More on Different Graph Covering Numbers
Structural, Extremal and Algorithmic Results

Master Thesis of

Peter Stumpf

At the Department of Informatics
Institute of Theoretical Computer Science

Reviewers:	Prof. Dr. Dorothea Wagner
	Prof. Dr. Peter Sanders
	Prof. Maria Axenovich Ph.D.
Advisors:	Torsten Ueckerdt
	Marcel Radermacher

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Statement of Authorship

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Abstract

A *covering number* measures how "difficult" it is to cover all edges of a *host graph* with *guest graphs* of a given *guest class.* E.g., the global covering number, which has received the most attention, is the smallest number k such that the host graph is the union of k guest graphs from the guest class. In this thesis, we consider the recent framework of the *global*, the *local* and the *folded covering number* (Knauer and Ueckerdt, *Discrete Mathematics* **339** (2016)). The local covering number relaxes the global covering number by counting the guest graphs only locally at every vertex. And the folded covering number relaxes the local covering number even further, by allowing several vertices in the same guest graph to be identified, at the expense of counting these with multiplicities.

More precisely, we consider a *cover* of a host graph H with regards to a guest class $\mathcal{G}$ to be a finite set of guest graphs $S = \{G_1, \ldots, G_m\} \subset \mathcal{G}$ paired with an edge-surjective homomorphism $\phi\colon V(G_1 \uplus \ldots \uplus G_m) \to V(H)$. The cover is called *guest-injective*, if for $i = 1, \ldots, m$ its restricted homomorphism $\phi|_{G_i}$ is vertex-injective. The global covering number is thus the smallest number k such that there is a guest-injective cover with $|S| = k$. The local covering number is the smallest number k such that there is a guest-injective cover with $|\phi^{-1}(v)| \leq k$ for every vertex $v \in H$. And the folded covering number is the smallest number k such that there is a (not necessarily guest-injective) cover with $|\phi^{-1}(v)| \leq k$ for every vertex $v \in H$. In this thesis we investigate the relations between these numbers.

As one result we show that the local covering number of any shift graph with regards to bipartite graphs is at most 2, while the corresponding global covering numbers can be arbitrarily large. This is the first known *separation* of local and global covering number with a subgraph-hereditary guest class. This concludes the study of separation results, as a separation of folded and local covering number with such a guest class was already known, as well as the fact that such a separation is not possible for union-closed topological-minor-closed guest classes.

Furthermore, we investigate for $a, b \in \mathbb{N}_0$ with $b < 2a$ the classes of (a, b)-sparse graphs as guest classes. For these the local and the global covering number always coincide. This generalizes existing results for forests and pseudo-forests. We prove that the global covering number with regards to these (a, b)-sparse graphs always matches a fairly simple lower bound given by a variation of the host graph's maximum average degree. We moreover provide an efficient algorithm to calculate corresponding optimal covers.

While most attention in the given framework focuses on properties for the guest classes, we also investigate graph classes of host graphs without fixing the guest class. Namely, we introduce a property called *cover resistance.* We call a class $\mathcal{H}$ of host graphs $(f/l/g)$-cover resistant, if the (folded/local/global) covering numbers for these host graphs usually become arbitrarily large. I.e., it is $(f/l/g)$-cover resistant, if for each union-closed induced-hereditary guest class $\mathcal{G}$ we have host graphs in $\mathcal{H}$ with arbitrarily large (folded/local/global) covering numbers, unless $\mathcal{G}$ contains $\mathcal{H}$. For the classes of host graphs the relaxations of folded and local covering number are inverse, i.e., the f-cover resistance implies l-cover resistance which in turn implies g-cover resistance. We give examples for each of these resistances. As a result of our investigation, we characterize the induced-hereditary guest classes with bounded folded covering number as those containing all bipartite graphs. We further show the class of all graphs is the only induced-hereditary guest class with bounded local covering number.

Deutsche Zusammenfassung

Eine *Überdeckungszahl* misst wie "schwer" es ist alle Kanten eines *Gastgebergraphen* mit *Gastgraphen* einer gegebenen *Gastklasse* zu überdecken. Die globale Überdeckungszahl ist z.B. die kleinste Zahl k für die der Gastgebergraph die Vereinigung von k Gastgraphen ist. Sie hat bisher am meisten Aufmerksamkeit erhalten. In dieser Arbeit betrachten wir ein junges Rahmenkonzept welches die *globale*, die *lokale* und die *gefaltete Überdeckungszahl* umfasst. (Knauer und Ueckerdt, *Discrete Mathematics* **339** (2016)). Die locale Überdeckungszahl relaxiert die globale Überdeckungszahl, indem sie die Gäste nur lokal an jedem einzelnem Knoten zählt. Und die gefaltete Überdeckungszahl wiederum relaxiert die lokale Überdeckungszahl, indem sie es zulässt mehrere Knoten des gleichen Gastgraphen zu identifizieren, die dafür jedoch auch mehrfach gezählt werden.

Genauer gesagt betrachten wir eine Überdeckung eines Gastgebergraphen H bezüglich einer Gastklasse $\mathcal{G}$ als Paar einer endliche Menge von Gastgraphen $S = \{G_1, \ldots, G_m\} \subset \mathcal{G}$ zusammen mit einem Kanten-surjektiven Homomorphismus $\phi\colon V(G_1 \uplus \ldots \uplus G_m) \to V(H)$. Wir bezeichnen die Überdeckung als *Gast-injektiv*, wenn für $i = 1, \ldots, m$ der eingeschränkte Homomoprhismus $\phi|_{G_i}$ Knoten-injektiv ist. In diesem Formalismus ist die globale Überdeckungszahl die kleinste Zahl k, sodass es eine Gast-injektive Überdeckung mit $|S| = k$ gibt. Die lokale Überdeckungszahl ist die kleinste Zahl k, sodass es eine Gast-injektive Überdeckung gibt mit $|\phi^{-1}(v)| \leq k$ für jeden Knoten $v \in H$. Und die gefaltete Überdeckungszahl ist die kleinste Zahl k, sodass es eine (nicht notwendigerweise Gast-injektive) Überdeckung gibt mit $|\phi^{-1}(v)| \leq k$ für jeden Knoten $v \in H$. In dieser Arbeit untersuchen wir die Beziehungen unter diesen Überdeckungszahlen.

Eines unserer Resultate ist, dass bezüglich bipartiter Graphen die lokale Überdeckungszahl jedes Shift Graphs höchstens 2 beträgt, während die entsprechende globale Überdeckungszahl beliebig groß wird. Dies stellt die erste bekannte *Separierung* der lokalen und der globalen Überdeckungszahl mit einer Subgraph-hereditären Gastklasse dar. Damit schließen wir die Studie der Separierungen ab, da eine entsprechende Separierung der gefalteten und der lokalen Überdeckungszahl bereits bekannt war, und wir zudem wissen, dass solche Separierungen nicht mit Gastklassen möglich sind, die abgeschlossen unter disjunkter Vereinigung und dem Nehmen von topologischen Minoren sind.

Wir untersuchen außerdem für $a, b \in \mathbb{N}_0$ mit $b < 2a$ die Klasse der (a, b)-dünnbesetzten Graphen als Gastklasse. Für diese Klassen fallen die lokale und die globale Überdeckungszahl zusammen. Damit verallgemeinern wir bestehende Resultate für Wälder und Pseudowälder. Wir zeigen dass die globale Überdeckungszahl bezüglich dieser (a, b)-dünnbesetzten Graphen immer mit einer sehr einfachen unteren Schranke zusammenfällt, die als Variation des maximalen Durschnittsgrades des Gastgebergraphens gegeben is. Darüberhinaus beschreiben wir einen effizienten Algorithmus mit dem man entsprechende optimale Überdeckungen erhält.

Während der größte Teil der Aufmerksamkeit im gegebenen Rahmenkonzept auf Eigenschaften der Gastklassen liegt, untersuchen wir auch Graphklassen von Gastgebergraphen ohne die Gastklasse festzulegen. Genauer gesagt führen wir die Eigenschaft der Überdeckungsresistenz ein. Eine Klasse $\mathcal{H}$ heißt (f,l,g)-überdeckungsresistent, wenn die (gefaltete/lokale/globale) Überdeckungszahl für diese Gastgebergraphen gewöhnlich sehr groß werden. Genauer, die Klasse is (f,l,g)-überdeckungsresistent, wenn wir für jede Gastklasse $\mathcal{G}$, die abgeschlossen unter disjunkter Vereinigung und induziert-hereditär ist, Gastgebergraphen mit beliebig großen Überdeckungszahlen finden, es sei denn $\mathcal{G}$ enthält $\mathcal{H}$ bereits. Für die Gastgeberklassen ist die Reihenfolge der Relaxierungen der gefalteten und der lokalen Überdeckungszahl umgekehrt. Das heißt, die f-Überdeckungsresistenz impliziert die l-Überdeckungsresistenz, welche wiederung die g-Überdeckungsresistenz impliziert. Wir geben für jede dieser Resistenzen Beispiele. Als Resultat erhalten wir die Charakterisierung der induziert-hereditären Gastklassen mit beschränkter gefalteter Überdeckungszahl als jene Klassen, die alle bipartiten Graphen enthalten. Wir zeigen außerdem, dass die Klasse aller Graphen die einzige Gastklasse mit beschränkter lokaler Überdeckungszahl ist.

Contents

1. Introduction

In general a *covering number* states how "difficult" it is to cover all edges of a *host graph* H by *guest graphs* from a given *guest class* $\mathcal{G}$. We consider the recent framework of the *global*, the *local* and the *folded* covering number that was introduced by Knauer and Ueckerdt [KU16].

The *global covering number* has been investigated the most. It is the minimum number k such there are k *guests* $G_1, \dots, G_k \in \mathcal{G}$ with $G_1 \cup \dots \cup G_k = H$. We write in short $c_g^{\mathcal{G}}(H) = k$ (the index indicates the kind of covering number). Already in 1891 Petersen showed that the global covering number of a $2k$-regular host graph H with regards to 2-regular guest graphs equals k [Pet91]. Note that he actually provided a decomposition of H into 2-regular graphs which means the guests $G_1, \dots, G_k$ are edge-disjoint. This is not required in the setting of covers. However, if the guest class is closed under taking subgraphs, we can transform a cover into a decomposition by removing duplicated edges from guests. A well known global covering number is the *arboricity* which was investigated by Nash-Williams and is just the global covering number with regards to forests [NW64]. Nash-Williams showed that the arboricity always meets a natural lower bound similar to the maximum average degree. Other examples are the *track number* introduced by Gyárfás and West [GW95] and the *thickness* introduced by Aggarwal et al. [AKL$^+$85] corresponding to the guest classes of interval graphs and planar graphs respectively.

The *local covering number* is a relaxation of the global covering number. It is the minimum number k such that there are some guests $G_1, \dots, G_m \in \mathcal{G}$ with $G_1 \cup \dots \cup G_m = H$ such that there are locally only k guests, i.e., every vertex $v \in V(H)$ is contained in at most k of these guests. We write in short $c_l^{\mathcal{G}}(H) = k$. Fishburn and Hammer introduced the *bipartite degree* which is the local covering number with regards to complete bipartite graphs [FH96]. They showed the bipartite degree can be arbitrarily large. They also considered the corresponding global covering number called *bipartite dimension* and examined forbidden induced subgraphs for graphs where the bipartite dimension (bipartite degree) is bounded. Pinto recently showed that the bipartite degree and its decomposition variant coincide on complete graphs, while the decomposition variant of the bipartite dimension of complete graphs differs heavily from the bipartite dimension itself [Pin14].

The *folded covering number* is in turn a relaxation of the local covering number. We call the operation of identifying non-adjacent vertices in a graph *folding*. The folded covering number is the minimum number k such that there are some guests $G_1, \dots, G_m \in \mathcal{G}$ such that H is the result of folding vertices in graph $G = G_1 \mathbin{\dot\cup} \dots \mathbin{\dot\cup} G_m$ and every vertex

$v \in V(H)$ is the result of folding at most k vertices. We write in short $c_f^{\mathcal{G}}(H) = k$. Eppstein et al. recently examined the *planar split thickness* which is the folded covering number with regards to planar graphs [EKK+16]. They most notably examined the planar split thickness of complete graphs and complete bipartite graphs, and they showed determining the planar split thickness is $\mathcal{NP}$-hard. The folded covering number with regards to interval graphs was introduced by Trotter and Harary as the *interval-number* and has been extensively studied as pointed out by Butman et al. [TH79] [BHLR10].

For more examples of considered covering numbers see Knauer and Ueckerdts paper [KU16]. They examined a variety of forest classes and interval graphs as guest classes. Examples for applications of different covering numbers are provided in the Bachelor Thesis of Stumpf [Stu15].

Since the local and the folded covering number are relaxations of the global covering number, we obtain for a host class H and a guest class $\mathcal{G}$ that $c_f^{\mathcal{G}}(H) \leq c_l^{\mathcal{G}}(H) \leq c_g^{\mathcal{G}}(H)$. These inequalities give motivation to compare those parameters. Let $\overline{\mathcal{G}}$ be the *union-closure* of $\mathcal{G}$, i.e., let $\overline{\mathcal{G}}$ be the closure under taking disjoint unions. It is easy to show that $c_f^{\mathcal{G}}(H) = c_f^{\overline{\mathcal{G}}}(H)$ and $c_l^{\mathcal{G}}(H) = c_l^{\overline{\mathcal{G}}}(H)$. For a meaningful comparison to the global covering number Knauer and Ueckerdt considered only *union-closed* guest classes, i.e., guest classes that are closed under taking disjoint unions. However, there are also meaningful global covering numbers with regards to not union-closed guest classes. For example the size of a minimum vertex cover of H is just the global covering number with regards to the guest class of stars. To distinct those two cases we introduce the *union covering number*.

The *union covering number* of a host class H with regards to a guest class $\mathcal{G}$ is the minimum number k such that $c_g^{\overline{\mathcal{G}}}(H) = k$ where $\overline{\mathcal{G}}$ is the union-closure of $\mathcal{G}$. We write in short $c_u^{\mathcal{G}}(H) = c_g^{\overline{\mathcal{G}}}(H)$. The union covering number fits neatly into the existing framework. Every global covering number with regards to a union-closed guest class is also a union covering number. Another example is the *union-boxicity*. The *boxicity* of a graph H was introduced by Roberts [Rob69] and is the smallest number k such that H is the intersection of k interval graphs. Cozzens and Roberts observed that this is just the global covering number of the complement H^c with regards to cointerval graphs [CR83]. Blaesius, Ueckerdt and Stumpf introduced the *local* and *union* boxicity which correspond to the local and union covering number [BSU16]. In this example following the framework of Knauer and Ueckerdt generated new parameters with meaningful geometric interpretations.

Let H be a host graph and let $\mathcal{G}$ be a guest class. Given the inequalities $c_f^{\mathcal{G}}(H) \leq c_l^{\mathcal{G}}(H) \leq c_u^{\mathcal{G}}(H) \leq c_g^{\mathcal{G}}(H)$, the question arises how much these covering numbers may differ. One approach is to consider *separations*. For $i = f, l, u, g$ we define the covering number of a host class $\mathcal{H}$ as $c_i^{\mathcal{G}}(\mathcal{H}) = \sup_{H \in \mathcal{H}} c_i^{\mathcal{G}}(H)$. A *separation* is a pair of a host class $\mathcal{H}$ and a guest class $\mathcal{G}$ such that for $i, j = f, l, u, g$ we have $c_i^{\mathcal{G}}(\mathcal{H}) = 2$ while $c_j^{\mathcal{G}}(\mathcal{H}) = \infty$. Knauer and Ueckerdt provided the guest class $\mathcal{K}$ of complete graphs and the host class $\mathcal{L}$ of line graphs as separation of the local and the union covering number [KU16]. Note that here the guest class is *induced-hereditary*, i.e., it is closed under taking induced subgraphs. Stumpf provided the guest class of $\mathcal{B}ip$ bipartite graphs and the host class $\mathcal{A}$ of all graphs (or the host class $\mathcal{K}$) as separation of the folded and the local covering number in his bachelor thesis [Stu15]. Note that $\mathcal{B}ip$ is even *subgraph-hereditary*, i.e., it is closed under taking subgraphs. Stumpf also showed that there is no separation of the folded and the union covering number with a topological-minor closed guest class. Therefore the question remains whether there is a separation of the local and the global covering number with a subgraph-hereditary guest class. In this thesis we answer this question positively by providing the guest class of bipartite graphs and the host class of *shift graphs* as separation of the local and the global covering number. Shift graphs were introduced by Erdős and Hajnal [EH66].

Stumpf showed in his Bachelor Thesis that we have for any induced-hereditary guest class $\mathcal{G}$ that $c_l^{\mathcal{G}}(\mathcal{K}) \in \{0, 1, \infty\}$ [Stu15]. For other host classes of separations we provide similar results. Note that the host graphs of the separations were not chosen to have this property and it is not necessary for a separation. This motivates an investigation of host graphs with this property which we introduce as the *cover resistance*. Let $i = f, l, u, g$ and let $\mathcal{H}$ be a host graph. We say that $\mathcal{H}$ is *i-cover resistant*, if we have for every induced-hereditary guest class $\mathcal{G}$ that $c_i^{\mathcal{G}}(\mathcal{H}) \in \{0, 1, \infty\}$. The name is justified by following observation: For every host class $\mathcal{H}$ we have an induced-hereditary guest class $\mathcal{G}$ with $\mathcal{H} \subseteq \mathcal{G}$. For every such guest class $\mathcal{G}$ we obtain $c_g^{\mathcal{G}}(\mathcal{H}) \leq 1$, since every graph $H \in \mathcal{H}$ can be covered by itself. If $\mathcal{H}$ is g-cover resistant, then we obtain for every guest classes $\mathcal{G}'$ with $\mathcal{H} \nsubseteq \mathcal{G}'$ (modulo isolated vertices) that $c_g^{\mathcal{G}'}(\mathcal{H}) = \infty$. For $i = f, l, u$ we obtain similar results for the i-cover resistance.

The investigation of cover resistances is related to the Induced Ramsey Theory. We say a graph class $\mathcal{H}$ has the *Induced Ramsey Property*, if for every graph $H \in \mathcal{H}$ there is a *Ramsey graph* $H' \in \mathcal{H}$ such that for every 2-colouring of the edges of H' there is a monochromatic induced subgraph of H' that is a copy of H. This property can be considered as the K_2-Ramsey property which is a special case of the A-Ramsey property whose formalism was developed by Leeb [Lee73] and Nešetřil and Rödl [NR] as cited by Nešetřil and Rödl [NR84]. If the copy of H is not required to be an induced subgraph of H' but only required to be an induced subgraph within its own colour, we instead speak of the *Weak Induced Ramsey Property*. We provide characterizations for the cover resistances that are very similar. As a result we obtain that for a host class $\mathcal{H}$ the Induced Ramsey Property implies the g-cover resistance which in turn implies the Weak Induced Ramsey Property.

With the inequalities $c_f^{\mathcal{G}}(H) \leq c_l^{\mathcal{G}}(H) \leq c_u^{\mathcal{G}}(H) \leq c_g^{\mathcal{G}}(H)$, we obtain a similar but reversed hierarchy of relaxations for the cover resistances. We show for $i = f, l, g$ that if a host class $\mathcal{H}$ is not i-cover resistant, then there is an induced-hereditary guest class $\mathcal{G}$ with $c_i^{\mathcal{G}}(\mathcal{H}) = 2$. If $\mathcal{H}$ is at the same time j-cover resistant for some $j = f, l, u, g$, then $\mathcal{H}$ and $\mathcal{G}$ provide a separation of the covering numbers c_i and c_j. Further we show that the class of all graphs $\mathcal{A}$ is l-cover resistant which means the only induced-hereditary guest class $\mathcal{G}$ with $c_l^{\mathcal{G}}(\mathcal{A}) < \infty$ is the class of all graphs itself. Similarly we show that the class of bipartite graphs $\mathcal{B}ip$ is f-cover resistant. This means every induced-hereditary guest class $\mathcal{G}$ with $c_f^{\mathcal{G}}(\mathcal{A}) < \infty$ must contain all bipartite graphs. Since we have $c_f^{\mathcal{B}ip}(\mathcal{A}) = 2$ this implies $c_f^{\mathcal{G}}(\mathcal{A}) \leq 2$.

In contrast to separations, we also consider guest classes where the local and global covering number coincide. Let $a, b \in \mathbb{N}_0$ with $b < 2a$. Lee and Streinu introduced the notion of *(a, b)-spare graphs* [LS08]. A graph H is called (a, b)-sparse, if we have for every subgraph $G \subseteq H$ with $|V(G)| \geq 2$ that $|E(G)| \leq a|V(G)| - b$. Forests are for example exactly the $(1, 1)$-sparse graphs. The upper bounds on the number of edges provide lower bounds for the covering number. As indicated earlier, Nash-Williams showed that in case of $(1, 1)$-sparse graphs this lower bound is always met for the union covering number [NW64]. We generalize this result for the class of (a, b)-sparse graphs. As a result we obtain that the local and global covering number with regards to (a, b)-sparse guest graphs coincide.

Let H be a host graph. We denote the class of (a, b)-sparse graphs with $\mathcal{G}(a, b)$. We further provide an algorithm that provides an optimal global $\mathcal{G}(a, b)$-cover for a given graph $H = (V, E)$ with a runtime in $\mathrm{O}(|V| \cdot |E|^2)$ for a constant a. This algorithm makes use of the pebble game algorithm introduced by Lee and Streinu which allows to determine whether a given graph $G = (V, E)$ is (a, b)-sparse with a runtime in $\mathrm{O}(a|E|^2)$ [LS08]. We can also use the pebble game algorithm to determine the global covering number of H without computing a cover. For $k \in \mathbb{N}$ we observe that $c_g^{\mathcal{G}(a,b)}(H) \leq k$ is equivalent to H being (ka, kb)-sparse. With this observation we can use the pebble game algorithm of Lee and Streinu to determine $k = c_g^{\mathcal{G}(a,b)}(H)$ with a runtime in $\mathrm{O}(k \log(k)|E|^2) \subseteq \mathrm{O}(|E|^3/|V|(\log(|E|) - \log(|V|)))$ for a constant a. Streinu and Theran advanced the pebble game algorithm to additionally provide an (a, b)-sparsity certifying decomposition [ST09]. However, the guests in that decomposition are all $(1, 0)$-sparse. Their main results characterize their decompositions of (a, b)-tight graphs. Streinu and Theran make use of *augmenting paths* that were introduced by Edmonds for the general matroid setting and are also used in our algorithm [Edm65].

This thesis is organized as follows.

In **Chapter 2** we give basic definitions considered in this thesis.

In **Chapter 3** we state alternative definitions of the global, the local and the folded covering number in terms of edge-surjective graph homomorphisms and give related definitions. We give a more detailed motivation for the introduction of the union covering number and also introduce the notion of union covers. Further, we show that the union covering number fits neatly into the framework of the global, the local and the folded covering umber. We finally establish basic results used in this thesis.

In **Chapter 4** we first present old and new separations of covering numbers. We especially show that for the class $\mathcal{S}$ of shift graphs and the class $\mathcal{B}ip$ of bipartite graphs we have $c_l^{\mathcal{B}ip}(\mathcal{S}) = 2$ while $c_u^{\mathcal{B}ip}(\mathcal{S}) = \infty$. On the other side, we show that for a host class of bounded chromatic number such separations are not possible for any induced-hereditary guest class, since the union covering number is bounded by a function of the folded covering number. We further extend known results by showing for $a, b \in \mathbb{N}_0$ with $b < 2a$ that the local and global covering number with regards to the guest class of (a, b)-sparse graphs coincide.

In **Chapter 5** we provide for $a, b \in \mathbb{N}_0$ with $b < 2a$ an algorithm to compute an optimal global (a, b)-sparse cover of a given graph $H = (V, E)$ with runtime in $\mathrm{O}(|V| \cdot |E|^2)$. This algorithm builds on the pebble algorithm for detection of(a, b)-sparse graphs that was introduced by Lee and Streinu [LS08]. This pebble algorithm is introduced first.

In **Chapter 6** we shortly consider covering numbers with regards to guest classes of bounded degree. We provide a lower bound and show it is always met by the folded covering number while sometimes not met by the global covering number.

In **Chapter 7** we introduce the cover resistances corresponding to the four kinds of covering numbers. Then we characterize the cover resistances similar to the Induced Ramsey Property. As a result we obtain that the Induced Ramsey Property implies the g-cover resistance which in turn implies the Weak Induced Ramsey Property. Then we provide results that, given a cover resistant host class, allow to construct other cover resistant host classes. We analyse relations between the different cover resistances, i.e., for a host class $\mathcal{H}$ we show which cover resistance implies which cover resistance. For each combination of cover resistances that are possible for one host class by this implications, we provide an example of a host class which has exactly the cover resistances of the combination. To obtain some examples, we establish that a g-cover resistant host class $\mathcal{H}$ is also l-cover resistant, if it is closed under adding universal vertices, and it is f-cover resistant if it has bounded chromatic number. Let $\mathcal{G}$ be an induced-hereditary guest class. We obtain as result that the folded covering number with regards to $\mathcal{G}$ is unbounded, unless $\mathcal{G}$ contains all bipartite graphs. Similarly, the local covering number with regards to $\mathcal{G}$ is only bounded, if $\mathcal{G}$ is the class of all graphs.

2. Preliminaries

For any set S and any number n we define $\binom{S}{n} = \{T \subseteq S : |T| = n\}$. For any mapping $\phi : A \to B$ and any subset $C \subseteq A$ we write $\phi(C)$ for $\{\phi(c) : c \in C\}$. Further, we notate the restriction of ϕ as $\phi|_C : C \to B, x \mapsto \phi(x)$. For any $k \in \mathbb{N}$ we write $[k]$ for $\{i \in \mathbb{N} : i \leq k\}$. Note that $0 \notin [k]$.

In this thesis a *graph* G is a tuple (V, E) with the finite vertex set V and the edge set $E \subseteq \binom{V}{2}$ and $\forall e \in E : \exists u, v \in V : u \neq v, e = \{u, v\}$. The graph G is said to be a graph *on* V. Elements of V are called *vertices* and elements of E are called *edges*. An edge $\{v, u\}$ is shortly denoted as vu. By $V(G)$ we denote the vertex set of G and by $E(G)$ its edge set. Given a finite set E of sets containing two elements each, we write $G(E)$ for the graph (V, E), where $V = \{v \mid \{u, v\} \in E\}$. We often write $v \in G$ for $v \in V(G)$ and $uv \in G$ for $uv \in E(G)$. We call $|V|$ the *order* of G and it is denoted by $|G|$. We call $|E|$ the *size* of G and it is denoted by $||G||$. A graph G is called *trivial* if $|G| \leq 1$.

For two edges $vu, wv \in E$ we call the vertices v and u *adjacent*, we call the edges uv and vw *adjacent* and we call the vertex v and the edge uv incident. We say vu connects v and u.

For any vertex $v \in V$ its *neighbourhood* is defined as $\{u \in V : vu \in E\}$ and denoted by $\mathrm{N}(v)$. Its elements are called *neighbours* of v. The *degree* of v is defined as $|\mathrm{N}(v)|$ and is denoted by $\deg_G(v)$ or simply $\deg(v)$. If $\deg(v) = 0$, then v is called *isolated*. If $\deg(v) = 1$, then v is called a *leaf*. The *maximum degree* of G is defined as $\Delta(G) := \max_{v \in V} \deg(v)$ and the *minimum degree* of G is defined as $\delta(G) := \min_{v \in V} \deg(v)$. If $\delta(G) = \Delta(G) =: r$, then G is called *r-regular* and has *regularity* r. The *average degree* $\mathrm{avd}(G)$ of a graph G is $\frac{\Sigma_{v \in V(G)} \deg(v)}{|V(G)|} = \frac{2|E(G)|}{|V(G)|}$.

A graph $G' = (V', E')$ is called *subgraph* of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. The graph G is then called a *supergraph* of G' and said to *contain* G' and we write in short $G' \subseteq G$. The graph G' is called *induced subgraph* of G if $V' \subseteq V$ and $E' = E \cap \binom{V'}{2}$. We denote this shortly by $G \sqsubseteq H$. We say V' *induces* G' in G and denote G' by $\langle V' \rangle_G$. The graph G' is called *spanning* in G if $G' \subseteq G$ and $V' = V$. A subset $W \subseteq V$ is called *independent* if it induces a graph without edges. The graph induced by W is then also called *independent set*. The *chromatic number* of G is the smallest number k such that $V(G)$ can be partitioned into k independent sets. We write $\chi(G)$ for the chromatic number of G. If $\chi(G) \leq n$ we call G an *n-partite* graph. In case of $n = 2$ we call G a *bipartite* graph. If its two independent sets are fully connected, the graph G is called *complete bipartite*. We say a subset $F \subset E$ induces (U, F) in G where U is the set of all vertices incident to an edge of F in G ($U = \bigcup F$). A set of pairwise non-adjacent edges is called *matching*. We also call a graph a *matching* if its induced by a matching. A spanning matching is called *perfect*. A k-regular spanning subgraph is called *k-factor*. By the *maximum average degree* $\mathrm{mad}(G)$ of a graph G we denote the maximum average degree of all induces subgraphs of G, i.e., $\mathrm{mad}(G) = \max_{H \subseteq G} \mathrm{avd}(H)$. The *intersection* of two graphs G_1 and G_2 is defined as $(V(G_1) \cap V(G_2), E(G_1) \cap E(G_2))$ and denoted by $G_1 \cap G_2$. The *union* of two graphs G_1 and G_2 is defined as $(V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$ and denoted by $G_1 \cup G_2$.

The union of disjoint sets is itself called *disjoint*. If we speak of the *disjoint union of graphs*, we assume their vertex sets to be disjoint (Formally, for a family $\{G_i : i \in I\}$ of graphs with index set I we replace for every $i \in I$ every vertex v in the vertex set V_i (and all edges) of G_i by (v, i) and speak of (v, i) $(V(G_i) \times \{i\})$ as v in G_i $(V(G_i))$.

Let G and H be graphs. The *disjoint union* of G and H is denoted by $G \mathbin{\dot\cup} H$ and is defined as the graph with vertex set $V(G) \mathbin{\dot\cup} V(H)$ and edge set $E(G) \mathbin{\dot\cup} E(H)$.

The *complement* G^c of G is defines as the graph $(V, \binom{V}{2} \setminus E)$.

If $E = \binom{V}{2}$ then G is called *complete*, a *clique* and to be a K_n with $n = |G|$. The graph G is called a *path* if up to relabelling of vertices $V = \{v_0, \ldots, v_{n-1}\}$ and $E = \{v_i v_{i+1} : 0 \leq i < n\}$. Graph G is then denoted by P_n and its *ends* are v_0 and v_{n-1}. In a graph G we say two vertices are *connected by a path* if they are the ends of a subgraph of G that is a path. The *length* of P_n is $n - 1$ for $n \in \mathbb{N}$ and denoted by $||P_n||$. A cycle C_n is a graph received from a path P_{n+1} with $n \geq 3$ by identifying its ends. Note that a cycle is 2-regular. A graph is called *forest* if it has no cycle as subgraph.

We say G is *connected* if any two vertices of G are connected by a path. If G' is an inclusion-maximal connected subgraph of G, then it is called a *(connected) component* of G. Connected forests are called *trees*. A tree with one vertex adjacent to all other vertices is called a *star*. A star with t leaves is denoted as S_t.

The *line graph* $\mathrm{L}(G)$ of G is defined as the graph (E, F) with edge set

$$F = \{\{e_1, e_2\} : e_1, e_2 \in E, e_1 \text{ and } e_2 \text{ are adjacent in } G\}.$$

A *planar graph* is a graph G with an embedding into the plane. An *embedding into the plane* is an injective mapping from the vertices of G to elements of $\mathbb{R}^2$ and a mapping from the edges of G to Jordan curves such that the ends of the Jordan curves are the images of the ends of the corresponding edges and the Jordan curves do neither intersect otherwise nor contain images of vertices otherwise.

Let G and H be graphs. A function $\phi\colon V(G) \to V(H)$ is called a *graph homomorphism* if $vu \in E(G)$ implies $\phi(v)\phi(u) \in E(H)$. We sometimes write in short $\phi\colon G \to H$. Note that two adjacent vertices $u, v \in G$ must not be mapped onto the same vertex $x \in H$, since this would imply an edge from x to itself, which is not possible in graphs in this thesis. If for every $xy \in E(H)$ there is an edge $vu \in E(G)$ with $\phi(v) = x$ and $\phi(u) = y$, then ϕ is called *edge-surjective*. If this edge $vu \in E(G)$ is unique for every edge $xy \in E(H)$, then ϕ is called *edge-bijective*. Let $G' \subseteq G$. We write $\phi(G')$ for the graph $(\phi(V(G')), F)$ where $F = \{\phi(e) : e \in E(G')\}$.

