exclusively of diagonal edges. Since C'' covers all non-K_4-edges along P'_i, the K_4 edge on the path, $e'' = (u'', v'')$, is not covered by C''. u'' and v'' have even distances to u'_i, v'_i along P'_i, so they are in the same set as u'_i, v'_i.

Since exactly two vertices in K''_4 are in S'' and T'', the other two vertices in K''_4 are in the same set as u_i, v_i. However, the edge between K'_4 and K''_4 is not covered by C'' in that case.

The edge between K'_4 and K''_4 is not part of P_j, and is therefore distinct from e_j. Consequently $|C'_{max}| - |C''| \geq 2$, in contradiction to $|C''| > |C'|$. □

Lemma 5.14. *Let $C = (S,T)$ be a cut in an unmodified crossing subgraph G_c (Figure 5.1a), and C_{max} be a maximum cut of G_c. Let $v_{ne}, v_{nw}, v_{se}, v_{sw}$ be the top-right, top-left, bottom-right and bottom-left vertices in G_c. Let $v'_{ne}, v'_{nw}, v'_{se}, v'_{sw}$ be the top-right, top-left, bottom-right and bottom-left vertices in the full modified crossing subgraph, G'_c (Figure 5.2). Let $C' = (S',T')$ be a cut in G'_c such that $v'_i \in S' \iff v_i \in S$ ($i \in \{ne, nw, se, sw\}$), and let C'_{max} be a maximum cut of G'_c.*

Then $|C_{max}| - |C| \leq |C'_{max}| - |C'|$, and there exists a C' such that equality holds.

Proof. Let $v''_{ne}, v''_{nw}, v''_{se}, v''_{sw}$ be the top-right, top-left, bottom-right and bottom-left vertices in the subgraph $G''_c = (V''_c, E''_c)$ of G'_c (Figure 5.3). We observe the subgraph $G^c_{\subset}$ of G'_c, with $G^c_{\subset} = G'_c \setminus (G''_c \setminus \{v''_{ne}, v''_{nw}, v''_{se}, v''_{sw}\})$, highlighted red in Figure 5.8. Consider the four connected components of $G^c_{\subset}$, each of which contains exactly one vertex from $\{v'_{ne}, v'_{nw}, v'_{se}, v'_{sw}\}$.

The given cut C of G_c determines which set (S' or T') the vertices $v'_{ne}, v'_{nw}, v'_{se}, v'_{sw}$ are assigned to in C'. We construct the remainder of the cut C' of G'_c as follows:

For each $v'_i, i \in \{ne, nw, se, sw\}$, let the vertices in $G^c_{\subset}$ with even distance to v'_i be in the same set as v'_i, let those with odd distance to v'_i be in the other set.

The distance between v'_{nw} and v''_{nw}, and between v'_{se}, v''_{se}, is even, while the distance between v'_{ne} and v''_{ne}, and between v'_{sw}, v''_{sw} is odd. This means v''_{nw}, v''_{se} are assigned to the same set (S' or T') if and only if v'_{ne}, v'_{se} are assigned to the same set. Similarly, v''_{ne}, v''_{sw} are assigned the same set if and only if v'_{ne}, v'_{sw} are assigned the same set.

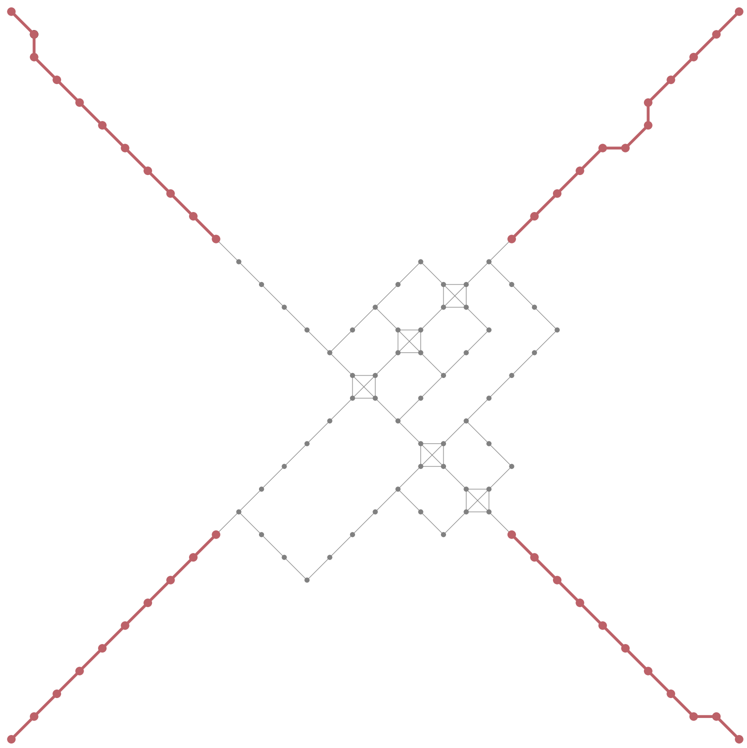


Figure 5.8: $G^c_{\subset}$, subgraph of G'_c

Let the remaining vertices, which are exactly the vertices in $G''_c \setminus \{v'_{ne}, v'_{nw}, v'_{se}, v'_{sw}\}$, be assigned to S', T' as described in Construction 5.6.

All edges in $G^c_{\subset}$ are covered by C'. Consider the cut $C'' = (S' \cap V'', T' \cap V'')$, and let C''_{max} be a maximum cut of G''_c. Due to Lemmas 5.7, 5.10, 5.13, $|C''_{max}| - |C''| = |C_{max}| - |C|$. Hence, $|C'_{max}| - |C'| = |C_{max}| - |C|$.

For the sake of contradiction, assume $C_2 = (S_2, T_2)$ is a cut of G'_c, such that $v'_i \in S' \iff v_i \in S$ $(i \in \{ne, nw, se, sw\})$ and $|C_2| > |C'|$. Since C' covers all edges in $G^c_{\subset}$, C_2 would need to cover more edges than C' in G''_c. Then, due to Lemmas 5.7, 5.10, 5.13, this is only possible by changing the sets assigned to vertices from $\{v''_{ne}, v''_{nw}, v''_{se}, v''_{sw}\}$. This also requires the additional assumption that C' does not already cover G''_c optimally, which is the case if and only if the cut C of G_c was not optimal.

Consider a connected component of $G^c_{\subset}$. The component is simply a path, and therefore has a unique maximum cut $C_c = (S_c, T_c)$ (the only change that can be made to C_c without decreasing its size is swapping the sets S_c and T_c). Therefore, in every connected component of $G^c_{\subset}$ where C_2 partitions vertices differently than C', C_2 covers fewer edges than C'.

If v''_{ne} and v''_{sw} are assigned to the same set in C', C_2 can cover at most one edge more than C' in G''_c by assigning v''_{ne} and v''_{sw} to different sets S_2 and T_2. This is a consequence of Lemmas 5.7, 5.10, 5.13. However, this also means C_2 covers at least one edge less than C' in $G^c_{\subset}$. The case where v''_{nw} and v''_{se} are assigned to the same set in C' is analogous. C_2 can cover one edge more than C' in G''_c by assigning v''_{ne} and v''_{sw} to different sets, but then C_2 covers fewer edges on $G^c_{\subset}$ than C'.

As a result, any changes C_2 makes to the partition of $\{v''_{ne}, v''_{nw}, v''_{se}, v''_{sw}\}$ do not result in $|C_2| > |C'|$, which means $|C_2| \leq |C'|$. □

Theorem 5.15. *MAXIMUM CUT is $\mathcal{NP}$-complete on strong grid graphs.*

Proof. Since MAXIMUM CUT on arbitrary graphs is $\mathcal{NP}$-complete, it is clear that MAXIMUM CUT on strong grid graphs is in $\mathcal{NP}$.

To prove $\mathcal{NP}$-hardness, we perform a polynomial-time reduction from MAXIMUM CUT on cubic graphs to MAXIMUM CUT on strong grid graphs.

Let (G, k) be an instance of MAXIMUM CUT on cubic graphs, with a cubic graph $G = (V, E)$ and $k \in \mathbb{N}$. We begin by constructing the strong grid graph $G_{grid} = (V_{grid}, E_{grid})$ from G as follows:

We perform Construction 2.7 on G, to obtain the partial strong grid G'''_{grid}. We transform G'''_{grid} to G''_{grid} by performing five resolution doubling operations (Observation 2.3). Then, we replace every crossing subgraph in G''_{grid} with the graph G'_c depicted in Figure 5.2, to obtain the strong grid graph $G'_{grid} = (V'_{grid}, E'_{grid})$.

Finally, we consider the paths in G'_{grid}, whose endpoints are either vertices $v \in V \subseteq V'_{grid}$, or vertices $v^c_{ne}, v^c_{nw}, v^c_{se}, v^c_{sw}$ in the corners of a G'_c subgraph (labeled in Figure 5.2), such that the path does not contain any other vertices from V or other vertices within G'_c subgraphs. Along all such paths which have even length, we perform the same subgraph replacement as described in Construction 2.7 (see Figure 2.9). The resulting paths have odd length. By carrying out this final step we obtain the result of the transformation, $G_{grid} = (V_{grid}, E_{grid})$.

Edges that are part of a K_4 subgraph within G'_c will be referred to as *K_4-edges*, the other edges as *non-K_4-edges.* Analogously, vertices can be classified as *K_4-vertices* and *non-K_4-vertices.*

The five resolution doubling operations, as well as Construction 2.7, can be performed in polynomial time in $n := |V|$. Subgraph replacement can be carried out in constant time, and is done for every crossing in G, of which there is a polynomial amount. Hence G_{grid} can be computed in polynomial time in n. Since only a constant amount of resolution doubling operations is carried out, $|V_{grid}|$ is polynomial in n.

Claim 5.16. *Let $C = (S, T)$ be a maximum cut of G, with $|C| \geq k_o$, $k_o \in \mathbb{N}$. Let $c(G)$ be the number of crossings in G. Let $G' = (V', E')$ be the partial strong grid obtained by applying Construction 2.7 to G. Let $C' = (S', T')$ be a maximum cut of G'.*

Then there exists a cut $C_{grid} = (S_{grid}, T_{grid})$ of G_{grid} with $|C_{grid}| \geq k_{grid} := k_o + \sum_{e \in E} |s(G', e)| + 31(|E'| - 2c(G)) + 112c(G)$.

Proof. Using the arguments from Theorem 5.5, since $|C| \geq k_o$, it follows that $|C'| \geq k' := k_o + \sum_{e \in E} |s(G', e)|$.

Recall that G_{grid} is obtained by computing G' from G, performing resolution doubling operations (Observation 2.3) and replacing crossing subgraphs. Note that $V' \subseteq V_{grid}$. Further subgraph replacements are carried out to ensure specific paths in G_{grid} are even. By construction, G_{grid} is an even subdivision of G, where crossing subgraphs were replaced with the modified crossing subgraph G'_c (Figure 5.2).

We define the cut $C_{grid} = (S_{grid}, T_{grid})$ as follows:

Let all vertices $v_o \in V_{grid} \cap V'$ be in the set S_{grid} if $v_o \in S'$, or to T_{grid} if $v_o \in T'$. Let $e_1 = (u_1, v_1) \in E'$ be an edge that is not crossed by another edge in G'. Let the vertices in $s(G_{grid}, e_1)$ with even distance to u_1 be in the same set as u_1, and let those with odd distance be in the other set.

Let $e_c^1 = (u_c^1, v_c^1), e_c^2 = (u_c^2, v_c^2) \in E'$ be edges that form a crossing in G'. Let $C'_c = (S' \cap \{u_c^1, v_c^1, u_c^2, v_c^2\}, T' \cap \{u_c^1, v_c^1, u_c^2, v_c^2\}$. C'_c is a cut on $G_c = (\{u_c^1, v_c^1, u_c^2, v_c^2\}, \{e_c^1, e_c^2\})$. Let C'_{max} be a maximum cut on G_c.

In G_{grid}, the crossing with e_c^1, e_c^2 was replaced by a modified crossing subgraph $G'_c = (V'_c, E'_c)$. The top-left, top-right, bottom-left and bottom-right vertices of G'_c are in $V_{grid} \cap V'$, and were therefore already assigned a set.

Let the remaining vertices in G'_c be assigned to S_{grid}, T_{grid}, as described in Lemma 5.14. Let $C_{maxgrid}$ be a maximum cut of G'_c, and let $C^c_{grid} = (S_{grid} \cap V'_c, T_{grid} \cap V'_c)$. C^c_{grid} is a cut of G'_c. Due to Lemma 5.14, $|C'_{max}| - |C'_c| = |C_{maxgrid}| - |C^c_{grid}|$.

All vertices in G_{grid} are assigned to exactly one set S_{grid} or T_{grid}, resulting in the cut C_{grid}.

For every edge $e_{cov} \in E'$ covered by C', which does not have another edge crossing it, there are $|s(G_{grid}, e_{cov})| + 1$ edges covered by C_{grid} along the subdivision of e_{cov} in G_{grid}. For every edge $e_{unc} \in E'$ not covered by C', which does not have another edge crossing it, there are $|s(G_{grid}, e_{unc})|$ edges covered by C_{grid} along the subdivision of e_{unc} in G_{grid}.

Since any edge in E' that is not part of a crossing was subdivided 5 times, $|s(G_{grid}, e_{cov})| = |s(G_{grid}, e_{unc})| = 31$, which results in C_{grid} covering $31(|E'| - 2c(G'))$ additional edges along the subdivisions of these edges, compared to C'. Construction 2.7 ensures $c(G') = c(G)$.

For two edges $e_1, e_2 \in E'$ that cross each other, neither of which is covered by C', 112 edges will be covered by C_{grid} in the subgraph G'_c. If exactly one of e_1, e_2 is covered by C', 113 edges will be covered by C_{grid} in G'_c. If both e_1, e_2 are covered by C', C_{grid} will cover 114 edges in G'_c. Consequently, C_{grid} covers 112 additional vertices on each crossing subgraph, compared to C'.

Hence $|C_{grid}| = |C'| + 31(|E'| - 2c(G)) + 112c(G) \geq |C| + \sum_{e \in E} |s(G', e)| + 31(|E'| - 2c(G)) + 112c(G) = k_{grid}$. □

Claim 5.17. *Let $G' = (V', E')$ be the graph obtained when Construction 2.7 is applied to G. Let $C_{grid} = (S_{grid}, T_{grid})$ be a maximum cut of G_{grid}, and let $k_{grid} \in \mathbb{N}$. Let $|C_{grid}| \geq k_{grid}$.*

Then there exists a cut $C = (S, T)$ of G, with $|C| \geq k_o := k_{grid} - (\sum_{e \in E} |s(G', e)| + 31(|E'| - 2c(G)) + 112c(G))$.

Proof. Recall that G_{grid} is obtained by computing G' from G, performing resolution doubling operations (Observation 2.3) and replacing crossing subgraphs. Note that $V' \subseteq V_{grid}$. Further subgraph replacements are carried out to ensure specific paths in G_{grid} are even. By construction, G_{grid} is an even subdivision of G, where crossing subgraphs were replaced with the modified crossing subgraph G'_c (Figure 5.2).

Let $C' = (S_{grid} \cap V', T_{grid} \cap V')$. Then, C' is a cut of G'.

Let $e_{cov} = (u_{cov}, v_{cov}) \in E'$ be an edge in G' which is not part of a crossing, such that one of its endpoints is in S_{grid}, and the other in T_{grid}. Then, e_{cov} is covered by C' in G'. Since C_{grid} is a maximum cut, $|s(G_{grid}, e_{cov})| + 1 = 32$ edges are be covered by C_{grid} along the subdivision of e_{cov} in G_{grid}.

