

Algorithms on the Strong Product of Graphs

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Abstract

The strong product of two graphs is a graph product that combines the Cartesian and tensor product of two graphs. In this thesis, we investigate the structure of the strong product of two graphs and how it can be exploited to derive solutions for graph problems based on the factor graphs. Specifically, we focus on three fundamental problems: the maximum independent set problem, the minimum dominating set problem, and, as the main focus of this thesis, the minimum feedback vertex set problem. We first analyze the maximum independent set problem and the minimum dominating set problem on the strong product of an arbitrary graph with a path, giving constructions for maximum independent sets and minimum dominating sets of the strong product based on independent sets and dominating sets of the factor graph. Then, we focus on the minimum feedback vertex set problem and investigate the strong product of paths, trees and different combinations of these graph types.

Zusammenfassung

Das starke Produkt von zwei Graphen ist ein Graphprodukt, das das kartesische und tensorielle Produkt von zwei Graphen kombiniert. In dieser Arbeit untersuchen wir die Struktur des starken Produkts von zwei Graphen und wie sie ausgenutzt werden kann, um Lösungen für grundlegende Graphprobleme basierend auf der Struktur der Faktorgraphen zu entwickeln. Insbesondere konzentrieren wir uns auf drei grundlegende Probleme: das Maximum Independent Set Problem, das Minimum Dominating Set Problem, und als Hauptfokus dieser Arbeit, das Minimum Feedback Vertex Set Problem. Zunächst analysieren wir das Maximum Independent Set Problem und das Minimum Dominating Set Problem auf dem starken Produkt eines beliebigen Graphen mit einem Pfad, indem wir Konstruktionen für maximale independent Sets und minimale dominating Sets des starken Produkts basierend auf independent Sets und dominating Sets des Faktorgraphen geben. Dann konzentrieren wir uns auf das Minimum Feedback Vertex Set Problem und untersuchen das starke Produkt von Pfaden, Bäumen und verschiedenen Kombinationen dieser Graphentypen.

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1 Introduction

Products of graphs are used to form complex graphs from simpler ones. Knowing the properties of the factor graphs can help to understand the structure of the product graph and to derive solutions for graph problems on the product graph. The *strong product* ($\boxtimes$) of two graphs forms a graph consisting of vertices for all pairs of vertices from the two factor graphs, and edges between vertices that are adjacent in either one or both of the factor graphs. A detailed definition of the strong product is given in Section 2.4.

The importance of the strong product of graphs has been highlighted by recent results in *product structure theory*, as it has been shown that graphs of relevant graph classes are subgraphs of the strong product of simpler graphs. One example of such a class is planar graphs, which are subgraphs of the strong product of a graph of treewidth at most 6 and a path [Duj+20|UWY22]. For them, a decomposition into the factor graphs can even be computed in polynomial time [Mor21].

Motivated by these results and the fact that graphs of some complex graph classes can be embedded into the strong product of simpler graphs, in this thesis we focus on the fundamental properties of the strong product itself. Specifically, we focus on graphs of the form $G = H_1 \boxtimes H_2$, where H_1 and H_2 are simpler graphs, for example paths or trees. The goal of this thesis is to understand if and how the structure of graphs formed by the strong product of simpler graphs can be exploited to derive solutions for graph problems on G based on solutions for the same problem or other problems on the factor graphs. To achieve this, we investigate three fundamental graph problems that are NP-hard on general graphs. The *maximum independent set problem* asks for a maximum set of vertices such that no two vertices in the set are adjacent. The *minimum dominating set problem* asks for a minimum set of vertices such that every vertex in the graph is either in the set or adjacent to a vertex in the set. Lastly and most importantly, the *minimum feedback vertex set problem* asks for a minimum set of vertices such that the graph obtained by removing these vertices and all edges incident to them is acyclic.

We show that for graphs of the form $G = H \boxtimes P_\ell$, where H is an arbitrary graph and P_ℓ is a path, for some problems we can derive optimal solutions for G directly from optimal solutions of H . For the maximum independent set problem and the minimum dominating set problem we rely on the factor's solution of the same problem to construct an optimal solution for G .

The main focus and most interesting problem of this thesis is the minimum feedback vertex set problem on the strong product of graphs of different well-known graph classes, such as paths and trees. Unlike independent sets and dominating sets, this problem does not allow for a direct, optimal construction of a solution for G from the feedback vertex sets of the factor graphs. Similar to the other problems, this problem has been studied in the past for the Cartesian product of graphs, for example on grids [BV97], but there are not many results for the strong product of graphs. To address this, we study the structure of cycles in the strong product graphs and derive solutions for G based on matchings and vertex covers of the factor graphs.

1.1 Related Work

In this section we give an overview of related work on graph products and the investigated graph problems.

1.1.1 Work on Different Graph Products

A lot of research has been done on the structure of Cartesian products of graphs, mostly because of its natural construction and connection to grid networks. For grid graphs, defined as the Cartesian product of two paths, $P_n \square P_m$, the maximum independent set is straightforward to determine since they are bipartite, but the minimum dominating set and the minimum feedback vertex set are more complex and have been investigated in the past. The exact size of the minimum dominating set on grid graphs was first proposed as a conjecture by Chang and Clark who show formulas on specific cases of grids [Cha92 | CC93 | CCH94] and later Gonçalves, Pinlou, Rao, and Thomassé [GPRT11] confirm that for every $16 \leq n \leq m$, the domination number of $P_n \square P_m$ is exactly $\gamma(P_n \square P_m) = \lfloor \frac{(n+2)(m+2)}{5} \rfloor - 4$. The most important motivation for the study of domination on Cartesian products is Vizing's conjecture, which states that $\gamma(G \square H) \geq \gamma(G)\gamma(H)$ for all graphs G and H [Viz68]. While this conjecture is still open, the best known bound is $\gamma(G \square H) \geq \frac{1}{2}\gamma(G)\gamma(H)$, which Clark and Suen prove [CS00]. For chordal graphs, which are graphs that do not contain induced cycles of length at least 4, the conjecture holds [AS09]. Further investigations include results on Cartesian products of cycles by Klavžar and Seifter [KS95] and paths with cycles by Pavlic and Zerovnik [PZ13]. The independence number of Cartesian products that are not grids was investigated by Abay-Asmerom, Hammack, Larson, and Taylor. They show that for the radius $r(G)$ of a graph G it holds that $\alpha(G \square H) \geq 2r(G)r(H)$ and that this bound is tight for specific graphs [AHLT11]. The feedback vertex number of $P_n \square P_m$, i.e., the size of a minimum feedback vertex set of the Cartesian product of two paths, was shown to be at least $\frac{(n-1)(m-1)+1}{3}$ by Luccio [Luc+98]. Later Lien, Fu and Shih [LFS14] show $\lceil \frac{(n-1)(m-1)+1}{3} \rceil \leq \nabla(P_n \square P_m) \leq \lceil \frac{(n-1)(m-1)+1}{3} \rceil + 1$ and some exact values for specific n and m , where $\nabla(G)$ denotes the feedback vertex number of a graph G . Pike and Zou show an exact formula for the feedback vertex number of the Cartesian product of two cycles, $C_n \square C_m$ in [PZ05]. More recent work on the feedback vertex set includes results by Ghalavand, Klavžar, and Yang [GKY25], where they show that if T and T' are trees with n and n' vertices, and S_n and $S_{n'}$ are stars with n and n' vertices, then $\nabla(T \square T') \geq \nabla(S_n \square S_{n'})$.

1.1.2 Work on the Strong Product

The most important motivation for the study of strong products is recent results in graph product structure theory, introduced by Dujmović, Joret, Micek, Morin, Ueckerdt, and Wood in 2020 [Duj+20] and further developed by others [UWY22 | ISW24], which show that complicated graph classes, such as planar graphs or K_t -minor-free graphs, can be structurally decomposed into subgraphs of the strong product of simpler graphs. Planar graphs, for example, are subgraphs of the strong product of a graph with treewidth at most 6 and a path [UWY22] and K_t -minor-free graphs are subgraphs of the strong product of a graph with treewidth $\text{tw}(H) \leq t-2$ and a complete graph of size $\text{tw}(G)+1$ [ISW24]. Additionally, bounded treewidth graphs in general are subgraphs of the strong product of a graph with smaller treewidth and a bounded-size complete graph [Cam+24]. A lot more results on the product structure of various graph classes have been shown in the past years [DMW23 | Dvo+21 | HJMW24 | DW23].

The problem of finding maximum independent sets on the strong product of graphs has roots in information theory. The Shannon capacity of a graph, introduced by Claude Shannon in 1956 to determine the zero-error capacity of a noisy communication channel [Sha56], is defined using the independence number α of the strong product of a graph with itself over multiple powers, expressed as $\Theta(G) = \sup_k \sqrt[k]{\alpha(G^k)}$, with G^k being the strong product of G with itself k times. As the computational complexity of the Shannon capacity is not known, studying the behavior of the independent set on strong products is of great interest. The most famous result in this area is the Lovász theta function, which provides a polynomially computable upper bound on the Shannon capacity and was used by Lovász to show that $\Theta(C_5) = \sqrt{5}$, where C_5 is the cycle on 5 vertices [Lov79]. This result shows that the independence number of the strong product of C_5 with itself is $\alpha(C_5 \boxtimes C_5) = 5$. The minimum dominating set problem on strong grids has been investigated by Thangam, Venkataraman and Thinakaran [TVT25], but variations of the problem, such as the total domination number, are a lot more prevalent in the literature.

Not only fundamental structural properties of graphs have been investigated on the strong product of graphs, but also game-related problems, such as the cop number of a graph, which is the minimum number of cops needed to catch a robber in a game played on the vertices of a graph. The cop number of the strong product of two graphs was investigated by Nowakowski and Winkler [NN98], where they show that for the strong product of two graphs G and H , the cop number $c(G \boxtimes H)$ is at most $c(G) + c(H) + 1$. They also investigate the Cartesian product of different graphs.

The feedback vertex set is generally less covered for the strong product. A notable exception is the work by Xie [Xie07], who investigates the feedback vertex number on the strong product of two paths and Xie, Chen and Lai [XCL14], who analyze the feedback vertex number on strong products of cycles. This still leaves more room for investigation towards more combinations of graph classes and a deeper understanding of the structure of feedback vertex sets on strong products, which is the main focus of this thesis.

1.2 Outline and Contribution

We start with some preliminaries on graph theory and the definition of the strong product in Chapter 2. As an introduction to the topic, in Chapter 3 and Chapter 4 we investigate the maximum independent set and the minimum dominating set on graphs of the form $G = H \boxtimes P_\ell$ for an arbitrary graph H and a path P_ℓ . We show in Theorem 3.2 that for the maximum independent set problem for G , it holds that the independence number is $\alpha(G) = \alpha(H) \cdot \lceil \frac{\ell}{2} \rceil$ and that there exists a maximum independent set of G that consists of all vertices corresponding to vertices in a maximum independent set of H and P_ℓ .

For the minimum dominating set problem for G , similar to the maximum independent set problem, we show that there exists a minimum dominating set of G that consists of the copies of a minimum dominating set of H , with the size $\gamma(G) = \gamma(H) \cdot \lceil \frac{\ell}{3} \rceil$ in Theorem 4.2.

The main contribution of this thesis is the investigation of the minimum feedback vertex set problem on the strong product of graphs from different, simple graph classes. In Chapter 5 we first give two constructions of a feedback vertex set of $G = H \boxtimes P_\ell$. We then compare the two constructions and show in Corollary 5.2 that for $G = H \boxtimes P_{2k}$ a feedback vertex set based on a minimum vertex cover of H is always smaller or equal to the other construction.

After this first investigation, we focus on the minimum feedback vertex set problem on the strong product of two paths, $G = P_a \boxtimes P_b$, also known as the *strong grid* graph. An exact formula for the feedback vertex number of G was previously established by Xie [Xie07]. We independently derive the same exact formula using an alternative proof, which is based on local K_4 subgraphs of G and provides a different structural perspective in Theorem 5.6, Theorem 5.7 and Theorem 5.10. We show that there always exists a minimum feedback vertex set of G that consists of either all vertices on every second row or all vertices on every second column of the grid.

Following this, we investigate the minimum feedback vertex set problem on the strong product of a tree and a path, $G = T \boxtimes P_\ell$. Here, we learn that if one of the two factors has a perfect matching, we can derive a minimum feedback vertex set of G directly with a minimum vertex cover of the other factor. We use this to also derive exact formulas for the feedback vertex number of the strong product of two trees, $G = T_1 \boxtimes T_2$ when one of the two trees has a perfect matching.

To further analyze the strong product of a tree and a path, where neither of the two factors has a perfect matching, we investigate the subclass *caterpillars* of trees and show that for a strong product $G = T \boxtimes P$ of a caterpillar T and a path P , we can derive a minimum feedback vertex set of G based on the structure of the caterpillar in Theorem 5.17. There we cannot give a closed formula for the feedback vertex number of G , but we give a recursive algorithm with linear running time in the size of T to compute a minimum feedback vertex set of G .

We then show that a comparable odd subtree structure also exists for general trees with a single unmatched vertex, that allows us to give a lower bound in Theorem 5.21 applicable to the size of a minimum feedback vertex set of $G = T \boxtimes P$ for trees T with a single unmatched vertex. Using Conjecture 5.23 on the structure of minimum feedback vertex sets of the strong product of a tree and a path, we give a linear time dynamic programming algorithm to solve the minimum feedback vertex set problem for $G = T \boxtimes P$ for general trees T and paths P in Section 5.3.3.

As a final contribution, we investigate the minimum feedback vertex set problem on the strong product of more than two paths in Section 5.5 and Section 5.6, as a generalization of the strong grid graph in higher dimensions, i.e., on graphs of the form $G = P_{x_1} \boxtimes P_{x_2} \boxtimes \cdots \boxtimes P_{x_d}$ with $x_1, \dots, x_d \in \mathbb{N}$. We show exact formulas for the strong product of three paths, as long as at least one of the paths has an even number of vertices in Theorems 5.30 to 5.32 and for the strong product of d paths, as long as at least one of the paths has an even number of vertices in Theorem 5.37, we give a recursive formula for the feedback vertex number of G .

Finally, we give an outlook on open problems and potential future research directions in Chapter 6.

2 Preliminaries

In this chapter we give an introduction to the necessary graph theoretic concepts and definitions that are used in this thesis. First, in Section 2.1 we give some basic definitions and properties of graphs. Then, in Section 2.2 we introduce some specific graph classes that we refer to in the following chapters. In Section 2.3 we introduce the graph problems that we investigate in this work and finally in Section 2.4 we give the definitions of graph products and in particular the strong product, which is the main focus of this work.

2.1 Graphs

Let $G = (V, E)$ be a graph with vertex set V and edge set E . We only consider simple, undirected and finite graphs in this work, so V is a finite set of vertices and E is a subset of $\binom{V}{2}$. To refer to the vertex set and edge set of a graph G , we write $V(G)$ and $E(G)$, respectively. For edges, we write uv or vu to denote the edge between the vertices u and v . Given a subset of vertices $S \subseteq V(G)$, the *induced subgraph* of G on S is the graph with vertex set S and edge set $E(G) \cap \binom{S}{2}$ and is denoted as $G[S]$. A *cycle* is a sequence of vertices $v_1, v_2, \dots, v_k$ with $k \geq 3$ such that $v_i v_{i+1} \in E(G)$ for $i \in [k-1]$ and $v_k v_1 \in E(G)$. A graph is called *acyclic* if it does not contain any cycles. Two graphs G_1 and G_2 are called *isomorphic* if there exists a bijection $f : V(G_1) \rightarrow V(G_2)$ such that for every pair of vertices $u, v \in V(G_1)$, it holds that $uv \in E(G_1)$ if and only if $f(u)f(v) \in E(G_2)$. We use the notation $G_1 \cong G_2$ to denote that G_1 and G_2 are isomorphic. Given a graph G and a vertex $v \in V(G)$, we write $G - v$ to denote the graph obtained from G by removing the vertex v and all edges incident to v . Given a graph G and a subset of vertices $S \subseteq V(G)$, we write $G - S$ to denote the graph obtained from G by removing all vertices in S and all edges incident to any vertex in S .

2.2 Graph Classes

We introduce some specific graph classes that we refer to in the following chapters. A *path* P_n is a graph with vertex set $\{p_1, p_2, \dots, p_n\}$ and edge set $\{p_i p_{i+1} \mid i \in [n-1]\}$, where $[n]$ denotes the set $\{1, 2, \dots, n\}$. We call a path with an even number of vertices an *even path* and a path with an odd number of vertices an *odd path*. A *complete graph* K_n is a graph with vertex set $\{v_1, v_2, \dots, v_n\}$ and edge set $\{v_i v_j \mid i, j \in [n], i \neq j\}$ where every pair of distinct vertices is connected by an edge. A *tree* is a connected acyclic graph and a *forest* is a graph whose connected components are trees. A *caterpillar* is a tree such that the removal of all leaves results in a path. For caterpillars, we refer to a maximal path in the caterpillar as a *spine* of the caterpillar and to the vertices on the spine as *spine vertices*.

2.3 Graph Problems

In this thesis we investigate three fundamental graph problems, the *Maximum Independent Set (MIS) Problem*, the *Minimum Dominating Set (MDS) Problem* and the *Minimum Feedback Vertex Set (MFVS) Problem*. An *independent set* of a graph G is a set of vertices $I \subseteq V(G)$ such that for every pair of distinct vertices $u, v \in I$, it holds that $uv \notin E(G)$. The *independence number* $\alpha(G)$ of a graph G is the size of a maximum independent set of G . A *dominating set* is a set of vertices $D \subseteq V(G)$ such that for every vertex $v \in V(G)$, either $v \in D$ or there exists a vertex $u \in D$ such that $uv \in E(G)$. The *domination number* $\gamma(G)$ of a graph G is the size of a minimum dominating set of G . Finally, a *feedback vertex set* of a graph G is a set of vertices $F \subseteq V(G)$ such that the graph obtained from G by removing all vertices in F and all edges incident to any vertex in F , i.e. $G \setminus F$, is acyclic. We call the size of a minimum feedback vertex set of G the *feedback vertex number* and we denote it as $\nabla(G)$. Finding a maximum independent set, a minimum dominating set or a minimum feedback vertex set of a graph is NP-hard in general.

Some other graph problems that we refer to in the following chapters are the *vertex cover* problem and the *matching* problem. A *vertex cover* of a graph G is a set of vertices $C \subseteq V(G)$ such that for every edge $uv \in E(G)$, at least one of the endpoints u or v is in C . The *vertex cover number* $\tau(G)$ of a graph G is the size of a minimum vertex cover of G . The complement of a vertex cover is an independent set and $\tau(G) = |V(G)| - \alpha(G)$ for every graph G . A *matching* of a graph G is a set of edges $M \subseteq E(G)$ such that for every pair of distinct edges $e_1 \neq e_2$ in M , it holds that e_1 and e_2 do not share a common endpoint. Considering the vertices covered by the edges in a matching M , we can also refer to M as a set of vertices, which we call the *vertex set* of the matching. A *maximum matching* of a graph G is a matching of maximum size and a *perfect matching* of a graph G is a matching that covers every vertex of G exactly once. According to König's theorem, in bipartite graphs the size of a maximum matching is equal to the size of a minimum vertex cover [Kön31].