A bijective, edge-bijective homomorphism is called an *isomorphism*. A graph G is called *isomorphic* to a graph H if there is an isomorphism $\phi : G \to H$. In this case we write $G \simeq H$. Note that $\simeq$ is an equivalence relation.

In this thesis we assume all graph classes to be closed under taking isomorphic graphs.

2.1 Hypergraphs

A *hypergraph* is a tuple (V, E) where V is a finite set and E is a set of subsets of V. A hypergraph (V, E) is called *d-uniform*, if every set in E has exactly d elements.

2.2 Hyper Ramsey Theory

An a-hypergraph G is a tuple (V, E) where V is a set of vertices and $E \subseteq \binom{V}{a}$. By K_n^a we denote a complete a-hypergraph, i.e., a graph (V, E) where $|V| = n$ and $E = \binom{V}{a}$. For $a, k, c \in \mathbb{N}$ the *hypergraph Ramsey number* $R(a, k, c)$ is the smallest number n such that every proper colouring of K_n^a using c colours induces a monochromatic subgraph K_k^a. The hypergraph Ramsey Theorem ensures that $R(a, k, c)$ exists [Ram30].

2.3 Matroids

A *matroid* is a tuple $(E, \mathcal{I})$ where E is a finite set and $\mathcal{I}$ is a set of *independent sets* that are subsets of E with the following properties:

1. $\emptyset \in \mathcal{I}$.
2. $\forall I \in \mathcal{I} : \forall J \subseteq I : J \in \mathcal{I}$. (*Hereditary Property*)
3. $A, B \in \mathcal{I} \land |B| \leq |A| \Rightarrow \exists x \in A \setminus B : B \cup \{x\} \in \mathcal{I}$. (*Augmentation Property*)

Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid. A subsets $J \subseteq E$ that is not an independent sets is called *dependent*. A minimal dependent subset is called *circuit*.

A maximal independent set of $\mathcal{M}$ is called a *base*. All bases of $\mathcal{M}$ have the same number of elements called the *rank* of $\mathcal{M}$. For $F \subseteq E$ the tuple $(F, \mathcal{J})$ with $\mathcal{J} = \{J \in I \mid J \subseteq F\}$ is a matroid called *submatroid* of $\mathcal{M}$. The *rank* of set $F \subseteq E$ is the rank of the submatroid induced by F.

2.4 Directed Graphs

A *directed graph*, short *digraph*, is a tuple $D = (V, E)$ where V is a finite (vertex) set and for the set of edges we have $E \subseteq \{(u, v) \in V^2 \mid u \neq v\}$. We write in short uv for the edge (u, v). An edge $uv \in E$ is said to *start* in u and *end* in v. We denote the corresponding *undirected graph* with $G(D)$, i.e., we define $G(D) = (V, \{\{u, v\} \in \binom{V}{2} \mid uv \in E\})$. For some purposes, we identify a digraph $D = (V, E)$ with graph $G(D)$. However, a digraph $D = (V, E)$ is only called *subgraph of a digraph* $H = (W, F)$, if $V \subseteq W$ and $E \subseteq F$. For a digraph $D = (V, E)$ and a vertex $v \in V$ we define the in-Neighbourhood $\mathrm{N}^-{}_D(v) = \{u \mid uv \in E\}$ and the out-Neighbourhood $\mathrm{N}^+{}_D(v) = \{u \mid vu \in E\}$. Correspondingly we define the *indegree* $\deg_D^-(v) = |\mathrm{N}^-{}_D(v)|$ and the *outdegree* $\deg^+(v)_D(v) = |\mathrm{N}^+{}_D(v)|$. Further we define for $U \subseteq V$ indegree $\deg^-(U) = \sum_{u \in U} \deg^-(u)$ and outdegree $\deg^+(U) = \sum_{u \in U} \deg^+(u)$. Finally we define for any directed graph G indegree $\deg_D^-(G) = |\{uv \in E \mid v \in G \text{ and } u \notin G\}|$ and outdegree $\deg_D^+(G) = |\{uv \in E \mid v \notin G \text{ and } u \in G\}|$.

2.5 Graph Classes

Let $\mathcal{G}$ be a graph class. We call $\mathcal{G}$ *induced-hereditary* if it is closed under taking induced subgraphs. We call $\mathcal{G}$ *subgraph-hereditary* if it is even closed under taking subgraphs. We call $\mathcal{G}$ *union-closed* if it is closed under taking disjoint unions. We denote the *closure under taking disjoint unions* (or shorter *union-closure*) of $\mathcal{G}$ by $\overline{\mathcal{G}}$. It contains every disjoint union of a finite number of graphs in $\mathcal{G}$. Correspondingly, the *induced-hereditary closure* of $\mathcal{G}$

is denoted by $\hat{\mathcal{G}}$ and contains for every $G \in \mathcal{G}$ every subgraph $G' \subseteq G$. Graph class $\mathcal{G}$ is called *minor-closed*, if it is subgraph-hereditary and closed under identifying adjacent vertices. Graph class $\mathcal{G}$ is called *topological-minor-closed*, if it is subgraph-hereditary and closed under *smoothing* where smoothing is the operation of deleting a vertex of degree 2 and replacing its two incident edges by an edge between its two neighbours. Let G be a graph. We write $G \in_e \mathcal{G}$, if there is a graph $H \in \mathcal{G}$ such that $E(H) = E(G)$ and $V(G) \subseteq V(H)$. I.e. by removing isolated vertices of G one obtains a graph in $\mathcal{G}$. This will be helpful as we usually focus on edges. Similarly, we write for another graph class $\mathcal{H}$ that $\mathcal{G} \subseteq_e \mathcal{H}$, if for every graph $G \in \mathcal{G}$ we have $G \in_e \mathcal{H}$. Finally we usually overload functions with the domain of graphs to have also the domain of graph classes and mapping to the supremum over all graphs in that graph class. E.g., we define for a graph class $\mathcal{G}$ that $\chi(\mathcal{G}) = \sup\{\chi(H) \mid H \in \mathcal{H}\}$.

2.6 Easier Notation

Let G be a graph. We sometimes write $v \in G$ instead of $v \in V(G)$ and $e \in G$ instead of $e \in E(G)$ if it is clear that v is a vertex and e is an edge. In several cases we specify that something is defined with regards to a certain graph, e.g. the reach of U in graph G_i may be noted as $\mathrm{Reach}_{G_i}(U)$. To avoid stacked indices in such cases we instead directly use the index of that graph notation, e.g. we may write $\mathrm{Reach}_i(U)$ instead. Let S be a set. Then we write for example $\bigcup S$ in short for $\bigcup_{G \in S} G$.

3. Covering Numbers

Knauer and Ueckerdt introduced the framework of the folded, the local and the global covering number [KU16]. Let H be a graph and $\mathcal{G}$ be a graph class. In general a covering number measures how good all edges of H can be covered by graphs in $\mathcal{G}$.

To define covering numbers, we define covers which are defined in terms of graph homomorphisms.

Let G and H be graphs and let $\phi\colon V(G) \to V(H)$ be a homomorphism. Let $uv \in E(H)$. We say vertex $v \in V(H)$ is *covered* by G (with regards to ϕ), if we have $v \in \phi(V(G))$. We say edge uv is *covered* by G (with regards to ϕ), if we have $\{\phi^{-1}(u), \phi^{-1}(v)\} \in E(G)$. We call ϕ *edge-surjective*, if every edge in $E(H)$ is covered by G with regards to ϕ.

Let H be a graph and $\mathcal{G}$ be a graph class. A *$\mathcal{G}$-cover* of H is a tuple (S, ϕ) of a finite multiset $S = \{G_1, \dots, G_m\} \subseteq \mathcal{G}$ and an edge-surjective graph homomorphism $\phi\colon V(G_1 \mathbin{\dot\cup} \dots \mathbin{\dot\cup} G_m) \to V(H)$. Note that as graph homomorphism ϕ can not map two adjacent vertices to the same vertex in H. See Figure 3.1 for examples. The graph H is called *host graph*, the class $\mathcal{G}$ is called *guest class*, the graphs in $\mathcal{G}$ are called *guest graphs* and the graphs in S are called *guests*. We call $\mathcal{G}$-cover (S, ϕ) *guest-injective*, if we have for $G \in S$ that $\phi|_{V(G)}$ is injective. In that case we usually assume for $G \in S$ that $G \subseteq H$ and for $v \in G$ we have $\phi(v) = v$. This implies $\bigcup S = H$.

We call (S, ϕ) a *k-folded* $\mathcal{G}$-cover of H, if we have for $v \in V(H)$ that $\phi^{-1}(v) \leq k$. We call (S, ϕ) a *k-local* $\mathcal{G}$-cover of H, if it is guest-injective and we have for $v \in V(H)$ that $\phi^{-1}(v) \leq k$. We call (S, ϕ) a *k-global* $\mathcal{G}$-cover of H, if it is guest-injective and we have $|S| \leq k$.

Let H be a host graph and $\mathcal{G}$ be a guest class. The folded ($c_f^{\mathcal{G}}$), local ($c_l^{\mathcal{G}}$), and global ($c_g^{\mathcal{G}}$) covering number of H with regards to $\mathcal{G}$ are defined as follows, respectively.

$$
\begin{aligned}
c_f^{\mathcal{G}}(H) &= \min\{k \in \mathbb{N}_0 \mid \exists k\text{-folded } \mathcal{G}\text{-cover of } H\} \\
c_l^{\mathcal{G}}(H) &= \min\{k \in \mathbb{N}_0 \mid \exists k\text{-local } \mathcal{G}\text{-cover of } H\} \\
c_g^{\mathcal{G}}(H) &= \min\{k \in \mathbb{N}_0 \mid \exists k\text{-global } \mathcal{G}\text{-cover of } H\}
\end{aligned}
$$

Note that in the original definition of Knauer and Ueckerdt a cover (S, ϕ) was only a folded cover, if one had $|S| = 1$. However, they considered only union-closed guest classes. If the guest class $\mathcal{G}$ is union-closed and we have a k-folded $\mathcal{G}$-cover (S, ϕ) of H, then we obtain

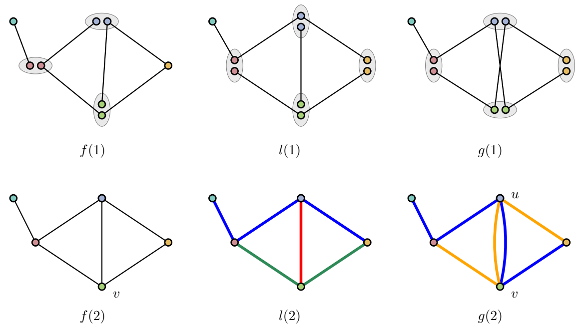

Figure 3.1: A host graph H is presented in $f(2)$. We consider the guest class of paths denoted as $\mathcal{G}$. The three columns represents from the left to the right three $\mathcal{G}$-covers (S_1, ϕ_1), (S_2, ϕ_2) and (S_3, ϕ_3). In the top row the corresponding guests are presented. For $i = 1, 2, 3$ we indicate that ϕ_i maps multiple vertices to the same vertex of H by enclosing them by an ellipse. Only non-adjacent vertices may be mapped to the same vertex. Cover (S_1, ϕ_1) is a 2-folded $\mathcal{G}$-cover of H. Note that at most two vertices are mapped to the same guest. Further note that the two vertices that are mapped to v are from the same guest in S_1. Therefore (S_1, ϕ_1) is not guest-injective. Cover (S_2, ϕ_2) is a 2-local 3-global $\mathcal{G}$-cover of H. First note that no two vertices of the same guest are mapped to the same vertex. Thus cover (S_2, ϕ_2) is guest-injective and we can represent H as union of its guests as done in $l(2)$. Since it has 3 guests, we obtain that (S_2, ϕ_2) is a 3-global $\mathcal{G}$-cover. We note that ϕ_2 maps at most two vertices to the same vertex of H. Thus (S_2, ϕ_2) is a 2-local $\mathcal{G}$-cover. Cover (S_3, ϕ_3) is by analogous argumentation a 2-global $\mathcal{G}$-cover of H. By Proposition 3.1 it is thus also a 2-local and 2-folded $\mathcal{G}$ cover. Note that edge uv is covered by two guests. The union of the guests is represented in $g(2)$. Multiple edges covering the same edge are represented by a multi-colouring of the covered edge (here presented as multi-edge). Since H is not a union of paths, we obtain by Proposition 3.5 that the folded $\mathcal{G}$-covering number of H is at least 2. By the existence of a 2-global $\mathcal{G}$-cover we obtain $c_g^{\mathcal{G}} \leq 2$. With Proposition 3.1 this implies $c_f^{\mathcal{G}}(H) = c_l^{\mathcal{G}}(H) = c_g^{\mathcal{G}}(H) = 2$.

$G = \bigcup S \in \mathcal{G}$ and therefore tuple $(\{G\}, \phi)$ is a k-folded $\mathcal{G}$-cover of H with $|\{G\}| = 1$. In this sense that requirement had no effect. We dismiss it, to achieve that also for a not union-closed guest class $\mathcal{G}$ a k-local cover is also a k-folded cover. Let H be a host graph and $\mathcal{G}$ be a guest class. Note that in a guest-injective cover (S, ϕ) of H every vertex $v \in H$ is covered by every guest at most once. Therefore every k-local cover is also a k-global cover.

Knauer and Ueckerdt proved these and other findings in their paper [KU16].

Proposition 3.1 (Knauer, Ueckerdt [KU16])
Let $\mathcal{G}$ and $\mathcal{G}'$ be guest classes, let $\mathcal{H}$ and $\mathcal{H}'$ be host classes and let H be a host graph. Let $k \in \mathbb{N}_0$. Then each of the following holds:

(i) For H any k-global $\mathcal{G}$-cover is also a k-local $\mathcal{G}$-cover. Especially $c_l^{\mathcal{G}}(H) \leq c_g^{\mathcal{G}}(H)$.

(ii) For H any k-local $\mathcal{G}$-cover is also a k-folded $\mathcal{G}$-cover. Especially $c_f^{\mathcal{G}}(H) \leq c_l^{\mathcal{G}}(H)$.

(iii) If $\mathcal{H} \subseteq \mathcal{H}'$, then $c_i^{\mathcal{G}}(\mathcal{H}) \leq c_i^{\mathcal{G}}(\mathcal{H}')$ for $i = f, l, g$.

(iv) If $\mathcal{G} \subseteq \mathcal{G}'$, then $c_i^{\mathcal{G}'}(\mathcal{H}) \leq c_i^{\mathcal{G}}(\mathcal{H})$ for $i = f, l, g$.

3.1 The Union Covering Number

When Knauer and Ueckerdt introduced the notions of the folded, the local and the global covering number, they considered only union-closed guest classes, i.e., classes closed under taking vertex-disjoint unions [KU16]. However, in some cases it is natural to consider non-union-closed guest classes. For example, the vertex cover number is equal to the global covering number with regards to stars (the centres of the stars correspond to the vertices of the vertex cover). Another example is the intersection number which is equal to the global covering number with regards to complete graphs [Rob85].

A third example is the *boxicity*. The boxicity of a graph H is the smallest number d such that H is an intersection graph of d-dimensional axis-aligned boxes in $\mathbb{R}^d$ and denoted by $\operatorname{box}(H)$. Interestingly, we have $\operatorname{box}(H) = c_g^{\mathcal{I}^c}(H^c)$ where $\mathcal{I}^c$ denotes the class of cointerval graphs (i.e., complements of interval graphs) [CR83]. Bläsius, Stumpf and Ueckerdt considered the *local-boxicity* and *union-boxicity* as the corresponding covering numbers $c_l^{\mathcal{I}^c}(H^c)$ and $c_g^{\overline{\mathcal{I}^c}}(H^c)$ where $\overline{\mathcal{I}^c}$ denotes the class of vertex-disjoint unions of cointerval graphs [BSU16].

Those variants of boxicity have their own geometric interpretations. Thus, both considerations - of union-closures as guest classes as well as of non-union-closed guest classes for the global covering number - are legitimated. We hence introduce the notion of *union covering numbers* as global covering numbers with regards to union-closed guest classes in general. However, we first note that for the folded and local covering number it does not matter whether the guest class is union-closed. This means only for the global covering number a difference occurs.

Proposition 3.2
Let H be a host graph and $\mathcal{G}$ be a guest class. Let $\overline{\mathcal{G}}$ denote the closure under taking vertex-disjoint unions of $\mathcal{G}$. Then every k-(folded/local) $\overline{\mathcal{G}}$-cover of H induces a k-(folded/local) $\mathcal{G}$-cover of H. For $i = f, l$ we have especially $c_i^{\mathcal{G}}(H) = c_i^{\overline{\mathcal{G}}}(H)$.

Proof. By Proposition 3.1(iv) we know $c_i^{\overline{\mathcal{G}}}(H) \leq c_i^{\mathcal{G}}(H)$. Now consider a k-(folded/local) $\overline{\mathcal{G}}$-cover (S, ϕ) of H with $S = \{G_1, \ldots, G_m\}$ as set of guests and $k = \max_{v \in V(H)}(\phi^{-1}(v))$. For every $j = 1, \ldots, m$, guest G_j is the disjoint union of some graphs $G'_{j,1}, \ldots, G'_{j,r(j)}$ in $\mathcal{G}$. We set $S' = \{G'_{1,1}, \ldots, G'_{1,r(1)}, \ldots, G'_{m,1}, \ldots, G'_{m,r(m)}\}$. By definition of the guests in S', we have $\dot\bigcup S' = \dot\bigcup S$. Hence, (S', ϕ) is a k-folded $\mathcal{G}$-cover of H.

If the cover (S, ϕ) is a local cover, it is guest-injective. Then the guests

$$G'_{1,1}, \ldots, G'_{1,r(1)}, \ldots, G'_{m,1}, \ldots, G'_{m,r(m)} \in S'$$

are subgraphs of the guests $G_1, \ldots, G_m \in S$. Thus, the restriction of ϕ on each graph $G \in S'$ is vertex-injective, since the restriction of ϕ on a certain supergraph of G is already vertex-injective. Hence, the cover (S', ϕ) is guest-injective and thus a k-local cover of H. This concludes the proof. □

Hence, we can only get a different value for the global covering number.

Definition 3.3
Let $\mathcal{G}$ be a guest class and let H be a host graph. A guest-injective $\mathcal{G}$-cover (S, ϕ) of H is a k-union $\mathcal{G}$-cover of H, *if S can be partitioned into k sets $S_1, \ldots, S_k$ such that $\phi|_{\dot\bigcup S_i}$ is vertex-injective for $i = 1, \ldots, k$. We call those partition sets* guest-unions. *See Figure 3.2 for an example. We define the corresponding* union $\mathcal{G}$-covering number of H *as*

$$c_u^{\mathcal{G}}(H) = \min\{k \in \mathbb{N}_0 \mid \textit{there is a } k\textit{-union } \mathcal{G}\textit{-cover of } H\}.$$

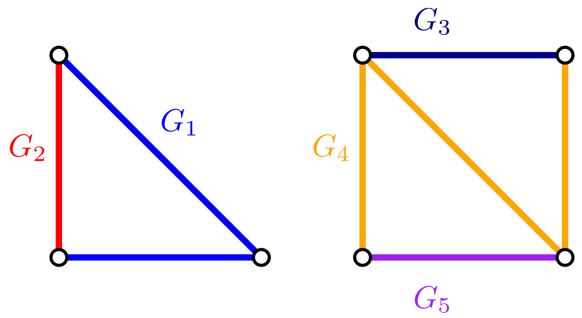

Figure 3.2: A host graph H with two components. Let $\mathcal{G}$ denote the class of paths. We see a 5-global $\mathcal{G}$-cover (S, ϕ) of H. We define the sets $S_1 = \{G_1, G_3, G_5\}$ and $S_2 = \{G_2, G_4\}$ and observe that for $i = 1, 2$ the elements of S_i are pairwise vertex-disjoint. Thus S_1 and S_2 are guest-unions and (S, ϕ) is a 2-union $\mathcal{G}$ cover of H. We further note that for $S' = \{\bigcup S_1, \bigcup S_2\}$ the tuple (S', ϕ) is a 2-global $\overline{\mathcal{G}}$-cover of H.

Note that, if $\overline{\mathcal{G}}$ denotes the closure of $\mathcal{G}$ under taking vertex-disjoint unions, then the sets $S_1, \ldots, S_k$ correspond to k graphs $\overline{G_1}, \ldots, \overline{G_k}$ in $\overline{\mathcal{G}}$.

The following proposition shows that the union covering number fits neatly between the local and the global covering number and provides a characterization in terms of the global covering number.

Proposition 3.4
Let $\mathcal{G}$ be a guest class, let $\overline{\mathcal{G}}$ be its closure under taking vertex-disjoint unions and let H be a host graph. Let $k \in \mathbb{N}_0$. Then each of the following holds:

(i) Any k-global $\mathcal{G}$-cover is also a k-union $\mathcal{G}$-cover. Especially we have $c_u^{\mathcal{G}}(H) \leq c_g^{\mathcal{G}}(H)$.

(ii) Any k-union $\mathcal{G}$-cover is also a k-local $\mathcal{G}$-cover. Especially we have $c_l^{\mathcal{G}}(H) \leq c_u^{\mathcal{G}}(H)$.

(iii) Any k-union $\mathcal{G}$-cover corresponds to a k-global $\overline{\mathcal{G}}$-cover and vice versa. Especially we have $c_u^{\mathcal{G}}(H) = c_g^{\overline{\mathcal{G}}}(H)$.

(iv) If $\mathcal{G} = \overline{\mathcal{G}}$, then we have $c_u^{\mathcal{G}}(H) = c_g^{\mathcal{G}}(H)$.

(v) Let $\mathcal{H}$ be a host class and let $\overline{\mathcal{H}}$ be its closure under taking vertex-disjoint unions. Then we have $c_u^{\mathcal{G}}(\mathcal{H}) = c_u^{\mathcal{G}}(\overline{\mathcal{H}})$.

Proof. For any k-global $\mathcal{G}$-cover (S, ϕ) with $S = \{G_1, \ldots, G_k\}$, multiset S can be partitioned into k sets $\{G_1\}, \ldots, \{G_k\}$. The elements of each of them are pairwise disjoint, since they contain only one graph each. The cover (S, ϕ) is thereby also a k-union $\overline{\mathcal{G}}$-cover. This implies $c_u^{\mathcal{G}}(H) \leq c_g^{\mathcal{G}}(H)$.

Let (S, ϕ) be a k-union $\mathcal{G}$-cover. Then S can be partitioned into k sets $S_1, \ldots, S_k$ of pairwise disjoint graphs. Hence, every vertex v in H is covered by at most one guest of each set S_i for $i = 1, \ldots, k$. This implies (S, ϕ) is a k-local $\mathcal{G}$-cover. Hence, we have $c_l^{\mathcal{G}}(H) \leq c_u^{\mathcal{G}}(H)$.

Let (S, ϕ) be a k-union $\mathcal{G}$-cover of H. Then S can be partitioned into k sets $S_1, \ldots, S_k$ of pairwise disjoint graphs. We obtain a k-global $\overline{\mathcal{G}}$-cover of H by using mapping ϕ and considering the new guests $\overline{G_1}, \ldots, \overline{G_k}$, where $\overline{G_i}$ is the disjoint union of the graphs in S_i for $i = 1, \ldots, k$. On the other hand, let (S', ϕ') be a k-global $\overline{\mathcal{G}}$-cover of H where $S' = \{G'_1, \ldots, G'_k\}$. Then for $i = 1, \ldots, k$, guest G'_i is the disjoint union of some graphs $G'_{i,1}, \ldots, G'_{i,m(i)}$ in $\mathcal{G}$ by definition of $\overline{\mathcal{G}}$. We consider $S'' = \{G'_{i,1}, \ldots, G'_{i,m(i)}, \ldots, G'_{k,1}, \ldots, G'_{k,m(k)}\}$ as new multiset of guests together with the same mapping ϕ' as new cover (S'', ϕ') of H. The sets

$S'_i = \{G'_{i,1}, \dots, G'_{i,m(i)}\}$ for $i = 1, \dots, k$ provide a partition of S'' proving (S'', ϕ') is a k-union $\mathcal{G}$-cover. This implies $c_u^{\mathcal{G}}(H) = c_g^{\overline{\mathcal{G}}}(H)$.

If $\mathcal{G} = \overline{\mathcal{G}}$, then we have $c_u^{\mathcal{G}}(H) = c_g^{\overline{\mathcal{G}}}(H) = c_g^{\mathcal{G}}(H)$ by Item (iii).

Let $\mathcal{H}$ be a host class and $\overline{\mathcal{H}}$ be its closure under taking vertex-disjoint unions. If $c_u^{\mathcal{G}}(\mathcal{H}) = \infty$ then we have by Item (iii) and Proposition 3.1(iii) that $c_u^{\mathcal{G}}(\mathcal{H}) = \infty = c_u^{\mathcal{G}}(\overline{\mathcal{H}})$. Hence, let $c_u^{\mathcal{G}}(\mathcal{H}) = k < \infty$. Further, let $H = H_1 \mathbin{\dot\cup} \dots \mathbin{\dot\cup} H_m \in \overline{\mathcal{H}}$. Then for $i = 1, \dots, m$ the host graph H_i has a k-union $\mathcal{G}$-cover (S_i, ϕ_i) with k guest-unions $S_{i,1}, \dots, S_{i,k}$. We consider the guest-injective $\mathcal{G}$-cover (S, ϕ) of H with $S = S_1 \mathbin{\dot\cup} \dots \mathbin{\dot\cup} S_m$ and $\phi = \phi_1 \mathbin{\dot\cup} \dots \mathbin{\dot\cup} \phi_m$, i.e., homomorphism $\phi\colon \dot\bigcup S \to H$ is defined by $\phi|_{\dot\bigcup S_i} = \phi_i$ for $i = 1, \dots, m$. The set S can be partitioned into k guest-unions $S = S^1 \mathbin{\dot\cup} \dots \mathbin{\dot\cup} S^k$ by setting $S^i = S_{1,i} \mathbin{\dot\cup} \dots \mathbin{\dot\cup} S_{m,i}$ for $i = 1, \dots, k$. Hence, it is a k-union cover which proves $c_u^{\mathcal{G}}(\mathcal{H}) = c_u^{\mathcal{G}}(\overline{\mathcal{H}})$. □

For a guest class $\mathcal{G}$ and a host graph H we can thus write in short

$$c_f^{\mathcal{G}}(H) \le c_l^{\mathcal{G}}(H) \le c_u^{\mathcal{G}}(H) \le c_g^{\mathcal{G}}(H).$$

The union covering number fits thereby neatly into the framework of the other three covering numbers.

3.2 Basic Results

If a host graph H has a covering number of 1 with regards to a guest class $\mathcal{G}$ this roughly means it must be covered by itself and is thus contained in $\mathcal{G}$. We give a precise characterization in Proposition 3.5. We use the notions of $\subseteq_e$ and $\in_e$ in terms of graph classes and graphs: Those are roughly speaking the relations $\subseteq$ and $\in$, but ignoring independent vertices. More precisely, we write $\mathcal{G} \subseteq_e \mathcal{H}$ if for every graph $G \in \mathcal{G}$ we have $G \in_e \mathcal{H}$, i.e., for every graph $G \in \mathcal{G}$ there is a graph $H \in \mathcal{H}$ with $H = (V(G) \mathbin{\dot\cup} W, E(G))$ for some vertex set W.

Proposition 3.5
Let $\mathcal{G}$ be a guest class. Let $\overline{\mathcal{G}}$ be its union-closure and H be a host graph. Let $i = f, l, u, g$. We have:

(i) $c_i^{\mathcal{G}}(H) = 0 \Leftrightarrow$ *host graph H is an independent set.*

(ii) For $i = f, l, u, g$, we have $c_i^{\mathcal{G}}(H) \le 1 \Leftarrow H \in_e \mathcal{G}$.

(iii) $c_g^{\mathcal{G}}(H) \le 1 \Leftrightarrow H \in_e \mathcal{G}$.

(iv) For $i = f, l, u$, we have $c_i^{\mathcal{G}}(H) \le 1 \Leftrightarrow H \in_e \overline{\mathcal{G}}$.

Proof. If $c_i^{\mathcal{G}}(H) = 0$, then host graph H is covered without using a guest. Therefore, no edge is covered meaning H contains no edge. On the other hand, if no edge is contained in H, its (empty) set of edges is covered without using a single guest.

If $H \in_e \mathcal{G}$, then host graph H can be covered using H itself (after removing corresponding isolated vertices) as the only guest. Hence, we have $c_i^{\mathcal{G}}(H) = 1$.

If on the other hand, we have $c_g^{\mathcal{G}}(H) \le 1$, then host graph H can be covered using a single guest G. This means G contains all edges of H and may differ only by missing isolated vertices. We thus have $H \in_e \mathcal{G}$.

If $c_i^{\mathcal{G}}(H) \le 1$ for $i = f, l, u$, then host graph H can be covered such that every vertex is covered at most once. All guests are pairwise vertex-disjoint. Their union G is thus contained in $\overline{\mathcal{G}}$, while G also contains all edges of H. Analogously to the previous case, we get $H \in_e \overline{\mathcal{G}}$. □

Note that for a guest class $\mathcal{G}$ containing the trivial graph $T = (\{v\}, \emptyset)$ and a host graph H we have $H \in_e \overline{\mathcal{G}} \Rightarrow H \in \overline{\mathcal{G}}$, since missing isolated vertices can be added by disjoint union with T.

Let $\mathcal{G}$ be a guest class and let (S, ϕ) be a $\mathcal{G}$-cover of a host graph H. Then (S, ϕ) covers all edges of H and thereby also all edges of any subgraph $H' \subseteq H$. If $\mathcal{G}$ allows the guests to be restricted accordingly, we obtain a cover of H'.

Remember that for two graphs G and H we write $G \sqsubseteq H$ to describe G is an induced subgraph of H. Further note that for $U \subseteq V(H)$ we write $\langle U \rangle_H$ for the subgraph induced by U in graph H. We sometimes replace a graph in an index by its own index to avoid index stacking.

Proposition 3.6
Let $\mathcal{G}$ be a guest class and let H and H' be host graphs.

- *If $\mathcal{G}$ is induced-hereditary and $H' \sqsubseteq H$, then $c_i^{\mathcal{G}}(H') \leq c_i^{\mathcal{G}}(H)$ for $i = f, l, u, g$.*
- *If $\mathcal{G}$ is subgraph-hereditary and $H' \subseteq H$, then $c_i^{\mathcal{G}}(H') \leq c_i^{\mathcal{G}}(H)$ for $i = f, l, u, g$.*

Proof. Let $\mathcal{G}$ be induced-hereditary and Let $H' \sqsubseteq H$. Further, let tuple (S, ϕ) be a k-(folded/local/union/global) cover of H with $S = \{G_1, \ldots, G_m\}$. We obtain a k-(folded/local/union/global) cover of H' as follows. For $1 \leq i \leq m$ let G_i' denote the graph gained by removing all vertices of G_i that are not mapped to H'. I.e., let $G_i' = \langle \{v \in G_i \mid \phi(v) \in H'\} \rangle_i$. Since $G_1', \ldots, G_m'$ are induced subgraphs of $G_1 \ldots, G_m$, we have $G_1', \ldots, G_m' \in \mathcal{G}$. Let $S' = G_1', \ldots, G_m'$. The restriction $\phi' = \phi_{G_1' \cup \ldots \cup G_m'}$ covers all edges of H', since only edges to vertices in $H \setminus H'$ are removed. Thus, we have with (S', ϕ') a cover of H' which is also k-(folded/local/union/global).

The case where $\mathcal{G}$ is subgraph-hereditary can be argued analogously using the guests $G_i'' = (V_i, E_i)$ with $V_i = \{v \in G_i \mid \phi(v) \in H'\}$ and $E_i = \{vu \in G_i \mid \phi(v)\phi(u) \in H'\}$ for $1 \leq i \leq m$ instead. ☐

4. Separability and Non-Separability

Remember we have for a host graph H and a guest class $\mathcal{G}$ the inequalities

$$c_f^{\mathcal{G}}(H) \leq c_l^{\mathcal{G}}(H) \leq c_u^{\mathcal{G}}(H) \leq c_g^{\mathcal{G}}(H).$$

This rises the question, whether two different covering numbers can differ arbitrarily. In the strongest case there is for $i, j \in \{f, l, u, g\}$ a host class $\mathcal{H}$ and a guest class $\mathcal{G}$ with $c_j^{\mathcal{G}}(\mathcal{H}) = \infty$, while $c_i^{\mathcal{G}}(\mathcal{H}) \leq 2$. In this case we speak of a *separation* of the covering numbers c_i and c_j.