Let $e_{unc} = (u_{unc}, v_{unc}) \in E'$ be an edge in G' which is not part of a crossing, such that both of $\{u_{unc}, v_{unc}\}$ were assigned to the same set S_{grid} or T_{grid}, in C_{grid}. Then, e_{unc} is not covered by C' in G', and $|s(G_{grid}, e_{unc})| = 31$ edges are covered by C_{grid} along the subdivision of e_{cov} in G_{grid}.

Therefore, in total C_{grid} covers $31(|E'| - 2c(G'))$ edges more on these edges, compared to C'. Construction 2.7 ensures $c(G) = c(G')$.

Let $G'_c = (V'_c, E'_c)$ be a modified crossing subgraph in G_{grid} (Figure 5.2), which replaced edges $e^1_c = (u^1_c, v^1_c), e^2_c = (u^2_c, v^2_c) \in E'$ during the construction of G_{grid}. $C^c_{grid} = (S_{grid} \cap V'_c, T_{grid} \cap V'_c)$ is a cut of G'_c. Let C^c_{max} be a maximum cut of G'_c.

Let $C'_c = (S_{grid} \cap \{u^1_c, v^1_c, u^2_c, v^2_c\}, T' \cap \{u^1_c, v^1_c, u^2_c, v^2_c\})$. C'_c is a cut of $G_c = (\{u^1_c, v^1_c, u^2_c, v^2_c\}, \{e^1_c, e^2_c\})$. Let $C'_{c\,opt}$ be a maximum cut of G_c.

Due to Lemma 5.14, $|C^c_{max}| - |C^c_{grid}| = |C'_{c\,opt}| - |C'_c|$. Consequently $|C^c_{grid}| = 114 - (|C'_{c\,opt}| - |C'_c|)$. In G' exactly $|C'_c|$ edges are covered by C' on the corresponding crossing subgraph, which is $114 - |C'_c\; opt| = 112$ fewer edges than C_{grid} covers on G'_c.

Consequently, C' is a cut of G', such that $|C'| = |C_{grid}| - 31(|E' - 2c(G)) - 112c(G)$. G' is the result of applying Construction 2.7 to G. Due to the arguments in Theorem 5.5, there exists a cut C of G, with $|C| \geq k_o = |C'| - \sum_{e \in E} |s(G', e)| = |C_{grid}| - \sum_{e \in E} |s(G', e)| - 31(|E'| - 2c(G)) - 112c(G)$. □

To prove MAXIMUM CUT is $\mathcal{NP}$-hard on strong grid graphs, we perform a polynomial-time reduction from MAXIMUM CUT on cubic graphs to MAXIMUM CUT on strong grid graphs. Recall that we defined an instance (G, k) of MAXIMUM CUT on cubic graphs for this purpose at the start of this theorem, and constructed the strong grid graph G_{grid} from G in polynomial time.

Let $c(G)$ be the number of crossings in G.

Let $k' = k + \sum_{e \in E} |s(G', e)| + 31(|E'| - 2c(G)) + 112c(G)$. We define an instance (G_{grid}, k') of MAXIMUM CUT on strong grid graphs.

If (G, k) is a YES-instance, (G_{grid}, k') is a YES-instance, due to Claim 5.16.

If (G, k) is a NO-instance, (G_{grid}, k') is a NO-instance. If (G_{grid}, k') were a YES-instance, (G, k) would also be a YES-instance due to Claim 5.17.

Hence (G, k) is a YES-instance if and only if (G_{grid}, k') is a YES-instance. □

6. Vertex Coloring

Let $G = (V, E)$ be a graph, and $k \in \mathbb{N}$.

A function $c : V \to (1, \dots, k)$, such that $\forall e = (u, v) \in E : c(u) \neq c(v)$, is called a *k-vertex-coloring* (or simply *k-coloring*) of G. G is called k-colorable if it has a k-coloring. The *chromatic number* of G, $\chi(G)$, is the smallest k such that G is k-colorable.

The k-COLORABILITY problem, with a constant $k \in \mathbb{N}$, is defined as follows: Given a graph $G = (V, E)$, it should be decided whether $\chi(G) \leq k$. If $\chi(G) \leq k$, G is a YES-instance, else it is a NO-instance.

All partial strong grids are 4-colorable, so 4-COLORABILITY is trivial on subgraphs of strong grids (see Observation 6.1). While 2-COLORABILITY is simple to solve in polynomial time in general (see Observation 6.2), 3-COLORABILITY is $\mathcal{NP}$-complete on planar graphs of maximum degree 4 [GJS76]. On partial grids and grid graphs, 3-COLORABILITY is trivial, as these graphs are bipartite.

Hence 3-COLORABILITY on strong grid graphs and partial strong grids are the focus of this chapter.

Observation 6.1. *Let $G = (V, E)$ be a partial strong grid. Then G is 4-colorable.*

Proof. We define a 4-coloring $c : V \to \{1, 2, 3, 4\}$. For every $v \in V$, consider its grid coordinates v_x, v_y.

- If $v_x \bmod 2 = v_y \bmod 2 = 0$, let $c(v) = 1$.
- If $v_x \bmod 2 = v_y \bmod 2 = 1$, let $c(v) = 2$.
- If $v_x \bmod 2 = 0$ and $v_y \bmod 2 = 1$, let $c(v) = 3$.
- Else, let $c(v) = 4$.

No adjacent vertices in V are assigned the same color by c, hence c is a 4-coloring of G. $\square$

(a) G^1_{3vc} (b) G^2_{3vc}

Figure 6.1: G^1_{3vc} and G^2_{3vc}, with example 3-colorings

Observation 6.2. *Let $G = (V, E)$ be a graph. It can be decided in polynomial time whether G is 2-colorable.*

Proof. Let $v_0 \in V$ be a vertex, we set $c(v_0) = 1$. We perform a Breadth First Search (BFS) starting from v_0. We set $c(v_1) = 2$ for all $v_1 \in V$ with odd distance to v_0, and $c(v_2) = 1$ for all $v_2 \in V$ with even distance to v_0. If, after completion of the BFS, there exist two adjacent vertices v, v' with $c(v) = c(v')$, we can conclude that G is not 2-colorable. Else, c is a 2-coloring of G. □

The upcoming observation discusses two strong grid graphs, whose structure ensures that specific pairs of vertices within them cannot be assigned different colors in any 3-coloring. We later intend to utilize a combination of these graphs, to replace the edges of a given planar graph G with maximum degree 4. The resulting graph G_{3vc} will be structured in a way, that ensures vertices which were adjacent in G are assigned different colors in any 3-coloring of G_{3vc}.

Observation 6.3. *In any 3-coloring c_1 of the strong grid graph $G^1_{3vc} = (V_{3vc}, E_{3vc})$ depicted in Figure 6.1, the left and right vertices v_ℓ, v_r receive the same color, $c_1(v_\ell) = c_1(v_r)$.*

Similarly, the central vertex v_c and the top-left, top-right, bottom-left and bottom-right vertices $v_{nw}, v_{ne}, v_{sw}, v_{se}$ of G^2_{3vc}, labeled in Figure 6.1, are all assigned the same color by any 3-coloring c_2 of G^2_{3vc}.

Proof. Without loss of generality assume $c_1(v_\ell) = 1$. v_ℓ has two neighbors $v_1, v_2 \neq v_r$, and there exists an edge $(v_1, v_2) \in E_{3vc}$. Hence we can assume without loss of generality $c_1(v_1) = 2, c_1(v_2) = 3$. Since v_1, v_2 are also neighbors of v_r, $c_1(v_r) = 1 = c_1(v_\ell)$.

G^2_{3vc} contains multiple graphs isomorphic to G^1_{3vc} as subgraphs. Analogous arguments applied to these subgraphs show $v_c, v_{nw}, v_{ne}, v_{sw}, v_{se}$ all receive the same color in any 3-coloring c_2 of G^2_{3vc}. □

Construction 6.4. Let $G = (V, E)$ be a planar graph of maximum degree 4. We aim to construct a graph G_{3vc}, which is a strong grid graph, and is 3-colorable if and only if G is 3-colorable.

We use Construction 2.5 to obtain a strong grid graph $G_{strgrid}$, which is a subdivision of G. We transform $G_{strgrid}$ to $G'_{strgrid} = (V'_{strgrid}, E'_{strgrid})$ by carrying out 3 resolution doubling operations (Observation 2.3) on $G_{strgrid}$. This ensures there is sufficient space for the upcoming modifications. It follows that $\forall e \in E_{strgrid} : \; |s(G'_{strgrid}, e)| = 7$.

(a) G^1_{ne} (b) G^1_{nw}

Figure 6.2: Replacement subgraphs G^1_{ne} and G^1_{nw}

(a) G^d_{ne} (b) G^d_{nw}

Figure 6.3: Replacement subgraphs G^d_{ne} and G^d_{nw}

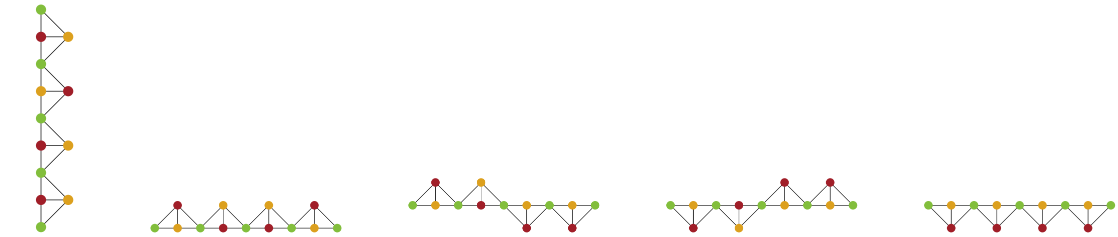
Observe a path P between two original vertices $o_1, o_2 \in V$ in $G_{strgrid}$, that does not contain another original vertex $o' \notin \{o_1, o_2\}$. P is the subdivision of an edge in E. For every such path in $G_{strgrid}$, we perform the following modifications in $G'_{strgrid}$, to obtain the desired graph G_{3vc}:

Let $e_1 = (v_1, o_1) \in E_{strgrid}$ be the edge on P adjacent to the original vertex $o_1 \in V$. Let G_{e_1} be the subgraph of $G'_{strgrid}$ induced by $\{v_1, o_1\} \cup s(G'_{strgrid}, e_1)$. Let $((v_1)_x, (v_1)_y), ((o_1)_x, (o_1)_y)$ be the grid coordinates of v_1 and o_1. Since e_1 is diagonal by construction, $(v_1)_x \neq (o_1)_x, (v_1)_y \neq (o_1)_y$. Let $v_{<x} \in \{v_1, o_1\}$ be the vertex with the smaller x-coordinate, $v_{>x} \in \{v_1, o_1\}$ is the other vertex. If $v_{<x}$ also has a smaller y-coordinate than $v_{>x}$, we replace G_{e_1} with the graph G^1_{ne} in $G'_{strgrid}$. Else, G_{e_1} is replaced with the graph G^1_{nw} in $G'_{strgrid}$. G^1_{nw}, G^1_{ne} are depicted in Figure 6.2.

Let $e_d = (u_d, v_d) \neq e_1$ be a diagonal edge on P, and G_{e_d} the subgraph of $G'_{strgrid}$ induced by $\{u_d, v_d\} \cup s(G'_{strgrid}, e_d)$. Let $((u_d)_x, (u_d)_y), ((v_d)_x, (v_d)_y)$ be the grid coordinates of u_d and v_d, without loss of generality assume $(u_d)_x < (v_d)_x$. If $(u_d)_y < (v_d)_y$, we replace G_{e_d} with the graph G^d_{ne} in $G'_{strgrid}$. Else we replace it with the graph G^d_{nw}. G^d_{nw} and G^d_{ne} are depicted in Figure 6.3.

Let $e_{nd} = (u_{nd}, v_{nd})$ be a non-diagonal edge on P, and $G_{e_{nd}}$ the subgraph of $G'_{strgrid}$ induced by $\{u_{nd}, v_{nd}\} \cup s(G'_{strgrid}, e_{nd})$. Let $((u_{nd})_x, (u_{nd})_y), ((v_{nd})_x, (v_{nd})_y)$ be the grid coordinates of u_{nd} and v_{nd}. If $(u_{nd})_x = (v_{nd})_x$, we replace $G_{e_{nd}}$ with the graph G_{ver} in $G'_{strgrid}$.

Else, we replace it with one of the graphs $G^1_{hor}, G^2_{hor}, G^3_{hor}, G^4_{hor}$. The case distinction is made to minimize the number of problematic induced edges between replacement subgraphs that are placed next to each other. Let $e_{ndl} = (u_{ndl}, u_{nd})$, $e_{ndr} = (v_{nd}, v_{ndr})$ be the edges adjacent to e_{nd} on P. Let $((u_{ndl})_x, (u_{ndl})_y)$, $((v_{ndr})_x, (v_{ndr})_y)$ be the grid coordinates of u_{ndl}, v_{ndr}. We assume without loss of generality $(u_{ndl})_x < (u_{nd})_x < (v_{nd})_x < (v_{ndr})_x$.



(a) G^{ver} (b) G^1_{hor} (c) G^2_{hor} (d) G^3_{hor} (e) G^4_{hor}

Figure 6.4: Replacement subgraphs $G_{ver}, G^1_{hor}, G^2_{hor}, G^3_{hor}, G^4_{hor}$

- If $(u_{ndl})_y < (u_{nd})_y$, $G_{e_{nd}}$ is replaced with one of G^1_{hor}, G^2_{hor}.
- If $(u_{ndl})_y > (u_{nd})_y$, $G_{e_{nd}}$ is replaced with one of G^3_{hor}, G^4_{hor}.
- If $(v_{ndr})_y < (v_{nd})_y$, $G_{e_{nd}}$ is replaced with one of G^1_{hor}, G^3_{hor}.
- If $(v_{ndr})_y > (v_{nd})_y$, $G_{e_{nd}}$ is replaced with one of G^2_{hor}, G^4_{hor}.

Let $x = min_i \in \{1, 2, 3, 4\}$ such that G^i_{hor} fulfills the conditions listed above. There always exists at least one $i \in \{1, 2, 3, 4\}$ such that G^i_{hor} fulfills all conditions. We replace $G_{e_{nd}}$ with G^x_{hor}. $G_{ver}, G^1_{hor}, G^2_{hor}, G^3_{hor}, G^4_{hor}$ are shown in Figure 6.4.

By performing all the discussed subgraph replacements we obtain the result of the transformation, G_{3vc}. In general, it is possible that edges are induced between replacement subgraphs in G_{3vc}, or that two replacement subgraphs have more than one vertex in common. These situations will be addressed in Lemma 6.11.

The number of paths between two original vertices $o_1, o_2 \in V$ in $G_{strgrid}$, that do not contain another original vertex $o' \notin \{o_1, o_2\}$, is linear in $|E| \leq n^2$. The length of the corresponding paths in $G'_{strgrid}$ is constant, since for each edge in E, only a constant number of subdivision vertices is introduced during the construction of $G'_{strgrid}$. Hence, the number of subgraph replacements (which can be performed in constant time) necessary to obtain G_{3vc} is polynomial in $n := |V|$, and G_{3vc} can be constructed in polynomial time in n. Since a constant number of resolution doubling operations is carried out in the construction of G_{3vc} and in Construction 2.5, $|V_{3vc}|$ is polynomial in n.

We claim that, given a planar graph G with maximum degree 4, the result G_{3vc} of applying Construction 6.4 to G is 3-colorable if and only if G is 3-colorable. We first show G is 3-colorable if G_{3vc} is 3-colorable, with the upcoming two lemmas. We then prove multiple lemmas, which we use to show G_{3vc} is 3-colorable if G is 3-colorable.

Lemma 6.5. *Let $G = (V, E)$ be a planar graph of maximum degree 4. Let $G_{strgrid} = (V_{strgrid}, E_{strgrid})$ be the strong grid graph obtained when Construction 2.5 is applied to G. Let $G'_{strgrid}$ be the result of carrying out 3 resolution doubling operations (Observation 2.3) on $G_{strgrid}$. Let G_{3vc} be the strong grid graph obtained when applying Construction 6.4 to G.*

Let $P = (V_P, E_P)$ be a path between two original vertices $o_1, o_2 \in V$ in $G_{strgrid}$, that does not contain another original vertex $o' \notin \{o_1, o_2\}$. In any 3-coloring c_{3vc} of G_{3vc}, $c(o_1) \neq c(o_2)$.