2.4 Graph Products

To define the strong product of two graphs, we first introduce the Cartesian product and the tensor product of two graphs. Given two graphs $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$, their *Cartesian product* $G = H_1 \square H_2$ is defined as follows. The vertex set of G is the Cartesian product of the two vertex sets, $V(G) = V_1 \times V_2$. Two vertices (u_1, v_1) and (u_2, v_2) in G are adjacent if and only if either $u_1 = u_2$ and $v_1v_2 \in E_2$, or $v_1 = v_2$ and $u_1u_2 \in E_1$. Given two graphs $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$, their *tensor product* $G = H_1 \times H_2$ has the same vertex set as the Cartesian product, $V(G) = V_1 \times V_2$, but two vertices (u_1, v_1) and (u_2, v_2) in G are adjacent if and only if $u_1u_2 \in E_1$ and $v_1v_2 \in E_2$.

Finally, the *strong product* of two graphs H_1 and H_2 , denoted as $G = H_1 \boxtimes H_2$, shares the same vertex set as the Cartesian and tensor product, $V(G) = V_1 \times V_2$. Its edge set is the union of the edge sets of the Cartesian and tensor product, so two vertices (u_1, v_1) and (u_2, v_2) in G are adjacent if and only if either $u_1 = u_2$ and $v_1v_2 \in E_2$, or $v_1 = v_2$ and $u_1u_2 \in E_1$, or $u_1u_2 \in E_1$ and $v_1v_2 \in E_2$. Similar to both other products, the strong product is commutative and associative. An example for all graph products, including $G = H \boxtimes P_\ell$ for an arbitrary graph H , is shown in Figure 2.1.

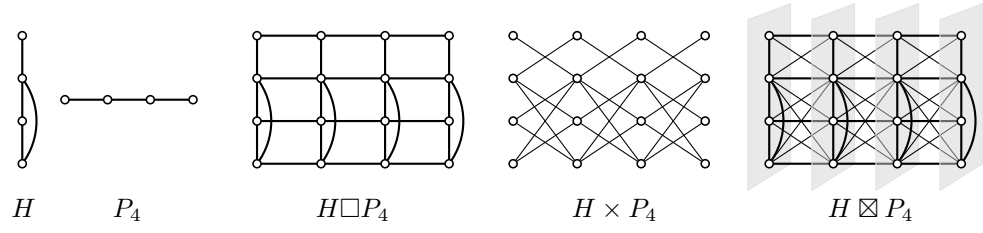


Figure 2.1: An example graph H , a path P_4 and from left to right, the Cartesian product $H \square P_4$, the tensor product $H \times P_4$ and the strong product $H \boxtimes P_4$ of the two graphs. The layers in the strong product graph are highlighted in gray.

In the following chapters, we often consider graphs of the form $G = H \boxtimes P_\ell$ for an arbitrary graph H and a path P_ℓ . The structure of this graph consists of ℓ copies of H and we refer to the induced subgraph of the vertices $(V(H) \times \{p_i\})$ corresponding to the i -th vertex p_i of P_ℓ as the i -th *layer* of G . This layer can also be denoted as $H \boxtimes \{p_i\}$, or $H \square \{p_i\}$. For a vertex $v \in V(H)$, we call the vertex $(v, p_i) \in V(G)$ the *copy* of vertex v in layer i . We refer to the edges between vertices on the same layer as *intra-layer edges* or edges contained in a layer and the edges from the tensor product as *tensor edges*.

3 Maximum Independent Set

3.1 Maximum Independent Set

As a first problem, we look at the maximum independent set problem on strong products of the form $G = H \boxtimes P_\ell$ for an arbitrary graph H . It is a well studied property on graphs in general and we show that the structure of the strong product allows us to derive a maximum independent set of G directly from a maximum independent set of H .

We start by considering the subgraph induced by two consecutive layers of G , which is isomorphic to $H \boxtimes P_2$.

Lemma 3.1: *Let $G = H \boxtimes P_2$ and H be an arbitrary graph. Then, it holds that $\alpha(G) = \alpha(H)$.*

Proof. Let $G = H \boxtimes P_2$ with an arbitrary graph H . Assume that there exists an independent set $I \subseteq V(G)$ of G with $|I| > \alpha(H)$. By the definition of the independent set, it holds for each pair of distinct vertices $v_i, v_j \in I$ that $v_i v_j \notin E(G)$. Following the definition of the strong product, for every vertex $v \in V(H)$ and a vertex p_i of the path, we get a corresponding vertex in G , so $(v, p_1) \in (V(H) \times \{p_1\})$ and $(v, p_2) \in (V(H) \times \{p_2\})$ are the copies of v in layer 1 and layer 2, respectively. These two vertices are connected by an edge in G . In addition, for every edge $uv \in E(H)$ there are edges $(u, p_1)(v, p_1)$ and $(u, p_2)(v, p_2)$ in G , as well as the tensor edges $(u, p_1)(v, p_2)$ and $(v, p_1)(u, p_2)$. The direct connection $(v, p_1)(v, p_2) \in E(G)$ implies that for every vertex $v \in V(H)$, either the vertex (v, p_1) or its copy (v, p_2) in layer 2 is not part of the independent set I . Since (v, p_1) and (v, p_2) share the same neighborhood in $G - \{(v, p_1), (v, p_2)\}$, we can arbitrarily decide whether to include (v, p_1) or (v, p_2) in the independent set I . Using this, by selecting the layer i vertex for every $v \in I$ we get an independent set I' , solely on layer i consisting of the vertices $(V(H) \times \{p_i\})$. This results in an independent set on the Graph H with size strictly larger than $\alpha(H)$ which is a contradiction. Therefore, there cannot exist an independent set of G with size larger than $\alpha(H)$ and since H is an induced subgraph of G , there exists an independent set of G with size at least $\alpha(H)$, so it holds that $\alpha(G) = \alpha(H)$. $\blacksquare$

This lemma implies that for any two consecutive layers of $G = H \boxtimes P_\ell$ there can only ever be an independent set of size at most $\alpha(H)$, since the two layers together are isomorphic to $H \boxtimes P_2$. We now use this observation to show that there is a minimum independent set of G , which consists of the copies of a maximum independent set of H on every odd layer of G . This is illustrated in Figure 3.1 for an example graph H and $G = H \boxtimes P_5$.

Theorem 3.2: *Let I_H be a maximum independent set of a graph H and $G = H \boxtimes P_\ell$ be the strong product of H with a path of ℓ vertices. Then, there exists a maximum independent set I_G of G with size $\alpha(G) = \lceil \frac{\ell}{2} \rceil \cdot \alpha(H)$.*

Proof. First we show that there exists an independent set of G with size $\alpha(G) = \lceil \frac{\ell}{2} \rceil \cdot \alpha(H)$.

We define the set $I_G = \{(v, p_i) \mid v \in I_H, i \in [\ell], i \text{ is odd}\}$ as the set of all copies of I_H on every odd layer of G .

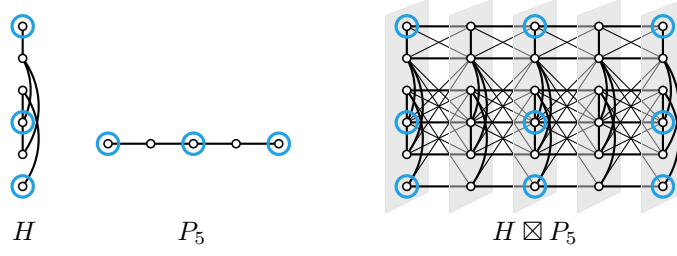


Figure 3.1: $G = H \boxtimes P_5$ for an example graph H . The maximum independent sets of H , P_5 and G are highlighted in blue.

If $\ell = 1$, G is isomorphic to H and the set I_G is exactly the maximum independent set of G with size $\alpha(H) = \lceil \frac{1}{2} \rceil \cdot \alpha(H)$.

Now, let $\ell > 1$. To show that I_G is indeed an independent set of G , we note that any vertex $v \in V(G)$ on a layer $H \square \{p_i\}$ is only adjacent to vertices on the same layer $H \square \{p_i\}$ and the neighboring layers $H \square \{p_{i-1}\}$ and $H \square \{p_{i+1}\}$ for $i \in [\ell - 1], i > 1$. All edges between vertices on the same layer $H \square \{p_i\}$ correspond to edges in H and since I_H is an independent set of H , there are no edges between any two vertices $v, w \in I_G$ on the same layer $v, w \in V(H \square \{p_i\})$.

Any two vertices (v, p_i) and (w, p_j) on different layers $H \square \{p_i\}$ and $H \square \{p_j\}$ with $|i - j| > 1$ are not adjacent in G , since by the definition of the strong product, (v, p_i) and (w, p_j) are only adjacent if $i = j$ or $|i - j| = 1$ and $vw \in E(H)$.

Since I_G only contains vertices (v, p_i) with odd i , there are also no edges between any two vertices in I_G that are not on the same layer.

To show that I_G is a maximum independent set of G , we partition the layers of G into $\lfloor \frac{\ell}{2} \rfloor$ pairs of consecutive layers $(H \square \{p_1\}, H \square \{p_2\})$, $(H \square \{p_3\}, H \square \{p_4\})$, $\dots$, and, if ℓ is odd, a single remaining layer $H \square \{p_\ell\}$. If ℓ is even, G consists of exactly $\frac{\ell}{2}$ such disjoint pairs. From Lemma 3.1, we know that for any pair of consecutive layers, any independent set can contain at most $\alpha(H)$ vertices. Summing up this upper bound for each pair, the maximum size of an independent set in G with even ℓ is at most $(\frac{\ell}{2}) \cdot \alpha(H)$. The constructed set I_G with $\alpha(H)$ on each odd layer achieves exactly this size, so it is optimal.

If ℓ is odd, G consists of $(\ell - 1)/2$ pairs and one single layer. The pairs contribute at most $(\frac{\ell-1}{2}) \cdot \alpha(H)$ vertices. The last remaining layer $H \square \{p_\ell\}$ is again isomorphic to H , so it can contribute at most $\alpha(H)$ vertices to the independent set. Since $p_{\ell-1}$ is even, $H \square \{p_{\ell-1}\}$ is not part of I_G and thus does not contain any vertices of I_G . Therefore, the last layer $H \square \{p_\ell\}$ can contribute at most $\alpha(H)$ vertices to the independent set without violating the independence property of I_G . Thus, the total size of an independent set in G with odd ℓ is at most $(\frac{\ell-1}{2}) \cdot \alpha(H) + \alpha(H) = \lceil \frac{\ell}{2} \rceil \cdot \alpha(H)$.

The set I_G with $\alpha(H)$ vertices on each odd layer achieves exactly this size, so it is optimal. ■

Note that this set $I_G = \{(v, p_i) \mid v \in I_H, i \in [\ell], i \text{ is odd}\}$ matches the set $I_H \times I_{P_\ell}$ for a maximum independent set I_H of H and I_{P_ℓ} of the path P_ℓ . This means that the maximum independent set of G is a direct application of the maximum independent sets of the factor graphs H and P_ℓ .

It is known that the property of $\alpha(G) = \alpha(H) \cdot \alpha(P_\ell)$ does not hold for the strong product of two arbitrary graphs H_1 and H_2 , as the example of $C_5 \boxtimes C_5$ was shown to have an independence number of 5 while $\alpha(C_5) = 2$ through results of Shannon, and Lovász [Sha56 | Lov79].

3.2 Connection to Minimum Vertex Cover

As a direct consequence of the relation between the independence number and the vertex cover number of a graph, since a maximum independent set of a graph G is the complement of a minimum vertex cover of G , we can also derive a minimum vertex cover of $G = H \boxtimes P_\ell$ directly from a minimum vertex cover of H .

In this case, for a minimum vertex cover C_H of H and a minimum vertex cover C_{P_ℓ} of P_ℓ , by the complement relation, a minimum vertex cover of G is given by $C_G = C_{P_\ell} \times C_H \cup (V(P_\ell) \setminus C_{P_\ell}) \times C_H$. For at least $\ell \geq 2$, this is also always a *connected vertex cover* of G , which is another known problem connected to vertex covers, where the vertex cover is required to induce a connected subgraph of G .

4 Minimum Dominating Set

In this chapter, we analyze the minimum dominating set problem on the strong product of an arbitrary graph H and a path P_ℓ . We show that we can construct a minimum dominating set of $G = H \boxtimes P_\ell$ efficiently by strategically combining copies of a minimum dominating set of H .

Similarly to the approach for the maximum independent set problem, we first look at a product graph $G = H \boxtimes P_3$ consisting of three layers and show that the copy of a minimum dominating set of H on the middle layer of G is a minimum dominating set of G .

Lemma 4.1: *Let H be an arbitrary graph and let $G = H \boxtimes P_3$. Let $D_H \subseteq V(H)$ be a minimum dominating set of H . Then, $D_H \times \{p_2\}$ is a minimum dominating set of G .*

Proof. Let D_H be a minimum dominating set of H and define $D = D_H \times \{p_2\}$ as the set containing the copies of vertices in D_H on the second layer of G . It follows that D is a minimum dominating set of $H \boxtimes \{p_2\}$. By definition, for every vertex $(v, p_2) \in V(H \boxtimes \{p_2\})$, either $(v, p_2) \in D$ or there is a neighbor $(u, p_2) \in D$ such that $(v, p_2)(u, p_2) \in E(H \boxtimes \{p_2\})$. It holds that for the vertices $(v, p_1) \in V(H \boxtimes \{p_1\})$, either their copy (v, p_2) is in D or there is a neighbor of (v, p_2) , $(u, p_2) \in D$, so by the definition of the strong product, (v, p_1) is adjacent to either (v, p_2) or (u, p_2) and thus dominated by D . An analogous argument holds for the vertices on layer 3. Therefore, D is a dominating set of G . To see that D is a minimum dominating set of G , we note that any dominating set of G must contain at least $\gamma(H)$ vertices to dominate the middle layer, as all layers are isomorphic to H and a vertex on the middle layer is only connected to copies of its neighbors on the same layer and the neighboring layers. Since D contains exactly $\gamma(H)$ vertices, it is a minimum dominating set of G . ■

We use this lemma to show that for any number of layers ℓ , there is a minimum dominating set of $G = H \boxtimes P_\ell$, which consists of the copies of a minimum dominating set of H on every third layer of G starting from layer 2. This is illustrated in Figure 4.1 for an example graph H and $G = H \boxtimes P_6$.

Theorem 4.2: *Let H be an arbitrary graph and let $G = H \boxtimes P_\ell$. Then, G has a minimum dominating set of size $\lceil \frac{\ell}{3} \rceil \cdot \gamma(H)$.*

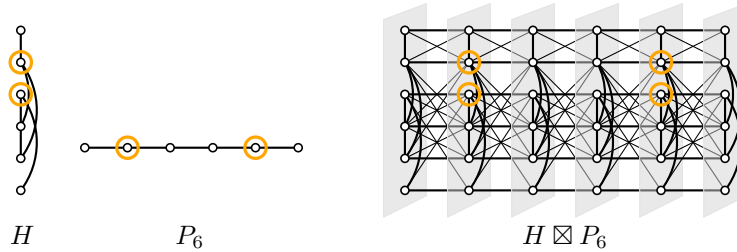


Figure 4.1: $G = H \boxtimes P_6$ for an example graph H . The minimum dominating sets of H , P_6 and G are highlighted in orange.

Proof. Let D_H be a minimum dominating set of H , with size $\gamma(H)$. If $\ell \leq 2$, then either $\ell = 1$ and $G \cong H$, making D_H a minimum dominating set for G , or $\ell = 2$ and by the same argument as in Lemma 4.1, $D_H \times \{p_1\}$ is a minimum dominating set of G . In both cases, the size is $\gamma(H) = \lceil \frac{\ell}{3} \rceil \cdot \gamma(H)$.

First, we show that there exists a dominating set of G with size $\lceil \frac{\ell}{3} \rceil \cdot \gamma(H)$. For $\ell \geq 3$, we show that a dominating set D_G for G exists, containing only copies of D_H on specific layers. Let $D_j = \{(v, p_j) \mid v \in D_H\}$ denote the copy of D_H on layer j . It follows from Lemma 4.1, that a copy D_j on an inner layer j dominates the layers $j - 1$, j , and $j + 1$. Let $D' = \{D_j \mid j \equiv 2 \pmod{3}, j \leq \ell\}$ be the set of copies of D_H on every third layer starting from layer 2.

In the case $\ell \equiv 0 \pmod{3}$, we partition the layers into $\ell/3$ groups of three consecutive layers. As each part $G_i := G[V(H \square \{p_{i-1}\}) \cup V(H \square \{p_i\}) \cup V(H \square \{p_{i+1}\})]$ is isomorphic to $H \boxtimes P_3$, and $D' \cap V(G_i) = D_i$, by Lemma 4.1, D' dominates each partition and thus the whole graph G . In the case $\ell \equiv 2 \pmod{3}$, we use the same partitioning except for the last part $G[V(H \square \{p_{\ell-1}\}) \cup V(H \square \{p_\ell\})]$ of two layers. Since D' contains D_ℓ it again dominates all parts and thus the whole graph G . In the case $\ell \equiv 1 \pmod{3}$, let D_ℓ be the copy of D_H on the last layer ℓ . As D' dominates $G - V(H \square \{p_\ell\})$ and D_ℓ dominates $H \square \{p_\ell\}$, the set $D' \cup D_\ell$ dominates the whole graph G .

Let D_G be D' for $\ell \equiv 0, 2 \pmod{3}$ and $D' \cup D_\ell$ for $\ell \equiv 1 \pmod{3}$. We note that in all cases, D_G contains $\lceil \frac{\ell}{3} \rceil \cdot \gamma(H)$ vertices, since it contains $\lceil \frac{\ell}{3} \rceil$ copies of D_H .

We now show the lower bound, by showing that the set D_G is a minimum dominating set of G . To show that D_G is a minimum dominating set of G , we note that for any three consecutive layers of G , by Lemma 4.1, a dominating set must contain at least $\gamma(H)$ vertices to dominate the middle layer, which is isomorphic to H . Thus, for $\ell \equiv 0 \pmod{3}$, D_G is optimal.

For the remaining cases $\ell \equiv 1 \pmod{3}$ and $\ell \equiv 2 \pmod{3}$, we use induction on ℓ to show that D_G is optimal. For the first base case $\ell = 4$, we know that at least $\gamma(H)$ vertices are necessary to dominate $H \square \{p_1\}$ and at least $\gamma(H)$ vertices are necessary to dominate $H \square \{p_4\}$. Since $H \square \{p_1\}$ and $H \square \{p_4\}$ do not share any neighbors, at least $2 \cdot \gamma(H)$ vertices are necessary to dominate G . For the second base case $\ell = 5$, in the same way, to dominate $H \square \{p_1\}$ and $H \square \{p_5\}$, the dominating set must contain at least $2 \cdot \gamma(H)$ vertices. The set D_G is valid and contains $2 \cdot \gamma(H)$ vertices, so it is optimal.