The separation of the union and the global covering number is quite easy. In Proposition 4.1 we make use of a non-union-closed guest class, in a way that is in most cases possible. Note that by Proposition 3.4 such a separation with a union-closed guest class is not possible.

Proposition 4.1
For the guest class $\{K_2\}$ and the host class $\mathcal{M}$ of all matchings, we have $c_u^{\{K_2\}}(\mathcal{M}) = 1$ and $c_g^{\{K_2\}}(\mathcal{M}) = \infty$.

Proof. Every matching $M \in \mathcal{M}$ can be covered using another K_2 for every edge. Since those guests are pairwise disjoint, we get $c_l^{\{K_2\}}(\mathcal{M}) = 1$. Actually, we must cover every edge of M using another K_2 resulting in $c_g^{\{K_2\}}(M) = ||M||$. Hence, we get $c_u^{\{K_2\}}(\mathcal{M}) = 1$ and $c_g^{\{K_2\}}(\mathcal{M}) = \infty$. □

Since this separation is easy, we are less interested in the global covering number.

In the paper that introduced the investigated framework of covering numbers, Knauer and Ueckerdt presented a separation of the local- and the union-covering number with an induced-hereditary guest class.

Theorem 4.2 (Knauer and Ueckerdt [KU16])
For the guest class $\mathcal{K}$ of complete graphs and the host class $\mathcal{L}$ of line graphs, we have $c_u^{\mathcal{K}}(\mathcal{L}) = \infty$ and $c_l^{\mathcal{K}}(\mathcal{L}) = 2$.

Stumpf provided in his Bachelor‘s Thesis a separation of the folded- and the local-covering number with a guest class that is even subgraph-hereditary.

Theorem 4.3 (Stumpf [Stu15])
For the guest class $\mathcal{B}ip$ of bipartite graphs and the host class $\mathcal{H}$ of all graphs, we have $c_l^{\mathcal{B}ip}(\mathcal{H}) = \infty$ and $c_f^{\mathcal{B}ip}(\mathcal{H}) = 2$.

He further showed a separation is not possible for stronger restrictions of the guest class.

Theorem 4.4 (Stumpf [Stu15])
Let $\mathcal{G}$ be a topological minor-closed class of graphs. Let $\mathcal{H}$ be a class of graphs such that $c_f^{\mathcal{G}}(\mathcal{H}) = c < \infty$. Then there is a constant $d = d(\mathcal{G}, c)$ with $c_u^{\mathcal{G}}(\mathcal{H}) \leq d$.

However, this left open the question, whether there is a separation of local- and union-covering number with a subgraph-hereditary guest class. We answer this question positively in Section 4.1. A result of non-separability is provided in Section 4.2. In Section 4.3 we give an example of a family of guest classes for which folded, local and global covering number even coincide.

4.1 Separation of Local- and Union-Covering Number with Regards to a Subgraph-Hereditary Guest Class

Let $\mathcal{B}ip$ denote the class of bipartite graphs. Let $\mathcal{C}$-$\mathcal{B}ip$ denote the class of complete bipartite graphs.

Ueckerdt raised the following two questions in the CGI Workshop [Uec15].

Question 4.5 (Ueckerdt [Uec15])
Is there a function ϕ such that for every host graph H we have $\chi(H) \leq \phi(c_f^{\mathcal{C}\text{-}\mathcal{B}ip}(H))$?

Question 4.6 (Ueckerdt [Uec15])
Are local and global $\mathcal{C}$-$\mathcal{B}ip$-covering number for every host graph H the same?

The answer to both questions is no, as we show in this section by consideration of the host class of *shift graphs*. Shift graphs were introduces by Erdős and Hajnal [EH66]. As in line graphs, vertices are edges of a given graph G and they are only adjacent if they are adjacent in G. However, for shift graphs the graph G is directed and the edges additionally need to be adjacent with different ends, i.e., an edge uv is adjacent to vw_1 and w_2u but not to uw_3 or w_4v (see Figure 4.1). I.e.: Let $G = (V, E)$ be a directed graph. The shift graph $\mathrm{S}(G)$ of G is the graph (E, F) where $F = \{((u,v),(v,w)) \in \binom{E}{2} \mid (u,v),(v,w) \in E\}$. We denote the class of all shift graphs by $\mathcal{S}$.

Note that for every vertex $v \in G$ the incident edges in G can be divided into *incoming* and *outgoing* edges by considering all edges directed towards the larger vertex. Those edges induce a complete bipartite graph $B_v(G)$ in $\mathrm{S}(G)$ with incoming and outgoing edges as the two partition sets. We write B_v in short if G is given. By this observation we obtain the following lemma.

Lemma 4.7
Let $G = (V, E)$ be a directed graph. Then we have $c_f^{\mathcal{C}\text{-}\mathcal{B}ip}(\mathrm{S}(G)) \leq c_l^{\mathcal{C}\text{-}\mathcal{B}ip}(\mathrm{S}(G)) \leq 2$ as well as $c_f^{\mathcal{B}ip}(\mathrm{S}(G)) \leq c_l^{\mathcal{B}ip}(\mathrm{S}(G)) \leq 2$.

Proof. It suffices to prove $c_l^{\mathcal{C}\text{-}\mathcal{B}ip}(\mathrm{S}(G)) \leq 2$. Then the remaining results follow by Proposition 3.1(ii) and (iv). For a 2-local cover, we consider for every vertex $v \in G$ the complete bipartite graph B_v as guest. This covers all edges of $\mathrm{S}(G)$ since every edge $\{uv, vw\} \in F$ is covered by B_v. And every edge $uv \in E$ is only contained in B_u and B_v. Thus every vertex $uv \in \mathrm{S}(G)$ is covered by at most two guests. Hence $c_l^{\mathcal{C}\text{-}\mathcal{B}ip}(\mathrm{S}(G)) \leq 2$ which concludes the proof. □

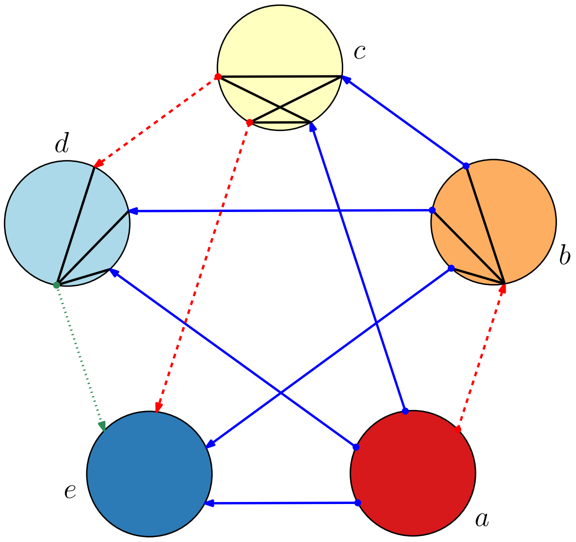

Figure 4.1: A representation of the ordered complete graph K_5^o where the circles represent the vertices and the arrows represent the edges. The vertex order is $a < b < c < d < e$. It also represents $\mathrm{S}(K_5^o)$ when the arrows represent the vertices and the other lines represent the edges. For the given colouring of $\mathrm{S}(K_5^o)$ the colours for $V(K_5^o)$ are $\emptyset$ for a, $\{red(dashed)\}$ for b, $\{blue\}$ for c, $\{red, blue\}$ for d and $\{red, blue, green(dotted)\}$ for e. Within every circle we have a complete bipartite graph B_v as guest for the local cover (in a and e those are independent sets and can be ignored). The edges ab, bc, cd, de and bd induce a cycle C_5 in $\mathrm{S}(K_5^o)$.

Shift graphs can have arbitrarily large chromatic number, as the following well known lemma shows.

Lemma 4.8 (e.g. Lovász [Lov93][Problem 9.26])
For every directed graph G we have $\chi(\mathrm{S}(G)) \geq \log(\chi(G))$.

Proof. Let $\mathrm{S}(G)$ be properly coloured with k colours. We construct a proper colouring of G using at most 2^k colours as follows. We colour every vertex v in G with a set c_v which contains exactly those colours that were used for incoming edges (see Figure 4.1). First note there are only 2^k different sets of this kind. Hence, we used at most 2^k colours. Now consider an edge uv in G. The colour c of uv is contained in c_v since it is incoming for v. However, colour c can not be contained in c_u since all incoming edges are adjacent to uv in $\mathrm{S}(G)$. Hence, vertices u and v have different colours. Therefore, the colouring of G is proper which concludes the proof. □

With Lemma 4.8, we can answer Question 4.5 negatively in Corollary 4.9.

Corollary 4.9
For every $k \geq 2$ we have $\chi(\mathrm{S}(K_{2^k+1})) > k$ and $c_f^{\mathcal{C}\text{-}\mathcal{B}ip}(\mathrm{S}(K_{2^k+1})) = c_f^{\mathcal{B}ip}(\mathrm{S}(K_{2^k+1})) = 2$.

Proof. Follows directly by Lemma 4.7, Lemma 4.8, Proposition 3.1(ii) and Proposition 3.5, since C_5 is a non-bipartite induced subgraph of $\mathrm{S}(K_5^o)$, as shown in Figure 4.1. □

To answer Question 4.6, we need to conclude from high chromatic number to high global covering number. Therefore we investigate the relationship between the chromatic number of a host graph and its guests in a global cover.

From a global cover with properly coloured guests we obtain a proper colouring of the host graph, by combining the colours of the guests. This relation between chromatic number of guests and host graph is shown in Proposition 4.10.

Proposition 4.10
Let $\mathcal{G}$ be a guest class and let H be a host graph. Let (S, ϕ) be a k-global $\mathcal{G}$-cover of H with $S = \{G_1, \ldots, G_k\}$. Then $\chi(H) \leq r^k$ where $r = \max\{\chi(G_i) \mid i = 1, \ldots, k\}$.

Proof. For $i \in [k]$, let guest G_i be properly coloured using the colours $1, \ldots, r$. We obtain a colouring c of H, by colouring every vertex $v \in H$ by colour $(c_1(v), \ldots, c_k(v))$ where for $i \in [k]$ the colour $c_i(v)$ is the colour of v in G_i (if v is not covered by G_i, then an arbitrary colour is chosen). Consider two vertices v and w of the same colour in H. By definition of colouring c, the vertices v and w have the same colour within every guest. Hence, they are not adjacent in any guest and thus not adjacent in H. Therefore c is a proper colouring. The colours are chosen from set $[r]^k$ which has r^k elements. This yields $\chi(H) \leq r^k$. □

If the guest class is not further restricted, then the reverse direction to Proposition 4.10 is also true, allowing a characterization of the global covering number as stated in Lemma 4.11. This especially provides a lower bound of the global covering number in terms of its chromatic number.

Lemma 4.11
Let $r \geq 2$. Let $\mathcal{C}_r$ be the guest class $\{G \mid \chi(G) \leq r\}$ of all graphs witch chromatic number at most r. Let H be a host graph. Then $c_g^{\mathcal{C}_r}(H) = \lceil \log_r(\chi(H)) \rceil$.

Proof. Since $\mathcal{C}_r$ contains at least K_2, there is a $c_g^{\mathcal{C}_r}(H)$-global $\mathcal{C}_r$-cover of H. By Proposition 4.10, we obtain $\chi(H) \leq r^{c_g^{\mathcal{C}_r}(H)}$. Hence, we have $\lceil \log_r(\chi(H)) \rceil \leq c_g^{\mathcal{C}_r}(H)$.

It remains to show that $\lceil \log_r(\chi(H)) \rceil$ guests suffice. For $\chi(H) = r^k$ we apply induction over k. For $k = 1$ the host graph H can be covered by itself. Now assume $k \geq 2$ and every graph H' with $\chi(H') \leq r^{k-1}$ can be globally covered by $(k-1)$ graphs in $\mathcal{C}_r$. Let H be a graph properly coloured with $\chi(H) = r^k$ colours. We can partition $V(H)$ into r sets $P_1, \ldots, P_2$ having only r^{k-1} colours each. By induction hypothesis, we can cover $\langle P_i \rangle_H$ globally using $k-1$ guests $G_1^i, \ldots, G_{k-1}^i$, for $i = 1, \ldots, r$. As the induced subgraphs $\langle P_1 \rangle_H, \ldots, \langle P_r \rangle_H$ are pairwise vertex-disjoint, we can cover $\langle P_1 \rangle_H \cup \ldots \cup \langle P_r \rangle_H$ with $k-1$ guests $G_1^1 \cup \ldots \cup G_1^r, \ldots, G_{k-1}^1 \cup \ldots \cup G_{k-1}^r$. Let R denote the graph of the remaining edges in H, i.e., let $R = (V(H), E(R))$ where $E(R) = E(H) \setminus (\langle P_1 \rangle_H \cup \ldots \cup \langle P_r \rangle_H)$. Graph R can be properly coloured using r colours by colouring every vertex v with i, where $v \in P_i$. We thus obtain $R \in \mathcal{C}_r$, and we can cover H with the $\lceil \log_r(\chi(H)) \rceil = k$ guests $G_1^1 \cup \ldots \cup G_1^r, \ldots, G_{k-1}^1 \cup \ldots \cup G_{k-1}^r, R$. This concludes the induction and we know for every k and every graph H that $\chi(H) = r^k \Rightarrow c_g^{\mathcal{C}_r}(H) \leq k$. Every graph H is induced subgraph of a graph H' with $\chi(H') = r^{\lceil \log \chi(H) \rceil}$ (to get such a graph H' just repeatedly add a vertex adjacent to all other vertices). With Proposition 3.1(iii) this concludes the proof. □

As a well known special case, we obtain Lemma 4.12.

Lemma 4.12 (e.g. Ueckerdt [Uec15])
Let H be a graph. Then $c_g^{\mathcal{B}ip}(H) = \lceil \log(\chi(H)) \rceil$.

Proof. Direct consequence of Lemma 4.11. □

With Lemma 4.12 we can answer Question 4.6 negatively.

Corollary 4.13
For the class $\mathcal{S}$ of shift graphs, we have

$$c_l^{\mathcal{B}ip}(\mathcal{S}) = c_l^{\mathcal{C}\text{-}\mathcal{B}ip}(\mathcal{S}) = 2$$

and

$$c_g^{\mathcal{C}\text{-}\mathcal{B}ip}(\mathcal{S}) = c_u^{\mathcal{C}\text{-}\mathcal{B}ip}(\mathcal{S}) = c_g^{\mathcal{B}ip}(\mathcal{S}) = c_u^{\mathcal{B}ip}(\mathcal{S}) = \infty.$$

Proof. The first equation follows directly by Lemma 4.7, Corollary 4.9 and Proposition 3.1(iv). The second equation follows directly by Lemma 4.8, Lemma 4.12 and Proposition 3.4(iv) observing $\mathcal{B}ip$ is union-closed. □

With Theorem 4.3 we now have a guest class for which folded and local as well as local and union covering number can be separated.

This is the first separation of local and union covering number not directly using line graphs as host graphs. More importantly, this is the first separation using a guest class that is even subgraph-hereditary.

Note that for the guest class of bipartite graphs Erdős et al. [EFH+86] stated an in some sense stronger result. They considered graphs with proper colourings where the number of different colours within every neighbourhood is bounded by some number r. They constructed graphs with such colourings for $r = 2$ and with arbitrarily large chromatic number. Those graphs actually also were shift graphs, implying the same upper bounds on local covering numbers. From such a local colouring we obtain an upper bound on the local bipartite covering number, more precisely that $c_l^{\mathcal{B}ip}(H) \leq r$, as follows. If such a colouring is given, use a guest for every pair of colours (a, b), containing all edges between a-coloured and b-coloured vertices. Those guests are bipartite, since they are properly coloured using only two colours. Every vertex v is covered by only r guests, as there are only r different colours used for $N(v)$.

4.1.1 Shift Graphs are u-Cover Resistant

In this context we provide Theorem 4.15 as a result for the later Chapter 7. Namely, we show that certain shift graphs are, in a sense, hard to cover.

As a preparation, we make an observation which points out a similarity between shift graphs and line graphs:

Observation 4.14
Let G and H be graphs with $G \subseteq H$. Then shift graph $\mathrm{S}(G)$ is an induced subgraph of shift graph $\mathrm{S}(H)$. In short, we have $\mathrm{S}(G) \sqsubseteq \mathrm{S}(H)$.

An *ordered graph* $G = (V, E)$ is a digraph with a total order $<$ on V such that the edges in E respect order $<$, i.e., for two adjacent vertices $v, u \in V$ we have $v < u \Rightarrow vu \in E$. For $n \in \mathbb{N}$ the *ordered complete graph* K_n^o is the graph $([n], E)$ where $E = \{ij \in \binom{[n]}{2} \mid i < j\}$. See Figure 4.1 as an example.

We can now prove Theorem 4.15.

Theorem 4.15
Let $\mathcal{OS}$ denote the class of shift graphs of ordered graphs. Then for any induced-hereditary guest class $\mathcal{G}$ we have either $\mathcal{OS} \subseteq \mathcal{G}$ or $c_g^{\mathcal{G}}(\mathcal{OS}) = \infty$.

Proof. This proof is inspired by the separation of interval and track-number by Milans et al. [MSW12]. Assume there is a $k \in \mathbb{N}$ with $c_g^{\mathcal{G}}(\mathcal{OS}) = k$. Let G be an arbitrary directed graph with $|G| = n$. Let $r = R(3, n, k)$ (the hypergraph Ramsey number exists [Ram30]). Then there is a global $\mathcal{G}$-cover of $\mathrm{S}(K_r^o)$ using only k guests. This induces an edge-colouring of $\mathrm{S}(K_r^o)$ using k colours by choosing one covering guest for every edge. This in turn induces an edge-colouring of K_r^3 where K_r^3 is the complete 3-uniform hypergraph, if the following bijection is considered: $b : E(\mathrm{S}(K_r^o)) \to E(K_r^3), \{uv, vw\} \mapsto \{u, v, w\}$. By definition of r as hypergraph Ramsey number, this induces a monochromatic copy of K_n^o in K_r^o. The inverse image of this copy then is a monochromatic copy of $\mathrm{S}(K_n^o)$. It is further an induced subgraph of $\mathrm{S}(K_r^o)$ as all edges between the contained vertices are contained. Thus, there is a guest G' with $\mathrm{S}(K_n^o) \sqsubseteq G'$. Since G is a directed subgraph of K_n^o, Observation 4.14 implies also $\mathrm{S}(G) \sqsubseteq G'$. As G was chosen arbitrarily, we obtain $\mathcal{OS} \subseteq \mathcal{G}$. $\square$

4.2 Non-Separability for Host Classes of Bounded χ

In this section we show that a separation of folded- and union-covering number is not possible for host classes of bounded chromatic number. Basically, we aim to construct a union cover from a k-folded cover (S, ϕ), by partitioning the guests into subgraphs of the host graph H to make ϕ guest-injective on those subgraphs.

In order to do so, we enumerate the (sub-)vertices $v_1, \ldots, v_k$ mapped to any vertex $v \in H$ from 1 to k. To obtain a subgraph of H, we may only chose one subvertex v_i for every vertex $v \in H$. However, without further knowledge, we need to cover for every edge $uv \in H$ all $k \cdot k$ possible subvertex combinations of u_i and v_j, to ensure the edge $u_i v_j$ mapped to uv is contained in one created guest. If we could choose two subvertices of $v \in H$ in each created guest, then we could just use one created guest for every tuple $(i, j) \in [k]^2$. That created guest would be the subgraph of $\bigcup S$ induced by the vertex set obtained by choosing v_i and v_j for every vertex $v \in H$.

However, we must chose only one subvertex per vertex and thus partition the vertices of H for that created guest into one set choosing subvertex v_i and another set choosing subvertex u_j. To ensure that still for every edge the right corresponding subvertices are contained in a common created guest, we make use of a proper colouring of H. This allows us to partition the vertices according to their colour. We need to have for any two colours and any two subvertex-indices a created guest assigning the given indices to the given colours. We aim to minimize the number of created guests.

A corresponding assignation is provided in Lemma 4.16. Here the functions $f_1, \ldots, f_t$ correspond to the mappings from colour to subvertex index in t created guests.

Lemma 4.16
Let $a, b \in \mathbb{N}$. Then there are $t = a^2 \lceil \log(b) \rceil$ functions $f_1, \ldots, f_t : [b] \to [a]$ such that for any $i, j \in [a]$ and $m, n \in [b]$ with $m \neq n$, there is a $k \in [t]$ such that $f_k(m) = i$ and $f_k(n) = j$.

Proof. For $l \in [\lceil \log(b) \rceil]$ we define $d_l : [b] \to \{0, 1\}$ as a function mapping a number y to its l'th digit in binary representation (i.e., we set $d_l(y) = 1$ if $(y \mod 2^l) \geq 2^{l-1}$ and $d_l(y) = 0$ otherwise).

For $i, j \in [a]$ and $l \in [\lceil \log(b) \rceil]$ we define function $f_{i,j,l} : [b] \to [a]$ as follows: We set
$$f_{i,j,l}(m) = \begin{cases} i, & \text{if } d_l(m) = 1 \\ j, & \text{otherwise} \end{cases}.$$

That are $a^2 \lceil \log(b) \rceil$ functions. We show they fulfil the statement.

Let $i, j \in [a]$ and $m, n \in [b]$ such that $m \neq n$. Then there is a number $l \in [\lceil \log(b) \rceil]$ such that the binary representations of m and n differ in the l'th digit (otherwise they had the same representation and were therefore equal). If $d_l(m) = 1$, then we have $d_l(n) = 0$ and $f_{i,j,l}(m) = i$ as well as $f_{i,j,l}(n) = j$. Otherwise, we have $d_l(m) = 0$, then we have $d_l(n) = 1$ and $f_{j,i,l}(m) = i$ as well as $f_{i,j,l}(n) = j$. This concludes the proof. □

We use Lemma 4.16 in Theorem 4.17 to reduce the number of unions needed for our construction of union covers from folded-covers. This construction makes use of a proper colouring of the host graph H and is thus dependent on $\chi(H)$.

Theorem 4.17
Let $\mathcal{G}$ be an induced-hereditary guest class and let H be a host graph. Let (S, ϕ) be a k-folded $\mathcal{G}$-cover of H with $S = \{G_1, \ldots, G_m\}$. Then there is a $(k^2\lceil \log(\chi(H)) \rceil)$-union $\mathcal{G}$-cover (S', ϕ') of H, such that every guest in S' is induced subgraph of one of the guests in S. Especially we have $c_u^{\mathcal{G}}(H) \leq c_f^{\mathcal{G}}(H)^2 \lceil \log(\chi(H)) \rceil$.

Proof. Let r be a proper colouring of H using $\chi(H)$ colours. For every vertex v in H denote the up to k vertices of $G_1, \ldots, G_m$ in $\phi^{-1}(v)$ as $v_1, \ldots, v_{a(v)}$. Every two adjacent vertices v and w in H have two different colours $r(v)$ and $r(w)$. The edge vw is covered by at least one edge xy in some guest G_j with $\phi(x) = v$ and $\phi(y) = w$. Let $x = v_s$ and $y = w_t$. Then we associate vw with the 5-tuple $(r(v), r(w), s, t, j)$. By covering all edges associated to any such 5-tuple, we cover all edges of host graph H.

Note that the edge-set corresponding to the same 5-tuple forms an induced subgraph of a guest and as such covers every vertex of H at most once. The colours are necessary to decide for every vertex v, whether v_s or v_t is used.

An easy way to deal with all those 5-tuples is to use for every of them another guest which contains the corresponding edges, resulting in $\chi(H)^2k^2m$ guests. This way only pre-images of vertices of H having two specific colours are used in a guest. The impact of $\chi(H)$ on the number of guests (and sets in their pairwise disjoint partition) can be drastically reduced by using all colours in every guest. It is then necessary to assign the numbers corresponding to colours in a clever way, e.g. by using Lemma 4.16.

By Lemma 4.16 we obtain $t = k^2\lceil \log(\chi(H)) \rceil$ functions $f_1, \ldots, f_t : [\chi(H)] \to [k]$ such that: for $i, j \in [k]$ and $r_1, r_2 \in [\chi(H)]$ with $r_1 \neq r_2$, there is an $l \in [t]$ such that $f_l(r_1) = i$ and $f_l(r_2) = j$.

For each $1 \leq i \leq t$ let $D_i = \{v_x \in G_1 \uplus \ldots \uplus G_m \mid x = f_i(r(v))\}$. This definition especially ensures $\phi^{-1}_{|D_i}(v) = 1$, for every $v \in H$ and $1 \leq i \leq t$.

For each $1 \leq i \leq t$ and $1 \leq j \leq m$ we define $G_{j,i} = \langle D_i \cap V(G_j) \rangle_{G_j}$. Let $S' = \{G_{1,1}, \ldots, G_{t,1}, \ldots, G_{1,m}, \ldots, G_{t,m}\}$ be the set of all those graphs. Further, let ϕ' be the corresponding function using ϕ as mapping, i.e. let

$$\phi' : G_{1,1} \uplus \ldots \uplus G_{t,1} \uplus \ldots \uplus G_{1,m} \uplus \ldots \uplus G_{t,m} \to H, (v, l) \mapsto \phi(v).$$

We now show (S', ϕ') is a $(k^2\lceil \log(\chi(H)) \rceil)$-union $\mathcal{G}$-cover of H. Since $G_{j,i} \sqsubseteq G_j$ and $\mathcal{G}$ is induced-hereditary, we also get $G_{j,i} \in \mathcal{G}$. Let $vw \in E(H)$. Then there is a guest G_j in S containing an edge v_xw_y mapped to vw by ϕ. Since v and w are adjacent, we have $r(v) \neq r(w)$. Hence, there is an $1 \leq i \leq t$ such that $f_i(r(v)) = x$ and $f_i(r(w)) = y$. This means that v_x and w_y are both contained in D_i and thus in $G_{j,i}$. Guest $G_{j,i}$ must therefore contain edge v_xw_y which is mapped to $\phi(v_x)\phi(w_y) = vw$. The tuple (S', ϕ') is therefore a $\mathcal{G}$-cover of H.

Since $|\phi'^{-1}_{|G_{j,i}}(v)| \leq |\phi'^{-1}_{|D_i}(v)| = |\phi^{-1}_{|D_i}(v)| = 1$ for every $G_{j,i} \in S'$, the cover is injective. Finally, we can partition S' into $t = k^2\lceil\log(\chi(H))\rceil$ sets $\{G_{1,1}, \dots, G_{1,m}\}, \dots, \{G_{t,1}, \dots, G_{t,m}\}$. Those contain each only pairwise disjoint guests since the supergraphs $G_1, \dots, G_m$ are already pairwise disjoint (in their disjoint union). The cover is therefore a $(k^2\lceil\log(\chi(H))\rceil)$-union cover. By definition of the guests, each of them is induced subgraph of one of the graphs $G_1, \dots, G_m$. □

We thereby obtain that a separation of folded- and union-covering number is not possible for host classes with bounded chromatic number.

Corollary 4.18
Let $\mathcal{G}$ be an induced-hereditary guest class. Let $\mathcal{H}$ be a host class such that $c_f^{\mathcal{G}}(\mathcal{H}) = c < \infty$ and let there be a number $r \in \mathbb{N}$ such that for any graph $H \in \mathcal{H}$ we have $\chi(H) \leq r$. Then we have $c_u^{\mathcal{G}}(\mathcal{H}) \leq c^2\lceil\log(r)\rceil$.

Proof. Direct consequence of Theorem 4.17. □

4.3 Non-Separability of (a, b)-Sparse Graphs

To cover a graph, every edge must be covered by at least one edge of one of the guests. The maximum number of edges in a guest therefore provides lower bounds on the covering numbers, as Proposition 4.19 shows.

Proposition 4.19
Let $\mathcal{G}$ be an induced-hereditary guest class and H be a host graph. Further let $\alpha\colon \mathbb{N} \to \mathbb{N}$, $\beta\colon \mathbb{N} \to \mathbb{R}$ and $\gamma\colon \mathbb{N} \to \mathbb{N}$ be three weakly monotonically increasing functions such that for $n \in \mathbb{R}$ we have

$$\begin{aligned}
\alpha(n) &\geq \max\{||G|| : G \in \mathcal{G}, |G| \leq n\},\\
\beta(n) &\geq \max\{\operatorname{avd}(G) : G \in \mathcal{G}, |G| \leq n\},\\
\gamma(n) &\leq \min\{|G| : G \in \overline{\mathcal{G}}, ||G|| \geq m\}
\end{aligned}$$

Then we have

(i)
$$c_g^{\mathcal{G}}(H) \geq \max_{H' \subseteq H} \left\lceil \frac{||H'||}{\alpha(|H'|)} \right\rceil$$

(ii)
$$c_l^{\mathcal{G}}(H) \geq \max_{H' \subseteq H} \left\lceil \frac{\operatorname{avd}(H')}{\beta(|H'|)} \right\rceil$$

(iii)
$$c_f^{\mathcal{G}}(H) \geq \max_{H' \subseteq H} \left\lceil \frac{\gamma(||H'||)}{|H'|} \right\rceil$$

Proof. Note that, speaking of guests, the function α provides an upper bound for the number of edges, function β provides an upper bound for twice the number of edges per vertex (which can be interpreted as the local number of edges), and $\gamma(H)$ is a lower bound for the number of vertices needed to provide enough edges to cover H.

"(i)": Let (S, ϕ) be a k-global $\mathcal{G}$-cover of H. Since (S, ϕ) is guest-injective, we have for $G \in S$ that $|G| \leq |H|$. With the definition of α we have $||G|| \leq \alpha(|G|) \leq \alpha(|H|)$. Since

ϕ is edge-surjective, we have $||H|| \leq ||\dot\bigcup S|| \leq k\alpha(|H|)$. Noting that k is an integer, we obtain $k \geq \left\lceil \frac{||H||}{\alpha(|H|)} \right\rceil$.

With $\mathcal{G}$ being induced-hereditary, the cover (S,ϕ) induces a k-global $\mathcal{G}$-cover on every induced subgraph of H. Hence, we have $c_g^{\mathcal{G}}(H) \geq \max_{H' \sqsubseteq H} \left\lceil \frac{||H'||}{\alpha(|H'|)} \right\rceil$. Finally, observe that for every subgraph $H' \subseteq H$ we have $\frac{||H'||}{\alpha(|H'|)} \leq \frac{||\langle H' \rangle_H||}{\alpha(|\langle H' \rangle_H|)}$. Thereby Statement (i) holds.

"(ii)": Let (S,ϕ) be a k-local $\mathcal{G}$-cover of H. Then we have $|\dot\bigcup S| \leq k|H|$. For $G \in S$ we have $|G| \leq |H|$, since (S,ϕ) is guest-injective. Since β is by definition weakly monotonically increasing, we can now calculate

$$\operatorname{avd}(\dot\bigcup S) \leq \max_{G \in S} \operatorname{avd}(G) \leq \beta(|G|) \leq \beta(|H|).$$

Next note that, for $v \in H$ we have $\deg(v) \leq \sum_{u \in \phi^{-1}(v)} \deg(u)$, since ϕ is an edge-surjective homomorphism. We obtain

$$\sum_{v \in H} \deg(v) \leq \sum_{v \in H} \sum_{u \in \phi^{-1}(v)} \deg(u) \leq \sum_{u \in \dot\bigcup S} \deg(u).$$

We can now easily calculate

$$\operatorname{avd}(H) = \frac{\sum_{v \in H} \deg(v)}{|H|} \leq \frac{\sum_{u \in \dot\bigcup S} \deg(u)}{|H|} \leq k \frac{\sum_{u \in \dot\bigcup S} \deg(u)}{|\dot\bigcup S|} = k \operatorname{avd}(\dot\bigcup S) \leq k\beta(|H|)$$

Hence, we have $k \geq \left\lceil \frac{\operatorname{avd}(H)}{\beta(|H|)} \right\rceil$. As in the proof of Statement (i) we can conclude Statement (ii) holds.

