Karlsruhe Institute of Technology

Conflict-Free Colorings

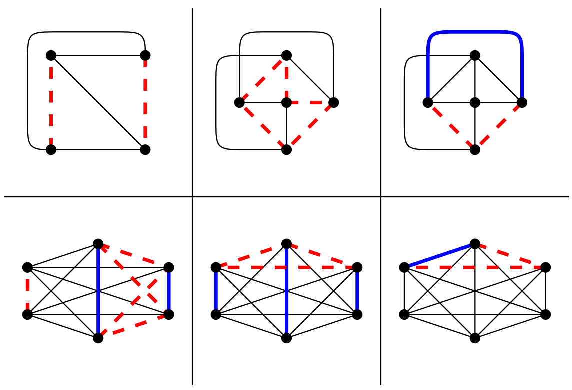
- Of Graphs and Hypergraphs -

Diploma-Thesis
of

Jonathan Rollin

Department of Mathematics
Institute for Algebra and Geometry
Discrete Mathematics

Department of Informatics
Institute of Theoretical Informatics
Algorithmics I



Reviewer: Prof. Maria Axenovich Ph.D.
Reviewer: Prof. Dr. Dorothea Wagner

November 2012 – May 2013

Abstract

Conflict-free colorings are known as vertex-colorings of hypergraphs. In such a coloring each hyperedge contains a vertex whose color is not assigned to any other vertex within this edge. In this thesis the notion of conflict-free colorings is translated to edge-colorings of graphs. For graphs G and H a conflict-free coloring of G ensures an edge of unique color in each copy of H in G. The minimum number of colors in such a coloring is denoted by $f(G, H)$ and is called conflict-free chromatic index of G. Since total multicolorings are conflict-free, $f(G, H)$ is well defined. In accordance with the chromatic-index of a graph it is NP-hard to determine $f(G, H)$ in general. Most results of the thesis are concerned with the asymptotic behavior of the function $f(G, H)$ for fixed H and large G. The cases of arbitrary graphs G and H, of paths in trees and of subcliques of complete graphs are studied in detail. Several constructive as well as probabilistic upper bounds are established in these settings, some of which are tight for certain graphs. Some lower bounds are determined, as well. Moreover, tight relations between the two notions of conflict-free colorings and between conflict-free colorings and (inverse) Ramsey numbers are encountered.

Erklärung

Hiermit erkläre ich, dass ich die vorliegende Diplomarbeit selbstständig und ohne die Verwendung anderer als der angegeben Quellen verfasst habe.

Ort, Datum *Jonathan Rollin*

Acknowledgments

Prof. Maria Axenovich introduced me to the interesting field of extremal graph theory and Ramsey theory in particular. I want to thank her for lots of time for discussions, all the comments, and challenging questions. Along with her Dr. Torsten Ueckerdt was a great advisor. He pointed me to the already existing notion of conflict-free colorings of hypergraphs. In addition, he suggested to consider paths in trees what is now one of the main parts of the thesis.

Many thanks go to Prof. Dr. Dorothea Wagner, Dr. Ignaz Rutter and Tanja Hartmann for supporting the topic of the thesis, although it does not fit properly into their field of research.

My wife-to-be Rafaela is my greatest support for almost four years now. I owe her many thanks for all the understanding, for all the happiness we have together and for breaking the proofs, if necessary.

Fabian proofread parts of the text and provided useful remarks during the last steps of writing. There are lots of other people which supported me during my whole studies with discussions, ideas, motivation or distraction. Many thanks go to all of them.

All this would have been impossible without my parents, who encouraged me since I was a child to go along my own way.

Contents

1. Introduction

Graphs have an immanent power to model various sorts of problems and applications in a simple and easily understandable way. Assigning colors, or other labels, to the edges or vertices improves this ability even more. Therefore coloring problems are among the most famous problems in graph theory. Especially the four color theorem is referred to as one of the results from graph theory which is best known outside of this field of research [9]. One of the first formulations of the corresponding problem is the question whether four colors are sufficient to color all countries on a map, such that neighbors are of different color. It was already stated in 1852 by Guthrie [32]. Colorings of this kind, with two elements of the same color not being adjacent, are called proper colorings and attracted lots of attention. Particularly the chromatic number of graphs is studied a lot, which denotes the minimum number of colors sufficient for a proper vertex-coloring of a graph. Compared to the first conjectures on proper colorings the concept of conflict-free colorings is fairly new.

Conflict-Free Colorings: In 2003 Even, Lotker, Ron and Smorodinsky set up the following constraints on a coloring [13]. Given a vertex-coloring of a hypergraph, a *vertex is of unique color* within a hyperedge, if its color is not assigned to any other vertex in this edge. The whole vertex-coloring is called *conflict-free*, if every hyperedge contains a uniquely colored vertex. It is easy to see that the problem of constructing such colorings gets easier the more colors are available. So conflict-free colorings with few colors are of particular interest. In accordance with the notion of the (ordinary) chromatic number the *conflict-free chromatic number* of a hypergraph denotes the minimum number of colors used by a conflict-free coloring. The left part of Figure 1.1 shows an example of a conflict-free coloring of a hypergraph. There are two different main approaches for studying such chromatic numbers. One possibility is to determine or approximate the explicit (conflict-free) chromatic number of a fixed hypergraph. The other possibility is to consider some parameters of the hypergraph as variables and analyze the asymptotic behavior of the (conflict-free) chromatic number. Possible parameters are the number of vertices, the number of edges or the maximum edge degree, for example. This thesis is mainly concerned with the extremal questions in the second approach.

In their introductionary work Even et. al. had a concrete application in mind [13]. Frequencies of a wireless network shall be assigned to base stations, such that the existence of a base station of unique frequency is guaranteed within the reach of a mobile client. Then, conflict-free communication is possible. For this reason they studied hypergraphs arising from geometric settings, particularly from points and discs in the plane. Following studies concentrated on conflict-free colorings of

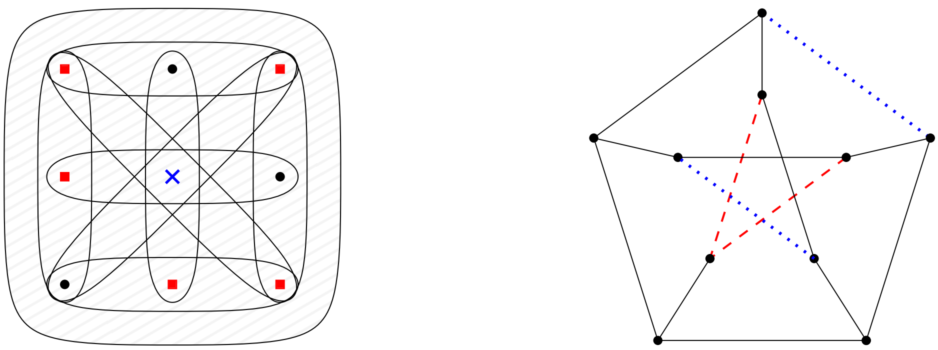


Figure 1.1.: The left hand side shows a conflict-free (vertex) coloring of a hypergraph (on nine vertices and nine hyperedges) in three colors (black disk, square red, cross blue). The right hand side shows a conflict-free (edge) coloring of the Petersen graph with respect to cycles on five edges using three colors (solid black, dashed red, dotted blue).

differently but still geometrically induced hypergraphs, as well. First studies on conflict-free colorings of general hypergraphs were presented in 2009 by Pach and Tardos [23]. A detailed review of the results on conflict-free colorings is presented in Section 2.2. Therefore concrete results are omitted here. The extensive review is due to tight relations of the conflict-free colorings of hypergraphs to the colorings studied in this thesis. These relations are established in the beginning of Section 2.2.

This thesis is concerned with a certain kind of edge-colorings of graphs. They were introduced by Radoš Radoičić and Veselin Jungić (personal communication) and are very similar to conflict-free colorings of hypergraphs. Given an edge-coloring an *edge is uniquely colored* within a set of edges, if there is no other edge of the same color in this set. Originally, the edges of a complete graph should be colored, such that all complete subgraphs of fixed order contain an edge of unique color. Since a total multicoloring always satisfies this condition, the minimum number of sufficient colors is studied. In particular, the extremal behavior of this number is of interest in case of increasing order of the complete graph whose edges get colored. Of course, the problem is not restricted to colorings of complete graphs with respect to complete subgraphs. Edge-colorings of arbitrary graphs G are studied that ensure a uniquely colored edge in each copy of another arbitrary but fixed graph H in G. Such an edge-coloring is called *conflict-free with respect to* H. An exemplary conflict-free coloring of the Petersen graph with respect to cycles on five edges is shown in the right part of Figure 1.1. Like for conflict-free colorings of hypergraphs the minimum number of sufficient colors is of particular interest here, as well. The function $f(G, H)$ denotes this number and is called *conflict-free chromatic index* of G with respect to H. Studying the behavior of f in various settings is the main topic of this thesis. Colorings of arbitrary graphs with respect to arbitrary (but fixed) subgraphs are studied in Chapter 2, first. Several basic results are presented together with a general upper bound on f. Afterwards, edge-colorings of trees with respect to paths of given length are considered in Chapter 3. Here, an interesting dependence of the behavior of f on the parity of the given length is discovered. Furthermore, it is shown that the height of the tree is not essential. Several constructive upper bounds are presented which do not need more colors on trees with larger height than on smaller trees. Some of the bounds are tight for certain classes of trees. The original setting of complete graphs K_n and their complete subgraphs is studied in Chapter 4. The main concern of that chapter is to analyze the extremal behavior of $f(K_n, K_p)$ in case of increasing $n \in \mathbb{N}$ and fixed $p \in \mathbb{N}$. Finally, Chapter 5 contains a summary of results and concluding remarks.

Extremal Graph Theory: With the questions formulated above these studies are part of extremal graph theory and combinatorics, an area of discrete mathematics. Extremal graph theory covers various problems and theories, most of which are interested in global conditions of graphs forcing certain properties. A very fundamental example is to study the largest or smallest number of edges of a graph on a fixed number of vertices, which admits a partition of the edges into a given number of classes of certain kind. Colorings naturally correspond to partitions by considering all elements of the same color as so called color classes (or the other way round). With this notion a proper coloring induces a partition into independent sets, i.e. a partition with no part containing two adjacent elements. Other structural conditions on the color classes lead to different variants of coloring, respectively partition problems. Among these, especially Ramsey type problems have tight relations and interconnections to conflict-free colorings. The following part gives a brief outline of the related topics.

A first extremal result in this field of research is due to Mantel who proved an upper bound on the number of edges in a triangle free graph [32]. This idea is extended by Turán to K_p-free graphs for $p \in \mathbb{N}$. Turán's theorem presents graphs of this kind with maximum number of edges (extremal graphs), the so called Turán graphs [29].

A slightly different line of extremal graph theory is due to Ramsey, who studied the following relaxed condition on the color classes. Given a positive integer $p \in \mathbb{N}$, no color class of an edge-coloring of a complete graph should contain a complete graph K_p on p vertices. Due to Ramsey's theorem this is not possible, if only few colors are available. In his original theorem Ramsey considered a more general variant of this. For an $r \in \mathbb{N}$ the hyperedges of a complete r-uniform hypergraph $\mathcal{H}$ are colored (i.e. partitioned). Note that a graph is just a 2-uniform hypergraph. Ramsey proved, that for a fixed number of colors and given size $p \in \mathbb{N}$, one of the color classes contains a complete r-uniform sub-hypergraph of $\mathcal{H}$ on p vertices, if $\mathcal{H}$ is sufficiently large [26]. Hence, the larger the complete graph is the larger is the number of colors necessary to avoid a monochromatic complete subgraph. This gives reason for the notion of *Ramsey numbers*. Depending on the number of available colors and the size p, they denote the smallest order of a complete graph which does not admit an edge-coloring without monochromatic copy of K_p. Lots of generalizations of Ramsey theory were established since then. Consider an arbitrary but fixed graph H and a given number of colors $k \in \mathbb{N}$. The so called *graph Ramsey number* denotes the smallest order of a complete graph which does not admit an edge-coloring without monochromatic copy of H. The existence (i.e. finiteness) of this number follows from the existence of the (classical) Ramsey number. A monochromatic complete subgraph of sufficient size contains the graph H. Furthermore, distinct graphs can be associated with the different colors. For example, an edge-coloring of a sufficiently large complete graph in two colors cannot avoid an entirely red cycle and an entirely blue star (of fixed size) at the same time. Again, finiteness in this (non-symmetric) case follows from Ramsey's theorem. For all these variants of Ramsey numbers one may ask the other way round. Given a graph and a subgraph (or several subgraphs), how many colors does an edge-coloring without monochromatic copy of the subgraph need? This number of colors is usually denoted as *inverse Ramsey number*.

Like for chromatic numbers there are two different main approaches for studying Ramsey numbers. Either fixed values for the parameters and explicit Ramsey numbers are considered or the asymptotic behavior for some varying parameters is analyzed. For the first case large tables exist with exact values or best known bounds respectively. For example the survey of Radziszowski [25] lists lots of results and several asymptotic bounds, and is updated from time to time. Only few exact results are known for Ramsey numbers. In case of more than two colors the following value of a Ramsey number is the only exact value known, yet. The smallest complete graph which does not admit a coloring in three colors without monochromatic triangles has 17 vertices [25]. More exact values are known for two colors or certain graph Ramsey numbers. In the asymptotic case only few exact results are known, as well. In case of two colors let $p \in \mathbb{N}$ denote the order of the clique that should not occur monochromatically. With a constant $c \in \mathbb{R}$ an initial lower bound on the classic Ramsey number of $c \cdot p \cdot 2^{\frac{p}{2}}$ was proved by Erdős [10]. This bound was improved later, but not significantly. An upper bound of $\binom{2p}{p} \cdot p^{\frac{-\bar{c}\log(p)}{\log\log(p)}}$, with a constant $\bar{c} \in \mathbb{R}$, is due to Conlon [8].

In the theory presented above certain unavoidable, monochromatic substructures of colorings are considered. One of several generalizations of this theory is due to Erdős [11] who considers the following stronger fact. For each $q \in \mathbb{N}$ there are graphs, such that certain substructures on less than q colors are unavoidable in any edge-coloring, if the number of available colors is to small. This reduces to the notion introduced above for $q = 2$. One of the results in this field of research due to Erdős and Gyárfás [12] is used in this thesis.

Next, notations and definitions are given. Afterwards, the main results of the thesis are summarized in Section 1.3 together with an outline of the following parts.

1.1. Definitions

All definitions and notions are consistent with those in West's introduction to graph theory [32]. The only exception is the notion of P_m denoting a path on $m \in \mathbb{N}$ edges (instead of vertices). Furthermore, some additional definitions are introduced.

For this whole work let G and H denote finite, simple and undirected graphs. For a graph G let $V(G)$ denote its vertex set and $E(G) \subseteq \binom{V(G)}{2}$ its edge set, where $\binom{V(G)}{2}$ denotes the set of all subsets of $V(G)$ of size 2. Both G and H are assumed to contain at least one edge. The *maximum degree* of G is denoted by $\Delta(G)$. If H is a subgraph of G, the term $H \subseteq G$ is used. For a subset $V' \subset V(G)$ the *subgraph induced by* V' is defined as $G[V'] := (V', E \cap \binom{V'}{2})$. In words, the subgraph induced by V' is obtained from G by removing all vertices in $V \setminus V'$ and all edges incident to these vertices. Similarly to this, define $G - \bar{V} := G[V(G) \setminus \bar{V}]$ for $\bar{V} \subset V(G)$. The same notion is used for edges. No vertices are removed here, so for $E' \subseteq E(G)$ define the subgraph $G - E' := (V(G), E')$. A subgraph of G which is isomorphic to H is called *copy of H in G*. The *maximum H-degree* $\Delta_H(G)$ of a graph G is the maximum number of copies of H in G sharing an edge with a fixed copy of H in G. Formally, $\Delta_H(G) := \max\{d \in \mathbb{N} \mid \exists H_0, H_1, \ldots, H_d \subseteq G, H_i \text{ copy of } H \text{ for all } i \leq d : E(H_0) \cap E(H_i) \neq \emptyset\}$. An *$H$-blocking set* of a graph G is a set $B \subseteq E(G)$, such that each copy of H in G contains at least one edge from B.
Some parts of the thesis work with hypergraphs. For a hypergraph $\mathcal{H}$ let $V(\mathcal{H})$ denote its vertex set and $E(\mathcal{H}) \subseteq 2^{V(\mathcal{H})}$ its edge set. Furthermore the *dependence* or the *degree* of an edge $E \in E(\mathcal{H})$ is the number of edges in $E(\mathcal{H})$ intersecting E. The *maximum edge degree* of $\mathcal{H}$ is denoted by $D(\mathcal{H})$. A *blocking set* (also known as *hitting set*) of a hypergraph $\mathcal{H}$ is a set $B \subseteq V(\mathcal{H})$, such that $E \cap B \neq \emptyset$ for all $E \in E(\mathcal{H})$. A blocking set (or an *H-blocking set* of graph G) is called *almost simple*, if each edge in $E(\mathcal{H})$ (respectively each copy of H in G) either contains exactly one element from B or exactly one of its elements is not contained in B. A blocking set is called *simple*, if each edge in $E(\mathcal{H})$ (respectively no copy of H in G) contains exactly one element from B.

Colorings: The thesis deals with certain *colorings* of G. An *edge-coloring* of G is map $c : E(G) \to \mathbb{N}$ (vertex-colorings are defined analogously). The notion of a *coloring assigning a color to an edge* is used in contrast to this definition but with the same meaning. A coloring is called *proper*, if adjacent elements do not have the same color. The minimum number of colors that is used by a proper edge-coloring of G is called *chromatic index* and denoted by $\chi'(G)$. It is known from Vizing's theorem [30], that $\chi'(G) \in \{\Delta(G), \Delta(G) + 1\}$.This divides graphs into two classes. *Class I* contains all graphs with chromatic index equal to their maximum degree and *Class II* all other graphs. If G' is a subgraph of G and c is an edge-coloring of G, the *induced edge-coloring* c' on G' is just the restriction of c from $E(G)$ to $E(G')$, i.e. $c'(e) := c(e)$ for $e \in E(G') \subseteq E(G)$. A *multicoloring* of a graph (respectively its edges), is an edge-coloring that assigns no color to more than one edge. In other words, each edge receives its own color. If all edges are colored with the same color, the coloring is called *monochromatic*.
The definitions are used for vertex-colorings of hypergraphs. A vertex-coloring of a hypergraph is called *conflict-free*, if each hyperedge contains a vertex whose color is not assigned to any other vertex in this hyperedge. The *conflict-free chromatic number* $\chi_{\mathrm{cf}}(\mathcal{H})$ is the minimum number of colors used by a conflict-free coloring of $\mathcal{H}$. Formally, $\chi_{\mathrm{cf}}(\mathcal{H}) := \min\{k \in \mathbb{N} \mid \text{there is a conflict-free coloring of } \mathcal{H} \text{ in } k \text{ colors}\}$.

Extremal Graph Theory: Two notions from extremal graph theory are used in the thesis: extremal functions and Ramsey numbers as well as several generalizations. For a graph H the *extremal number* $\mathrm{ex}(n, H)$ denotes the largest number of edges of a graph G on n vertices and without a subgraph isomorphic to H.
Classical Ramsey theory deals with edge-colorings of complete graphs. Given two graphs H_1 and H_2 the *Ramsey number* $n := R(H_1, H_2)$ denote the smallest integer, such that every edge-coloring of K_n with colors 1 and 2 induces a monochromatic copy of H_1 in K_n in color 1 or of H_2 in color 2. This definition is extended to an arbitrary number k of colors and graphs H_i. Then $R(H_1, H_2, \dots, H_k)$ is called *multicolor Ramsey number*. An edge-coloring without monochromatic copy of all H_i is called *Ramsey coloring*. Since distinct graphs H_i won't be considered in this work, define for $k \in \mathbb{N}$ colors and one fixed graph H the shorthand $R_k(H) := R(H, H, \dots, H)$ (where H is repeated k times). Instead of a fixed number of colors, one may fix the size n of the complete graph, whose edges get colored. Then the question is, how many colors are necessary for a Ramsey coloring of this graph. This number is called *inverse (multicolor) Ramsey number* and is denoted by $R_H^{inv}(n) = \min\{k \in \mathbb{N} \mid R_k(H) > n\}$. If H is a complete graph K_p these terms are rewritten as $R_k(p)$ and $R_p^{inv}(n)$. For an arbitrary graph G the inverse multicolor Ramsey number $R_H^{inv}(G)$ is analogously defined as the minimum number of colors used by an edge-coloring of G without monochromatic copy of H. Note that it is not an actual inverse of another function.
In classic Ramsey theory it is looked for colorings without monochromatic subgraphs of certain kind. In particular, at least two colors should be assigned to the edges of each copy of H. This is generalized as follows. For an integer $q \leq |E(H)|$ an edge-coloring of G is called *q-good* (with respect to H), if there are at least q distinct colors assigned to the edges of each copy of H in G. Within this thesis the size of the smallest complete graph which does not admit a q-good coloring is called *generalized multicolor Ramsey number* (with respect to H) and denoted by $\widetilde{R}_k(H; q)$. Again, the inverse $\widetilde{R}_H^{inv}(n; q) := \min\{k \in \mathbb{N} \mid \widetilde{R}_k(H; q) > n\}$ is of interest as well. In case of $H = K_p$, the same notation like above is used. Like above, this inverse number can be studied for arbitrary graphs G.

Trees and Other Graph Classes: Several well known special graphs are considered. Most frequently used are *complete graphs* on $n \in \mathbb{N}$ vertices, denoted by K_n, and (simple) *paths* on $m \in \mathbb{N}$ edges, denoted by P_m. For a path P_m, the number of edges m is called its *length*. The (two) vertices of degree 1 in P are called *endpoints* and all other vertices are called *internal vertices* of P. Two vertices are *connected by a path P*, if they are endpoints of P. The *distance* between two vertices in G is the length of a shortest path connecting the two vertices. The *diameter* $\mathrm{diam}(G)$ of a graph G is the largest distance between two vertices in G. For a vertex $v \in V(G)$ and an integer $a \in \mathbb{N}$ the maximum number $d_a(v)$ of edge-disjoint copies of P_a in G with v as an endpoint is called *a-degree of v*. The maximum a-degree of G is denoted by $\Delta_a(G) := \max_{v \in V(G)}(d_a(v))$.
In Chapter 3, trees are considered. A tree T is called *rooted*, if a fixed vertex is marked as its root. Then the *height* $h(T)$ of a rooted tree T is defined as the length

of a longest path connecting the root to some leaf. Most of the time, rooted trees are considered *leveled*. All vertices of the same distance i from the root form the vertex level of index i. All edges connecting vertices in levels $i-1$ and i form the edge level of index i. The root of a tree is considered to be its topmost vertex. All other levels occur below, ordered according to their index. A level is said to be *above* (respectively *below*) each level of larger index (respectively smaller index). A vertex $u \in V(T)$, which is adjacent (incident) to another vertex $v \in V(T)$ (an edge $e \in E(T)$) from above is called *parent vertex* of v (of e). The vertex v is called *child* of u. The *down degree* of a vertex v in rooted tree is defined as its number of children. A *downward subtree* of T is a rooted subtree of T whose root is its topmost vertex in T. Any vertex in T which is contained in a downward path in T connecting a vertex v (or an edge e) and the root of T is an *ancestor* of v (of e).
Three classes of trees of special structure are of particular interest. A *spider* is a tree with at most one vertex of degree larger than 2. If it exists, this particular vertex is called the *head* of the spider. Otherwise the *head* may be any of the vertices of degree 2. The paths connecting the head of a spider to its leafs are called *legs*. If every leg of a spider contains exactly one edge, the graph is called *star* and its head is called its *center*. In contrast to these trees with few vertices of degree larger than 2, (rooted) *complete d-ary trees* are considered. For $d \in \mathbb{N}$ each vertex in such a tree which is not the root and not a leaf has degree $d+1$. The root has degree d and all paths connecting the root to a leaf have the same length.

1.2. Problem Formulation

We are looking for the following colorings.

Definition 1.1. *Let G and H be graphs. An edge-coloring $c : E(G) \to \mathbb{N}$ is called* conflict-free *(with respect to H) or* UCE-H *coloring (*Uniquely Colored Edge-H*), if in each subgraph of G isomorphic to H there is a color class consisting of a single edge. The* conflict-free chromatic index *of G (with respect to H) is defined as follows.*

$$f(G,H) := \min\{k \in \mathbb{N} \mid \textit{there is a UCE-}H\textit{ coloring of } G \textit{ in } k \textit{ colors}\}.$$

A UCE-H coloring c is called minimum*, if it uses $f(G,H)$ colors.*

The map f is well defined as it is shown in Lemma 2.1. According to this definition, an edge-coloring of G is called *good (with respect to H)*, if it is a UCE-H coloring. Otherwise it is called *bad*. Given an arbitrary edge-coloring of G, a copy of H in G is called *good* (respectively *bad*), if it contains a uniquely colored edge (or not). According to this definition, the following decision problem is defined.

Problem 1 (UCE-H Problem)**.** *For $k \in \mathbb{N}$ and two graphs G and H the following decision problem is called the k-*UCE-H problem for G*.*

Is there a UCE-H coloring of G using exactly k colors?