For the inductive step, assume that D_G is optimal for all $\ell < k$ where $k > 5$ and $\ell \equiv 1 \pmod{3}$ or $\ell \equiv 2 \pmod{3}$. For $\ell = k$, we look at the last three layers of G and note that at least $\gamma(H)$ vertices are necessary to dominate the last layer. Since $H \square \{p_{\ell-3}\}$ and $H \square \{p_\ell\}$ do not share any neighbors, the vertices dominating $H \square \{p_\ell\}$ cannot be adjacent to any vertex on layer $H \square \{p_{\ell-3}\}$ and thus do not contribute to dominating the layers $H \square \{p_1\}, H \square \{p_2\}, \dots, H \square \{p_{\ell-3}\}$. Therefore, we reduce the problem to dominating $G' = H \boxtimes P_{\ell-3}$, which by the inductive hypothesis has a minimum dominating set of size $\lceil \frac{\ell-3}{3} \rceil \cdot \gamma(H)$. As $\lceil \frac{\ell-3}{3} \rceil \cdot \gamma(H) + \gamma(H) = \lceil \frac{\ell}{3} \rceil \cdot \gamma(H)$, the set D_G is optimal for $\ell = k$ as well.

Therefore, the set D_G is a minimum dominating set of G , and $\gamma(G) = \lceil \frac{\ell}{3} \rceil \cdot \gamma(H)$. $\blacksquare$

Similar to the maximum independent set problem, note that the minimum dominating set of G is a direct application of the minimum dominating sets of the factor graphs H and P_ℓ . For a minimum dominating set D_H of H and a minimum dominating set D_{P_ℓ} of P_ℓ , a minimum dominating set of G is given by $D_G = D_H \times D_{P_\ell}$. This is a consequence of the fact that for the path P_ℓ , there is a minimum dominating set containing only vertices on every third layer starting from layer 2.

5 Minimum Feedback Vertex Set

This chapter is dedicated to the minimum feedback vertex set problem and forms the main part of this thesis. First, we show two general constructions for a feedback vertex set of a graph $G = H \boxtimes P_\ell$, then we analyze the strong product of specific graph types.

5.1 Two General Feedback Vertex Set Constructions

Before we show the two constructions for a feedback vertex set of $G = H \boxtimes P_\ell$, we want to give some intuition on the structure of cycles in G .

The edges of G can be partitioned into three types. First, the intra-layer edges $(u, p_i)(v, p_i)$ with $uv \in E(H)$ and $i \in [\ell]$, which are the edges contained in the layers of G . Second, the layer-connecting edges, $(u, p_i)(u, p_{i+1})$ connecting a vertex to its copy on the neighboring layer for every $u \in V(H)$ and $i \in [\ell - 1]$. And third, the tensor edges, $(u, p_i)(v, p_{i+1})$ and $(v, p_i)(u, p_{i+1})$ connecting neighbors in H across neighboring layers for every edge $uv \in E(H)$ and $i \in [\ell - 1]$.

For each edge uv in H and two adjacent vertices p_i and p_{i+1} in P_ℓ , the four vertices (u, p_i) , (v, p_i) , (u, p_{i+1}) and (v, p_{i+1}) in G are pairwise adjacent, and therefore induce a K_4 in G .

5.1.1 Alternating Layer Construction

A natural first approach to remove all cycles in $G = H \boxtimes P_\ell$ is to remove the cycles within each layer completely, and then address the cycles spanning across layers. This leads to our first construction, which conceptually utilizes a vertex cover of the path P_ℓ to isolate the remaining intra-layer structures from each other.

Let F_H be a minimum feedback vertex set of H . To break all intra-layer cycles, we define $F' = \{(v, p_i) \mid v \in F_H, i \in [\ell]\}$ as the set containing the copies of F_H on every layer of G . However, the graph $G - F'$ still contains cycles. In particular, for any edge uv remaining in the acyclic graph $H - F_H$, the corresponding copies on any two consecutive layers still induce a K_4 . To guarantee that all layer-crossing cycles are broken, we use an alternating layer approach, where we take all remaining vertices on every second layer. Equivalent to using a minimum vertex cover of P_ℓ , we define $F'' = \{(v, p_i) \mid v \in V(H) \setminus F_H, i \in [\ell], i \text{ is even}\}$.

Let $F = F' \cup F''$. The removal of F leaves only the induced forests of the odd layers, which are completely disconnected from each other. These parts are acyclic, so the F is a valid feedback vertex set of G . Because F contains F_H on all the $\lceil \frac{\ell}{2} \rceil$ odd layers and all $|V(H)|$ vertices on the $\lfloor \frac{\ell}{2} \rfloor$ even layers, this feedback vertex set has a total size of $\lceil \frac{\ell}{2} \rceil \cdot \nabla(H) + \lfloor \frac{\ell}{2} \rfloor \cdot |V(H)|$.

5.1.2 Factor Vertex Cover Construction

For our second construction, instead of utilizing the vertex cover of the path P_ℓ , we rely entirely on a minimum vertex cover of the arbitrary factor graph H .

Let C_H be a minimum vertex cover of H . Since the removal of a vertex cover from a graph leaves an independent set, $H - C_H$ contains no edges and therefore no cycles. In particular, a vertex cover of a graph is always a valid feedback vertex set on the same graph as well.

We define $F = \{(v, p_i) \mid v \in C_H, i \in [\ell]\}$ as the set of all copies of C_H on each layer of G . Clearly, the removal of F breaks all cycles within the layers of G . By covering all edges uv in H , the tensor edges in G , $(u, p_i)(v, p_{i+1})$ and $(v, p_i)(u, p_{i+1})$ for every $i \in [\ell - 1]$, are also covered by F , since either u or v is in the vertex cover C_H .

Therefore, the only edges remaining in $G - F$ are $E(G - F) = \{(u, p_i)(u, p_{i+1}) \mid u \in (V(H) \setminus C_H), i \in [\ell]\}$. As a result, the graph $G - F$ is a collection of vertex-disjoint paths and thus acyclic. The set F is thus a valid feedback vertex set of G with size $\tau(H) \cdot \ell$.

5.1.3 Comparison of the Two Constructions

We show that for $G = H \boxtimes P_2$ the size of the factor vertex cover based feedback vertex set is always at most the size of the alternating layer feedback vertex set. To show this, we derive a relation between the feedback vertex number $\nabla(H)$ and the vertex cover number $\tau(H)$ of an arbitrary graph H .

Lemma 5.1: *Let H be an arbitrary graph. Then, it holds that $2\tau(H) \leq \nabla(H) + |V(H)|$.*

Proof. Let f be the size of a maximum induced forest M of H . Since forests are bipartite, we have $f \leq 2\alpha(M) \leq 2\alpha(H)$. Using this and the relation $\tau(H) = |V(H)| - \alpha(H)$, we derive the following inequality:

$$\begin{aligned} 2\tau(H) &= 2(|V(H)| - \alpha(H)) \\ &\leq 2|V(H)| - f = 2|V(H)| - (|V(H)| - \nabla(H)) = |V(H)| + \nabla(H). \end{aligned}$$

Thus, $2\tau(H) \leq \nabla(H) + |V(H)|$ always holds. ■

Corollary 5.2: *Let H be an arbitrary graph and $G = H \boxtimes P_{2k}$ for some $k \in \mathbb{N}^+$. Then, the factor vertex cover based feedback vertex set of G of size $\tau(H) \cdot 2k$ is smaller or equal to the layer alternating feedback vertex set of G with size $\nabla(H) \cdot k + |V(H)| \cdot k$.*

Proof. To show that the argument still holds for any even number of layers, we partition the layers of G into k pairs of consecutive layers. By applying Lemma 5.1 k times, to each pair of consecutive layers, we conclude that the factor vertex cover based feedback vertex set is smaller or equal to the layer alternating feedback vertex set on each partition of two layers and thus also for the whole graph G . ■

5.2 Strong Product of Paths

In this section we focus on graphs of the form $G = P_a \boxtimes P_b$ for some $a, b \in \mathbb{N}^+$, called *strong grid*. Such graphs have grid-like structure with the additional tensor product edges in each grid cell, as illustrated in Figure 5.1. For easier understanding we talk about a strong grid of a rows with b columns. To refer to the vertices of G we use the notation $v_{i,j}$ with $i \in [a], j \in [b]$ for the vertex in row i and column j . We determine the exact feedback vertex number for the strong grid. The exact value was first proven by Xie [Xie07] using a different inductive approach. In the following, we present an independent, alternative proof.

The following general observation for the feedback vertex number is a key tool to derive lower bounds in this chapter.

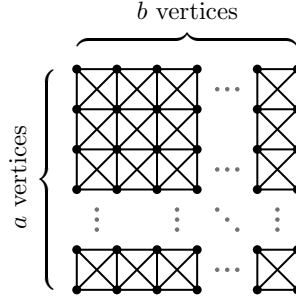


Figure 5.1: The strong product of two paths P_a and P_b with $a, b \in \mathbb{N}^+$.

Lemma 5.3: Let G be a graph and $H_1, \dots, H_k$ be pairwise vertex-disjoint subgraphs of G . Then it holds that $\nabla(G) \geq \sum_{i=1}^k \nabla(H_i)$.

Proof. Let F be a minimum feedback vertex set of G . Since $H_1, \dots, H_k$ are pairwise vertex-disjoint subgraphs of G , the set F can be partitioned into k subsets $F_1, \dots, F_k$, where $F_i = F \cap V(H_i)$ for every $i \in [k]$. Since F is a feedback vertex set of G , the set F_i is a feedback vertex set of H_i for every $i \in [k]$. Therefore, we have $|F_i| \geq \nabla(H_i)$ for every $i \in [k]$. Summing up these inequalities for all $i \in [k]$, we get $\nabla(G) = |F| = \sum_{i=1}^k |F_i| \geq \sum_{i=1}^k \nabla(H_i)$. ■

In the following we prove an exact formula for the feedback vertex number of G .

To do that, we first focus on the K_4 subgraphs of G that are induced by the edges of P_a and P_b . These help us to partition the graph into small subgraphs with a known lower bound on the feedback vertex number, which we can then use to derive a lower bound for the whole graph. First we show that there are exactly $\frac{a}{2} \cdot \frac{b}{2}$ vertex-disjoint K_4 subgraphs in G for even a and b .

Lemma 5.4: Let P_a and P_b be two paths with $a, b \in \mathbb{N}^+$. Then, the graph $G = P_a \boxtimes P_b$ contains $\frac{a}{2} \cdot \frac{b}{2}$ vertex-disjoint copies of K_4 .

Proof. The path P_a contains $\frac{a}{2}$ vertex-disjoint edges, $\{v_i v_{i+1} \mid i \in \{1, 3, \dots, a-1\}\}$. Similarly, the path P_b contains $\frac{b}{2}$ vertex-disjoint edges. By the definition of the strong product, every pair of an edge uv in P_a and an edge xy in P_b forms a complete graph of size four, with the vertices $(u, x), (u, y), (v, x), (v, y)$. Since there are $\frac{a}{2}$ vertex-disjoint edges in P_a and $\frac{b}{2}$ vertex-disjoint edges in P_b , there exist $\frac{a}{2} \cdot \frac{b}{2}$ vertex-disjoint K_4 copies in G . ■

The K_4 subgraphs of G are illustrated in Figure 5.2 for the case of even a and b .

To help derive a lower bound for the feedback vertex number of G , we now look at the feedback vertex number of the K_4 subgraphs of G .

Lemma 5.5: A minimum feedback vertex set of the complete graph K_4 on 4 vertices has size 2.

Proof. Let K_4 be the complete graph on the vertices $v_1, \dots, v_4$. Since $v_i v_j \in E(K_4)$ for every $i, j \in [4]$ with $i \neq j$, any three vertices v_i, v_j, v_k with i, j, k pairwise distinct induce a triangle in K_4 . Any two vertices v_i, v_j induce a single edge, which is acyclic. Therefore, each minimum feedback vertex set of K_4 contains exactly 2 vertices. ■

With the two previous lemmas we now derive an exact formula for the feedback vertex number of $G = P_a \boxtimes P_b$. We have to distinguish between the cases of a and b being even or odd, since the number of vertex-disjoint K_4 subgraphs in G as we have seen, only partitions the graph exactly for even a and b .

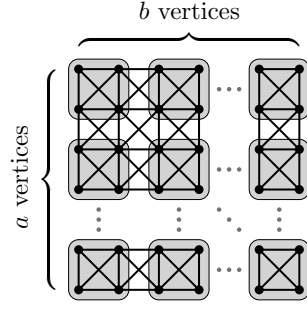


Figure 5.2: The strong product of two paths P_a and P_b with $a, b \in \mathbb{N}^+$ both even and the $\frac{a}{2} \cdot \frac{b}{2}$ vertex-disjoint K_4 copies highlighted in gray.

Theorem 5.6: Let $G = P_a \boxtimes P_b$ with $a, b \in \mathbb{N}^+$ both even. Then, every minimum feedback vertex set of G has size $\nabla(G) = \frac{ab}{2}$.

Proof. From Lemma 5.4 we know that G contains $\frac{a}{2} \cdot \frac{b}{2}$ vertex-disjoint K_4 subgraphs. By Lemma 5.5 we also know that every K_4 has a minimum feedback vertex set of size 2. Since the K_4 subgraphs are vertex-disjoint, by Lemma 5.3 the sum of the feedback vertex numbers of all K_4 subgraphs is a lower bound for the feedback vertex number of G . Thus, the size of a feedback vertex set of G is at least $2 \cdot \frac{a}{2} \cdot \frac{b}{2} = \frac{ab}{2}$.

To show that $\frac{ab}{2}$ is also an upper bound, we show that a feedback vertex set of G with size $\frac{ab}{2}$ exists. The set $F = \{v_{i,j} \in V(G) \mid i \text{ is even}\}$ marked in orange in Figure 5.3 has exactly size $\frac{ab}{2}$ and separates the graph G into paths and thus $G - F$ is acyclic. F is therefore a valid feedback vertex set of size $\frac{ab}{2}$, which is minimal. Alternatively, the set $F' = \{v_{i,j} \in V(G) \mid j \text{ is even}\}$ is also a valid and minimal feedback vertex set of size $\frac{ab}{2}$ containing every even column instead of every even row. ■

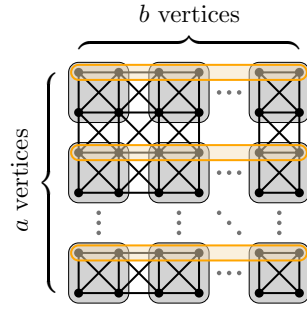


Figure 5.3: A minimum feedback vertex set of the strong product of two paths P_a and P_b with $a, b \in \mathbb{N}^+$ both even marked in orange.

In the case of one of either a or b being odd, using the commutativity of the strong product, we can assume without loss of generality that a is odd and b is even. Therefore, we only show one proof for the case of a odd and b even, which also covers the case of a even and b odd.

Theorem 5.7: Let $G = P_a \boxtimes P_b$ with $a \in \mathbb{N}^+$ odd and $b \in \mathbb{N}^+$ even. Then, every minimum feedback vertex set of G has size $\nabla(G) = \lfloor \frac{a}{2} \rfloor \cdot b$.

Proof. First, we use Lemma 5.3 again to derive a lower bound for the feedback vertex number of G . The graph G has $P_{a-1} \boxtimes P_b$ as a subgraph, which is a strong product of two even paths. From Theorem 5.6 we know that $P_{a-1} \boxtimes P_b$ contains $\frac{a-1}{2} \cdot \frac{b}{2}$ vertex-disjoint K_4 subgraphs. Summing up their lower bounds results in the lower bound for G of at least $2 \cdot \frac{a-1}{2} \cdot \frac{b}{2} = \lfloor \frac{a}{2} \rfloor \cdot b$.

To prove the upper bound, we define the set $F = \{v_{i,j} \in V(G) \mid i \text{ is even}\}$ of size $\lfloor \frac{a}{2} \rfloor \cdot b$. F separates the graph G into disjoint paths with b vertices each. This is again illustrated in Figure 5.4. Therefore, F is a valid feedback vertex set of size $\lfloor \frac{a}{2} \rfloor \cdot b$, which is minimal. ■

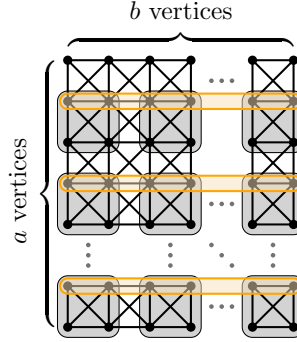


Figure 5.4: A minimum feedback vertex set of the strong product of two paths P_a and P_b with a odd and b even marked in orange.

Now the only remaining case is the one of a and b both being odd. In this case, the feedback vertex set of $P_{a-1} \boxtimes P_{b-1}$ is not a valid feedback vertex set of G anymore. Thus, we use a different approach to derive the lower bound for the feedback vertex number of G .

For this case, we first look at the K_4 subgraph of G induced by the vertices $v_{1,1}$, $v_{1,2}$, $v_{2,1}$ and $v_{2,2}$.

Lemma 5.8: *Let $G = P_a \boxtimes P_b$ with $a, b \in \mathbb{N}^+$, $a, b > 2$ and $v_{1,1}$, $v_{1,2}$, $v_{2,1}$ and $v_{2,2}$ be the vertices of the K_4 subgraph of G in the top left corner. Then, there either exist a minimum feedback vertex set of G containing $v_{1,2}$ and $v_{2,2}$, or a minimum feedback vertex set of G containing $v_{2,1}$ and $v_{2,2}$.*

Proof. The vertices $v_{1,1}$, $v_{1,2}$, $v_{2,1}$ and $v_{2,2}$ induce a K_4 in G . By Lemma 5.5, at least two of these four vertices are in every feedback vertex set of G .

We look at the different cases of which two vertices of the K_4 are in a minimum feedback vertex set of G . If $v_{1,1}$ and $v_{1,2}$ are in a feedback vertex set of G , then replacing $v_{1,1}$ with $v_{2,2}$ results in a feedback vertex set that is not larger than the original one. As the graph $G - \{v_{1,2}, v_{2,2}\}$ is a subgraph of $G - \{v_{1,1}, v_{1,2}\}$, the new feedback vertex set is at least as good as the original one. Similarly, $G - \{v_{1,2}, v_{2,2}\}$ is a subgraph of $G - \{v_{1,2}, v_{2,1}\}$, so replacing $v_{2,1}$ with $v_{2,2}$ also keeps the validity and is at least as good as the original feedback vertex set. In the remaining case of $v_{1,1}$ and $v_{2,2}$ being in a feedback vertex set of G , replacing $v_{1,1}$ with $v_{1,2}$ again keeps the validity as $G - \{v_{1,2}, v_{2,2}\}$ is a subgraph of $G - \{v_{1,1}, v_{2,2}\}$. Again the new feedback vertex set is at least as good as the original one. These cases are visually shown in Figure 5.5, with the remaining case of $v_{2,1}$ and $v_{2,2}$ being in a feedback vertex set of G being symmetric to the case of $v_{1,2}$ and $v_{2,2}$. Therefore, there either exists a minimum feedback vertex set of G containing $v_{1,2}$ and $v_{2,2}$, or a minimum feedback vertex set of G containing $v_{2,1}$ and $v_{2,2}$. ■

From this result, we want to further analyze the structure of such a minimum feedback vertex set of G .