Proof. The graphs G_r in the set of replacement subgraphs G_{rep}, from Construction 6.4 (Figures 6.2, 6.3, 6.4) all contain several subgraphs which are isomorphic to G^1_{3vc} (Figure 6.1). Hence the same arguments from Observation 6.3 can be applied to these graphs.

For every edge $e = (u, v) \in V_{strgrid}$, let G_e be the graph induced by $\{u, v\} \cup s(G'_{strgrid}, e)$ in $G'_{strgrid}$. When G_e is replaced with one of $G^d_{nw}, G^d_{ne}, G_{ver}, G^1_{hor}, G^2_{hor}, G^3_{hor}, G^4_{hor}$, any 3-coloring c of the resulting graph fulfills $c(u) = c(v)$. When G_e is replaced with G^1_{nw} or G^1_{ne}, any 3-coloring c of the resulting graph fulfills $c(u) \neq c(v)$.

Let $e = (o_1, o_2)$ be the edge connecting o_1, o_2 in G. Such an edge must exist for the path P to exist. Let $k = |s(G_{strgrid}, e)|$, it follows that $|V_P| = k + 2$. Let $\{o_1, v_1, \ldots, v_k, o_2\}$ be the vertices in V_P, where $v_i, i \in (1, \ldots, k)$ is the subdivision vertex with distance i to o_1. Let $e_1 = (o_1, v_1), e_{k+1} = (v_k, o_2)$ and $\{e_i = (v_{i-1}, v_i) \mid i \in \{2, \ldots, k\}\}$ be the edges in E_P.

To obtain G_{3vc} from $G'_{strgrid}$, the subgraph $G_{e'}$ of $G'_{strgrid}$ is replaced with a graph from G_{rep} for every edge $e' \in E_{strgrid}$. Let c_{3vc} be a 3-coloring of G_{3vc}. The subgraph G_{e_1} of $G'_{strgrid}$, corresponding to $e_1 = (o_1, v_1) \in E_{strgrid}$, was replaced by G^1_{nw} or G^1_{ne} in G_{3vc}, so $c_{3vc}(o_1) \neq c_{3vc}(v_1)$. Let $e_i = (u_i, v_i) \neq e_1$ be an edge in E_P. In G_{3vc} the subgraph G_{e_i} of $G'_{strgrid}$ was replaced by a graph $G'_{e_i} \in G_{rep} \setminus \{G^1_{nw}, G^1_{ne}\}$, so $c_{3vc}(u_i) = c_{3vc}(v_i)$.

It follows that $c_{3vc}(o_1) \neq c_{3vc}(v_1) = c_{3vc}(o_2)$. □

Lemma 6.6. *Let $G = (V, E)$ be a planar graph of maximum degree 4. Let $G_{strgrid}$ be the strong grid graph obtained by applying Construction 2.5 to G. Let $G_{grid} = (V_{grid}, E_{grid})$ be the graph obtained when Construction 6.4 is applied to G. Let c' be a 3-coloring of G_{grid}.*

Then there exists a 3-coloring c of G.

Proof. Let $e = (u, v) \in E$ be an edge. By construction, there exists a path P between in u, v in $G_{strgrid}$, which does not contain vertices $v' \in V$ with $v' \notin \{u, v\}$. Due to Lemma 6.5, $c'(u) \neq c'(v)$. So c' assigns different colors to all vertices $v_1, v_2 \in V$ that are adjacent to each other. By setting $c(v') = c'(v')$ for all vertices $v' \in V$, we obtain a 3-coloring c of G. □

Lemma 6.7. *Let $G_r = (V, E) \in \{G_{rep}\}$ be one of the replacement subgraphs defined in Construction 6.4. Let the vertices $v \in V$ be partitioned into sets $G, R, Y \subset V$ based on their color in the corresponding Figure 6.2, 6.3, 6.4. The set G contains all vertices colored green in the figure, R contains the red vertices, and Y the yellow vertices. The set ($\in \{G, R, Y\}$) which a vertex $v \in V$ is in, is denoted by $set(v)$.*

Let $c : V \to \{1, 2, 3\}$ be a function, such that $\forall v_1, v_2 \in V \mid v_1 \neq v_2 : c(v_1) = c(v_2) \iff set(v_1) = set(v_2)$. Then c is a 3-coloring of G_r.

Proof. All vertices $v \in V$ are colored red, yellow or green in Figure 6.2, and no adjacent vertices have the same color. Hence c is defined for all vertices of G_r, and $\forall e = (v_1, v_2) \in E : c(v_1) \neq c(v_2)$. It follows that c is a 3-coloring of G_r. □

Lemma 6.8. *Let $G = (V, E)$ be a planar graph of maximum degree 4, let $G_{grid} = (V_{grid}, E_{grid})$ be the result of applying Construction 2.5 to G. Then there are limitations to what orientations adjacent edges in G_{grid} can have.*

Proof. Due to the subgraph replacements carried out during Construction 2.5, near bends and original vertices, there are no vertical and a horizontal edges adjacent to each other in G_{grid}.

Let G'_{grid} be the graph obtained when Construction 2.4 is performed on G. To construct G_{grid}, the subgraph replacements mentioned above are carried out on G'_{grid}. No other changes are made to G'_{grid}. Since neither G'_{grid}, nor the replacement subgraphs for bends

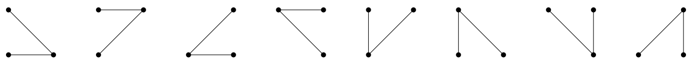


Figure 6.5: Edge transitions not possible in G_{grid}

(a) Before replacement (b) After replacement

Figure 6.6: Subgraph with bends not possible G_{grid}

and original vertices that are used during its construction, contain the combinations of adjacent edges depicted in Figure 6.5, these do not occur in G_{grid} either.

Finally, two adjacent diagonal edges, which are orthogonal to each other, can only occur when the shared vertex is an original vertex $v \in V$.

The only diagonal vertices in G_{grid} are part of the aforementioned replacement subgraphs. The replacement subgraph for original vertices does not contain adjacent diagonal edges that are orthogonal to each other, and none of the diagonal edges in that subgraph have neighbors outside of the subgraph in G_{grid}. For the replacement subgraphs of orthogonal bends to result in adjacent and orthogonal diagonal edges, two vertices where a bend occurs would need to be just two grid units apart (see Figure 6.6). Due to the resolution doubling operations performed before bend subgraphs are replaced, this is never the case. ☐

Recall that we want to show that, given a planar graph G with maximum degree 4, the result G_{3vc} of applying Construction 6.4 to G is 3-colorable if and only if G is 3-colorable. In order to prove that G_{3vc} is 3-colorable if G is 3-colorable, we first want to show there are several options to 3-color specific subgraphs of G_{3vc}. We consider modified path graphs P_{3vc}, which are subgraphs of G_{3vc}, that correspond to subdivisions of edges in E. We show that, for any fixed colors assigned to the endpoints of P_{3vc}, we are still free to assign either of the two valid colors to the non-diagonal neighbors of these endpoints in a 3-coloring of P_{3vc}.

Lemma 6.9. *Let $G = (V, E)$ be a planar graph of maximum degree 4, and $G_{strgrid} = (V_{strgrid}, E_{strgrid})$ the result of applying Construction 2.5 to G. Let $G'_{strgrid} = (V'_{strgrid}, E'_{strgrid})$ be the result of performing 3 resolution doubling operations (Observation 2.3) on $G_{strgrid}$.*

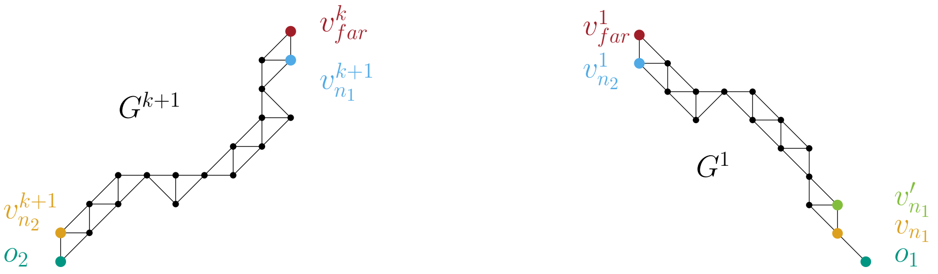
Let $P = (V_P, E_P)$ be a path between two original vertices $o_1, o_2 \in V$ in $G_{strgrid}$, which does not contain other original vertices $v' \in V \setminus \{o_1, o_2\}$. P is the subdivision of an edge $e \in E$. Let $P_{3vc} = (V_{3vc}, E_{3vc})$ be the graph obtained when Construction 6.4 is applied to P.

Let o_1 have degree one and o_2 have degree two in P_{3vc}.

Let $v_{n_2} \in V_{3vc}$ be the neighbor of o_2 in P_{3vc} which it is not diagonal to. Let $v_{n_1} \in V_{3vc}$ be the single neighbor of o_1 in P_{3vc}. Let $v'_{n_1} \neq o_1$ be the non-diagonal neighbor of v_{n_1}.

Then there exists a 3-coloring c of P_{3vc} for every possible combination of $c(v'_{n_1}) \in \{1, 2, 3\} \setminus \{c(o_2)\}$ and $c(v_{n_2}) \in \{1, 2, 3\} \setminus \{c(o_2)\}$.

Proof. For the path P to exist, there must be an edge $e = (o_1, o_2) \in E$. Let $k = |s(G_{strgrid}, e)|$, it follows that P has length $k + 1$.



(a) G^{k+1} (b) G^1

Figure 6.7: Examples of G^{k+1}, G^1

Let $c(o_1) = c_1, c(o_2) = c_2$, with $c_1, c_2 \in \{1, 2, 3\}$. Let $c_3 \in \{1, 2, 3\} \setminus \{c_1, c_2\}$ be the remaining color.

Let G_{rep} be the set of replacement subgraphs from Construction 6.4. Let $G^{k+1} \in G_{rep} \setminus \{G^1_{nw}, G^1_{ne}\}$ be the replacement subgraph of P_{3vc} that contains o_2. By construction, exactly one such subgraph exists. Let v^k_{far} be the vertex with the greatest distance to o_2 in G^{k+1}. Let $v^{k+1}_{n_2} = v_{n_2}$ be the non-diagonal neighbor of o_2 in G^{k+1}, and $v^{k+1}_{n_1}$ the non-diagonal neighbor of v^k_{far} in G^{k+1}. An example of this is shown in Figure 6.7a.

Let $G^1 = (V^1, E^1) \in \{G^1_{nw}, G^1_{ne}\}$ (Figure 6.2) be the subgraph of P_{3vc}, which contains o_1. By construction, exactly one such subgraph exists. Let v^1_{far} be the vertex in G^1 with the greatest distance to o_1. Let $v^1_{n_2}$ be the non-diagonal neighbor of v^1_{far}. An example is illustrated in Figure 6.7b.

We show there exist two different 3-colorings c_{k+1}, c'_{k+1} of G^{k+1}, with $c_{k+1}(v^{k+1}_{n_1}) = c_{k+1}(v^{k+1}_{n_2})$, and $c'_{k+1}(v^{k+1}_{n_1}) \neq c'_{k+1}(v^{k+1}_{n_2})$.

We then show there exist two different 3-colorings c_{1-k}, c'_{1-k} of $P_{3vc} \setminus G^{k+1}$, such that $c_{1-k}(v'_{n_1}) = c(o_1) = c_1$ and $c'_{1-k}(v'_{n_1}) = c_3$.

Given these colorings, we can select a coloring from c_{1-k}, c'_{1-k} to color the vertices in $V \setminus V^{k+1}$, and a coloring from c_{k+1}, c'_{k+1} to color the vertices in V^{k+1}. When we ensure the coloring for V^{k+1} assigns a color to v^k_{far} that is not shared by its neighbors in $V \setminus V^{k+1}$, we receive a 3-coloring of P_{3vc}.

Depending on the choice of c_{1-k}, c'_{1-k} and c_{k+1}, c'_{k+1}, we can obtain 4 possible 3-colorings c of P_{3vc}, one for every possible combination of $c(v'_{n_1}) \in \{1, 2, 3\} \setminus \{c(o_2)\}$ and $c(v_{n_2}) \in \{1, 2, 3\} \setminus \{c(o_2)\}$.

First, we define a 3-coloring $c_{1-k} : V \setminus V^{k+1} \to \{1, 2, 3\}$ with $c_{1-k}(v'_{n_1}) = c_1$.

For all vertices $v^1_1 \in V^1$, which have the same color (green, red, yellow) as o_1 in Figure 6.2, let $c_{1-k}(v^1_1) = c_1$. This includes v'_{n_1}.

Let v^1_{far} be the vertex with the greatest distance to o_1 in G^1, let $c_{1-k}(v^1_{far}) = c_2$. For all $v^1_2 \in V^1$, which have the same color as v^1_{far} in Figure 6.2, let $c_{1-k}(v^1_2) = c_2$.

For all vertices $v^1_3 \in V^1$ which do not have the same color as o_1, v^1_{far} in Figure 6.2, let $c_{1-k}(v^1_3) = c_3$. Due to Lemma 6.7, c_{1-k} is a 3-coloring of G^1.

A 3-coloring c'_{1-k} of G^1 with $c'_{1-k}(v'_{n_1}) = c_3$ can be obtained using analogous arguments, using the alternative coloring of G^1 depicted in Figure 6.8 instead of the one in Figure 6.2.

We now expand c_{1-k} to be a 3-coloring of $V \setminus V^{k+1}$.

Consider the subgraph $G^2 = (V^2, E^2) \in G_{rep}$ of P_{3vc}, with $v^1_{far} \in V^2$ and $o_1 \notin V^2$. G^2 is illustrated in one of the Figures 6.3, 6.4. For all $v^2_1 \in V^2$, which have the same color as

(a) G^1_{ne} (b) G^1_{nw}

Figure 6.8: Alternative 3-colorings of G^1_{ne} and G^1_{nw}

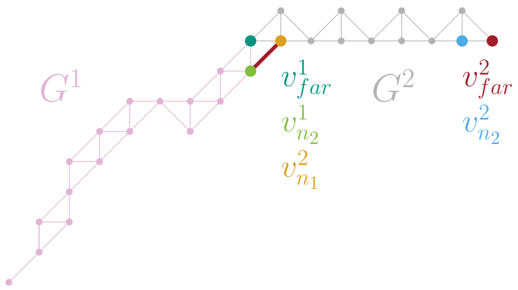


Figure 6.9: Example induced edge between G^1, G^2

v^1_{far} in the Figure showing G^2, let $c_{1-k}(v^2_1) = c_2$. This includes the vertex v^2_{far} with the greatest distance to v^1_{far} in G^2.

If v^1_{far} has a neighbor n_1 in G^1 and n_2 in G^2, such that v^1_{far} is not diagonal to n_1, n_2, but n_1, n_2 are diagonal to each other, an edge is induced between n_1, n_2 in P_{3vc}. No edges other than $e_n = (n_1, n_2)$ are induced between V^1, V^2 in P_{3vc}.

v^1_{far} has exactly one neighbor $v^2_{n_1}$ it is not diagonal to in G^2. Similarly v^2_{far} has exactly one non-diagonal neighbor $v^2_{n_2}$ in G^2. In G^1, v^1_{far} has a neighbor $v^1_{n_2}$ it is not diagonal to. An edge can only be induced between $v^1_{n_2}$ and $v^2_{n_1}$, an example of this is shown in Figure 6.9.