A big part of this work considers complete graphs and its complete subgraphs. Hence the notation $f(n,p) := f(K_n, K_p)$ is used for $n, p \in \mathbb{N}$ with $n, p \geq 2$.

1.3. Results

In the following the main theorems and results of this thesis are presented. The proofs are contained in the referenced sections together with minor results and connections in between. The general setting of arbitrary graphs G and H is studied in Chapter 2. First of all, several basic results are presented in Section 2.1. For example, there are marginal results for very large and very small subgraphs H. Afterwards, known results on conflict-free colorings are reviewed in Section 2.2. Moreover, a tight relation between good vertex-colorings of hypergraphs and good edge-colorings of ordinary graphs is established. This relation allows to translate some of the known results on hypergraphs to the setting of edge-colorings of graphs. Then, connections to Ramsey numbers are studied in Section 2.3. Using a generalized version of Ramsey theory, the following theorem is obtained with probabilistic methods.

Theorem 1. *Let G, H be graphs and $m := |E(H)|$. Then*

$$f(G,H) \leq (e \cdot (\Delta_H(G)+1))^{\frac{2}{m}} \cdot \frac{m^2}{4}.$$

The proof of this Theorem is used to obtain following corollary on the conflict-free chromatic number of uniform hypergraphs. It improves upon the best result known so far.

Corollary 2.18. *Let $r \in \mathbb{N}$ and $\mathcal{H}$ be an r-uniform hypergraph. Then*

$$\chi_{cf}(\mathcal{H}) \leq (e \cdot (D(\mathcal{H})+1))^{\frac{2}{r}} \cdot \frac{r^2}{4}.$$

Finally, the complexity of determining the conflict-free chromatic index is analyzed in Section 2.4. Besides two algorithms, the following NP-hardness result is obtained.

Theorem 2. *The decision problem k-UCE-H is NP-hard.*

Chapter 3 considers paths of fixed length in trees. In the beginning, an easy construction shows that minimum good colorings of path (i.e. trees on two leaves) with respect to another path use at most two colors. For general trees the parity of the length of the path is essential for the behavior of the conflict-free chromatic index $f(T, P_m)$. On account of this fundamental difference these two cases are handled in two distinct sections. Paths on an odd number of edges are considered in Section 3.2. For spiders there are good colorings (with respect to a path on an odd number of edges) using only two colors as well. They are presented in Subsection 3.2.1. The conflict-free chromatic index with respect to a path is bounded for a general tree in terms of the length of the path only. In particular, there is an upper bound which is independent of the given tree. It is stated in the following theorem.

Theorem 3. *Let $m \in \mathbb{N}$ be odd and T be a tree with diameter $\rho := diam(T) \geq m$. Then*

$$f(T, P_m) \leq \min\left\{\left\lceil \frac{\rho}{2} \right\rceil - \left\lfloor \frac{m}{2} \right\rfloor + 1, \left\lceil \frac{m}{2} \right\rceil\right\}.$$

If T is a complete d-ary tree of height at least $\lceil \frac{m}{2} \rceil$ and if $d \geq 2$ and $d > (\lceil \frac{m}{2} \rceil)^m$, then the upper bound is attained.

Afterwards, paths on an even number $m \in \mathbb{N}$ of edges are studied in Section 3.3. Again, spiders are considered first in Subsection 3.3.1. The following theorem is proven there. Recall the definition of the maximum a-degree $\Delta_a(T)$ of a tree T. For $a \in \mathbb{N}$ it is defined to be the maximum number of edge-disjoint copies of P_a in T with a common endpoint.

Theorem 4. *Let $m \in \mathbb{N}$, with $m \geq 4$ even, and S be a spider. Further let $\mathbb{I}_S := 1$ indicate the existence of legs on less than $\frac{m}{2}$ edges in S and $\mathbb{I}_S := 0$ the opposite. Then*

$$\begin{aligned} \lceil \log_2(\Delta_{\frac{m}{2}}(S)) \rceil \leq\ & \min\{k \in \mathbb{N} \mid \sum_{i=2}^{m/2} \tbinom{k}{i} + 1 \geq \Delta_{\frac{m}{2}}(S)\} \\ \leq\ & f(S, P_m) \\ \leq\ & \min\{k \in \mathbb{N} \mid \sum_{i=0}^{m/2} \tbinom{k}{i} \geq \Delta_{\frac{m}{2}}(S) + \mathbb{I}_S\} + 1. \end{aligned}$$

In particular if $\log_2(\Delta_{\frac{m}{2}}(S) + \mathbb{I}_S) \leq \frac{m}{2}$, then

$$\lceil \log_2(\Delta_{\frac{m}{2}}(S)) \rceil \leq f(S, P_m) \leq \lceil \log_2(\Delta_{\frac{m}{2}}(S) + \mathbb{I}_S) \rceil + 1.$$

For each even $m \in \mathbb{N}$ and every integer $k \in \mathbb{N}$ there is a tree T, such that all UCE-P_m colorings of T use more than k colors. Furthermore, the maximum $\frac{m}{2}$-degree of any graph G equals the maximum number of legs of a spider contained in G whose legs have at least $\frac{m}{2}$ edges each. Hence, the theorem provides a lower bound for arbitrary graphs in terms of their maximum $\frac{m}{2}$-degree, too. An upper bound on f for general trees is established in Subsection 3.3.2. It is tight for complete $(\Delta - 1)$-ary trees and $m = 4$.

Theorem 5. *Let $m \in \mathbb{N}$ even and let T be a tree of maximum degree Δ. Then*

$$f(T, P_m) \leq \frac{m}{2} + \Delta - 1.$$

Two constructions that are independent from the parity of m are presented in Section 3.4. Finally, all results are summarized in the last Section 3.5. Questions for maximum and minimum values of $f(T, P_m)$ are answered there for the case of fixed $m \in \mathbb{N}$ for all T with fixed maximum degree or fixed number of vertices (when applicable).

Chapter 4 is concerned with complete graphs. For $n, p \in \mathbb{N}$ good edge-colorings of K_n with respect to K_p are studied. The general results from Chapter 2 give a first insight into the behavior of the conflict-free chromatic index $f(K_n, K_p)$. Section 4.1 and Section 4.3 contain these information together with some specific results. In Section 4.2 constructive upper bounds on $f(K_n, K_p)$ are presented. They are linear in n and tight for a certain range of n.

Theorem 7. *Let $n, p \in \mathbb{N}$ with $3 \leq p \leq n$. If $n \leq \frac{5}{4}p - 2$, then $f(n, p) = n - p + 2$.*

The best general upper bound is obtained in Section 4.3 as a corollary to Theorem 1. For fixed $p \in \mathbb{N}$ with $p > 4$ it improves the linear upper bounds given before asymptotically, but is not constructive. In case $p \leq 4$ specific constructions yield sublinear bounds as well.

Theorem 8. *Let $n, p \in \mathbb{N}$ with $2 \leq p \leq n$. There is a constant $c_p \in \mathbb{R}$ depending on p but not on n, such that*

$$f(n, p) \leq c_p \cdot n^{\frac{4}{p}}.$$

A general lower bound is established in Section 4.4. It is obtained from large monochromatic stars.

Theorem 9. *For $p \in \mathbb{N}$ exist constants $c_p \in \mathbb{R}$ and $n_p \in \mathbb{N}$, such that for all $n \geq n_p$*

$$f(n, p) \geq c_p \cdot \frac{\ln(n)}{\ln \ln(n)}.$$

Finally, the results are summarized in the Conclusion 5 again. They are stated together with several open questions. Moreover, a comparison between the setting of paths in trees and complete subgraphs of complete graphs is given.

2. Colorings of Arbitrary Graphs

This chapter deals with UCE-H colorings of arbitrary graphs G and H. First of all, the next lemma states that the conflict-free chromatic index $f(G, H)$ is well defined, i.e. that a good coloring always exists. Afterwards, some basic results are presented in Section 2.1. A connection between conflict-free vertex-colorings of hypergraphs and conflict-free edge-colorings of graphs is established in Section 2.2. Known results on conflict-free colorings of hypergraphs are analyzed there and translated to conflict-free edge-colorings of graphs, if possible. The subsequent Section 2.3 considers relations to (generalized) inverse Ramsey numbers. Finally, Section 2.4 deals with complexity results and algorithms.

Lemma 2.1. *Let G and H be graphs. A total multicoloring of G is a UCE-H coloring of G. In particular*

$$f(G, H) \leq |E(G)| < \frac{|V(G)|^2}{2}.$$

Proof. Consider a total multicoloring c of the edges of G, i.e. an edge-coloring with no two edges of the same color. Since all color classes of G consist of a single edge only, every subgraph of G has a uniquely colored edge. Hence, the coloring c is a UCE-H coloring of G for all graphs H. Because only simple graphs are considered, the complete graph K_n has most edges for a fixed number n of vertices. Hence, $f(G, H) \leq |c(E(G))| = |E(G)| \leq |E(K_{|V(G)|})| < \frac{|V(G)|^2}{2}$. □

2.1. Basic Results

This section contains some basic facts about the behavior of the conflict-free chromatic index f in general. Mostly, it deals with marginal results like the following lemma, which states exact values for the smallest and largest subgraphs possible.

Lemma 2.2. *Let G be a graph.*

1. $f(G, K_2) = 1$.
2. *If* $|E(G)| > 1$, *then* $f(G, G) = 2$.

Proof. A coloring c that assigns the same color to all edges of G is a UCE-K_2 coloring of G, since K_2 consists of one single edge. This proves the first equality.

However, the coloring c is not a UCE-G coloring of G, if G has more than one edge. But changing the color of one edge $e \in E(G)$ yields a UCE-G coloring of G, because every copy of G in G contains this edge e. Hence, $f(G, G) = 2$. □

Furthermore, it is easy to see that one color is sufficient in two special cases only.

Lemma 2.3. *Let G, H be graphs. A UCE-H coloring of G in one color exists, if and only if H has at most one edge or H is not contained in G.*

Proof. In case of H having at most one edge the existence of the coloring follows from Lemma 2.2. If H is not contained in G, every edge-coloring of G is a UCE-H coloring and hence one color is sufficient.
The other way round, suppose there is a UCE-H coloring of G in one color. In this coloring each copy of H in G has only one color class containing all the edges of H. Hence, by definition of UCE-H colorings, this color class has to contain at most one edge, or there is no copy of H in G at all. □

There is no equivalent statement to Lemma 2.3 for two colors in general. There are families of arbitrarily large graphs G_n, such that for a certain graph H each G_n admits a UCE-H coloring in two colors. Figure 2.1 shows such an example. The family G_n consists of two stars which are joined by an edge e at their centers u, respectively v. Coloring e in a different color than all other edges yields an edge-coloring c of G_n in two colors. Let H be a graph that contains two adjacent vertices of degree 3. Then every copy of H in G_n has to contain e and hence c is a UCE-H coloring.

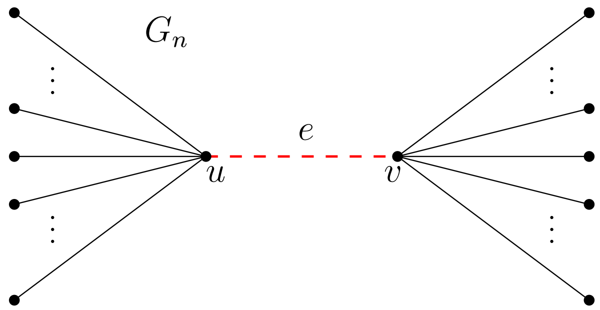


Figure 2.1.: A family of graphs together with a UCE-H coloring in two colors for all subgraphs H containing two vertices of degree 3.

But the case of two colors being necessary and sufficient can be characterized as follows. Recall the definition of an almost simple H-blocking set. It is a set B of edges of a graph G, such that every copy of a graph H in G either contains exactly one edge from B or exactly one edge of this copy is not contained in B. Furthermore, every edge in B must be contained in a copy of H in G.

Lemma 2.4. *Let G and H be graphs with $|E(H)| \geq 2$. A non-empty and almost simple H-blocking set of G exists, if and only if $f(G, H) = 2$.*

Proof. Let B be a non-empty, almost simple H-blocking set of G. Define an edge-coloring c of G by $c(e) := 1$, if $e \in B$, and $c(e) := 0$ otherwise. Then c is a UCE-H coloring, since each copy of H in G has exactly one edge from B or exactly one edge is not contained in B. Hence, $f(G, H) \leq 2$. Since B is non-empty, there is at least one edge in B. Since B is almost simple, this edge must be contained in a copy of H in G. Then $f(G,H) \geq 2$ due to Lemma 2.3, since $|E(H)| \geq 2$. Altogether, $f(G,H) = 2$ holds.

The other way round let c be a UCE-H coloring of G using two colors. Let B denote the set of all those edges of one of the color classes, that are contained in a copy of H in G. Since $|E(H)| \geq 2$ and $f(G,H) = 2$, there is at least one copy of H in G and it contains edges of both colors. Hence, B is non-empty. Since each copy of H contains an edge of unique color, either exactly one, or all but one edge of each copy of H in G are contained in B. Hence, B is almost simple.

□

Because every simple H-blocking set is almost simple as well, the existence of a simple and non-empty H-blocking set guarantees $f(T,H) = 2$, if $|E(H)| \geq 2$. But the question, whether the existence of a simple blocking set is necessary as well, remains open.

Last but not least there is a very useful tool for analysis of the behavior of the conflict-free chromatic index. Moreover, it confirms the intuition that one does not need more colors to color small graphs than are sufficient for larger graphs. If a graph arises from another graph by deleting vertices and edges, i.e. it is a subgraph, one needs at most the same number of colors for the smaller graph as are necessary for a UCE-H coloring of the larger graph.

Lemma 2.5. *Let G, H be graphs, $v \in V(G)$, $e \in E(G)$ and c a UCE-H coloring of G. The induced colorings of $G - v$ and of $G - e$ are UCE-H colorings.*

Proof. Every copy of H in $G - v$ is contained in G too. Hence, there is a color that is assigned by c to a single edge of this copy in G. The induced coloring does the same in $G - v$. Hence, it is good as well. Exactly the same argument holds for an edge $e \in E(G)$. □

Note that the lemma above does apply for deletion of vertices and edges only. Another common manipulation of graphs, the contraction of edges, does not have this property. For example consider a path P_5 and a subpath P_3. The path P_5 has five edges. Coloring the middle one different from all other edges yields a UCE-P_3 coloring of P_5 in two colors. By contracting this edge one obtains a monochromatic path P_4. Hence, the induced coloring is not a UCE-P_3 coloring of P_4. Note that the conflict-free chromatic index does not change in this case.
Another example is the following. Consider a graph G with large girth and a smaller cycle C. One color is sufficient for UCE-C colorings of G. The same holds when edges are contracted as long as the smaller cycle does not fit. If it fits after a contraction was made, at least two colors are necessary. Further contractions lead to a graph that does contain C again and one color is sufficient. In this example the conflict-free chromatic index is changed by the contraction.
The question whether contraction changes the conflict-free chromatic index of a graph G, if and only if the contraction removes the last or adds the first occurence of H in G remains open.

2.2. Conflict-Free Colorings of Hypergraphs

For vertex-colorings of hypergraphs the notion of conflict-free colorings was established in the introduction. A vertex-coloring is called conflict-free, if there is a vertex of unique color within each edge. So this sort of coloring is very similar in definition to the colorings studied in this thesis. The minimum number of colors used by a conflict-free coloring of a hypergraph $\mathcal{H}$ is called *conflict-free chromatic number* and denoted by $\chi_{cf}(\mathcal{H})$. If the unique color should be the maximum color in an edge, too, then the coloring is called *unique-maximum coloring* A tight relation of conflict-free colorings and UCE-H colorings is established next. Afterwards, a summary of known results on conflict-free colorings of hypergraphs is given. Some of the results are presented in detail, if they apply to UCE-H colorings for certain G and H.

Since conflict-free colorings of hypergraphs are vertex-colorings, there is no direct application of the known results to the colorings studied in this thesis. Nevertheless, a class of hypergraphs is established in the following, such that each UCE-H coloring of a graph G corresponds to a conflict-free (vertex) coloring of a hypergraph.

Definition 2.1. *Let G and H be graphs.*
With the set of hyperedges $E_H(G) := \{E \subseteq E(G) \mid E = E(K), K \subseteq G \text{ copy of } H\}$ the hypergraph $\mathcal{F}_{G,H} := (E(G), E_H(G))$ is called H-line graph *of G.*

The H*-line graph* of G is constructed as follows. The vertices in $\mathcal{F}_{G,H}$ correspond to the edges in G. Each copy of H in G induces a hyperedge, consisting of all edges of this copy (respectively vertices in $\mathcal{F}_{G,H}$). It is easy to see that the different notions of colorings are equivalent using this construction.

Lemma 2.6. *Let G and H be graphs and let $\mathcal{F}$ be the H-line graph of G. Then*

$$f(G, H) = \chi_{cf}(\mathcal{F}).$$

Proof. Let c be a minimum UCE-H coloring of G. Construct a vertex-coloring c' of $\mathcal{F}$ by assigning the same color to a vertex, that is assigned to its corresponding edge in G by c. The edges in G corresponding to the vertices of any hyperedge e in $\mathcal{F}$ form a copy of H in G. There is an edge of unique color in this copy and hence the color of the corresponding vertex is unique in e. Thus c' is conflict-free and $\chi_{cf}(\mathcal{F}) \leq f(G, H)$. The proof of $\chi_{cf}(\mathcal{F}) \geq f(G, H)$ is analogous. □

For graphs G and H the H-line graph of G has some nice properties. Clearly, the number of vertices is $|V(\mathcal{F}_{G,H})| = |E(G)|$ and the number of hyperedges $|E(\mathcal{F}_{G,H})|$ is the number of copies of H in G. Furthermore, $\mathcal{F}_{G,H}$ is $|E(H)|$-uniform and its maximum edge degree is $D(\mathcal{F}_{G,H}) = \Delta_H(G)$. The maximum degree $\Delta(\mathcal{F}_{G,H})$ is the maximum number of copies of H in G with a common edge.
The following interesting question remains open. Are there graphs G and H for each uniform hypergraph $\mathcal{H}$, such that $\mathcal{H}$ is isomorphic to the H-line graph of G?

Known Results on Conflict-Free Colorings

Even, Lotker, Ron and Smorodinsky started the research on conflict-free colorings [13]. They came up with this combinatorial problem from an assignment problem in wireless networks. As few frequencies as possible should be assigned to some base stations distributed on a map, such that within the reach of a mobile network client there is always a base station of unique frequency. This is necessary to avoid interference when communicating. The setting is modeled as a hypergraph by taking a vertex for each base station and an edge for each maximum set of base stations that are jointly reached by the mobile device from a location on the map. Then, a conflict-free coloring of the resulting hypergraph yields a useful assignment of frequencies by interpreting the colors as distinct frequencies. Since ranges of wireless networks are typically modeled as discs, this hypergraph is induced by intersections of certain discs in the plane. In case of n base stations (respectively discs) Even et al. showed an upper bound on the conflict-free chromatic number χ_{cf} of $O(\log(n))$ colors for this setting [13]. Pach and Toth proved that this bound is tight for all possible distributions of n points in the plane, i.e. there is always a configuration of discs in the plane such that $\Theta(\log(n))$ colors are necessary [24]. In a survey on conflict-free colorings [28] Smorodinsky mentions further results where the hypergraph under consideration is induced by some geometric setting, for example by rectangles in the plane [27] and pseudodiscs [15].

Besides these geometrically induced hypergraphs other classes of hypergraphs were considered. Pach and Tardos consider in [23] the so called *conflict-free chromatic parameter* of a graph G. This is defined as the conflict-free chromatic number of the hypergraph which has the same vertex set as G and a hyperedge for each vertex together with its neighborhood in G (its *closed neighborhood*). There is also a version with hyperedges consisting of the neighborhoods of vertices but not the vertices themselves (i.e. their *open neighborhood*). In this case the conflict-free chromatic number is called *pointed conflict-free chromatic parameter* of G. Initially this and lots of related questions were studied by Cheilaris [5]. A (poly-)logarithmic upper bound is proven on the conflict-free chromatic parameter as well as on the pointed version. Similar to this, hypergraphs are considered whose hyperedges are induced by all simple paths (of any lengths) of a graph [6, 7]. Unique-Maximum colorings of these hypergraphs are called *vertex rankings*.

These path induced hypergraphs are studied for edge-colorings as well. An edge-coloring is called *edge ranking*, if the largest color in every path connecting two edges of the same color is unique. Originally, these colorings were introduced by Iyer et al. [17]. Of course, such a coloring of a graph G corresponds to a unique-maximum coloring of a hypergraph whose vertices are the edges of G and whose hyperedges are induced by all paths in G. Edge rankings were considered for several classes of graphs. For complete graphs a lower bound quadratic in the number of vertices is known [2]. For trees a linear time algorithm is known to compute a minimum edge ranking [21].

There are several generalizations of conflict-free colorings. For example one might require more than one vertex of unique color in each edge. The conflict-free condition also be weakened by only bounding the size of the smallest color class in each edge by some integer. These generalization are considered by Smorodinsky [28]. Furthermore, there are other types of edge-colorings with certain structural conditions. For example so called *parity colorings* ensure that no color appears on any path an even number of times [4].

General Hypergraphs: So far, only results for certain classes of hypergraphs were presented. For some graphs G and H the H-line graph of G might fit into such a category and good colorings can be obtained, though a general approach (e.g. for a geometric interpreting) is not known. But there are some results for general hypergraphs. Since these are directly related to UCE-H colorings due to Lemma 2.6, they are stated in detail. Pach and Tardos prove several bounds on the conflict-free chromatic number depending on different parameters of the hypergraph [23]. Two theorems are restated in the following lemmas. They are translated to the setting of conflict-free edge-colorings of graphs with respect to other graphs using Lemma 2.6. The first one yields good results for hypergraphs with few edges or small maximum degree.

Lemma 2.7 (Theorem 1, [23]). *Let $s \in \mathbb{N}$ and $\mathcal{H}$ be a hypergraph with $|E(\mathcal{H})| < \binom{s}{2}$. Then the following upper bounds hold and are tight for certain hypergraphs*

$$\begin{aligned} \chi_{cf}(\mathcal{H}) &\leq \Delta(\mathcal{H}) + 1, \\ \chi_{cf}(\mathcal{H}) &< s, \\ \chi_{cf}(\mathcal{H}) &\leq \tfrac{1}{2} + \sqrt{2 \cdot |E(\mathcal{H})| + \tfrac{1}{4}}. \end{aligned}$$

By plugging in the corresponding parameters of the H-line graph, the following corollary holds.

Corollary 2.8. *Let $s \in \mathbb{N}$ and G, H graphs with $n_H(G) < \binom{s}{2}$ copies of H in G. Further let D denote the maximum number of copies H in G with a common edge. Then*

$$\begin{aligned} f(G,H) &\leq D + 1, \\ f(G,H) &< s, \\ f(G,H) &\leq \tfrac{1}{2} + \sqrt{2 \cdot n_H(G) + \tfrac{1}{4}}. \end{aligned}$$

The second theorem provides an upper bound that is sublinear in the maximum edge-degree $D(\mathcal{H})$, i.e. the maximum number of hyperedges intersecting a given hyperedge. A review of the proof is presented in the appendix in section A.1.

Lemma 2.9 (Theorem 2, [23]). *Let $t \in \mathbb{N}$ and let $\mathcal{H}$ be a hypergraph whose edges have at least $2t - 1$ vertices each. There is a constant $c \in \mathbb{R}$, such that*

$$\chi_{cf}(\mathcal{H}) \leq c \cdot t \cdot D(\mathcal{H})^{\frac{1}{t}} \cdot \log(D(\mathcal{H})).$$

Since the H-line graph is $|E(H)|$-uniform, $\lceil \frac{|E(H)|}{2} \rceil$ is the largest integer satisfying the condition on t in the lemma above. Hence, the following corollary holds.

Corollary 2.10. *Let G, H be graphs and $m := |E(H)|$. There is a constant $c \in \mathbb{R}$, such that*

$$f(G,H) \leq c \cdot \lceil \tfrac{m}{2} \rceil \cdot \Delta_H(G)^{\frac{2}{m}} \cdot \log(\Delta_H(G)).$$

Of course, the maximum H-degree $\Delta_H(G)$ of a graph G is smaller than the total number $n_H(G)$ of copies H in G. As a corollary, this yields $f(G,H) < c \cdot \lceil \frac{m}{2} \rceil \cdot n_H(G)^{\frac{2}{m}} \cdot \log(n_H(G))$.

Another work dealing with general hypergraphs is due to Kostochka, Kumbhat and Luczak [20]. They are concerned with hypergraphs on few edges. The first result is the following.

Lemma 2.11 (Lemma 2.3, [20]). *Let $r \in \mathbb{N}$ with $r \geq 3$ and $\mathcal{H}$ an r-uniform hypergraph with $|E(\mathcal{H})| \leq 6$. Then*

$$\chi_{cf}(\mathcal{H}) \leq 3.$$

This means, that every graph G admits a UCE-H coloring using at most three colors, if there are at most six copies of the graph H in G and $|E(H)| \geq 3$. A somewhat more general result is obtained next.