"(iii)": Let $c_f^{\mathcal{G}}(H) = k < \infty$. Then we also have $c_f^{\overline{\mathcal{G}}}(H) = k$ by Proposition 3.2. Hence, let (S,ϕ) be a k-folded $\overline{\mathcal{G}}$-cover of H. Since ϕ is edge-surjective, we have $||H|| \leq ||\dot\bigcup S||$. By definition, function γ is weakly monotonically increasing.

Then we have
$$\gamma(||H||) \leq \gamma(||\dot\bigcup S||) \leq |\dot\bigcup S| \leq k|H|.$$

Hence, we have $k \geq \left\lceil \frac{\gamma(||H||)}{|H|} \right\rceil$. As in the proof of Statement (i) we can conclude Statement (iii) holds. $\square$

The lower bounds provided by Proposition 4.19 are only meaningful, if the number of edges in guest graphs is significantly bounded. Strong bounds on the number of edges are given for sparse graphs. For those we show that these lower bounds are tight and that the local and the global covering number coincide. In some cases also the folded covering number coincides with the local and the global covering number.

4.3.1 Covers with Regards to (a,b)-Sparse Graphs

Let $a,b \in \mathbb{N}_0$. Lee and Streinu introduced the notion of (a,b)-sparse graphs [LS08]. A graph G is called *(a,b)-sparse*, if for every subgraph $H = (V,E) \subseteq G$ with $|V| \geq 2$ we have $|E| \leq a|V| - b$. In that case, if $||G|| = a|G| - b$ then G is called *(a,b)-tight*. Note that every subgraph of an (a,b)-sparse graph is by definition also (a,b)-sparse. Further note that we only consider simple graphs, while Lee and Streinu considered more general multigraphs.

For example, forests are $(1,1)$-sparse, pseudoforests are $(1,0)$-sparse and outer-planar graphs are $(2,3)$-sparse. However, not all $(2,3)$-sparse graphs are outer-planar. E.g., the forbidden minor $K_{2,3}$ is $(2,3)$-sparse. Note that by this definition planar graphs are not $(3,6)$-sparse, since for any subgraph H of a $(3,6)$-sparse graph with 2 vertices, we get

$||H|| \leq 3|H| - 6 = 0$. Therefore, $(3,6)$-sparse graphs may not contain any edges. Indeed, the parameters are basically always required to satisfy $b < 2a$.

Let G be a graph. A subgraph $H \subseteq G$ is called an (a,b)-*block*, if it is (a,b)-tight. The subgraph H is called an (a,b)-*component* of G if it is an inclusion-maximal (a,b)-block. I.e., subgraph H is an (a,b)-component if it is an (a,b)-block and for every (a,b)-block H' in G with $H \subseteq H'$ we have $H = H'$. Examples of blocks are given in Figure 4.2 and examples of components are given in Figure 4.3. We denote the class of all (a,b)-sparse graphs by $\mathcal{G}(a,b)$.

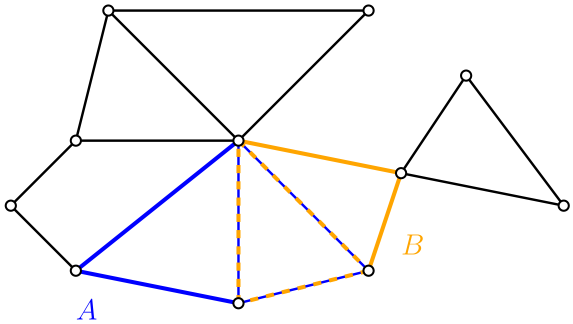

Figure 4.2: A $(2,3)$ sparse graph and two $(2,3)$-blocks A and B. Also the subgraphs $A \cap B$ and $A \cup B$ are $(2,3)$-blocks by Theorem 4.20, since A and B share at least 2 vertices.

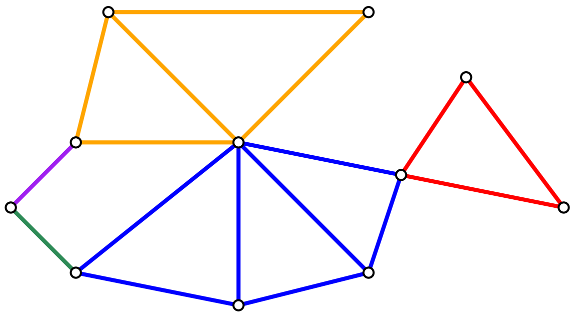

Figure 4.3: A $(2,3)$ sparse graph decomposed into its $(2,3)$-components. Since subgraphs induced by single edges already are blocks, here every edge is contained in a component.

A *matroid* is a tuple $(E, \mathcal{I})$ where E is a finite set and $\mathcal{I}$ is a set of *independent sets* that are subsets of E with the following properties:

1. $\emptyset \in \mathcal{I}$.
2. $\forall I \in \mathcal{I} : \forall J \subseteq I : J \in \mathcal{I}$. (*Hereditary Property*)
3. $A, B \in \mathcal{I} \wedge |B| \leq |A| \Rightarrow \exists x \in A \setminus B : B \cup \{x\} \in \mathcal{I}$. (*Augmentation Property*)

In a matroid $(E, \mathcal{I})$ we call subsets of E that are not independent sets *dependent*.

Let H be a host graph. Lee and Streinu proved that the (a,b)-sparse subsets of H are the independent sets of a matroid on $E(H)$. We prove the same result focusing on simple graphs and following their results. We will use the matroid property to apply the Matroid Base Cover Theorem by Edmonds [Edm65] to provide an upper bound for the global covering number with regards to $\mathcal{G}(a,b)$. This upper bound matches the lower bound provided by Proposition 4.19 which will show the local and global covering number coincide.

As a first step, we establish the decomposition of (a, b)-sparse graphs into (a, b)-components in Corollary 4.21. We first show Theorem 4.20 which allows to merge blocks to obtain larger blocks (see Figure 4.2 for an example).

Theorem 4.20 (Lee and Streinu [LS08])
Let $a, b \in \mathbb{N}_0$ with $b < 2a$. Let H be an (a, b)-sparse graph. Let H_1 and H_2 be non-empty (a, b)-blocks in H. Let one of the following statements hold:

(i) The blocks H_1 and H_2 have at least one common vertex and $b \leq a$.

(ii) The blocks H_1 and H_2 have at least two common vertices,

Then $H_1 \cup H_2$ and $H_1 \cap H_2$ are also (a, b)-blocks in G.

Proof. We first observe $|H_1 \cup H_2| = |H_1| + |H_2| - |H_1 \cap H_2|$. And further $||H_1 \cup H_2|| = ||H_1|| + ||H_2|| - ||H_1 \cap H_2||$. We can then calculate:

$$\begin{aligned} ||H_1 \cup H_2|| + ||H_1 \cap H_2|| &= ||H_1|| + ||H_2|| = a(|H_1| + |H_2|) - 2b \\ &= a(|H_1 \cup H_2| + |H_1 \cap H_2|) - 2b \\ &= a|H_1 \cup H_2| - b + a|H_1 \cap H_2| - b \end{aligned} \tag{$\star$}$$

We apply case distinction. If $|H_1 \cap H_2| \leq 1$ then the first assumption must hold and we thus have $|H_1 \cap H_2| = 1$ and $b \leq a$. This implies

$$\begin{aligned} ||H_1 \cup H_2|| &= ||H_1 \cup H_2|| + ||H_1 \cap H_2|| = a|H_1 \cup H_2| - b + a|H_1 \cap H_2| - b \\ &= a|H_1 \cup H_2| - b + a - b \\ &\geq a|H_1 \cup H_2| - b \end{aligned} \tag{$\star\star$}$$

If $|H_1 \cup H_2| \leq 1$, then we have $H_1 \cup H_2 = H_1 = H_1 \cap H_2$ and the statement holds. Hence, assume $|H_1 \cup H_2| \geq 2$. Since $H_1 \cup H_2$ is further a subgraph of H and thereby sparse, this means with ($\star\star$) that $H_1 \cup H_2$ is an (a, b)-block and that $a = b$. Note a subgraph on a single vertex is a block if and only if $a = b$, since $0 = a \cdot 1 - b \Leftrightarrow a = b$. Hence, the graph $H_1 \cap H_2$ is also a block.

Next, if $|H_1 \cap H_2| \geq 2$, then we have $||H_1 \cup H_2|| \leq a|H_1 \cup H_2| - b$ and $||H_1 \cap H_2|| \leq a|H_1 \cap H_2| - b$, since $H_1 \cup H_2$ and $H_1 \cap H_2$ are subgraphs of the (a, b)-sparse graph H and contain at least 2 vertices. With ($\star$) this implies $H_1 \cup H_2$ and $H_1 \cap H_2$ are (a, b)-blocks. This concludes the proof. □

The decomposition of (a, b)-sparse graphs into (a, b)-components as stated in Corollary 4.21 is a direct consequence.

Corollary 4.21 (Decomposition into Components; Lee and Streinu [LS08])
Let $a, b \in \mathbb{N}_0$ with $b < 2a$. Let H be an (a, b)-sparse graph. Then H is decomposed into (a, b)-components as well as edges and vertices, that are not contained in any block.

I.e., any two (a, b)-components of H have at most one common vertex and every edge or vertex that is not contained in any (a, b)-component of H is not contained in any (a, b)-block of H.

Proof. Direct consequence of the definition of (a, b)-components and Theorem 4.20. □

To prove the Augmentation Property of matroids, we need a characterization of the cases in which (a, b)-sparsity is preserved when adding an edge. The following theorem gives a characterization in terms of (a, b)-components.

Theorem 4.22 (Lee and Streinu [LS08])
Let $a, b \in \mathbb{N}_0$ with $b < 2a$. Let H be an (a,b)-sparse graph. Further, let u and v be non-adjacent vertices in H. Then $H + uv$ is (a,b)-sparse if and only if no (a,b)-component of H contains u and v.

Proof. Assume an (a,b)-component C of H contains u and v. Since it is tight, we have $||C|| = a|C| - b$. The graph $C + uv$ is subgraph of $H + uv$ and has

$$||C + uv|| = ||C|| + 1 > a|C + uv| - b = a|C| - b$$

edges. Hence, $H + uv$ is not (a,b)-sparse.

Next assume there is no (a,b)-component of H containing u and v. Then there is not even an (a,b)-block B of H containing u and v since otherwise by definition of (a,b)-components the block B must be contained in an (a,b)-component that would then contain u and v.

We therefore have for every subgraph $G \subseteq H$ with $u, v \in G$ that

$$||G + uv|| = ||G|| + 1 \leq (a|G| - b - 1) + 1 = a|G + uv| - b.$$

Further, for every subgraph $G' \subseteq H$ with $u \notin H$ or $v \notin H$, we have

$$||G' + uv|| = ||G'|| \leq a|G'| - b = a|G' + uv| - b.$$

Hence, graph $H + uv$ is (a,b)-sparse. This concludes the proof. $\square$

We can now show that the edge-sets of (a,b)-sparse subgraphs are the independent sets in a matroid.

Corollary 4.23 (Lee and Streinu [LS08])
Let $a, b \in \mathbb{N}_0$ with $b < 2a$. Let further H be a graph with $||H|| > 0$. Let $\mathcal{S}$ denote the set of all edge-sets of (a,b)-sparse subgraphs of H. More precisely, let

$$\mathcal{S} = \{E(G) \mid G \subseteq H \textit{ and } G \textit{ is } (a,b)\textit{-sparse}\}$$

Then $(E(H), \mathcal{S})$ is a matroid.

Proof. First note the trivial graph $(\{v\}, \emptyset)$ on one vertex is subgraph of H. Hence, we have $\emptyset \in \mathcal{S}$.

Let $G \subseteq H$ be a subgraph with $E(G) \in \mathcal{S}$. This means G is (a,b)-sparse. Let $uv \in E(G)$. Then $G - e$ is also (a,b)-sparse, since $G - e$ itself and all of its subgraphs are subgraphs of G. Hence, we have $E(G) \setminus \{e\} = E(G - e) \in \mathcal{S}$. Therefore, the hereditary property is fulfilled.

Let $E \in \mathcal{S}$ and $F \in \mathcal{S}$ with $|E| < |F|$. Let $G \subseteq H$ be a subgraph with edge-set $E(G) = E$ and let $G' \subseteq H$ be a subgraph with edge-set $E(G') = F$. Let $C_1, \ldots, C_c$ be the (a,b)-components of G. By Corollary 4.21 these components are pairwise edge-disjoint and the set R of the remaining edges contains no edge of those components.

We have at least $a \sum_{i=1}^{c} |C_i| - cb + |R|$ edges in G, since every component is tight. On the other hand, graph G' is (a,b)-sparse. Thus, for $i \in [c]$ graph G' has at most $a|C_i| - b$ edges in $\langle V(C_i) \rangle_H$, and it has at most $|R|$ edges in R. Since we have $|F| > |E|$, by pigeon hole principle there must be an edge $uv \in F \setminus E$ such that u and v are not contained in the same component of G'. By Corollary 4.22 follows $G + uv$ is an (a,b)-sparse subgraph of H. Hence, we have $E \cup \{uv\} \in \mathcal{S}$ and the augmentation property is fulfilled. This concludes the proof. $\square$

We aim to use the Matroid Base Cover Theorem of Edmonds [Edm65]. Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid. A maximal independent set of $\mathcal{M}$ is called a *base*. All bases of $\mathcal{M}$ have the same number of elements called the *rank* of $\mathcal{M}$. For $F \subseteq E$ the tuple $(F, \mathcal{J})$ with $\mathcal{J} = \{J \in \mathcal{I} \mid J \subseteq F\}$ is a matroid called *submatroid* of $\mathcal{M}$. The *rank* of set $F \subseteq E$ is the rank of the submatroid induced by F.

Theorem 4.24 (Matroid Base Cover Theorem; Edmonds [Edm65])
Let $(E, \mathcal{S})$ be a matroid. Then E can be partitioned into at most k sets in $\mathcal{S}$ if and only if for every subset $F \subseteq E$ we have $|F| \leq k\, \mathrm{r}(F)$, where $\mathrm{r}(F)$ is the rank of F.

In order to use the Matroid Base Cover Theorem, we need to show the corresponding inequality. Let $H = (V, E)$ be a graph and $F \subseteq E$. Then we denote the subgraph of H induced by F as $G(F)$, i.e., we say $G(F) = (U, F)$ where $U = \{v \in V \mid \exists u \in V : uv \in F\}$.

Lemma 4.25
Let $a, b \in \mathbb{N}_0$ with $b < 2a$. Let H be a graph with $||H|| > 0$ and let $(E(H), \mathcal{S})$ be the matroid with the (a, b)-sparse subgraphs of H as independent sets as given in Corollary 4.23. Let

$$k = \max_{\substack{H' \subseteq H \\ |H'| \geq 2}} \left\lceil \frac{||H'||}{a|H'| - b} \right\rceil .$$

Then for $F \subseteq E(H)$ we have $|F| \leq \mathrm{r}(F)k$, where $\mathrm{r}(F)$ is the rank of F.

Proof. We first proof the statement implicitly for $F \subseteq E(H)$ with $\mathrm{r}(F) = a|G(F)| - b$, i.e., those edge-sets spanned by an (a, b)-block. We then use the decomposition into (a, b)-components of a base edge-set F' to calculate that the rank is high enough for any set of edges $F \subseteq E(H)$.

Let $F \subseteq E(H)$. By the definition of $\mathrm{r}(F)$, there is a maximal subset $F' \subseteq F$ such that $G(F')$ is (a, b)-sparse with $|F'| = \mathrm{r}(F)$.

Let C be an (a, b)-component of $G(F')$. We consider $G = \langle V(C) \rangle_H$, the induced subgraph of H with the same vertex set as C. Since C is tight, we have $||C|| = a|C| - b = a|G| - b$. By definition of k we have $||G||/(a|G| - b) \leq k$ and thus $|E(G)| \leq k(a|G| - b) = k||C||$.

Let $C_1, \ldots, C_n$ denote the (a, b)-components of $G(F')$. By Corollary 4.21, these have pairwise at most one vertex in common, and the edges in set $R = F' \setminus E(C_1 \cup \ldots \cup C_n)$ are contained in no (a, b)-component of $G(F')$. Let $G_1, \ldots, G_n$ denote the subgraphs of H induced by the vertex sets $V(C_1), \ldots, V(C_n)$. Since those graphs have the same vertex sets as the components, they also have pairwise at most one vertex in common. Especially, they are edge-disjoint.

Since F' is maximal, it follows by Corollary 4.22 that for every other edge $uv \in F \setminus F'$ both vertices u and v are contained in a common (a, b)-component of $G(F')$. Therefore, every edge of F is contained in $G_1 \cup \ldots \cup G_n \cup R$. I.e., we have $F = E(G_1) \uplus \ldots \uplus E(G_n) \uplus R$. We can therefore calculate

$$|F| = |E(G_1) \uplus \ldots \uplus E(G_n) \uplus R| \leq k||C_1|| + \cdots + k||C_n|| + |R| \leq k|F'| = k\, \mathrm{r}(F).$$

□

We can now characterize the global covering number of a host graph H with regards to (a, b)-sparse graphs in terms of numbers of edges in the subgraphs of H. It coincides with the local and union covering number. An efficient algorithm providing best-possible $\mathcal{G}(a, b)$-covers is given in Chapter 5.

Theorem 4.26
Let $a, b \in \mathbb{N}_0$ with $b < 2a$. Let H be a host graph. Then we have

$$c_g^{\mathcal{G}(a,b)}(H) = c_u^{\mathcal{G}(a,b)}(H) = c_l^{\mathcal{G}(a,b)}(H) = \max_{\substack{H' \subseteq H \\ |H'| \geq 2}} \left\lceil \frac{||H'||}{a|H'| - b} \right\rceil = k.$$

Proof. By Corollary 4.23 and Lemma 4.25 there is a k-global $\mathcal{G}(a,b)$-cover of H. With Proposition 3.1 we obtain $c_l^{\mathcal{G}(a,b)}(H) \leq c_u^{\mathcal{G}(a,b)}(H) \leq c_g^{\mathcal{G}(a,b)}(H) \leq k$.

We define function $\beta \colon \mathbb{N} \to \mathbb{R}, n \mapsto \max(0, \frac{2(an-b)}{n})$. By definition, β is weakly monotonically increasing and one easily verifies for $n \in \mathbb{N}$ that $\beta(n) \geq \max\{\text{avd}(G) : G \in \mathcal{G}(a,b), |G| \leq n\}$. With Proposition 4.19 we calculate

$$c_l^{\mathcal{G}(a,b)}(H) \geq \max_{\substack{H' \subseteq H \\ |H'| \geq 2}} \left\lceil \frac{\text{avd}(H')}{\frac{2(a|H'| - b)}{|H'|}} \right\rceil = \max_{\substack{H' \subseteq H \\ |H'| \geq 2}} \left\lceil \frac{\frac{2||H'||}{|H'|}}{\frac{2(a|H'| - b)}{|H'|}} \right\rceil = \max_{\substack{H' \subseteq H \\ |H'| \geq 2}} \left\lceil \frac{||H'||}{a|H'| - b} \right\rceil = k$$

Hence, we obtain $k \leq c_l^{\mathcal{G}(a,b)}(H) \leq c_u^{\mathcal{G}(a,b)}(H) \leq c_g^{\mathcal{G}(a,b)}(H) \leq k$. This concludes the proof. □

With Proposition 4.19 we obtain that for $b = 0$ also the folded covering number coincides with the global covering number. This generalizes the corresponding result of Knauer and Ueckerdt for pseudoforests ($(1,0)$-sparse graphs) [KU16]. We first state the general lower bound obtained with Proposition 4.19.

Lemma 4.27
Let $a, b \in \mathbb{N}_0$ with $b < 2a$. Let H be a host graph. Then we have

$$c_f^{\mathcal{G}(a,b)}(H) \geq \max_{\substack{H' \subseteq H \\ |H'| \geq 2}} \left\lceil \frac{||H'|| + b}{a|H'|} \right\rceil.$$

Proof. Let $\gamma \colon \mathbb{N} \to \mathbb{R}, m \mapsto (m+b)/a$. This is obviously a weakly monotonically increasing function. Further, we have for every (a,b)-sparse graph G with $|G| \geq 2$ that $||G|| \leq a|G| - b$. This means for $m \leq ||G||$ we obtain $m \leq a|G| - b$. This is equivalent to $(m+b)/a \leq |G|$. We can therefore apply Proposition 4.19 and obtain the inequality we want to prove. □

With Theorem 4.26 and Lemma 4.27 we can conclude that all $\mathcal{G}(a,0)$-covering numbers coincide.

Corollary 4.28
Let $a \in \mathbb{N}$. Let H be a host graph. Then we have

$$c_f^{\mathcal{G}(a,0)}(H) = c_l^{\mathcal{G}(a,0)}(H) = c_u^{\mathcal{G}(a,0)}(H) = c_g^{\mathcal{G}(a,0)}(H) = \max_{\substack{H' \subseteq H \\ |H'| \geq 2}} \left\lceil \frac{||H'||}{a|H'|} \right\rceil.$$

Proof. Direct consequence of Theorem 4.26, Lemma 4.27 and Proposition 3.1(ii). □

The following theorem provides an easy example where those covering numbers do not coincide.

Theorem 4.29
Let $a = b = 1$ and let H be the graph obtained by removing one edge from K_5. Then we have $c_f^{\mathcal{G}(1,1)}(H) = 2 < 3 = c_g^{\mathcal{G}(1,1)}(H)$.

Proof. With Theorem 4.26 one easily verifies that

$$c_g^{\mathcal{G}(1,1)}(H) = \max_{\substack{H' \subseteq H \\ |H'| \geq 2}} \left\lceil \frac{||H'||}{a|H'| - b} \right\rceil = \left\lceil \frac{||9||}{1|5| - 1} \right\rceil = 3.$$

On the other hand there is a 2-folded $\mathcal{G}(1,1)$-cover of H as can be seen in Figure 4.4. □

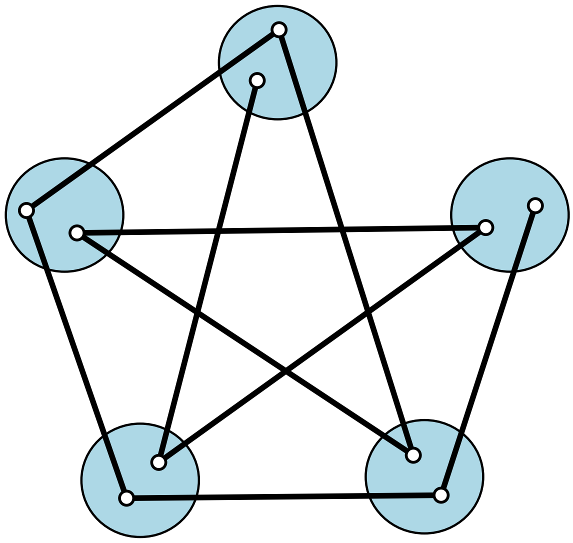

Figure 4.4: A 2-folded $\mathcal{G}(1,1)$-cover of $K_5 - e$.

5. Computing (a, b)-Sparse Covers

In Chapter 4 Section 4.3 we provide a formula for the exact global covering number with regards to (a, b)-sparse graphs and see that the local, the union and the global covering number coincide for all host graphs. For definitions concerning (a, b)-sparse graphs look in that section. In this section we provide an efficient algorithm that returns optimal global $\mathcal{G}(a, b)$-covers of a given host graph H. It makes heavy use of the pebble game algorithm provided by Lee and Streinu that identifies (a, b)-sparse graphs [LS08]. We first restate their algorithm in a version that is restricted to simple graphs and only able to identify (a, b)-sparse graphs. Their original algorithm was enhanced to provide additional results with the same runtime.

5.1 Detecting (a, b)-Sparse Graphs

The basic idea is to preserve a subgraph $D \subseteq H$, initially an independent set, in a good state, ensuring it is (a, b)-sparse, while adding the edges of H one by one until no more edge can be added. If this happens before all edges are added, then graph H is not (a, b)-sparse. The good state is realized by assigning pebbles to the vertices of D.

Recall that we define for a digraph D and a subgraph $G \subseteq D$ the outdegree of G as $\deg^+_D(G) = |\{uv \in D \mid u \in G, v \notin G\}|$.

Definition 5.1
Let $a, b \in \mathbb{N}_0$ with $b < 2a$. Let $D = (V, E)$ be a directed graph and let $p\colon V \to [a]$ be any map. For $U \subseteq V$ we define $p(U) = \sum_{u \in U} p(u)$. We depict function p by putting $p(v)$ pebbles *onto every vertex $v \in V$. These pebbles in some sense count how many edges may still be added.*

*We call digraph D (a, b)-*pebbled *by p, if we have:*

(i) $\forall v \in V : \deg^+(v) + p(v) = a$

(ii) $\forall G \subseteq D$ *with* $|G| \geq 2 : p(V(G)) + \deg^+(G) \geq b$

An example of a $(2, 3)$-pebbled digraph is given in Figure 5.1.

We first observe that every (a, b)-pebbled digraph is (a, b)-sparse. Roughly speaking, Property (i) ensures there are at most a edges per vertex and Property (ii) ensures in every non-trivial subgraph G there are b pebbles/edges reserved. Thereby $||G|| \leq a|G| - b$ is guaranteed.

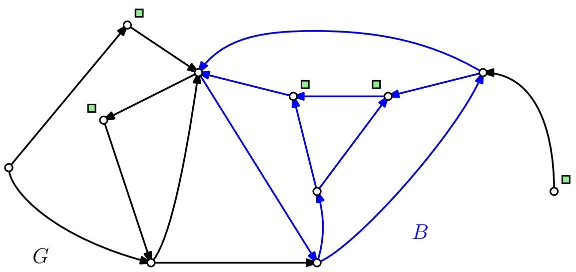

Figure 5.1: A $(2,3)$-pebbled digraph G where pebble function p is represented by pebbles: Each square represents a pebble and every vertex v has $p(v)$ pebbles. For the induced subgraph B we have $p(V(B)) = 2$ and $\deg^+(B) = 1$. By Lemma 5.3 it is thus a $(2,3)$-block of G.

Lemma 5.2 (Lee and Streinu [LS08])
Let $a, b \in \mathbb{N}_0$ with $b < 2a$. Let $D = (V, E)$ be a digraph that is (a, b)-pebbled by function $p \colon V \to [a]$. Then D is (a, b)-sparse.

Proof. Let $G \subseteq D$ be a non-trivial subgraph. Then we have by Property (i), that

$$\deg^+(V(G)) = a|G| - p(V(G)).$$

With Property (ii) we obtain that

$$||G|| = \deg^+(V(G)) - \deg^+(G) \leq (a|G| - p(V(G))) - (b - p(V(G))) = a|G| - b.$$

This concludes the proof. □

If in an execution of the pebble game algorithms all edges are added successfully, then we have $D = H$. And since we keep D always (a, b)-pebbled, by Lemma 5.2 the given graph H is (a, b)-sparse.

We next provide operations required to add edges to D. Let $G = (V, E)$ be an (a, b)-sparse graph and let $uv \in \binom{V}{2} \setminus E$. Graph $G + uv$ is (a, b)-sparse, if no block of G contains u and v. In that case we call set $\{u, v\}$ *free* in G. The (a, b)-blocks of (a, b)-pebbled digraphs are characterized by Lemma 5.3 (see Figure 5.1 for an example). It allows us to identify a set $\{u, v\}$ as free if $p(u) + p(v) \geq b + 1$.

Lemma 5.3
Let $a, b \in \mathbb{N}_0$ with $b < 2a$. Let $D = (V, E)$ be a digraph that is (a, b)-pebbled by function $p \colon V \to [a]$. Let $G \subseteq D$.

Then subgraph G is (a, b)-tight if and only if $G \sqsubseteq D$ and $\deg^+_D(G) + p(V(G)) \leq b$.

Proof. First note that if G is not an induced subgraph, then there is an (induced) subgraph H with more edges with the same vertex set $V(G)$. By Lemma 5.2 subgraph H is (a, b)-sparse and we have $||G|| < ||H|| \leq a|H| - b = a|G| - b$. Therefore G is not (a, b)-tight and the equivalence holds.

If G is an induced subgraph of D than we have with Property (i) of (a, b)-pebbled digraphs that $||G|| = \deg^+(V(G)) - \deg^+(G) = a|G| - p(V(G)) - \deg^+(G)$. By Property (ii) we have $p(V(G)) + \deg^+(G) \geq b$. We therefore have $||G|| = a|G| - b$ if and only if $\deg^+_D(G) + p(V(G)) \leq b$. □

If we have ensured a set $\{u, v\}$ is free, we only need to pay one pebble of u to add edge uv.

Lemma 5.4 (Lee and Streinu [LS08])
Let $a, b \in \mathbb{N}_0$ with $b < 2a$. Let $D = (V, E)$ be a digraph that is (a, b)-pebbled by function $p\colon V \to [a]$. Further let u and w be two non-adjacent vertices with $p(u) \geq 1$ and

(i) $\{u, w\}$ is free in D or

(ii) $p(u) + p(w) \geq b + 1$.

Then digraph $D' = (V, E \cup \{uw\})$ is (a, b)-pebbled by function

$$p'\colon V \to [a],\ v \mapsto \begin{cases} p(v) - 1 & , \textit{ if } v = u \\ p(v) & , \textit{ otherwise} \end{cases}.$$

Proof. First note we have for every vertex $v \in V \setminus \{u\}$ that $p'(v) + \deg^+_{D'}(v) = p(v) + \deg^+_D(v) = a$ by Property (i) of D. Further, for vertex u we have

$$p'(u) + \deg^+_{D'}(u) = (p(u) - 1) + (\deg^+_D(u) + 1) = p(u) + \deg^+_D(u) = a$$

by Property (i) of D. Hence, also D' has Property (i).

Next Assume $\{u, w\}$ is free in D. Then no block G of D contains u and w. Therefore by Lemma 5.3 we have $\deg^+_D(G) + p(G) \geq b + 1$. If u and w are contained in G we have $p'(V(G)) + \deg^+_{D'}(G) = (p(V(G)) - 1) + \deg^+_D(G) \geq b$. Otherwise we have:

$$\begin{aligned} p'(V(G)) + \deg^+_{D'}(G) &= \begin{cases} (p(V(G)) - 1) + (\deg^+_D(G) + 1) & , \text{ if } u \in G \text{ and } w \notin G \\ p(V(G)) + \deg^+_D(G) & , \text{ otherwise} \end{cases} \\ &= p(V(G)) + \deg^+_D(G) \geq b \end{aligned}$$

by Property (ii) of D.

Hence, also D' has Property (ii).

Next assume $p(u) + p(w) \geq b + 1$. For every subgraph $G \subseteq D$ with $u, w \in G$ this implies $\deg^+_D(G) + p(V(G)) \geq b + 1$. Therefore by Lemma 5.3 no block contains u and w and thus $\{u, w\}$ is free in D. Since the statement is proven for this case, this concludes the proof. ☐

This means we can add an edge if the corresponding vertices provide enough pebbles. Requirement (ii) is easily verified. By Lemma 5.5 it is possible to remove an edge without special requirements.

Lemma 5.5
Let $a, b \in \mathbb{N}_0$ with $b < 2a$. Let $D = (V, E)$ be a digraph that is (a, b)-pebbled by function $p\colon V \to [a]$.

Further let $uw \in D$.

Let $D' = (V, E \setminus \{uw\})$ be the digraph obtained from D by removing the edge uw. Then D' is (a, b)-pebbled by

$$p'\colon V \to [a],\ v \mapsto \begin{cases} p(v) + 1 & , \textit{ if } v = u \\ p(v) & , \textit{ otherwise} \end{cases}.$$

Proof. First note we have for every vertex $v \in V \setminus \{u\}$ that $p'(v) + \deg^+_{D'}(v) = p(v) + \deg^+_D(v) = a$ by Property (i) of D. Further, for vertex u we have

$$p'(u) + \deg^+_{D'}(u) = (p(u) + 1) + (\deg^+_D(u) - 1) = p(u) + \deg^+_D(u) = a$$

by Property (i) of D. Hence, also D' has Property (i).

Next let $G \subseteq D'$ be a non-trivial subgraph. Then we have

$$\begin{aligned} p'(V(G)) + \deg^+_{D'}(G) &= \begin{cases} (p(V(G)) + 1) + (\deg^+_D(G) - 1) & \text{, if } u \in G \text{ and } w \notin G \\ p(V(G)) + \deg^+_D(G) & \text{, otherwise} \end{cases} \\ &= p(V(G)) + \deg^+_D(G) \geq b \end{aligned}$$

by Property (ii) of D. Hence, also D' has Property (ii). This concludes the proof. □

After removing an edge xy, set $\{x, y\}$ is free and the reverse edge yx can be inserted as stated in the following Lemma (also see Figure 5.2). Such edge reversions can be used repeatedly to move pebbles along edges to collect them on two vertices u and v such that we can check whether $\{u, v\}$ is free.