Let $i = max\{x \in \{1,2,3\} | c_x \neq c_{1-k}(v^1_{n_2}) \wedge c_x \neq c_{1-k}(v^1_{far})\}$, and let $c_{1-k}(v^2_{n_1}) = c_i$. This ensures $v^2_{n_1}$ receives a different color from $v^1_{n_2}$ and v^1_{far}. Since c_{1-k} is based on the coloring in Figure 6.2 on G^1, $c_{1-k}(v^1_{n_2}) = c_1$ holds, which results in $i = 3$.

For all vertices $v^2_2 \in V^2$ with the same color as $v^2_{n_1}$ in the figure showing G^2 (Figure 6.3 or 6.4), let $c_{1-k}(v^2_2) = c_i$. This includes $v^2_{n_2}$, since in all colorings of graphs in Figures 6.3, 6.4, $v^2_{n_1}$ and $v^2_{n_2}$ have the same color.

Let c'_3 be the color in $\{1,2,3\} \setminus \{c_{1-k}(v^1_{far}), c_{1-k}(v^2_{n_1})\}$. For all vertices $v^2_3 \in V^2$, that have a different color than $v^1_{far}, v^2_{n_1}$ in the figure of G^2, let $c_{1-k}(v^2_3) = c'_3$.

Recall that it is not possible for adjacent diagonal edges in $G_{strgrid}$ to be orthogonal to each other, unless they share an endpoint in an original vertex $v \in V$, due to Lemma 6.8. Lemma 6.8 also states that certain orientations of adjacent diagonal and vertical or horizontal edges are impossible. As a consequence of this, and due to the case distinction made in Construction 6.4 between the 4 variants $G^1_{hor}, G^2_{hor}, G^3_{hor}, G^4_{hor}$ of graphs corresponding to horizontal edges, two subgraphs G^i, G^{i+1} $(i \in \{1, \dots, k\})$ of P_{3vc} share exactly one vertex. Any combinations of graphs in G_{rep}, that would result in these graphs sharing more than one vertex when placed adjacent to each other, do not occur in P_{3vc}.

Since c_{1-k} assigns different colors to the endpoints of any edges induced between G_1 and G_2 in P_{3vc}, and due to Lemma 6.7, c_{1-k} is a 3-coloring of $G^1 \cup G^2$.

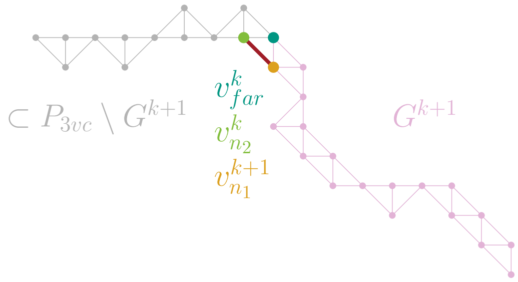


Figure 6.10: Example induced edge between $P_{3vc} \setminus G^{k+1}$ and G^{k+1}

The same arguments as for the coloring of G^2 can be repeated for the remaining $k-1$ subgraphs $G^i \in G_{rep}, i \in \{3, \ldots, k\}$, where G^i contains v^{i-1}_{far} but not v^{i-2}_{far}. This results in $c_{1-k}(v^i_{far}) = c_2$ for all i, and since $v^{k+1}_{far} = o_2$, this is consistent with $c(o_2) = c_2$. The resulting function c_{1-k} is a 3-coloring of $P_{3vc} \setminus G^{k+1}$.

For all $i \in \{1, \ldots, k-1\}$, the color $c_{1-k}(v^i_{n_1})$ in G^i is different from $c_{1-k}(v^{i+1}_{n_1})$ in G^{i+1}, and $c_{1-k}(v^i_{n_1}), c_{1-k}(v^{i+1}_{n_1}) \neq c_2$. Recall that $c_{1-k}(v^2_{n_1}) = c_3$. Therefore $c_{1-k}(v^k_{n_1}) = c_{1-k}(v^k_{n_2}) = c_3$ if k is even, else $c_{1-k}(v^k_{n_1}) = c_{1-k}(v^k_{n_2}) = c_1$.

To complete the 3-coloring c'_{1-k} on $P_{3vc} \setminus G^{k+1}$, analogous arguments as for c_{1-k} can be applied. We defined c'_{1-k} such that $c'_{1-k}(v^2_{n_1}) = c_1$. This results in $c'_{1-k}(v^k_{n_2}) = c_1$ if k is even, else $c'_{1-k}(v^k_{n_2}) = c_3$.

Hence we obtain two 3-colorings c_{1-k}, c'_{1-k} of $P_{3vc} \setminus G^{k+1}$, with $c_{1-k}(v'_{n_1}) = c(o_1) \neq c'_{1-k}(v'_{n_1})$.

We now consider the subgraph $G^{k+1} = (V^{k+1}, E^{k+1})$ of P_{3vc}, which contains o_2 and v^k_{far}. We define a color assignment $c_{k+1} : V^{k+1} \to \{1, 2, 3\}$.

Let $c_{k+1}(v^k_{far}) = c_2$, which does not result in conflict with the colorings c_{1-k}, c'_{1-k} on P_{3vc}. Consider the figure (Figure 6.3 or 6.4) which shows G^{k+1}. For all vertices v^{k+1}_1 which have the same color as v^k_{far} in the Figure, let $c_{k+1}(v^{k+1}_1) = c_2$.

In P_{3vc} an edge could be induced between the non-diagonal neighbor $v^k_{n_2}$ of v^k_{far} in $P_{3vc} \setminus G^{k+1}$, and the non-diagonal neighbor $v^{k+1}_{n_1}$ of v^k_{far} in G^{k+1}. An example of this is shown in Figure 6.10.

Depending on the length of P, and whether the 3-coloring c_{1-k} or c'_{1-k} is used to color the graph $P_{3vc} \setminus G^{k+1}$, it is possible $c(v^k_{n_2}) = c_1$ or $c(v^k_{n_2}) = c_3$. Let $c_n = c(v^k_{n_2})$, we assign a color to $v^{k+1}_{n_1}$ such that $c_{k+1}(v^{k+1}_{n_1}) \in \{1, 2, 3\} \setminus \{c_2, c_n\}$. This ensures that, in case an edge $(v^k_{n_2}, v^{k+1}_{n_1}) \in E_{3vc}$ is induced, c_{k+1} assigns different colors to its endpoints.

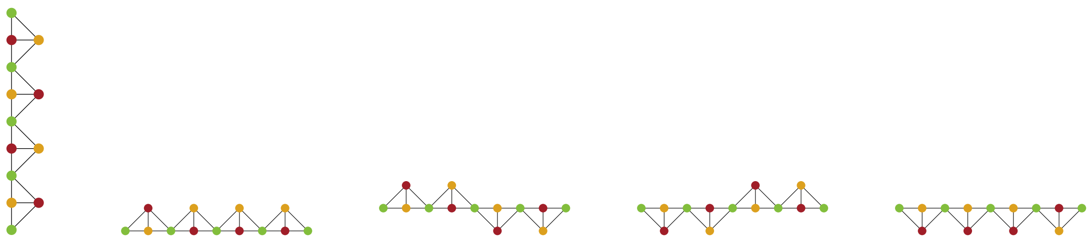
For all vertices $v^{k+1}_2 \in V^{k+1}$, which have the same color as $v^{k+1}_{n_1}$ in the Figure of G^{k+1}, let $c_{k+1}(v^{k+1}_2) = c_n$. Let $c'_n \in \{1, 2, 3\} \setminus \{c_2, c_n\}$ be the remaining color. For all $v^{k+1}_3 \in V^{k+1}$, which do not have the same color as $v^{k+1}_{n_1}$ or v^k_{far} in the Figure of G^{k+1}, let $c_{k+1}(v^{k+1}_3) = c'_n$.

Due to Lemma 6.7, c_{k+1} is a 3-coloring of G^{k+1}. By design, c_{k+1} is not in conflict with a given 3-coloring of $P_{3vc} \setminus G^{k+1}$.

Consider the non-diagonal neighbor v_{n_2} of o_2 in P_{3vc}. An alternative 3-coloring c'_{k+1} of G^{k+1}, which is also constructed to be without conflict with a given 3-coloring of $P_{3vc} \setminus G^{k+1}$, with $c'_{k+1}(v_{n_2}) \neq c_{k+1}(v_{n_2})$ exists. To show this, analogous arguments to those for the existence of c_{k+1} can be used, but where Figures 6.11, 6.12 are used to obtain a 3-coloring of G^{k+1}, instead of Figures 6.3, 6.4.

(a) G^d_{ne} (b) G^d_{nw}

Figure 6.11: Alternative 3-colorings of G^d_{ne} and G^d_{nw}



(a) G^{ver} (b) G^1_{hor} (c) G^2_{hor} (d) G^3_{hor} (e) G^4_{hor}

Figure 6.12: Alternative 3-colorings of $G_{ver}, G^1_{hor}, G^2_{hor}, G^3_{hor}, G^4_{hor}$

We can obtain 4 different 3-colorings c of P_{3vc} using one of the functions c_{1-k}, c'_{1-k} to color vertices in $P_{3vc} \setminus G^{k+1}$, and one of the functions c_{k+1}, c'_{k+1} to color G^{k+1}.

Depending on the choice of c_{1-k} or c'_{1-k}, and the choice of c_{k+1} or c'_{k+1}, all combinations of $c(v'_{n_1}) \in \{1,2,3\} \setminus \{c(o_2)\}$ and $c(v_{n_2}) \in \{1,2,3\} \setminus \{c(o_2)\}$ can be attained. □

Lemma 6.10. *Let $G = (V, E)$ be a planar graph of maximum degree 4, and $G_{strgrid} = (V_{strgrid}, E_{strgrid})$ the result of applying Construction 2.5 to G. Let $G'_{strgrid}$ be the result of performing 3 resolution doubling operations (Observation 2.3) on $G_{strgrid}$.*

Let $P = (V_P, E_P)$ be a path between two original vertices $o_1, o_2 \in V$ in $G_{strgrid}$, which does not contain other original vertices $v' \in V$, such that $v' \notin \{o_1, o_2\}$. Let $P_{3vc} = (V, E)$ be the graph obtained when Construction 6.4 is applied to P.

Let o_1, o_2 both have degree two in P_{3vc}.

Let $v_{n_1} \in V_P$ be the neighbor of o_1 in P_{3vc} which it is not diagonal to, and v_{n_2} the neighbor of o_2 in P_{3vc} which it is not diagonal to.

Then there exists a 3-coloring c of P_{3vc} for every possible combination of $c(v_{n_1}) \in \{1,2,3\} \setminus \{c(o_1)\}$, and $c(v_{n_2}) \in \{1,2,3\} \setminus \{c(o_2)\}$.

Proof. This follows from analogous arguments to those in Lemma 6.9. We use the same definitions for G^1, G^{k+1} as in Lemma 6.9.

The difference in this lemma is the degree of o_1, which is 1 in Lemma 6.9, but is 2 here. As a consequence, we want to be able to select different colors $\neq c(o_1)$ for the non-diagonal neighbor v_{n_1} of o_1, instead of focusing on the non-diagonal neighbor $v'_{n_1} \neq o_1$ of v_{n_1}, like in Lemma 6.9.

We repeat the same approach as Lemma 6.9. Showing there exist two different 3-colorings c_{k+1}, c'_{k+1} of G^{k+1}, with $c_{k+1}(v^{k+1}_{n_1}) = c_{k+1}(v^{k+1}_{n_2})$, and $c'_{k+1}(v^{k+1}_{n_1}) \neq c'_{k+1}(v^{k+1}_{n_2})$ is equivalent. This is because the only difference between Lemma 6.9 and this lemma is in the structure of G^1.

There also exist two different 3-colorings c_{1-k}, c'_{1-k} of $P_{3vc} \setminus G^{k+1}$, such that $c_{1-k}(v_{n_1}) = c_2$ and $c'_{1-k}(v_{n_1}) = c_3$. The existence of these two colorings can be proven using analogous arguments to those used in Lemma 6.9, where a different variation of the existence of two different 3-colorings of $P_{3vc} \setminus G^{k+1}$ was proven.

By combining one of the colorings c_{1-k}, c'_{1-k} of $P_{3vc} \setminus G^{k+1}$, with one of the colorings c_{k+1}, c'_{k+1} of G^{k+1}, we can obtain a 3-coloring c of G_{3vc} for every possible combination of $c(v_{n_1}) \in \{1,2,3\} \setminus \{c(o_1)\}$, and $c(v_{n_2}) \in \{1,2,3\} \setminus \{c(o_2)\}$. □

Lemma 6.11. *Let $G = (V, E)$ be a planar graph of maximum degree 4. Let $G_{strgrid}$ be the strong grid graph obtained by applying Construction 2.5 to G. Let $G_{3vc} = (V_{3vc}, E_{3vc})$ be the graph obtained when Construction 6.4 is applied to G. Let c be a 3-coloring of G.*

Then there exists a 3-coloring of G_{3vc}.

Proof. For any edge $e = (o_1, o_2) \in E$, let $P_e = (V_{P_e}, E_{P_e})$ be the path between $o_1, o_2 \in V$ in $G_{strgrid}$, which does not contain other original vertices $v' \in V$, such that $v' \in \{o_1, o_2\}$. Then $\bigcup_{e \in E} P_e = G_{strgrid}$.

Let $P^e_{3vc} = (V^e_{P3vc}, E^e_{P3vc})$ be the result of applying Construction 6.4 to P_e. Then G_{3vc} contains P^e_{3vc} as a subgraph, such that the vertices $o_1, o_2 \in V$ are contained in that subgraph.

We claim we can select 3-colorings c_e of the subgraph P^e_{3vc} of G_{3vc}, for every edge $e = (u, v) \in E$, that fulfill the following conditions:

- For all original vertices $v \in V$, the chosen 3-colorings c_e of all subgraphs P^e_{3vc} that include v, fulfill $c_e(v) = c(v)$.
- Any vertices $v_s \notin V$ shared by two subgraphs $P^{e'}_{3vc}, P^{e''}_{3vc}$ receive the same color in the 3-colorings $c_{e'}$ of $P^{e'}_{3vc}$ and $c_{e''}$ of $P^{e''}_{3vc}$, so $c_{e'}(v_s) = c_{e''}(v_s)$.
- If an edge $e_i = (u_i, v_i) \in E_{3vc}$ is induced between subgraphs $P^{e'}_{3vc}, P^{e''}_{3vc}$, such that $u_i \in V^{e'}_{3vc}, v_i \in V^{e''}_{3vc}$, then $c_{e'}(u_i) \neq c_{e''}(v_i)$.

$E_{3vc} \setminus (\bigcup_{e \in E} E^e_{P3vc})$ is exactly the set of edges induced between the subgraph P^e_{3vc} corresponding to edges $e \in E$. Also, $\bigcup_{e \in E} V^e_{P3vc} = V_{3vc}$. Consequently, the individual colorings of the subgraphs $P^e_{3vc} = (V^e_{P3vc}, E^e_{P3vc})$ $(e \in E)$ can be combined to a full 3-coloring of G_{3vc}, if they fulfill the listed conditions.

For the graph P^e_{3vc} corresponding to an edge $e = (u, v) \in E$, the colors of u, v can be chosen freely when defining a 3-coloring of P^e_{3vc}, which means the first of the conditions listed above is fulfilled. The existence of the four different 3-colorings shown in Lemmas 6.9, 6.10 implies that, once a color for u, v has been assigned, all valid combinations of colors remain available for the neighbors of u, v.

It is possible that for adjacent edges $e, e' \in E$ with shared endpoint $v_s \in V$, the graphs $P^e_{3vc}, P^{e'}_{3vc}$ have a second vertex in common, if v_s has degree 2 in both $P^e_{3vc}, P^{e'}_{3vc}$. The second shared vertex is the vertical neighbor v'_s of v_s. Let $v_e, v_{e'}$ be the other endpoints of e, e' respectively. An example of this is shown in Figure 6.13.