Lemma 2.12 (Theorem 3.2. i, [20]). *Let $r \in \mathbb{N}$ and let $\mathcal{H}$ be an r-uniform hypergraph. If $D(\mathcal{H})$ is sufficiently large and $D(\mathcal{H}) \leq 2^{\frac{r}{2}}$, then*

$$\chi_{cf}(\mathcal{H}) \leq 120 \cdot \mathit{ln}(D(\mathcal{H})).$$

Translated to edge-colorings with uniquely colored edges this yields the following corollary.

Corollary 2.13. *Let G, H be graphs and $m := |E(H)|$. If $\Delta_H(G)$ is sufficiently large and $\Delta_H(G) \leq 2^{\frac{m}{2}}$, then*

$$f(G,H) \leq 120 \cdot \mathit{ln}(\Delta_H(G)).$$

Under the corollary's condition $\Delta_H(G) \leq 2^{\frac{|E(H)|}{2}}$, this implies $f(G,H) \leq 60 \cdot |E(H)|$, if $\Delta_H(G)$ is sufficiently large.

Finally, Smorodinsky's survey states a result connecting the classical chromatic number for hypergraphs with the conflict-free chromatic number [28].

Lemma 2.14 (Theorem 1.5, [28]). *Let $\mathcal{H}$ be a hypergraph. Further let $k \in \mathbb{N}$, such that $\chi(\mathcal{H}') \leq k$ for all induced subhypergrpahs $\mathcal{H}' \subseteq \mathcal{H}$. Then*

$$\chi_{cf}(\mathcal{H}) \leq \log_{1+\frac{1}{k-1}}(|V(\mathcal{H})|).$$

2.3. Ramsey Theory

The problem considered in this work is not only inspired by Ramsey theory, but there are also direct relations like the following ones. In some extent it means that the UCE-H property is stronger than the Ramsey property not to be monochromatic.

Lemma 2.15. *Let G, H be graphs with $|E(H)| \geq 2$. Then*

$$f(G,H) \geq R_H^{inv}(G).$$

In case $|E(H)| \leq 3$ this bound is tight, i.e. $f(G,H) = R_H^{inv}(G)$.

Proof. Let c be a UCE-H coloring of G. Since there is a color class in each copy of H in G consisting of one single edge and since H has at least two edges, there must be another color in each copy of H. Hence, there is no monochromatic copy of H in G. Particularly $f(G,H) \geq R_H^{inv}(G)$.

Consider the case $|E(H)| \leq 3$. It is not possible to color the edges of H in two or more colors, such that each color class has at least two edges. Hence, every coloring avoiding monochromatic copies of H in G is a UCE-H coloring as well and hence $f(G,H) = R_H^{inv}(G)$. □

Using this lemma, one can obtain a lower bound on the conflict-free chromatic index f for sufficiently large and dense graphs G. The following corollary states the result.

Corollary 2.16. *Let G and H be graphs with $|E(H)| \geq 2$ and let $n := |V(G)|$, $m := |E(G)|$, $p := |V(H)|$. Then*

$$f(G,H) > \max\{k \in \mathbb{N} \mid m > \mathrm{ex}(n, R_k(p))\}.$$

Proof. If $m > \mathrm{ex}(n, R_k(p))$, then there is a copy of $K_{R_k(p)}$ contained in G by definition of the extremal number. Due to the definition of the multicolor Ramsey number $R_k(p)$, there is a monochromatic copy of K_p in each coloring of G in at most k colors. Hence, $R_H^{inv}(G) \geq R_p^{inv}(G) > k$. Lemma 2.15 implies $f(G,H) > k$. □

From (classical) multicolor Ramsey theory lower bounds are obtained. Using a generalization of Ramsey theory from Erdős [11] it is possible to obtain an upper bound on the conflict-free chromatic index f as well. Colorings are considered which do not only avoid monochromatic subgraphs, but for some $q \in \mathbb{N}$ avoid certain subgraphs on less than q colors. Recall the definition of the generalized Ramsey number $\widetilde{R}_k(H;q)$, which denotes the order of the smallest complete graph that does not admit an edge-coloring with this property with respect to H using k colors. The following lemma establishes a relation to the minimum number of colors of such a coloring for certain choice of q.

Lemma 2.17. *Let G, H be graphs. Then*

$$f(G,H) \leq \widetilde{R}_H^{inv}(G; \left\lfloor \tfrac{|E(H)|}{2} \right\rfloor + 1).$$

Proof. Let c be a coloring of G that assigns at least $\lfloor \frac{|E(H)|}{2} \rfloor + 1$ distinct colors to each copy of H in G. Assume that every color class in such a copy of H consists of more than one color. Let $m := |E(H)|$. Then there are at least $2 \cdot \lfloor \frac{m}{2} \rfloor + 2 \geq 2 \cdot (\frac{m}{2} - \frac{1}{2}) + 2 = m + 1$ edges in this copy of H, contradicting the assumption. Hence, c is a UCE-H coloring of G. In particular $f(G, H) \leq \widetilde{R}_H^{inv}(G; \lfloor \frac{m}{2} \rfloor + 1)$. $\square$

Using the Lovasz Local Lemma [1] one can determine an upper bound on the conflict-free chromatic index from the lemma above. There are several versions of the Local Lemma. The symmetric one, which is used here, has the following statement. Let $p \in \mathbb{R}$ with $0 \leq p \leq 1$ and $d \in \mathbb{N}$. Further let Ω denote a set of events, such that each event occurs with probability at most p and depends on at most d of the other events. If $e \cdot p \cdot (d + 1) < 1$, then the probability that none of the events in Ω occurs is positive.
The proof of the following theorem is similar to (and inspired by) the proof of Theorem 1 in [12] from Erdős and Gyárfás.

Theorem 1. *Let G, H be graphs and $m := |E(H)|$. Then*

$$f(G, H) \leq (e \cdot (\Delta_H(G) + 1))^{\frac{2}{m}} \cdot \frac{m^2}{4}.$$

Proof. Using the Lovasz Local Lemma an upper bound on $\widetilde{R}_H^{inv}(G; \lfloor \frac{m}{2} \rfloor + 1)$ is obtained. Let c be a random edge-coloring of G in k colors, assigning the colors independently and uniformly to the edges. The probability P that a copy K of H has less than $\lfloor \frac{m}{2} \rfloor + 1$ colors assigned to its edges is overestimated by $\binom{k}{\lfloor \frac{m}{2} \rfloor} \cdot (\frac{\lfloor \frac{m}{2} \rfloor}{k})^m < k^{-\frac{m}{2}} \cdot (\frac{m}{2})^m$. If $k \geq (e \cdot (\Delta_H(G) + 1))^{\frac{2}{m}} \cdot \frac{m^2}{4}$ holds, then $e \cdot (\Delta_H(G) + 1) \cdot k^{-\frac{m}{2}} \cdot (\frac{m}{2})^m \leq 1$ holds too. Furthermore, $\Delta_H(G)$ is at least as large as the number of copies of H in G that are not independent from the coloring of K by definition. Hence, there is a coloring of G using $(e \cdot (\Delta_H(G) + 1))^{\frac{2}{m}} \cdot \frac{m^2}{4}$ colors that assigns at least $\lfloor \frac{m}{2} \rfloor + 1$ colors to each copy of H in G due to Lovasz Local Lemma. Thus $\widetilde{R}_H^{inv}(G; \lfloor \frac{m}{2} \rfloor + 1) \leq (e \cdot (\Delta_H(G) + 1))^{\frac{2}{m}} \cdot \frac{m^2}{4}$. Finally, this theorem follows from Lemma 2.17. $\square$

Two corollaries are derived from this theorem. The first one translates the result into the setting of conflict-free vertex-colorings of hypergraphs. It holds, because exactly the same proof can be applied to a vertex-coloring of a uniform hypergraph. The result improves the best known upper bound from Lemma 2.9 in case of uniform hypergraphs by a factor of $\log(D(\mathcal{H}))$.

Corollary 2.18. *Let $r \in \mathbb{N}$ and $\mathcal{H}$ be an r-uniform hypergraph. Then*

$$\chi_{cf}(\mathcal{H}) \leq (e \cdot (D(\mathcal{H}) + 1))^{\frac{2}{r}} \cdot \frac{r^2}{4}.$$

The second corollary restates the upper bound of the theorem in terms of the number of vertices in the graph. Using a very rough estimate on $\Delta_H(G)$ one obtains the following.

Corollary 2.19. *Let G, H be graphs and $n := |V(H)|$, $p := |V(H)|$, $m := |E(H)|$. Then there is a constant $c_H \in \mathbb{R}$ depending on H, such that*

$$f(G,H) \leq c_H \cdot n^{\frac{2\cdot(p-2)}{m}}.$$

Proof. For each edge in G the number of copies of H in G containing this edge is at most the number of possible mappings of the given edge and $p-2$ of the remaining vertices of G into H. The number of such mappings is at most $m \cdot (n-2)_{p-2} = m \cdot (n-2) \cdot (n-3) \cdots (n-p+1)$. Since H has m edges, $\Delta_H(G)+1 \leq m^2 \cdot (n-2)_{p-2}$. Due to Theorem 1,

$$f(G,H) \leq (e \cdot m^2 \cdot (n-2)_{p-2})^{\frac{2}{m}} \cdot \frac{m^2}{4} \leq (e \cdot m^2 \cdot (n-2)^{p-2})^{\frac{2}{m}} \cdot \frac{m^2}{4} \leq c_H \cdot n^{\frac{2\cdot(p-2)}{m}}.$$

□

2.4. Complexity and Algorithms

So far, some values and different bounds on the conflict-free chromatic index f were determined. Within this section it is proven, that deciding the k-UCE-H problem is NP-hard in general. Nevertheless, there are algorithms that calculate UCE-H colorings for all graph G and H and yield colorings using less colors than are used by a total multicoloring of G.

Complexity

For a special choice of H there is an exact result for $f(G,H)$ in terms of the chromatic index $\chi'(G)$ of G.

Lemma 2.20. *Let G be a graph. Then*

$$f(G,P_2) = \chi'(G).$$

Proof. The path P_2 consists of exactly two adjacent edges. Hence, the copies of P_2 in G are exactly all pairs of adjacent edges. An edge of P_2 has a unique color within P_2, if and only if both edges of P_2 are of distinct colors. Hence, an edge-coloring c is a UCE-P_2 coloring of G, if and only if it is a proper coloring. □

This directly leads to the following complexity result, since deciding whether a graph is class I or class II is NP-complete [16]. The proof shows another reduction argument to see this.

Theorem 2. *The decision problem k-UCE-H is NP-hard.*

Proof. Another reduction of the k-UCE-H problem is used instead of the determination of the chromatic index of G. Let $n := |V(G)|$ and let C_n denote the simple cycle on n vertices. Then $f(G,C_n) > 1$, if and only if G contains a Hamiltonian cycle. Deciding the presence of a Hamiltonian cycle in a graph is NP-complete, as well [19]. □

Algorithms

Next, two algorithms are presented which calculate good colorings. Because the k-UCE-H problem is NP-hard, there is no efficient algorithm calculating optimal colorings (if $P \neq NP$). So the colorings are not minimum or the running time is not polynomial.

Greedy Algorithm: The first construction is kind of a greedy algorithm. Let c be an edge-coloring of a graph G. For another graph H let $g(c, H)$ denote the number of good copies of H in G. For a color θ and an edge $e \in E(G)$ define the edge-coloring c'_e by $c'_e(e) := \theta$ and $c'_e(e') := c(e')$ for all other edges $e' \in E(G) \setminus \{e\}$. The *$\theta$-weight* of an edge $e \in E(G)$ is defined as $w_H(\theta, e) := g(c'_e, H) - g(c, H)$. In words, the weight of an edge e (with respect to a certain color θ and a given coloring c) is the growth of the number of good copies of H in G when assigning color θ to the edge e.
The basic greedy algorithm is described in the following. For several colors (including a new one) the weight function is calculated for all edges. The color which induces the largest weight on some edge is assigned to this edge. Note that the weights change during the execution of the construction. Moreover, the weights depend on the choices made by the algorithm before.

Construction 2.1. *Let G and H be graphs.*

s1) *Initially, color 0 is assigned to all edges of G.*

s2) *Let c be the current edge-coloring of G and let $T \subset \mathbb{N}_0$ with $|T \setminus c(E(G))| = 1$. Choose $\hat{\theta} \in T$ and $\hat{e} \in E(G)$ with $w_H(\hat{\theta}, \hat{e}) \geq w_H(\theta, e)$ for all $\theta \in T$, $e \in E(G)$. If $w_H(\hat{\theta}, \hat{e}) > 0$, define $c'(\hat{e}) := \hat{\theta}$ and $c'(e) := c(e)$ for all $e \in E(G) \setminus \{\hat{e}\}$ and repeat this step with the coloring c'. Otherwise, the construction is finished.*

The running time of this algorithm mainly depends on the calculation of the weight function. In the worst case it is necessary to find all copies of H in G (containing the edge which was colored last) and check if they became good or not. But this problem itself is NP-hard. Nevertheless, the algorithm is correct as stated in the next lemma.

Lemma 2.21. *Let G, H be graphs. Construction 2.1 yields a UCE-H coloring of G.*

Proof. Since the number of copies of H in G is finite and since the number of good copies increases in each execution of the second construction step, the algorithm stops after a finite number of steps.
It remains to shows correctness of the algorithm. If it stops, there is no color which may be assigned to an edge of G, such that the number of good copies of H in G is increased. In each step the set of colors T contains a color θ which is currently not assigned to any edge. If there is a bad copy of H in G, it becomes good, if color θ is assigned to one of its edges e. Furthermore, no currently good copy of H in G is bad afterwards, since the edge e is uniquely colored in all copies of H in G containing it. Hence, there is an edge e and a color θ with $w_H(\theta, e) > 0$, if there is a bad copy of H in G. Thus, the algorithm does not stop before a UCE-H coloring of G is obtained. $\square$

Within the repeated second step of Construction 2.1 there are two choices to be made. First of all, the set T of those colors whose weights are evaluated in each step contains a new color together with an arbitrary choice of already used colors. Restricting the number of already used colors improves the running time.
Furthermore, among all combinations of colors and edges with maximum weight an arbitrary one is chosen. Preferring already used colors to the new color may decrease the total number of colors used. This is of particular importance due to the following observation. Consider the graph G together with an edge-coloring c and let $\theta \in c(E(G))$, $\lambda \in \mathbb{N}_0 \setminus c(E(G))$. Then $w_H(\lambda, e) \geq w_H(\theta, e)$ for all $e \in E(G)$, because every copy of H in G that becomes good by assigning θ to e becomes good by assigning λ to e, as well. Hence, the new color is always among those colors with maximum gain of good copies of H in G. One may use the following modification of the second construction step to avoid this. It is divided into two cases. First of all, evaluate the weights of all edges for already used colors only. If there is a color with positive weight on an edge, choose a color and an edge with maximum weight. Otherwise, calculate the weights for all edges and a new color. If there is an edge with positive weight, assign the new color to an edge with maximum weight. Otherwise, the construction is finished and the coloring is good, due to the same arguments as in Lemma 2.21 above.

Separator Algorithm: The second algorithm works with separating sets of edges. It is inspired by work in [18]. Good colorings of the components of a graph induced by a separating set can be extended to a good coloring of the whole graph as follows. Choose colors that are not used in the partial colorings. The edges of the separating set $\mathcal{E}$ are colored in such a way, that in every copy of H containing edges from $\mathcal{E}$ one of these edges is of unique color. A multicoloring of $\mathcal{E}$ satisfies this condition in any case. Note that the colorings of distinct components may use the same colors. The choice of the separating set and its coloring have great impact on the total number used. The larger the separating set, the less colors are necessary for the components. But on the other hand more colors are needed for the edges in the separator, though a total multicoloring is not necessary in general.

3. Paths in Trees

In this chapter UCE-P_m colorings of trees are studied. Given a length $m \in \mathbb{N}$ and a tree T, edge-colorings of T are considered that ensure a uniquely colored edge in every path of length m. First of all, paths as a very special class of trees are considered in Section 3.1. It is shown that two colors are sufficient for any good coloring of a path with respect to a smaller path. For general trees it turns out that the parity of m is essential. On account of this fundamental difference these two cases are handled in two distinct sections. First of all, Section 3.2 deals with paths on an odd number of edges. The class of spiders is studied in Subsection 3.2.1, first. As it is true for paths, every spider has a good coloring (with respect to paths on an odd number of edges) which uses only two colors. Afterwards, general trees are considered. A construction is given which uses at most $\lceil \frac{m}{2} \rceil$ colors, independently of T. It is shown that this upper bound on the conflict-free chromatic index is tight for complete d-ary trees and sufficiently large $d \in \mathbb{N}$. Afterwards, paths on an even number $m \in \mathbb{N}$ of edges are considered in Section 3.3. As mentioned in the beginning, there are increasing lower bounds on the function f depending on the tree in this case. They are established on spiders, although there are good colorings with two colors for every spider in the previous case. So spiders are considered first in Subsection 3.3.1. Recall the definition of the a-degree of a vertex v for an $a \in \mathbb{N}$. It is the maximum number of edge-disjoint copies of P_a with endpoint v. A lower bound is given which is logarithmic in the maximum $\frac{m}{2}$-degree of a spider. Moreover, there is a construction which almost attains this bound. One key observation of this result is the following. For each even $m \in \mathbb{N}$ and every integer $k \in \mathbb{N}$ there is a tree T, such that all UCE-P_m colorings of T use more than k colors. A second lower bound is stated afterwards. Furthermore, every graph contains spiders as subgraphs. Hence, the results provide lower bounds for all graphs in terms of their maximum $\frac{m}{2}$-degree, too. An upper bound on the conflict-free chromatic index $f(T, P_m)$ for general trees T and even m is established in Subsection 3.3.2. It depends on m and the maximum degree of T. It is tight for complete $(\Delta-1)$-ary trees and $m = 4$. Two constructions that are independent from the parity of m are presented in Section 3.4. Finally, all results are summarized in the last Section 3.5. For all trees T with fixed maximum degree or fixed number of vertices questions for maximum and minimum values of $f(T, P_m)$ are answered there for the case of fixed $m \in \mathbb{N}$ (when applicable).

3.1. Big Path and Small Path

First of all trees with only two leaves, i.e. paths, are considered. In this setting the values of f are given exactly. For $n, m \in \mathbb{N}$ the following construction provides a UCE-P_m coloring of the path P_n as it is proven in the subsequent lemma. This

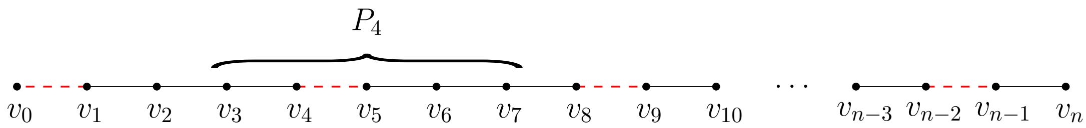


Figure 3.1.: A UCE-P_4 coloring of a larger path P_n in two colors for $\delta = 0$.

coloring is used later to construct UCE-P_m colorings of arbitrary trees.

Definition 3.1. *Let $n, m, \delta \in \mathbb{N}$ with $0 \leq \delta \leq m - 1$ and let $P_n = v_0 v_1 \dots v_n$ with $v_0, \dots, v_n \in V(P_n)$. The following edge-coloring is called* path-coloring *of P_n.*

$$c_\delta : E(P_n) \to \{0, 1\}, \; c_\delta(\{v_i, v_{i+1}\}) := \begin{cases} 1 & , \text{ if } m|(i - \delta), \\ 0 & , \text{ otherwise.} \end{cases}$$

Edges of color 1 are called special edges. *The color 1, which might be substituted by another color distinct from 0, is called* special color. *The integer δ causes a cyclic shift of the coloring along the path P_n and is called* translation.

Figure 3.1 shows an example of the path coloring defined above in case $m = 4$ and $\delta = 0$.

Lemma 3.1. *For all $n, m \in \mathbb{N}$ and each $\delta \in \mathbb{N}$ with $0 \leq \delta \leq m - 1$ the path coloring c_δ of P_n is a UCE-P_m coloring.*

Proof. In the case $m = 1$, color 0 is assigned to all edges. Since any edge-coloring is good with respect to P_1, the path coloring is good in this case.

The case $m \geq 2$ remains.
Here, the coloring c_δ assigns color 0 to the first $\delta \geq 0$ edges of P_n and color 1 to the subsequent edge. Afterwards, each m'th edge receives color 1 and all other edges color 0. Hence, each copy of P_m (a path on m edges) contains exactly one of the edges of color 1, since $\delta \leq m - 1$. Therefore c_δ is a UCE-P_m coloring. □

The following lemma is based on this construction.

Lemma 3.2. *Let $n, m \in \mathbb{N}$. Then*

$$f(P_n, P_m) = \begin{cases} 1 & , \text{ if } m = 1 \text{ or } m > n, \\ 2 & , \text{ if } 2 \leq m \leq n. \end{cases}$$

Proof. Lemma 2.2 implies the first equality $f(P_n, P_m) = 1$, in case $m = 1$ or $m > n$. In case $2 \leq m \leq n$, one concludes from Lemma 2.3 that $f(P_n, P_m) > 1$. For each $\delta \in \mathbb{N}$ with $0 \leq \delta \leq m - 1$ the path coloring c_δ from Definition 3.1 is a UCE-P_m coloring of P_n in two colors, due to Lemma 3.1. Hence $f(P_n, P_m) = 2$ in this case. □

3.2. Paths with Odd Number of Edges

3.2.1. Spiders and Paths with Odd Number of Edges

The path coloring defined in Section 3.1 provides for all $m \in \mathbb{N}$ a UCE-P_m coloring for trees on two leaves using two colors only. The following construction shows that for every given $\Delta \in \mathbb{N}$ there is an arbitrarily large tree with maximum degree Δ that admits a UCE-P_m coloring using only two colors as well.

Construction 3.1. *Let $m \in \mathbb{N}$ be odd and S be a spider. On each leg $L = vu_1u_2 \dots u_L$, with $v, u_1, \dots, u_L \in V(S)$, the path coloring from Definition 3.1 with translation $\delta = \lfloor \frac{m}{2} \rfloor$ is used.*

Lemma 3.3. *Let $m \in \mathbb{N}$ odd and S be a spider. Construction 3.1 yields a UCE-P_m coloring of S.*

Proof. Let c be a coloring obtained by Construction 3.3. On each leg of S, the path coloring is used with translation $\lfloor \frac{m}{2} \rfloor$. Hence, there are $\lfloor \frac{m}{2} \rfloor$ edges between the head of S and the first special edge. Thus, there are exactly $2 \cdot \lfloor \frac{m}{2} \rfloor = m-1$ edges between two special edges that are nearest to the head of S. So the path colorings of two distinct legs of S fit together and are a path coloring of the path that is formed by the two legs. Since a copy of P_m in S is contained in at most two distinct legs, the coloring c is a UCE-P_m coloring of the spider. □

Using the coloring constructed above and some basic facts, the following holds.

Lemma 3.4. *Let $m \in \mathbb{N}$ odd and S be a spider with diameter $\rho > 0$. Then*

$$f(S, P_m) = \begin{cases} 1 & \text{, if } \rho < m \text{ or } m = 1, \\ 2 & \text{otherwise.} \end{cases}$$

Proof. If $\rho < m$ or $m = 1$, one color is sufficient due to Lemma 2.2. Otherwise Construction 3.1 yields a UCE-P_m coloring of S in two colors. Due to Lemma 2.3 this bound is sharp, since P_m contains more than one edge if $m > 1$. □

3.2.2. Arbitrary Trees and Paths with Odd Number of Edges

This section deals with UCE-P_m colorings of general trees for odd $m \in \mathbb{N}$. As mentioned in the beginning there is a construction providing a UCE-P_m coloring for any tree. Moreover the number of colors used is bounded from above by a term depending on m but not on the tree. This construction is described next, followed by a proof of its correctness. Afterwards, it is shown that this coloring uses minimum number of colors on complete d-ary trees for sufficiently large $d \in \mathbb{N}$.

Construction 3.2. *Let $m \in \mathbb{N}$ be odd and let T be a tree rooted at (one of) the middle vertices of a longest path in T.*

s1) *Color 1 is assigned to all edges in the first $\lfloor \frac{m}{2} \rfloor$ edge levels of T.*

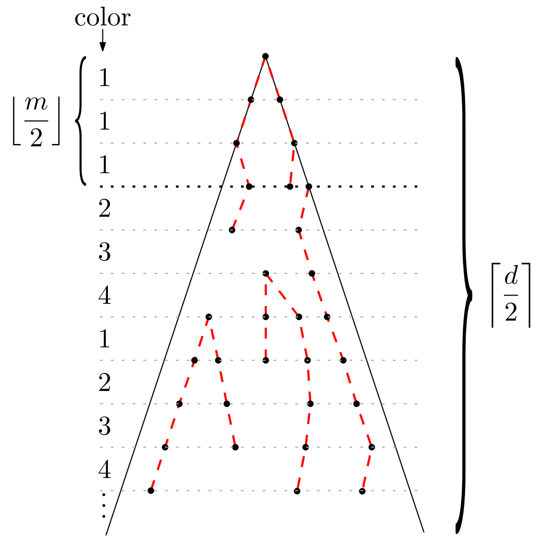


Figure 3.2.: A UCE-P_7 coloring of a tree in 4 colors. The colors written on the left hand side are assigned to all edges in the corresponding level.

s2) *In an edge level of index $j \geq \lceil \frac{m}{2} \rceil$ color $i \in \mathbb{N}$ is assigned to all edges, if $(j+1) \equiv (i-1) \mod \lceil \frac{m}{2} \rceil$. In words, the colors from 1 up to $\lceil \frac{m}{2} \rceil$ are used cyclically one after the other, the same color assigned to all edges of a level (if color $\lceil \frac{m}{2} \rceil$ is reached, it starts with color 1 again).*

An example of the coloring is given in Figure 3.2.
The next lemma shows that the coloring is a UCE-P_m coloring.