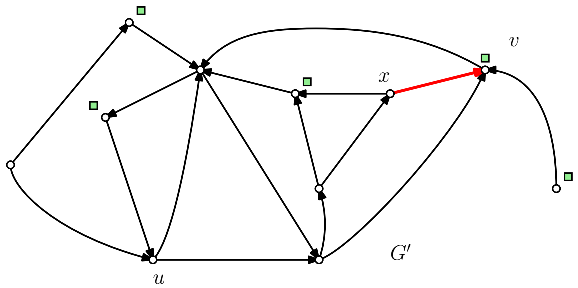

Figure 5.2: Result of reversing edge vx in graph G of Figure 5.1. Note that a pebble moved from x to v.

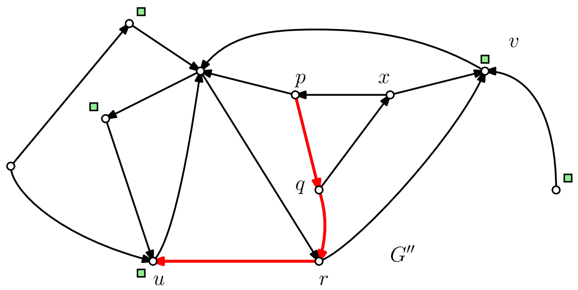

Figure 5.3: Result of reversing the edges qp, rq and ur in graph G' of Figure 5.2. Note that a pebble moved from p to u.

Lemma 5.6 (Lee and Streinu [LS08])
Let $a, b \in \mathbb{N}_0$ with $b < 2a$. Let $D = (V, E, d)$ be a digraph that is (a, b)-pebbled by function $p\colon V \to [a]$. Further let $uw \in D$.

Let $D' = (V, E \setminus uw \cup wu)$ be the digraph obtained from D by reversing the direction of edge uw and let $p(w) \geq 1$. Then D' is (a, b)-pebbled by

$$p'\colon V \to [a],\ v \mapsto \begin{cases} p(v) + 1 & \text{, } \textit{if } v = u \\ p(v) - 1 & \text{, } \textit{if } v = w \\ p(v) & \text{, } \textit{otherwise} \end{cases} .$$

Proof. First note the outdegree of a vertex v changes only from within D to within D', if $v = u$ or $v = w$. Those changes are counterbalanced by the changes from p to p'. Therefore digraph D' has Property (i).

Next let $G \subseteq D'$ be a non-trivial subgraph. Then we have

$$p'(V(G)) + \deg^+_{D'}(G) = \begin{cases} (p(V(G)) + 1) + (\deg^+_D(G) - 1) & \text{, if } u \in G \wedge w \notin G \\ (p(V(G)) - 1) + (\deg^+_D(G) + 1) & \text{, if } u \notin G \wedge w \in G \\ p(V(G)) + \deg^+_D(G) & \text{, otherwise} \end{cases}$$
$$= p(V(G)) + \deg^+_D(G) \geq b$$

by Property (ii) of D. Hence, also D' has Property (ii). This concludes the proof. □

We can apply Lemma 5.6 repeatedly to reverse directed paths ending at some pebbles (see Figure 5.3), bringing one pebble to the start vertex of the path. If a pebble can be reached, such a path can be found by a depth-first search.

Let $a, b \in \mathbb{N}_0$ with $b < 2a$. Let $D = (V, E)$ be a digraph that is (a,b)-pebbled by function $p\colon V \to [a]$. Further let $U \subseteq V$. Then the *reach* of U is the set of all vertices in V that can be reached by starting in U and following directed edges. More precisely, we define the reach by

$$\mathrm{Reach}(U) = \{v \in V \mid \exists \text{ directed path } P \subseteq D : \exists u \in U : P = (u, \ldots, v)\}.$$

We write $\mathrm{Reach}_D(U)$ to specify that the reach in digraph D is meant. See Figure 5.4 for an example.

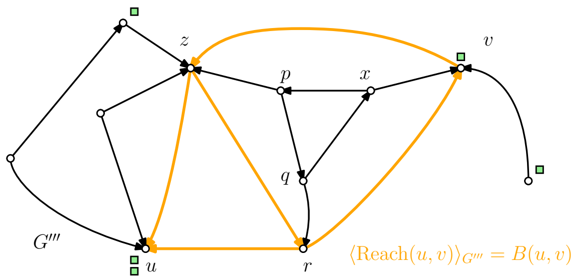

Figure 5.4: Digraph G''' resulting from collecting pebbles on u and v in graph G of Figure 5.1. We have $\mathrm{Reach}(u,v) = \{u, v, r, z\}$. Since $p(\mathrm{Reach}(u,v)) = p(\{u,v\}) = b$ we have by Theorem 5.8 that the subgraph induced by this reach is the minimum block $B(u,v)$ containing u and v.

If the vertices in U are the only vertices in $\mathrm{Reach}(U)$ with pebbles, then no more pebbles can be moved from $V \setminus U$ to U by applying Lemma 5.6. However, if $p(U) \leq b$ in that situation, then the subgraph induced by U is already (a,b)-tight, as shown in Lemma 5.7. While Lee and Streinu considered only the case $|U| = 2$, we consider also greater sets for later use.

Lemma 5.7
Let $a, b \in \mathbb{N}_0$ with $b < 2a$. Let $D = (V, E)$ be a digraph that is (a,b)-pebbled by function $p\colon V \to [a]$. Further let $U \subseteq V$ with $|U| \geq 2$ and let $|p(\mathrm{Reach}(U))| \leq b$.

Then $\langle \mathrm{Reach}(U) \rangle_D$ is (a,b)-tight.

Proof. By the definition of $\mathrm{Reach}(U)$, the digraph $H = \langle \mathrm{Reach}(U) \rangle_D$ has no outgoing edge, i.e., we have $\deg^+(H) = 0$. Therefore we obtain $\deg^+_D(H) + p(H) \leq b$ and by Lemma 5.3 graph H is (a,b)-tight. □

We strengthen this result in Theorem 5.8. To this end, we introduce the notion of *minimum blocks*. An example of a minimum Block is given in Figure 5.4.

Let $a, b \in \mathbb{N}_0$ with $b < 2a$. Let H be an (a, b)-sparse graph with vertex set U such that $|U| \geq 2$ and let there be a block B with $U \subseteq V(B)$. Then we define the *minimum block* containing all vertices in U as $B_H(U) = \bigcap\{G \subseteq H \mid G \text{ is block}\}$. If it is clear which H is meant we just write $B(U)$. By Theorem 4.20 the graph $B(U)$ is actually a block. In case that $U = \{u, v\}$ we usually write $B(u, v)$ instead of $B(\{u, v\})$. For a directed or undirected edge $e = uv$ we also write $B(e)$ instead of $B(u, v)$.

Theorem 5.8
Let $a, b \in \mathbb{N}_0$ with $b < 2a$. Let $D = (V, E)$ be a digraph that is (a, b)-pebbled by function $p \colon V \to [a]$. Further let $U \subseteq V$ with $|U| \geq 2$, let $p(\mathrm{Reach}(U) \setminus U) = 0$ and let $p(U) \leq b$.

Then we have $B(U) = \langle \mathrm{Reach}(U) \rangle_D$.

Proof. By Lemma 5.7 we know $B = \langle \mathrm{Reach}(U) \rangle_D$ is a block. By Property(ii) of (a, b)-pebbled digraphs we obtain $p(U) = b$. For any subgraph $G \subsetneq B$ with $U \subseteq G$ we have $p(G) = b$ and if $V(G) \subsetneq V(B)$ then $\deg^+(G) \geq 1$. With Lemma 5.3 this implies G is not (a, b)-tight, unless $G = B$. □

When edge uv is next to be added in the pebble game algorithm and there are at most b pebbles in $\mathrm{Reach}(\{u, v\})$, then by Lemma 5.7 graph H is not (a, b)-sparse, since no edge may be added to a tight graph. This means if $p(\mathrm{Reach}(\{u, v\}) \setminus \{u, v\}) = 0$, then we can apply either Lemma 5.7 to show that H is not (a, b)-sparse, or we can apply Lemma 5.4 to add uv.

As indicated earlier, we can search repeatedly for paths to pebbles in $\mathrm{Reach}(\{u, v\}) \setminus \{u, v\}$ and reverse them to move pebbles to u and v until $p(\mathrm{Reach}(\{u, v\}) \setminus \{u, v\}) = 0$ as described by Algorithm 5.1.

Note that given a digraph $D = (V, F)$ we denote the corresponding undirected graph by $G(D)$, i.e, $G(D) = (V, E)$ where $E = \{\{u, v\} \in \binom{V}{2} \mid (u, v) \in F\}$.

Theorem 5.9
Let $a, b \in \mathbb{N}_0$ with $b < 2a$. Let $D = (V, E)$ be a digraph that is (a, b)-pebbled by function $p \colon V \to [a]$. Further let $U \subseteq V$.

Then a digraph D' with $G(D') = G(D)$ and a function $p' \colon V \to [a]$ such that D' is (a, b)-pebbled by p' with $p'(\mathrm{Reach}_{D'}(U) \setminus U) = 0$ can be computed with a runtime in $\mathrm{O}(a|U||E|)$.

Proof. A depth-first search starting from all vertices in U simultaneously allows to find a path P from a vertex $u \in U$ to a vertex $v \in \mathrm{Reach}(\{u, v\}) \setminus U$ with $p(v) \geq 1$, if such a vertex exists. Otherwise, no such path is found and we know $p(\mathrm{Reach}_D(U) \setminus U) = 0$. This depth-first search is possible with runtime in $\mathrm{O}(|E|)$.

If no other vertex with a pebble was found, we can return D and p. Hence, assume such a path P was found.

By repeatedly applying Lemma 5.6, we can reverse every edge in P starting in v and move a pebble from v along P to u. This can also be realized with runtime in $\mathrm{O}(|E|)$. This increases $p(U)$ by one. Note that we keep as invariant that D is (a, b)-pebbled by p. We can therefore repeat these steps to further increase $p(U)$.

Since there can be at most $a|U|$ pebbles on U while D is (a, b)-pebbled, no new pebble can be reached after at most $a|U|$ path reversions. We thus obtain D' and p' such that D' is (a, b)-pebbled by p' with $p'(\mathrm{Reach}_{D'}(U) \setminus U) = 0$. □

This yields a corollary for later use.

Corollary 5.10
Let $a, b \in \mathbb{N}_0$ with $b < 2a$. Let $D = (V, E)$ be a digraph that is (a, b)-pebbled by a function $p\colon V \to [a]$. Let $\{u, v\} \in \binom{V}{2}$ be not free in D.

Then we have $B_D(u, v) \subseteq \langle \mathrm{Reach}_D(u, v)\rangle_D$.

Proof. By Theorem 5.9 we obtain a graph D' that is (a, b)-pebbled by a function $p'\colon V \to [a]$ with $G(D) = G(D')$ and $p'(\mathrm{Reach}_{D'}(u, v) \setminus \{u, v\}) = 0$. By Lemma 5.3 we know $p'(\{u, v\}) \leq b$. With Theorem 5.8 this implies $G(B_D(u, v)) = G(B_{D'}(u, v)) \subseteq G(\langle \mathrm{Reach}_{D'}(u, v)\rangle_D)$. Digraph D' is obtained from D by reversing paths starting in u or v alone. Since all vertices in such a path could be reached from u or v before, no new edge ends at a vertex out of reach. Therefore no new vertex can be reached from u or v in D'. I.e., we obtain $V(B_D(u, v)) \subseteq \mathrm{Reach}_{D'}(u, v) \subseteq \mathrm{Reach}_D(u, v)$ and thus $B_D(u, v) \subseteq \langle \mathrm{Reach}_D(u, v)\rangle_D$. ☐

Algorithm 5.1: Pebble-Collection on (a, b)-pebbled digraph D

Data: A digraph D that is (a, b)-pebbled by a function $p\colon V \to [a]$. And a subset $U \subseteq V$ where pebbles shall be collected.

Result: A digraph D with $G(\mathsf{D}) = G(D)$ that is (a, b)-pebbled by a function $\mathsf{p}\colon V \to [a]$ such that $\mathsf{p}(\mathrm{Reach}_{\mathsf{D}}(U) \setminus U)) = 0$

Function collect(D, p, U)
- $\mathsf{D} \leftarrow D$;
- $\mathsf{p} \leftarrow p$;
- **while** *$\exists$path P from $u \in U$ to $v \in V \setminus U$ with $\mathsf{p}(v) \geq 1$* **do**
  - reverse every edge in P;
  - $\mathsf{p}(u) \leftarrow \mathsf{p}(u) + 1$;
  - $\mathsf{p}(v) \leftarrow \mathsf{p}(v) - 1$;
- **return** (D, p);

With Theorem 5.9 we can detect (a, b)-sparse graphs: Given a graph $H = (V, E)$ we start with a digraph $D = (V, \emptyset)$ where every vertex holds a pebbles. For every edge $\{u, v\} \in E$ we collect pebbles on u and v applying Theorem 5.9. If at most b pebbles could be collected, then graph D is not (a, b)-sparse by Lemma 5.7. Since $G(D) \subseteq H$ the given graph H is also not (a, b)-sparse which is then returned as the result of the algorithm. Otherwise add uv or vu by applying Lemma 5.4. If all edges could be added, graph D is (a, b)-pebbled and $G(D) = H$. Therefore graph H is (a, b)-sparse. This is described in Algorithm 5.2.

Theorem 5.11 (Lee and Streinu [LS08])
Detecting whether a given graph $H = (V, E)$ is (a, b)-sparse is possible with runtime in $\mathrm{O}(a|E|^2) \subseteq \mathrm{O}(a^3|V|^2)$.

Proof. We have already described an algorithm to detect (a, b)-sparse graphs. It remains to show that its runtime is in $\mathrm{O}(a^3|V|^2)$. By Theorem 5.9 we have for every edge $e \in E$ a runtime in $\mathrm{O}(a|E|)$ to collect the pebbles. Note that e can be added in constant time. We therefore have a runtime in $\mathrm{O}(a|E|^2)$. Further note we have to consider at most $a|V| - b + 1$ edges, since any subgraph with more than $a|V| - b$ edges is not (a, b)-sparse. Thus, we have a runtime in $\mathrm{O}(a^3|V|^2)$. ☐

Algorithm 5.2: Algorithm deciding whether a graph H is (a, b)-sparse

Data: A graph $H = (V, E)$

Result: *true* if H is (a, b)-sparse. *false* otherwise

$\mathsf{D} \leftarrow (V, \emptyset)$;
$\mathsf{p} \leftarrow (V \to [a], v \mapsto a)$;
for $\{\mathsf{u}, \mathsf{v}\} \in E$ **do**
  $(\mathsf{D}, \mathsf{p}) \leftarrow \texttt{collect}(\mathsf{D}, \mathsf{p}, \{\mathsf{u}, \mathsf{v}\})$;
  if $\mathsf{p}(\mathsf{u}) + \mathsf{p}(\mathsf{v}) \geq b + 1$ **then** // enough pebbles
    if $\mathsf{p}(\mathsf{u}) \geq 1$ **then** // add edge uv by Lemma 5.4
      $\mathsf{p}(u) \leftarrow \mathsf{p}(u) - 1$;
      $\mathsf{D} \leftarrow \mathsf{D} + uv$;
    else // add edge vu by Lemma 5.4
      $\mathsf{p}(v) \leftarrow \mathsf{p}(v) - 1$;
      $\mathsf{D} \leftarrow \mathsf{D} + vu$;
  else `// at most` b `pebbles on u and v:` $\langle \mathrm{Reach}_{\{\mathsf{u},\mathsf{v}\}} \rangle_{\mathsf{D}}$ `already tight`
    return *false*;
`// all edges have been added to D and` $\mathsf{D} = H$ `is` (a, b)`-pebbled`
return true;

5.2 Computing Optimal Global (a, b)-Sparse Covers

For a given $k \in \mathbb{N}$ and a given host graph H we can already use Algorithm 5.2 to decide $c_g^{\mathcal{G}(a,b)}(H) \leq k$ by deciding whether host graph H is (ka, kb)-sparse, as the following lemma shows.

Lemma 5.12
Let $a, b \in \mathbb{N}_0$ with $b < 2a$. Let H be a host graph. Further let $k \in \mathbb{N}$. Then the following three statements are equivalent:

(i) H *is* (ka, kb)*-sparse*

(ii) $k \geq \max_{\substack{H' \subseteq H \\ |H'| \geq 2}} \left\lceil \frac{||H'||}{a|H'| - b} \right\rceil$.

(iii) $c_g^{\mathcal{G}(a,b)}(H) \leq k$

Proof. Graph H is by definition (ka, kb)-sparse if and only if for $H' \subseteq H$ with $|H'| \geq 2$ we have $||H'|| \leq ka|H'| - kb$ which is equivalent to $\frac{||H'||}{a|H'|-b} \leq k$. This in turn is equivalent to

$$k \geq \max_{\substack{H' \subseteq H \\ |H'| \geq 2}} \left\lceil \frac{||H'||}{a|H'| - b} \right\rceil$$

Statement (ii) in turn is equivalent to Statement (iii) by Theorem 4.26. This concludes the proof. □

So we can check whether $c_g^{\mathcal{G}(a,b)}(H) \leq k$ by checking whether H is (ka, kb)-sparse using Lee and Streinu‘s Pebble Game Algorithm with runtime in $\mathrm{O}((ka)|E|^2) \subseteq \mathrm{O}(|E|^3/|V|)$. But this does not give a k-global cover.

We provide an algorithm that returns a best-possible global $\mathcal{G}(a, b)$-cover of H with runtime in $\mathrm{O}(|V| \cdot |E|^2)$. The algorithm is based on the Pebble Game Algorithm used to detect

(ka, kb)-sparse graphs where $k = c_g^{\mathcal{G}(a,b)}(D)$, i.e., parameter k is increased during the execution. An edge-partition of D into k guests $G_1, \ldots, G_k$ that are (a, b)-pebbled is maintained. In the end this partition provides an optimal (a, b)-sparse cover of H.

Let $\{u, v\} \in E$ be an edge that is about to be added to the digraph D. If we can collect $b+1$ pebbles on u and v within a single guest G_i, then we can apply Lemma 5.4 to add $\{u, v\}$ to G_i and proceed with the next edge. Otherwise, we encounter complications. Edge $\{u, v\}$ can not be added to any guest and we have only kb pebbles on u and v after collecting pebbles only within the guests. However, there may still be a vertex w with a pebble in $\mathrm{Reach}_D(\{u, v\}) \setminus \{u, v\}$, since all paths from u and v to w may be partitioned into more than one guest. This is a problem, since in this case we have no single (a, b)-sparse graph containing that path so we can not apply Lemma 5.6. Indeed, there are cases where such a path can not be reversed correspondingly (see Figure 5.5 and 5.6).

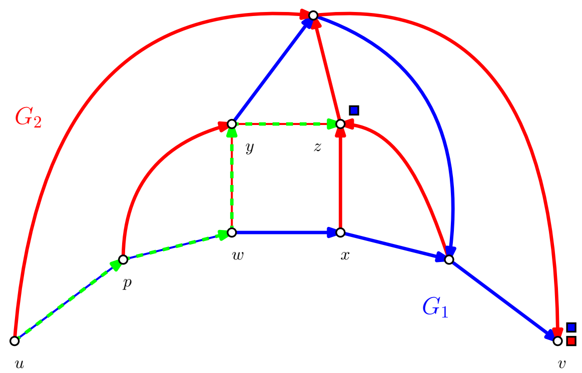

Figure 5.5: A $(2, 2)$-sparse graph H covered by two $(1, 1)$-sparse graphs (forests) and a global pebble path $P = (u = x_1^1, p = x_2^1, w = x_3^1 = x_1^2, y = x_2^2, z = x_3^2 = x_1^3)$ of global length 3. With guest function π such that $\pi(1) = \pi(3) = 1$ and $\pi(2) = 2$.

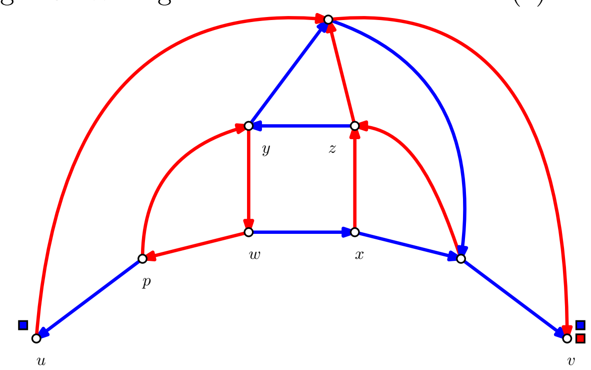

Figure 5.6: Result of reversing path P in the guests of graph H in Figure 5.5. The edges have to belong to the guest containing the corresponding pebble (eventually obtained from the succeeding edge). Note that p, y and w induce a cycle in one guest which is thus no longer $(1, 1)$-sparse.

Instead of reversing such a path directly, we use it to find an *augmenting path* as introduced by Edmonds [Edm65] and described by Gabow and Westermann [GW92]. Augmenting paths are placed in the setting of matroids. We therefore at least implicitly use the (a, b)-sparsity matroid structure on H as provided by Corollary 4.23.

Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid. Then a set $J \subset E$ is called *circuit* in $\mathcal{M}$, if it is minimal with $J \notin \mathcal{I}$. Let $(E, \mathcal{I})$ be a matroid, let $I \subseteq E$ and let $S = \{I_1, \ldots, I_k\} \subseteq \mathcal{I}$ with $\bigcup S = I$. Roughly speaking, an augmenting path is a finite sequence of edges $(e_0, \ldots, e_s)$ that allows to add e_0 by exchanging these edges between the independent sets in S: inserting e_0 into some partition set I_q creates a circuit, which is broken by removing e_1 from I_q; inserting

e_1 into a different partition set creates a circuit which is broken by removing e_2. This pattern continues until e_s is inserted into a partition set and does not create a circuit. If the sequence has no shortcut, i.e., if for $i \in [s-2]$ the circuit created by adding element e_i to a partition set does not contain any e_j with $j > i+1$, then applying all these exchanges adds e_0 while keeping the partition of $I + \{e_0\}$ into independent sets.

If no such augmenting path exists, then the number of guests does not suffice and a new guest may be added.

Consider the (a, b)-sparsity matroid of a graph H. Let $G \subseteq H$ be an (a, b)-sparse subgraph and let $uv \in E(H) \setminus E(G)$. If $E(G) + uv$ is dependent, then there is a block containing u and v in G. In that case the created circuit consists of the edge uv and the minimum block containing u and v which we called $B(u, v)$. By definition of $B(u, v)$, after removing any edge of $B(u, v)$ there is no block left containing u and v. Therefore, edge uv can be added to H as replacement for an arbitrary edge in $B(u, v)$. This fact can be used to compute an optimal $\mathcal{G}(a, b)$-cover of a graph $G = (V, E)$ in the way described for general matroids by Edmonds [Edm65]. We define augmenting paths accordingly:

Let $a, b \in \mathbb{N}_0$ with $b < 2a$. Let further $G_1, \dots, G_k$ be pairwise edge-disjoint (a, b)-sparse graphs. Let $u, v \in V(G_1 \cup \dots \cup G_k)$ with $u \neq v$ and $uv \notin G_1 \cup \dots \cup G_k$.

We call a finite sequence $\sigma = (uv = e_0, \dots, e_l)$ a *pre-augmenting path* (with regards to $G_1, \dots, G_k$) if:

(i) $\exists i \in [k]$: for $xy = e_l$ the set $\{x, y\}$ is free in G_i

(ii) $\forall i \in [l] : \exists j \in [k] : e_i \in B_j(e_{i-1})$

We call l the *length* of σ. We call a tuple $(i, j) \in [l]^2$ a *shortcut* of σ, if we have $j - i > 1$ and that $\exists r \in [k] : e_j \in B_r(e_i)$ holds. A pre-augmenting path σ is an *augmenting path*, if it has no shortcut. See Figure 5.7 for an example.

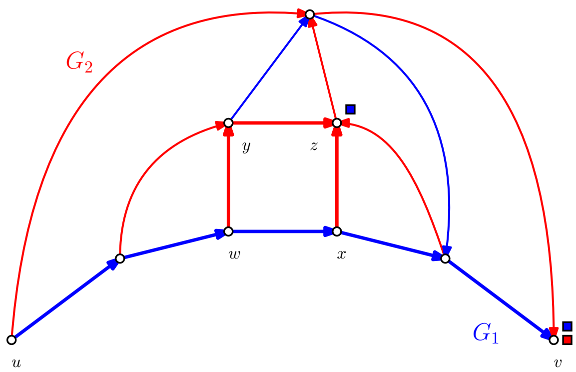

Figure 5.7: The minimum block $B_1(u, v)$ contains all edges of the path in G_1 from u to v, especially edge wx. The minimum block $B_2(w, x)$ contains the edges wy,yz and xz. The edge yz is free in G_1. We obtain the augmenting path $\sigma = (uv, wx, yz)$. For sequence $\pi = (1, 2, 1)$ we have $R(\pi, \{G_1, G_2\}, uv, 1) = B_1(u, v)$ and in this case $R(\pi, \{G_1, G_2\}, uv, 2) = G_2$. Since $R_\pi(uv, 2)$ contains an edge that is free in G_1, it is not nested in G_1 and thus $R(\pi, \{G_1, G_2\}, uv, 3)$ does not exist.

We introduce pre-augmenting paths, since they are easier to find but still yield augmenting paths.

Lemma 5.13
Let $a, b \in \mathbb{N}_0$ with $b < 2a$. Let further $G_1, \dots, G_k$ be pairwise edge-disjoint (a, b)-sparse graphs. Let $u, v \in V(G_1 \cup \dots \cup G_k)$ with $u \neq v$ and $uv \notin G_1 \cup \dots \cup G_k$. Let $\sigma' = (uv = e'_0, \dots, e'_l)$ be a pre-augmenting path.

Then σ' is an augmenting path or there is an augmenting path $\sigma = (uv = e_0, \ldots, e_j)$ with $j < l$.

Proof. We apply induction on l. If $l = 0$, then σ' can not contain a shortcut and is thus an augmenting path.

Next let $j \in \mathbb{N}$ and assume as induction hypothesis that the statement holds for $l \in \{0, \ldots, j\}$. Let $\sigma' = (uv = e'_0, \ldots, e'_{j+1})$ be a pre-augmenting path. If σ' contains no shortcut, then it is an augmenting path and the statement holds for σ'. Otherwise, there is a shortcut (c, d) of σ'. Then $\sigma'' = (uv = e'_0, \ldots, e'_c, e'_d, \ldots, e'_{j+1})$ is also a pre-augmenting path. It is shorter than σ' and therefore we can conclude with induction hypothesis that there is an augmenting path $\sigma = (uv = e_0, \ldots, e_m)$ with $m < j + 1$. This concludes the induction and thus the proof. □

A direct adaptation of Edmonds approach to provide an optimal global (a, b)-sparse cover of a given host graph $H = (V, E)$ results in a runtime in $\mathrm{O}(|E|^3)$. For the outline of this algorithm we reference Lemma 5.15 and Lemma 5.19 which we will show later. We start with an empty set S of guests represented as (a, b)-pebbled digraphs. We add all edges of H one by one as follows. Lets say we are about to add edge uv. Let D be the subgraph of H realized by the current guests. We implicitly consider a digraph in which we search for the augmented path: We define $M = ((E(D) \cup \{uv\}), F)$ where $F = \{ef \in E(M)^2 \mid \exists j \in [k] : f \in B_j(e)\}$. We apply a breath-first search for an edge that is free in some guest and start that search in uv. By Theorem 5.8 and Lemma 5.3, we can for a given edge xy and $j \in [k]$ either verify xy is free in G_j, or we have $p_j(\mathrm{Reach}(x, y)) = p_j(x, y) \leq b$ and can thus compute the block $B_j(x, y)$ with a simple depth-first search. By Theorem 5.9 this state can be reached with runtime in $\mathrm{O}(|E(G_j)|)$ (for constant a). Thus, one such breath-first search has a runtime in $\mathrm{O}(|E|^2)$. If a free edge is found, the search path $\sigma = (e_0 = uv, \ldots, e_l)$ is a pre-augmenting path by construction and has no shortcut since it has minimum length. Thus we found an augmenting path. By Lemma 5.15 we can use σ to add uv with a runtime in $\mathrm{O}(|V| \cdot |E|) \subseteq \mathrm{O}(|E|^2)$. If no free edge is found, we need at least $k + 1$ guests to cover H by Lemma 5.19. We can thus add a new guest G_{k+1} which contains edge uv. The algorithm finishes if all edges have been added. In that case the computed guests provide an optimal global $\mathcal{G}(a, b)$-cover of H.

Note that the correctness of this algorithm was already established by Edmonds [Edm65]. We prove the needed lemmas in the setting of (a, b)-sparsity matroids and use them for an improved algorithm for this setting. To prove Lemma 5.15, we first introduce Lemma 5.14 which ensures that the exchange of edges preserves (a, b)-sparsity and does not change blocks that are still needed later in the process.

Lemma 5.14
Let $a, b \in \mathbb{N}_0$ with $b < 2a$. Let H be an (a, b)-sparse graph and let $u, v \in H$ with $uv \notin H$. Let $\{u, v\}$ be not free in H and let $xy \in B_H(u, v)$. Further let $U \subseteq V(H)$ with $|U| \geq 2$.

Then $H' = H - xy + uv$ is (a, b)-sparse. Further we have that

(i) if we have $\{u, v\} \not\subseteq U$ and $\{x, y\} \not\subseteq U$, then $\langle U \rangle_H$ is (a, b)-tight if and only if $\langle U \rangle_{H'}$ is (a, b)-tight

(ii) $\langle U \rangle_H$ is an (a, b)-component of H if and only if $\langle U \rangle_{H'}$ is an (a, b)-component of H'.

Proof. Let $G = \langle U \rangle_H$ and $G' = \langle U \rangle_{H'}$.

If we have $u, v \in U$ and $\{x, y\} \not\subseteq U$, then G is not (a, b)-tight, since it would otherwise contain $B_G(u, v)$ and thus also x and y. In this case we obtain $||G'|| \leq ||G|| + 1 \leq$

$a|G| - b - 1 + 1 = a|G'| - b$. Otherwise we have $\{u, v\} \not\subseteq U$ or $x, y \in U$, and then we obtain $||G'|| \leq ||G||$ which implies $||G'|| \leq ||G|| = a|G| - b = a|G'| - b$. We can conclude H' is (a, b)-sparse.

Assume G is (a, b)-tight. If we have $u, v \in U$ then we also have $xy \in G$ by definition of $B_H(u, v)$. Therefore we can compute $||G'|| = ||G - uv + xy|| = ||G|| = a|G| - b = a|G'| - b$ meaning G' is also (a, b)-tight. Especially, we know that subgraph $B = \langle V(B_H(u, v))i\rangle_{H'}$ is a block in H' containing u, v, x and y. If we have $\{x, y\} \not\subseteq U$, then we obtain $||G'|| \geq ||G||$ meaning G' is also (a, b)-tight.

Next assume G' is (a, b)-tight. If we have $\{u, v\} \not\subseteq U$ and $\{x, y\} \not\subseteq U$, then we have $G = G'$ and thus G is also (a, b)-tight. Therefore Statement (i) holds.

We next show that if G is a component, then G' is (a, b)-tight and vice versa.

Assume G is a component. If we have $\{u, v\} \not\subseteq U$ and $\{x, y\} \not\subseteq U$, then we know by Statement (i) that G' is (a, b)-tight. Otherwise we have $\{u, v\} \subseteq U$ or $\{x, y\} \subseteq U$, and then we have by Theorem 4.20 that $B_H(u, v) \subseteq G$. In that case we have $||G'|| = ||G||$ meaning G' is (a, b)-tight. For the case that G' is a component we argue analogously using B instead of $B_H(u, v)$ to show G is (a, b)-tight.

We finally show Statement (ii). Assume G is a component. Then G' is a block and thus there is a component C' of H' with $G' \subseteq C'$. We obtain that $C = \langle V(C')\rangle_{H'}$ is (a, b)-tight. Since G is a component and we have $U \subseteq V(C')$, this implies $U = V(C')$ and therefore G' is the component C'. For the case that G' is a component we analogously show that G is a component. We thus obtain that Statement (ii) holds. $\square$

With Lemma 5.14 we can exchange the last edge of an augmenting path and obtain that the remaining path is then an augmenting path. By iterating from the end to uv finally $\{u, v\}$ is free in some graph and can be added.