In that case, we select 3-colorings of $P^e_{3vc}, P^{e'}_{3vc}$ that assign the same color to the shared vertex v'_s. The choice of colors for the neighbors of the other endpoints $v_e, v_{e'}$ of e, e' is not constrained further due to this decision, as a consequence of Lemmas 6.9, 6.10. This means the second condition listed above is also fulfilled, and there remain two options for the color of the neighbors of endpoints $\neq v_s$.

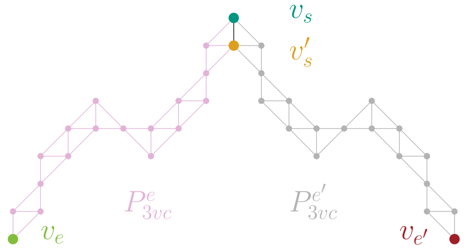


Figure 6.13: Two shared vertices in P^e_{3vc} and $P^{e'}_{3vc}$

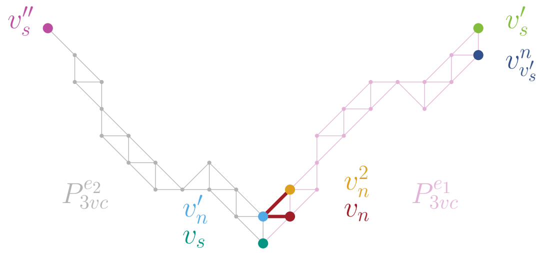


Figure 6.14: Vertices in P^e_{3vc} and $P^{e'}_{3vc}$ with induced edges

Let edges $e^1, e^2 \in E$ share an endpoint $v_s \in V$, and let edges be induced between the corresponding subgraphs $P^{e_1}_{3vc} = (V^{e_1}_{P3vc}, E^{e_1}_{P3vc}), P^{e_2}_{3vc} = (V^{e_2}_{P3vc}, E^{e_2}_{P3vc})$ of G_{3vc}. This can only occur if v_s has degree 1 in one of $P^{e_1}_{3vc}, P^{e_2}_{3vc}$, and degree 2 in the other. Without loss of generality let v_s have degree 1 in $P^{e_1}_{3vc}$. Since v_s has degree 1, the subgraph in G_{rep} of $P^{e_1}_{3vc}$ that includes v_s is one of $\{G^1_{nw}, G^1_{ne}\}$ (Figure 6.2) by construction.

Let v_n be the (diagonal) neighbor of v_s in $P^{e_1}_{3vc}$. Let $v'_s \neq v_s$ be the vertex in $V_{strgrid} \cap V^{e_1}_{3vc}$, that is closest to v_s in G_{3vc}. Analogously let $v''_s \neq v_s$ be the vertex in $V_{strgrid} \cap V^{e_2}_{3vc}$, that is closest to v_s in G_{3vc}. Let $v^n_{v'_s}$ be the vertical neighbor of v'_s. Let v^2_n be the vertical neighbor of v_n in $P^{e_1}_{3vc}$, and v'_n the vertical neighbor of v_s in $P^{e_2}_{3vc}$.

Then the edges $e_i = (v'_n, v_n)$ and $e'_i = (v'_n, v^2_n)$ are induced. Therefore, to ensure the 3-coloring c_{e_1} of $P^{e_1}_{3vc}$ and c_{e_2} of $P^{e_2}_{3vc}$ are not in conflict, $c_{e_1}(v_n) \notin \{c_{e_2}(v_s), c_{e_2}(v'_n)\}$, and $c_{e_1}(v^2_n) = c_{e_2}(v_s)$ has to hold. An example of this situation is illustrated in Figure 6.14.

As previously discussed, Lemmas 6.9, 6.10 imply that the colors of v_n, v'_n can be chosen freely in a 3-coloring of $P^{e_1}_{3vc}$ or $P^{e_2}_{3vc}$, under the constraint that they are distinct from the color assigned to v_s. This does not affect the choice of color for v'_s, v''_s in the 3-colorings of $P^{e_1}_{3vc}, P^{e_2}_{3vc}$.

It follows that for any adjacent edges $e^1, e^2 \in E$ with shared endpoint $v_s \in V$, the graphs $P^{e_1}_{3vc}, P^{e_2}_{3vc}$ can be 3-colored, such that the endpoints of any edge that is induced between them are assigned different colors by the 3-colorings of $P^{e_1}_{3vc}, P^{e_2}_{3vc}$. This means the third and final one of the conditions listed above is also fulfilled.

We have shown that the options to color the graph P^e_{3vc} corresponding to an edge $e \in E$ can be constrained at both ends of the path, by shared vertices or induced edges with other graphs $P^{e'}_{3vc}$, $P^{e''}_{3vc}$ ($e', e'' \in E \setminus \{e\}$). Even when that is the case, due to the previous arguments and Lemmas 6.9, 6.10, a 3-coloring c_e of P^e_{3vc} exists that assigns the same colors to shared vertices as given 3-colorings of $P^{e'}_{3vc}$, $P^{e''}_{3vc}$, and c_e assigns different colors to the endpoints of induced edges between P^e_{3vc} and $P^{e'}_{3vc}$, $P^{e''}_{3vc}$.

Hence we can select 3-colorings c_e of the subgraph P^e_{3vc} for every edge $e = (u, v) \in E$, such that the three conditions listed above are fulfilled. We can construct a complete 3-coloring of G_{3vc} by combining the colorings of the subgraphs P^e_{3vc} for all edges $e \in E$. $\square$

Theorem 6.12. *3-COLORABILITY is $\mathcal{NP}$-complete on strong grid graphs.*

Proof. Since 3-COLORABILITY on arbitrary graphs is $\mathcal{NP}$-complete, it is clear that 3-COLORABILITY on strong grid graphs is in $\mathcal{NP}$.

To prove $\mathcal{NP}$-hardness, we perform a polynomial-time reduction from 3-COLORABILITY on planar graphs of maximum degree 4 to 3-COLORABILITY on strong grid graphs.

Let G be a planar graph of maximum degree 4. Let G_{grid} be the graph obtained when Construction 6.4 is applied to G.

Due to Lemmas 6.11, 6.6, G is 3-colorable if and only if G_{grid} is 3-colorable.

When we consider G an instance of 3-COLORABILITY on planar graphs of maximum degree 4, and G_{grid} an instance of 3-COLORABILITY on strong grid graphs, G is a YES-instance if and only if G_{grid} is a YES-instance. Since G can be transformed to G_{grid} in polynomial time, it follows that 3-COLORABILITY on strong grid graphs is $\mathcal{NP}$-complete. □

7. Edge Coloring

Let $G = (V, E)$ be a graph, and $k \in \mathbb{N}$.

For a vertex $v \in V$, let $e(v)$ denote the set $\{e = (v_1, v_2) \in E \mid v \in \{v_1, v_2\}\}$.

A function $c : E \to (1, \ldots, k)$, such that for all vertices $v \in V$, there do not exist two edges $e_1, e_2 \in e(v)$ with $c(e_1) = c(e_2)$, is called a *k-edge-coloring* of G. In other words, no two adjacent edges are assigned the same color by c.

G is called k-edge-colorable if it has a k-edge-coloring. The *chromatic index* of G, $\chi'(G)$, is the smallest k such that G is k-edge-colorable.

Vizing's theorem states, for any graph G, that $\chi'(G) \in \{\Delta(G), \Delta(G)+1\}$ [Viz64]. Graphs G_1 with $\chi'(G_1) = \Delta(G_1)$ are referred to as *Class I* graphs, graphs G_2 with $\chi'(G_2) = \Delta(G) + 1$ are referred to as *Class II* graphs.

7.1 3-Edge-Colorability

The k-Edge-Colorability problem, with a constant $k \in \mathbb{N}$, is defined as follows: Given a graph $G = (V, E)$, it should be decided whether $\chi'(G) \leq k$. If $\chi'(G) \leq k$, G is a yes-instance, else it is a no-instance.

2-Edge-Colorability can be solved in polynomial time on arbitrary graphs. All graphs G_1 with $\Delta(G_1) = 1$ are Class I, and any graph G_2 with $\Delta(G_2) = 2$ is 2-edge-colorable if and only if it does not contain an odd cycle.

3-edge-colorability is $\mathcal{NP}$-complete on cubic graphs [Hol81]. The computational complexity of 3-edge-colorability on planar graphs of maximum degree 3 is open [GJU24].

Observation 7.1. *Consider the graph $G_s = (V_s, E_s)$ depicted in Figure 7.1. Any 3-edge-coloring c of G_s assigns the same color to the edges e_1, e_2 labeled in the figure.*

Proof. Let $e_1 = (u_1, v_1)$, without loss of generality, assume $c(e_1) = 1$. Let e_1' and e_1'' be the two edges adjacent to e_1. Then $c(e_1') \neq c(e_1'')$, and $c(e_1'), c(e_1'') \in \{2, 3\}$.

Let the vertices $v', v'' \in V_s \setminus \{u_1, v_1\}$ be endpoints of e_1' and e_1'' respectively. The edge $e' = (v', v'')$ in G_s is adjacent to e_1', e_1'', hence $c(e') = c(e_1) = 1$.

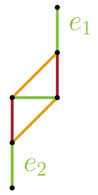


Figure 7.1: 3-edge-coloring of G_s

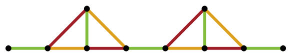


Figure 7.2: G_s^2

Let e_2', e_2'' be the edges adjacent to e_2. Then $c(e_2') \neq c(e_2'')$, and since e_2', e_2'' are both adjacent to e', $c(e_2'), c(e_2'') \in \{2, 3\}$.

Consequently, $c(e_2) = 1 = c(e_1)$. □

Definition 7.2. A number $k \in \mathbb{N}$ of instances of the graph $G_s = (V_s, E_s)$ (Observation 7.1) can be connected together to form a "chain" $G_s^k = (V_s^k, E_s^k)$ of k instances of G_s graphs. We define G_s^k inductively:

The graph $G_s = G_s^1$ has two edges adjacent to degree 1 vertices. We can chain two G_s graphs together by letting them share one of these edges. The resulting graph G_s^2, shown in Figure 7.2, again has two edges adjacent to degree one vertices.

Additional G_s graphs can be attached to a chain G_s^c of $c \geq 2$ instances of G_s graphs in the same way, to obtain the chain G_s^{c+1} of $c + 1$ instances of G_s graphs.

We call the embedding in the grid of G_s^k the *standard vertical embedding*, if all G_s graphs are embedded as shown in Figure 7.3a. Analogously, the *standard horizontal embedding* embeds all G_s graphs as shown in Figure 7.3b.

Observation 7.3. *Let $G_s^k = (V_s^k, E_s^k)$ be a chain of k G_s graphs (7.2). The two edges $e, e' \in E_s^k$ which are adjacent to degree one vertices, fulfill $c(e) = c(e')$ in any 3-edge-coloring c of G_s^k.*

Proof. G_s^k contains the graph G_s as a subgraph multiple times. Due to Observation 7.1, the edges adjacent to degree 1 vertices in the graph G_s both receive the same color in any 3-edge-coloring c_s of G_s. Hence, these edges also both receive the same color in any 3-edge-coloring c of G_s^k.

When this argument is applied to all G_s subgraphs in G_s^k, it follows that $c(e) = c(e')$, where $e, e' \in E_s^k$ are the two edges adjacent to degree 1 vertices in G_s^k. □

(a) Standard vertical embedding (b) Standard horizontal embedding

Figure 7.3: Standard embeddings of G_s

(a) Before replacement (b) Replacement graph G_o

Figure 7.4: Replacement of subgraph near $v \in V$

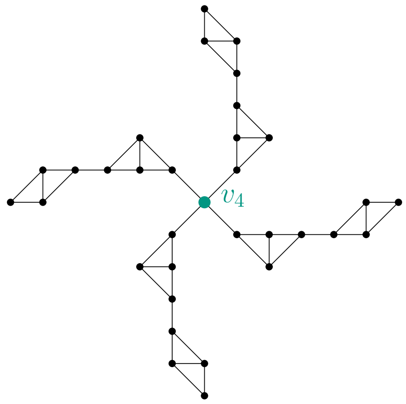


Figure 7.5: Replacement of subgraphs around degree 4 vertex $v_4 \in V$

Theorem 7.4. *3-EDGE-COLORABILITY is $\mathcal{NP}$-complete on partial strong grids.*

Proof. Let $G = (V, E)$ be a cubic graph. We construct the partial strong grid $G_{3ec} = (V_{3ec}, E_{3ec})$ from G as follows:

We obtain a partial grid G_{grid} by applying Construction 2.2 to G. Let G'_{grid} be the result of performing 5 resolution doubling operations (Observation 2.3) on G_{grid}.

The following subgraph replacements are carried out on G'_{grid} to obtain a new graph G'_{3ec}:

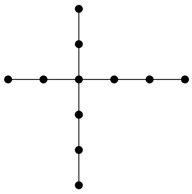
Let $e = (u, v) \in E$ be an edge in G. In G'_{grid}, the subgraph induced by the vertices in $s(G'_{grid}, e) \cup \{u\}$ with distance ≤ 6 to u, including u, is replaced by the graph G_o (Figure 7.4). The same replacement is carried out for v. We rotate G_o appropriately, as shown in Figure 7.5, such that there are no edges induced between the 4 replacement subgraphs G_o around a degree 4 vertex v_4. Then no edges are induced between these replacement subgraphs near lower degree vertices either.

If G is non-planar, let e_1, e_2 be two edges that cross each other in the given embedding of G. In G'_{grid}, there are no crossings, since it was obtained by applying Construction 2.2 to G, and subsequently carrying out resolution doubling operations. Instead, the subgraph in G'_{grid} corresponding to a crossing in G will be the graph G'_c shown in Figure 7.6a. We replace all subgraphs in G'_{grid} corresponding to crossings in G with the graph G_c (Figure 7.6b).

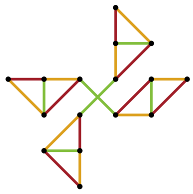
Finally, let v_b be a vertex at a bend in G'_{grid}. We replace the subgraph in G'_{grid} induced by the vertices with distance ≤ 3 to v_b (including v_b), with G_b (Figure 7.7).

G'_{3ec} is the result of carrying out the subgraph replacements described above on G'_{grid}. In G'_{3ec}, subgraphs around original vertices and bends, and subgraphs corresponding to crossings in G, have been replaced with strong grid graphs, whose structure makes the upcoming lemmas and theorem possible. There are paths in G'_{3ec} between these replacement subgraphs, which still need to be transformed to obtain the desired graph G_{3ec}.

Let $G_{\subset} = (V_{\subset}, E_{\subset})$ be the subgraph of G'_{3ec} which is not part of any of the replacement subgraphs G_b, G_c, G_o. $G_{\subset}$ consists of a set of disconnected paths. We carry out the following subgraph replacements in G'_{3ec} on these paths to obtain G_{3ec}.

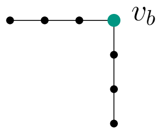


(a) Subgraph G'_c corresponding to crossing in G'_{grid}

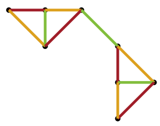


(b) Replacement graph G_c

Figure 7.6: Replacement of crossing subgraph



(a) Before replacement



(b) Replacement graph G_b

Figure 7.7: Replacement of bend subgraph

Let $P = (V_P, E_P)$ be a path in $G_\subset$. Due to the 5 resolution doubling operations performed on G_{grid} to obtain G'_{grid}, crossings, bends and original vertices in G'_{grid} will be at least 32 units apart.

The distance to a crossing is measured from the vertex in G_{grid} that was placed at the intersection of the two crossing edges. The distance to a bend is measured from the vertex with both a horizontal and vertical edge adjacent to it. The largest distance a replacement subgraph has to the original vertex, bend, or crossing it replaced, is 6 vertices. An additional 2 edges have one endpoint in V_P and one in $V \setminus V_\subset$, and are therefore not part of P. It follows that P has a length of at least $32 - 2 \times 6 - 2 = 18$. Along P, either all edges are horizontal, or all edges are vertical.