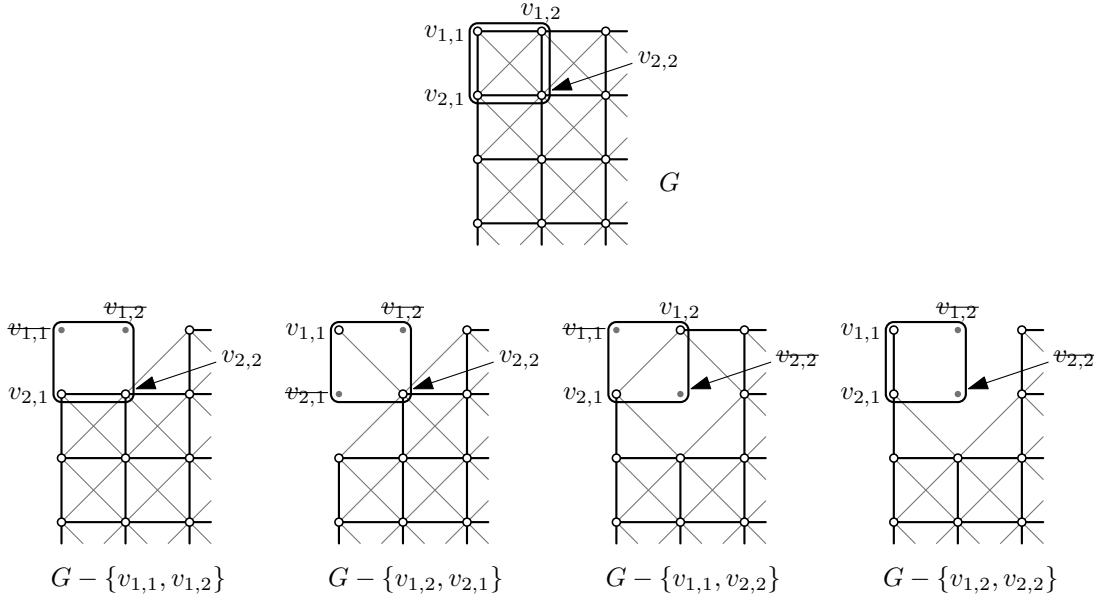


Figure 5.5: From left to right, for a graph $G = P_a \boxtimes P_b$ we have $G - \{v_{1,1}, v_{1,2}\}$, $G - \{v_{1,2}, v_{2,1}\}$, $G - \{v_{1,1}, v_{2,2}\}$ and $G - \{v_{1,2}, v_{2,2}\}$ which is a subgraph of all the other three graphs.

Lemma 5.9: Let $G = P_a \boxtimes P_b$ with $a, b \in \mathbb{N}^+$, $a, b > 2$. Then, there either exist a minimum feedback vertex set of G containing all vertices $v_{i,j}$ with $i = 2$ and $j \in [b]$, or a minimum feedback vertex set of G containing all vertices $v_{i,j}$ with $i \in [a]$ and $j = 2$.

Proof. From Lemma 5.8, we know that there exists either a minimum feedback vertex set of G containing $v_{1,2}$ and $v_{2,2}$ or a minimum feedback vertex set of G containing $v_{2,1}$ and $v_{2,2}$. We show, that if a minimum feedback vertex set of G contains $v_{1,2}$ and $v_{2,2}$, then there also exists a minimum feedback vertex set of G containing all vertices $v_{i,j}$ with $i \in [a]$ and $j = 2$. The other case of a minimum feedback vertex set of G containing $v_{2,1}$ and $v_{2,2}$ is symmetric.

Let F be a minimum feedback vertex set of G containing $v_{1,2}$ and $v_{2,2}$. The vertices $v_{2,1}, v_{2,2}, v_{3,1}, v_{3,2}$ induce a K_4 subgraph in G . Since $v_{2,1}$ is only part of a single triangle with $v_{3,1}$ and $v_{3,2}$ in the remaining graph, any for any feedback vertex set of G containing $v_{1,2}, v_{2,2}$ and $v_{2,1}$, there is a feedback vertex set of at most the same size containing $v_{1,2}, v_{2,2}$ and either $v_{3,1}$ or $v_{3,2}$ instead of $v_{2,1}$. As the graph $G - \{v_{1,2}, v_{2,2}\} - v_{3,2}$ is isomorphic to a subgraph of $G - \{v_{1,2}, v_{2,2}\} - v_{3,1}$, any feedback vertex set of G containing $v_{1,2}, v_{2,2}$ and $v_{3,1}$ cannot be smaller than a feedback vertex set of G containing $v_{1,2}, v_{2,2}$ and $v_{3,2}$. Visually this is shown in Figure 5.6. This argument repeats for every K_4 copy in the first two columns of the strong grid, resulting in $\{v_{i,2} \mid i \in [a]\}$ being part of the minimum feedback vertex set F .

Therefore, we can conclude from Lemma 5.8 that there either exists a minimum feedback vertex set of G containing all vertices $v_{i,j}$ with $i = 2$ and $j \in [b]$, or a minimum feedback vertex set of G containing all vertices $v_{i,j}$ with $i \in [a]$ and $j = 2$. ■

We now have a very specific structure for a minimum feedback vertex set of G , which we can use to derive the lower bound for the feedback vertex number of G . We only show the case of $a \leq b$ since the other case is symmetric by the commutativity of the strong product.

Theorem 5.10: Let $G = P_a \boxtimes P_b$ with $a, b \in \mathbb{N}^+$ both odd and $a \leq b$. Then, every minimum feedback vertex set of G has size $\nabla(G) = \lfloor \frac{a}{2} \rfloor \cdot b$.

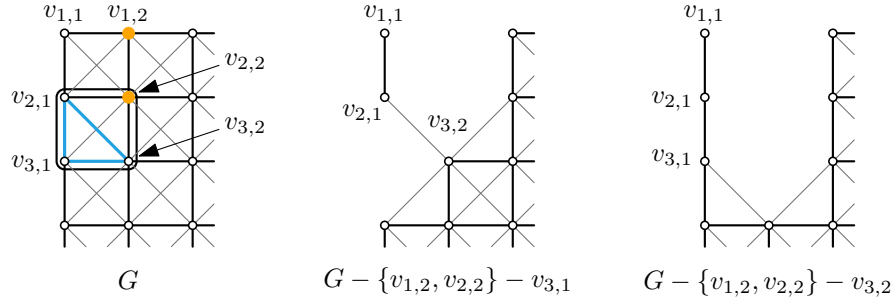


Figure 5.6: G with the triangle of $v_{2,1}$, $v_{3,1}$ and $v_{3,2}$ highlighted in blue and next to it the graphs of removing either $v_{3,1}$ or $v_{3,2}$ in addition to $v_{1,2}$ and $v_{2,2}$ to show the subgraph relation.

Proof. First we show that $\lfloor \frac{a}{2} \rfloor \cdot b$ is a valid upper bound. Removing the set $F = \{v_{i,j} \in V(G) \mid i \text{ is even}\}$ with size $\lfloor \frac{a}{2} \rfloor \cdot b$ separates the graph G into $\lceil \frac{a}{2} \rceil$ disjoint, disconnected P_b copies, which are acyclic. Therefore, $\lfloor \frac{a}{2} \rfloor \cdot b$ is an upper bound for the feedback vertex number of G .

To show that $\lfloor \frac{a}{2} \rfloor \cdot b$ is also a lower bound, we proceed by induction on a and b .

For the **base case**, let $a = 1$ and $b \geq 1$. Here the graph G is isomorphic to the path graph P_b , which is acyclic. Thus, the size of the minimum feedback vertex set is 0, which matches our formula since $\lfloor \frac{1}{2} \rfloor \cdot b = 0 \cdot b = 0$.

For the **inductive step**, assume that the statement holds for all strictly smaller graphs. That is, for any graph of the form $G' = P_{a'} \boxtimes P_{b'}$ with odd a', b' where $a' \leq b'$ and G' is strictly smaller than G , every minimum feedback vertex set F' of G' has size at least $\lfloor \frac{a'}{2} \rfloor \cdot b'$.

Consider now a graph $G = P_a \boxtimes P_b$ with odd a, b , where $a > 1$ and $a \leq b$, and let F be a minimum feedback vertex set of G .

Case: $a < b$. We can now distinguish two subcases based on the structure of F , since we know from Lemma 5.9, that either all vertices in the second column, $v_{i,2}$ with $i \in [a]$ and $j = 2$, or all vertices of the second row, $v_{i,j}$ with $i = 2$ and $j \in [b]$, are in a minimum feedback vertex set of G . We calculate the lower bound for both cases and show that in both cases the feedback vertex set has size at least $\lfloor \frac{a}{2} \rfloor \cdot b$.

Subcase 1: $v_{i,2} \in F$ for every $i \in [a]$. Removing the vertices $v_{i,2}$ for all $i \in [a]$ from G separates the graph into a component isomorphic to P_a and a component isomorphic to $P_a \boxtimes P_{b-2}$. The path P_a has an empty feedback vertex set and since $a < b$ holds by assumption, with a and b both odd, it follows, that the other component still has the property of $a \leq b - 2$. By induction, we conclude that there is a minimum feedback vertex set F' of size $\lfloor \frac{a}{2} \rfloor \cdot (b - 2)$ in the second component and therefore, there is a feedback vertex set $F = F' \cup \{v_{i,2} \mid i \in [a]\}$ of size $a + \lfloor \frac{a}{2} \rfloor \cdot (b - 2)$ for G . As $a + \lfloor \frac{a}{2} \rfloor \cdot (b - 2) > \lfloor \frac{a}{2} \rfloor \cdot b \Leftrightarrow a > 2\lfloor \frac{a}{2} \rfloor = a - 1$ we know that this choice results in a feedback vertex set of size greater than $\lfloor \frac{a}{2} \rfloor \cdot b$, so the lower bound holds in this case.

Subcase 2: $v_{2,j} \in F$ for every $j \in [b]$. Removing the vertices $v_{2,j}$ for all $j \in [b]$ from G separates the graph into a component isomorphic to P_b and a component isomorphic to $P_{a-2} \boxtimes P_b$. The path P_b has an empty feedback vertex set and since $a < b$ holds by assumption, with a and b both odd, it follows, that the other component still has the property of $a - 2 \leq b$.

By induction, we conclude that there is a minimum feedback vertex set F' of size $\lfloor \frac{a-2}{2} \rfloor \cdot b$ in the second component and therefore, there is a feedback vertex set $F = F' \cup \{v_{2,j} \mid j \in [b]\}$ of size $b + \lfloor \frac{a-2}{2} \rfloor \cdot b = \lfloor \frac{a}{2} \rfloor \cdot b$ for G . Thus, the lower bound holds tight in this case as well.

As we have shown that either of these cases is a minimum feedback vertex set of G and in both cases the size of the feedback vertex set is at least $\lfloor \frac{a}{2} \rfloor \cdot b$, we conclude that $\lfloor \frac{a}{2} \rfloor \cdot b$ holds in the case of $a < b$.

Case: $a = b$. Similar to the other case we know, that either all the second column or second row vertices are part of a minimum feedback vertex set as shown in Lemma 5.9. As we have a symmetric case, we can without loss of generality assume the second row of b vertices in $\{v_{2,j} \mid j \in [b]\}$ being in a minimum feedback vertex set F . Removing these b vertices from G separates the graph into two components, one isomorphic to P_b and acyclic and the other isomorphic to $P_{a-2} \boxtimes P_b$ with $a-2 < b$. By induction this results in $|F| = b + \lfloor \frac{a-2}{2} \rfloor \cdot b = \lfloor \frac{a}{2} \rfloor \cdot b$.

Thus, the lower bound of the size of a feedback vertex set on a graph $G = P_a \boxtimes P_b$ with $a \leq b$ is always $\lfloor \frac{a}{2} \rfloor \cdot b$. Since we also have an upper bound of $\lfloor \frac{a}{2} \rfloor \cdot b$ by removing every second row, we conclude that every minimum feedback vertex set of G has size $\nabla(G) = \lfloor \frac{a}{2} \rfloor \cdot b$. ■

5.3 Strong Product of Tree and Path

In this section we generalize the graphs we look at to the strong product of an arbitrary tree T and a path P_ℓ . Graphs such as $G = T \boxtimes P_\ell$ have a structure of ℓ layers isomorphic to T with connecting edges between every two consecutive layers. Similar to the strong grid, vertices with an edge in the factor graphs induce K_4 subgraphs in the product. We use this and the fact that $\tau(P_\ell) = \lfloor \frac{\ell}{2} \rfloor$ to derive a lower bound for the feedback vertex number of G . As notation for the vertices of G we use the notation (v, p_i) with $v \in V(T)$ and $p_i \in V(P_\ell)$ for $i \in [\ell]$ to refer to the copy of v in the layer corresponding to p_i .

Lemma 5.11: *Let T be a tree and $G = T \boxtimes P_\ell$. Further, let C be a minimum vertex cover of T . Then, both $F_1 = \{(v, p_i) \in V(G) \mid v \in C, p_i \in V(P_\ell)\}$ and $F_2 = \{(v, p_i) \in V(G) \mid v \in V(T), p_i \in V(P_\ell), i \text{ is even}\}$ are valid feedback vertex sets of G .*

Proof. The proof is split into two parts, one for each of the two feedback vertex sets F_1 and F_2 .

First, we show that F_1 is a valid feedback vertex set of G . For each edge uv in T , as C is a vertex cover of T , at least one of u and v is in C . Therefore, for every edge uv in T and every vertex p_i in P_ℓ , with $i \in [\ell]$, all the edges $(u, p_i)(v, p_i)$, contained in the layers of G , and for $i \in [\ell-1]$ the edges $(u, p_i)(v, p_{i+1})$ and $(u, p_{i+1})(v, p_i)$, contained in the connections between the layers of G , are removed by F_1 . The resulting graph $G - F_1$ is a forest of disjoint paths isomorphic to P_ℓ , which is acyclic, so F_1 is a valid feedback vertex set of G .

We now show that F_2 is a valid feedback vertex set of G . For the path P_ℓ the minimum vertex cover, $C_P = \{p_i \in V(P_\ell) \mid i \text{ is even}\}$, has size $\tau(P_\ell) = \lfloor \frac{\ell}{2} \rfloor$ and covers all edges of P_ℓ . The graph $G - F_2$ is isomorphic to $T \boxtimes P_\ell - C_P$, which is a forest of disjoint trees isomorphic to T and thus acyclic, so F_2 is a valid feedback vertex set of G . ■

Corollary 5.12: *Let T be a tree and let $G = T \boxtimes P_\ell$. Then, the minimum feedback vertex set of G has size at most $\min\{\tau(T) \cdot \ell, \lfloor \frac{\ell}{2} \rfloor \cdot |V(T)|\}$.*

Proof. This follows directly from Lemma 5.11 and the fact that $\tau(P_\ell) = \lfloor \frac{\ell}{2} \rfloor$. As F_1 and F_2 are valid feedback vertex sets of G and $|F_1| = \tau(T) \cdot \ell$ and $|F_2| = \lfloor \frac{\ell}{2} \rfloor \cdot |V(T)|$, it follows that $\nabla(G) \leq \min\{\tau(T) \cdot \ell, \lfloor \frac{\ell}{2} \rfloor \cdot |V(T)|\}$. ■

Theorem 5.13: Let T be a tree and let $G = T \boxtimes P_\ell$ be the strong product of T and a path of ℓ vertices. If ℓ is even, then the minimum feedback vertex set has size $\tau(T) \cdot \ell$. If ℓ is odd and T has a perfect matching, then the minimum feedback vertex set has size $\lfloor \frac{\ell}{2} \rfloor \cdot |V(T)|$.

Proof. Since Corollary 5.12 already shows an upper bound for the feedback vertex number of G , we only show that the stated values are also lower bounds for the feedback vertex number of G to conclude that they are the exact values. We use Lemma 5.3 and vertex disjoint K_4 subgraphs in G to derive a lower bound for the feedback vertex number of G . First we prove the case of ℓ even and then the case of ℓ odd and T having a perfect matching.

Case 1: ℓ is even According to the theorem of König the number of edges in a maximum matching of a bipartite graph H is equal to the number of vertices in a minimum vertex cover of H [Kön31]. Therefore, for a tree T with a minimum vertex cover of size $\tau(T)$, there is a maximum matching with exactly $\tau(T)$ vertex-disjoint edges. As we know, any edge uv in T and any edge $p_i p_{i+1}$ in P_ℓ with $i \in [\ell - 1]$ results in a K_4 subgraph of G with the vertices $(u, p_i), (u, p_{i+1}), (v, p_i), (v, p_{i+1})$. Since P_ℓ with even ℓ has $\frac{\ell}{2}$ vertex-disjoint edges and T has $\tau(T)$ vertex-disjoint edges, there are $\frac{\ell}{2} \cdot \tau(T)$ vertex-disjoint K_4 subgraphs in G . As per Lemma 5.5, a K_4 has a minimum feedback vertex set of size 2, resulting in a minimum feedback vertex set of size at least $2 \cdot \frac{\ell}{2} \cdot \tau(T) = \ell \cdot \tau(T)$ vertices for G . So in this case it holds that $\nabla(G) \geq \tau(T) \cdot \ell$ and by Lemma 5.11 we also know that $F_1 = \{(v, p_i) \in V(G) \mid v \in C, p_i \in V(P_\ell)\}$ for a minimum vertex cover C of T is a valid feedback vertex set of G with size $\tau(T) \cdot \ell$. Thus, it holds that $\nabla(G) = \tau(T) \cdot \ell$ for even ℓ and F_1 is a minimum feedback vertex set of G in this case. This is visually shown in Figure 5.7.

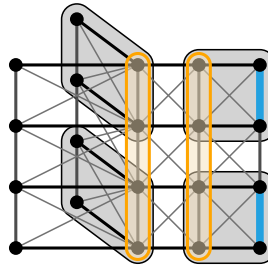


Figure 5.7: A minimum feedback vertex set in orange on the product of a tree without perfect matching and an even path with a perfect matching marked in blue.

Case 2: ℓ is odd and T has a perfect matching Similar to the first case, we know that for a tree T with a minimum vertex cover of size $\tau(T)$ there is a maximum matching with exactly $\tau(T)$ vertex-disjoint edges. Since T has a perfect matching, $\tau(T) = \frac{n}{2}$ where $n = |V(T)|$. For odd ℓ the path P_ℓ has $\lfloor \frac{\ell}{2} \rfloor$ vertex-disjoint edges. It follows that there are $\lfloor \frac{\ell}{2} \rfloor \cdot \frac{n}{2}$ vertex-disjoint K_4 subgraphs in G . As per Lemma 5.5 again, a minimum feedback vertex set of G has at least size $2 \cdot \lfloor \frac{\ell}{2} \rfloor \cdot \frac{n}{2} = \lfloor \frac{\ell}{2} \rfloor \cdot |V(T)|$. So in this case it holds that $\nabla(G) \geq \lfloor \frac{\ell}{2} \rfloor \cdot |V(T)|$ and since by Lemma 5.11 we also know that $F_2 = \{(v, p_i) \in V(G) \mid v \in V(T), p_i \in V(P_\ell), i \text{ is even}\}$ is a valid feedback vertex set of G with size $\tau(P_\ell) \cdot |V(T)| = \lfloor \frac{\ell}{2} \rfloor \cdot |V(T)|$. Thus, it holds that $\nabla(G) = \lfloor \frac{\ell}{2} \rfloor \cdot |V(T)|$ for odd ℓ and trees T with a perfect matching, and F_2 is a minimum feedback vertex set of G in this case. This is visually shown in Figure 5.8. ■

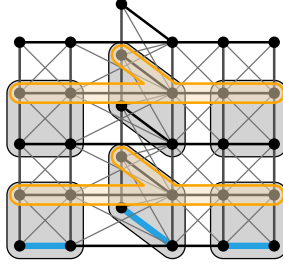


Figure 5.8: A minimum feedback vertex set in orange on the product of a tree with a perfect matching marked in blue and an odd path.