Lemma 5.15
Let $a, b \in \mathbb{N}_0$ with $b < 2a$. Let further $G_1, \ldots, G_k$ be pairwise edge-disjoint digraphs with $V = V(G_1) = \cdots = V(G_k)$. Let $u_0, v_0 \in V$ with $u_0 \neq v_0$ and $u_0v_0 \notin H = (V, E) = G_1 \cup \ldots \cup G_k$.

For $i \in [k]$ let G_i be (a, b)-pebbled by a function $p_i \colon V \to [a]$. Let there be an augmenting path $\sigma = (u_0v_0 = e_0, \ldots, u_lv_l = e_l)$. And a function $\pi \colon [l + 1] \to [k]$ such that we have $\forall i \in [l] : e_i \in B_{\pi(i)}(u_{i-1}, v_{i-1})$ and e_l is free in $G_{\pi(l+1)}$.

Then we can compute a set of (a, b)-pebbled digraphs ${G'}_1, \ldots, {G'}_k$ such that

$$G({G'}_1 \cup \ldots \cup {G'}_k) = G((V, E \cup \{u_0v_0\})$$

with runtime in $O(|V|)$ per edge in σ for constant a.

Proof. See Figure 5.8, 5.9 and 5.10 for a sketched example execution of the used algorithm. By Lemma 5.3 there is a path $X = (x_1, \ldots, x_m) \subseteq G_{\pi(l+1)}$ with $x_1 \in \{u_l, v_l\}$ and $p_{\pi(i)}(x_m) \geq 1$. Such a path can be found with a depth-first search in $O(|V|)$. By repeated use of Lemma 5.6 we can move a pebble from x_m to x_1 by reversing the edges of X. Lets assume without loss of generality that $x_1 = u_l$. Then we can add edge u_lv_l to the resulting guest $G''_{\pi(l+1)}$ by applying Lemma 5.4.

We apply induction on l. For $l = 0$ the steps above suffice to prove the statement. Hence, assume as induction hypothesis that for any augmenting path $\sigma' = (e'_0, \ldots, e'_{l'})$ with $l' = l - 1 \in \mathbb{N}_0$ the corresponding statement holds. Then we proceed after the steps above. Since the resulting guest ${G'}_{\pi(l+1)}$ covers edge e_l, it can be removed from $G_{\pi(l)}$ by applying

Lemma 5.5. By Lemma 5.14 we obtain that sequence $\sigma' = (u_0v_0 = e_0, \dots, u_{l-1}v_{l-1} = e_{l-1})$ is an augmenting path and we can apply the induction hypothesis. This concludes the induction and thereby the proof. $\square$

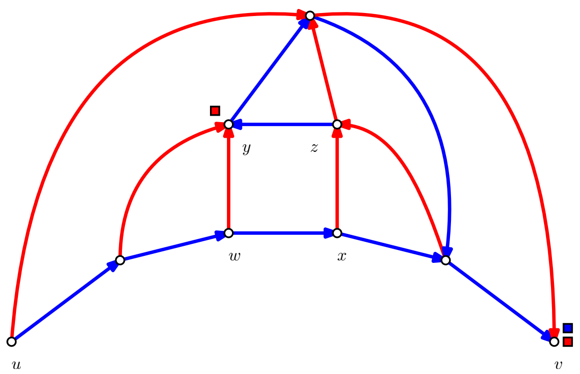

Figure 5.8: Graph $G_1 \cup G_2$ of Figure 5.7 after processing yz as last edge of augmenting path $\sigma = (uv, wx, yz)$ as described in the proof of Lemma 5.15.

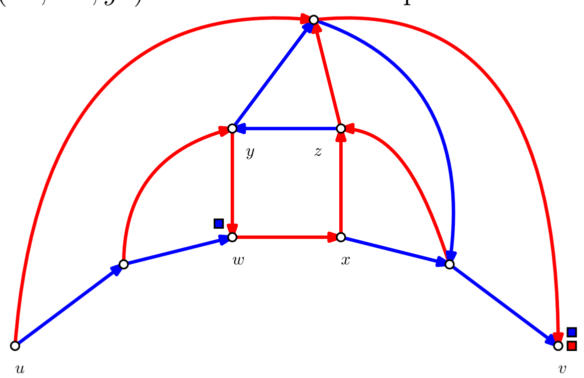

Figure 5.9: Result of processing wx in Figure 5.8.

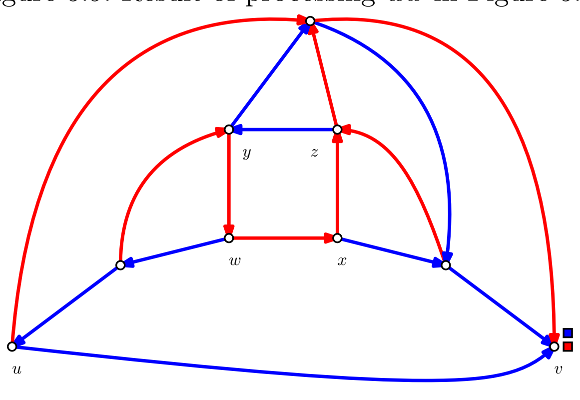

Figure 5.10: Result of processing uv in Figure 5.9.

To prove Lemma 5.19 we first introduce the notion of *range graphs*. In our improved algorithm we will first obtain a sequence $\pi \in [k]^l$ such that there is an augmenting path $\sigma = (e_0 = uv, \dots, e_l)$ such that for $i \in [l]$ we have $e_i \in G_{\pi(i)}$. This sequence is then used to find an augmenting path by constructing a sequence of range graphs. These range graphs roughly correspond with the levels of a breath-first search respecting sequence π of guests for the minimum blocks considered.

Let $a, b \in \mathbb{N}_0$ with $b < 2a$. Let G and H be (a, b)-sparse graphs such that for every edge $xy \in H$ the set $\{x, y\}$ is not free in G. Then we call H *nested* in G and we define the *range graph* $R(G, H) = \bigcup_{xy \in H} B_G(x, y)$.

Let further $S = \{G_1, \ldots, G_k\}$ be a set of (a, b)-sparse graphs. Let $u, v \in V(G_1 \cup \ldots \cup G_k)$ with $u \neq v$ and $uv \notin G_1 \cup \ldots \cup G_k$. Let $l \in \mathbb{N}$ and let π be a finite sequence in $[k]^l$. Then we define $R(\pi, S, uv, 0)$ as the graph $(\{u, v\}, \{uv\})$ and for $i \in [l]$, if for $j \in [i]$ graph $R(\pi, S, uv, (j-1))$ is nested in $G_{\pi(j)}$, we define

$$R(\pi, S, uv, i) = R(G_{\pi(i)}, R(\pi, S, uv, (i-1))).$$

See Figure 5.7 for an example. If π, S and uv are clear, we usually just write $R(i)$ instead of $R(\pi, S, uv, i)$.

For $i \in [k]$ we denote the last range graph contained in G_i by $R_i(\pi, S, uv)$, i.e., we set $t = \max\{j \in [l] \mid \pi(j) = t\}$ and $R_i(\pi, S, uv) = R(\pi, S, uv, t)$. The sequence π is called *terminated*, if adding more elements to the sequence does not change for any $i \in [k]$ the range graph $R_i(\pi, S, uv)$. I.e., sequence π is terminated, if for $i, j \in [k]$ with $i \neq j$ we have $R_i(\pi, S, uv) = R(G_i, R_j(\pi, S, uv))$. Note that by definition for every range graph $R(\pi, S, uv, i)$ any edge $xy \in R(\pi, S, uv, i)$ is contained in a block of $R(\pi, S, uv, i)$. Therefore any two consecutive elements of π can be assumed to be different. We usually write R_j instead of $R_j(\pi, S, uv)$ if π, S and uv are clear.

Note that by construction we have $V(H) \subseteq V(R(G, H))$, since for $xy \in H$ we have $x, y \in B_G(x, y)$.

Let $i \in [l]$. We have that $R(i) = \bigcup_{xy \in R(i-1)} B_{\pi(i)}(x, y)$. One can therefore think of $R(i)$ as the result of replacing each edge xy in $R(i-1)$ by $B_{\pi(i)}(x, y)$. These minimum blocks may share edges. However, by Corollary 4.21 the (a, b)-components of $R(i)$ are edge-disjoint and every edge is contained in such a component. Further, every edge xy of $R(i-1)$ is spanned by $B_{\pi(i)}(x, y)$ and thus by a component in $R(i)$. We can thus obtain $R(i)$ from $R(i-1)$ by replacing for every component C of $R(i)$ the edges of $R(i)$ that are spanned by C by the edges of C and eventually adding missing vertices of C. Lemma 5.16 shows this process only worsens the relation between the number of edges and the number of vertices.

Lemma 5.16
Let $a, b \in \mathbb{N}_0$ with $b < 2a$. Let H be a graph with $||H|| \geq a|H| - b$. Let G be an (a, b)-tight graph. Let $U = V(H) \cap V(G)$. Let $X = (H - \langle U \rangle_H) \cup G$ be the graph obtained by removing all edges from H that are spanned by G and adding all new edges and vertices of G. Let $||\langle U \rangle_H|| \leq a|U| - b$.

Then we have $||X|| \geq a|X| - b$.

Proof. We compute

$$||X|| = ||H|| + ||G|| - ||\langle U \rangle_H|| \geq a|H| - b + a|G| - b - ||\langle U \rangle_H|| \geq a|X| - b.$$

□

As a consequence we obtain that, if $R(i-1)$ is (a, b)-tight, then range graph $R(i)$ is also (a, b)-tight. Note that range graph $R(1) = B_{\pi(1)}(u, v)$ is (a, b)-tight, if it exists.

Lemma 5.17
Let $a, b \in \mathbb{N}_0$ with $b < 2a$. Let H be an (a, b)-tight graph and let G be an (a, b)-sparse graph with $V(H) \subseteq V(G)$ such that H is nested in G.

Then $R(G, H)$ is (a, b)-tight and the minimum block $B_G(V(H))$.

Proof. Since G is (a, b)-sparse, also $R = R(G, H)$ is (a, b)-sparse. By Corollary 4.21 the components $C_1, \ldots, C_m$ of R are edge-disjoint.

We define $H_0 = H$ and for $i \in [m]$ we define $H_i = (H_{i-1} - \langle V(C_i) \cap V(H_{i-1})\rangle_{H_{i-1}}) \cup C_i$. Using Lemma 5.16 and the fact that the components $C_1, \ldots, C_m$ are edge-disjoint, we obtain by induction for $i = 0, \ldots, m$ that $||H_i|| \geq a|H_i| - b$ and

$$H_i = (H - \langle V(C_1 \cup \ldots \cup C_i) \cap V(H)\rangle_H) \cup C_1 \cup \ldots \cup C_i.$$

Therefore we have $||H_m|| \geq a|H_m| - b$ and since every edge is spanned by a block and thus by a component, we obtain

$$H_m = (H - \langle V(C_1 \cup \ldots \cup C_m) \cap V(H)\rangle_H) \cup C_1 \cup \ldots \cup C_m = C_1 \cup \ldots \cup C_m.$$

By definition of R, every vertex of R is contained in a block and thus in a component. I.e, we have $V(R) = V(C_1 \cup \ldots \cup C_m) = V(H_m)$. We can now calculate

$$||R|| \geq ||H_m|| \geq a|H_m| - b = a|R| - b.$$

Therefore R is (a, b)-tight.

For every edge $xy \in H$ the block $B_G(x, y)$ must be subgraph of any block containing $V(H)$ and thereby x and y. Thus R is by construction the minimum block containing all vertices in $V(H)$. □

We will use Lemma 5.18 in the proof of Lemma 5.19 to show H is (ka, kb)-tight in absence of an augmenting path.

Lemma 5.18
Let $a, b \in \mathbb{N}_0$ with $b < 2a$. Let further $S = \{G_1, \ldots, G_k\}$ be a set of pairwise edge-disjoint (a, b)-sparse graphs. Let $u, v \in V(G_1 \cup \ldots \cup G_k)$ with $u \neq v$ and $uv \notin G_1 \cup \ldots \cup G_k$. Further let $l \in \mathbb{N}$ and let $\pi \in [k]^l$ be a terminated sequence.

Then $R_1 \cup \ldots \cup R_k$ is (ka, kb)-tight.

Proof. For $i, j \in [k]$ with $i \neq j$ we have $R_i = R(G_i, R_j)$. This implies $V(R_j) \subseteq V(R_i)$. Therefore we have $V(R_1) = \cdots = V(R_k)$. Set $n = V(R_1)$. By Lemma 5.17 we have that for $i \in [k]$ reach graph R_i is (a, b)-tight, i.e., we have that $||R_i|| = an - b$. Thus, we obtain $||R_1 \cup \ldots \cup R_k|| = kan - kb = ka|R_1 \cup \ldots \cup R_k| - kb$. With Lemma 5.12 we conclude $R_1 \cup \ldots \cup R_k$ is (ka, kb)-tight. □

We can now show that we always find an augmenting path, if an edge can be added without increasing the number of guests.

Lemma 5.19
Let $a, b \in \mathbb{N}_0$ with $b < 2a$. Let further $G_1, \ldots, G_k$ be pairwise edge-disjoint (a, b)-sparse graphs. Let $u, v \in V(G_1 \cup \ldots \cup G_k)$ with $u \neq v$ and $uv \notin G_1 \cup \ldots \cup G_k$.

Then graph $G_1 \cup \ldots \cup G_k \cup (\{u, v\}, \{uv\})$ is (ka, kb)-sparse if and only if there is an augmenting path $\sigma = (uv = e_0, \ldots, e_l)$.

In this case we define for $j \in [k]$ the guest $G'_j = (G - S_j) \cup T_j$ where $S_j = \{e_i \in \sigma \mid e_i \in G_j\}$ and $T_j = \{e_i \in \sigma \mid e_{i+1} \in G_j$ or $(i = l$ and $j = \min\{r \in [k] \mid e_l$ is free in $G_r\})\}$.

Then we have $G'_1 \cup \ldots \cup G'_k = G_1 \cup \ldots \cup G_k \cup (\{u, v\}, \{uv\})$ and for $i \in [k]$ the graph G'_i is (a, b)-sparse.

Proof. We consider a finite sequence $\pi \in [k]^l$ that repeats the sequence $(1, \ldots, k)$ at least $|G_1 \cup \ldots \cup G_k| + 2$ times. If π is terminated, then by Lemma 5.18 graph $R_1 \cup \ldots \cup R_k$ is (ka, kb)-tight. In that case graph $G_1 \cup \ldots \cup G_k \cup (\{u, v\}, \{uv\})$ is not (ka, kb)-sparse.

First, let $s \in [l - 2k + 1]$. Consider the range graphs $R(s), \ldots, R(s + 2k - 1)$. We have $V(R(s)) \subseteq \cdots \subseteq V(R(s + 2k - 1))$.

If we have $V(R(s)) = V(R(s + 2k - 1))$, then we set $U = V(R(s))$ and obtain for $i \in [k]$ that $R(s - 1 + i) = R(s - 1 + k + i) = B_i(U)$. An easy induction shows that in this case $R_i = B_i(U)$ and thus π is terminated.

Hence, assume that for $s \in [l-2k+1]$ we have $V(R(s))+1 \leq V(R(s+2k-1))$. Then there either exists an $i \in [l]$ and a $j \in [k]$ with $i \neq j$ such that $R(i)$ is not nested in G_j, or for $r \in [l - k]$ that $|V(R(r))| + 1 \leq |V(R(r + k)|$. In the second case we obtain a contradiction since it implies $V(R(l)) \geq |G_1 \cup \ldots \cup G_k| + 1 > |G_{\pi(l)}|$ while $R(l) \subseteq G_{\pi(l)}$. Therefore, we have the first case and there is an edge $e_l = xy \in R(i)$ such that $\{x, y\}$ is free in G_j.

For $i \in [l]$ we have by definition of $R(i)$ that for every edge $e_i \in R(i)$ there is some edge $e_{i-1} \in R(i - 1)$ with $e_i \in B_{\pi(i)}(e_{i-1})$, unless $i = 1$, in which case $R(i) = B_{\pi(1)}(u, v)$ or $\{u, v\}$ is itself free in $G_i = G_1$. By induction sequence $(e_1, \ldots, e_l)$ is a pre-augmenting path.

Since we have $l \in \mathbb{N}_0$, and there is at least one pre-augmenting path, there is by Lemma 5.13 also an augmenting path.

Assume there is an augmenting path $\sigma = (uv = e_0, \ldots, e_l)$ and let for $j \in [k]$ the graph G' be defined as above.

First note that

$$G'_1 \cup \ldots \cup G'_k = (G_1 \cup \ldots \cup G_k - (S_1 \cup \ldots \cup S_k)) \cup T_1 \cup \ldots \cup T_k = G_1 \cup \ldots \cup G_k \cup (\{u, v\}, \{uv\}).$$

Let $j \in [k]$. We finally show guest G'_j is (a, b)-sparse. We define a sequence of graphs which represent the exchanges of single edges leading from $G = G_j$ to $G' = G'_j$. We define

$$G^0 = \begin{cases} G + e_l & \text{, if } e_l \in T_j \\ G & \text{, otherwise} \end{cases}.$$

Further, for $i \in [l]$ we define

$$G^i = \begin{cases} G^{i-1} - e_{l-i+1} + e_{l-i} & \text{, if } e_{l-i} \in T_j \\ G^{i-1} & \text{, otherwise} \end{cases}$$

First note that $G^l = G'$. We apply induction to show for $n = 0, \ldots, l + 1$ that

(i) G^n is (a, b)-sparse.

(ii) for $xy = e_m \in T_j$ we have that if $m < l - n$ holds, then we have $B_G(x, y) \subseteq G^n$ and $B_{G^n}(x, y) = B_G(x, y)$.

For $n = 0$ this statement follows from the definitions of G^0 and T_j and Requirement (ii) of (pre-)augmenting path σ: If $e_l \in T_j$ then we have that e_l is free in G^0. Therefore graph $G + e_l$ is (a, b)-sparse. Further Property (ii) is preserved by adding edges. We conclude the statement holds for $n = 0$.

Now assume the statement holds for $n = 0, \ldots, l$. Let $xy = e_{l-(n+1)}$. If $e_{l-(n+1)} \notin T_j$, then we have $G^{n+1} = G^n$, i.e., the statement holds. Otherwise we have with Property (ii) that $wz = e_{l-n} \in B_{G^n}(x, y) = B_G(x, y)$ and $G^{n+1} = G^n - wz + xy$. With Lemma 5.14 we obtain that G^{n+1} is (a, b)-sparse and for $U \subseteq V(G)$ with $\{x, y\} \nsubseteq U$ and $\{wz\} \nsubseteq U$ we have that $\langle U\rangle_{G^{n+1}}$ is (a, b)-tight if and only if $\langle U\rangle_{G^n}$ is (a, b)-tight. Let $m \in \mathbb{N}_0$ with $m < l - (n + 1)$ and $pq = e_{m+1} \in S_j$.

Assume that $wz \in B_G(p, q)$. In that case, the sequence $\sigma = (uv = e_0, \ldots, e_c = pq, \ldots, e_d = wz, \ldots, e_l)$ has shortcut (c, d). I.e., we have a contradiction. Thus, $wz \notin B_G(p, q)$. Next assume $xy \in B_G(p, q)$. Then by definition of $B_G(x, y)$ we also have $wz \in B_G(x, y) \subseteq B_G(p, q)$. This is a contradiction to $wz \notin B_G(p, q)$.

Since blocks must be induced subgraphs, we obtain $\{x, y\} \nsubseteq V(B_G(p, q))$ and $wz \notin B_G(p, q)$. This implies $B_G(p, q) \subseteq G^{n+1}$. Further, let $U \subseteq V(G)$ such that $\langle U\rangle_{G^{n+1}}$ is (a, b)-tight, but $\langle U\rangle_{G^n}$ is not. Since xy is the only new edge, this implies $\{x, y\} \subseteq U$ and thus $\langle U\rangle_{G^{n+1}} \nsubseteq B_G(p, q)$. By definition of minimum blocks we obtain $B_{G^n}(x, y) = B_G(x, y)$. This concludes the induction.

Hence, graph ${G'}_j = G^l$ is (a, b)-sparse which concludes the proof. □

With Lemma 5.19 we have proven all lemmas that we used to show the correctness of the direct adaption of Edmonds algorithm.

Let $a, b \in \mathbb{N}_0$ with $b < 2a$. Let $H = (V, E)$ be a graph for which we want to find an optimal global $\mathcal{G}(a, b)$-cover. We aim to improve the runtime of the direct adaptation of Edmonds algorithm from $\mathrm{O}(|E|^3)$ to $\mathrm{O}(|V| \cdot |E|^2)$. We apply the outlined algorithm with the difference, that we take another approach to find an augmenting path $\sigma = (e_0 = uv, \ldots, e_l)$ when we are about to insert edge uv. In the direct approach, we compute for every insertion for possibly nearly every edge e in every guest the minimum block. The general idea of our new approach is to compute for e only the minimum block in one guest rather than the minimum blocks in every guest. Since a single guest has at most $a|V| - b$ edges, this improves the runtime for e in one insertion from a runtime in $\mathrm{O}(|E|)$ to a runtime in $\mathrm{O}(|V|)$.

Let $G_1, \ldots, G_k$ be the current guests and let uv be the edge we are about to insert. Let $D = G_1 \cup \ldots \cup G_k$. Assume we know a sequence $\pi \in [k]^l$ such that there is an augmenting path $\sigma = (e_0 = uv, \ldots, e_l)$ of minimum length such that we have for $i \in [l]$ that $e_i \in B_{\pi(i)}(e_{i-1})$. We can then again apply breath-first search on graph $M = ((E(D) \cup \{uv\}), F)$ where $F = \{ef \in E(M)^2 \mid \exists j \in [k] : f \in B_j(e)\}$. However, we only traverse an edge $ef \in E(M)$, if we have $f \in B_{\pi(i)}(e)$ where i is the level of e in the breath-first search-tree. In this way we find σ or another augmenting path. As indicated before, the levels of this search-tree correspond with the range graphs $R(1), \ldots, R(l)$. Let $i \in [l]$. More precisely range graph $R(i)$ contains all edges of level i and prior levels in guest $G_{\pi(i)}$. We first note those range graphs actually exist.

Lemma 5.20
Let $a, b \in \mathbb{N}_0$ with $b < 2a$. Let further $S = \{G_1, \ldots, G_k\}$ be a set of pairwise edge-disjoint digraphs with $V = V(G_1) = \cdots = V(G_k)$. Let $u, v \in V$ with $u \neq v$ and $uv \notin H = G_1 \cup \ldots \cup G_k$.

For $i \in [k]$ let G_i be (a, b)-pebbled by a function $p_i \colon V \to [a]$. Let $\pi \in [k]^{l+1}$ be a sequence such that there is an augmenting path $\sigma = (e'_0 = uv, \ldots, e'_l)$ of minimum length such that e'_l is free in $G_{\pi(l+1)}$ and for $i \in [l]$ we have $e'_i \in B_{\pi(i)}(e'_{i-1})$.

Then we have for $i \in [l]$ and $m \in [k]$ that range graph $R(i-1)$ is nested in G_m. Especially, we have that $R(i)$ exists. Further $R(l)$ is not nested in $G_{\pi(l+1)}$.

Proof. First observe for $i \in [l]$, that if $R(i)$ exists, then we have $e'_i \in R(i)$ by an easy induction. Therefore, if $R(l)$ exists, we have $e'_l \in R(l)$ and $R(l)$ is not nested in $G_{\pi(l+1)}$.

We can therefore define $i \in [l]$ such that for $j \in [i]$ the range graph $R(i-1)$ is nested in $G_{\pi(i)}$ and $R(i)$ is not nested in $G_{\pi(i+1)}$. I.e., range graph $R(i+1)$ does not exist. Note that for $j \in [i]$ and $e \in R(j)$ there is an edge $f \in R(j-1)$ with $e \in B_{\pi(j)}(f)$. Further note, since $R(i)$ is not nested in $G_{\pi(i)}$, there is an edge $xy \in R(i)$ such that $\{x, y\}$ is free in $G_{\pi(i+1)}$. An easy induction shows there is a pre-augmenting path $(e_0 = uv, \dots, e_j = xy)$ with $j \leq i$. By Lemma 5.13 we know there is also an augmenting path $(f_0 = uv, \dots, f_d)$ with $d \leq i$. If we have $i < l$, then this contradicts the definition of σ as an augmenting path with minimum length. Hence, we have $i = l$. If there is an $j \in [l-1]$ and an $m \in [k]$ such that $R(j)$ is not nested in G_m we can apply the same argumentation for a contradiction. We finally note that, since $e'_l \in R(l)$ is not free in $G_{\pi(l+1)}$, we have that $R(l)$ is not nested in $G_{\pi(l+1)}$. □

On the other site, we obtain a pre-augmenting path from a certain sequence of range graphs.

Lemma 5.21
Let $a, b \in \mathbb{N}_0$ with $b < 2a$. Let further $S = \{G_1, \dots, G_k\}$ be a set of pairwise edge-disjoint digraphs with $V = V(G_1) = \dots = V(G_k)$. Let $u, v \in V$ with $u \neq v$ and $uv \notin H = G_1 \cup \ldots \cup G_k$. For $i \in [k]$ let G_i be (a, b)-pebbled by a function $p_i \colon V \to [a]$.

Let $\pi \in [k]^{l+1}$ be a sequence such that $R(l)$ is not nested in $G_{\pi(l+1)}$ and for $i \in [l]$ range graph $R(i-1)$ is nested in $G_{\pi(i)}$.

Then there is a pre-augmenting path $\sigma = (e_0 = uv, \dots, e_l)$ such that e_l is free in $G_{\pi(l+1)}$ and for $i \in [l]$ we have $e'_i \in B_{\pi(i)}(e'_{i-1})$.

Proof. Since $R(l)$ is not nested in $G_{\pi(l+1)}$, there is an edge $e_l \in R(l)$ such that e_l is free in $G_{\pi(l+1)}$. For $i \in [l]$ we have that for every edge $e'_i \in R(i)$ there is an edge $e'_{i-1} \in R(i-1)$ with $e_i \in B_{\pi(i)}(e_{i-1})$. By induction we obtain that for $j \in \{0, \dots, l\}$ there is a pre-augmenting path $(e_{l-j}, \dots, e_l)$ such that $e_{l-j} \in R(l-j)$ and e_l is free in $G_{\pi(l+1)}$ and for $i \in \{l-j+1, \dots, l\}$ we have $e'_i \in B_{\pi(i)}(e'_{i-1})$. This yields the statement. □

A deeper analysis of the structure of the range graphs helps us to find a sequence π as required in Lemma 5.20. With such a sequence, Lemma 5.23 allows us then to construct the range graphs quite efficient.

Lemma 5.22
Let $a, b \in \mathbb{N}_0$ with $b < 2a$. Let further $G_1, \dots, G_k$ be pairwise edge-disjoint digraphs with $V = V(G_1) = \dots = V(G_k)$. Let $u, v \in V$ with $u \neq v$ and $uv \notin H = G_1 \cup \ldots \cup G_k$.

For $i \in [k]$ let G_i be (a, b)-pebbled by a function $p_i \colon V \to [a]$ such that $p_i(\{u, v\}) = b$ and $p_i(\mathrm{Reach}_i(u, v) \setminus \{u, v\}) = 0$. Let $\pi \in [k]^l$ be a sequence such that for $i \in [l]$ the range graph $R(i-1)$ is nested in $G_{\pi(i)}$.

Then we have for $i \in [l]$ that

$$R(i) = B_{G\pi(i)}(V(R(i-1))) = \langle \mathrm{Reach}_{\pi(i)}(R(i-1)) \rangle_{\pi(i)}$$

and $p(V(R(i)) \setminus \{u, v\}) = 0$.

Proof. We apply induction on i. For $i=1$ we have by the definition of range graphs and by Theorem 5.8 that

$$\begin{aligned}R(1) &= R(G_{\pi(1)}, R(0)) = R(G_{\pi(1)}, (\{u,v\},\{uv\})) \\ &= \bigcup_{xy\in(\{u,v\},\{uv\})} B_{G_{\pi(1)}}(x,y) = B_{G_{\pi(1)}}(u,v) = \langle \mathrm{Reach}_{\pi(i)}(R(0))\rangle_{\pi(i)}.\end{aligned}$$

By assumption we obtain that $p(V(R(1))\setminus\{u,v\}) = p(\mathrm{Reach}_{\pi(1)}(u,v))\setminus\{u,v\}) = 0$.

Next assume $i\in[l-1]$ and the statement holds for i. Then we have by Lemma 5.17, Lemma 5.3 and Theorem 5.8 that

$$R(i+1) = R(G_{\pi(i+1)}, R(i)) = B_{\pi(i+1)}(R(i)) = \langle \mathrm{Reach}_{\pi(i+1)}(R(i))\rangle_{\pi(i+1)}.$$

Note that Lemma 5.3 implies $p(V(R(i+1))\setminus\{u,v\}) = p(\mathrm{Reach}_{\pi(i+1)}(R(i)))\setminus\{u,v\}) = 0$, since $R(G_{\pi(i+1)}, R(i))$ is (a,b)-tight. □

This means we can iteratively compute for $i\in[l]$ the range graph $R(i)$ by a depth-first search starting in all vertices of $R(i-1)$.

Lemma 5.23
Let $a,b\in\mathbb{N}_0$ with $b<2a$. Let $G_1,\dots,G_k$ be pairwise edge-disjoint digraphs with $V = V(G_1)=\dots=V(G_k)$. Let $u,v\in V$ with $u\neq v$ and $uv\notin H=(V,E)=G_1\cup\ldots\cup G_k$. Let $k\le|V|$.

For $i\in[k]$ let G_i be (a,b)-pebbled by a function $p_i\colon V\to[a]$ Let $l\in[|V|]$ and let $\pi\in[k]^l$ be a sequence such that for $i\in[l]$ the range graph $R(i-1)$ is nested in $G_{\pi(i)}$.

Then we can compute the range graphs $R(1)\dots R(l)$ with a runtime in $\mathrm{O}(|V|^2)$ for constant a. We further obtain edge-sets $T_1\dots,T_l$ with $T_i = E(R(i))\setminus\bigcup_{j<i}E(R(j))$.

Proof. We represent the range graphs by mapping every edge $e\in E$ to the smallest number $r(e)$ with $e\in R(r(e))$. I.e., we set $r(e)$ such that $e\in T_{r(e)}$. For $e\in E$ we initialize $r(e)=\infty$. Further we initialize the edge sets $T'_1=\dots=T'_{l-1}=\emptyset$. For $i=0$ we initialize $U_0=\{u,v\}=V(R(0))$.

First we collect in every guest pebbles on u and v as described in Theorem 5.9. This is possible with a runtime in $\mathrm{O}(k|V|)\subseteq\mathrm{O}(|V|^2)$. Note that with Lemma 5.17 and Lemma 5.3 we obtain for $i\in[k]$ that $p_i(\{u,v\})=b$ and $p_i(\mathrm{Reach}_i(u,v)\setminus\{u,v\})=0$.

By Lemma 5.22 we can then compute for $i\in[l-1]$ the set

$$U_i = V(R(i)) = \mathrm{Reach}_{\pi(i)}(V(R(i-1))$$

by a depth-first search starting in all vertices in U_{i-1}. In this search we traverse every edge in $R(i)=\langle\mathrm{Reach}_{\pi(i)}(V(R(i-1))\rangle_{\pi(i)}$. Therefore, for every edge wz traversed the first time, we can set $r(wz)=i$ and add wz to T'_i. As invariant we keep that for $j\in[i]$ we have $T'_j=T_j$ and the contained edges are labelled correctly. If an edge $wz\in G_j$ was traversed, then also all other outgoing edges of w in G_j were traversed. We thus know that $\mathrm{N}^+{}_{\pi(i)}(w)\subseteq U_i\subseteq U_{i+1}\subseteq\dots$. As a consequence, vertex w can be considered visited in every following depth-first search in G_j. Hence, if an edge wz with $r(wz)<\infty$ is about to be traversed (in a run after the i'th run), we instead stop processing vertex w which is then considered visited. Except for these repeated traversals, every edge is traversed at most once. Therefore at most $\mathrm{O}(|E|)\subseteq\mathrm{O}(|V|^2)$ true edge-traversals are executed. For every $i\in[l-1]$ there is at most one repeated traversal per vertex. We thus have at most $\mathrm{O}(l|V|)=\mathrm{O}(|V|^2)$ false edge-traversals. We finally obtain a runtime in $\mathrm{O}(|V|^2)$ and our invariants ensure correctness. □

After computing the range graphs we can use them to actually find an augmenting path. We do so by finding the edges of an augmenting path in reverse order. By knowing the range graphs we can restrict the number of possible preceding edges in the augmenting path. We then just compute for every possible preceding edge the corresponding minimum block and check whether it contains the current edge. A breath-first search as indicated earlier works as well and does not require a separate computation of the range graphs. However, this approach appears more promising for optimization.