Assume P is a horizontal path, let v_ℓ be the vertex in V_P with the lowest x-coordinate.

If the number of vertices in V_P is divisible by 3, let $k = |V_P|/3$. In that case we replace P with a modified chain G_s^k of k instances of G_s graphs (see Observation 7.3), where the two degree one vertices and their adjacent edges are removed. Recall that there are 2 edges with exactly one endpoint in V_P in G'_{3ec}. The degree 2 vertices at the ends of the modified G_s^k chain replace the endpoints in V_P of these 2 edges. We use the standard horizontal embedding of G_s^k, described in Observation 7.3. An example of such a modified chain, for $k = 3$, is depicted in Figure 7.8b.

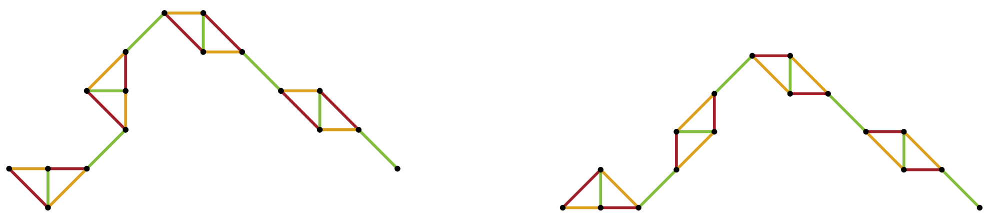
If $|V_P| \bmod 3 = 1$, consider the subgraph $G_{10} = (V_{10}, E_{10})$ of P, induced by the vertices with distance ≤ 10 to v_ℓ. We replace G_{10} with the graph G_1, shown in Figure 7.9a. Then the number of vertices in $P \setminus G_{10}$ is divisible by 3, let $k_2 = |V_P \setminus V_{10}|/3$. $P \setminus G_{10}$ is replaced with a chain $G_s^{k_2}$ of k_2 instances of G_s graphs, using the standard horizontal embedding.

If $|V_P| \bmod 3 = 2$, consider the subgraph $G_{11} = (V_{11}, E_{11})$ of P, induced by the vertices with distance ≤ 11 to v_ℓ. We replace G_{11} with G_2 (Figure 7.9b). Then the number of vertices in $P \setminus G_{11}$ is divisible by 3, let $k_3 = |V_P \setminus V_{11}|/3$. $P \setminus G_{11}$ is replaced with a chain $G_s^{k_3}$ of k_3 instances of G_s graphs, using the standard horizontal embedding.

(a) G_s^3

(b) G_s^3, removed degree 1 vertices

Figure 7.8: Chain of three G_s graphs



(a) G_1 (b) G_2

Figure 7.9: Replacement graphs for different path lengths

Analogous modifications can be performed on a vertical path: a vertical path is simply a horizontal path rotated by 90 degrees. When chains G_s^k are used in this case, we use the standard vertical embedding we defined in Observation 7.3.

We perform the discussed modifications on every path P in $G_\subset$. The resulting graph G_{3ec} connects all vertices in V via graphs that are isomorphic to chains of G_s graphs, where the degree 1 vertices were not removed. This follows from the structure of the replacement subgraphs in $G \setminus G_\subset$. For every edge $e = (u, v) \in E$, there is exactly one chain of G_s graphs in G_{3ec} which contains u, v but no other original vertices in $V \setminus \{u, v\}$. The union of the G_s chains corresponding to e, across all edges $e \in E$, makes up all of G_{3ec}.

Since G_s chains, as well as the graphs G_o, G_v are all strong grid graphs, G_{3ec} is a strong grid graph if G is planar.

The number of crossings, bends and original vertices in G'_{grid} is polynomial in $n := |V|$. The number of paths in $G_\subset$ is linear in $|E|$. Since subgraph replacements can also be carried out in polynomial time, it follows that the construction of G_{3ec} can be performed in polynomial time in n. The number of resolution doubling operations carried out in the construction of G_{3ec}, and in Construction 2.2, is constant, so $|V_{3ec}|$ is polynomial in n.

Claim 7.5. *Let c' be a 3-edge-coloring of G_{3ec}.*

Then there exists a 3-edge-coloring c of G.

Proof. For every edge $e = (u, v) \in E$, let $G_s^e = (V_s^e, E_s^e)$ be the subgraph of G_{3ec}, which is isomorphic to a chain of G_s graphs, such that the endpoints u, v of e have degree 1 in G_s^e, and $\nexists v' \in V \cap V_s^e : v' \notin \{u, v\}$. Such a graph exists in G_{3ec} by construction. Let $e', e'' \in E_s^e$ be the edges adjacent to a degree one vertex in G_s^e. Due to Lemma 7.3, $c'(e') = c'(e'')$. We define a color assignment $c : E \to \{1, 2, 3\}$, such that $c(e) = c'(e')$ for all edges $e \in E$. We claim that c is a 3-edge-coloring of G.

Consider two adjacent edges $e_1 = (v_1, v_s), e_2 = (v_s, v_2) \in E$, with shared endpoint $v_s \in V$.

G_{3ec} contains a subgraph $G_s^{e_1} = (V_s^{e_1}, E_s^{e_1})$, which is isomorphic to a chain of G_s graphs, such that the endpoints v_1, v_s of e_1 have degree 1 in $G_s^{e_1}$, and $\nexists v' \in V \cap V_s^{e_1} : v' \notin \{v_1, v_s\}$. Analogously, G_{3ec} also contains such a subgraph $G_s^{e_2} = (V_s^{e_2}, E_s^{e_2})$ corresponding to e_2.

Let $e'_1, e''_1 \in E_s^{e_1}$ be the two edges adjacent to v_1 or v_s in $G_s^{e_1}$. Similarly, let $e'_2, e''_2 \in E_s^{e_2}$ be adjacent to v_2 or v_s in $G_s^{e_2}$. Let e''_1, e''_2 be the edges with endpoint v_s. Due to Lemma 7.3, $c'(e'_1) = c'(e''_1)$, and $c'(e'_2) = c'(e''_2)$.

Since e''_1, e''_2 have an endpoint in common, $c'(e''_1) \neq c'(e''_2)$. It follows that $c(e_1) \neq c(e_2)$.

Hence, c assigns different colors to every pair of adjacent edges in E, and is therefore a 3-edge-coloring of G. □

Claim 7.6. *Let c be a 3-edge-coloring of G. Then there exists a 3-edge-coloring c' of G_{3ec}.*

Proof. Let $e = (u, v) \in E$ be an edge, with $c(e) = c_e \in \{1, 2, 3\}$.

We define a color assignment $c' : E_{3ec} \to \{1, 2, 3\}$ as follows:

Consider the subgraph G_s^e of G_{3ec} which is isomorphic to a chain of G_s graphs, such that the endpoints u, v of e have degree 1 in G_s^e, and $\nexists v' \in V \cap V_s^e : v' \notin \{u, v\}$. Let e', e'' be the two edges adjacent to u, v in G_s^e.

Let $c'(e') = c_e$. Consider the subgraph G_s^1 of G_s^e, isomorphic to G_s (Figure 7.1), which contains e'. Let e'_{n_1}, e'_{n_2} be the neighbors of e' in G_s^1. We assign two different colors c_{n_1}, c_{n_2} from $\{1, 2, 3\} \setminus \{c_e\}$ to e'_{n_1}, e'_{n_2} respectively, let $c'(e_{mid}) = c_e$. Let $e^2_{n_1}, e^2_{n_2} \notin \{e'_{n_1}, e'_{n_2}\}$ be the other edges adjacent to e_{mid}. Let $e^2_{n_1}$ be adjacent to e'_{n_1}, then $e^2_{n_2}$ is adjacent to e'_{n_2}. Let $c'(e^2_{n_1}) \in \{1, 2, 3\} \setminus \{c_e, c'(e'_{n_1})\}$, and $c'(e^2_{n_2}) \in \{1, 2, 3\} \setminus \{c_e, c'(e'_{n_2})\}$. Let $e^2 \neq e_{mid}$ be the other edge adjacent to both $e^2_{n_1}, e^2_{n_2}$, let $c(e^2) = c_e$. Then c' is a 3-edge-coloring of G_s^1.

Now observe the subgraph G_s^2 of G_s^e, isomorphic to G_s (Figure 7.1), which contains e^2 but not e'. We can define c' on G_s^2 using the same approach as G_s^1, resulting in a 3-edge-coloring c' of $G_s^1 \cup G_s^2$.

The same can be repeated for the remaining subgraphs of G_s^e, which are isomorphic to G_s and contain an edge from the previously colored G_s subgraph. The result is a 3-edge-coloring c' of G_s^e.

We define c' using this approach, on the corresponding subgraphs G_s^e for all edges $e \in E$. We claim c' is a 3-edge-coloring of G_{3ec}.

It is only necessary to show that, for two edges $e_1, e_2 \in E$ that are adjacent to each other in G, with shared endpoint $v_s \in V$, the two edges adjacent to v_s in G_{3ec} are assigned different colors by c'. This is because the only vertex in $G_s^{e_1} \cap G_s^{e_2}$ is v_s.

For the edge e'_1 adjacent to v_s in $G_s^{e_1}$, $c'(e'_1) = c(e_1)$. For the edge e'_2 adjacent to v_s in $G_s^{e_2}$, $c'(e'_2) = c(e_2) \neq c(e_1) = c'(e'_1)$. Hence, c' is a 3-edge-coloring on $\bigcup_{e \in E} G_s^e$.

By construction, $\bigcup_{e \in E} G_s^e = G_{3ec}$, so c' is a 3-edge-coloring of G_{3ec}. □

Since 3-EDGE-COLORABILITY on arbitrary graphs is $\mathcal{NP}$-complete, it is clear that 3-EDGE-COLORABILITY on partial strong grids is in $\mathcal{NP}$.

To prove 3-EDGE-COLORABILITY is $\mathcal{NP}$-hard on partial strong grids, we perform a polynomial-time reduction from 3-EDGE-COLORABILITY on cubic graphs to 3-EDGE-COLORABILITY on partial strong grids. Recall that we defined the cubic graph G for this purpose at the start of this theorem, and constructed the partial strong grid G_{3ec} from G in polynomial time.

Due to Claims 7.5, 7.6, G is 3-edge-colorable if and only if G_{3ec} is 3-edge colorable.

When we consider G an instance of 3-EDGE-COLORABILITY on cubic graphs, and G_{3ec} an instance of 3-EDGE-COLORABILITY on partial strong grids, G is a YES-instance if and only if G_{3ec} is a YES-instance. Since G can be transformed to G_{3ec} in polynomial time, it follows that 3-EDGE-COLORABILITY on partial strong grids is $\mathcal{NP}$-complete. □

Theorem 7.7. *3-EDGE-COLORABILITY on planar graphs of maximum degree 3 is polynomially reducible to 3-EDGE-COLORABILITY on strong grid graphs.*

Proof. Let $G = (V, E)$ be a planar graph of maximum degree 3. We construct a strong grid graph G_{3ec} from G, using the same method as described at the start of Theorem 7.4. Since G contains no crossing, G_{3ec} is a strong grid graph. G_{3ec} can be computed from G in polynomial time, and the number of vertices in G_{3ec} is polynomial in $|V|$.

Due to the Claims 7.5, 7.6 in Theorem 7.4, G is 3-edge-colorable if and only if G_{3ec} is 3-edge colorable.

When we consider G an instance of 3-EDGE-COLORABILITY on planar graphs of maximum degree 3, and G_{3ec} an instance of 3-EDGE-COLORABILITY on strong grid graphs, G is a YES-instance if and only if G_{3ec} is a YES-instance. Since G can be transformed to G_{3ec} in polynomial time, it follows that 3-EDGE-COLORABILITY on planar graphs of maximum degree 3 is polynomially reducible to 3-EDGE-COLORABILITY on strong grid graphs. □

Theorem 7.8. *3-EDGE-COLORABILITY on strong grid graphs is polynomially reducible to 3-EDGE-COLORABILITY on planar graphs of maximum degree 3.*

Proof. Graphs with a maximum degree greater than 3 are not 3-edge-colorable, those with maximum degree less than 3 are always 3-edge-colorable. We therefore consider strong grid graphs of maximum degree 3.

Any crossing in a strong grid graph induces a K_4. A K_4 is 3-regular, so no additional edges can be adjacent to it in a strong grid graph of maximum degree 3. Since a K_4 is a planar graph, it follows that all strong grid graphs with maximum degree 3 are planar.

As a consequence, the polynomial-time reduction from 3-EDGE-COLORABILITY on strong grid graphs to 3-EDGE-COLORABILITY on planar graphs of maximum degree 3 is trivial, since strong grid graphs with maximum degree 3 are a subset of planar graphs with maximum degree 3.

□

7.2 Class I and II graphs

In this chapter we discuss whether there exist Class II strong grid graphs and partial strong grids with maximum degree k for each $k \in \{1, \ldots, 8\}$.

For subgraphs of strong grids G with $\Delta(G) \in \{1, 2\}$ this is trivial. All graphs of maximum degree 1 are Class I. A simple example of a Class II strong grid graph, with maximum degree 2, is a cycle with 3 vertices.

Theorem 7.9. *There exist strong grid graphs G_i with $\Delta(G_i) = 3$, which are Class II.*

Proof. Consider the graph G_3 in Figure 7.10. G_3 contains subgraphs isomorphic to the graph G_s (Figure 7.1) multiple times. By applying Observation 7.1 to these subgraphs, we can conclude that the edges colored red in Figure 7.10 are all assigned the same color in any 3-edge-coloring of G_3. Since two of these edges share an endpoint, G_3 is not 3-edge-colorable. □

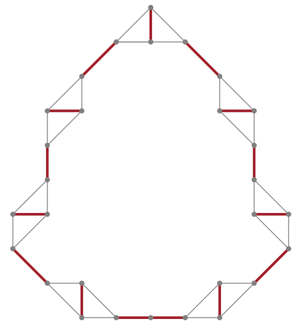


Figure 7.10: Class II strong grid graph G_3, $\Delta(G_3) = 3$

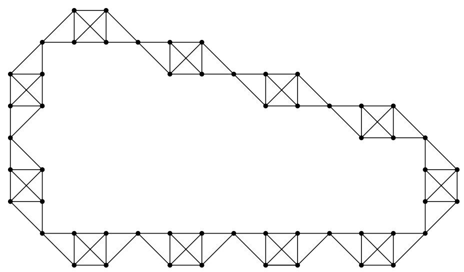


Figure 7.11: Class II partial strong grid G_4, $\Delta(G_4) = 4$

Observation 7.10. *In a k-edge-coloring c of the graph $G = (V, E)$, the set of edges which are assigned a color $c_i \in \{1, \ldots, k\}$ by c is a matching of G.*

If G is k-regular, this matching is a perfect matching.

Proof. Since no two edges which receive the same color in the edge-coloring c are adjacent, the set of edges which receive the color c_i is a matching of G.

If G is k-regular, each vertex has k adjacent edges, which are all assigned distinct colors by c. Hence, there is an edge with the color c_i adjacent to every vertex in G. It follows that the set of edges which receive the color c_i in the edge-coloring c is a perfect matching of G. □

Theorem 7.11. *There exist partial strong grids G_p with $\Delta(G_p) = 4$, which are Class II.*

Proof. Consider the graph G_4 in Figure 7.11. Note that this is not a strong grid graph: edges would be induced at the orthogonal bends in G_4. G_4 is 4-regular, and contains an odd number of vertices.

It is a known theorem that k-regular graphs with an odd number of vertices cannot be k-edge-colored. This is a consequence of Observation 7.10, since graphs with an odd number of vertices do not have a perfect matching.

Hence, G_4 is a Class II graph. □

Lemma 7.12. *All partial grids can be 4-edge-colored.*

Proof. Let $G = (V, E)$ be a partial grid. We construct a function $c : E \to \{1, 2, 3, 4\}$.