Lemma 3.5. *Let T be a tree and let $m \in \mathbb{N}$ be odd. Construction 3.2 yields a UCE-P_m coloring of T.*

Proof. The tree T is considered rooted at (one of) the middle vertices of one of its longest paths. Except for the first $\lfloor \frac{m}{2} \rfloor$ levels, distinct colors are assigned to $\lceil \frac{m}{2} \rceil$ consecutive levels. Consider a copy P of P_m in T. All its edges are contained in consecutive edge levels. Furthermore, there are at most two edges from P in each level. At least the first or the last edge of P must be contained in a level of index larger than $\lfloor \frac{m}{2} \rfloor$ where no other edge of P is contained in, since the number of edges in P is odd. The coloring c is a UCE-P_m coloring, if one of these edges is contained in a level of unique color among all levels containing edges from P. All other edges of P are contained in edge levels above. Since $\lceil \frac{m}{2} \rceil$ consecutive levels have distinct colors and colors are used in fixed order, it is not possible that all the colors get repeated in P (note that $m < 2 \cdot \lceil \frac{m}{2} \rceil$ since m is odd). Hence c is a UCE-P_m coloring of T. □

Using this coloring an upper bound on the conflict-free chromatic index is proven. Morever, the coloring is a minimum good coloring for certain trees, as well. The following theorem states both facts.

Theorem 3. *Let $m \in \mathbb{N}$ be odd and T be a tree with diameter $\rho := diam(T) \geq m$. Then*

$$f(T, P_m) \leq \min\left\{\left\lceil \frac{\rho}{2} \right\rceil - \left\lfloor \frac{m}{2} \right\rfloor + 1, \left\lceil \frac{m}{2} \right\rceil\right\}.$$

If T is a complete d-ary tree of height at least $\lceil \frac{m}{2} \rceil$ and if $d \geq 2$ and $d > (\lceil \frac{m}{2} \rceil)^m$, then the upper bound is attained.

Proof. Let c denote a coloring of T obtained from Construction 3.2. It is a UCE-P_m coloring due to Lemma 3.5. Distinct colors are assigned to $\lceil \frac{m}{2} \rceil$ consecutive levels, except for the first $\lfloor \frac{m}{2} \rfloor$ levels whose edges all have the same color. The rooted tree T has height $\lceil \frac{\rho}{2} \rceil$. Hence, the coloring c uses $\lceil \frac{\rho}{2} \rceil - \lfloor \frac{m}{2} \rfloor + 1$ colors, if $\lceil \frac{\rho}{2} \rceil \leq m - 1$, and $\lceil \frac{m}{2} \rceil$ colors otherwise.

It remains to proof the tightness of the bound. Let $d \in \mathbb{N}$ with $d \geq 2$ and $d > (\lceil \frac{m}{2} \rceil)^m$, and let T a complete d-ary tree. Note that $h(T) = \lceil \frac{\text{diam}(T)}{2} \rceil$ holds for T. Hence, the lower bound $f(T, P_m) \geq \min\left\{h(T) - \lfloor \frac{m}{2} \rfloor + 1, \lceil \frac{m}{2} \rceil\right\}$ needs to be proven for the case $h(T) \geq \lceil \frac{m}{2} \rceil$. If $m = 1$, this lower bound evaluates to $f(T, P_m) \geq 1$, which is true due to Lemma 2.2. Consider the case $m \geq 3$ and an edge-coloring c of T which uses $k \leq \lceil \frac{m}{2} \rceil$ colors. A complete binary subtree $T' \subseteq T$ is called *bad*, if it has the same root as T, if each of the first m edge levels of T' is colored monochromatically, and if each leaf of T' is a leaf of T (in other words $h(T) = h(T')$).

First of all, it is proven by induction on the height $h(T)$ that T contains such a bad subtree. The key observation is the following. Since $d > (\lceil \frac{m}{2} \rceil)^m$ and $\lceil \frac{m}{2} \rceil \geq k$, there is a monochromatic star on at least $\lceil \frac{d}{k} \rceil > (\lceil \frac{m}{2} \rceil)^{m-1} > 2$ edges incident to the root of T. Let Z be the set of vertices of this star without the root of T. In the basic case $h(T) = 1$, choosing two edges of such a monochromatic star yields a bad complete binary subtree of T. If $h(T) > 1$, consider all those complete d-ary subtrees of T whose roots are in Z and have height $h(T) - 1$. By induction hypothesis, each of these trees contains a bad complete binary subtree. Since $|Z| > (\lceil \frac{m}{2} \rceil)^{m-1} \geq k^{m-1}$ and because the first $\frac{m}{2}$ edge levels of these bad subtrees are colored monochromatically, there are two of these bad subtrees whose first $m - 1$ edge levels are colored identically, due to pigeonhole principle. Since these two bad subtrees are joined by two edges of the same color within the monochromatic star incident to the root of T, the tree T contains a bad complete binary subtree.

It remains to show the following. If $h(T) \geq \lceil \frac{m}{2} \rceil$, then the existence of a bad subtree T' forces the existence of a bad copy of P_m in $T' \subseteq T$. Let T be a complete d-ary tree with $h(T) \geq \lceil \frac{m}{2} \rceil$ and $T' \subseteq T$ a bad complete binary subtree of T. A copy $P \subseteq T$ of P_m is called *balanced*, if the (unique) topmost vertex of P is one of the middle vertices of P. In the following let P be a balanced copy of P_m in the bad subtree T'. An edge incident to a leaf of P is called *leaf edge*. Since m is odd, exactly one edge e of P is contained in an edge level Λ of T' which does not contain any other edge from $E(P)$. Furthermore, e is a leaf edge of P. The level Λ is located below all other levels in T' containing edges from $E(P)$. If P is not bad, the color of all edges in Λ is distinct from all colors in the $\lfloor \frac{m}{2} \rfloor$ edge levels above, because T' is bad and therefore has monochromatic edge levels. Because each edge level in T' of index at least $\lceil \frac{m}{2} \rceil$ contains such a leaf edge of a balanced copy of P_m in T', this is true for all these levels. Hence, either there is a bad copy of P_m in T' or $\lceil \frac{m}{2} \rceil$ consecutive edge levels of T' have distinct colors, except for the first $\lfloor \frac{m}{2} \rfloor$ levels. In case $h(T) < m$ this means c is bad, if $k < h(T) - \lfloor \frac{m}{2} \rfloor + 1$, since the last $h(T) - \lfloor \frac{m}{2} \rfloor < \lceil \frac{m}{2} \rceil$ edge levels of T' cannot have distinct colors then. In case $h(T) \geq m$ there are $\lceil \frac{m}{2} \rceil$ consecutive edge levels in T' below the $\lfloor \frac{m}{2} \rfloor$ first levels. Hence c is bad in this case, if $k < \lceil \frac{m}{2} \rceil$. Altogether, c is bad, if $k < \min \left\{ h(T) - \lfloor \frac{m}{2} \rfloor + 1, \lceil \frac{m}{2} \rceil \right\}$. □

3.3. Paths with Even Number of Edges

UCE-P_m coloring of trees are studied in this section for even number of edges $m \in \mathbb{N}$. First of all, the case of a path on two edges, i.e. $m = 2$, is considered. Since the chromatic index of a tree equals its maximum degree, the following corollary to Lemma 2.20 holds.

Corollary 3.6. *Let T be a tree. Then $f(T, P_2) = \Delta(T)$.*

Proof. Let $N = \{1, \ldots, \Delta(T)\}$ be a set of colors. Start with an arbitrary leaf of T and color its adjacent edge with an arbitrary color from N. Iteratively choose an edge $e \in E(T)$ which is incident to an already colored edge of T. There is a color in N which is not assigned to any edge adjacent to e. Assign this color to e and choose the next uncolored edge until all edges are colored. Since a tree is acyclic, this greedy construction is well defined and yields a proper coloring of T using $\Delta(T)$ colors. According to Lemma 2.20, the chromatic index of G equals the conflict-free chromatic index of G with respect to P_2. □

The value of the conflict-free chromatic index is directly related to the maximum degree of the tree, here. First of all, a similar relation is established for all paths on a larger but even number of edges by considering a certain class of trees, the spiders. It was shown in the previous section, that in case of odd m two colors are sufficient. This is not true for even m. It is shown that the necessary number of colors increases with increasing number of legs. Afterwards, colorings of arbitrary trees are considered.

3.3.1. Spiders and Paths with Even Number of Edges

Spiders are of particular interest here, since every tree contains spiders as subtrees. Especially there is spider on Δ legs in a tree of maximum degree Δ. The following construction yields a UCE-P_m coloring of a spider.

Construction 3.3. *Let $m \in \mathbb{N}$ be even, S be a spider and $\mathcal{L}$ be the set of legs of S on at least $\frac{m}{2}$ edges. For each leg $L \in \mathcal{L}$ let N_L be a set of colors with $|N_L| \leq \frac{m}{2}$ and $0 \notin N_L$, such that $N_L \neq N_J$ for all distinct legs $L, J \in \mathcal{L}$. If there is a leg in S on less than $\frac{m}{2}$ edges, the sets N_L must not be empty. Define $k := |\bigcup_{L \in \mathcal{L}} N_L|$.*

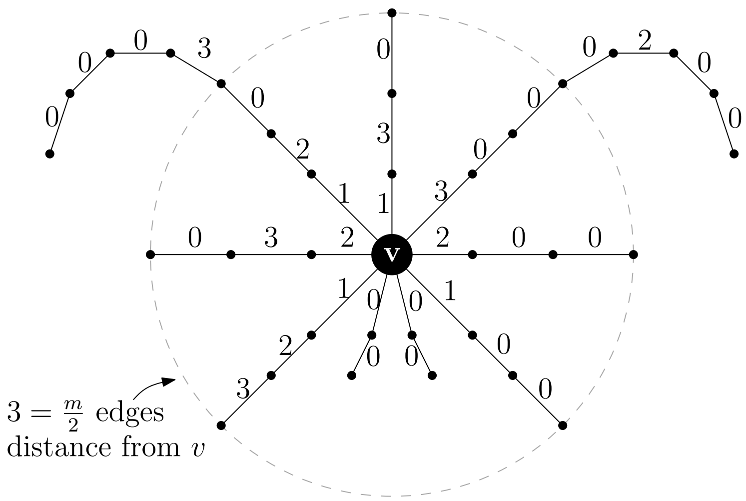


Figure 3.3.: A UCE-P_6 coloring in 4 colors of a spider with 7 legs on more than 3 edges and 2 legs on less edges.

s1) *Initially, assign color* 0 *to all edges of S.*

s2) *On each leg $L \in \mathcal{L}$ assign the colors in N_L to the $|N_L|$ edges closest to the head of S.*

s3) *Color the remaining edges of a leg L using the path-coloring from Section 3.1 (starting from the head of S). If $|N_L| < k$ and $|N_L| < \frac{m}{2}$, the special color in this path coloring is an arbitrary color in $(\bigcup_{J \in \mathcal{L}} N_J) \setminus N_L$ and the translation $\delta_L := m - |N_L| - 1$. Otherwise, the special color is the color from N_L assigned to the edge of index $|N_L|$ and $\delta_L := |N_L| - 1$.*

In Figure 3.3 a spider together with the coloring from Construction 3.3 is shown. The coloring obtained by this construction is well defined. Furthermore, it is a UCE-P_m coloring of a spider S for even $m \in \mathbb{N}$. Both claims are proven in the following lemma.

Lemma 3.7. *Let $m \in \mathbb{N}$ be even and S be a spider. The coloring obtained by Construction 3.3 is a UCE-P_m coloring of S.*

Proof. Due to the first step of the construction, a color is assigned to every edge in S. Hence, the coloring is well defined. Let P be a copy of P_m in S. Either P is completely contained in some leg L or not. Consider the first case. Since P has m edges, L has at least m edges and $L \in \mathcal{L}$. On the first $|N_L|$ edges every color in N_L is assigned to exactly one edge. All other edges of the leg L are colored according to a path coloring. The first special edge of this path coloring is the edge of index $\delta_L + 1$ by definition of the translation δ_L. The translation δ_L is chosen in such a way that there are at most $m-1$ edges between the first special edge and the last edge colored according to N_L. Hence, P contains the special color θ used by the path coloring or at least one color from N_L. If $\theta \notin N_L$, there is a uniquely colored edge. Otherwise $\delta_L = |N_L| - 1$ due to the construction. In this case the first special edge of the path coloring is the edge of index $|N_L|$. Since the special color of the path coloring θ is the same color assigned to this edge in the second step before, P contains an edge of unique color.

It remains to consider the case that P is part of two distinct legs L and J. At most one of the legs L and J has less than $\frac{m}{2}$ edges. W.l.o.g. assume $L \in \mathcal{L}$. If $J \notin \mathcal{L}$, let $N_J := \emptyset$. Hence, $N_L \neq N_J$ holds (regardless of whether $J \in \mathcal{L}$ or not). Let $x_L > 0$ denote the number of edges of P in L and x_J that in the leg J. Then $x_L + x_J = m$. Further let N'_L denote the set of colors assigned to these edges in L and N'_J the corresponding colors in J, without color 0.
If $x_L = x_J = \frac{m}{2}$, then $N'_L = N_L$ and $N'_J = N_J$ since the path-coloring of any leg L assigns additional colors to edges of index at least $m - |N_L| > \frac{m}{2}$ only. There is an edge of unique color in P, because $N_L \neq N_J$ and each color from N_L and N_J is assigned to exactly one edge in L respective J.
If $x_L > x_J$, then $N_L \subseteq N'_L$ and $N'_J \subseteq N_J$ (note the contrary relations here, there are more edges from L and less edges from J). Again, there is an edge of unique color in P, if $N'_L \neq N'_J$. If $|N_L| \geq |N_J|$, the set N'_L contains a color which is not in N'_J due to the relations above (N'_L contains at least all the elements from $N_L \neq N_J$). So consider the case $|N_L| < |N_J|$. In particular $|N_L| < a$ and $|N_L| < t$. The set N'_L differs by at most one color from N_L, namely the special color used by the path coloring. Due to the choice of the translation δ_L, this color is assigned to the edge of index $m - |N_L|$ in L. Hence, if $x_L < m - |N_L|$, then $|N'_L| = |N_L|$. But in this case $x_J > |N_L|$ (because $x_L + x_J = m$) and therefore $|N'_J| > |N_L| = |N'_L|$. If $x_L \geq m - |N_L|$, then $|N'_L| > |N_L| \geq |N'_J|$, since $x_J \leq |N_L|$. In both cases the two sets differ. The case $x_L < x_J$ is analogous. $\square$

Due to the lemma above, Construction 3.3 yields an upper bound on the conflict-free chtomatic index $f(T, P_m)$ for odd $m \in \mathbb{N}$. It is stated in the following theorem. Moreover, the coloring from Construction 3.3 is almost a minimum UCE-P_m coloring of a spider. This is proven by establishing a lower bound on f. It is obtained by considering the sets of colors on distinct legs.

Theorem 4. *Let $m \in \mathbb{N}$, with $m \geq 4$ even, and S be a spider. Further let $\mathbb{I}_S := 1$ indicate the existence of legs on less than $\frac{m}{2}$ edges in S and $\mathbb{I}_S := 0$ the opposite. Then*

$$\begin{aligned}\lceil \log_2(\Delta_{\frac{m}{2}}(S)) \rceil &\leq \min\{k \in \mathbb{N} \mid \sum_{i=2}^{m/2} \tbinom{k}{i} + 1 \geq \Delta_{\frac{m}{2}}(S)\} \\ &\leq f(S, P_m) \\ &\leq \min\{k \in \mathbb{N} \mid \sum_{i=0}^{m/2} \tbinom{k}{i} \geq \Delta_{\frac{m}{2}}(S) + \mathbb{I}_S\} + 1.\end{aligned}$$

In particular if $\log_2(\Delta_{\frac{m}{2}}(S) + \mathbb{I}_S) \leq \frac{m}{2}$, *then*

$$\lceil \log_2(\Delta_{\frac{m}{2}}(S)) \rceil \leq f(S, P_m) \leq \lceil \log_2(\Delta_{\frac{m}{2}}(S) + \mathbb{I}_S) \rceil + 1.$$

Proof. An edge-coloring of S obtained by Construction 3.3 is a UCE-P_m coloring due to Lemma 3.7. Let $k \in \mathbb{N}$ such that $\sum_{i=0}^{m/2} \binom{k}{i} \geq \Delta_{\frac{m}{2}}(S) + \mathbb{I}_S$ and $N := \{1, \ldots, k\}$. The set N has at least $\Delta_{\frac{m}{2}}(S) + \mathbb{I}_S$ distinct subsets of size at most $\frac{m}{2}$. Thus, the sets of colors N_L in Construction 3.3 can be chosen as subsets of N for all legs L on at least $\frac{m}{2}$ edges. With this choice the construction uses $k+1$ colors in total (counting color 0). Choosing a minimum $k \in \mathbb{N}$ which satisfies the given constraint yields the upper bound in the theorem. If $\log_2(\Delta_{\frac{m}{2}}(S) + \mathbb{I}_S) \leq \frac{m}{2}$, the number $\lceil \log_2(\Delta_{\frac{m}{2}}(S) + \mathbb{I}_S) \rceil$ satisfies the constraint on k and is minimum, too.

The lower bound is proven with an idea similar to the construction by considering distinct sets of colors on distinct legs. Let v be the head of the spider S. Consider a UCE-P_m coloring c of S and let k denote the number of colors used by c in total. A copy of P_m in S which has v as its middle vertex is called *centered path in S*. For each pair of distinct legs $L, J \in \mathcal{L}$ there is a centered path in S, which has $\frac{m}{2}$ edges in L and the other $\frac{m}{2}$ edges in J. For each leg $L \in \mathcal{L}$ let N_L be the set of colors which are assigned to the $\frac{m}{2}$ edges closest to the head of S. Since c is a UCE-P_m coloring, each centered path contains a uniquely colored edge. Hence, all the sets N_L are distinct. In particular, there are at least $\Delta_{\frac{m}{2}}(S)$ distinct subsets of the set of all k colors, each of size at most $\frac{m}{2}$. Furthermore, at most one of the sets N_L contains only one color, since in this case all $\frac{m}{2}$ edges of L closest to the head of S are of the same color. Hence, all other legs must have at least two colors assigned to these edges, because there is an edge of unique color among these edges in each leg (and $m \geq 4$). Altogether, $\Delta_{\frac{m}{2}}(S) \leq \sum_{i=2}^{m/2} \binom{k}{i} + 1 \leq 2^k$. This implies $f(S, P_m) \geq \min \left\{ k \in \mathbb{N} \mid \sum_{i=2}^{m/2} \binom{k}{i} + 1 \geq \Delta_{\frac{m}{2}}(S) \right\} \geq \lceil \log_2(\Delta_{\frac{m}{2}}(S)) \rceil$. □

Another lower bound is given in the following lemma. If a spider has sufficiently many legs on more than $\frac{m}{2}$ edges, it improves upon the lower bound of theorem above.

Lemma 3.8. *Let $m \in \mathbb{N}$ be even and S be a spider without legs on less than $\frac{m}{2}$ edges. Then*

$$f(S, P_m) \geq (\Delta(S))^{\frac{2}{m}}.$$

Proof. Let c be an edge-coloring of S, and let k denote the number of colors used by c. Consider S rooted at its head. For $i \in \mathbb{N}$ two legs L, J of S are called *equally colored up to edge level i*, if L and J have at least i edges each and for each index $j \leq i$ the same color is assigned to the edges of index j in L and J. Let l_i denote the maximum number of legs of S which are equally colored up to level i. The following induction on i proves $l_i \geq \frac{\Delta(S)}{k^i}$, if $i \leq \frac{m}{2}$. Due to the pigeonhole principle, there are at least $\frac{\Delta(S)}{k}$ edges of the same color incident to the head of S. Hence, $l_1 \geq \frac{\Delta(S)}{k}$. For $i < \frac{m}{2}$ there is a set $\mathcal{L}$ of l_i legs of S which are equally colored up to edge level i. Since each le has at least $\frac{m}{2}$ edges by assumption, at least $\frac{l_i}{k}$ of the legs in $\mathcal{L}$ have the same color assigned to their edge of index $i+1$. Due to the induction hypothesis, $l_{i+1} \geq \frac{l_i}{k} \geq \frac{\Delta(S)}{k^{i+1}}$.

If two legs L and J of S are equally colored up to edge level $\frac{m}{2}$, consider the copy of P_m in S which has $\frac{m}{2}$ edges in L and the other $\frac{m}{2}$ edges in J. This path has no edge of unique color, because it contains exactly the $\frac{m}{2}$ edges of L as well as of J closest to the head of S. Hence, c is bad with respect to P_m, if $l_{\frac{m}{2}} \geq 2$. If c is a minimum UCE-P_m coloring, $k = f(S, P_m)$. Thus, $\frac{\Delta(S)}{f(S,P_m)^{\frac{m}{2}}} \leq l_{\frac{m}{2}} \leq 1$ holds in this case. This yields $f(S, P_m) \geq (\Delta(S))^{\frac{2}{m}}$. □

This lower bound does not contradict the upper bound of $\lceil \log_2(\Delta_{\frac{m}{2}}(S) + \mathbb{I}_S)) \rceil + 1$ from Theorem 5, since this particular upper bound in the theorem is proven for the case $\Delta_{\frac{m}{2}}(S) \leq \frac{m}{2}$ only.

The maximum $\frac{m}{2}$-degree $\Delta_{\frac{m}{2}}(G)$ of a graph G indicates the existence of a spider as a subgraph of G which has $\Delta_{\frac{m}{2}}(G)$ legs on at least $\frac{m}{2}$ edges each. Hence, the following corollary to both lower bounds holds for arbitrary graphs.

Corollary 3.9. *Let $m \in \mathbb{N}$ even and G be a graph. Then*

$$f(G, P_m) \geq \max\{\log_2(\Delta_{\frac{m}{2}}(G)), (\Delta_{\frac{m}{2}}(G))^{\frac{2}{m}}\}.$$

Unfortunately, there is no nice way of extending the coloring of the spider to the whole graph known yet. So the question for which graphs this lower bound is tight remains open.

3.3.2. Arbitrary Trees and Paths with Even Number of Edges

In the previous subsection lower bounds on the conflict-free chromatic index were revealed, which depend on the maximum $\frac{m}{2}$-degree $\Delta_{\frac{m}{2}}(T)$ of a tree T. Recall that $\Delta_{\frac{m}{2}}(T)$ is the maximum number of edge-disjoint copies of $P_{\frac{m}{2}}$ in T with a common endpoint. In this section, a constructive upper bound on $f(T, P_m)$ is established, which depends on the maximum degree but not on the height of T. This is done by the following construction of a UCE-P_m coloring that uses at most $\frac{m}{2} + \Delta - 1$ colors. The construction is minimum on complete d-ary trees in case $m = 4$ for sufficiently large $d \in \mathbb{N}$.

Construction 3.4. *Let $m \in \mathbb{N}$, with $m \geq 4$ even, and let T be a rooted tree.*

- **s1)** *Color 1 is assigned to all edges in the first $\frac{m}{2} - 1$ edge levels of T.*
- **s2)** *The vertex levels in T are considered one after the other starting with the root and going down in the tree. In each vertex level all vertices are considered independently of each other.*
  - **s2.1)** *If a vertex v is considered, let $u_1, \ldots, u_d$ denote all children of v in T. Further let $\theta_1, \ldots, \theta_d$ be distinct colors, such that no color θ_i is already assigned to an edge in the first $\frac{m}{2}-1$ edge levels in the maximum downward subtree of T with root v. In addition, if there is an edge incident to v from above, the colors θ_i must be distinct from the color of this edge.*

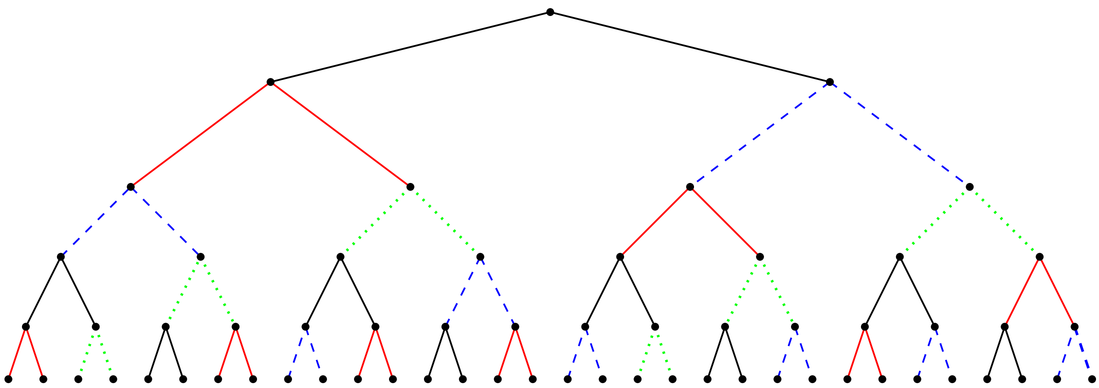


Figure 3.4.: A UCE-P_4 coloring in 4 colors of a complete binary tree of height 5.

s2.2) *For each child u_i, consider the maximum downward subtree of T with root u_i. Color θ_i is assigned to all edges in the edge level of index $\frac{m}{2} - 1$ in this tree.*

An example of this construction on a complete binary tree is shown in Figure 3.4. The next lemma states that the coloring obtained by this construction is well defined. Moreover, it is a good coloring with respect to P_m for even $m \in \mathbb{N}$.