Lemma 5.24
Let $a, b \in \mathbb{N}_0$ with $b < 2a$. Let further $G_1, \dots, G_k$ be pairwise edge-disjoint digraphs with $V = V(G_1) = \dots = V(G_k)$. Let $u, v \in V$ with $u \neq v$ and $uv \notin H = (V, E) = G_1 \cup \ldots \cup G_k$.

For $i \in [k]$ let G_i be (a, b)-pebbled by a function $p_i \colon V \to [a]$. Let $\pi \in [k]^{l+1}$ be a sequence such that there is an (unknown) augmenting path $\sigma' = (f_0 = uv, \dots, f_l)$ of minimum length such that f_l is free in $G_{\pi(l+1)}$ and for $i \in [l]$ we have $f_i \in B_{\pi(i)}(f_{i-1})$.

Then we can compute an augmenting path $\sigma = (e_0 = uv, \dots, e_l)$ with runtime in $\mathrm{O}(|V| \cdot |E|)$ for constant a.

Proof. By Lemma 5.20 we have for $i \in [l]$ and $m \in [k]$ that range graph $R(i-1)$ is nested in G_m. Further $R(l)$ is not free in $G_{\pi(l+1)}$. By Lemma 5.23 we can compute edge-sets $T_0, \dots, T_l$ with $T_i = E(R(i)) \setminus \bigcup_{j<i} E(R(j))$ in $\mathrm{O}(|V|^2)$.

For $xy \in T_l$ we collect pebbles on x and y in $G_{\pi(l+1)}$ as in Theorem 5.9. If this results in at least $b+1$ pebbles on x and y in $G_{\pi(l+1)}$, then by Lemma 5.3 edge xy is free in $G_{\pi(l+1)}$ and we can set $e_l = xy$ and continue by searching a preceding edge. Otherwise we know by the same lemma that xy is not free in $G_{\pi(l+1)}$. Since $R(l)$ is not nested in $G_{\pi(l+1)}$ but for $i \in [l-1]$ the range graph $R(i)$ is nested in $G_{\pi(l-1)}$, there actually must be an edge in $T_l = E(R(l)) \setminus \bigcup_{j<l} E(R(j))$ that is free in $G_{\pi(l+1)}$.

For $i = l-1, \dots, 0$ we obtain $e_i \in T_i$ of our future augmenting path $\sigma = (e_0, e_1, \dots, e_{l-1})$ as follows. Let $p'q' = e_{i+1}$. For $pq \in T_i$ we collect pebbles on p and q in $G_{\pi(i+1)}$ as in Theorem 5.9. If $\mathrm{Reach}_{\pi(i+1)}(p, q)$ then contains p' and q', then we set $e_i = pq$ and continue with the search for the next edge of the augmenting path. Edge $pq \in T_i \subseteq R(i)$ can not be free in $G_{\pi(i+1)}$, since $R(i)$ is nested in $G_{\pi(i+1)}$. With Theorem 5.8 we obtain $\mathrm{Reach}_{\pi(i+1)}(p, q) = B_{\pi(i+1)}(p, q)$. Thus, our choice ensures $e_i \in B_{\pi(i+1)}(p, q)$.

For correctness of this algorithm it remains to show that there is actually an edge $pq \in T_i$ with $e_{i+1} \in B_{\pi(i)}(p, q)$. By definition of $R(i+1)$, there exists an edge $g \in R(i)$ with $e_{i+1} \in B_{\pi(i+1)}(g)$. Assume we have a $\xi < i$ with $g \in R(\xi)$. Then by repeatedly choosing a predecessor we can construct a sequence $(g_0 = uv, \dots, g_\xi = g)$ such that for $j \in [i]$ we have some $n \in k$ with $e'_j \in B_n(e'_{j-1})$. Together with the edges $e_{i+1}, \dots, e_l$, we obtain a pre-augmenting path $(g_0 = uv, \dots, g_\xi = g, e_{i+1}, \dots, e_l)$. By Lemma 5.13 we obtain an augmenting path with length less than l. This is a contradiction to the definition of σ'. Therefore we have $g \in E(R(i)) \setminus \bigcup_{j<i} E(R(j)) = T_i$. By Theorem 5.8 we know we actually find g or another suiting edge as the new edge e_i. Since T_0 contains only edge uv we finally also obtain $uv = e_0$. By this construction we ensure $\sigma = (e_0 = uv, \dots, e_l)$ is an augmenting path.

Since the edge-sets $T_0, \dots, T_l$ are pairwise disjoint, we apply Theorem 5.9 at most once per edge. In Theorem 5.9 we have a runtime in $\mathrm{O}(|E(G_i)|)$ where G_i is the corresponding guest. For $i \in [k]$ we have $|E(G_i)| \leq a|G| - b$ and therefore $|E(G_i)| \in \mathrm{O}(|V(G_i)|) = \mathrm{O}(|V|)$. We thus have a runtime in $\mathrm{O}(|E| \cdot |V|)$ for the computation of the howl augmenting path σ. □

We can now efficiently find an augmenting path, if we have a sequence π as required in Lemma 5.24. We find such a sequence by searching for a certain *global pebble path*.

Let $a, b \in \mathbb{N}_0$ with $b < 2a$. Let further $G_1, \ldots, G_k$ be pairwise edge-disjoint digraphs with $V = V(G_1) = \cdots = V(G_k)$. For $i \in [k]$ let G_i be (a, b)-pebbled by a function $p_i \colon V \to [a]$. Let

$$P = (v_1^1, \ldots, v_{n(1)}^1 = v_1^2, \ldots, v_{n(2)}^2 = v_1^3, \ldots, v_{n(l-1)}^{l-1} = v_1^l, \ldots, v_{n(l)}^l) = (w_1, \ldots, w_t) \subseteq H$$

be a directed path. For $i \in [l]$ we define the directed path $P_i = (v_1^i, \ldots, v_{n(i)}^i)$.

We call P a *global path* starting in $U \subseteq V(H)$ with the *guest function* $\pi \colon [l] \to [k]$, if all of the following properties are fulfilled:

(i) $w_1 \in U$ and $w_t \notin U$

(ii) $\forall i \in [l] : P_i \subseteq G_{\pi(i)}$

(iii) $\forall i \in [l] : \forall j \in [n(i)] : p_{\pi(i)}(v_j^i) > 0 \Rightarrow v_j^i \in w_1, w_t$

The number l is called the *global length* of P. We call P a *global pebble path* if we have $p_{\pi(l)}(w_t) \geq 1$. See Figure 5.5 for an example.

Note that if P is a global pebble path, then P_l may only contain a single vertex to match the guest in which w_t has that pebble. However, it still counts towards the global length (which is l). We will usually consider global pebble paths starting in one of the two vertices u or v where uv is an edge to be inserted by our algorithm.

We first observe that every vertex in a range graph $R(uv, i)$ can be reached by a global path starting in $\{u, v\}$ with global length at most i. This can be used to construct other global paths.

Lemma 5.25
Let $a, b \in \mathbb{N}_0$ with $b < 2a$. Let further $S = \{G_1, \ldots, G_k\}$ be a set of pairwise edge-disjoint digraphs with $V = V(G_1) = \cdots = V(G_k)$. Let $u, v \in V$ with $u \neq v$ and $uv \notin H = G_1 \cup \ldots \cup G_k$.

For $i \in [k]$ let G_i be (a, b)-pebbled by a function $p_i \colon V \to [a]$ such that $p_i(\{u, v\}) = b$ and $p_i(\mathrm{Reach}_i(u, v) \setminus \{u, v\}) = 0$. Let $\pi \in [k]^l$ be a sequence such that for $i \in [l]$ the range graph $R(i-1)$ is nested in $G_{\pi(i)}$.

Then for $i \in [l]$ and $x \in R(i)$ there is a global path $P = (w_1, \ldots, w_t)$ of global length at most i with $w_1 \in \{u, v\}$ and $w_t = x$.

Proof. We apply induction on i. Let $i = 1$ and let $x \in R(1)$. By Lemma 5.22 we have

$$x \in R(1) = \langle \mathrm{Reach}_{\pi(1)}(R(0)) \rangle_{G_{\pi(1)}} = \langle \mathrm{Reach}_{\pi(1)}(u, v) \rangle_{\pi(1)}.$$

This means $x \in \mathrm{Reach}_{\pi(1)}(u, v)$. By definition this means there is a directed path $P_1 = (w_1, \ldots, w_t = x) \subseteq R(1) \subseteq G_{\pi(i)}$ with $w_1 \in \{u, v\}$. This is a global path of global length 1.

Next assume the statement holds for some $i \in [l-1]$ and let $x \in R(i+1)$. By Lemma 5.22 we have $x \in R(i+1) = \langle \mathrm{Reach}_{\pi(i+1)}(R(i)) \rangle_{\pi(i+1)}$. We obtain a path $(z_1, \ldots, z_s = x) \subseteq R(i+1)$ with $z_1 \in R(i) \subseteq G_{\pi(i+1)}$. By Lemma 5.22 we also have $z_1 \in \mathrm{Reach}_{\pi(i)}(R(i-1))$. If $z_1 \in \{u, v\}$, then Z is a global path of global length $1 \leq i + 1$. Otherwise there is an edge $wz_1 \in \langle R(i-1) \rangle_{\pi(i)} = R(i) \subseteq G_{\pi(i)}$. By induction hypothesis there is a global path $P' = (w_1, \ldots, w_s = w, z_1)$ of global length at most i with $w_1 \in \{u, v\}$. We obtain the global path $P = (w_1, \ldots, w_s = w, z_1, \ldots, z_s = x)$ with global length at most $i + 1$. This concludes the induction and thereby the proof. $\square$

On the other side, a global pebble path with minimum global length has a guest function that provides a sequence of range graphs as required for Lemma 5.21 which provides a pre-augmenting path.

Lemma 5.26
Let $a, b \in \mathbb{N}_0$ with $b < 2a$. Let further $S = \{G_1, \dots, G_k\}$ be a set of pairwise edge-disjoint digraphs with $V = V(G_1) = \cdots = V(G_k)$. Let $u, v \in V$ with $u \neq v$ and $uv \notin H = G_1 \cup \ldots \cup G_k$.

For $i \in [k]$ let G_i be (a,b)-pebbled by a function $p_i \colon V \to [a]$ such that $p_i(\{u,v\}) = b$ and $p_i(\mathrm{Reach}_i(u,v) \setminus \{u,v\}) = 0$. Let

$$P = (u = v_1^1, \dots, v_{n(1)}^1, \dots, v_1^l, \dots, v_{n(l)}^l) = (w_1, \dots, w_t)$$

be a global pebble path starting in $\{u,v\}$ with guest function π. Let P be of minimum global length with these properties. Let $j = \pi(l)$.

Then we have for $i \in [l-1]$ that range graph $R(i-1)$ is nested in $G_{\pi(i)}$. Further there is an edge $xy \in R(l-1)$ such that $\{x,y\}$ is free in G_j.

Proof. We first show for $i \in [l-1]$ that range graph $R(i)$ is well defined. Assume otherwise, i.e., assume there is an $i \in [l-1]$ such that $R(i-1)$ is not nested in $G_{\pi(i)}$. Then there is an edge $xy \in R(i-1)$ with $\{x,y\}$ being free in $G_{\pi(i)}$. By Lemma 5.3 this implies there is a path $X = (x_1, \dots, x_m) \subseteq G_{\pi(i)}$ with $x_1 \in \{x,y\}$ and $x_m \notin \{u,v\}$ with $p_{\pi(i)}(x_m) \geq 1$.

By Lemma 5.25 there is a global path $P = (z_1, \dots, z_t)$ of global length at most $i - 1 \leq l - 2$ with $z_1 \in \{u,v\}$ and $z_t = x_1$. We obtain the global pebble path $(z_1, \dots, z_t = x_1, \dots, x_m)$ with global length at most $l-1$. This is a contradiction to P having minimum global length. Therefore, for $i \in [l-1]$ graph $R(i-1)$ is nested in $G_{\pi(i)}$ and we also have range graph $R(i)$.

Assume there is no edge $xy \in R(l-1)$ that is free in G_j. Then $R(l-1)$ is nested in G_j. By Lemma 5.17 we obtain that $R(l)$ is a block. However, we have $p_j(V(R(l))) \geq p_j(u) + p_j(v) + p_j(w_t) \geq b + 1$. By Lemma 5.3 this is a contradiction to $R(l)$ being a block. Therefore there is an edge $xy \in R(l-1)$ such that $\{x,y\}$ is free in G_j. ☐

With the next lemma we obtain a global pebble path from an augmenting path. We obtain a boundary on the global length. We will use the contraposition to show that all augmenting paths have at least a certain length. This fact can then be used to show a pre-augmenting path of a certain length is actually an augmenting path (whose existence we want to show).

Lemma 5.27
Let $a, b \in \mathbb{N}_0$ with $b < 2a$. Let further $S = \{G_1, \dots, G_k\}$ be a set of pairwise edge-disjoint digraphs with $V = V(G_1) = \cdots = V(G_k)$. Let $u, v \in V$ with $u \neq v$ and $uv \notin H = G_1 \cup \ldots \cup G_k$.

For $i \in [k]$ let G_i be (a,b)-pebbled by a function $p_i \colon V \to [a]$ such that $p_i(\{u,v\}) = b$ and $p_i(\mathrm{Reach}_i(u,v) \setminus \{u,v\}) = 0$. Let there be an augmenting path $\sigma = (uv = e_0, \dots, e_l)$.

Then there is a global pebble path $P = (w_1, \dots, w_t)$ starting in $\{u,v\}$ with global length $m \leq l + 1$.

Proof. We define the function $\pi \colon [l] \to [k]$ such that for $i \in [l]$ we have $e_i \in G_{\pi(i)}$. We consider the range graphs for π. Assume that for $i \in [l]$ the range graph $R(i-1)$ is nested in $G_{\pi(i)}$. Then also the range graph $R(i)$ exists. Further, since σ is an augmenting path we obtain by induction that for $i \in [l]$ we have $e_i \in R(i)$. Therefore $R(l)$ contains the edge e_l which is for some $j \in [k]$ free in G_j.

Otherwise, we obtain an $i \in [l]$ for which $R(i-1)$ contains an edge that is for some $j \in [k]$ free in G_j. By Theorem 5.8 we know $i \neq 1$.

Hence, there is an $i \in [l]$ such that $R(i)$ contains an edge pq that is for some $j \in [k]$ free in G_j. By Lemma 5.3 there is a path $X = (x_1, \ldots, x_n) \subseteq G_j$ with $p_j(x_n) \geq 1$. By Lemma 5.25 there is a global path $(w_1, \ldots, w_s = x_1)$ with $w_1 \in \{u, v\}$ and global length at most i. We obtain the global pebble path $(w_1, \ldots, w_s = x_1, \ldots, x_n)$ of global length at most $i + 1 \leq l + 1$. This concludes the proof. □

For our preparation it remains to show that a global pebble path with minimum global length can be found fast.

Lemma 5.28
Let $a, b \in \mathbb{N}_0$ with $b < 2a$. Let further $G_1, \ldots, G_k$ be pairwise edge-disjoint digraphs with $V = V(G_1) = \cdots = V(G_k)$. Let $u, v \in V$ with $u \neq v$ and $uv \notin H = G_1 \cup \ldots \cup G_k$.

For $i \in [k]$ let G_i be (a, b)-pebbled by a function $p_i \colon V \to [a]$ such that $p_i(\{u, v\}) = b$ and $p_i(\mathrm{Reach}_i(u, v) \setminus \{u, v\}) = 0$.

Then we can compute a global pebble path $P = (w_1, \ldots, w_t)$ with a guest function π and global length m such that $w_1 \in \{u, v\}$ and P has minimum global length, or determine no such path exists with a runtime in $\mathrm{O}(|E(H)|)$.

Proof. We use a variant of a depth-first search. We define the set $L_0 = \{u, v\}$. In general we will put a vertex that is reachable with a global path of global length i into set L_i. We start with $i = 1$. For $j \in [k]$ we compute $R_{i,j} = \mathrm{Reach}_j(L_{i-1}) \setminus \bigcup_{l \in [i-2]} L_l$ using a normal depth-first search starting in all vertices of L_{i-1} that considers all vertices in $M_i = \bigcup_{l \in [i-2]} L_l$ as visited. This is correct, since all vertices reachable from M_i in G_j are already contained in L_{i-1}. We check whether $p_j(x) \geq 1$ when a vertex x is reached. If we have $p_j(x) \geq 1$, then the algorithm returns the search path to x. If no such vertex was found for $j \in [k]$, then we define $L_i = R_{i,1} \cup \ldots \cup R_{i,k}$. If $L_i = \emptyset$, then we return $\emptyset$ implying no such global pebble path exists. If $L_i \neq \emptyset$, then we repeat these steps with i increased by one.

It is easy to verify the invariant, that for $i \in \mathbb{N}$ and $x \in L_i$ there is a global path of global length i starting in $\{u, v\}$ and ending in x (if L_i was defined or a search path was returned in round i). On the other hand, it is also easy to verify the invariant, that if there is a global path of length $i \in \mathbb{N}$ starting in $\{u, v\}$ and ending in x, then we have $x \in L_i \cup M_i$. If x with $p_j(x) \geq 1$ was found in round i, this implies there is no such global pebble path of length less than i. We also obtain that there is no such global pebble path, if no search path is returned.

Every edge is considered at most once like in a normal depth-first search. Further, if a vertex x has no outgoing edge in some guest G_j, then we have $p_j(x) = a \geq 1$. Therefore the runtime for checking whether a vertex has a pebble is dominated by the runtime for processing the edges. We thus obtain a runtime in $\mathrm{O}(|E|)$. □

We finally have everything needed to compute optimal global $\mathcal{G}(a, b)$-covers efficiently.

Theorem 5.29
Computing an optimal global $\mathcal{G}(a, b)$-cover of a given graph $H = (V, E)$ is possible with runtime in $\mathrm{O}(|E|^2 \cdot |V|)$ for constant a.

Proof. Let $H = (V, E)$ be a graph. Roughly speaking, our approach is to insert the edges of H one by one into the guest graphs $G_1, \dots, G_k$ and increase the number of guests if needed. The guests are represented as (a, b)-pebbled digraphs. Eventually it is necessary to exchange edges between the guests along an augmenting path. We do obtain this augmenting path using Lemma 5.24. However, we first need a sequence π for which an augmenting path with certain properties exists. We obtain such a sequence π by finding a global pebble path P.

We keep as invariants that our guests are (a, b)-sparse, cover all edges added so far, and we only have as many guests as needed to cover all edges added so far. We start with an empty set $S = \emptyset$ of guests and we insert the edges of H one by one.

Let uv denote the edge we are about to insert. Let $H' = \bigcup S + uv$ and let $k = |S|$.

First we collect in every guest pebbles on u and v as described in Theorem 5.9. If $\{u, v\}$ is free in some guest G_i, then we add edge uv (or vu) to G_i as described in Lemma 5.4 and continue with the next edge.

Next, we search for a minimum global pebble path $P = (w_1, \dots, w_t)$ with global length l and a guest function π as described in Lemma 5.28. By Lemma 5.12 we know $c_g^{\mathcal{G}(a,b)}(H') \leq k$ holds only if H' is (ka, kb)-sparse. By Lemma 5.19 we know this is only the case, if there is an augmenting path starting in $\{u, v\}$. And by Lemma 5.27 in turn we know that, if there is such an augmenting path, then we also have a global pebble path starting in $\{u, v\}$. Therefore, if there is no such global pebble path, then we have $c_g^{\mathcal{G}(a,b)}(H') \geq k + 1$. In this case we add another guest $G_{k+1} = (V, \emptyset)$ to S with pebble function $p_{k+1}\colon V \to [a], x \mapsto a$. We then add edge uv by applying Lemma 5.4 and continue with the next edge.

Hence, assume we found such a global pebble path P with guest function π. Obviously we have $l < 2|V|$.

We aim to prove the mere existence of an augmenting path as required for Lemma 5.24. By Lemma 5.26 we have for $i \in [l-1]$ that range graph $R(i-1)$ is nested in $G_{\pi(i)}$ and $R(l-1)$ is not nested in $G_{\pi(l)}$. With Lemma 5.21 we obtain that there is a pre-augmenting path $\sigma = (e_0 = uv, \dots, e_{l-1})$ such that e_{l-1} is free in $G_{\pi(l)}$ and for $i \in [l-1]$ we have $e'_i \in B_{\pi(i)}(e'_{i-1})$. By the definition of P with l as minimum global length, there is no augmenting path $(uv, e'_1, \dots, e'_m)$ with $m < l - 1$, since otherwise we would obtain a shorter global pebble path by Lemma 5.27. With Lemma 5.13 this implies σ itself is an augmenting path. We have shown σ fulfils all requirements for Lemma 5.24. We can thus compute an augmenting path $\tau = (f_0 = uv, \dots, f_m)$ with a runtime in $\mathrm{O}(|V| \cdot |E|)$.

By Lemma 5.15 we can finally use augmenting path τ to insert edge uv (or vu) in our guests with runtime in $\mathrm{O}(|V|^2)$. Then we proceed with the next edge in H unless we already added all edges.

Our invariants guarantee that our guests provide in the end a k-global $\mathcal{G}(a, b)$-cover of H with $k = c_g^{\mathcal{G}(a,b)}(H)$. In total we obtain a runtime in $\mathrm{O}(|E|^2 \cdot |V|)$. This concludes the proof. □

6. Guests of Bounded Degree

In this section we consider graphs of bounded degree as guests.

Let $\mathcal{D}_d$ denote the class of all graphs with degree at most d. As a natural lower bound on the $\mathcal{D}_d$-covering number of a host graph H we have $c_i^{\mathcal{D}_d}(H) \geq \lceil \Delta(H)/d \rceil$ for $i = f, l, g$. By the following lemma it suffices to consider only regular host graphs for the question whether this lower bound is always met.

Lemma 6.1 (Stumpf [Stu15])
Every graph H with maximum degree Δ is an induced subgraph of a Δ-regular graph.

There is always an optimal folded cover considering this lower bound as following lemma shows.

Lemma 6.2
Let $\mathcal{D}_d$ be the guest class of all graphs with degree at most d. Let H be a host graph with maximum degree Δ. Then $c_f^{\mathcal{D}_d}(H) = \lceil \Delta/d \rceil$.

Proof. We obtain a suiting guest graph G by dividing every vertex v into $\lceil \deg(v)/d \rceil$ new vertices and distributing the incident edges evenly among them. By mapping every vertex of G to the corresponding original vertex of H we obtain a folded cover covering every vertex at most $\lceil \Delta(H)/d \rceil$ times. This concludes the proof. □

For regular host graphs of degree kd a global cover with k guests is equivalent to a partition into d-factors. Such partition does not always exist. E.g. it is $\mathcal{NP}$-hard to decide whether a given regular graph can be partitioned into 1-factors perfect matchings [Hol81]. Another example are regular host graphs of degree $2d$ with an odd number of edges [Pet91]. However, for an even maximum degree d we can always find such partition (and thereby an optimal global cover for all host graphs by Lemma 6.1) [Pet91].

In the local case it is not known (to the best of the author‘s knowledge) whether there is always an optimal cover considering the lower bound. For $d = 1$ we can use each edge as another guest providing an optimal cover. Further, we can optimally cover all complete graphs as following lemma shows. This is not possible with global covers, as for odd maximum degree d we have $||K_{2d+1}|| = \binom{2d+1}{2} = (2d+1)(2d)/2 = (2d+1)d$ which is an odd number of edges.

Lemma 6.3
For every $d \in \mathbb{N}$ we have $c_l^{\mathcal{D}_d}(K_{kd+1}) = k$.

Proof. Partition the vertex set of K_{kd+1} into one vertex x and k sets $A_1, \ldots, A_k$ of d vertices each. Consider the subgraphs induced by $A_i \cup \{x\}$ for $i = 1, \ldots, k$ as guests. Further, for $1 \leq i \leq j \leq k$ consider the subgraphs induced by $A_i \cup A_j$ as guests. The first kind of guests are copies of K_{d+1} covering all edges to x and between vertices of the same set. The second kind of guests are copies of $K_{d,d}$ covering all edges between vertices of different sets. Hence, all guests are edge-disjoint, d-regular and cover all edges. Further, vertex x is covered by the first k guests while every other vertex is covered by one guest of the first kind and $k-1$ guests of the second kind. Thus, our cover actually covers every vertex k times. □

Such an optimal local cover of a kd-regular host graph implies a d-regular subgraph since the guests must be vertex-disjoint d-regular subgraphs to provide enough degree in every vertex. The existence of such a subgraph was proven by Taskinov [Tas88].

7. Cover Resistance

7.1 Motivation

The research on covering numbers so far focused on covering numbers for fixed guest classes, for which different host graphs or host classes where considered. In this chapter we instead focus on covering numbers for fixed host classes, while we consider different guest classes.

Let $\mathcal{H}$ be a host class. We investigate for $i = f, l, u, g$ the possible values for $c_i^{\mathcal{G}}(\mathcal{H})$ by choosing different guest classes $\mathcal{G}$.

Without restriction, we usually obtain a large number of possible values for the covering numbers: For $i = g$, let k be the number of edges of some host graph $H_k \in \mathcal{H}$. Consider the guest class $\mathcal{G}'$ containing K_2 and every host graph $H \in \mathcal{H}$ with $||H|| > k$. The host graph H_k must be covered using k copies of K_2 as guests. This implies $c_g^{\mathcal{G}'}(\mathcal{H}) \geq k$. As every host graph H'' with $||H''|| > k$ can be covered by itself, and every other host graph can be covered using at most k copies of K_2 as guests, we obtain $c_g^{\mathcal{G}'}(\mathcal{H}) = k$.

We can find an analogous construction of a guest class for $i = f, l, u$ by considering the maximum degree of host graphs instead of their size. Hence, it is easy to find host classes for which every positive integer is a possible covering number.

Therefore we consider, on the other side, host classes with as few as possible values for their covering numbers. However, the constructions above strongly restrict such host classes, allowing only a small number of sizes or maximum degrees which is undesirable.

We thus consider only induced-hereditary guest classes, preventing those constructions above.

7.2 Cover Resistance

Remember that we write $\mathcal{G} \subseteq_e \mathcal{H}$ if for every graph $G \in \mathcal{G}$ we have $G \in_e \mathcal{H}$, i.e., for every graph $G \in \mathcal{G}$ there is a graph $H \in \mathcal{H}$ with $H = (V(G) \mathbin{\dot\cup} W, E(G))$ for some vertex set W. In other words it denotes a subclass property where additional vertices in graphs of the subclass are ignored.

By Proposition 3.5 we know that for $i = f, l, u, g$ we have $c_i^{\mathcal{G}}(\mathcal{H}) \leq 1$ if $\mathcal{H} \subseteq_e \mathcal{G}$. Thus, a covering number of 0 or 1 is always possible considering the induced-hereditary closure of $\mathcal{H}$ as guest class. We consider the host classes with the least number of possible values for some covering variant. To this end, we introduce the notion of cover resistances.

Definition 7.1
*Let $i = f, l, u, g$. A host class $\mathcal{H}$ is called i-*cover resistant*, if for every induced-hereditary guest class $\mathcal{G}$ we have $\mathcal{H} \not\subseteq_e \mathcal{G} \Leftrightarrow c_i^{\mathcal{G}}(\mathcal{H}) = \infty$ for $i = g$ and $\mathcal{H} \not\subseteq_e \overline{\mathcal{G}} \Leftrightarrow c_i^{\mathcal{G}}(\mathcal{H}) = \infty$ for $i = f, l, u$. By Proposition 3.5 this is equivalent to $c_i^{\mathcal{G}}(\mathcal{H}) > 1 \Rightarrow c_i^{\mathcal{G}}(\mathcal{H}) = \infty$ and to the contraposition $c_i^{\mathcal{G}}(\mathcal{H}) < \infty \Rightarrow c_i^{\mathcal{G}}(\mathcal{H}) \leq 1$.*

I.e., a host class $\mathcal{H}$ is cover resistant, if its covering number with regards to any induced-hereditary guest class $\mathcal{G}$ is only finite, if $\mathcal{G}$ already contains $\mathcal{H}$. In this sense, host class $\mathcal{H}$ resists to be covered.

By Theorem 4.15 we know the class $\mathcal{OS}$ of shift graphs of ordered graphs is g-cover resistant. And by a result of Stumpf in his Bachelor‘s Thesis, we know the class $\mathcal{K}$ of all complete graphs is l-cover resistant.

Theorem 7.2 (Stumpf [Stu15])
Let $\mathcal{K}$ denote the class of all complete graphs and let $\mathcal{G}$ be an induced-hereditary class of graphs and r the smallest natural number with $K_r \notin \mathcal{G}$. Let $r < \infty$. Then $c_l^{\mathcal{G}}(\mathcal{K}) = \infty$.

An example for an f-cover resistant class is the class $\mathcal{S}t$ of stars.

Theorem 7.3
The class $\mathcal{S}t$ of stars is f-cover resistant.

Proof. Let $\mathcal{G}$ be an induced-hereditary guest class with $c_f^{\mathcal{G}}(\mathcal{S}t) > 1$. Then there is a star S_r with r leaves that is not contained in $\mathcal{G}$. The guest class $\mathcal{G}$ does then not contain any star with more than r leaves, as it is induced-hereditary. Let k be any positive number. Then, a folded cover of star S_{rk}, with x as non-leaf vertex, contains at least k stars as guests, each covering x. This implies $c_f^{\mathcal{G}}(\mathcal{S}t) = \infty$. □

An overview over further results on cover resistances is provided in Table 7.1.

7.3 Relation to Induced Ramsey Theory

The investigation of cover resistance is related to Induced Ramsey Theory, as we will show in Theorem 7.5 and Corollary 7.6 below. Roughly speaking, a host class $\mathcal{H}$ is cover resistant, if and only if for every graph $H \in \mathcal{H}$ there is a graph $H' \in \mathcal{H}$ such that every relevant cover of H' has H as induced subgraph of its guests. In this sense the cover resistance of a host class is a self-similarity.

For the g-cover resistance we think in terms of edge-colourings, allowing as usual multiple colours per edge. The class $\mathcal{H}$ is g-cover resistant if and only if for every graph $H \in \mathcal{H}$ there is a graph $H' \in \mathcal{H}$ such that every colouring of H' using two colours has H as induced subgraph of one colour. A difference to Induced Ramsey Theory is that edges can be coloured using multiple colours. We actually use two results of the Induced Ramsey Theory to show the f-cover resistance of bipartite graphs in Corollary 7.24 and the l-cover resistance of the class of all graphs in Corollary 7.28.