Consider a vertical edge $e_{ver} = (u_{ver}, v_{ver}) \in E$. Let v_{ver} be the endpoint of e_{ver} with the lower y-coordinate. If $(v_{ver})_y \bmod 2 = 0$, let $c(e_{ver}) = 1$. Else, let $c(e_{ver}) = 2$.

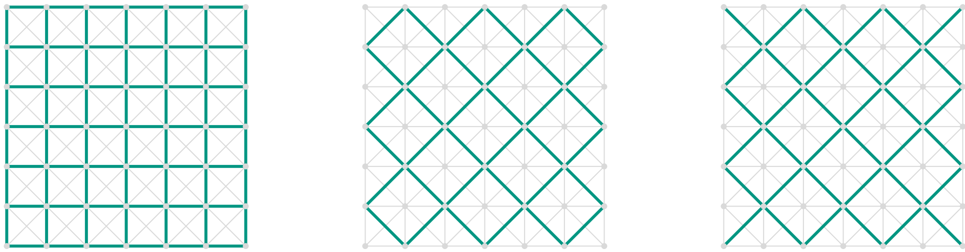


Figure 7.12: Decomposition of strong grid into 3 partial grids

Consider a horizontal edge $e_{hor} = (u_{hor}, v_{hor}) \in E$. Let v_{hor} be the endpoint of e_{hor} with the lower x-coordinate. If $(v_{hor})_x \bmod 2 = 0$, let $c(e_{hor}) = 3$. Else, let $c(e_{hor}) = 4$. Then every edge in E is assigned a color by c.

Let e_1, e_2 be two adjacent edges in G. If one of e_1, e_2 is horizontal and one is vertical, $c(e_1) \neq c(e_2)$. If both are vertical, let v_s be the shared endpoint of e_1, e_2. For one of e_1, e_2, v_s is the endpoint with the lower y-coordinate. For the other, the endpoint with the lower y-coordinate is the vertex v_b one grid unit below v_s. Since $(v_b)_y \bmod 2 \neq (v_s)_y \bmod 2$, $c(e_1) \neq c(e_2)$. If e_1, e_2 are both vertical, analogous arguments can be applied to the shared endpoint v_s of e_1, e_2, and the vertex one grid unit to the left of v_s.

In any case $c(e_1) \neq c(e_2)$, hence c is a 4-edge-coloring of G. □

Theorem 7.13. *Let $G = (V, E)$ be an $n \times m$ strong grid. Then G is 8-edge-colorable.*

Proof. G can be decomposed into 3 edge-disjoint partial grids: one partial grid that consists exclusively non-diagonal edges, and two partial grids that consist exclusively of diagonal vertices. An example of this is shown in Figure 7.12.

All partial grids can be 4-edge-colored (Lemma 7.12). The two diagonal partial grids do not share any vertices or edges, so these subgraphs of G can both be edge-colored using the same 4 colors. The remaining 4 colors can be used to 4-edge-color the non-diagonal partial grid.

This results in an 8-edge-coloring of G. □

Partial strong grids are subgraphs of strong grids, which are 8-edge-colorable due to Theorem 7.13. Hence partial strong grids are also 8-edge-colorable. It follows that all partial strong grids G with $\Delta(G) = 8$ are Class I.

7.2.1 Δ-Matchings

Let $G = (V, E)$ be a graph, with maximum degree $\Delta(G)$. The set $\Lambda(G) \subseteq V$ denotes the set of vertices in G, which have degree $\Delta(G)$. We refer to a matching, which covers all vertices in $\Lambda(G)$, as a Δ-*matching* of G.

Observation 7.14. *Let $G = (V, E)$ be a graph, which has a Δ-matching $M \subseteq E$. If $G' = (V, E \setminus M)$ is Class I, then G is Class I.*

Proof. We define $k := \Delta(G)$, it holds that $\Delta(G') = k - 1$. If G' has a $(k-1)$-edge-coloring $c' : V \to \{1, \ldots, k-1\}$, a k-coloring c of G can be obtained by setting $c(e) = k$ for all $e \in M$, and $c(e') = c'(e')$ for all $e' \in E \setminus M$. □

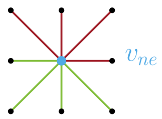


Figure 7.13: Possible adjacent edges at v_{ne}

Δ-matchings are a relevant structural property for Class I/II classification. Assume there always exists a Δ-matching for partial strong grids G with maximum degree $\Delta(G) = k$, and all partial strong grids G' with $\Delta(G') = k - 1$ are Class I. Then all partial strong grids with maximum degree k are also Class I, due to Observation 7.14.

The same statement does not hold for strong grid graphs, since removing a matching from a strong grid graph does not necessarily result in a strong grid graph.

However, a similar statement does hold: Assume there always exists a Δ-matching for strong grid graphs G with $\Delta(G) = k$, and all partial strong grids G' with $\Delta(G') = k - 1$ are Class I. Then all strong grid graphs with maximum degree k are also Class I.

If all partial strong grids G with $\Delta(G) = k$ have a Δ-matching, this naturally implies all strong grid graphs G_i with $\Delta(G_i) = k$ also have a Δ-matching.

The upcoming definitions and lemmas are in preparation for proving the existence of Δ-matchings in partial strong grids and strong grid graphs with a specific maximum degree.

Lemma 7.15. *There are no k-regular partial strong grids for $k \geq 5$.*

Proof. For the sake of contradiction, assume $G = (V, E)$ is a k-regular partial strong grid ($k \geq 5$). Observe the top-right vertex v_{ne} of V (Definition 2.1.11). Since v_{ne} has $k \geq 5$ edges, at least one of its neighbors in G has an x-coordinate of at least $(v_{ne})_x$ and a y-coordinate of at least $(v_{ne})_y$. This is in contradiction to v_{ne} being the top-right vertex of V. The situation is illustrated in Figure 7.13. □

Definition 7.16. *Let $G = (V, E)$ be a graph. If $|V|$ is even, we define $\overline{G} := (V, E \cup \{(u, v) \mid u \neq v \wedge u, v \in \Lambda(G)\})$.*

If $|V|$ is odd, we first define the graph $G' := (V \cup \{v\}, E)$, which has an additional isolated vertex $v \notin V$ but is otherwise identical to G. Then $\overline{G} := \overline{G'}$.

Lemma 7.17. *Let $G = (V, E)$ be a graph. If there exists a perfect matching of $\overline{G} = (\overline{V}, \overline{E})$, then there exists a Δ-matching in G.*

Proof. Let M be a perfect matching of $\overline{G}$. By definition no edges $e \notin E$, which are adjacent to vertices in $\Lambda(G)$, are present in $\overline{E}$. The edges in M adjacent to vertices in $\Lambda(G)$ are therefore also in E. Hence, $M \cap E$ is a Δ-matching of G. □

Theorem 7.18. ***Tutte-Berge formula*** *[Tut47]: Let $G = (V, E)$ be a graph, with maximum matching M. For any graph G', let $odd(G')$ denote the number of connected components with an odd number of vertices in G'.*

Then $|M| = \frac{1}{2} \min_{U \subseteq V}(|U| - odd(G \setminus U) + |V|)$.

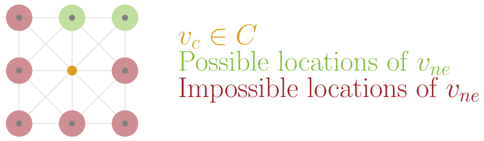


Figure 7.14: Possible locations of v_{ne} relative to v_c

Corollary 7.19. *A graph $G = (V, E)$ has a perfect matching if and only if it holds that* $\min_{U \subseteq V}(|U| - odd(G \setminus U)) = 0$.

Proof. A perfect matching is equivalent to matching of size $|V|/2$. $\frac{1}{2}\min_{U \subseteq V}(|U| - odd(G \setminus U) + |V|) = |V|/2 \iff \min_{U \subseteq V}(|U| - odd(G \setminus U)) = 0$. □

Observation 7.20. *Let $G = (V, E)$ be a graph. Let $U \subseteq V$. Let C_Δ be the set of all connected components in $\overline{G} \setminus U$, which consist exclusively of vertices from $\Lambda(G)$.*

If, $\forall U \subseteq V : |\bigcup_{C=(V_c,E_c) \in C_\Delta} N(\overline{G}, V_c)| > |C_\Delta|$, then $\overline{G}$ has a perfect matching.

Proof. Since $V \setminus \Lambda(G)$ is a clique in $\overline{G}$, there can be at most one connected component in $\overline{G} \setminus U$ that contains vertices from $V \setminus \Lambda(G)$.

So, in $\overline{G}$, there are at most $|C_\Delta| + 1$ connected components. To separate these components from each other in $\overline{G} \setminus U$, the vertices in $\bigcup_{C=(V_c,E_c) \in C_\Delta} N(\overline{G}, V_c)$ must be in U.

If $|\bigcup_{C=(V_c,E_c) \in C_\Delta} N(\overline{G}, V_c)| > |C_\Delta|$, it follows that $odd(G \setminus U) \leq |U|$. Then there exists no subset $U \subseteq V$, such that the removal of U results in more than $|U|$ separated components in $\overline{G} \setminus U$. Due to Corollary 7.19, it follows that $\overline{G}$ has a perfect matching. □

Theorem 7.21. *Let $G = (V, E)$ be a partial strong grid, with $\Delta(G) = 7$. Then G has a Δ-matching.*

Proof. We observe the graph $\overline{G}$. Let $U \subseteq V$.

Let C_Δ be the set of connected components in $\overline{G} \setminus U$ which consist exclusively of vertices in $\Lambda(G)$.

Claim 7.22. *The positions the top-right neighbor $v_{ne} \in N(\overline{G}, C)$ (Definition 2.1.11) of a component $C = (V_c, E_c) \in C_\Delta$ can have relative to a vertex $v_c \in V_c$ are limited.*

Figure 7.14 shows the possible positions v_{ne} can have relative to a vertex v_c in C.

Proof. Since all vertices in C have degree 7, v_c has a vertex in at least one of the two locations marked as *possible* in Figure 7.14. All other locations in the neighborhood of v_c either have a smaller y-coordinate than these two locations, or the same y-coordinate but a smaller x-coordinate, and can therefore not be the location of v_{ne}. □

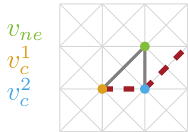


Figure 7.15: Impossible edges adjacent to v_c^2

Claim 7.23. *Let $v_{ne}^1 \in N(\overline{G}, V_c^1), v_{ne}^2 \in N(\overline{G}, V_c^2)$ be the top-right neighbors of the two components $C_1 = (V_c^1, E_c^1), C_2 = (V_c^2, E_c^2) \in C_\Delta$, with $C_1 \neq C_2$, respectively. Then $v_{ne}^1 \neq v_{ne}^2$.*

Proof. Assume for the sake of contradiction $v_{ne}^1 = v_{ne}^2$. We define $v_{ne} := v_{ne}^1$. Let $v_c^1 \in V_c^1$ and $v_c^2 \in V_c^2$ be adjacent to v_{ne}.

From Claim 7.22 we can conclude $|(v_c^1)_x - (v_c^2)_x| = 1$, and $(v_c^1)_y = (v_c^2)_y$. Else v_{ne} would be in a location relative to v_c^1 or v_c^2, which was identified as impossible (see Figure 7.14). We assume without loss of generality $(v_c^1)_x < (v_c^2)_x$.

Figure 7.15 highlights edges adjacent to v_c^2 that cannot exist. The existence of one of these edges would either imply v_{ne} is not the top-right neighbor of v_c^2, or the edge would connect v_c^1, v_c^2, who are part of distinct connected components. So the vertex v_c^2 can have at most 6 adjacent edges, which is in contradiction to $v_c^2 \in V_c^2$, since all vertices in C_2 have degree 7. □

For every $C = (V_c, E_c) \in C_\Delta$, it holds $N(\overline{G}, V_c) \neq \emptyset$, due to Lemma 7.15. Since two distinct components in C_Δ cannot have the same top-right neighbor, it follows that $|\bigcup_{C \in C_\Delta} N(\overline{G}, C)| \geq |C_\Delta|$.

Let $C_{sw} = (V_{sw}, E_{sw})$ be the component that contains a vertex v_b with the lowest y-coordinate in C_Δ. Since v_b has degree 7, it must be adjacent to a vertex $v_b' \in N(V_{sw})$ with a lower y-coordinate. Claim 7.22 implies that v_b' cannot be the top-right neighbor of another component in C_Δ.

Hence $|\bigcup_{C=(V_c,E_c) \in C_\Delta} N(\overline{G}, V_c)| > |C_\Delta|$. Due to Observation 7.20, it follows that $\overline{G}$ has a perfect matching. Then, due to Lemma 7.17, G has a Δ-matching. □

Theorem 7.24. *Let $G = (V, E)$ be a strong grid graph, with $\Delta(G) = 6$. Then G has a Δ-matching.*

Proof. We observe the graph $\overline{G}$. Let $U \subseteq V$.

Let C_Δ be the set of connected components in $\overline{G} \setminus U$ which consist exclusively of vertices in $\Lambda(G)$.

Claim 7.25. *The positions the top-right neighbor $v_{ne} \in N(\overline{G}, V_c)$ (Definition 2.1.11) of a component $C = (V_c, E_c) \in C_\Delta$ can have relative to a vertex $v_c \in V_c$ are limited.*

Figure 7.16 shows the possible positions v_{ne} can have relative to a vertex v_c in C.

Proof. Since all vertices in C have degree 6, v_c has a vertex in at least one of the three locations marked as *possible* in Figure 7.16. All other locations in the neighborhood of v_c have a smaller y-coordinate than these three locations, and can therefore not be the location of the top-right neighbor of C. □

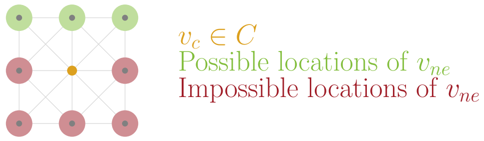
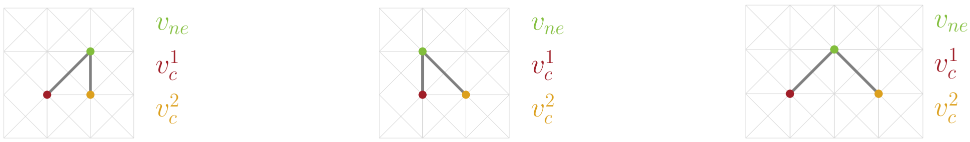


Figure 7.16: Possible locations of v_{ne} relative to v_c



(a) Only v_c^1 diagonal to v_{ne} (b) Only v_c^2 diagonal to v_{ne} (c) v_c^1, v_c^2 diagonal to v_{ne}

Figure 7.17: Possible locations of v_c^1, v_c^2 relative to v_{ne}

Claim 7.26. *Let $v_{ne}^1 \in N(\overline{G}, V_c^1), v_{ne}^2 \in N(\overline{G}, V_c^2)$ be the top-right neighbors of the two components $C_1 = (V_c^1, E_c^1), C_2 = (V_c^1, E_c^1) \in \mathcal{C}_\Delta$, with $C_1 \neq C_2$, respectively.*

Then, if $v_{ne}^1 = v_{ne}^2$ holds, there is a vertex v_n' in $N(\overline{G}, V_c^1) \cup N(\overline{G}, V_c^2)$ that is not the top-right vertex of any component in $\mathcal{C}_\Delta$, and $(v_n')_x = (v_{ne}^1)_x \wedge (v_n')_y = (v_{ne}^1)_y - 1$.

Proof. We define $v_{ne} := v_{ne}^1 = v_{ne}^2$.

Let $v_c^1 \in V_c^1$ and $v_c^2 \in V_c^2$ be adjacent to v_{ne}. Due to Claim 7.25, there are only three possible configurations of positions v_c^1, v_c^2, v_{ne} can have relative to each other. These configurations are shown in Figure 7.17. We assume without loss of generality $(v_c^1)_x < (v_c^2)_x$.