With this theorem we have shown that for even ℓ and for odd ℓ with T having a perfect matching, a minimum feedback vertex set of $G = T \boxtimes P_\ell$ can be constructed in constant time from the minimum vertex cover of one of the factor graphs.

We can even generalize this result to the case of B being an arbitrary bipartite graph instead of a tree in the case of ℓ even.

Corollary 5.14: *Let B be a bipartite graph and let $G = B \boxtimes P_\ell$ be the strong product of B and an even path of ℓ vertices. Then, the minimum feedback vertex set of G has size $\tau(B) \cdot \ell$.*

Proof. This follows directly from Theorem 5.13 and the fact that we only used the theorem of König for the case of ℓ even, which holds for all bipartite graphs. ■

This is especially interesting since the minimum feedback vertex set problem is NP-hard on bipartite graphs [Yan81], but for the strong product of a bipartite graph and an even path, the minimum feedback vertex set can be computed in polynomial time.

The cases of odd ℓ and trees T without a perfect matching are more complex. In Figure 5.9 we see that for an odd path and a tree without a perfect matching, the minimum feedback vertex set of $G = T \boxtimes P_\ell$ does not necessarily have size $\nabla(G) = \min\{\tau(T) \cdot \ell, \lfloor \frac{\ell}{2} \rfloor \cdot |V(T)|\}$. In such cases, the optimal solution may require a more complex combination of vertex removals across layers, as seen on the rightmost example in Figure 5.9, and the simple construction of taking vertex covers or entire layers is a valid approximation, but may not yield the minimum feedback vertex set.

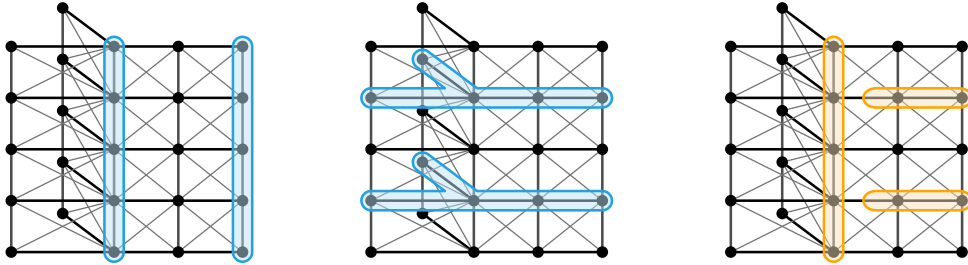


Figure 5.9: A graph $G = T \boxtimes P_\ell$ with odd ℓ and a tree T without a perfect matching. On the left, two different feedback vertex sets of size $\tau(T) \cdot \ell$ and $\lfloor \frac{\ell}{2} \rfloor \cdot |V(T)|$ are shown in blue. On the right, a minimum feedback vertex set of size 9 is shown in orange, which is smaller than both of the previous feedback vertex sets.

5.3.1 Caterpillars

To further understand the case of odd ℓ and trees T without a perfect matching, we look at caterpillars.

First, we present a sufficient condition for a caterpillar to have a perfect matching, which is also the “if” direction of Remark 6.19 in [GHSW26]. We provide a proof for this condition, which is not included in the paper, and then show that if this condition is not met, there is a leaf-to-leaf odd path in the caterpillar, which we use to show a lower bound for the feedback vertex set of the strong product of such a caterpillar with an odd path. A caterpillar with a perfect matching is shown in Figure 5.10 and a caterpillar without a perfect matching is shown in Figure 5.11.

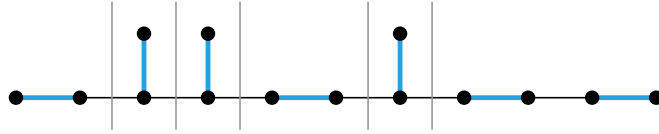


Figure 5.10: A caterpillar with a perfect matching. The matching edges are shown in blue.

Lemma 5.15: *Let T be a caterpillar with the spine S . T has a perfect matching if*

1. *every vertex has degree at most three and*
2. *if the degree three vertices of T are deleted from S , the subpaths of S that remain (if any) each have an even number of vertices.*

Proof. Let T be a caterpillar with the spine S , with every vertex having degree at most three and with the property that if the degree three vertices of T are deleted from S , the subpaths of S that remain (if any) each have an even number of vertices. Let B be the set of degree three vertices in T . We show that there is a perfect matching M of T .

Since the degree of every vertex in T is at most three, every vertex in B has exactly one leaf neighbor. For every vertex $v \in B$, let l_v be the unique leaf neighbor of v and let each vl_v be in the set M_l .

Let $S_1, S_2, \dots, S_k$ be the subpaths of S that remain after deleting the vertices in B from S . Since the degree three vertices of T are deleted from S , the subpaths $S_1, S_2, \dots, S_k$ are vertex-disjoint. For each subpath S_i with the vertices $s_1, s_2, \dots, s_{2m}$ where $m \in \mathbb{N}^+$ and s_1, s_{2m} are the endpoints of S_i , there is a perfect matching M_i that matches s_k with s_{k+1} for every odd $k \in [2m - 1]$.

Therefore, there is a perfect matching M of T that is the union of the perfect matchings M_i on the subpaths $S_1, S_2, \dots, S_k$ and the matching M_l of the vertices in B with their unique leaf neighbors. ■

Lemma 5.16: *Let T be a caterpillar with the spine S that does not satisfy the conditions of Lemma 5.15. Then, T contains a leaf-to-leaf odd path $Q = (q_1, q_2, \dots, q_{2m+1})$ for some $m \in \mathbb{N}^+$ such that the inner vertices of Q , namely $q_3, \dots, q_{2m-1}$, have degree exactly two in T .*

Proof. Let T be a caterpillar with the spine S .

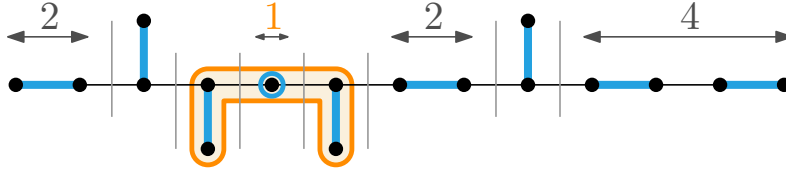


Figure 5.11: A caterpillar without a perfect matching. A maximum matching is marked in blue. The odd path Q is shown in orange.

If there is a vertex v in T with degree at least four, then v has at least two leaf neighbors. Thus, there is a path $Q = (q_1, v, q_3)$ between these two leaf neighbors. Here, Q has an odd number of vertices of 3, with $m = 1$. Since there are no vertices between $q_2 = v$ and $q_{2m} = v$, the condition on the inner vertices holds.

Assume every vertex in T has degree at most three and let B be the set of degree three vertices in T . Let $S' = (s_1, \dots, s_k)$ be a subpath of S that remains after deleting the vertices of B and has an odd number of vertices, $k \in \mathbb{N}^+$. Since the vertices of S' are on the spine but not in B , they have degree at most two in T . The subpath S' is bounded by vertices $u, w \in B$ on the spine, or one or both of the endpoints of S' are the endpoints of S .

In the first case, because $u, w \in B$, they each have exactly one leaf neighbor, say l_u and l_w . We define Q as the path $(l_u, u, s_1, \dots, s_k, w, l_w)$, which is a path between the leaf neighbors of u and w . An example of this case is shown in Figure 5.11 with the path Q in orange. Because S' has k vertices, where k is odd, Q has $k + 4$ vertices, which is also odd. By the definition, the vertices strictly between u , which is q_2 , and w , which is q_{2m} , are exactly the vertices of S' , which have degree two in T .

In the second case, if one of the endpoints of S' are the endpoints of S , since the spine S is maximal, the endpoints of S are leaves in T . Assume without loss of generality that s_1 is an endpoint of S' and S and thus a leaf in T and that the other endpoint s_{2m+1} of S' is not an endpoint of S . Then, we define Q as the path $(s_1, \dots, s_k, w, l_w)$, where $w \in B$ is the neighbor of s_{2m+1} on the spine. This Q is a leaf-to-leaf odd path with $k + 2$ vertices.

Similarly, if both endpoints of S' are endpoints of S , then both are leaves in T and $Q = S'$ is a leaf-to-leaf odd path of k vertices.

In all cases, the inner vertices of Q , $q_3, \dots, q_{2m-1}$, are exactly the vertices of S' , which have degree two in T . ■

Theorem 5.17: Let T be a caterpillar graph and P_ℓ be an odd path. Then, $G = T \boxtimes P_\ell$ has a minimum feedback vertex set F of the following form:

1. If T has a perfect matching, $F = V(T) \times \{v_i \mid i \in [\ell], i \text{ is even}\}$.
2. If T does not have a perfect matching, let Q be the shortest odd path in T , with vertices $q_1, \dots, q_{2m+1}$, where $m \in \mathbb{N}^+$ and q_1, q_{2m+1} are leaves in T and all inner vertices $q_3, \dots, q_{2m-1}$ have degree two in T . Then, $F = F_Q \cup F'$, where F' is the union of a minimum feedback vertex set for each of the components of $(T - Q) \boxtimes P_\ell$, and

$$F_Q = \begin{cases} \{q_i \mid i \in [2m+1], i \text{ is even}\} \times V(P_\ell) & \text{if } \ell \geq 2m+1, \\ V(Q) \times \{v_i \mid i \in [\ell], i \text{ is even}\} & \text{if } \ell < 2m+1. \end{cases}$$

Proof. We prove the validity of F by induction on the size of T .

In the base case that T has a perfect matching, the set $F = V(T) \times \{v_i \mid i \in [\ell], i \text{ is even}\}$ is a valid feedback vertex set of G . The validity and optimality of F follows from Lemma 5.11 and Theorem 5.13 as a caterpillar is a tree.

In the inductive step, if T does not have a perfect matching, by Lemma 5.16 T contains a leaf-to-leaf odd path Q whose internal vertices $q_3, \dots, q_{2m-1}$ have degree two in T . Let Q be the shortest such path in T with the vertices $q_1, q_2, \dots, q_{2m+1}$ where $m \in \mathbb{N}^+$ and q_1, q_{2m+1} are leaves in T . If Q consists of three vertices, because q_2 is a vertex of degree at least four on the spine of T , then removing the vertices of Q from T results in at most two connected components, corresponding to spine neighbors of q_2 and more possible single vertex components if q_2 has more than two leaf neighbors. We do not have to consider the single vertex components as for $2m+1=3$ with $\ell > 1$, the set F_Q already contains all vertices of $\{q_2\} \times V(P_\ell)$. If Q consists of more than three vertices, removing the vertices of Q from T splits $T - Q$ into at most two connected components, as only q_2 and q_{2m} can have neighbors in $T - Q$ and neither q_2 nor q_{2m} can have multiple leaf neighbors since Q is the shortest odd leaf-to-leaf path in T . By the inductive hypothesis, the smaller caterpillar components have explicitly constructed valid feedback vertex sets. Let F' be the union of these valid feedback vertex sets for $(T - Q) \boxtimes P_\ell$.

From Theorem 5.10, F_Q is a valid feedback vertex set for $Q \boxtimes P_\ell$. To show $F = F_Q \cup F'$ is a valid feedback vertex set of G , it suffices to verify that no cycle spans across the partition into $Q \boxtimes P_\ell - F_Q$ and $(T - Q) \boxtimes P_\ell - F'$.

If $\ell \geq 2m+1$, F_Q fully contains the columns $\{q_2, q_{2m}\} \times V(P_\ell)$. Therefore, the entirely removed columns act as a barrier, leaving no remaining vertices at the designated attachment points q_2 and q_{2m} in $Q \boxtimes P_\ell - F_Q$. Thus, no edges exist that connect the two subgraphs.

If $\ell < 2m+1$, F_Q contains all even layers $V(Q) \times \{v_i \mid i \in [\ell], i \text{ is even}\}$. Let $u \in V(T - Q)$ be a neighbor of q_2 or q_{2m} . By the inductive construction of F' , the connection vertices u in $T - Q$ are either part of an odd path in their component of $T - Q$ that fulfills the case of $\ell \geq 2m+1$ and thus, $\{q_2, q_{2m}\} \times V(P_\ell) \in F'$, or the odd path in $T - Q$ that contains u fulfills the case of $\ell < 2m+1$ or does not exist and thus $\{u\} \times \{v_i \mid i \in [\ell], i \text{ is even}\} \in F'$. Hence, any remaining vertices in the columns $q_2 \times V(P_\ell)$, $q_{2m} \times V(P_\ell)$, and their corresponding neighbor columns in $T - Q$ strictly belong to the odd layers of P_ℓ . Since any valid feedback vertex set in the construction does not leave any path between distinct vertices in $\{u\} \times \{v_i \mid i \in [\ell], i \text{ is odd}\}$ the edges between $Q \boxtimes P_\ell - F_Q$ and $(T - Q) \boxtimes P_\ell - F'$ are not part of any cycle in $G - (F_Q \cup F')$. Therefore, $F_Q \cup F'$ is a valid feedback vertex set of G .

We now prove the minimality of F . In the base case that T has a perfect matching, we know that F is a minimum feedback vertex set of G by Theorem 5.13. In the inductive step, if T does not have a perfect matching, we know that F_Q is a minimum feedback vertex set of $Q \boxtimes P_\ell$ by Theorem 5.10. We also know that F' is a minimum feedback vertex set of $(T - Q) \boxtimes P_\ell$ by the inductive hypothesis. By Lemma 5.3, any feedback vertex set of G must contain at least $|F_Q| + |F'|$ vertices, so F is a minimum feedback vertex set of G . ■

5.3.2 Structural Properties of General Trees

We now try to apply the insights from caterpillars to general trees. First we show a more general version of Lemma 5.16. For this we use a construction comparable to the Dulmage-Mendelsohn decomposition of a bipartite graph [DM58].

Theorem 5.18: *Let T be a tree without a perfect matching. Then, there exists an induced subgraph T' of T , with $|V(T')|$ odd, such that for every two leaves l_1, l_2 in T' , the path from l_1 to l_2 in T is odd and all vertices in T' , adjacent to vertices in $T - T'$, are part of a minimum vertex cover of T' .*

Proof. Since T does not have a perfect matching, let M be a maximum matching of T . As M is not perfect, there is at least one vertex r in T that is not matched by M . We use r as a root and grow the tree T' from r by exploring all maximal matching-alternating paths in a search starting from r .

Specifically, while performing the matching-alternating breadth-first-search we partition the vertices of T' into the set X with even distance to r and Y with odd distance to r . In particular r is in X and all direct neighbors of r are in Y . Since r is not matched in T' , by construction for any edge uv in T' if $u \in X$ and $v \in Y$ then $uv \notin M$ and if $u \in Y$ and $v \in X$ then $uv \in M$.

We now verify that the subtree T' has all the stated properties. First we show, that $|V(T')|$ is indeed odd, using the matching in T' . We verify that every vertex in Y is matched in T' and thus matched to a unique vertex in $X \setminus \{r\}$. Assume, for the sake of contradiction, that there is a vertex $v \in Y$ that is not matched in T' . In this case the path from r to v is an augmenting alternating path, meaning that switching the matched and unmatched edges along this path would yield a larger matching than M in T , which is a contradiction to the maximality of M . Therefore, every vertex in Y is matched in T' and since the search-construction of T' would continue along the next matched edge from v , every vertex $v \in Y$ is matched to a unique vertex in $X \setminus \{r\}$. Since r is not matched, we have $|X| = |Y| + 1$, so $|V(T')| = |X| + |Y| = 2|Y| + 1$ which is odd.

To determine the distance of the leaves in T' we look at the partition they are part of. Since we confirmed that every vertex in Y is matched and reached from r via an unmatched edge, every vertex $v \in Y$ has at least one child in T' that it is matched to. Therefore, no vertex in Y can be a leaf in T' and all leaves of T' are in X . Since T' is bipartite with the bipartite sets X, Y , the path between any two vertices in X , like two leaves, has an odd number of vertices.

We now show that all vertices in T' adjacent to vertices in $T - T'$ are part of a minimum vertex cover of T . To do this, we first show that all vertices in T' adjacent to vertices in $T - T'$ are in Y . For this, let $u \in X$ and assume, for the sake of contradiction, that u is adjacent to a vertex w in $T - T'$. Since $u \in X$ is reached from r via a matched edge, the edge uw cannot be in M . This is also true in the case that u is r , since r is not matched. This would mean that w is reached from r via an unmatched edge, so w would be part of T' and Y in particular, and thus not in $T - T'$, which contradicts the assumption. Therefore, all vertices in T' adjacent to vertices in $T - T'$ are in Y .

In this bipartite graph T' with the bipartite sets X, Y , with $|X| = |Y| + 1$, where every vertex in Y has a unique matched neighbor in X and another neighbor in X via an unmatched edge, the set Y is a minimum vertex cover since every edge in the maximum matching has to be covered by at least one vertex and the vertices in Y are sufficient to cover all edges in T' . Since all vertices in T' adjacent to vertices in $T - T'$ are in Y and Y is a minimum vertex cover of T' , all vertices in T' adjacent to vertices in $T - T'$ are part of a minimum vertex cover of T' .

This concludes the proof that T' has all the stated properties. ■

An example of the construction of the tree T' in Theorem 5.18 is shown in Figure 5.12.

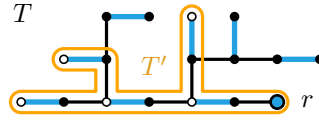


Figure 5.12: Odd alternating subtree T' highlighted in orange of a tree T with the matching M marked blue.

A useful property of the tree T' is that all vertices in Y have degree at most two in T' . Seeing this is straightforward, since all vertices in Y are reached from r via an unmatched edge and are matched to at most one vertex in X via a matched edge, so they cannot have more than two neighbors in T' . This allows us to apply arguments shown before to such trees.

In the following we refer to the tree T' with the properties stated in the previous theorem as an *odd M -alternating subtree* of T with respect to the maximum matching M of T and with the bipartite sets X and Y with $|X| = m + 1$ and $|Y| = m$ and the root r in X .

We first show that for an odd M -alternating subtree T' of T , we can modify the maximum matching M of T to guarantee that the root r of T' is a leaf in T' , which is useful for the later analysis of the minimum feedback vertex set of $T' \boxtimes P_\ell$.

Lemma 5.19: *Let T be a tree without a perfect matching M and let T' be an odd M -alternating subtree of T with the bipartite sets X and Y as described in the previous theorem. If the unmatched root r of T' is not a leaf in T' , then for any leaf l in T' with the path L from r to l in T , there is a maximum matching M' of T such that $M \cap E(T - L) = M' \cap E(T - L)$ and $M' \cap E(L) = E(L) \setminus M$ making l the unmatched vertex in T' instead of r .*

Proof. Let T be a tree without a perfect matching M and let T' be an odd M -alternating subtree of T with the bipartite sets X and Y as described in Theorem 5.18. Let r be the unmatched root of T' and let l be a leaf in T' with the path L from r to l in T .