Lemma 3.10. *Let $m \in \mathbb{N}$, with $m \geq 4$ even, and let T be a tree. Construction 3.4 yields a UCE-P_m coloring of T.*

Proof. First of all, it is shown that all edges in T get colored exactly once, i.e. that the coloring is well defined. Let $e \in E(T)$ and distinguish two cases. If e is in one of the first $\frac{m}{2} - 1$ edge levels of T, it is colored in the first step of the construction. Moreover, there is no vertex $u \in V(T)$, such that e is in the edge level of index $\frac{m}{2} - 1$ in a maximum downward subtree of T whose root is a child of u. Hence, no other color is assigned to e. If e is not in one of the first $\frac{m}{2} - 1$ edge levels of T, there is a unique vertex $u \in V(T)$, such that e is in the edge level of index $\frac{m}{2} - 1$ in a maximum downward subtree of T whose root is a child of u. In this case, a color is assigned to e, if and only if u is considered in the Step **s2.1)** of the construction. Since each vertex of T is considered there, a color is assigned to e exactly once.

It remains to prove that the coloring is good with respect to P_m. Consider a color θ that is assigned to an edge e in a level of index larger than $\frac{m}{2} - 1$. Then it was assigned in Step **s2.2)**. Due to the constraints on the colors in Step **s2.1)**, the following holds. Let u be the unique vertex in T, such that e is contained in an edge level of index $\frac{m}{2} - 1$ in a maximum downward subtree T' of T whose root is a child of u. All edges above e are colored in an execution of Step **s2.2)** before e. Hence, θ is not assigned to any other edge of smaller index in T' than e, not to the edge connecting the root of T' to u and not to an edge which is incident to u from above. Let y be the lower endpoint of e. Because all edges below e are colored in a later execution of Step **s2.2)** than e, color θ is not assigned to any edge in any of the first $\frac{m}{2}$ edge levels of the maximum downward subtree with root y.

Let $P = v_1 v_2 \dots v_{m+1}$ be a copy of P_m in T with $v_1, \dots, v_{m+1} \in V(T)$ and v be the (unique) topmost vertex of P. Two cases are distinguished.
If v is an endpoint of P, the path contains edges from m consecutive levels, exactly one edge per level. W.l.o.g. assume $v = v_1$ and consider the edge $e := \{v_{\frac{m}{2}+1}, v_{\frac{m}{2}+2}\}$. It is contained in the edge level of index $\frac{m}{2} - 1$ in the maximum downward subtree of T with root v_3. Since v_3 is a child of v_2, the color assigned to e is distinct from all colors assigned to the $\frac{m}{2}$ edges in P above e, due to the arguments above. All edges below e in P are contained in one of the first $\frac{m}{2}$ edge levels of the maximum downward subtree with root $v_{\frac{m}{2}+2}$. As argued above, all colors assigned to these edges are distinct from the color assigned to e.
If v is not an endpoint of P, consider the maximum downward subtree D of T with root v. Exactly one or two edges of P are contained in the edge level $\mathcal{E}$ of index $\frac{m}{2}$ in D. These edges are colored in Step **s2.2)** of the construction, when v (respective one of its children) is considered. If there is exactly one edge e of P in this level $\mathcal{E}$, the color of e differs from all other colors assigned to the edges above e in D. Furthermore, there are at most $\frac{m}{2} - 1$ edges of P below e in T. Hence, the color of e is not assigned to any of these edges, since they are contained in the maximum downward subtree with the lower endpoint of e as its root. Thus, the color of e is unique in P. If there are two edges e_1 and e_2 of P contained in the edge level $\mathcal{E}$ of D, the whole path P is contained in the first $\frac{m}{2}$ edge levels of D. In this case, both edges e_1 and e_2 are incident to an endpoint of P. Moreover, the edges e_1 and e_2 have distinct children of v as its ancestors. Hence, distinct colors are assigned to e_1 and e_2 by construction. Furthermore, these colors are distinct from all other colors assigned to edges in edge levels above $\mathcal{E}$ in D. Hence, both colors are unique in P. $\square$

So Construction 3.4 yields a good coloring with respect to P_m for even $m \geq 4$. With an appropriate choice of colors the coloring yields the following upper bound on the conflict-free chromatic index $f(T, P_m)$ for every tree T.

Theorem 5. *Let $m \in \mathbb{N}$ even and let T be a tree of maximum degree Δ. Then*

$$f(T, P_m) \leq \frac{m}{2} + \Delta - 1.$$

Proof. In case $m = 2$ Corollary 3.6 yields an even better upper bound. If $\Delta = 1$, there is exactly one edge in T and one color is sufficient. For the remaining proof let $m \geq 4$, $\Delta \geq 2$ and T rooted, such that its maximum down degree is $\Delta - 1$. It is shown that all colors in Construction 3.4 applied to T can be chosen from the set $\{1, \dots, \frac{m}{2} + \Delta - 1\}$. In the first step of the construction only color 1 is used. All other colors are chosen according to the constraints given in Step **s2.1)**. If vertex $v \in V(T)$ is considered, the colors differ from all colors assigned to the edges in the first $\frac{m}{2} - 1$ edge levels in the maximum downward subtree T' of T with root v. In addition, if there is an edge incident to v from above, the colors are distinct from the color of this edge.

The following induction on the index of the vertex level which contains v proves that each of the first $\frac{m}{2} - 1$ edge levels of T' is monochromatic. If v is the root of T, color 1 is assigned to all edges in these levels of the tree T'. In particular, all levels are monochromatic. If v is not the root, let x be its parent vertex in T. The tree T' is one of the maximum downward subtrees of T whose roots are children of x. Hence, the same color is assigned to all edges in the edge level of index $\frac{m}{2}-1$ in T' in Step **s2.2)** of the construction. In particular, this level is colored monochromatically. Each edge level of smaller index in T' is part of the edge level of index $\frac{m}{2}-1$ of a maximum downward subtree whose root is a child vertex of a vertex above v in T. By induction hypothesis, the same color is assigned to all edges in this level. Thus, each of the first $\frac{m}{2}-1$ edge levels of T' is monochromatic.

In particular, there are at most $\frac{m}{2}-1$ different colors assigned to these edges. Together with the color of the edge incident to v from above (if it exists) at most $\frac{m}{2}$ colors are not available in Step **s2.1)**. Since v has at most $\Delta-1$ children, at most that many further colors are necessary. Hence, the colors can be chosen from the set $\{1, \ldots \frac{m}{2}+\Delta-1\}$ in each execution of Step **s2.1)**. □

So Construction 3.4 establishes an upper bound on $f(T, P_m)$ for any tree T and even $m \in \mathbb{N}$. Due to the constructions for spiders and paths from the previous sections, it is known that the resulting coloring is not minimum for every tree . But it is minimum for complete d-ary trees in case $m = 4$.

Theorem 6. *Let $d \in \mathbb{N}$ with $d \geq 2$. Then there is $h_d \in \mathbb{N}$, such that for all complete d-ary trees T of height at least h_d*

$$f(T, P_4) = d + 2.$$

Proof. Let T be a complete d-ary tree. Theorem 5 yields $f(T, P_4) \leq d+2$ as an upper bound. Assume c is a UCE-P_4 coloring of T using only $d+1$ colors.
Recall the definition of a downward star in T which is a star whose center is located above all other vertices of the star. A contradiction is established in three steps by considering such monochromatic downward stars. The existence of a monochromatic downward star on at least two edges within the first three edge levels of T is proven first. Afterwards it is shown, that the existence of a monochromatic downward star on s edges in T forces the existence of an even larger star below. This leads to a contradiction, since a monochromatic downward star on $d+1$ edges is encountered. In each step of the proof it is assumed that the height of T is at least as large as necessary for the argument to hold.

For the first step it is proven that there are two adjacent edges of the same color within the first two edge levels of T. Let a and a' denote two edges in T incident to the root of T. If one of the lower endpoints of a or a' is incident to two edges of the same color, these are the desired edges. Otherwise, each of these endpoints is incident to an edge of every color used by c (because they have degree $d+1$). So either $c(a) = c(a')$ (and they are the desired edges), or there is an edge of color $c(a')$ incident to a from below and an edge of color $c(a)$ incident to a' from below. But these four edges form a bad copy of P_4. Hence, there are two adjacent edges e and e' of the same color λ within the first two edge levels of T. Let u denote the endpoint of e not incident to e' and v the endpoint of e' not incident to e.

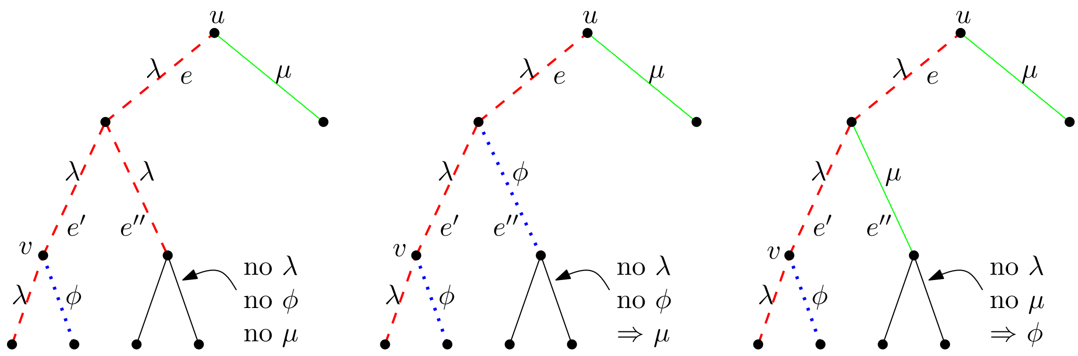


Figure 3.5.: The picture shows the first two edge levels of a binary tree, together with a UCE-P_4 coloring using three distinct colors λ (dashed, red), ϕ (dotted, blue) and μ (solid, green). The first case, yields a contradiction. In the other cases the same color is assigned to the two edges adjacent to e'' from below. Hence, they form a monochromatic downward star.

These two edges force the existence of a monochromatic downward star on two edges as follows. Either the two edges e and e' already form such a star, or not. In the latter case, one of the edges is in the first edge level (and incident to the root of T) and the other one in the second level. W.l.o.g. suppose e is the upper edge and hence u is the root of T. Each of the edges incident to u must not be of the same color as an edge incident to v from below, because they form a bad P_4 together with e and e' otherwise. There are d edges incident to v from below and $d-1$ edges, distinct from e, incident to u. Hence, there are $2 \cdot d + 1$ edges in total.
If $d \geq 3$, then $2d - 1 > d + 1$ holds. Hence, there is a monochromatic downward star on at least two edges incident to u or v due to pigeonhole principle.
If $d = 2$, this argument does not hold. But the following case distinction proves the existence of the desired star in this case. If color λ is assigned to the second edge incident to the root as well, this edge together with e forms the monochromatic downward star. Otherwise, one of the edges incident to v from below is of color λ (or there is the desired star, because only one color is available). Either the other edge incident to v from below is of color λ as well (and the desired star is there), or a different color ϕ is assigned. Then, a third color $\mu \neq \phi$ is assigned to the edge incident to u distinct from e. The situation is shown in Figure 3.5 for the following three cases Let e'' denote the (unique) edge which is incident to e and e' (in the middle). If color λ is assigned to e'', there is a bad copy of P_4 regardless of which color is assigned to an edge adjacent to e'' from below. If color ϕ is assigned to e'', both edges adjacent to e'' from below must be of color μ (and form the desired star), because there is a bad copy of P_4 otherwise. If color μ is assigned to e'', both edges adjacent to e'' from below must be of color ϕ (and form the desired star), because there is a bad copy of P_4 otherwise. Altogether, there is a monochromatic downward star on at least two edges in T.

Finally it is proven, that the existence of monochromatic downward star S forces the existence of an even larger star within the four levels below S in T. Let S

denote a monochromatic downward star with $2 \leq s \leq d$ edges. As above, distinct colors are assigned to edges which are incident from below to distinct leafs in S, since there is a bad copy of P_4 otherwise. Hence, there is a leaf in S which has at most $\left\lfloor \frac{d+1}{s} \right\rfloor$ distinct colors assigned to its edges incident from below, due to the pigeonhole principle. Let S' denote a monochromatic downward star with a leaf of S as root and maximum number s' of edges. Another application of pigeonhole principle yields

$$s' \geq \left\lceil \frac{d}{\left\lfloor \frac{d+1}{s} \right\rfloor} \right\rceil \geq \frac{d}{d+1} \cdot s \geq \frac{s}{s+1} \cdot s > \frac{s^2-1}{s+1} = s-1.$$

Note the strict inequality which proves $s' \geq s$. If $s' > s$, S' is the desired star. If $s' = s$, let θ denote the color assigned to the edges in S and θ' denote the color assigned to the edges in S'. The situation is shown in Figure 3.6. A copy of P_4 which contains two edges from S, one edge from S' and an edge incident to S' from below of color θ' is bad. Hence, color θ' is not assigned to any edge incident to S' from below. As above, distinct colors are assigned to edges incident to S' from below, if they are incident to distinct parent vertices. Let s'' denote the maximum number of edges of a monochromatic downward star with root in S'. The same calculation as above yields $s'' \geq s'$. Again, there is the desired star, if $s'' > s' = s$. The case $s'' = s' = s$ remains. Since θ' is not assigned to any edge incident to S' from below, there are at most d colors assigned to these edges. If there are less than d colors, there are at most $(d-1) \cdot s$ edges, since each monochromatic downward star has at most s edges. But S' has s edges and each of these edges is incident to d edges from below. Hence, there are $d \cdot s$ edges in total. In consequence, each of the remaining colors is assigned to at least one edge. A similar argument shows that each of the monochromatic downward stars incident to S' has exactly s edges. Two cases are distinguished to finish the proof.

The case $\theta \neq \theta'$ is called *final case for* S'. In this case, color θ is assigned to the edges of one of the monochromatic downward stars S'' incident to S' from below (because there is such a star for every remaining color). Then, neither color θ nor color θ' is assigned to edges incident to S'' from below. Because only $d-1$ colors are available, the same arguments as above prove the existence of a monochromatic downward star on more than s edges.

If $\theta = \theta'$, let S'' denote an arbitrary monochromatic downward star incident to S' from below. A color $\theta'' \neq \theta'$ is assigned to its edges. All arguments which are applied to S' above are applied to S'' (and are true). But the final case is reached for S'' always, because $\theta'' \neq \theta'$. Hence, the desired star exists two levels below S''.

Altogether, this yields a contradiction, since the size of the monochromatic downward stars is at most d. Hence, a UCE-P_4 coloring of T using only $d+1$ colors does not exist.

□

Unfortunately, the proof of the previous theorem could not be extended to larger, even $m \in \mathbb{N}$ yet.

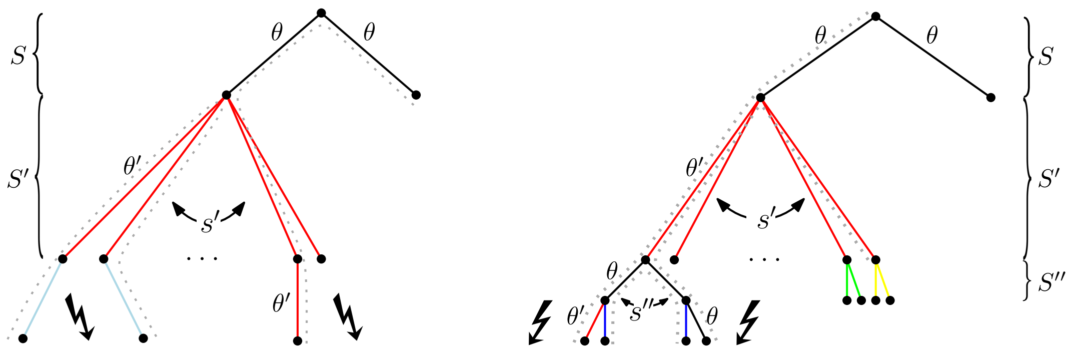


Figure 3.6.: A UCE-P_4 coloring of a complete d-ary tree in d+1 colors is studied. The monochromatic downward stars S and S' forbid certain configurations of colors on the edges below S''. The left hand side shows, that no incident edge has color θ', and edges with distinct parent vertices have distinct colors. So there are at most d colors available. The right hand side shows that color θ must not occur in the next level, as well. If $\theta \neq \theta'$, there are only $d-1$ colors available for the edges below S''. This forces a star on more than s'' edges below. The dotted paths indicate copies of P_4 without uniquely colored edge.

3.4. General Constructions

In this section two constructions are presented that are independent of the parity of the number of edges in the paths. The first one replaces certain structures in the tree by smaller pieces without any impact on the value of f. The second one recursively uses the path-coloring to obtain a good coloring of the whole tree.

3.4.1. Reduction to Smaller Trees

First of all a reduction is described that contracts certain small subtrees to single paths. It applies not only to trees but also to arbitrary graphs. It is inspired by Construction 3.3 for spiders where all legs on few edges (less edges than half of the edges of the path) were colored identically.

Definition 3.2. *Let $m \in \mathbb{N}$. An induced subtree T' of a graph G is called P_m-*reducible*, if the following conditions hold.*

- **c1)** *The tree T' has diameter at most $m-1$.*
- **c2)** *Exactly one vertex $v_0 \in V(T')$ has smaller degree in T' than in G.*
- **c3)** *The tree T' has at least 2 leaves distinct from v_0.*

The vertex v_0 defined in **c2)** *is considered to be the* root *of T'.*
A P_m-reducible subtree T' of G is called maximum*, if there is no other P_m-reducible subtree of G containing T'.*

The following lemma describes an equivalent characterization of P_m-reducible subtrees.

Lemma 3.11. *Let $m \in \mathbb{N}$ and T' be an induced subtree of a graph G. Then T' is P_m-reducible, if and only if the following conditions hold.*

c1)' *No copy of P_m is contained in T'.*

c2)' *Every path containing an edge from $E(T')$ has at least one endpoint in T'.*

c3)' *The tree T' is not a path having its root as an endpoint.*

c4)' *There is at least one edge in T'.*

Proof. The Conditions **c1)** and **c1)'** are equivalent by definition.

The same holds for the next conditions **c2)** and **c2)'** as is shown next. Suppose there are two vertices $u, v \in V(T')$ that have smaller degree in T' than in G. Let P be a path connecting these two vertices within T'. Since T' is a tree, such a path exists and contains at least one edge. Then P can be extended by two edges in $E(G) \setminus E(T')$, one incident to u and the other incident to v. Since T' is an induced tree, both endpoints of the extended path are not in $V(T')$.
The other way round, let P be a path which has both endpoints not in T' and contains an edge from $E(T')$. Then there are two distinct vertices in P, each incident to an edge in $E(G) \setminus E(T')$ as well as to an edge in $E(T')$. Hence, these vertices are of smaller degree in T' than in G.

It remains to show that condition **c3)** in Definition 3.2 is equivalent to the last two conditions **c3)'** and **c4)'** in this lemma. If the tree T' has at least two leaves distinct from a vertex v_0, the set of edges $E(T')$ is non-empty. Since the two endpoints of a paths are exactly its leaves, the tree T' is either not a path or v_0 is not an endpoint. If $E(T')$ is non-empty, there are at least two leaves. If the number of leaves is at least 3, two of them are distinct from a fixed vertex v_0. If there are exactly two, the tree T' is a path. Hence, its endpoints and therefore its leaves are distinct from its given root v_0. □

The next lemma states that P_m-reducible subtrees can be considered as paths, when UCE-P_m colorings are studied. The corresponding reduction is described in the following construction.

Construction 3.5. *For a graph G its P_m-*reduced graph $\textsc{red}_m(G)$ *is obtained by replacing each maximum P_m-reducible subtree T' in G by a path on $h(T')$ edges starting at v.*

Lemma 3.12. *The P_m-reduced graph of a graph G is unique (up to isomorphism) and does not contain any P_m-reducible subtree.*

Proof. Since paths are not P_m-reducible due to the leaf constraint **c3)**, there is no P_m-reducible subtree contained in $\text{RED}_m(G)$.

As long as there are no P_m-reducible subtrees with more than one vertex in common, it is easy to see that the P_m-reduced graph of G is unique (if such a subtree is substituted by a path, there is no P_m-reducible subtree containing an edge of this path).
Consider two distinct maximum P_m-reducible trees T_1 and T_2 in G that have more than one vertex in common. One of these vertices v is not the root of T_2. Due to the degree constraint **c2)**, all its neighbors in G are contained T_2 as well. Either v is the root of T_1 or not. In the latter case all its neighbors in G are contained T_1 as well. Since the same argument applies iteratively to all vertices in T_2, but T_1 is not completely contained in T_2, the root of T_1 is contained in T_2. Symmetrically, the root of T_2 is contained in T_1. Since only the roots of the subtrees have smaller degree within their trees than in G, the roots are distinct, because the trees have at least two vertices in common. Thus, there are at most two such P_m-reducible trees (and no other P_m-reducible tree) in G. Then $\text{RED}_m(G)$ is a (unique) path (on less than $2 \cdot (m-1)$ edges). □

Lemma 3.13. *Let $m \in \mathbb{N}$ and let G be a graph. Then*

$$f(G, P_m) = f(\text{RED}_m(G), P_m).$$

Proof. Let c be a UCE-P_m coloring of G and T' be a P_m-reducible subtree of G. The tree T' contains a path P on $h(T')$ edges connecting the root of T' to one of its leaves. The substitution of T' described in Construction 3.5 can be realized by deleting all vertices in T' that are not contained in P. Hence, the coloring c induces a UCE-P_m coloring of $\text{RED}_m(G) \subseteq G$ due to Lemma 2.5. In particular $f(\text{RED}_m(G), P_m) \leq f(G, P_m)$.
The other way round, let c be a UCE-P_m coloring of $\text{RED}_m(G) \subseteq G$. A UCE-$P_m$ coloring of G is constructed as follows. All edges in $E(G) \cap E(\text{RED}_m(G))$ receive the same color assigned by c. For the other edges consider each path P which substitutes a subtree T' of G. It connects the root of T' to one of its leaves in G. The color assigned by c to an edge in P is assigned to all edges in the same edge level of the rooted tree T'.
This construction yields a UCE-P_m coloring of G due to the properties of P_m-reducible trees as follows. It is easy to see that copies of P_m that do not contain an edge of any P_m-reducible subtree are good because c is a good coloring. Moreover, a P_m-reducible subtree does not contain any copy of P_m entirely due to Condition **c1)**. Hence, every copy of P_m containing edges of a P_m-reducible subtree T' corresponds to a copy of P_m in $\text{RED}_m(G)$, because it needs to contain the root of T' due to Condition **c2)**. Thus, it has an edge of unique color, since c is a UCE-P_m coloring of $\text{RED}_m(G)$. Altogether $f(G, P_m) \leq f(\text{RED}_m(G), P_m)$. □

3.4.2. Leaf Construction

Another construction of a UCE-P_m coloring is described next which works for all paths and all trees. It uses the known path coloring from Definition 3.1 on iteratively chosen paths in the tree. Each of these paths ends in a leaf. Hence, the total number of colors is equal to the number of leaves in the tree.

Construction 3.6. *Let $m \in \mathbb{N}$ and T be a tree. Further let τ be a labeling of the leafs of T, such that $\tau(v) \neq \tau(u) \neq 0$ for distinct leaves $u, v \in V(T)$.*

s1) *Let u be a leaf of T. Let P be a path in T connecting u to another vertex $v \in V(T)$, such that a vertex $w \in V(P)$ has degree 2 in T, if and only if $w \notin \{u, v\}$.*

s2) *Apply the path coloring of Definition 3.1 to P with special color $\tau(u)$, such that v is incident to an edge of color $\tau(u)$.*

s3) *Let e denote the unique edge in P which is incident to v and T' denote the connected component of $T - e$ containing v.*

s4) *If $E(T') \neq \emptyset$, apply the construction recursively to T' with the restriction of the labeling τ to the remaining leaves in T'.*

The following lemma states that a coloring obtained by construction is well-defined and good with respect to P_m for all $m \in \mathbb{N}$.

Lemma 3.14. *Let $m \in \mathbb{N}$ and T be a tree. Then Construction 3.6 yields a UCE-P_m coloring of T.*

Proof. The lemma is proven by induction on the number t of leaves of T. Consider a tree T with 2 leaves. This is the basic step, because in case $t \leq 1$ there is no edge in T. Since every vertex which is not a leaf in T has degree 2, the path chosen in Step **s1)** is the whole tree. Thus, the path-coloring is applied to the whole tree T in the next step. This coloring is a UCE-P_m coloring due to Lemma 3.1. Since there is only one edge e incident to v in T, the vertex v is of degree 0 in $T - e$. Thus, there is no edge in the connected component T' in Step **s3)**. Hence, the construction finishes and yields a good coloring for trees on two leaves.