Consider the configurations depicted in Figures 7.17a, 7.17b. Since the vertices v_c^1, v_c^2 are neighbors in the strong grid in both of these cases, an edge is induced between them in G. This would connect the components C_1, C_2, in contradiction to $C_1 \neq C_2$.

Consider the situation shown in Figure 7.17c. Figure 7.18 shows two edges, in the neighborhood of v_c^2 in the strong grid, to which v_c^2 cannot have edges. If v_c^2 had edges to these locations, they would result in v_{ne} not being the top-right neighbor of C_2. Since v_c^2 has degree 6, it has edges to all 6 remaining locations in its neighborhood in the strong grid.

Consider the vertex v_n', with $(v_n')_x = (v_{ne})_x \wedge (v_n')_y = (v_{ne})_y - 1$. An edge is induced between v_c^1 and v_n'. Then v_n' is adjacent to both v_c^1, v_c^2, and can therefore not have degree 6, as $C_1 = C_2$ would hold otherwise. It follows that $v_n' \in N(\overline{G}, V_c^1) \cap N(\overline{G}, V_c^2)$. This is illustrated in Figure 7.19.

Assume for the sake of contradiction that v_n' is the top-right neighbor of a component $C_3 = (V_c^3, E_c^3) \in \mathcal{C}_\Delta$, with $C_3 \notin \{C_1, C_2\}$. Let $v_c^3 \in V_c^3$ be adjacent to v_n'. Then v_c^3 has to be in one of the three locations marked in Figure 7.20. If v_c^3 is in any of these locations, an edge is induced between v_c^3, v_c^1, or between v_c^3, v_c^2, or both. This connects the component C_3 to at least one of C_1, C_2, in contradiction to $C_3 \notin \{C_1, C_2\}$.

Consequently, if $v_{ne}^1 = v_{ne}^2 := v_{ne}$, there is a vertex v_n' in $N(\overline{G}, V_c^1) \cup N(\overline{G}, V_c^2)$ that is not the top-right neighbor in $\overline{G}$ of any component in $\mathcal{C}_\Delta$, with $(v_n')_x = (v_{ne})_x \wedge (v_n')_y = (v_{ne})_y - 1$. $\square$

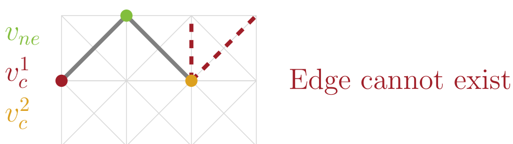


Figure 7.18: Impossible edges adjacent to v_c^2

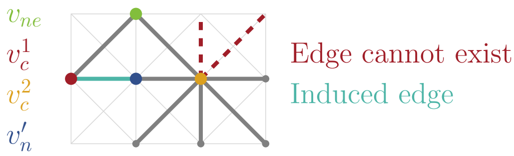


Figure 7.19: Induced edge between v_c^1, v_n'

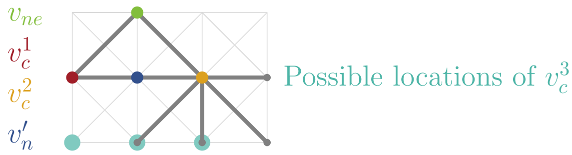


Figure 7.20: Possible locations of v_c^3

Claim 7.27. *It holds that* $|\bigcup_{C=(V_c,E_c)\in C_\Delta} N(\overline{G}, V_c)| > |C_\Delta|$.

Proof. For every $C = (V_c, E_c) \in C_\Delta$, it holds $N(\overline{G}, V_c) \neq \emptyset$, due to Lemma 7.15.

We construct a set $N \subseteq \bigcup_{C=(V_c,E_c)\in C_\Delta} N(\overline{G}, V_c)$.

We add the top-right neighbors in $\overline{G}$ of all $C \in C_\Delta$ to N. For each vertex v_{ne}, which is the top-right neighbor in $\overline{G}$ of two components $C_1 = (V_c^1, E_c^1), C_2 = (V_c^2, E_c^2) \in C_\Delta$, we add the vertex v_n' in $N(\overline{G}, V_c^1) \cup N(\overline{G}, V_c^2)$ to N, which is not the top-right neighbor in $\overline{G}$ of another component $C_3 \in C_\Delta$. Such a vertex v_n', with $(v_n')_y = (v_{ne})_y - 1$, always exists in this case, due to Claim 7.26. Since $(v_n')_y = (v_{ne})_y - 1$, there exists a vertex v_c' in $V_c^1 \cup V_c^2$ with $(v_c')_x = (v_n')_x$.

Then $|N| \geq |C_\Delta|$.

Let $C_{sw} = (V_{sw}, E_{sw})$ be the component that contains a vertex v_b with the lowest y-coordinate in C_Δ. Since v_b has degree 6, it must be adjacent to a vertex $v_b' \in N(\overline{G}, V_{sw})$ with a lower y-coordinate. Then every vertex in a component in C_Δ has a greater y-coordinate than v_b'. Therefore, all vertices in N have a greater y-coordinate than v_b'. It follows that $v_b' \notin N$.

The set $N' = N \cup \{v_b'\}$ is a subset of $\bigcup_{C=(V_c,E_c)\in C_\Delta} N(\overline{G}, V_c)$, with $|N'| > |C_\Delta|$. □

It follows that $|\bigcup_{C=(V_c,E_c)\in C_\Delta} N(\overline{G}, V_c)| > |C_\Delta|$. Observation 7.20 implies that $\overline{G}$ has a perfect matching. Then, due to Lemma 7.17, G has a Δ-matching. □

8. Conclusion

In this thesis we investigated the computational complexity of several well-known algorithmic problems on subgraphs of grids and strong grids.

We presented a method to obtain subdivisions of arbitrary graphs of maximum degree 4, which are partial strong grids. Variations of this allowed us to obtain subdivisions of planar graphs with maximum degree 4, which are induced subgraphs of grids or strong grids. By modifying these subdivisions further, we were able to transform input graphs to (induced) subgraphs of (strong) grids, which are equivalent to the original graph with respect to one of the studied decision problems.

Using this approach, we were able to show the $\mathcal{NP}$-completeness of MINIMUM VERTEX COVER, MAXIMUM CUT and 3-COLORABILITY on strong grid graphs. We proved that FEEDBACK VERTEX SET is $\mathcal{NP}$-complete on grid graphs, and 3-EDGE-COLORABILITY is $\mathcal{NP}$-complete on partial strong grids. Furthermore, we found that the problem 3-EDGE-COLORABILITY on planar graphs is polynomially equivalent to 3-EDGE-COLORABILITY on strong grid graphs.

Further research could investigate other problems which are $\mathcal{NP}$-complete for general graphs. Due to the simple structure of grids and strong grids, polynomial-time algorithms may exist when the problems are restricted to variations of subgraphs of grids. For the same reason, it may also be interesting to explore whether the runtime of problems, where polynomial-time algorithms are known for general graphs, can be improved when restricted to variations of subgraphs of grids.

While the focus of this thesis is on the Cartesian and strong products of two paths, there are many other types of product graphs that could be of interest. Another graph product operation that is commonly used is the tensor product, for example. One could also explore products which include graphs that are more complex than paths, such as the product of a tree and a path, or products with more than two graphs, such as 3-dimensional grids. The strong product of a path and a graph with bounded treewidth is especially interesting, due to the planar graph product structure theorem [DJM+20, UWY21]. While most problems we considered in this thesis are already $\mathcal{NP}$-complete when constrained to the simpler product of two paths, other problems may yield different results.

Another topic explored in this thesis is the question whether subgraphs of strong grids can be Class II, dependent on their maximum degree. We showed that strong grid graphs G_i with $2 \leq \Delta(G_i) \leq 3$ can be Class II, while those with $\Delta(G_i) \in \{1, 8\}$ are always Class I.

Similarly, we proved partial strong grids G with $2 \leq \Delta(G) \leq 4$ can be Class II, and those with $\Delta(G) \in \{1, 8\}$ are always Class I. It remains open whether there exist partial strong grids G with $\Delta(G) \in \{5, 6, 7\}$, and strong grid graphs G_i with $\Delta(G_i) \in \{4, 5, 6, 7\}$, which are Class II. For the open cases of strong grid graphs G_i with $\Delta(G_i) \in \{6, 7\}$ and partial strong grids G with $\Delta(G) = 7$, we show there always exists a Δ-matching. We believe that the existence of Δ-matchings can also be proven in the remaining open cases. This structural property provides some constraints for the open questions.

Further research could explore whether there exist Class II graphs in the open cases. It may also be interesting to investigate the computational complexity of determining a k-edge-coloring of (induced) subgraphs of strong grids, with $k \in \{4, 5, 6, 7\}$.

Bibliography

[BEU23] Therese Biedl, David Eppstein, and Torsten Ueckerdt. "On the complexity of embedding in graph products", 2023. doi:10.48550/arXiv.2303.17028.

[BGJR88] Francisco Barahona, Martin Groetschel, Michael Jünger, and Gerhard Reinelt. "An application of combinatorial optimization to statistical physics and circuit layout design". *Operations Research*, 36(3):493–513, 1988. doi:10.1287/opre.36.3.493.

[BK98] Therese Biedl and Goos Kant. "A better heuristic for orthogonal graph drawings". *Computational Geometry*, 9(3):159–180, 1998. doi:10.1016/S0925-7721(97)00026-6.

[BK99] Piotr Berman and Marek Karpinski. "On some tighter inapproximability results (Extended Abstract)". In Jirí Wiedermann, Peter van Emde Boas, and Mogens Nielsen, editors, *Automata, Languages and Programming*, pages 200–209, Berlin, Heidelberg, 1999. Springer Berlin Heidelberg. doi:10.1007/3-540-48523-6_17.

[BYGNR98] Reuven Bar-Yehuda, Dan Geiger, Joseph Naor, and Ron M. Roth. "Approximation algorithms for the feedback vertex set problem with applications to constraint satisfaction and Bayesian inference". *SIAM Journal on Computing*, 27(4):942–959, 1998. doi:10.1137/S0097539796305109.

[CAC+81] Gregory J. Chaitin, Marc A. Auslander, Ashok K. Chandra, John Cocke, Martin E. Hopkins, and Peter W. Markstein. "Register allocation via coloring". *Computer Languages*, 6(1):47–57, 1981. doi:10.1016/0096-0551(81)90048-5.

[Cou90] Bruno Courcelle. "The monadic second-order logic of graphs. I. Recognizable sets of finite graphs". *Information and Computation*, 85(1):12–75, 1990. doi:10.1016/0890-5401(90)90043-H.

[CZS+19] Chase Cook, Hengyang Zhao, Takashi Sato, Masayuki Hiromoto, and Sheldon X.-D. Tan. "GPU based parallel ising computing for combinatorial optimization problems in VLSI physical design", 2019. doi:10.48550/arXiv.1807.10750.

[DHHW22] Marc Distel, Robert Hickingbotham, Tony Huynh, and David R. Wood. "Improved product structure for graphs on surfaces". *Discrete Mathematics and Theoretical Computer Science*, 24(2), 2022. doi:10.48550/arXiv.2112.10025.

[DJM+20] Vida Dujmović, Gwenaël Joret, Piotr Micek, Pat Morin, Torsten Ueckerdt, and David R. Wood. "Planar graphs have bounded queue-number". *Journal of the ACM*, 67(4):1–38, 2020. doi:10.1145/3385731.

[DMW23] Vida Dujmović, Pat Morin, and David R. Wood. "Graph product structure for non-minor-closed classes". *Journal of Combinational Theory, Series B*, 162:34–67, 2023. doi:10.1016/j.jctb.2023.03.004.

[GJS76] Michael R. Garey, David S. Johnson, and Larry Stockmeyer. "Some simplified NP-complete graph problems". *Theoretical Computer Science*, 1:237–267, 1976. doi:10.1016/0304-3975(76)90059-1.

[GJU24] Miriam Goetze, Paul Jungeblut, and Torsten Ueckerdt. "Recognition complexity of subgraphs of k-connected planar cubic graphs", 2024. doi:10.48550/arXiv.2401.05892.

[GP95] Raymond Greenlaw and Rossella Petreschi. "Cubic graphs". *ACM Computing Surveys*, 27(4):471–495, 1995. doi:10.1145/234782.234783.

[GR17] Runa Ganguli and Siddhartha Roy. "A study on course timetable scheduling using graph coloring approach". *International Journal of Computational and Applied Mathematics*, 12(2):469–485, 2017.

[Gre89] Angelo Gregori. "Unit-length embedding of binary trees on a square grid". *Information Processing Letters*, 31(4):167–173, 1989. doi:10.1016/0020-0190(89)90118-X.

[Had75] Frank Hadlock. "Finding a maximum cut of a planar graph in polynomial time". *SIAM Journal on Computing*, 4(3):221–225, 1975. doi:10.1137/0204019.

[Hal80] William K. Hale. "Frequency assignment: Theory and applications". *Proceedings of the IEEE*, 68(12):1497–1514, 1980. doi:10.1109/PROC.1980.11899.

[Hol81] Ian Holyer. "The NP-completeness of edge-coloring". *SIAM Journal on Computing*, 10(4):718–720, 1981. doi:10.1137/0210055.

[Kar72] Richard M. Karp. "Reducibility among combinatorial problems". In Raymond E. Miller, James W. Thatcher, and Jean D. Bohlinger, editors, *Complexity of Computer Computations*, pages 85–103. Springer, Boston, MA, 1972. doi:10.1007/978-1-4684-2001-2_9.

[KUD14] Vedat Kavalci, Aybars Ural, and Orhan Dagdeviren. "Distributed vertex cover algorithms for wireless sensor networks", 2014. doi:10.48550/arXiv.1402.2140.

[MFKS13] Atsushi Mochizuki, Bernold Fiedler, Gen Kurosawa, and Daisuke Saito. "Dynamics and control at feedback vertex sets. II: A faithful monitor to determine the diversity of molecular activities in regulatory networks". *Journal of Theoretical Biology*, 335:130–146, 2013. doi:10.1016/j.jtbi.2013.06.009.

[Spe83] Ewald Speckenmeyer. "Untersuchungen zum Feedback Vertex Set Problem in ungerichteten Graphen". *Ph. D. thesis, Universität Paderborn*, 1983.

[Spe88] Ewald Speckenmeyer. "On feedback vertex sets and nonseparating independent sets in cubic graphs". *Journal of Graph Theory*, 12(3):405–412, 1988. doi:10.1002/jgt.3190120311.

[SZ01] Daniel P. Sanders and Yue Zhao. "Planar graphs of maximum degree seven are class I". *Journal of Combinatorial Theory, Series B*, 83(2):201–212, 2001. doi:10.1006/jctb.2001.2047.

[TTU12] Asahi Takaoka, Satoshi Tayu, and Shuichi Ueno. "On minimum feedback vertex sets in graphs". In *2012 Third International Conference on Networking and Computing (ICNC)*, pages 429–434, 2012. doi:10.1109/ICNC.2012.81.

[Tut47] William T. Tutte. "The factorization of linear graphs". *Journal of the London Mathematical Society*, 1(2):107–111, 1947. doi:10.1112/jlms/s1-22.2.107.

[UWY21] Torsten Ueckerdt, David R. Wood, and Wendy Yi. "An improved planar graph product structure theorem", 2021. doi:10.48550/arXiv.2108.00198.

[Viz64] Vadim G. Vizing. "On an estimate of the chromatic index of a ρ-graph". *Diskret. Analiz*, 3:25–30, 1964.

[Whi84] Sue H. Whitesides. "A method for solving certain graph recognition and optimization problems, with applications to perfect graphs". *North-Holland Mathematics Studies*, 88:281–297, 1984. doi:10.1016/S0304-0208(08)72941-4.