We show that switching the matched and unmatched edges along the path L yields a maximum matching M' of T . Since L is an odd path in T between two vertices in X , it has an even number of edges and thus an odd number of vertices. Since r is unmatched in M and l is a leaf in T' , the edges of L alternate between unmatched and matched edges in M . Therefore, switching the matched and unmatched edges along L yields a new matching M' of T that matches l and leaves r unmatched. All other vertices on L remain matched in M' so the size of M' is the same as the size of M , making M' a maximum matching of T . Moreover, the tree T' is still an odd M' -alternating subtree of T with the same vertices and the same bipartite sets X, Y but with the new unmatched vertex l as the root instead of r . ■

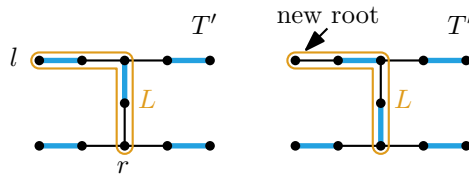


Figure 5.13: Odd alternating subtree T' of a tree T with the matching M on the left and M' on the right with the path L highlighted in orange and the matching edges colored blue.

This fact is also visually illustrated in Figure 5.13. We note that for an odd M -alternating subtree T' of T , with the root r in X , the tree $T' - r$ has a perfect matching, since r is the only unmatched vertex in T' .

We now show lower bounds on the size of the minimum feedback vertex set of $T' \boxtimes P_\ell$ for an odd M -alternating subtree T' of T . Note that $T' \boxtimes P_\ell$ is comparable to the strong product of an odd path and an odd path, which we analyzed in Theorem 5.10. Comparable to the strong grid, we analyze the two cases of $2m + 1 \geq \ell$ and $2m + 1 < \ell$ separately, where $|V(T')| = 2m + 1$.

We first show some helping lemmas that we later use in the proof of the lower bound.

Lemma 5.20: *Let T' be an odd M -alternating subtree of T with the maximum matching M and $|V(T')| = 2m + 1$ where $m \in \mathbb{N}^+$. Also let P_ℓ be an odd path with ℓ vertices and let X and Y be the bipartite sets of T' , with $|X| = m + 1$ and $|Y| = m$. Let r be the unmatched root and leaf of T' in X and let $v \in Y$ be the unique neighbor of r in T' . Then, there exists a minimum feedback vertex set of $T' \boxtimes P_\ell$ that contains either all vertices in $V(T') \times \{p_2\}$ or all vertices in $\{v\} \times V(P_\ell)$.*

Proof. We know that T' has at least two leaves, since T' is a tree with at least three vertices. By Lemma 5.19 we can assume that the unmatched root r of T' is a leaf in T' . Let $v \in Y$ be the unique neighbor of r in T' and let $w \in X$ be the unique neighbor of v in T' that is not r . The vertices (r, p_1) , (v, p_1) , (r, p_2) and (v, p_2) induce a K_4 subgraph in $T' \boxtimes P_\ell$. This part of the graph is similar to the vertices $v_{1,1}$, $v_{1,2}$, $v_{2,1}$ and $v_{2,2}$ in the strong product of two odd paths as analyzed in Lemma 5.8. Thus, we can conclude that there exists a minimum feedback vertex set of $T' \boxtimes P_\ell$ that contains either both vertices (r, p_2) and (v, p_2) or both vertices (v, p_1) and (v, p_2) .

More specifically, in the case that (v, p_1) and (v, p_2) are in the minimum feedback vertex set, by Lemma 5.9, we can conclude that all vertices in $\{v\} \times V(P_\ell)$ are in the minimum feedback vertex set.

The other case that (r, p_2) and (v, p_2) are in the minimum feedback vertex set is more complex as we are not working along the path factor, but the tree T' . However, we can still conclude that all vertices in $V(T') \times \{p_2\}$ are in the minimum feedback vertex set.

To see this, we first show that (w, p_2) is also in a minimum feedback vertex set of $T' \boxtimes P_\ell$ if (r, p_2) and (v, p_2) are in a minimum feedback vertex set of $T' \boxtimes P_\ell$. This follows from the fact that the vertices (v, p_1) , (w, p_1) , (v, p_2) and (w, p_2) also induce a K_4 subgraph in $T' \boxtimes P_\ell$, and as the neighborhood of (w, p_2) in $T' \boxtimes P_\ell$ is a superset of the neighborhood of (v, p_1) and (w, p_1) , if (r, p_2) and (v, p_2) are in a minimum feedback vertex set of $T' \boxtimes P_\ell$ then there must also be a minimum feedback vertex set of $T' \boxtimes P_\ell$ that contains (r, p_2) , (v, p_2) and (w, p_2) . Since we know that $T' - r$ has a perfect matching, the subgraph induced by the vertices $V(T' - r) \times \{p_1, p_2\}$ can be partitioned into m disjoint K_4 subgraphs, each having a minimum feedback vertex set size two. The set of vertices in $V(T') \times \{p_2\}$ contains exactly two vertices in each of these K_4 subgraphs and also contains all vertices in the subgraph induced by $V(T' - r) \times \{p_1, p_2\}$ with connecting edges to the remaining graph, except for (v, p_1) , which is not necessary as (r, p_2) is also in the minimum feedback vertex set.

Therefore, if (r, p_2) and (v, p_2) are in a minimum feedback vertex set of $T' \boxtimes P_\ell$ then there must also be a minimum feedback vertex set of $T' \boxtimes P_\ell$ that contains all vertices in $V(T') \times \{p_2\}$. ■

We use the previous lemma to show the following lower bound on the size of the minimum feedback vertex set of $T' \boxtimes P_\ell$ for an odd M -alternating subtree T' of T .

Theorem 5.21: *Let T' be an odd M -alternating subtree of T with the maximum matching M and $|V(T')| = 2m + 1$ where $m \in \mathbb{N}^+$. Also let P_ℓ be an odd path with ℓ vertices and let X and Y be the bipartite sets of T' , with $|X| = m + 1$ and $|Y| = m$. Then, the minimum feedback vertex set of $T' \boxtimes P_\ell$ has size exactly $\min\{\lfloor \frac{\ell}{2} \rfloor \cdot (2m + 1), m \cdot \ell\}$.*

Proof. We prove the lower bound only, since the upper bound follows from Lemma 5.11 as T' is a tree.

We prove the lower bound by induction on m and ℓ . For one base case, assume $\ell = 1$. Then, $T' \boxtimes P_\ell$ is isomorphic to T' and $\nabla(T') = 0$ since T' is a tree, so the base case holds and the minimum feedback vertex set has size $0 = \min\{\lfloor \frac{1}{2} \rfloor \cdot (2m + 1), m \cdot 1\}$ as stated. In the other base case, assume $m = 0$ so that $2m + 1 = 1$. Then, $T' \boxtimes P_\ell$ is isomorphic to P_ℓ and $\nabla(P_\ell) = 0$, also matching the lower bound of $\min\{\lfloor \frac{\ell}{2} \rfloor \cdot 1, 0 \cdot \ell\} = 0$ as stated.

In the inductive step let T' be an odd M -alternating subtree of a tree T with the maximum matching M and let $\ell \geq 3$ odd and $|V(T')| = 2m + 1 > 1$. Assume that the lower bound holds for any strong product of an odd matching-alternating tree T'' of size $2m' + 1$, and a path $P_{\ell'}$ with odd ℓ' , that is a strict subgraph of $T' \boxtimes P_\ell$.

By Lemma 5.19 we can assume that the unmatched root r of T' is a leaf in T' . Let $v \in Y$ be the unique neighbor of r in T' .

From Lemma 5.20, there exists a minimum feedback vertex set of $T' \boxtimes P_\ell$ that contains either all vertices in $V(T') \times \{p_2\}$ or all vertices in $\{v\} \times V(P_\ell)$ where v is the unique neighbor of the unmatched root r of T' .

Case 1: The vertices in $V(T') \times \{p_2\}$ are in the minimum feedback vertex set. Removing the vertices in $V(T') \times \{p_2\}$ from $T' \boxtimes P_\ell$ separates the graph into two components, one being isomorphic to T' with the vertex set $V(T') \times \{p_1\}$ and the other induced by the vertices $V(T') \times \{p_3, p_4, \dots, p_\ell\}$. Let T'' be the induced subgraph of T' with the vertex set $V(T'') = V(T') \times (V(P_\ell) \setminus \{p_1, p_2\})$. The graph T'' is isomorphic to $T' \boxtimes P_{\ell-2}$, so it is a strict subgraph of $T' \boxtimes P_\ell$ and thus by the inductive hypothesis, the minimum feedback vertex set of T'' has size at least $\min\{\lfloor \frac{\ell-2}{2} \rfloor \cdot (2m + 1), m \cdot (\ell - 2)\}$. Therefore, in this case the minimum feedback vertex set of $T' \boxtimes P_\ell$ has size at least $2m + 1 + \min\{\lfloor \frac{\ell-2}{2} \rfloor \cdot (2m + 1), m \cdot (\ell - 2)\} = \min\{\lfloor \frac{\ell}{2} \rfloor \cdot (2m + 1), m \cdot \ell + 1\}$ matching the lower bound in the first case. Since $m \cdot \ell + 1 > m \cdot \ell$, this lower bound is at least $\min\{\lfloor \frac{\ell}{2} \rfloor \cdot (2m + 1), m \cdot \ell\}$.

Case 2: The vertices in $\{v\} \times V(P_\ell)$ are in the minimum feedback vertex set. Let T'' be the induced subgraph of T' with the vertex set $V(T'') = (V(T') \setminus \{r, v\}) \times V(P_\ell)$. The graph T'' is isomorphic to $(T' - \{r, v\}) \boxtimes P_\ell$ which is a strict subgraph of $T' \boxtimes P_\ell$ and thus by the inductive hypothesis, the minimum feedback vertex set of T'' has size at least $\min\{\lfloor \frac{\ell}{2} \rfloor \cdot (2m - 1), (m - 1) \cdot \ell\}$. This results in a feedback vertex set size of at least $\ell + \min\{\lfloor \frac{\ell}{2} \rfloor \cdot (2m - 1), (m - 1) \cdot \ell\} = \min\{\lfloor \frac{\ell}{2} \rfloor \cdot (2m + 1) + 1, m \cdot \ell\}$ matching the lower bound in the second case. Since $\lfloor \frac{\ell}{2} \rfloor \cdot (2m + 1) + 1 > \lfloor \frac{\ell}{2} \rfloor \cdot (2m + 1)$, this lower bound is at least $\min\{\lfloor \frac{\ell}{2} \rfloor \cdot (2m + 1), m \cdot \ell\}$.

Therefore, since by Lemma 5.20 at least one of the two cases must apply for a minimum feedback vertex set of $T' \boxtimes P_\ell$, and both cases yield a lower bound of at least $\min\{\lfloor \frac{\ell}{2} \rfloor \cdot (2m + 1), m \cdot \ell\}$, the lower bound is confirmed. Together with the upper bound from Lemma 5.11, this concludes the proof of the theorem. $\blacksquare$

Corollary 5.22: *Let T' be an odd M -alternating subtree with $|V(T')| = 2m + 1$ and P_ℓ an odd path with $\ell = 2k + 1$. Then the minimum feedback vertex set of $T' \boxtimes P_\ell$ strictly follows one of two structural constructions, depending on the initial size ratio of $2m + 1$ and ℓ :*

1. *If $2m + 1 > \ell$, or $m > k$, then $(2m + 1)k < m(2k + 1)$, meaning the construction containing all vertices on even layers is strictly smaller.*

2. If $2m + 1 < \ell$, or $m < k$, then $m(2k + 1) < (2m + 1)k$, meaning the construction containing all vertices in the m columns corresponding to the vertex cover Y of T' is strictly smaller.
3. If $2m + 1 = \ell$, both constructions yield the exact same size of $m \cdot \ell = \lfloor \frac{\ell}{2} \rfloor \cdot (2m + 1)$, so both constructions are optimal.

Furthermore, as each structural reduction step recursively reduces either k or m while leaving the other constant, the strict inequality is preserved throughout all inductive subproblems. Thus, an optimal feedback vertex set never requires alternating between layers and columns.

For a tree T without a perfect matching, that contains a single unmatched vertex in a maximum matching, and thus a single odd M -alternating subtree T' , the previous theorem, combined with the result on trees with a perfect matching, already gives a complete characterization of the minimum feedback vertex set of $T \boxtimes P_\ell$. However, for a tree T without a perfect matching that contains multiple unmatched vertices in a maximum matching, and thus multiple odd M -alternating subtrees $T'_1, T'_2, \dots, T'_r$ for $r \geq 2$, this proof only holds if no odd M -alternating subtrees overlap.

5.3.3 Dynamic Programming Approach for General Trees

On all previous results and empirical experiments on the minimum feedback vertex set on the strong product of a tree and an odd path $T \boxtimes P_\ell$ on programmatically generated random instances, for a vertex $v \in V(T)$, we observed that a minimal feedback vertex set F of G either contains all vertices in the column $\{v\} \times V(P_\ell)$, it contains the vertices in the even layers $\{v\} \times \{p_i \in V(P_\ell) \mid i \text{ is even}\}$, or it contains no vertices in the column $\{v\} \times V(P_\ell)$. We conjecture that this observation holds for any tree T and any odd path P_ℓ .

Conjecture 5.23: *Let T be a tree and P_ℓ be an odd path. Let $G = T \boxtimes P_\ell$. Then, there exists a minimum feedback vertex set F of G such that for any vertex $v \in V(T)$, one of the following holds:*

1. $\{v\} \times V(P_\ell) \subseteq F$,
2. $\{v\} \times V(P_\ell) \cap F = \{v\} \times \{p_i \in V(P_\ell) \mid i \text{ is even}\}$, or
3. $\{v\} \times V(P_\ell) \cap F = \emptyset$.

Assuming the correctness of this conjecture, we can compute the minimum feedback vertex set of $G = T \boxtimes P_\ell$ efficiently using dynamic programming. By rooting the tree at an arbitrary vertex r , we process the tree from the leaves to the root. For each node $v \in V(T)$, we compute the minimal cost of the subtree rooted at v given that v takes state $S \in \{\text{all}, \text{even}, \text{none}\}$. We define the cost functions as follows:

$$\begin{aligned}
 c(v, \text{all}) &= \ell + \sum_{w \in \text{Children}(v)} \min\{c(w, \text{all}), c(w, \text{even}), c(w, \text{none})\} \\
 c(v, \text{even}) &= \lfloor \frac{\ell}{2} \rfloor + \sum_{w \in \text{Children}(v)} \min\{c(w, \text{all}), c(w, \text{even})\} \\
 c(v, \text{none}) &= 0 + \sum_{w \in \text{Children}(v)} c(w, \text{all})
 \end{aligned}$$

For a leaf node v , the children set is empty, yielding the base costs $c(v, all) = \ell$, $c(v, even) = \lfloor \frac{\ell}{2} \rfloor$, and $c(v, none) = 0$. The overall size of the minimum feedback vertex set for the entire graph is then given by $\min(c(r, all), c(r, even), c(r, none))$. Since computing the cost for each node relies only on the states of its children, assuming the conjecture holds, this dynamic program computes a minimal feedback vertex set size in linear time.

5.4 Strong Product of Trees

In the following, we consider the strong product of two trees T_1 and T_2 . $G = T_1 \boxtimes T_2$ has a structure that can be interpreted as $|V(T_2)|$ copies of T_1 with additional edges between neighboring copies of T_1 that come from the edges in T_2 , or equivalently as $|V(T_1)|$ copies of T_2 with additional edges between neighboring copies of T_2 that come from the edges in T_1 .

We generalize previous results on the strong product of a tree and a path to the strong product of two trees. We first show that the minimum feedback vertex set of G has size at most $\min\{\tau(T_1) \cdot |V(T_2)|, \tau(T_2) \cdot |V(T_1)|\}$ by showing that the sets $F_1 = C_1 \times V(T_2)$ and $F_2 = V(T_1) \times C_2$ are valid feedback vertex sets of G , where C_1 is a minimum vertex cover of T_1 and C_2 is a minimum vertex cover of T_2 .

Lemma 5.24: *Let T_1 and T_2 be two trees and let $G = T_1 \boxtimes T_2$. Let C_1 be a minimum vertex cover of T_1 and C_2 be a minimum vertex cover of T_2 . The sets $F_1 = C_1 \times V(T_2)$ and $F_2 = V(T_1) \times C_2$ are valid feedback vertex sets of G .*

Proof. First we show that F_1 is a valid feedback vertex set of G . Let $uv \in E(T_1)$ be an edge in T_1 . By definition, the vertex cover C_1 of T_1 contains at least one of the two vertices u or v . Therefore, in $G = T_1 \boxtimes T_2$, for any edge of the form $(u, t)(v, t') \in E(G)$ with $u, v \in V(T_1)$ and $t, t' \in V(T_2)$ where t and t' are either the same vertex or adjacent in T_2 , at least one of the vertices (u, t) and (v, t') is in F_1 since either u or v is in C_1 . The only remaining edges in $G - F_1$ are of the form $(u, t)(u, t')$ where t and t' are adjacent in T_2 , so the remaining graph is a forest of trees isomorphic to T_2 and thus contains no cycles.

The validity of F_2 as a feedback vertex set of G follows by symmetry, since the same argument applies with the roles of T_1 and T_2 swapped.

Since we have shown, that both $G - F_1$ and $G - F_2$ are acyclic, both F_1 and F_2 are valid feedback vertex sets of G . ■

Corollary 5.25: *Let T_1 and T_2 be two trees and let $G = T_1 \boxtimes T_2$. Then, the minimum feedback vertex set F has size at most $\min\{\tau(T_1) \cdot |V(T_2)|, \tau(T_2) \cdot |V(T_1)|\}$.*

Proof. This follows directly from Lemma 5.24, since the size of F_1 is $\tau(T_1) \cdot |V(T_2)|$ and the size of F_2 is $\tau(T_2) \cdot |V(T_1)|$. ■

In the case that one of the trees T_1 and T_2 has a perfect matching, the upper bound from Corollary 5.25 is tight and thus gives the exact size of the minimum feedback vertex set of G . We only show the case for T_1 having a perfect matching, since the other case is symmetric.

Theorem 5.26: *Let T_1 be a tree with a perfect matching and T_2 be any tree and let $G = T_1 \boxtimes T_2$. Then, G has a minimum feedback vertex set of size $|V(T_1)| \cdot \tau(T_2)$.*

Proof. The size of a maximum matching of a tree T is $\tau(T)$ according to the theorem of König, meaning that there are $\tau(T)$ vertex-disjoint edges in a tree T [Kön31].

Thus, similar to the strong product of a tree and a path, G has $\tau(T_1) \cdot \tau(T_2)$ copies of K_4 as subgraphs. Since T_1 has a perfect matching, $\tau(T_1) = \frac{|V(T_1)|}{2}$. Therefore, there are $\frac{|V(T_1)|}{2} \cdot \tau(T_2)$ vertex-disjoint K_4 subgraphs in G . As per Lemma 5.5, a K_4 has a minimum feedback vertex set of size 2, so any feedback vertex set of G must contain at least two vertices from each of these K_4 copies. Thus, any feedback vertex set of G contains at least $2 \cdot \frac{|V(T_1)|}{2} \cdot \tau(T_2) = |V(T_1)| \cdot \tau(T_2)$ vertices. The upper bound from Corollary 5.25 gives a valid feedback vertex set of this size, so the minimum feedback vertex set of G has size $|V(T_1)| \cdot \tau(T_2)$. ■

Note that we only used the theorem of König in the proof of the lower bound in Theorem 5.26 [Kön31]. Thus, it holds for any bipartite graphs B_1 and B_2 . However, the same bound does not hold as an upper bound if B_1 and B_2 are not acyclic. We therefore generalize the lower bound to the following corollary.