Consider a tree T with more than 2 leaves. Let c be the coloring of T obtained by Construction 3.6 for an arbitrary labeling τ of the leafs of T with distinct values. Since there are vertices of degree larger than 2 in T, the path P chosen in Step **s1)** connects a leaf u to a vertex v of degree at least 3 in T. Let e denote the unique edge in P incident to v. In particular, v is of degree at least 2 in $T - e$. Because all other inner vertices of P are of degree 2 in T, the connected component T' chosen in Step **s3)** has exactly one leaf less than T. It is a tree, since it is connected and T is a tree. It has at least two edges, because v has degree at least 2 in T'. Moreover, the labeling τ assigns a label to every leaf in T'. Hence, it can be restricted to a labeling of all remaining leaves in T',

Each edge in T is either contained in T' or in P. Due to the induction hypothesis, the construction applied to T' terminates and yields a UCE-P_m coloring of T'. All edges in P are colored with the path-coloring. Hence, exactly one color is assigned to every edge in T. Moreover, each copy of P_m in T which is entirely contained in P or in T' is good. Every copy P' of P_m which does contain edges from P as well as from T' contains the edge e. The construction assigns the color $\tau(u)$ to this edge. This color is assigned to no edge in T', because $\tau(u) \neq 0$ and the other colors used by the labeling τ are distinct from $\tau(u)$. Furthermore, there are $m-1$ edges between e and the next special edge in P by definition of the path coloring. Hence, the edge e is of unique color in P'. Altogether, the construction finishes and yields a good coloring of T with respect to P_m. □

Since the construction uses color 0 (in the path coloring) and all but one color from the labeling τ (in the last recursion step only the color associated with one of the two leafs is used) the following corollary holds.

Corollary 3.15. *Let $m, t \in \mathbb{N}$ and T be tree on t leaves. Then*

$$f(T, P_m) \leq t.$$

So this construction yields an upper bound on $f(T, P_m)$ in terms of the number of leaves of T. Compared to the constructions given in previous sections, this is worse on the considered classes of trees (spiders and complete d-ary trees). For example Theorem 3 and Theorem 5 yield upper bounds on $f(T, P_m)$ that do not depend on the size of the tree T. In particular, there are much better construction for spiders, since this construction yields a total multicoloring of all edges incident to the head. But its advantage is its independence from the number of edges in long subpaths. For trees with long paths and few branching edges this construction yields better results. Furthermore, it can be improved by using the ideas from above as follows. First of all, the reduction described in Construction 3.5 decreases the number of leaves. Furthermore, instead of paths, spiders which are joined to the tree at one vertex only can be colored using Construction 3.3 or 3.1 (depending on the parity of m). One needs to ensure here, that the edge connecting the spider and the rest of the tree has a special color distinct from other special colors used in the spider and the tree. Another improvement is due to the following observation. In the given description of the construction, distinct special colors are used for all leaves. This may not be necessary for leaves that are far apart, if there are special edges of other color in between.

3.5. Conclusion

Several different lower and upper bounds on the conflict-free chromatic index $f(T, P_m)$ were established in the previous sections. The aim of this part is to summarize all those results in a compact form. Therefore, two certain classes of trees are considered and the maximum and minimum value of f for all those trees is studied. On the one hand all trees with fixed maximum degree are considered and on the other hand all trees on a fixed number of vertices. Since the behavior of f depends essentially on the parity of m, these two cases are considered independently. In the following let $n, m, \Delta \in \mathbb{N}$ denote fixed positive integers.

Case m is odd: There are upper bounds not depending on the parameters of the trees in case of paths on an odd number of edges. The following upper bound holds due to Theorem 3 for all trees. In particular it holds for all trees with fixed maximum degree or given number of vertices. For very large Δ or n the same theorem states tightness of the upper bound. In case $\Delta \leq 2$ it is not tight due to the path coloring from Section 3.1. Similarly one color is sufficient, if $n \leq m$.

$$\begin{aligned} ?? \ &\leq \max_{T:\Delta(T)=\Delta} f(T, P_m) \ \leq \lceil \tfrac{m}{2} \rceil. \\ ?? \ &\leq \max_{T:\ |V(T)|=n} f(T, P_m) \ \leq \lceil \tfrac{m}{2} \rceil. \end{aligned}$$

As above, minimum values of f are known exactly and are attained by a star on Δ respectively $n-1$ edges.

$$\begin{aligned} \min_{T:\Delta(T)=\Delta} f(T, P_m) \ &= 1. \\ \min_{T:\ |V(T)|=n} f(T, P_m) \ &= \begin{cases} 0 & \text{, if } n = 1, \\ 1 & \text{, otherwise.} \end{cases} \end{aligned}$$

Case m is even: For all trees with maximum degree Δ the following holds.

$$\max\{\log_2(\Delta), \Delta^{\frac{2}{m}}\} \leq \max_{T:\Delta(T)=\Delta} f(T, P_m) \leq \frac{m}{2} + \Delta - 1.$$

The logarithmic lower bound follows from Theorem 4 and the other one from Lemma 3.8. The upper bound is stated in Theorem 5. In case $\Delta = 2$, a better upper bound of 2 is known due to the path coloring and the case $\Delta = 1$ is trivial. Furthermore, the upper bound is attained in case $m = 4$ for complete $(\Delta - 1)$-ary trees, see Theorem 6.
The following equations state the corresponding minimum values of f.

$$\min_{T:\Delta(T)=\Delta} f(T, P_m) = \begin{cases} \Delta & \text{, if } m = 2, \\ 1 & \text{, if } m \neq 2. \end{cases}$$

The first equality corresponds to Corollary 3.6. The second equality is realized by a star on Δ edges. One color is sufficient there, because a star does not contain a path on more than two edges.

Considering all trees on a fixed number of vertices, the following holds.

$$\log_2 \left(\left\lfloor \frac{2 \cdot (n-1)}{m} \right\rfloor \right) \leq \max_{T:\ |V(T)|=n} f(T, P_m) \leq \begin{cases} 1 & \text{, if } n < m+1, \\ n - m + 1 & \text{, otherwise.} \end{cases}$$

Again, the lower bound is deduced from Corollary 3.9. The bound given there is maximized by a spider on as many legs of length $\frac{m}{2}$ as possible. Since one vertex is needed for the head of the spider, there are at most $\left\lfloor \frac{2\cdot(n-1)}{m} \right\rfloor$ such legs. Right now it is not known how a worst case tree for a given number of vertices looks like.

The upper bound given above is deduced from Corollary 3.15, since the maximum number of leaves of a tree on n vertices and diameter at least m is $n - m + 1$ (and in case $n < m + 1$ one color is sufficient anyway).
Similar to the case of fixed maximum degree, minimum values are known exactly.

$$\min_{T:\ |V(T)|=n} f(T, P_m) = \begin{cases} 0 & \text{, if } n = 1, \\ 2 & \text{, if } m = 2 \text{ and } n \geq 3, \\ 1 & \text{, otherwise.} \end{cases}$$

If there is no edge, no color is needed. In case $m = 2$ the equalities are attained by a path on n vertices (and the values follow from Lemma 3.2). If $m > 2$, a star on $n - 1$ edges does not contain a copy of P_m and hence one color is sufficient.

4. Cliques and Subcliques

One of the most studied kind of graphs in Ramsey theory are complete graphs with complete subgraphs. Especially the original theorem by Ramsey is stated for complete graphs [26] (precisely not for graphs, but for r-uniform set systems). So this chapter deals with complete graphs K_n on $n \in \mathbb{N}$ vertices where all complete subgraphs K_p of fixed order $p \in \mathbb{N}$ should have an edge of unique color. First of all, the general results from Chapter 2 may be used to get some insight on the behavior of the conflict-free chromatic index $f(n,p) = f(K_n, K_p)$. Together with several easy but specific results this is presented in the first Section 4.1. Since all general upper bounds on the function f in Chapter 2 were obtained by probabilistic arguments, there are no explicit constructions (though the probabilistic ones might be derandomized). Section 4.2 provides several constructions whose number of colors used is at most linear in n and logarithmic in the case $p = 3$. The best known probabilistic upper bound from Section 2.3 is studied for this specific setting of complete graphs in Section 4.3. The same section contains an explicit and better construction for the case $p = 4$. Finally, a lower bound is established in Section 4.4 by considering large monochromatic stars. Some of the results are summarized in the following table. Additionally, a Brute-Force algorithm, that tested all possible colorings, provided some more values. The minimum colorings calculated by the algorithm are shown in Figure 4.2. If no exact value is known yet, all possible values are shown in brackets. Therefore, the upper bounds from Section 4.2 are used.

$f(n,p)$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$	$p=7$	$p=8$
$n=2$	1	-	-	-	-	-	-
$n=3$	1	2	-	-	-	-	-
$n=4$	1	2	2	-	-	-	-
$n=5$	1	2	3	2	-	-	-
$n=6$	1	3	3	3	2	-	-
$n=7$	1	3	$\{3,4\}$	$\{3,4\}$	3	2	-
$n=8$	1	3	$\{3,4,5\}$	$\{3,4,5\}$	$\{3,4\}$	3	2

Table 4.1.: Values of $f(n,p)$

4.1. Basic Results

First of all, the following corollary summarizes some results that are obtained by using the results from Chapter 2. Afterwards more specific results are shown.

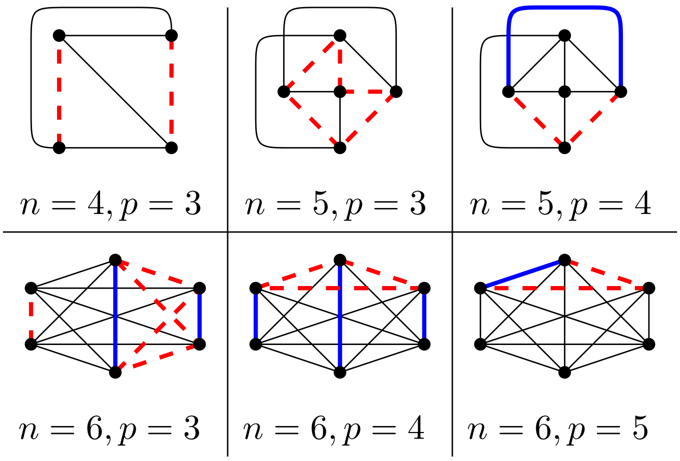


Figure 4.2.: Examples of minimum UCE-K_p colorings

Corollary 4.1. *Let $n, p \in \mathbb{N}$, with $p \geq 2$. Then the following statements hold.*

C1) $f(1, p) = 0$.

C2) $f(n, 2) = 1$.

C3) If $n \geq 3$, *then* $f(n, n) = 2$.

C4) $f(n, p) = 1 \Leftrightarrow p = 2$ *or* $p > n$.

C5) $f(n + 1, p) \geq f(n, p)$.

C6) If $p \geq 3$, *then* $f(n, p) \geq R_p^{inv}(n)$.

C7) $\forall p \in \mathbb{N}, p \geq 3, \ \forall k \in \mathbb{N} \ \exists n \in \mathbb{N} : f(n, p) > k$.

This means for fixed p the function $f(n, p)$ exceeds each given number of colors for large enough n.

Proof. The first three equations C1), C2) and C3) follow from Lemma 2.2 and Lemma 2.3 implies the equivalence C4).

The relation C5), $f(n + 1, p) \geq f(n, p)$, is an application of Lemma 2.5. Deleting a vertex in K_{n+1} yields a K_n. Hence, $f(n + 1, p) \geq f(n, p)$.

The lower bound C6) follows directly from Lemma 2.15. Ramsey's theorem states that for every $p, k \in \mathbb{N}$ there is an $n \in \mathbb{N}$, such that there is a monochromatic copy of K_p in every k-coloring of K_n [26]. Furthermore, if k colors are sufficient to avoid monochromatic copies of K_p in K_n, then it is easy to see that one can achieve the same with more than k colors. Just assign new colors to some edges. In other words it is not possible to avoid monochromatic copies by less than k colors, if it is not possible with k colors. Using these facts one obtains statement C7) in terms of $R_p^{inv}(n)$ and by C6) the same for $f(n, p)$. □

The first fact handles the pathological case of a graph without any edge. The next three facts characterize the conflict-free chromatic index f for the smallest and the largest possible value of p. The relation C5) shows that $f_p(n) := f(n,p)$ is not decreasing (for some fixed value p). Furthermore it won't converge, since it exceeds any given integer for large enough n due to C7).

Few possibilities for two colors: All UCE-H colorings of G which use only one color, or no color at all are easily characterized for general graphs in the beginning of this thesis. There are only very few possibilities where such colorings exist. Either there is no edge in G or H has at most one edge. In the setting of this chapter, the case of conflict-free chromatic index of 2 can be characterized as well. The few cases of $n, p \in \mathbb{N}$ with minimum UCE-K_p colorings of K_n using two colors only are stated in the following lemmas. In particular, if $n \geq 6$, the only possibility is $p = n$.

Lemma 4.2. *Let $n, p \in \mathbb{N}$ with $f(n,p) = 2$. A UCE-K_p coloring of K_n in two colors with one colors class consisting of a single edge exists, if and only if $n = p$.*

Proof. If $n = p$, color a single edge different from all other edges. Obviously this coloring is a UCE-K_p coloring in two colors.
Let $n, p \in \mathbb{N}$ with $f(n,p) = 2$. Consider a minimum UCE-K_p coloring of K_n with an edge e whose color is assigned to no other edge. If $n < p$ or $p = 2$, then $f(n,p) = 1$ holds due to Corollary 4.1. Hence, $p, n \geq 3$. Each K_p contains at least three edges, then. Hence, each K_p must contain e, because all other edges have the same color. This is only possible, if $n = p$. □

Lemma 4.3. *Let $n, p \in \mathbb{N}$ with $p < n$ and $f(n,p) = 2$. Then $n \leq 5$.*

Proof. Let c be UCE-K_p coloring of K_n in two colors. Due to Lemma 4.2, each color class contains at least two edges. Furthermore $p \geq 3$, because $f(n,2) = 1$, and hence $n \geq 4$. Consider a copy A of K_p with uniquely colored edge e_0. Let $e_1 \in E(K_n)$ be an edge with $c(e_1) = c(e_0)$. In particular, the edge e_1 is not in A.

Consider the case $p \geq 4$ first. There is another copy $B \subseteq K_n$ of K_p which contains e_0 and e_1. Since c is a UCE-K_p coloring with two colors, B contains exactly one edge $\tilde{e}_0$ with $c(\tilde{e}_0) \neq c(e_0)$. All other edges have the same color as e_0. Replacing one vertex from B by any vertex from A (except the endpoints of e_0) yields another copy of K_p. But this one contains at least two edges from each color. There are two edges from A (incident to the endpoints of e_0) with color different to the color of e_0. Furthermore there is the edge e_0 together with at least one edge from B of the same color. This is a contradiction because c is a UCE-K_p coloring.

In case $p = 3$, the conflict-free chromatic index equals the inverse multicolor Ramsey number, due to Lemma 2.15. It is a very common example in classical Ramsey theory to show $f(6,3) = R_3^{inv}(6) \geq 3$. Consider an arbitrary vertex v of K_6 and the coloring c. By pigeonhole principle there are three edges of the same color incident to v. Either one of the edges connecting their endpoints has the same color or all of these have the other color. In both cases there is a monochromatic copy of K_3. Hence, $f(n,3) \geq 3$ for all $n \geq 6$. □

Further values of f: The following lemma states another case, where the exact value of $f(n,p)$ is known.

Lemma 4.4. *Let $n \in \mathbb{N}$ with $n \geq 5$. Then $f(n, n-1) = 3$ holds.*

Proof. Since $n \geq 5$, there are two non adjacent edges e_1 and e_2. Coloring e_1 and e_2 in distinct colors and all other edges in another color yields a coloring in three colors. It is a UCE-K_{n-1} coloring, since each copy of K_{n-1} in K_n contains at least one of the two uniquely colored edges. Hence, $f(n, n-1) \leq 3$.

Lemma 4.3 states, that $f(n, n-1) \geq 3$, if $n \geq 6$. Thus it remains to show that $f(5,4) \geq 3$. Let c be a minimum UCE-K_4 coloring of K_5. Consider a copy K of K_4 in K_5. It contains an edge e_0 of unique color. Let $x := c(e_0)$. Either there are two other colors on the edges of K (and $f(5,4) \geq 3$) or all other edges in this K_4 are of color $y \neq x$. Consider the latter case. There is another copy of K_4 in K_5 which does not contain e_0 but three other edges of K. Furthermore, it has an edge e_1 of unique color $z \neq y$, because there are three edges of color y. But e_0 and e_1 are contained in a common K_4. It is easy to see that $x = z$ is not possible, because there are at least two edges of color y as well. Hence, $f(5,4) \geq 3$. □

4.2. Constructive Bounds

In this section some explicit constructions of UCE-K_p colorings of K_n, for arbitrary $p, n \in \mathbb{N}$ with $p \geq 2$, are presented. They use a number of colors linear in n, except in case $p = 3$, where a logarithmic upper bound is obtained. For small n, the linear upper bound is tight. Furthermore, colorings are considered which are good not only with respect to K_p for fixed p but for several values of p at the same time.

At Most One New Color for a New Vertex

A common question to ask is the following. Consider fixed $p, n \in \mathbb{N}$ and a UCE-K_p coloring c of K_n. How many new colors are necessary and sufficient to extend c to a UCE-K_p coloring of K_{n+1}? If the answer to this question is known, it is possible to deduce either lower bounds or constructive upper bounds on the conflict-free chromatic index $f(n,p)$ inductively. One can show that at most one new color is necessary for a good coloring of K_{n+1} using the following construction.

Construction 4.1. *Let $n, p \in \mathbb{N}$ with $p \geq 2$, let $V(K_n) = \{v_1, \ldots, v_n\}$, let c be a UCE-K_p coloring of K_n and let θ denote a color. Choose a vertex $v_i \in V(K_n)$ and define an edge-coloring $c_{v_i,\theta}$ of K_{n+1} with $V(K_{n+1}) = \{v_1, \ldots, v_n, v_{n+1}\}$ as follows. For $s, t \in \mathbb{N}$ with $s, t \leq n+1$ define*

$$c_{v_i,\theta}(\{v_s, v_t\}) := \begin{cases} c(\{v_s, v_t\}) & , \textit{if } s,t \leq n, \\ c(\{v_i, v_t\}) & , \textit{if } s = n+1 \textit{ and } t \neq i, \\ \theta & , \textit{if } s = n+1 \textit{ and } t = i. \end{cases}$$

Lemma 4.5. *Let $n, p \in \mathbb{N}$ and c be a UCE-K_p coloring of K_n. If $\theta \notin c(E(K_n))$, then the coloring $c_{v_i,\theta}$ of K_{n+1} obtained from c by Construction 4.1 is a UCE-K_p coloring for all $v_i \in V(K_n)$. In particular,*

$$f(n+1, p) \leq f(n, p) + 1.$$

Proof. Let $V(K_n) = \{v_1, \dots, v_n\}$ and $V(K_{n+1}) = \{v_1, \dots, v_n, v_{n+1}\}$. Further let c be a UCE-K_p coloring of K_n and $c' := c_{v_i,\theta}$ an edge-coloring of K_{n+1} obtained by Construction 4.1 from c. Then c' equals the coloring c on all edges connecting vertices from $\{v_1, \dots, v_n\}$. All edges connecting v_{n+1} to some vertex $v_j \neq v_i$ are colored like the corresponding edge connecting v_i and v_j. Hence, the edges incident to v_{n+1} look like a copy of the edges incident to v_i. Thus, c' equals the coloring c on all edges connecting vertices from $\{v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_{n+1}\}$. Finally, a new color $\theta \notin c(E(K_n))$ is assigned to the edge connecting v_{n+1} and v_i. Altogether, the coloring $c_{v_i,\theta}$ is a UCE-K_p coloring of K_{n+1}, which uses exactly one more color than c. Hence, $f(n+1, p) \leq f(n, p) + 1$, by choosing a minimum good coloring c of K_n. □

Note that $c_{v_i,\theta}$ does not need to be a minimum UCE-K_p coloring of K_{n+1}. Construction 4.1 may be iterated to receive a UCE-K_p coloring of an arbitrary large K_n, e.g. starting with the known good coloring on K_p. The number of colors that is used by the resulting coloring is stated in the following corollary.

Corollary 4.6. *Let $n, p \in \mathbb{N}$ with $2 \leq p \leq n$. Then $f(n, p) \leq n - p + 2$ holds.*

Proof. In case $p = 2$ the bound is trivial, since $f(n, 2) = 1$ due to Corollary 4.1. For fixed $p \geq 3$ the upper bound is proven by induction on n. Consider the basic case $n = p$. Due to Corollary 4.1, $f(p, p) = 2 = p - p + 2$ is known.
Suppose $f(n_0, p) \leq n_0 - p + 2$ holds for a fixed $n_0 \geq p$. Due to Lemma 4.5

$$f(n_0 + 1, p) \leq f(n_0, p) + 1 \leq n_0 - p + 2 + 1 = (n_0 + 1) - p + 2.$$

Hence, $f(n, p) \leq n - p + 2$ for all $n, p \in \mathbb{N}$ with $2 \leq p \leq n$. □

For a special choice of the copied vertex v_i in Construction 4.1, the following coloring is obtained. It is defined explicitly here. The equivalence to Construction 4.1 is shown in the subsequent lemma. The coloring is illustrated in Figure 4.3.

Construction 4.2. *Let $n, p \in \mathbb{N}$ with $3 \leq p \leq n$, $V(K_n) = \{v_1, \dots, v_n\}$ and $s, t \in \mathbb{N}$ with $1 \leq s < t \leq n$. Define an edge-coloring of K_n by*

$$c(\{v_s, v_t\}) := \begin{cases} 0 & , \textit{if } 1 \leq s \leq p-2, \\ s - p + 2 & , \textit{if } p-1 \leq s < n. \end{cases}$$

Lemma 4.7. *Let $n, p \in \mathbb{N}$ with $3 \leq p \leq n$. The coloring obtained by Construction 4.2 is a UCE-K_p coloring of K_n and uses $n - p + 2$ colors.*

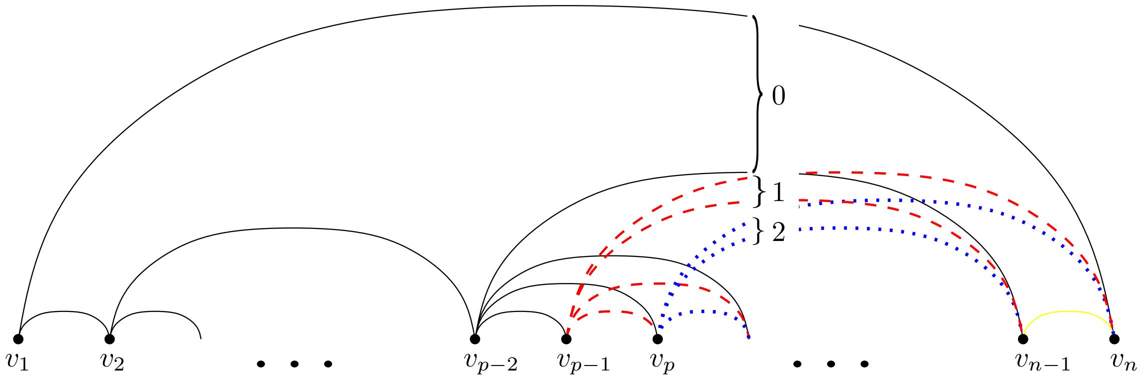


Figure 4.3.: A UCE-K_p coloring of K_n with $n-p+2$ colors.

Proof. Let $V(K_n) = \{v_1, \dots, v_n\}$ and let c denote a coloring obtained by Construction 4.2. The coloring c has the following structure. For each vertex $v_s \in V(K_n)$ all edges to vertices $v_t \in V(K_n)$, with $t > s$, are of the same color. If $s \leq p-2$, this is color 0, otherwise it is color $s-p+2$. Hence, this color is distinct for all vertices v_s with index greater than $p-2$. Thus the number of colors equals $(n-1)-(p-3) = n-p+2$ (note that there is no vertex with index lager than n and hence no edge from v_n to such a vertex).

In the remaining part it is shown that c is a UCE-K_P coloring of K_n. If $n = p$, the coloring has exactly one edge e of color 1 and all other edges are of color 0. Hence, c is good. If $n > p$, the coloring c is obtained from the coloring of the case before by iteratively applying Construction 4.1 as follows. In the first step choose one of the endpoints of e. Afterwards, choose the vertex which was added to the graph just in the step before. Let v_i be the vertex chosen in step i. A new color is introduced in each step. It is assigned to all edges connecting v_i to all vertices, that are introduced in the subsequent steps. Thus, both constructions are equal and c is a good coloring due to Lemma 4.5. □

Using Partitions

Another approach to construct a UCE-K_p coloring divides the vertices of K_n into two parts. The parts are chosen in such a way that there is one part that contains at least two vertices (and hence one edge) from each choice of p vertices of K_n. A coloring of this part which is good with respect to K_t for all $t \leq p$ can be extended to a UCE-K_p coloring for the whole K_n with only one additional color.