Corollary 5.27: *Let B_1 be a bipartite graph with a perfect matching and let B_2 be any bipartite graph. Then every feedback vertex set of $G = B_1 \boxtimes B_2$ has size at least $|V(B_1)| \cdot \tau(B_2)$.*

5.5 Strong Product of Three Paths

In the following, we consider the strong product of three paths P_x , P_y and P_z . A graph of the form $G = P_x \boxtimes P_y \boxtimes P_z$ has a grid-like cuboid structure, as shown in Figure 5.14. Since the strong product is commutative and associative, the order of the paths does not matter. Thus, when specifying the parity of some of the paths, we may assign parities without loss of generality. We use the following notation for vertices in such a graph. We denote by $v_{i,j,k}$ for $i \in [x]$, $j \in [y]$, $k \in [z]$ the vertex in $V(G)$ corresponding to the i -th vertex in P_x , the j -th vertex in P_y and the k -th vertex in P_z .

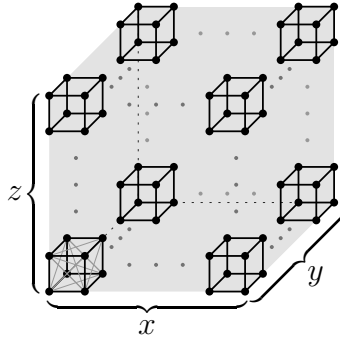


Figure 5.14: The cuboid structure of the graph $P_x \boxtimes P_y \boxtimes P_z$ (simplified by removing most tensor edges).

Lemma 5.28: *Let P_x , P_y and P_z be three even paths. Then, the graph $G = P_x \boxtimes P_y \boxtimes P_z$ contains $\frac{x}{2} \cdot \frac{y}{2} \cdot \frac{z}{2}$ pairwise vertex-disjoint copies of K_8 .*

Proof. By Lemma 5.4, the graph $P_x \boxtimes P_y$ contains $\frac{x}{2} \cdot \frac{y}{2}$ vertex-disjoint copies of K_4 as subgraphs. Furthermore, P_z has a partition into $\frac{z}{2}$ vertex-disjoint copies of K_2 since P_z is an even path.

By the definition of the strong product, for every edge $v_1 v_2 \in E(P_x \boxtimes P_y)$ and every edge $w_1 w_2 \in E(P_z)$, the vertices (v_1, w_1) , (v_1, w_2) , (v_2, w_1) , (v_2, w_2) induce a K_4 subgraph in G . Thus, for any copy of K_4 on the vertices $\{a_1, a_2, a_3, a_4\}$ in $P_x \boxtimes P_y$ and any copy of K_2 with vertices $\{b_1, b_2\}$ in P_z , the vertices corresponding to the Cartesian product of these two copies $\{(a_i, b_j) \mid i \in [4], j \in [2]\}$ induce a K_8 subgraph in G .

For that reason there are $\frac{x}{2} \cdot \frac{y}{2} \cdot \frac{z}{2}$ vertex-disjoint copies of K_8 in G , one for each distinct triple of edges in the matchings. This is the maximum number of vertex-disjoint copies of K_8 in G since G has $x \cdot y \cdot z$ vertices and $\frac{x}{2} \cdot \frac{y}{2} \cdot \frac{z}{2} \cdot 8 = x \cdot y \cdot z$. ■

Lemma 5.29: *A minimum feedback vertex set of the complete graph K_8 on 8 vertices has size 6.*

Proof. Let K_8 be the complete graph on the vertices $v_1, \dots, v_8$. Since $v_i v_j \in E(K_8)$ for every $i, j \in [8]$ with $i \neq j$, any three distinct vertices v_i, v_j, v_k with $i, j, k \in [8]$ pairwise distinct induce a triangle in K_8 . Any two vertices v_i, v_j induce a single edge, which is acyclic. Therefore, each minimum feedback vertex set of K_8 contains exactly 6 vertices. ■

Theorem 5.30: *Let P_x, P_y and P_z be three even paths. Then, every minimum feedback vertex set of $G = P_x \boxtimes P_y \boxtimes P_z$ has size $\frac{3}{4} \cdot x \cdot y \cdot z$ and there is a minimum feedback vertex set of G containing all vertices $v_{i,j,k}$ with $i = x$ and $k = z$.*

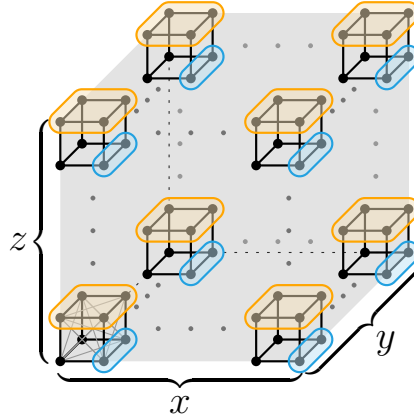


Figure 5.15: Construction of a minimum feedback vertex set of $P_x \boxtimes P_y \boxtimes P_z$ with x, y and z even (simplified). The orange vertices form F_1 , the blue vertices form F_2 , the union of the minimum feedback vertex sets of the remaining $P_x \boxtimes P_y$ copies in $G - F_1$.

Proof. First we show that every feedback vertex set of G has size at least $\frac{3}{4} \cdot x \cdot y \cdot z$. By Lemma 5.28, G has $\frac{x}{2} \cdot \frac{y}{2} \cdot \frac{z}{2}$ pairwise vertex-disjoint copies of K_8 . As per Lemma 5.29, a minimum feedback vertex set of K_8 has size 6. Then, Lemma 5.3 yields a lower bound of $6 \cdot \frac{x}{2} \cdot \frac{y}{2} \cdot \frac{z}{2} = \frac{3}{4} \cdot x \cdot y \cdot z$ vertices for any feedback vertex set of G .

We now show that there is a feedback vertex set of G with size $\frac{3}{4} \cdot x \cdot y \cdot z$. Let $F_1 = \{v_{i,j,k} \in V(G) \mid i \in [x], j \in [y], k \equiv 0 \pmod{2}\}$ be the set of vertices of G with even k -index. The set F_1 has size $\frac{z}{2} \cdot x \cdot y$ and the graph $G - F_1$ is the disjoint union of $\frac{z}{2}$ copies of $P_x \boxtimes P_y$. By Theorem 5.6, we know that the minimum feedback vertex set of $P_x \boxtimes P_y$ has size $\nabla(P_x \boxtimes P_y) = \frac{1}{2} \cdot x \cdot y$. Combining these feedback vertex sets for every copy of $P_x \boxtimes P_y$ in $G - F_1$ results in the set $F_2 = \{v_{i,j,k} \in V(G) \mid i \equiv 0 \pmod{2}, k \equiv 1 \pmod{2}\}$. Both F_1 and F_2 are displayed in Figure 5.15 with orange and blue vertices respectively. The set $F = F_1 \cup F_2$ is a valid feedback vertex set of G with size $\frac{z}{2} \cdot x \cdot y + \frac{z}{2} \cdot \nabla(P_x \boxtimes P_y) = \frac{z}{2} \cdot x \cdot y + \frac{z}{2} \cdot \frac{1}{2} \cdot x \cdot y = \frac{3}{4} \cdot x \cdot y \cdot z$. F contains all vertices $v_{i,j,k}$ with $i = x$ and $k = z$ since x and z are even. ■

Theorem 5.31: *Let P_x and P_y be even paths and let P_z be an odd path. Then, every minimum feedback vertex set of $G = P_x \boxtimes P_y \boxtimes P_z$ has size $\frac{3}{4} \cdot x \cdot y \cdot (z - \frac{1}{3})$ and there is a minimum feedback vertex set of G containing all vertices $v_{i,j,k}$ with $i = x$.*

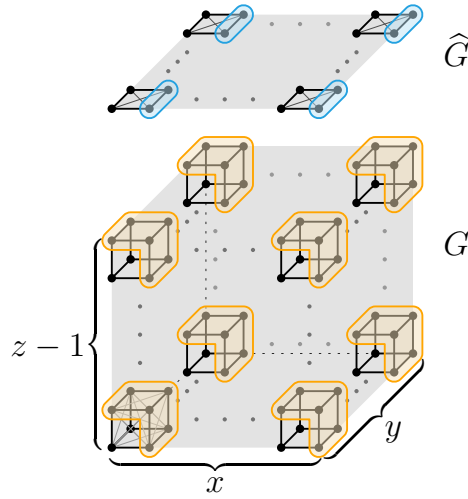


Figure 5.16: Construction of a minimum feedback vertex set of $P_x \boxtimes P_y \boxtimes P_z$ with x and y even and z odd (simplified). The orange vertices form $F_1 \cup F_2$, the blue vertices form the minimum feedback vertex set F_3 of $\widehat{G}$.

Proof. Similar to the previous case, we first show that every feedback vertex set of G has size at least $\frac{3}{4} \cdot x \cdot y \cdot (z - \frac{1}{3})$. By Theorem 5.30, every minimum feedback vertex set of the subgraph $G' = P_x \boxtimes P_y \boxtimes P_{z-1}$ of G with only even paths has size $\frac{3}{4} \cdot x \cdot y \cdot (z - 1)$. The remaining graph $\widehat{G} = G - G'$ is isomorphic to $P_x \boxtimes P_y$ and by Theorem 5.6 has a minimum feedback vertex set size of $\frac{1}{2} \cdot x \cdot y$. Since G' and $\widehat{G}$ are vertex-disjoint subgraphs of G , by Lemma 5.3, the sum of the lower bounds of the feedback vertex set sizes of G' and $\widehat{G}$ is a lower bound for the feedback vertex set size of G . This results in a minimum feedback vertex set size of at least $\frac{3}{4} \cdot x \cdot y \cdot (z - 1) + \frac{1}{2} \cdot x \cdot y = \frac{3}{4} \cdot x \cdot y \cdot (z - \frac{1}{3})$.

To show that the bound is tight, we construct the feedback vertex set $F' = F_1 \cup F_2$ of G' similarly to Theorem 5.30. This set has size $\frac{3}{4} \cdot x \cdot y \cdot (z - 1)$ and in particular contains all vertices $v_{i,j,k}$ with $i = x$ and $k = z - 1$, which are the only vertices in G' , adjacent to the vertices of $\widehat{G}$. Thus, there are no edges between the components of G' and $\widehat{G}$ in $G - F'$ and any minimum feedback vertex set of $\widehat{G} \cong P_x \boxtimes P_y$ in union with F' is a valid feedback vertex set of G . Again by Theorem 5.6 the set $F_3 = \{v_{i,j,z} \in V(\widehat{G}) \mid i \equiv 0 \pmod{2}\}$ is a minimum feedback vertex set of $\widehat{G}$ and has size $\frac{xy}{2}$. In Figure 5.16 the sets $F_1 \cup F_2$ and F_3 are displayed with orange and blue vertices, respectively. Therefore, $F = F_1 \cup F_2 \cup F_3$ is a valid feedback vertex set of G with size $\frac{3}{4} \cdot x \cdot y \cdot (z - 1) + \frac{1}{2} \cdot x \cdot y = \frac{3}{4} \cdot x \cdot y \cdot (z - \frac{1}{3})$, which is thus also a minimum feedback vertex set of G . In particular, F contains all vertices $v_{i,j,k}$ with $i = x$. ■

Theorem 5.32: Let P_x and P_z be odd paths and let P_y be an even path. Then, every minimum feedback vertex set of $G = P_x \boxtimes P_y \boxtimes P_z$ has size $\frac{1}{4} \cdot y \cdot (3xz - x - z - 1)$.

Proof. We first show that every feedback vertex set of G has size at least $\frac{1}{4} \cdot y \cdot (3xz - x - z - 1)$. To do so we look at two vertex-disjoint subgraphs of G . For this let $\widehat{G}$ be the graph induced by the vertices $v_{i,j,k} \in V(G)$ with $i = x$ and let $G' = G - \widehat{G} \cong P_{x-1} \boxtimes P_y \boxtimes P_z$ be the subgraph induced by the remaining vertices. As $x - 1$ and y are even and z is odd, by Theorem 5.31, every minimum feedback vertex set of G' has size at least $\frac{3}{4} \cdot (x - 1) \cdot y \cdot (z - \frac{1}{3})$. The graph $\widehat{G}$ is

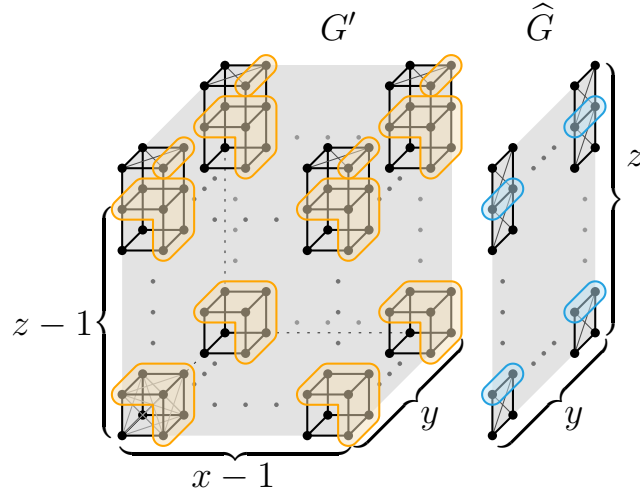


Figure 5.17: Construction of a minimum feedback vertex set of $P_x \boxtimes P_y \boxtimes P_z$ with y even and x and z odd (simplified). The orange vertices form $F_1 \cup F_2 \cup F_3$, a minimum feedback vertex set of $G' = P_{x-1} \boxtimes P_y \boxtimes P_z$, and the blue vertices form the minimum feedback vertex set F_4 of $\widehat{G} = G - G'$.

isomorphic to $P_y \boxtimes P_z$ with even y and odd z , so by Theorem 5.7, every feedback vertex set of $\widehat{G}$ has size at least $\frac{1}{2} \cdot y \cdot z$. The subgraphs G' and $\widehat{G}$ are vertex-disjoint, so by Lemma 5.3 every feedback vertex set of G has size at least $\frac{3}{4} \cdot (x-1) \cdot y \cdot (z - \frac{1}{3}) + \lfloor \frac{z}{2} \rfloor \cdot y = \frac{1}{4} \cdot y \cdot (3xz - x - z - 1)$.

To show that a valid feedback vertex set of G with size $\frac{1}{4} \cdot y \cdot (3xz - x - z - 1) = \frac{3}{4} \cdot (x-1) \cdot y \cdot (z-1) + \lfloor \frac{z}{2} \rfloor \cdot y$ exists, we use the construction of Theorem 5.31 for the subgraph $G' = P_{x-1} \boxtimes P_y \boxtimes P_z$, which has a minimum feedback vertex set of size $\frac{3}{4} \cdot (x-1) \cdot y \cdot (z - \frac{1}{3})$ that in particular contains every vertex $v_{i,j,k}$ with $i = x-1$, which are the only vertices in G' adjacent to vertices in $\widehat{G}$. This is also visible in Figure 5.17, where the orange vertices form $F_1 \cup F_2 \cup F_3$, and the blue vertices form a minimum feedback vertex set of $\widehat{G}$. Therefore, a minimal feedback vertex set F_4 of $\widehat{G}$ with size $\lfloor \frac{z}{2} \rfloor \cdot y$ together with this minimum feedback vertex set of G' is a valid feedback vertex set of G with size $\frac{3}{4} \cdot (x-1) \cdot y \cdot (z - \frac{1}{3}) + \lfloor \frac{z}{2} \rfloor \cdot y = \frac{1}{4} \cdot y \cdot (3xz - x - z - 1)$ and thus a minimum feedback vertex set of G . ■

5.6 Strong Product of d Paths

We now generalize the approach of Section 5.5 to the strong product of d paths where at least one path has an even number of vertices. The proof uses induction on both the number of paths d and the number of odd paths k .

To do so, we first generalize the existence of vertex-disjoint complete subgraphs.

Lemma 5.33: *Let $P_{x_1}, \dots, P_{x_d}$ be d paths with $x_1, \dots, x_d$ even. Then, the graph $G = P_{x_1} \boxtimes \dots \boxtimes P_{x_d}$ contains $\prod_{m=1}^d \frac{x_m}{2}$ pairwise vertex-disjoint copies of K_{2^d} .*

Proof. The proof is similar to the proof of Lemma 5.28. As there are $\frac{x_j}{2}$ vertex-disjoint copies of K_2 in each path P_{x_j} for $j \in [d]$, every combination of these disjoint copies of K_2 in the paths forms a complete graph on 2^d vertices in G . This results in $\prod_{m=1}^d \frac{x_m}{2}$ pairwise vertex-disjoint copies of K_{2^d} in G . ■

Lemma 5.34: A minimum feedback vertex set of the complete graph K_{2^d} on 2^d vertices has size $2^d - 2$.

Proof. Let K_{2^d} be the complete graph on the vertices $v_1, \dots, v_{2^d}$. Since $v_i v_j \in E(K_{2^d})$ for every $i, j \in [2^d]$ with $i \neq j$, any three pairwise distinct vertices v_i, v_j, v_k with $i, j, k \in [2^d]$ induce a triangle in K_{2^d} . Any two distinct vertices v_i, v_j induce a single edge, which is acyclic. Therefore, each minimum feedback vertex set of K_{2^d} contains exactly $2^d - 2$ vertices. ■

Theorem 5.35: Let $P_{x_1}, \dots, P_{x_d}$ be d even paths with $d \geq 3$ and let $G = P_{x_1} \boxtimes \dots \boxtimes P_{x_d}$. Then, there exists a minimum feedback vertex set of G of size $\frac{2^d-2}{2^d} \cdot \prod_{m=1}^d x_m = \frac{2^d-2}{2^d} \cdot |V(G)|$ containing all vertices $v_{i_1, \dots, i_d}$ with $i_j \equiv 0 \pmod{2}$ for at least one $j \in \{2, \dots, d\}$.

Proof. By Lemma 5.33, G has $\prod_{m=1}^d \frac{x_m}{2}$ pairwise vertex-disjoint copies of K_{2^d} . By Lemma 5.34, a minimum feedback vertex set of K_{2^d} has size $2^d - 2$. Thus, for each distinct copy of K_{2^d} in G , the feedback vertex set of G contains at least $2^d - 2$ vertices. This results in a lower bound of $(2^d - 2) \cdot \prod_{m=1}^d \frac{x_m}{2} = \frac{2^d-2}{2^d} \cdot \prod_{m=1}^d x_m$ vertices for any minimum feedback vertex set of G .

To show that there is a feedback vertex set of G with size $\frac{2^d-2}{2^d} \cdot \prod_{m=1}^d x_m$, that fulfills the additional property, we use induction on d .

For the base case of $d = 3$, the statement holds by Theorem 5.30, as the set $F = \{v_{i_1, i_2, i_3} \in V(G) \mid i_3 \text{ is even}\} \cup \{v_{i_1, i_2, i_3} \in V(G) \mid i_2 \text{ is even, } i_3 \text{ is odd}\}$ is a minimum feedback vertex set of $G = P_{x_1} \boxtimes P_{x_2} \boxtimes P_{x_3}$ with size $\frac{3}{4} \cdot x_1 \cdot x_2 \cdot x_3$. And since F contains all vertices v_{i_1, i_2, i_3} with at least one of i_2 or i_3 even, the required additional property holds.

For the induction step, let $d > 3$ and assume that the statement holds for $d - 1$ paths. Let $F_1 = \{v_{i_1, \dots, i_d} \in V(G) \mid i_d \equiv 0 \pmod{2}\}$ be the set of vertices of G with even i_d -index. The set F_1 has size $\frac{x_d}{2} \cdot \prod_{m=1}^{d-1} x_m$ and the graph $G - F_1$ is the disjoint union of $\frac{x_d}{2}$ copies of $P_{x_1} \boxtimes \dots \boxtimes P_{x_{d-1}}$. By the induction hypothesis, each of these copies has a minimum feedback vertex set of size $\frac{2^{d-1}-2}{2^{d-1}} \cdot \prod_{m=1}^{d-1} x_m$ containing all vertices $v_{i_1, \dots, i_{d-1}}$ with $i_j \equiv 0 \pmod{2}$ for at least one $j \in \{2, \dots, d-1\}$. We define the union of these feedback vertex sets for every copy of $P_{x_1} \boxtimes \dots \boxtimes P_{x_{d-1}}$ in $G - F_1$ as $F_2 = \{v_{i_1, \dots, i_d} \in V(G) \mid i_j \equiv 0 \pmod{2} \text{ for at least one } j \in \{2, \dots, d-1\}, i_d \equiv 1 \pmod{2}\}$. Then, $F = F_1 \cup F_2$ is a valid feedback vertex set of G with size $\frac{x_d}{2} \cdot (\prod_{m=1}^{d-1} x_m) + \frac{x_d}{2} \cdot \frac{2^{d-1}-2}{2^{d-1}} \cdot \prod_{m=1}^{d-1} x_m = \frac{2^d-2}{2^d} \cdot \prod_{m=1}^d x_m$. By the definition of F_1 and F_2 , F contains all vertices $v_{i_1, \dots, i_d}$ with $i_j \equiv 0 \pmod{2}$ for at least one $j \in \{2, \dots, d\}$ fulfilling the required additional property. ■

In the following, we give a general formula for the feedback vertex set size of the strong product of d paths with k odd paths. To state the formula, we first show that there always exists a valid partition of G into two simpler subgraphs.

Again by the commutativity of the strong product, we can order the paths such that the even paths are the first $d - k$ paths and the odd paths are the last k paths.

Lemma 5.36: Let $P_{x_1}, \dots, P_{x_d}$ be d paths with $d \geq 2$ and let exactly k of these be odd paths, where $0 \leq k \leq d - 2$. Let $G = P_{x_1} \boxtimes \dots \boxtimes P_{x_d}$ and let $P_{x_1}, \dots, P_{x_{d-k}}$ be the even paths in G . Then, there exists a minimum feedback vertex set of G containing all vertices $v_{i_1, \dots, i_d} \in V(G)$ with i_a even for at least one $a \in \{2, \dots, d - k\}$.

Proof. We prove the statement by induction on the number of odd paths k in G .

For the base case $k = 0$, all paths in G are even. By Theorem 5.35, there exists a minimum feedback vertex set of G containing all vertices $v_{i_1, \dots, i_d} \in V(G)$ that have an even index in at least one even path, except for P_{x_1} , which is the required property.

For the induction step, let $1 \leq k \leq d - 2$ with $P_{x_{d-k+1}}, \dots, P_{x_d}$ being the odd paths. Let $j = d - k + 1$ and let P_{x_j} be the first of the odd paths. We partition G into two vertex-disjoint subgraphs. The partition $\widehat{G}$ induced by the vertices with maximal j -index $i_j = x_j$, and G' induced by the vertices with $i_j \leq x_j - 1$. Both $\widehat{G}$ and G' contain at least two even paths, since G has at least two even paths. Further, $\widehat{G}$ is isomorphic to the strong product of $d - 1$ paths with $k - 1$ odd paths, and G' is isomorphic to the strong product of d paths with $k - 1$ odd paths, since $x_j - 1$ is even. As both subgraphs have at least two even path factors, including P_{x_1} and strictly fewer odd paths than G , we apply the induction hypothesis to both.

Thus, $\widehat{G}$ has a minimum feedback vertex set $\widehat{F}$ and G' has a minimum feedback vertex set F' , both containing all vertices that have an even i_a -index for at least one of the even path factors P_{x_a} , and $a \neq 1$.

Let $F = \widehat{F} \cup F'$. As $j - 1$ is even, F' and therefore F contains all vertices of G with an even i_j -index in particular. Thus, removing F from G entirely disconnects $\widehat{G}$ from G' . Since $\widehat{G}$ and G' are vertex-disjoint, by Lemma 5.3, F is a valid and minimal feedback vertex set for G with the required property. $\blacksquare$

Using this, we give the following recursive formula for the feedback vertex set size of the strong product of d paths with k odd paths.

Theorem 5.37: *Let $P_{x_1}, \dots, P_{x_d}$ be d paths with $d \geq 3$, and let exactly k of these paths have an odd number of vertices, where $1 \leq k \leq d - 1$. Let $G = P_{x_1} \boxtimes \dots \boxtimes P_{x_d}$ and let $P_{x_1}, \dots, P_{x_{d-k}}$ be the even paths in G with $j = d - k + 1$ being the index of the first odd path. Then, every minimum feedback vertex set of G has size $\nabla(P_{x_1} \boxtimes \dots \boxtimes P_{x_{j-1}} \boxtimes P_{x_{j-1}} \boxtimes P_{x_{j+1}} \boxtimes \dots \boxtimes P_{x_d}) + \nabla(P_{x_1} \boxtimes \dots \boxtimes P_{x_{j-1}} \boxtimes P_1 \boxtimes P_{x_{j+1}} \boxtimes \dots \boxtimes P_{x_d})$.*

Proof. We partition G into two vertex-disjoint subgraphs, $\widehat{G}$ and G' . Let $\widehat{G}$ be the subgraph of G induced by the vertices $v_{i_1, \dots, i_d} \in V(G)$ with maximal j -index $i_j = x_j$. Since this partition restricts the j -th path to a single vertex, $\widehat{G}$ is isomorphic to $P_{x_1} \boxtimes \dots \boxtimes P_{x_{j-1}} \boxtimes P_1 \boxtimes P_{x_{j+1}} \boxtimes \dots \boxtimes P_{x_d} \cong P_{x_1} \boxtimes \dots \boxtimes P_{x_{j-1}} \boxtimes P_{x_{j+1}} \boxtimes \dots \boxtimes P_{x_d}$. The remaining graph $G' = G - \widehat{G}$ is induced by the vertices with j -index $i_j \leq x_j - 1$, and is thus isomorphic to $P_{x_1} \boxtimes \dots \boxtimes P_{x_{j-1}} \boxtimes P_{x_{j-1}} \boxtimes P_{x_{j+1}} \boxtimes \dots \boxtimes P_{x_d}$, where the length of the j -th path is reduced by one, making it even.

Since $\widehat{G}$ and G' are vertex-disjoint, the sum of their minimum feedback vertex set sizes is a lower bound for the feedback vertex set size of G by Lemma 5.3. Thus, $\nabla(G) \geq \nabla(G') + \nabla(\widehat{G}) = \nabla(P_{x_1} \boxtimes \dots \boxtimes P_{x_{j-1}} \boxtimes P_{x_{j-1}} \boxtimes P_{x_{j+1}} \boxtimes \dots \boxtimes P_{x_d}) + \nabla(P_{x_1} \boxtimes \dots \boxtimes P_{x_{j-1}} \boxtimes P_1 \boxtimes P_{x_{j+1}} \boxtimes \dots \boxtimes P_{x_d})$.

To show that this bound is tight, let $\widehat{F}$ be a minimum feedback vertex set of $\widehat{G}$ with size $\nabla(P_{x_1} \boxtimes \dots \boxtimes P_{x_{j-1}} \boxtimes P_{x_{j+1}} \boxtimes \dots \boxtimes P_{x_d})$.

The only edges connecting $\widehat{G}$ and G' in G are those between vertices with $i_j = x_j$ and $i_j = x_j - 1$. Since x_j is odd, the number of vertices of the j -th path in G' is $x_j - 1$, which is even. By Lemma 5.36, there is a minimum feedback vertex set F' of G' that contains all vertices with an even i_j -index in G' .

Therefore, in $G - F'$, there are no edges between any vertex in $\widehat{G}$ and any vertex in G' . As a result, the union of F' and $\widehat{F}$ is a valid feedback vertex set for G with size $\nabla(P_{x_1} \boxtimes \dots \boxtimes P_{x_{j-1}} \boxtimes P_{x_{j-1}} \boxtimes P_{x_{j+1}} \boxtimes \dots \boxtimes P_{x_d}) + \nabla(P_{x_1} \boxtimes \dots \boxtimes P_{x_{j-1}} \boxtimes P_1 \boxtimes P_{x_{j+1}} \boxtimes \dots \boxtimes P_{x_d})$. $\blacksquare$

This concludes the results for the feedback vertex number of the strong product of d paths with k odd paths and thus also our results for the feedback vertex number on strong product graphs in general.

6 Conclusion

In this thesis, we considered various algorithmic problems on the strong product of graphs, in particular with paths as at least one factor. The main focus was on the feedback vertex set problem, where we derived bounds for the minimum feedback vertex set size of the strong product of different simple graphs, in particular of paths and trees with and without perfect matchings. We first looked at the strong product of two paths and then generalized our approach to the strong product of trees and more than two paths.

This provides a starting point for further structural investigations. The first direction is to explore whether these results or similar techniques map to subgraphs with removed edges, such as the Cartesian product.

Question 6.1: *How do the approaches and lower bounds for the feedback vertex set on the strong product of paths relate to their Cartesian product counterparts and general subgraphs with removed edges?*

Note that Cartesian products are subgraphs of the strong product of the same factors, with some edges removed. The second possible direction is to look at the strong product of other more complex graphs, such as k -trees or others. It is important to note that in contrast to the graphs we looked at in this thesis, those have a non-empty feedback vertex set in general.

Question 6.2: *Can we determine exact bounds for the minimum feedback vertex set size of the strong product of more complex graphs, such as k -trees?*

In this thesis, we also contributed to a better understanding of the strong product of trees. We did so by introducing exact formulas for $T_1 \boxtimes T_2$ and $T \boxtimes P_\ell$ when at least one factor has a perfect matching. We additionally derived bounds based on the odd paths in caterpillars that lack perfect matchings. However, the general case for arbitrary trees remains open.

Question 6.3: *What is the exact feedback vertex number of the strong product of two arbitrary trees?*

A natural generalization from trees and caterpillars are graphs with bounded treewidth. The vertex cover problem is fixed-parameter tractable for bounded treewidth graphs, which raises the question of whether this can help to efficiently bound the minimum feedback vertex set for the strong product of a bounded treewidth graph and a path.

Question 6.4: *Can we exploit the tractability of vertex cover on bounded treewidth graphs to efficiently bound the minimum feedback vertex set for the strong product of any bounded treewidth graph and a path?*

Finally, similar to our approach for the strong product of multiple paths, the strong product of more than two completely different graphs could be looked into as a final open problem for future research.

Question 6.5: *Can we determine the minimum feedback vertex set size of the strong product of more than two different graphs?*

Bibliography

- [AHLT11] Ghidewon Abay-Asmerom, R Hammack, C Larson, and D Taylor. “Notes on the independence number in the Cartesian product of graphs”. In: *Discussiones Mathematicae Graph Theory* Volume 31 (2011), pp. 25–35.
- [AS09] Ron Aharoni and Tibor Szabó. “Vizing’s conjecture for chordal graphs”. In: *Discrete Mathematics* Volume 309 (Apr. 2009), pp. 1766–1768. DOI: 10.1016/j.disc.2008.02.025.
- [BV97] Lowell W. Beineke and Robert C. Vandell. “Decycling graphs”. In: *Journal of Graph Theory* Volume 25 (1997), pp. 59–77.
- [Cam+24] Rutger Campbell, Katie Clinch, Marc Distel, J. Pascal Gollin, Kevin Hendrey, Robert Hickingbotham, Tony Huynh, Freddie Illingworth, Youri Tamitegama, Jane Tan, and et al. “Product structure of graph classes with bounded treewidth”. In: *Combinatorics, Probability and Computing* Volume 33 (2024), pp. 351–376. DOI: 10.1017/S0963548323000457.
- [CC93] Tony Chang and W. Clark. “The domination numbers of the $5 \times n$ and $6 \times n$ grid graphs”. In: *Journal of Graph Theory* Volume 17 (Mar. 1993). DOI: 10.1002/jgt.3190170110.
- [CCH94] Tony Yu Chang, W Edwin Clark, and Eleanor O Hare. “Domination numbers of complete grid graphs. I”. In: *Ars Combin* Volume 38 (1994), p. 994.
- [Cha92] Tony Yu Chang. *Domination numbers of grid graphs*. University of South Florida, 1992.
- [CS00] W. Edwin Clark and Stephen Suen. “Inequality Related to Vizing’s Conjecture”. In: *The Electronic Journal of Combinatorics* Volume 7 (May 2000), N4. DOI: 10.37236/1542.
- [DM58] A. L. Dulmage and N. S. Mendelsohn. “Coverings of Bipartite Graphs”. In: *Canadian Journal of Mathematics* Volume 10 (1958), pp. 517–534. DOI: 10.4153/CJM-1958-052-0.
- [DMW23] Vida Dujmović, Pat Morin, and David R Wood. “Graph product structure for non-minor-closed classes”. In: *Journal of Combinatorial Theory, Series B* Volume 162 (2023), pp. 34–67. DOI: 10.1016/j.jctb.2023.03.004.
- [Duj+20] Vida Dujmović, Gwenaél Joret, Piotr Micek, Pat Morin, Torsten Ueckerdt, and David R. Wood. “Planar Graphs Have Bounded Queue-Number”. In: *J. ACM* Volume 67 (Aug. 2020). ISSN: 0004-5411. DOI: 10.1145/3385731.
- [Dvo+21] Zdeněk Dvořák, Tony Huynh, Gwenaél Joret, Chun-Hung Liu, and David R. Wood. “Notes on Graph Product Structure Theory”. In: *2019-20 MATRIX Annals*. Edited by Jan de Gier, Cheryl E. Praeger, and Terence Tao. Cham: Springer International Publishing, 2021, pp. 513–533. ISBN: 978-3-030-62497-2. DOI: 10.1007/978-3-030-62497-2_32.

- [DW23] Zdeněk Dvořák and David R. Wood. *Product structure of graph classes with strongly sublinear separators*. 2023. arXiv: 2208.10074.
- [GHSW26] Himanshu Gupta, Leslie Hogben, Bryan Shader, and Tony Wong. *The Inverse Symplectic Eigenvalue Problem of a Graph*. 2026. arXiv: 2601.11912.
- [GKY25] Ali Ghalavand, Sandi Klavžar, and Ning Yang. “On Decycling and Forest Numbers of Cartesian Products of Trees”. In: *Bulletin of the Malaysian Mathematical Sciences Society* Volume 48 (Oct. 2025). ISSN: 2180-4206. DOI: 10.1007/s40840-025-01972-9.
- [GPRT11] Daniel Gonçalves, Alexandre Pinlou, Michaël Rao, and Stéphan Thomassé. “The Domination Number of Grids”. In: *SIAM Journal on Discrete Mathematics* Volume 25 (Jan. 2011), pp. 1443–1453. ISSN: 1095-7146. DOI: 10.1137/11082574.
- [HJMW24] Robert Hickingbotham, Paul Jungeblut, Laura Merker, and David R Wood. “The product structure of squaregraphs”. In: *Journal of Graph Theory* Volume 105 (2024), pp. 179–191. DOI: 10.1002/jgt.23008.
- [ISW24] Freddie Illingworth, Alex Scott, and David Wood. “Product structure of graphs with an excluded minor”. In: *Transactions of the American Mathematical Society, Series B* Volume 11 (Nov. 2024), pp. 1233–1248. ISSN: 2330-0000. DOI: 10.1090/btran/192.
- [Kön31] Dénes König. “Graphs and matrices (Gráfok és mátrixok)”. In: *Matematikai és Fizikai Lapok* Volume 38 (1931), pp. 116–119.
- [KS95] Sandi Klavžar and Norbert Seifter. “Dominating Cartesian products of cycles”. In: *Discrete applied mathematics* Volume 59 (1995), pp. 129–136.
- [LFS14] Min-Yun Lien, Hung-Lin Fu, and Chih-Huai Shih. “The decycling number of $P_m \square P_n$ ”. In: *Discrete Mathematics, Algorithms and Applications* Volume 6 (Sept. 2014). DOI: 10.1142/S1793830914500335.
- [Lov79] L. Lovasz. “On the Shannon capacity of a graph”. In: *IEEE Transactions on Information Theory* Volume 25 (1979), pp. 1–7. DOI: 10.1109/TIT.1979.1055985.
- [Luc+98] Flaminia L Luccio et al. “Almost exact minimum feedback vertex set in meshes and butterflies”. In: *Inf. Process. Lett.* Volume 66 (1998), pp. 59–64.
- [Mor21] Pat Morin. “A Fast Algorithm for the Product Structure of Planar Graphs”. In: *Algorithmica* Volume 83 (May 2021), pp. 1544–1558. ISSN: 1432-0541. DOI: 10.1007/s00453-020-00793-5.
- [NN98] S. Neufeld and R. Nowakowski. “A game of cops and robbers played on products of graphs”. In: *Discrete Mathematics* Volume 186 (1998), pp. 253–268. ISSN: 0012-365X. DOI: 10.1016/S0012-365X(97)00165-9.
- [PZ05] David Pike and Yubo Zou. “Decycling Cartesian Products of Two Cycles”. In: *SIAM J. Discrete Math.* Volume 19 (Jan. 2005), pp. 651–663. DOI: 10.1137/S089548010444016X.
- [PZ13] POLONA Pavlic and Janez Zerovnik. “A note on the domination number of the Cartesian products of paths and cycles”. In: *Kragujevac Journal of Mathematics* Volume 37 (2013), pp. 275–285.
- [Sha56] C. Shannon. “The zero error capacity of a noisy channel”. In: *IRE Transactions on Information Theory* Volume 2 (1956), pp. 8–19. DOI: 10.1109/TIT.1956.1056798.

-
- [TVT25] Kalaiselvi Thangam, Yegnanarayanan Venkataraman, and Rajermani Thinakaran. *On Domination and Total Domination Numbers on Strong Product of Two Path Graphs*. May 2025. DOI: [10.20944/preprints202505.1725.v2](https://doi.org/10.20944/preprints202505.1725.v2).
- [UWY22] Torsten Ueckerdt, David Wood, and Wendy Yi. “An Improved Planar Graph Product Structure Theorem”. In: *The Electronic Journal of Combinatorics* Volume 29 (June 2022), P2.51. DOI: [10.37236/10614](https://doi.org/10.37236/10614).
- [Viz68] Vadim G Vizing. “Some unsolved problems in graph theory”. In: *Russian Mathematical Surveys* Volume 23 (1968), pp. 125–141.
- [XCL14] Jinshan Xie, Xiao Chen, and Yongxuan Lai. “Decycling numbers of strong product graphs involving paths, circuits, stars and complete graphs.” In: *International Journal of Applied Mathematics and Statistics* Volume 52 (Oct. 2014), pp. 83–90.
- [Xie07] Jinshan Xie. “Decycling number of the box-cross product of two paths”. In: *Journal of Fuzhou University. Natural Science Edition* Volume 35 (Jan. 2007).
- [Yan81] M. Yannakakis. “Node-Deletion Problems on Bipartite Graphs”. In: *SIAM Journal on Computing* Volume 10 (1981), pp. 310–327. DOI: [10.1137/0210022](https://doi.org/10.1137/0210022).