Lemma 4.8. *Let $n, p, k \in \mathbb{N}$. If there is an edge-coloring of K_{n-p+2} which is a UCE-K_t coloring for all $t \in \mathbb{N}$ with $2 \leq t \leq p$ using k colors, then there is a UCE-K_p coloring of K_n using $k+1$ colors.*

Proof. Let c denote an edge-coloring of K_{n-p+2} which is a UCE-K_t coloring for all $t \in \mathbb{N}$ with $2 \leq t \leq p$. Consider an arbitrary copy $K \subseteq K_n$ of K_{n-p+2} and color the edges of K according to c. Color all other edges in $E(K_n) \setminus E(K)$ with a color not used by c. Let c' denote the resulting coloring of K_n. There are exactly $p-2$ vertices of K_n not contained in K. Hence, at least two vertices from each copy of K_p in K_n are contained in K. The coloring c provides a uniquely colored edge e for the part of K_p contained in K. Since all edges in $E(K_n) \setminus E(K)$ are colored differently from the edges in K, the edge e has unique color in the whole copy of K_p. Thus c' is a UCE-K_p coloring of K_n. $\square$

Of course, the requirement to be a UCE-K_t coloring for all $t \leq p$ is much more restrictive than to be a UCE-K_p coloring for p only. As it turns out, Construction 4.1 can be used to build colorings satisfying this condition.

Lemma 4.9. *Let $n, \lambda \in \mathbb{N}$ with $2 \leq \lambda \leq n$. There is an edge-coloring of K_n using at most $n-\lambda+2$ colors which is a UCE-K_t coloring of K_n for all $t \in \{\lambda, \ldots, n\}$.*

Proof. The lemma is proven by induction on n. In the basic case $n = 2$, there is only one edge to be colored. In particular, every coloring of this K_2 is a good coloring with respect to K_t for all $t \in \{\lambda, \ldots, n\}$, because $2 \leq \lambda \leq n = 2$.
Consider the case $n \geq 3$. Two cases are distinguished. If $\lambda = n$, it is sufficient to color one edge of K_n differently from all other edges. This coloring uses $2 = n-\lambda+2$ colors and is a UCE-K_t coloring for all $t \in \{\lambda, \ldots, n\} = \{n\}$.
If $\lambda < n$, there is an edge-coloring c of K_{n-1} using at most $(n-1)-\lambda+2$ colors which is a UCE-K_t coloring of K_{n-1} for all $t \in \{\lambda, \ldots, n-1\}$ by the induction hypothesis. Let $v \in V(K_{n-1})$ and let θ denote a color not used by c. Applying Construction 4.1 to c using the vertex v and the new color θ yields an edge-coloring c' of K_n. According to Lemma 4.5, the coloring c' is good with respect to all K_t with $t \in \{\lambda, \ldots, n-1\}$. Furthermore, color θ is assigned to exactly one edge in K_n. Hence, c' is a UCE-K_n coloring as well.
Since c uses at most $(n-1)-\lambda+2$ colors, the coloring c' uses at most $n-\lambda+2$ colors. $\square$

Using these results it is possible to construct a UCE-K_p coloring of K_n. As is turns out this coloring uses $n-p+2$ colors and hence yields the same upper bound on the conflict-free chromatic index f as the constructions before. With $\lambda := 3$, Lemma 4.9 provides an edge-coloring c of K_{n-p+2} using $n-p+1$ colors, which is a UCE-K_t coloring for all $t \in \{3, \ldots, n\}$. Since every edge-coloring is a UCE-K_2 coloring, c can be extended to a UCE-K_p coloring of K_n that uses $n-p+2$ colors due to Lemma 4.8.

Exact Value for Small n

So far, a constructive, linear upper bound on the conflict-free chromatic index f was determined. As it turns out this bound is tight, if n is sufficiently small. Due to Lemma 4.5 it is known, that $f(n, p)$ either increases by 1 or remains at the same value for $f(n+1, p)$. The next lemma characterizes the case that a new color is necessary. Afterwards, this characterization is used to show, that the upper linear bound $f(n, p) \leq n-p+2$ from Corollary 4.6 is tight, if $p \leq n \leq \frac{5}{4}p - 2$.

Lemma 4.10. *Let $n, p \in \mathbb{N}$ with $2 \leq p \leq n$. Then $f(n+1,p) = f(n,p)+1$, if and only if there is a minimum UCE-K_p coloring of K_{n+1} that has a star color class.*

Proof. Consider a minimum UCE-K_p coloring c of K_{n+1} in $k := f(n+1,p)$ colors with a star color class A with center $v \in V(K_{n+1})$. Removing v from K_{n+1} induces a UCE-K_p coloring c' of K_n in $k-1$ colors. Since c is minimum and the minimum number of sufficient colors may drop by at most 1 due Lemma 4.5, the induced coloring c' is minimum as well. Hence, $f(n+1,p) = f(n,p)+1$.

If $f(n+1,p) = f(n,p)+1$, let c denote a minimum UCE-K_p coloring of K_n. Construction 4.1 applied to c with a new color $\theta \notin c(E(K_n))$ provides a UCE-K_p coloring c' of K_{n+1}. The coloring c' uses exactly one color more than c. Hence, it is minimum. Furthermore, color θ is assigned exactly to one edge. This proves the lemma, since a single edge of own color is always a star color class. □

The lemma may be restated as follows. Each color class of a minimum UCE-K_p coloring of K_{n+1} needs to contain two non adjacent edges, if $f(n+1,p) = f(n,p)$. In particular, there is no color class consisting of a single edge only. But if the number of colors is small, two edges of each color may be put into a common copy of K_p. This is the key observation for the proof of the following theorem. Note that in case $p = 2$ one color is sufficient in any case.

Theorem 7. *Let $n, p \in \mathbb{N}$ with $3 \leq p \leq n$. If $n \leq \frac{5}{4}p - 2$, then $f(n,p) = n - p + 2$.*

Proof. If $n \leq \frac{5}{4}p-2$, the linear upper bound from Corollary 4.6 implies $f(n,p) \leq \frac{p}{4}$. Let c be a minimum UCE-K_p coloring of K_n. Assume each color class contains at least two edges. Then, two arbitrary edges from each color class may be put into a common copy of K_p, since c uses at most $\frac{p}{4}$ colors. But this copy of K_p in K_n would be bad. Hence, there is a color class consisting of a single edge only. Due to Lemma 4.10, $f(n,p) = f(n-1,p)+1$ holds for all $p \leq n \leq \frac{5}{4}p - 1$.
This yields $f(n,p) = n-p+2$ for all $p \leq n \leq \frac{5}{4}p-1$ by induction on n. In the basic case $n = p$, the value $f(n,p) = 2 = n-p+2$ is known due to Corollary 4.1, part C3) (since $p \geq 3$ by assumption). If $p < n \leq \frac{5}{4}p-1$, the value $f(n-1,p) = (n-1)-p+2$ is known due to the induction hypothesis. Hence, $f(n,p) = f(n-1,p)+1 = n-p+2$ holds. □

Logarithmic Upper Bound on Triangles

It was proven in Lemma 4.5 that every coloring $c_{v_i,\theta}$ obtained from a good coloring by Construction 4.1 is good, if θ is a new color. The key observation for the following lemma is that in case of triangles K_3 a new color is not necessary in every step.

Lemma 4.11. *Let $n \in \mathbb{N}$. Then $f(n,3) \leq \lceil \log_2(n) \rceil$.*

Proof. Let c be a UCE-K_3 coloring of K_n and $\theta \in c(E(K_n))$ a color assigned to an edge by c. First of all, the following claim is proven. If there is a vertex $v \in V(K_n)$ with $c(e) \neq \theta$ for all edges $e \in E(K_n)$ incident to v, then the coloring $c_{v,\theta}$ obtained by Construction 4.1 from c is good. For the proof consider a copy Δ of K_3 in K_{n+1}.

Let v' denote the new vertex which was added to K_n in the construction. If Δ does not contain v and v' together, it has a uniquely colored edge, since it corresponds to a triangle in the original K_n. If Δ contains both vertices, then let u denote the third vertex in $V(\Delta)$. By construction, $c_{v,\theta}$ assigns the same color to both edges incident to u in Δ. Furthermore, one of these two edges is incident to v and was contained in the original K_n, already. Since the same color is assigned to this edge by c and $c_{v,\theta}$, the color does not equal θ due to the choice of θ. Because θ is assigned to the edge $\{v, v'\} \in E(\Delta)$, this edge is of unique color in Δ. Altogether, $c_{v,\theta}$ is a UCE-K_3 coloring of K_{n+1}.

In particular, a new color is not necessary in every iteration of Construction 4.1. Each time if there is a vertex which is not incident to edges of every color, a color may be reused. The coloring $c_{v,\theta}$ of K_{n+1} obtained from Construction 4.1 assigns color θ exactly to one edge incident to v and the new vertex of K_{n+1}. The color is not assigned to any other edge to which it was not assigned before by c. Hence, if there are q vertices in K_n which are not incident to an edge of color θ, Construction 4.1 can be repeated q times using color θ. With this observation an upper bound $f(2^i, 3) \leq i$ for $i \in \mathbb{N}$ is proven next, by induction on i. In case $i = 1$, one color is sufficient for a good coloring of K_2 since there is no copy of K_3 in K_2. Consider the case $i = 2$. A minimum UCE-K_3 coloring of K_3 assigns color 1 to two edges and color 2 to the third edge. Hence, one vertex u of this K_3 is not incident to an edge of color 2. Thus, the coloring $c_{u,2}$ obtained by Constrcution 4.1 from this coloring is a good coloring of K_4. In particular, $f(4, 3) = f(2^2, 3) \leq 2$.
If $i > 2$, let c be a minimum good coloring of $K_{2^{i-1}}$. By induction hypothesis, it uses at most $i - 1$ colors. Let θ denote a color not used by c. Then Construction 4.1 can be applied 2^{i-1} times using color θ as argued above. The resulting UCE-K_3 coloring of K_{2^i} uses one color more than c. Hence, $f(2^i, 3) \leq i$.

Finally, a good coloring of K_n, for $n, i \in \mathbb{N}$ with $2^i < n < 2^{i+1}$, is obtained as an induced coloring from a good coloring of $K_{2^{i+1}}$. In particular $f(n, 3) \leq \lceil \log_2(n) \rceil$. □

The same upper bound as in the previous lemma can be obtained as follows. Consider two copies of K_n, each with the same UCE-K_3 coloring c. Let θ be a color not used by c. A good coloring of K_{2n} is created by assigning θ to every edge connecting two vertices of the distinct copies of K_n. Then every triangle is either entirely contained in one of the copies of K_n, or contains exactly one edge from one of these copies. In both cases it contains a uniquely colored edge. Since the number of vertices is doubled with only one new color, the same upper bound as above is obtained.

4.3. Ramsey Theory

Two different relations of conflict-free colorings to Ramsey theory were established in Section 2.3. Classical (inverse) multicolor Ramsey numbers yield a lower bound on the conflict-free chromatic index, since every UCE-H coloring is a (graph) Ramsey coloring with respect to H as well. This bound was already stated for complete graphs in Corollary 4.1, C6). Few general bounds on multicolor Ramsey numbers are known, but in case of triangles K_3 there are. Furthermore, the conflict-free chromatic index equals the inverse multicolor Ramsey number, due to Lemma 2.15 in this case, since K_3 has only three edges. Thus, also upper bounds on the conflict-free chromatic index $f(n,3)$ and upper bounds on $R_3^{inv}(n)$ are related. The following corollary states the best known results (as far as they are known to the author).

Corollary 4.12. *There is an $n_0 \in \mathbb{N}$, such that for all $n \geq n_0$*

$$\frac{\ln(n)}{\ln\ln(n)} < f(n,3) \leq \lceil \log_{3.199}(n) \rceil.$$

Proof. Due to Corollary 4.1, C6), $f(n,3)$ is bounded from below by the classic inverse multicolor Ramsey number. An upper bound on $R_k(3)$ is $\lambda \cdot k!$, for a constant λ with $2 \leq \lambda \leq e$ [31]. Each integer k with $R_k(3) \leq n$ is a lower bound on the inverse Ramsey number, i.e. $k < R_3^{inv}(n)$. Hence, the inverse of $k!$ is needed to use this upper bound. Bounds on this inverse are calculated in Section A.3 of the appendix. The best result is stated in Lemma A.7. With $n \in \mathbb{N}$ sufficiently large, $\lambda \cdot (\frac{ln(n)}{lnln(n)}))! \leq n$ holds. Hence, $f(n,3) > \frac{\ln(n)}{\ln\ln(n)}$ holds for an $n_0 \in \mathbb{N}$ and all $n \geq n_0$.

Due to Lemma 2.15, the inverse multicolor Ramsey number $R_3^{inv}(n)$ is an upper bound on $f(n,3)$ as well. In the literature there are such bounds which do improve the upper bound of $f(n,3) \leq \lceil \log_2(n) \rceil$ known from Lemma 4.11 above. They are obtained as inverses of lower bounds on the classical multicolor Ramsey number $R_k(3)$. The lower bound on the multicolor Ramsey number $R_k(3) > (3.199)^k$ is established asymptotically in k (using Schur numbers) [33]. Hence, there is an upper bound $f(n,3) = R_3^{inv}(n) \leq \lceil \log_{3.199}(n) \rceil$ for sufficiently large n. This holds, because $\lceil \log_{3.199}(n) \rceil$ is the smallest of all integers k with $(3.199)^k > n$. $\square$

A generalized version of Ramsey theory provides upper bounds on the conflict-free chromatic index. In Lemma 2.17 it is stated that a graph H with more than $\frac{|E(H)|}{2}$ distinct colors contains a uniquely colored edge. Hence, each coloring which satisfies this condition for all copies of K_p in K_n is a UCE-K_p coloring of K_n. This generalization of Ramsey theory was developed by Erdős in [11] and later together with Gyárfás in [12]. First, a probabilistic result is presented for the general case. Afterwards, an explicit construction for the case $p = 4$ is shown. The following theorem is obtained from Corollary 2.19 by plugging in the number of edges of K_p.

Theorem 8. *Let $n, p \in \mathbb{N}$ with $2 \leq p \leq n$. There is a constant $c_p \in \mathbb{R}$ depending on p but not on n, such that*

$$f(n,p) \leq c_p \cdot n^{\frac{4}{p}}.$$

Explicit Construction for K_4

A better upper bound is known in case of K_4, due to an explicit construction. Again, the fact from Lemma 2.17 is used that four colors on the six edges of K_4 guarantee an edge of unique color. A construction of such a coloring is given by Mubayi in [22] and is reviewed next. The coloring is created as a product coloring of two different colorings. One part is called *Symmetric Subset Ranking coloring* (SSR coloring) and described first. The other part is called *Algebraic aoloring* and presented afterwards.

Construction 4.3 (SSR coloring, [22])**.** *Let $N, r \in \mathbb{R}$ and G be a graph with vertices $V(G) \subseteq \binom{N}{r}$, i.e. each vertex is an r-element set of integers from $\{1, \dots, N\}$. For each set in $V(G)$ an arbitrary linear ordering of its proper subsets is fixed. For distinct vertices $A, B \in V(G)$, let $R \in \{A, B\}$ denote the set which contains the minimum element of their symmetric difference $A \triangle B := (A \setminus B) \cup (B \setminus A)$. Then let $S \neq R$ denote the other set in $\{A, B\}$. The SSR coloring c assigns a four dimensional vector (c_0, c_1, c_2, c_3) to an edge $\{A, B\} \in E(G)$ with*

- $c_0(\{A, B\}) := \min(A \triangle B)$,
- $c_1(\{A, B\})$ *is the rank of $A \cap B$ in the linear ordering of subsets of R,*
- $c_2(\{A, B\})$ *is an arbitrary element in $S \setminus R$,*
- $c_3(\{A, B\})$ *is the rank of $A \cap B$ in the linear ordering of subsets of S.*

Mubayi proves several properties of this coloring. First of all, a coloring of K_n uses less than $e^{2\sqrt{2\log(4)\log(n)\cdot(1+o(1))}}$ colors, for appropriate choice of $N, r \in \mathbb{N}$ in the construction. Furthermore, the coloring assigns at least three colors to every copy of K_4 in K_n. The last property is proven by discovering several configurations of colors on the edges of K_4 that do not occur in a SSR coloring.
The algebraic coloring is defined as follows.

Construction 4.4 (Algebraic coloring, [22])**.** *Let q be an odd prime power, $\mathcal{F}$ the field on q elements and G be a graph with $V(G) \subseteq \mathcal{F} \times \mathcal{F}$. For vertices $A = (a_1, a_2)$ and $B = (b_1, b_2)$ let $\delta(A, B) := 1$, if $a_1 = b_1$, and $\delta(A, B) := 0$ otherwise. The algebraic coloring of an edge $\{A, B\} \in E(G)$ is the two dimensional vector $c(\{A, B\}) := (a_1 b_1 - a_2 - b_2, \delta(A, B))$.*

Again several properties of this coloring are proven. Particularly, there are several configurations of colors on K_4 which do not occur. The coloring uses at most $(2 + o(1))\sqrt{n}$ colors on the edges of K_n. This coloring is modified to get rid of another certain configuration of colors in K_4 by dividing several color classes into two distinct ones. The resulting coloring is called *Divided Algebraic coloring.* It uses at most twice the number of colors used by the original algebraic coloring.

Finally, the product of these two colorings is considered. Let c denote a SSR coloring of K_n and c' a Divided Algebraic coloring of K_n. The product coloring of an edge e is defined as $c^*(e) := (c(e), c'(e))$. By considering the configurations of colors which do not occur, it is shown that this product coloring assigns at least four colors to the edges of each K_4 in K_n. Furthermore, it uses at most $\sqrt{n} \cdot e^{c\cdot\sqrt{\log(n)}}$ colors for an absolute constant $c \in \mathbb{R}$. This yields the following corollary.

Corollary 4.13 (Theorem 1, [22])**.** *There is a constant $c \in \mathbb{R}$, such that for all $n \in \mathbb{N}$*

$$f(n, 4) \leq \sqrt{n} \cdot e^{c\cdot\sqrt{\log(n)}}.$$

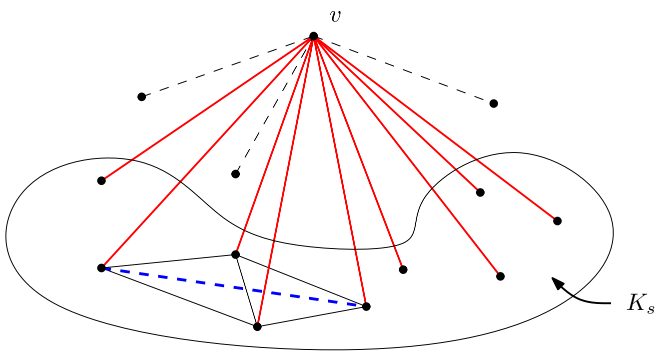


Figure 4.4.: A (red) monochromatic star on s edges with center v. If the coloring is a UCE-K_p coloring it is a UCE-K_{p-1} coloring on the complete graph K_s induced by the leaves of the monochromatic star.

4.4. Monochromatic Stars

During the last sections mainly upper bounds were determined. This section deals with an observation that allows derivation of lower bounds. The following is the key observation of this approach.

Lemma 4.14. *Let $n, p \in \mathbb{N}$ with $p \geq 3$ and c a UCE-K_p coloring of K_n. If there is a monochromatic star with $s \geq p-1$ leaves, then c uses at least $f(s, p-1)$ colors.*

Proof. Let K_s be the subgraph of K_n induced by the s leaves of a monochromatic star and v the center of this star. Further let K be a copy of K_{p-1} in K_s. Such a copy exists, since $s \geq p-1$. Then K forms a copy of K_p in K_n together with v and the edges connecting v to vertices in K. Denote this copy by K' and the set of edges connecting v to K by $E(v, K)$. Since c is a UCE-K_p coloring of K_n, there must be an edge e of unique color in K'. Because $p \geq 3$, there are more than two edges in $E(v, K)$ and all have the same color. Hence, e has to be contained in K. Moreover, it is of unique color within K too. Thus, c is a UCE-K_{p-1} coloring of K_s, due to the arbitrary choice of K. Hence, c uses at least $f(s, p-1)$ colors. □

Figure 4.4 shows an example of such a monochromatic star. Now, the task is to determine the size of monochromatic stars. There are different ways to estimate this number. Two similar ones are used in the following lemmas.

Lemma 4.15. *Let $p \in \mathbb{N}$ with $p \geq 3$. Then there is $n_0 \in \mathbb{N}$, such that for all $n \geq n_0$*

$$f(n,p) \geq f(\left\lceil \frac{n-1}{f(n,p)} \right\rceil, p-1).$$

Proof. Due to the pigeonhole principle, there is a monochromatic star on $\lceil \frac{n-1}{f(n,p)} \rceil$ edges in each minimum UCE-K_p coloring of K_n. The upper bound $f(n,3) \leq \lceil \log_2(n) \rceil$ is known from Lemma 4.11. The upper bound $f(n,4) \leq \sqrt{n} \cdot e^{c \cdot \sqrt{\log(n)}}$ is known from Lemma 4.13 for a constant $c \in \mathbb{R}$. Theorem 1 provides a sublinear upper bound on $f(n,p)$ for fixed $p \geq 5$. Altogether, there is an $n_0 \in \mathbb{N}$, such that $\frac{n-1}{f(n,p)} \geq p-1$ for all $n \geq n_0$ and $p \geq 3$. Hence, Lemma 4.14 implies $f(n,p) \geq f(\lceil \frac{n-1}{f(n,p)} \rceil, p-1)$ for all $n \geq n_0$. □

A very similar approach of estimating the size of monochromatic stars is stated in the following lemma. Note that the right hand side of the inequality stated there does not contain $f(n,p)$ any more.

Lemma 4.16. *Let $p \in \mathbb{N}$ with $p \geq 3$. There is $n_0 \in \mathbb{N}$, such that for all $n \geq n_0$*

$$f(n,p) \geq f(\left\lceil \frac{n-1}{f(n,p-1)} \right\rceil, p-1).$$

Proof. Either $f(n,p) > f(n,p-1)$, or not. The lemma holds in the first case, since $f(n,p-1) \geq f(\lceil \frac{n-1}{f(n,p-1)} \rceil, p-1)$, due to Corollary 4.1 C5). In the latter case, $\left\lceil \frac{n-1}{f(n,p)} \right\rceil \geq \left\lceil \frac{n-1}{f(n,p-1)} \right\rceil$ holds. Hence this lemma follows from Lemma 4.15. □

The following lower bound on the conflict-free chromatic index $f(n,p)$ is obtained from Lemma 4.15.

Theorem 9. *For $p \in \mathbb{N}$ exist constants $c_p \in \mathbb{R}$ and $n_p \in \mathbb{N}$, such that for all $n \geq n_p$*

$$f(n,p) \geq c_p \cdot \frac{\ln(n)}{\ln\ln(n)}.$$

Proof. In case $p = 3$ the lower bound is already known from Corollary 4.12.

The case $p > 3$ is proven by induction on p. Theorem 8 provides the upper bound $f(n,p) \leq c_p \cdot n^{\frac{4}{p}}$ for a constant $c_p \in \mathbb{R}$ and all $n \in \mathbb{N}$. The induction hypothesis yields a lower bound $f(n,p-1) \geq c\frac{\ln(n)}{\ln\ln(n)}$ for a constant $c \in \mathbb{R}$ and all $n \geq n_0$. Lemma 4.15 provides the following. For sufficiently large $n \in \mathbb{N}$ there are constants $c_p', \tilde{c_p} \in \mathbb{R}$, such that

$$f(n,p) \geq f(\left\lceil \frac{n-1}{f(n,p)} \right\rceil, p-1) \geq c \cdot \frac{\ln(\frac{n-1}{c_p \cdot n^{\frac{4}{p}}})}{\ln\ln(\frac{n-1}{c_p \cdot n^{\frac{4}{p}}})} \geq c \cdot \frac{\ln(c_p' \cdot n^{1-\frac{4}{p}})}{\ln\ln(n)} \geq \tilde{c_p} \cdot \frac{\ln(n)}{\ln\ln(n)}.$$

□

5. Conclusion

This diploma thesis deals with a new variant of conflict-free colorings. For a given graph H the edges of another graph G shall be colored, such that every copy of H in G contains an edge of unique color (i.e. no other edge in this copy of H is of the same color). If the number of available colors is too small, this is not possible in most cases. On the other hand, a total multicoloring of G is good in this sense in any case. Hence, the minimum number $f(G, H)$ of sufficient colors is of interest. In accordance to the (classical) chromatic index of a graph it is called conflict-free chromatic index (with respect to H).

Since the conflict-free chromatic index of a graph with respect to a path on two edges equals the (classical) chromatic index, it is NP-hard to decide whether k colors are sufficient in general. There is an efficient approximation of the (classical) chromatic index which uses at most one color more than an optimal solution. The question whether a similar result holds for the conflict-free chromatic index remains open.

Nevertheless, an upper bound of $f(G, H) \leq (\Delta_H(G)+1)^{\frac{2}{\lceil E(H)\rceil}}$ is established asymptotically in the maximum H-degree $\Delta_H(G)$ (i.e. the maximum number of copies of H in G sharing an edge with a fixed copy of H in G). It is determined using a generalization of Ramsey theory and probabilistic arguments. The question for a lower bound remains open in this general setting. In particular, it is not clear in which parameters of the graphs such a lower bound may be found.

This notion of conflict-free edge-colorings of graphs can be translated to the notion of conflict-free vertex-colorings of hypergraphs, which was introduced by Even et. al [13]. Therefore, a hypergraph is created with the edges of the original graph G as vertices and a hyperedge for each copy of H in G. Then, the conflict-free colorings of G with respect to H and conflict-free colorings of the hypergraph are equivalent. So lots of already known results for conflict-free colorings of hypergraphs apply to conflict-free colorings for certain graphs G and H and vice versa. In particular, the upper bound stated above improves upon the best known result for the conflict-free chromatic number of uniform hypergraphs.

Two specific settings are studied afterwards, paths as subgraphs of trees and complete graphs with complete subgraphs. These two are very different in structure. For paths in trees there are upper bounds on the conflict-free chromatic index that are independent from the height of the tree (and hence up to some extent from the number of vertices). For paths on an odd number $m \in \mathbb{N}$ of edges the upper bound $f(T, P_m) \leq \lceil \frac{m}{2} \rceil$ holds for all trees T. This bound is independent from the tree at all. Moreover, it is tight for sufficiently large $d \in \mathbb{N}$ on a complete d-ary tree

of sufficient height. For paths on an even number of edges $m \in \mathbb{N}$ the maximum degree is essential. In this case $f(T, P_m) \leq \frac{m}{2} + \Delta(T) - 1$ holds for all trees T. There are also lower bounds in terms of the maximum degree of the tree.
The independence from the height of the tree holds, because a path (of fixed length) in a tree covers only nearby edges and vertices. So the problem is very local in this case. Hence, the (local) structure of the tree is important and of more interest than the extremal behavior in global parameters.

For complete graphs the best known upper bound is derived from the general one given above. For $n, p \in \mathbb{N}$ and a constant $c_p \in \mathbb{R}$ it yields $f(K_n, K_p) \leq c_p \cdot n^{\frac{4}{p}}$. In particular, the maximum K_p-degree $\Delta_p(n)$ of K_n increases for fixed p, if n increases. This holds, because every choice of p vertices in $V(K_n)$ yields a copy of K_p in K_n. Hence, the problem is inherently global in this case. With a constant $c_p \in \mathbb{R}$ a lower bound of $f(K_n, K_p) \geq c_p' \cdot \frac{\ln(n)}{\ln\ln(n)}$ is derived by considering large monochromatic stars. The question how this large gap between upper and lower bound may be closed remains open. Since tight relations between (inverse) Ramsey numbers and the conflict-free chromatic index are established, answers to this question would have implications in Ramsey theory as well.

Following studies might consider paths in complete graphs. A general lower bound and an upper bound for paths on an odd number of edges are shown in the appendix in Section A.2.

A. Omitted Proofs

A.1. Pach and Tardos Theorem

In the following the proof of a Theorem by Pach and Tardos is reviewed. In particular it contains several calculations which are not presented in [23].

Lemma 2.9 (Theorem 2, [23]). *Let $t \in \mathbb{N}$ and let $\mathcal{H}$ be a hypergraph whose edges have at least $2t-1$ vertices each. There is a constant $c \in \mathbb{R}$, such that*

$$\chi_{cf}(\mathcal{H}) \leq c \cdot t \cdot D(\mathcal{H})^{\frac{1}{t}} \cdot \log(D(\mathcal{H})).$$

Proof. In the following the proof in [23] is summarized. If $t \geq D(\mathcal{H})$, a better upper bound is known due to Lemma2.7. Note that $\Delta(\mathcal{H}) \leq D(\mathcal{H})$. So assume $t < D(\mathcal{H})$ for the rest of the proof.

A geometric distribution is used to color the vertices and Lovasz's Local Lemma is used to prove the bound. When using the geometric distribution the total number of colors is not fixed. The coloring assigns integers to the vertices and this is done in the following way. Choose $p \in [0,1]$. Start with color 1 and flip a coin for each vertex, such that head comes up with probability p. If the coin shows head, color the vertex with color one, otherwise leave it uncolored. Then take the next color and process all yet uncolored vertices as in the step before. This is repeated until every vertex is colored. Altogether, the probability that a vertex is of color k is $p \cdot (1-p)^{k-1}$. Note that among all colors, color one has the highest probability.

The hypergraph $\mathcal{H}$ is colored in two steps. First of all the following subhypergraph is colored. Consider all edges in $E(\mathcal{H})$ one after the other and select $2t-1$ vertices in each edge. Let $V' \subseteq V(\mathcal{H})$ denote the set of all selected vertices and $\mathcal{H}' := (V', E(\mathcal{H}) \cap V')$ the hypergraph induced by V'. A conflict-free coloring of $\mathcal{H}'$ can be extended to a conflict-free coloring of $\mathcal{H}$ with one additional color. An edge in $\mathcal{H}'$ contains at most $(2t-1)(D(\mathcal{H})+1)$ vertices.

Let $T := c \cdot t \cdot D(\mathcal{H})^{\frac{1}{t}} \cdot \log(D(\mathcal{H}))$ denote the desired bound on the number colors. In order to proof the lemma the following two bad events need to be considered. Let P_{bad} denote the probability that a given hyperedge receives no unique color. Further let $P_{>T}$ the probability that a color larger than T is assigned to a given vertex. Note that both events may occur simultaneously. In the following upper bounds in terms of $D(\mathcal{H})$ and t are derived on both probabilities. Choose $p = \frac{1}{D(\mathcal{H})^{\frac{1}{t}} \cdot 30t}$.

There is a constant $c \in \mathbb{R}$, such that

$$P_{>T} = (1-p)^T \leq (1 - \frac{1}{(D(\mathcal{H}))^{\frac{1}{t}} \cdot 30t})^{c \cdot t \cdot (D(\mathcal{H}))^{\frac{1}{t}} \cdot \log(D(\mathcal{H}))} \leq e^{\frac{-c \cdot t \cdot (D(\mathcal{H}))^{\frac{1}{t}} \cdot \log(D(\mathcal{H}))}{(D(\mathcal{H}))^{\frac{1}{t}} \cdot 30t}}$$
$$= e^{\frac{-c \cdot \log(D(\mathcal{H}))}{30}} = \frac{1}{(D(\mathcal{H}))^{\frac{c}{30}}} < \frac{1}{30 \cdot (D(\mathcal{H}))^3}.$$

Next, an upper bound on P_{bad} is established. For a given hyperedge $\mathcal{E} \in E(\mathcal{H})'$ two cases are distinguished based on the number N of vertices in $\mathcal{E}$.

Consider the first case $N = 2t - 1$. If there are more than $t - 1$ distinct colors in $\mathcal{E}$, there is at least one vertex of unique color. Hence, P_{bad} may be bounded from above by the probability that there are at most $t - 1$ colors in $\mathcal{E}$. Let P_k denote the probability that a geometric coloring c of $\mathcal{E}$ uses exactly k colors. In order to estimate this probability all possible partitions of N vertices into k color classes are considered. This number is at most $\frac{k^N}{k!} \leq (\frac{e}{k})^k \cdot k^N$ ("labeled vertices, unlabeled color classes"). Let P_{part} denote the probability that a given partition $V_1 \dot\cup \ldots \dot\cup V_k$ corresponds to the color classes of the geometric coloring c. Then

$$P_k \leq \left(\frac{e}{k}\right)^k \cdot k^N \cdot P_{\text{part}}.$$

Fix a vertex v_i in each partition V_i. Noter that the sum occurring in the following calculation in finite for a given coloring.

$$\begin{aligned}
P_{\text{part}} &\leq P(\forall i \in 1, \ldots, k \forall v \in V_i : c(v) = c(v_i)) \\
&= \prod_{i=1}^{k} \prod_{v \in V_i} P(c(v) = c(v_i)) \\
&= \prod_{i=1}^{k} \prod_{v \in V_i \setminus \{v_i\}} \sum_{x=1}^{\infty} P(c(v) = x \mid c(v_i) = x) \cdot P(c(v_i) = x) \\
&= \prod_{i=1}^{k} \prod_{v \in V_i \setminus \{v_i\}} \sum_{x=1}^{\infty} p(1-p)^{x-1} \cdot p(1-p)^{x-1} \\
&= \prod_{i=1}^{k} \prod_{v \in V_i \setminus \{v_i\}} \tfrac{p^2}{1-(1-p)^2} \\
&= \prod_{i=1}^{k} \prod_{v \in V_i \setminus \{v_i\}} \tfrac{p}{2-p} \\
&\leq \prod_{i=1}^{k} \prod_{v \in V_i \setminus \{v_i\}} p \\
&= p^{N-k}
\end{aligned}$$

Hence $P_k \leq e^k \cdot (kp)^{N-k} =: X_k$. In order to find an upper bound on the probability of assigning at most $t - 1$ colors to N vertices the sum over these P_k is considered. The following estimate on the upper bound is used.

$$\sum_{k=1}^{t-1} \frac{X_k}{X_{t-1}} \leq \sum_{k=1}^{t-1} e^{k-t+1} \cdot \frac{k^{N-k}}{(t-1)^{N-t+1}} \cdot p^{t-1-k} \leq \sum_{k=1}^{t-1} e^{k-t+1} = e^{2-t} \frac{e^{t-1} - 1}{e - 1} = 1$$

This calculation shows that $\sum_{k=1}^{t-2} X_k \leq X_{t-1}$. Hence,

$$P_{\text{bad}} \leq \sum_{k=1}^{t-1} P_k \leq 2 \cdot X_{t-1} \leq 2 \cdot e^{t-1} \cdot ((t-1)p)^{2t-1-(t-1)} \leq (etp)^t.$$

Consider the second case $N \geq 2t$. Let S denote the set of those $2t$ vertices in $\mathcal{E}$ that receive their color last. Only one of the colors assigned to these vertices may be assigned to vertices in $\mathcal{E} \setminus S$, because each color is assigned in exactly one step of the geometric coloring as it is described above. Hence, if there is more than one vertex of unique color in S, there is a vertex of unique color in the complete edge. Similar to the first case P_{bad} is bounded from above by the probability that there are at most t colors on the vertices on S. Considering the same probabilities P_k as above (but for $2t$ elements) yields $P_{\text{bad}} \leq 2(etp)^t$.

Altogether the calculations yield

$$P_{\text{bad}} \leq 2(\frac{e \cdot t}{(D(\mathcal{H}))^{\frac{1}{t}} \cdot 30t})^t \leq (\frac{2e}{30})^t \frac{1}{D(\mathcal{H})} < \frac{1}{5 \cdot D(\mathcal{H})}.$$

If $D(\mathcal{H}) > t \geq 1$, the joint probability that an edge is bad or a color larger than T is assigned to a vertex in this edge is bounded from above by

$$\begin{aligned} &P_{\text{bad}} + (2t-1)(D(\mathcal{H})+1) \cdot P_{>T} = \frac{1}{5 \cdot D(\mathcal{H})} + \frac{(2t-1)(D(\mathcal{H})+1)}{30 \cdot (D(\mathcal{H}))^3} \\ =&\frac{1}{D(\mathcal{H})+1} \cdot (\frac{1}{5}\frac{D(\mathcal{H})+1}{D(\mathcal{H})} + \frac{1}{30}\frac{(2t) \cdot (D(\mathcal{H})+1)}{(D(\mathcal{H}))^3}) \leq \frac{1}{D(\mathcal{H})+1} \cdot (\frac{1}{5} \cdot \frac{3}{2} + \frac{1}{30} \cdot \frac{6}{4}) \\ <&\frac{1}{D(\mathcal{H})+1} \cdot \frac{1}{e}. \end{aligned}$$

Thus there is a conflict-free coloring using at most T colors due to Lovasz Local Lemma. □

A.2. Paths in Complete Graphs

In this section edge-colorings of complete graphs are studied, which ensure an edge of unique color in each copy of a fixed path. First of all, a corollary to Lemma 2.20 for paths on two edges is given. The corollary holds, because the chromatic index of a complete graph on n vertices equals n if n is odd and $n-1$ if n is even [3].

Corollary A.1. *Let $n \in \mathbb{N}$ with $n \geq 2$. Then*

$$f(K_n, P_2) = \begin{cases} n & \text{, if } n \text{ odd} \\ n-1 & \text{, if } n \text{ even.} \end{cases}$$

Using the extremal number for paths (each color class does not contain a path) one can determine a linear lower bound. The following result is due to Faudree and Schelp [14].

Lemma A.2 (second Theorem, [14]). *Let $n, m \in \mathbb{N}$. Then*

$$ex(n, P_m) \leq (m-1)\frac{n}{2}.$$

If $m|n$, the extremal graphs are exactly the disjoint unions of K_m.

The following result is a direct consequence of Lemma A.2.

Corollary A.3. *Let $n, m \in \mathbb{N}$. Then*

$$f(n, P_m) \geq \frac{n-1}{m-1}.$$

Proof. Due to Lemma A.2 and the pigeonhole principle the following holds:

$$f(n, P_m) \geq \frac{|E(K_n)|}{ex(n, P_m)} \geq \frac{n-1}{m-1}.$$

□

For paths on an odd number of edges a construction from Chapter 4 can be used.

Lemma A.4. *Let $n, m \in \mathbb{N}$ with $n \geq 2$ and m odd. Then*

$$f(n, P_m) \leq n-1.$$

Proof. Let c denote a UCE-K_3 (i.e. $p := 3$) coloring of K_n with $V(K_n) = \{v_1, \ldots, v_n\}$ obtained by Construction 4.2. For each $i \in \mathbb{N}$ with $1 \leq i < n$ the same color is assigned to all edges $\{v_i, v_j\} \in E(K_n)$ in case $j > i$. Moreover, different colors are used for distinct indices i. It is proven next, that the coloring c is a UCE-P_m coloring of K_n. If $m \geq n$, this holds trivially, because there is no copy of P_m in K_n.

Consider the case $m > n$ and a copy P of P_m in K_n. The path P has a uniquely colored edge, if and only if there is a vertex $v_i \in V(K_n)$, such that there is exactly one vertex $v_j \in V(K_n)$ with $j > i$ and the edge connecting v_i and v_j is in P. The leaves of P are of degree 1 in P, all other vertices in P are of degree 2. Assume P is bad. Then, both leaves are adjacent in P to vertices of smaller index in $V(K_n)$ (otherwise the color of the connecting edge is unique). But every vertex in P which is adjacent in P to a vertex of larger index in $V(K_n)$ must be adjacent in P to another vertex of larger degree in $V(K_n)$. Since vertices are of degree at most 2 in P this is not possible since the number of edge in P is odd. Hence, P is good. □

Note that this construction does not work for paths on an even number of edges.

A.3. Lower Bounds on Inverse Factorial

To calculate Inverse Ramsey Numbers it may be necessary to find a lower bound on the inverse of the factorial. This happens if the Ramsey Number is bounded from above by some term including a factorial. Then the necessary number of colors is bounded from below by lower bounds on the inverse of this term.

Lemma A.5. *Let $\lambda > 0$ be a constant. For all $0 < c < 1$ exists $n_c \in \mathbb{N}$, such that*

$$\lambda \cdot (ln(n)^c)! \leq n \text{ for all } n \geq n_c.$$

Proof. Let $0 < c < 1$ be fixed. If n is large enough, the following holds.

$$\begin{aligned}
\lambda \cdot (ln\,(n)^c)! &\leq \lambda \cdot e^{1-ln(n)^c} \cdot ln\,(n)^{c \cdot ln(n)^c + \frac{c}{2}} \\
&\leq \lambda \cdot e^{1-ln(n)^c} \cdot e^{lnln(n)\cdot\left(c\cdot ln(n)^c + \frac{c}{2}\right)} \\
&= \lambda \cdot \exp\left(1 + c \cdot ln\,(n)^c \cdot lnln\,(n) + \tfrac{c}{2} \cdot lnln\,(n) - ln\,(n)^c\right) \\
&\leq \lambda \cdot \exp\left(1 + c \cdot ln\,(n)^c \cdot ln\,(n)^{1-c} + \tfrac{c}{2} \cdot lnln\,(n) - ln\,(n)^c\right) \\
&\leq \lambda \exp\left(1 + c \cdot ln\,(n)\right) \\
&\leq n
\end{aligned}$$

□

The following calculation shows, that $c \geq 1$ does not work anymore.

Lemma A.6. *For all $c \geq 1$ exists n_0, such that $(ln(n)^c)! > n$ for all $n \geq n_0$.*

Proof. For large enough n the following holds.

$$(ln(n)^c)! \geq (ln(n))! \geq \sqrt{2\pi}(ln(n))^{ln(n)+\frac{1}{2}} e^{-ln(n)} > (e^2)^{ln(n)} \frac{1}{n} \geq n^2 \frac{1}{n} \geq n$$

□

There are some functions (asymptotically) between $ln(n)$ and $ln(n)^c$ for $c < 1$. The following lemma shows a function that is a valid lower bound on the inverse factorial.

Lemma A.7. *Let $\lambda > 0$ be a constant. For all $0 < c \leq 1$ exists $n_c \in \mathbb{N}$, such that*

$$\lambda \cdot (c\frac{ln(n)}{lnln(n)})! \leq n \text{ for all } n \geq n_c.$$

Proof. Let $0 < c \leq 1$ be fixed. If n is large enough, the following holds.

$$\begin{aligned}
\lambda \cdot \left(c\tfrac{ln(n)}{lnln(n)}\right)! &\leq \lambda \cdot e^{1-c\frac{ln(n)}{lnln(n)}} \left(c\tfrac{ln(n)}{lnln(n)}\right)^{\frac{1}{2}+c\frac{ln(n)}{lnln(n)}} \\
&\leq \lambda \cdot e^{1-c\frac{ln(n)}{lnln(n)}} \left(ln\,(n)\right)^{\frac{1}{2}+c\frac{ln(n)}{lnln(n)}} \\
&= \lambda \cdot \exp\left(1 - c\tfrac{ln(n)}{lnln(n)} + lnln\,(n)\left(1 + c\tfrac{ln(n)}{lnln(n)}\right)\right) \\
&= \lambda \cdot \exp\left(1 + c \cdot ln\,(n) + lnln\,(n) - c\tfrac{ln(n)}{lnln(n)}\right) \\
&\leq \lambda \cdot \exp\left(1 + c \cdot ln\,(n) + \tfrac{c}{2}\tfrac{ln(n)}{lnln(n)} - c\tfrac{ln(n)}{lnln(n)}\right) \\
&\leq \lambda \cdot e \cdot n \cdot e^{\frac{-c}{2\cdot lnln(n)}} \\
&\leq n
\end{aligned}$$

□

For larger constant factors this is not true any more, as is shown next.

Lemma A.8. *For all $c > 1$ exists $n_c \in \mathbb{N}$, such that*

$$(c\frac{ln(n)}{lnln(n)})! \geq n \text{ for all } n \geq n_c.$$

Also the following function is too large.

Lemma A.9. *For all $c > 0$ exists $n_c \in \mathbb{N}$, such that*

$$(c\,(ln\,(n) - lnln\,(n)))! \geq n \text{ for all } n \geq n_c.$$

Bibliography

[1] Noga Alon and Joel H. Spencer. *The probabilistic method.* Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons Inc., Hoboken, NJ, third edition, 2008.

[2] Hans L. Bodlaender, Jitender S. Deogun, Klaus Jansen, Ton Kloks, Dieter Kratsch, Haiko Müller, and Zsolt Tuza. Rankings of graphs. *SIAM J. Discrete Math.*, 11(1):168–181 (electronic), 1998.

[3] Béla Bollobás. *Extremal graph theory*, volume 11 of *London Mathematical Society Monographs.* Academic Press Inc. [Harcourt Brace Jovanovich Publishers], London, 1978.

[4] David P. Bunde, Kevin Milans, Douglas B. West, and Hehui Wu. Parity and strong parity edge-coloring of graphs. In *Proceedings of the Thirty-Eighth Southeastern International Conference on Combinatorics, Graph Theory and Computing*, volume 187, page 193–213, 2007.

[5] Panagiotis Cheilaris. *Conflict-Free Coloring.* PhD thesis, City University of New York, 2009.

[6] Panagiotis Cheilaris, Balázs Keszegh, and Dömötör Pálvölgyi. Unique-Maximum and Conflict-Free Coloring for Hypergraphs and Tree Graphs. In *SOFSEM*, page 190–201, 2012.

[7] Panagiotis Cheilaris and Géza Tóth. Graph unique-maximum and conflict-free colorings. *Journal of Discrete Algorithms*, 9(3):241–251, 2011.

[8] David Conlon. A new upper bound for diagonal Ramsey numbers. *Ann. of Math. (2)*, 170(2):941–960, 2009.

[9] Reinhard Diestel. *Graph theory*, volume 173 of *Graduate Texts in Mathematics.* Springer-Verlag, Berlin, third edition, 2005.

[10] Paul Erdős. Some remarks on the theory of graphs. *Bull. Amer. Math. Soc.*, 53:292–294, 1947.

[11] Paul Erdős. Solved and unsolved problems in combinatorics and combinatorial number theory. In *Proceedings of the Twelfth Southeastern Conference on Combinatorics, Graph Theory and Computing, Vol. I (Baton Rouge, La., 1981)*, volume 32, page 49–62, 1981.

[12] Paul Erdős and András Gyárfás. A Variant of the Classical Ramsey Problem. *Combinatorica*, page 459–467, 1997.

[13] Guy Even, Zvi Lotker, Dana Ron, and Shakhar Smorodinsky. Conflict-free colorings of simple geometric regions with applications to frequency assignment in cellular networks. *SIAM J. Comput.*, 33(1):94–136 (electronic), 2003.

[14] Ralph J. Faudree and Richard H. Schelp. Path Ramsey numbers in multicolorings. *J. Combinatorial Theory Ser. B*, 19(2):150–160, 1975.

[15] Sariel Har-Peled and Shakhar Smorodinsky. Conflict-Free Coloring of Points and Simple Regions in the Plane. *Discrete & Computational Geometry*, 34(1):47–70, 2005.

[16] Ian Holyer. The NP-completeness of edge-coloring. *SIAM J. Comput.*, 10(4):718–720, 1981.

[17] Ananth V. Iyer, H. Donald Ratliff, and Gopalakrishnan Vijayan. On an edge ranking problem of trees and graphs. *Discrete Appl. Math.*, 30(1):43–52, 1991.

[18] Georgia Kaouri, Aris Pagourtzis, and Katerina Potika. Conflict-free Coloring for Connected Subgraphs of Trees and Trees of Rings. In *Proc. 11th Panhellenic Conference in Informatics*, page 643–650, 2007.

[19] Richard M. Karp. Reducibility among combinatorial problems. In *Complexity of computer computations (Proc. Sympos., IBM Thomas J. Watson Res. Center, Yorktown Heights, N.Y., 1972)*, page 85–103. Plenum, New York, 1972.

[20] Alexander Kostochka, Mohit Kumbhat, and Tomasz Luczak. Conflict-free colourings of uniform hypergraphs with few edges. *Comb. Probab. Comput.*, 21(4):611–622, July 2012.

[21] Tak Wah Lam and Fung Ling Yue. Optimal edge ranking of trees in linear time. *Algorithmica*, 30(1):12–33, 2001.

[22] Dhruv Mubayi. An explicit construction for a Ramsey problem. *Combinatorica*, 24(2):313–324, 2004.

[23] János Pach and Gábor Tardos. Conflict-free colourings of graphs and hypergraphs. *Combin. Probab. Comput.*, 18(5):819–834, 2009.

[24] János Pach and Géza Tóth. Conflict-free colorings. In *Discrete and computational geometry*, volume 25 of *Algorithms Combin.*, page 665–671. Springer, Berlin, 2003.

[25] Stanisław P. Radziszowski. Small Ramsey numbers. *Electron. J. Combin.*, 1:Dynamic Survey 1, 30 pp. (electronic), 1994.

[26] Frank P. Ramsey. On a Problem of Formal Logic. *Proc. London Math. Soc.*, S2-30(1):264–286, 1930.

[27] Shakhar Smorodinsky. On The Chromatic Number of Geometric Hypergraphs. *SIAM J. Discrete Math.*, 21(3):676–687, 2007.

[28] Shakhar Smorodinsky. Conflict-Free Coloring and its Applications. *CoRR*, abs/1005.3616, 2010.

[29] Paul Turán. On an extremal problem in graph theory. *Matematikai és Fizikai Lapok*, 48:435–452, 1941. in Hungarian.

[30] Vadim G. Vizing. On an estimate of the chromatic class of a p-graph. *Diskret. Analiz No.*, 3:25–30, 1964.

[31] Honghui Wan. Upper bounds for Ramsey numbers $R(3, 3, \cdots, 3)$ and Schur numbers. *J. Graph Theory*, 26(3):119–122, 1997.

[32] Douglas B. West. *Introduction to graph theory*. Prentice Hall Inc., Upper Saddle River, NJ, second edition, 2001.

[33] Xu Xiaodong, Xie Zheng, Geoffrey Exoo, and Stanisław P. Radziszowski. Constructive lower bounds on classical multicolor Ramsey numbers. *Electron. J. Combin.*, 11(1):Research Paper 35, 24 pp. (electronic), 2004.