To prove the cover resistance of a host class $\mathcal{H}$, we need to exclude all possible covering numbers except for $0, 1$ and ∞. Lemma 7.4 states that for all but the l-cover resistance only the covering number of 2 has to be excluded. In terms of edge-colourings for the g-cover resistance, this means only two colours need to be considered. This supports the similarity to Induced Ramsey Theory.

	g-cover res.	u-cover res.	l-cover res.	f-cover res.	
$\{K_2\}$	✓	✓	✓	✓	Thm. 7.19
matchings	✓	✓	✓	✓	Thm. 7.19
stars	✓	✓	✓	✓	Thm. 7.3
complete bipartite	✓	✓	✓	✓	Cor. 7.22
bipartite	✓	✓	✓	✓	Cor. 7.24
complete	✓	✓	✓	✗	Thm.7.2/[Stu15]
all graphs	✓	✓	✓	✗	Cor. 7.28
$\mathcal{OS}$	✓	✓	✗	✗	Thm. 4.15
$\Delta(\mathcal{H}) > 1, \lvert\mathcal{H}\rvert < \infty$	✗	✗	✗	✗	Pro. 7.18
$3 \leq \chi(\mathcal{H}) < \infty$	✗	✗	✗	✗	Pro. 7.17
planar	✗	✗	✗	✗	Pro. 7.17
outer-planar	✗	✗	✗	✗	Pro. 7.17
(a,b)-sparse	✗	✗	✗	✗	Pro. 7.17
forests	✗	✗	✗	✗	Pro. 7.17
$\chi(\mathcal{H}) = \infty$	-	-	-	✗	Thm. 4.3/[Stu15]
$\{K_2 \uplus K_2\}$	✗	✓	✓	✓	Pro. 7.18
$\overline{\mathcal{K} \cup \mathcal{S}t}$	✗	✓	✓	✗	Thm. 7.13
$\overline{\mathcal{K} \cup \mathcal{S}t \cup \mathcal{OS}}$	✗	✓	✗	✗	Thm. 7.13

Table 7.1: Host classes and their cover-resistances. Let $\mathcal{H}$ be a host class. (i) By Proposition 7.12 the f-cover resistance implies l-cover resistance which in turn implies u-cover resistance which, if all host graphs in $\mathcal{H}$ are connected, implies g-cover resistance. (ii) Further, by Proposition 7.10 and Proposition 7.11 for $i = f, l, u$, if $\mathcal{H}$ is i-cover resistant, then the union-closure $\overline{\mathcal{H}}$ and the induced-hereditary closure $\widehat{\mathcal{H}}$ of $\mathcal{H}$ are also i-cover resistant. The later holds even for $i = g$. (iii) Let $i = f, l, u$. By Theorem 7.9 for two i-cover resistant host classes $\mathcal{H}_1$ and $\mathcal{H}_2$ also $\mathcal{H}_1 \cup \mathcal{H}_2$ is i-cover resistant.

Lemma 7.4
Let $\mathcal{H}$ be a host class and $i = f, u, g$. Then $\mathcal{H}$ is i-cover resistant if and only if there is no induced-hereditary guest class $\mathcal{G}$ with $c_i^{\mathcal{G}}(\mathcal{H}) = 2$.

Proof. "$\Rightarrow$": Let $\mathcal{G}$ be an induced-hereditary guest class. Then by the i-cover resistance of $\mathcal{H}$ we have $c_i^{\mathcal{G}}(\mathcal{H}) \neq 2$.

"$\Leftarrow$"(Contraposition): Assume $\mathcal{H}$ is not i-cover resistant. Then there is an induced-hereditary guest class $\mathcal{G}$ with $1 < c_i^{\mathcal{G}}(\mathcal{H}) = k < \infty$. Let $\mathcal{G}^{k-1} = \{H \text{ graph} \mid c_i^{\mathcal{G}}(H) \leq k-1\}$. Note that the definition of $\mathcal{G}^{k-1}$ depends on i. Further note that $\mathcal{G}^{k-1}$ is induced-hereditary by Proposition 3.6, since $\mathcal{G}$ is induced-hereditary. We aim to prove $c_i^{\mathcal{G}^{k-1}}(\mathcal{H}) = 2$ which then concludes the proof. Since $c_i^{\mathcal{G}}(\mathcal{H}) = k$, we have $\mathcal{H} \not\subseteq_e \overline{\mathcal{G}^{k-1}}$ for $i = f, l, u$ and $\mathcal{H} \not\subseteq_e \mathcal{G}^{k-1}$ for $i = g$. With Proposition 3.5 we get $c_i^{\mathcal{G}^{k-1}}(\mathcal{H}) \geq 2$.

Let $H \in \mathcal{H}$ be a host graph and (S, ϕ) be a k-(folded/local/union/global) $\mathcal{G}$-cover of H. If $k \leq 2$ we are done. Hence, assume $k > 2$.

If $i = g$, then we have $|S| \leq k$. Without loss of generality, we assume every graph in S to be identical to its image with regards to ϕ. We arbitrarily choose a guest $G \in S$. As there are only $k-1$ guests in $S \setminus \{G\}$, their union $H' = \bigcup(S \setminus \{G\})$ is guest-injectively covered by them. Thus, we have $H' \in \mathcal{G}^{k-1}$ and $G \in \mathcal{G} \subseteq \mathcal{G}^{k-1}$. With $G \cup H' = H$ follows $c_g^{\mathcal{G}^{k-1}}(H) \leq 2$.

If $i = u$, then S can be partitioned into k guest-unions $S_1, \ldots, S_k$. Let H' denote the subgraph of H covered by $S_1, \ldots, S_{k-1}$, i.e., let $H' = (V, E)$ where

$$E = \{uv \in E(H) \mid \exists j \in \{1, \ldots, k-1\} : \exists G \in S_j : \exists xy \in E(G) : \phi(x) = u \text{ and } \phi(y) = v\}.$$

Let G denote the subgraph of H covered by S_k. By definition we have $H' \in \mathcal{G}^{k-1}$ and $G \in \overline{\mathcal{G}} \subseteq \mathcal{G}^{k-1}$. With $G \cup H' = H$ follows $c_u^{\mathcal{G}^{k-1}}(H) \leq c_g^{\mathcal{G}^{k-1}}(H) \leq 2$.

If $i = f$, then for every vertex $v \in H$ we have $|\phi^{-1}(v)| \leq k$. For every vertex $v \in H$ choose a vertex $z(v) \in \phi^{-1}(v)$. We define

$$\psi \colon \dot{\bigcup} S \to H', u \mapsto \begin{cases} \phi(u) & \text{, if } z(\phi(u)) \neq u \\ u & \text{, otherwise} \end{cases}.$$

Where H' is chosen such that ψ is an edge-surjective and vertex-surjective homomorphism. Then (S, ψ) is a $(k-1)$-folded $\mathcal{G}$-cover of H' and thus H' is contained in $\mathcal{G}^{k-1}$. We can then define cover $(\{H'\}, \tau)$ where

$$\psi \colon H' \to H, u \mapsto \begin{cases} \phi(u) & \text{, if } z(\phi(u)) = u \\ u & \text{, otherwise} \end{cases}.$$

Since every vertex v of H has only $z(v)$ and v itself in $\tau^{-1}(v)$, it is a 2-folded $\mathcal{G}^{k-1}$-cover of H. Hence, we have $c_f^{\mathcal{G}^{k-1}}(H) \leq 2$. □

An analogous proof of Lemma 7.4 for $i = l$ is not possible. It would require the guests $G_1, \ldots, G_m$ of a k-local cover of a host graph H to be merged to new guests $G'_1, \ldots, G'_n$ that provide a 2-local cover of H, without merging all guests that cover the same vertex v (since the guest G'_i covering v would not be $(k-1)$-locally covered by $G_1, \ldots, G_m$). This is not possible in general:

Consider the host class $\mathcal{H}$ that contains all line-graphs of 3-uniform complete hypergraphs. I.e., let $\mathcal{H}$ be the class containing for $r \in \mathbb{N}$ the graph $H_r = (V, E)$ with $V = \binom{[r]}{3}$ and

$E = \{(S,T) \in \binom{[r]}{3}^2 \mid S \cap T = \emptyset\}$. First note that for the class $\mathcal{K}$ of complete graphs we have $\mathcal{H} \not\subseteq_e \mathcal{K}$ since for $r \geq 6$ there are two disjoint vertices $\{1,2,3\}$ and $\{4,5,6\}$.

Let $H_r \in \mathcal{H}$. Then we define for $i \in [r]$ the complete guest graph G_i on vertex set $\{v \in \binom{[r]}{3} \mid i \in v\}$. These r guests induce a $\mathcal{K}$-cover (S,ϕ) of H_r since two adjacent vertices u and v share a common element i and are therefore both contained in guest G_i in which they are also adjacent. Cover (S,ϕ) is further 3-local since every vertex $\{h,i,j\}$ is only covered by the three guests G_h, G_i and G_j. Therefore, the class $\mathcal{H}$ is not l-cover resistant.

Let $\mathcal{K}^2 = \{H \text{ graph} \mid c_l^{\mathcal{K}}(H) \leq 2\}$. Since $\mathcal{K}$ is induced-hereditary, it follows by Proposition 3.6 that $\mathcal{K}^2$ is also induced-hereditary. Consider graph H_9. It induces the star S_3 with 3 leaves on the vertices $\{1,2,3\}, \{1,4,5\}, \{2,6,7\}$ and $\{3,8,9\}$. Since every edge must be covered by a different clique, vertex $\{1,2,3\}$ is covered by three guests. With Proposition 3.6 we obtain $c_l^{\mathcal{K}}(H_9) \geq c_l^{\mathcal{K}}(S_3) \geq 3$. Thus, we have for $n \geq 9$ that $H_n \notin \mathcal{K}^2$.

Now assume for the sake of contradiction that there is a 2-local $\mathcal{K}^2$-cover (S',ϕ) of H_{17} that is obtained by uniting guests of the corresponding $\mathcal{K}$-cover (S,ϕ) from above. There must be two guests $G_a, G_b \in S$ that are not united in S' since otherwise we have $H_{17} \in S' \subseteq \mathcal{K}^2$ which is a contradiction. Let ${G'}_a$ and ${G'}_b$ denote the guests in S' containing G_a and G_b. Consider a vertex in $V(H_{17})$ of the form $\{a,b,x\}$. It is covered by ${G'}_a$ and ${G'}_b$. Since (S',ϕ) is a 2-local cover, the guest G_x must be contained in ${G'}_a$ or ${G'}_b$. By pigeon hole principle one of the guests ${G'}_a$ and ${G'}_b$ must contain at least eight of the guests G_i with $i \in [17] \setminus \{a,b\}$. Without loss of generality, lets assume that it is the case for ${G'}_a$.

Then let $T = \{i \in [17] \mid G_i \subseteq {G'}_a\}$. Set T contains at least nine elements, since it also contains a itself. By definition of T, the induced subset $\langle T \rangle_{H_{17}} = H_9$ is covered by ${G'}_a$. Since we have ${G'}_a \in \mathcal{K}^2$ and $\mathcal{K}^2$ is induced-hereditary, it follows by Proposition 3.6 that $c_l^{\mathcal{K}^2}(H_9) = 1$. This in turn implies that $H_9 \in \mathcal{K}^2$ which is a contradiction.

Hence, an analogous proof is not possible. We did however not show that the analogous statement is false, since we have not considered general guest classes. The above defined host class $\mathcal{H}$ appears to be at least a good candidate for a falsification.

Let H and G be graphs. Then we write $H \sqsubseteq_e G$ if H is an induced subset of G ignoring isolated vertices, i.e., it means $H \sqsubseteq (V(G) \cup V(H), E(G))$.

Now we state Theorem 7.5, which characterises the covering resistances in a way similar to Induced Ramsey Theory.

Theorem 7.5
Let $\mathcal{H}$ be a host class and $\mathcal{A}$ be the host class of all graphs.

(i) The host class $\mathcal{H}$ is g-cover resistant if and only if the following statement holds:

$$\forall H \in \mathcal{H} : \exists H' \in \mathcal{H} : \forall 2\text{-global } \mathcal{A}\text{-cover } (S,\phi) \text{ of } H' : \exists G \in S : H \sqsubseteq_e G \qquad (\star)$$

(ii) For $i = f, u$, the host class $\mathcal{H}$ is i-cover resistant if and only if the following statement holds:

$$\forall H \in \mathcal{H} : \exists H' \in \mathcal{H} : \forall 2\text{-}i \ \mathcal{A}\text{-cover } (S,\phi) \text{ of } H' : H \sqsubseteq_e \dot{\bigcup} S, \qquad (\star\star)$$

where k-i means k-(folded/union) depending on i.

(iii) The host class $\mathcal{H}$ is l-cover resistant if and only if the following statement holds:

$$\forall k \geq 2 : \forall H \in \mathcal{H} : \exists H' \in \mathcal{H} : \forall k\text{-local } \mathcal{A}\text{-cover } (S,\phi) \text{ of } H' : H \sqsubseteq_e \dot{\bigcup} S \qquad (\star\star\star)$$

Remark
We need the extension to $\forall k \geq 2$ in (iii), since we do not have Lemma 7.4 for the l-cover resistance.

Proof. "(i)": Assume statement $(\star)$ does not hold. I.e., assume there is a host graph $H \in \mathcal{H}$ such that for every host graph $H' \in \mathcal{H}$ there is a 2-global cover (S, ϕ) of H' such that for every guest $G \in S$ we have $H \not\sqsubseteq_e G$.

Let $\mathcal{G} = \{G \text{ graph} \mid H \not\sqsubseteq_e G\}$. The guest class $\mathcal{G}$ is by definition induced-hereditary. Further, for each host graph H', our assumption provides a 2-global $\mathcal{G}$-cover of H. Hence, we have $c_g^{\mathcal{G}}(\mathcal{H}) \leq 2 < \infty$. Since $\mathcal{G}$ does by definition not contain H and is closed under adding isolated vertices, we obtain $\mathcal{H} \not\subseteq_e \mathcal{G}$. Therefore $\mathcal{H}$ is not g-cover resistant.

Next assume the statement $(\star)$ holds. Let $\mathcal{G}$ be an induced-hereditary guest class with $c_g^{\mathcal{G}}(\mathcal{H}) \leq 2$. Let $H \in \mathcal{H}$. By statement $(\star)$ there exists a host graph $H' \in \mathcal{H}$ such that for each 2-global $\mathcal{G}$-cover (S, ϕ) of H' we have a guest $G \in S$ with $H \sqsubseteq_e G \in S \subseteq \mathcal{G}$. Since $\mathcal{G}$ is induced-hereditary this implies $H \in_e \mathcal{G}$ and thus $c_g^{\mathcal{G}}(H) \leq 1$. With Lemma 7.4 we can conclude that $\mathcal{H}$ is g-cover resistant. This concludes the case.

"(ii)": This case can be proven analogously to (i). On the one hand, we note that for $\mathcal{G} = \{G \text{ graph} \mid H \not\sqsubseteq_e G\}$ we have $\mathcal{G} = \overline{\mathcal{G}}$. On the other hand, we note that, if we have a host graph H, an induced-hereditary guest class $\mathcal{G}$ and a set of guests $S \subseteq \overline{\mathcal{G}}$ such that $H \sqsubseteq_e \dot\bigcup S \in \overline{\mathcal{G}}$, then we have $H \in_e \overline{\mathcal{G}}$, since $\overline{\mathcal{G}}$ is induced hereditary.

"(iii)": Assume statement $(\star\star\star)$ does not hold. I.e., assume there is a number $k \geq 2$ and a host graph $H \in \mathcal{H}$ such that for every host graph $H' \in \mathcal{H}$ there is a k-local cover (S, ϕ) of H' with $H \not\sqsubseteq_e \dot\bigcup S$.

Let $\mathcal{G} = \{G \text{ graph} \mid H \not\sqsubseteq_e G\}$. The guest class $\mathcal{G}$ is by definition induced-hereditary. Further, for each host graph H', our assumption provides a k-local $\mathcal{G}$-cover of H. Hence, we have $c_l^{\mathcal{G}}(\mathcal{H}) \leq k < \infty$. Since $\mathcal{G}$ does by definition not contain H and is closed under adding isolated vertices, we obtain $\mathcal{H} \not\subseteq_e \mathcal{G} = \overline{\mathcal{G}}$. Therefore $\mathcal{H}$ is not l-cover resistant.

Next assume the statement $(\star\star\star)$ holds. Let $\mathcal{G}$ be an induced-hereditary guest class. Assume $c_l^{\mathcal{G}}(\mathcal{H}) = k < \infty$. Let $H \in \mathcal{H}$. By statement $(\star\star\star)$ there exists a host graph $H' \in \mathcal{H}$ such that for each k-local $\mathcal{G}$-cover (S, ϕ) of H' we have $H \sqsubseteq_e \dot\bigcup S \in \overline{\mathcal{G}}$. Since $\overline{\mathcal{G}}$ is induced-hereditary this implies $H \in_e \overline{\mathcal{G}}$ and thus $\mathcal{H} \subseteq_e \overline{\mathcal{G}}$. Therefore $\mathcal{H}$ is l-cover resistant which concludes the proof. □

We say a host class $\mathcal{H}$ has the *Induced Ramsey Property*, if for any graph $G \in \mathcal{H}$ we find a graph $H = (V, E) \in \mathcal{H}$ such that for every bipartition of E we find a copy of G as induced subgraph of H with all edges in the same partition set.

We say host class $\mathcal{H}$ has the *Weak Induced Ramsey Property*, it for any graph $G \in \mathcal{H}$ we find a graph $H = (V, E) \in \mathcal{H}$ such that for every bipartition (A, B) of E we find a copy of G as induced subgraph of (V, A) or (V, B).

There is a direct relation between these two properties and the g-cover resistance.

Corollary 7.6
Let $\mathcal{H}$ be a host class. If $\mathcal{H}$ has the Induced Ramsey Property, then it is g-cover resistant. And if $\mathcal{H}$ is g-cover resistant, then it has the Weak Induced Ramsey Property.

Proof. Assume $\mathcal{H}$ has the Induced Ramsey Property. We aim to use Theorem 7.5. Let $G \in \mathcal{H}$. Let $H \in \mathcal{H}$ be a graph provided by the Induced Ramsey Property for G. Consider a 2-global cover (S, ϕ) of H with $S = \{G_1, \ldots, G_2\}$. Every edge of H is covered by at least one of the guests. Thus, we can define a function $f\colon E(H) \to \{1, 2\}$ such that for $e \in H$ we have e is covered by $G_{f(e)}$. By definition of H there is a copy G' of G with $G' \sqsubseteq H$ and $(E(G') \subseteq E(G_1)$ or $E(G') \subseteq E(G_2))$. Without loss of generality assume $E(G') \subseteq E(G_1)$. Let F be the graph obtained from G' by removing isolated vertices. Then we obtain $F \subseteq G_1$. With $F \sqsubseteq H$ and $G_1 \subseteq H$ this implies $H' \sqsubseteq_e H$. Therefore $\mathcal{H}$ is g-cover resistant.

Next assume $\mathcal{H}$ is g-cover resistant. Let $H \in \mathcal{H}$ and $H' = (V, E) \in \mathcal{H}$ be a graph provided by Theorem 7.5 since $\mathcal{H}$ is g-cover resistant. Let (A, B) be a bipartition of E. We consider the 2-global cover of H' using the guests (V, A) and (V, B). By definition of H', we have either $H \sqsubseteq (V, A)$ or $H \sqsubseteq (V, B)$. Therefore $\mathcal{H}$ has the Weak Induced Ramsey Property. $\square$

7.4 Constructing Cover Resistant Host Classes

In this section we investigate the construction of new cover resistant classes from known cover resistant classes. We first show that the union of two cover resistant host classes is itself cover resistant, and that the union-closure of a cover resistant host class is also itself cover resistant. In order to do so, we first consider taking disjoint unions of covers. This is not possible for global covers if the guest class is not union-closed.

Lemma 7.7
Let $\mathcal{G}$ be a guest class and $\mathcal{H}$ be a host class. Let $i = f, l, u$ and $c_i^{\mathcal{G}}(\mathcal{H}) = k < \infty$. Let $\dot\bigcup \mathcal{H}$ denote the (possibly infinite) graph that is the disjoint union of all host graphs in $\mathcal{H}$ (considering isomorphic graphs as identical).

Then there is a k-(folded/local/union) $\mathcal{G}$-cover of $\dot\bigcup \mathcal{H}$.

Proof. Since we have $c_i^{\mathcal{G}}(\mathcal{H}) = k < \infty$, for every host graph $H \in \mathcal{H}$ there is a k-(folded/local/union) $\mathcal{G}$-cover (S^H, ϕ^H) of H. We construct a cover (S, ϕ) by merging all those covers as follows: We set $S = \bigcup_{H \in \mathcal{H}} S^H$ and we define

$$\phi\colon \dot\bigcup S \to \dot\bigcup \mathcal{H}, u \mapsto \phi^H(u), \text{ where } u \in \dot\bigcup S^H.$$

Less formally, we cover each graph H which is part of the disjoint union $\dot\bigcup \mathcal{H}$ using the guests in S^H in the way described by ϕ^H. Note that a copy H' of H may appear as subgraph of another host graph in $\dot\bigcup \mathcal{H}$, and that copy H' may be covered in another way than H. For this proof, whenever we write $H \subseteq \dot\bigcup \mathcal{H}$, we mean a graph H which is part of the disjoint union $\dot\bigcup \mathcal{H}$ and not just a subgraph of $\dot\bigcup \mathcal{H}$.

Note that by the definition of S, the function ϕ is well defined. Let vw be an edge in $\dot\bigcup S$. Then it is edge of a guest $G \in S$. By the definition of S, this guest G belongs to a guest-set S^H. Since ϕ^H is a homomorphism, we have

$$(\phi(v), \phi(w)) = (\phi^H(v), \phi^H(w)) \in E(H) \subseteq E(\dot\bigcup \mathcal{H}).$$

Therefore, the function ϕ is a homomorphism.

Let xy be an edge in $\dot\bigcup \mathcal{H}$. By the definition of $\dot\bigcup \mathcal{H}$ there is a graph $H \subseteq \dot\bigcup \mathcal{H}$ with $xy \in H$. Since (S^H, ϕ^H) is a cover, there are two vertices $v, w \in \dot\bigcup S^H$ with $\phi(v) = \phi^H(v) = x$ and $\phi(w) = \phi^H(w) = y$. Hence, the homomorphism ϕ is edge-surjective.

For every guest $G \in \dot\bigcup S$ we have a graph H with $G \in S^H$. We further have

$$\phi|_G = (\phi|_{\dot\bigcup S^H})|_G = \phi^H|_G$$

which is vertex-injective, if $i = l, u$. Hence, cover (S, ϕ) is guest-injective, if $i = l, u$.

Assume $i = f, l$ and consider a vertex $v \in \dot\bigcup \mathcal{H}$. By definition of $\dot\bigcup \mathcal{H}$, vertex v belongs to a graph $H \subseteq \dot\bigcup \mathcal{H}$. Then we have $|\phi^{-1}(v)| = |(\phi^H)^{-1}(v)| \leq k$, since only vertices of $\dot\bigcup S^H$ are mapped onto H. Hence, cover (S, ϕ) is indeed a k-(folded/local) $\mathcal{G}$-cover of $\dot\bigcup \mathcal{H}$.

Next assume $i = u$. Then for every graph $H \in \mathcal{H}$ the set S^H can be partitioned into k sets $S_1^H, \dots, S_k^H$ such that for $j \in [k]$ the restricted homomorphism $\phi^H|_{\dot\bigcup S_j^H}$ is vertex-injective.

For $j \in [k]$, we define $S_j = \dot\bigcup_{H \in \mathcal{H}} S_j^H$. Since for every graph $H \subseteq \dot\bigcup \mathcal{H}$ the vertices of $\dot\bigcup S^H$ are only mapped onto H, we obtain for every vertex $x \in \dot\bigcup \mathcal{H}$ with $x \in H$ that $|(\phi|_{\dot\bigcup S_j})^{-1}(x)| = |(\phi|_{\dot\bigcup S_j^H})^{-1}(x)| \leq 1$. Hence, the restricted homomorphism $\phi|_{\dot\bigcup S_j}$ is vertex-injective and (S, ϕ) is a k-union $\mathcal{G}$-cover of $\dot\bigcup \mathcal{H}$. This concludes the proof. □

Lemma 7.8 verifies that the covers provided by Lemma 7.7 are actually best-possible for finite disjoint unions. It thus directly provides the covering number of disjoint unions.

Lemma 7.8
Let H_1 and H_2 be host graphs and let $\mathcal{G}$ be an induced-hereditary guest class. Let $i = f, l, u$. Then $c_i^{\mathcal{G}}(H_1 \dot\cup H_2) = \max(c_i^{\mathcal{G}}(H_1), c_i^{\mathcal{G}}(H_2))$.

Proof. Since $\mathcal{G}$ is induced-hereditary, a k-(folded/local/union) $\mathcal{G}$-cover of $H_1 \dot\cup H_2$ induces a k-(folded/local/union) $\mathcal{G}$-cover of H_1 and another one of H_2. Hence, we have $c_i^{\mathcal{G}}(H_1 \dot\cup H_2) \leq \max(c_i^{\mathcal{G}}(H_1), c_i^{\mathcal{G}}(H_2))$.

On the other hand, if we have a k-(folded/local/union) $\mathcal{G}$-cover (S_1, ϕ_1) of H_1 and an m-(folded/local/union) $\mathcal{G}$-cover (S_2, ϕ_2) of H_2, we obtain a $\max(k, m)$-(folded/local/union) $\mathcal{G}$-cover by Lemma 7.7 considering the host class $\{H_1, H_2\}$. □

This allows us to conclude from the cover resistance of two host classes to the cover resistance of their union.

Theorem 7.9
Let $i = f, l, u$ and let $\mathcal{H}_1$ and $\mathcal{H}_2$ be i-cover resistant host classes. Then $\mathcal{H}_1 \cup \mathcal{H}_2$ is an i-cover resistant host class.

Proof. First consider $\mathcal{H}_1 \cup \mathcal{H}_2$. Let $\mathcal{G}$ be an induced-hereditary guest class. If $c_i^{\mathcal{G}}(\mathcal{H}_1) = \infty$ or $c_i^{\mathcal{G}}(\mathcal{H}_2) = \infty$, then with Proposition 3.1(iii) we obtain $c_i^{\mathcal{G}}(\mathcal{H}_1 \cup \mathcal{H}_2) = \infty$. Hence, assume $c_i^{\mathcal{G}}(\mathcal{H}_1) \leq 1$ and $c_i^{\mathcal{G}}(\mathcal{H}_2) \leq 1$. Now let $H \in \mathcal{H}_1 \cup \mathcal{H}_2$. Then there are two graphs $H_1 \in \mathcal{H}_1$ and $H_2 \in \mathcal{H}_2$ such that $H = H_1 \dot\cup H_2$. With Lemma 7.8 we have that

$$c_i^{\mathcal{G}}(H) = \max\{c_i^{\mathcal{G}}(H_1), c_i^{\mathcal{G}}(H_2)\} \leq \max\{c_i^{\mathcal{G}}(\mathcal{H}_1), c_i^{\mathcal{G}}(\mathcal{H}_2)\} \leq 1.$$

Hence, host class $\mathcal{H}_1 \cup \mathcal{H}_2$ is i-cover resistant. □

Lemma 7.8 also allows us to conclude from the cover resistance of one class to the cover resistance of its union-closure.

Theorem 7.10
Let $\mathcal{H}$ be a host class and let $\overline{\mathcal{H}}$ be its union-closure. Let $i = f, l, u$. Then $\mathcal{H}$ is i-cover resistant if and only if $\overline{\mathcal{H}}$ is i-cover resistant.

Proof. Let $\mathcal{G}$ be an induced-hereditary guest class. By Lemma 7.8 we obtain $c_i^{\mathcal{G}}(\mathcal{H}) = c_i^{\mathcal{G}}(\overline{\mathcal{H}})$. Since the covering numbers of $\mathcal{H}$ and $\overline{\mathcal{H}}$ coincide for induced-hereditary guest classes, so does their cover resistance. □

With Theorem 7.10 we only have to consider union-closed host classes for cover resistance. For other host classes their cover resistance is determined by the cover resistance of their union-closure.

Another interesting closure for cover resistant host classes is the induced-hereditary closure. This is especially the case, since the guest classes are required to be induced-hereditary.

Proposition 7.11
Let $\mathcal{H}$ be a host class and let $\hat{\mathcal{H}}$ be its induced-hereditary closure. Let $i = f, l, u, g$. Then $\mathcal{H}$ is i-cover resistant if and only if $\hat{\mathcal{H}}$ is i-cover resistant.

Proof. Assume $\mathcal{H}$ is i-cover resistant. Consider an induced-hereditary guest class $\mathcal{G}$ with $c_i^{\mathcal{G}}(\hat{\mathcal{H}}) > 1$. Then there is a graph $H \in \hat{\mathcal{H}}$ with $c_i^{\mathcal{G}}(H) > 1$. Since we have $H \sqsubseteq H'$ for some host graph $H' \in \mathcal{H}$, we obtain by Proposition 3.6 that $c_i^{\mathcal{G}}(H') \geq c_i^{\mathcal{G}}(H) > 1$. Since $\mathcal{H}$ is i-cover resistant, this implies $c_i^{\mathcal{G}}(\mathcal{H}) = \infty$. With Proposition 3.1(iii) and Proposition 3.4(iii) we obtain $c_i^{\mathcal{G}}(\hat{\mathcal{H}}) = \infty$.

Now assume $\hat{\mathcal{H}}$ is i-cover resistant. Consider an induced-hereditary guest class $\mathcal{G}$ with $c_i^{\mathcal{G}}(\mathcal{H}) > 1$. Then there is a graph $H \in \mathcal{H} \subseteq \hat{\mathcal{H}}$ with $c_i^{\mathcal{G}}(H) > 1$. Since $\hat{\mathcal{H}}$ is i-cover resistant, this implies $c_i^{\mathcal{G}}(\hat{\mathcal{H}}) = \infty$. Therefore, for any positive integer k, there is a graph $H_k \in \hat{\mathcal{H}}$ with $c_i^{\mathcal{G}}(H_k) \geq k$. As $H_k \sqsubseteq H'_k$ for some host graph $H'_k \in \mathcal{H}$, we obtain $c_i^{\mathcal{G}}(\mathcal{H}) \geq c_i^{\mathcal{G}}(H'_k) = k$. Hence $c_i^{\mathcal{G}}(\mathcal{H}) = \infty$. ☐

Analogously to Theorem 7.10, Theorem 7.11 allows us to only consider induced-hereditary host classes for cover resistance, since the cover resistances of other classes can be derived from those.

7.5 Relations Between Different Cover Resistances

For a fixed guest class $\mathcal{G}$, we have for any host class $\mathcal{H}$ that $c_f^{\mathcal{G}}(\mathcal{H}) \leq c_l^{\mathcal{G}}(\mathcal{H}) \leq c_u^{\mathcal{G}}(\mathcal{H}) \leq c_g^{\mathcal{G}}(\mathcal{H})$. These relations are reflected in implications for cover resistances, except for the g-cover resistance which requires some extra care.

Proposition 7.12
Let $\mathcal{H}$ be a host class.

(i) If $\mathcal{H}$ is f-cover resistant, then $\mathcal{H}$ is also l-cover resistant.

(ii) If $\mathcal{H}$ is l-cover resistant, then $\mathcal{H}$ is also u-cover resistant.

(iii) If $\mathcal{H}$ is u-cover resistant, then $\mathrm{Comp}\,(\mathcal{H})$ *is g-cover resistant, where* $\mathrm{Comp}\,(\mathcal{H}) = \{C \mid \exists H \in \mathcal{H} : C \text{ is component of } H\}$.

(iv) If $\mathcal{H}$ is u-cover resistant and every graph $H \in \mathcal{H}$ is connected, then $\mathcal{H}$ is also g-cover resistant.

(v) If $\mathcal{H}$ is g-cover resistant, then $\mathcal{H}$ is also u-cover resistant.

Proof. "(i)": Let $\mathcal{H}$ be f-cover resistant. Let $\mathcal{G}$ be an induced-hereditary guest class. Now assume $c_l^{\mathcal{G}}(\mathcal{H}) > 1$. Then there is some host graph $H \in \mathcal{H}$ such that $c_l^{\mathcal{G}}(H) = k > 1$. This means $H \notin_e \overline{\mathcal{G}}$ and with Proposition 3.5 we obtain $c_f^{\mathcal{G}}(\mathcal{H}) \geq c_f^{\mathcal{G}}(H) > 1$. Since $\mathcal{H}$ is f-cover resistant, with Proposition 3.1(ii) this implies $c_l^{\mathcal{G}}(\mathcal{H}) \geq c_f^{\mathcal{G}}(\mathcal{H}) = \infty$. Hence, host class $\mathcal{H}$ is l-cover resistant.

"(ii)": Can be proven analogously.

"(iii)": For $H \in \mathrm{Comp}(\mathcal{H})$ we have $H \notin_e \mathcal{G} \Rightarrow H \notin_e \overline{\mathcal{G}}$ since H is connected. Using this, item (iii) can be proven analogously.

"(iv)": Direct consequence of (iii), noting that $\mathrm{Comp}(\mathcal{H}) = \mathcal{H}$.

"(v)": Let $\mathcal{G}$ be an induced-hereditary guest class. Let $\mathcal{H}$ be g-cover resistant. Then we have $c_u^{\mathcal{G}}(\mathcal{H}) = c_g^{\overline{\mathcal{G}}}(\mathcal{H}) \in \{0, 1, \infty\}$. Therefore $\mathcal{H}$ is also u-cover resistant. ☐

With Proposition 7.12 it suffices to state the strongest cover resistance(s) of a host class. E.g., we know that the class $\mathcal{OS}$ of shift graphs of ordered graphs is g-cover resistant by Theorem 4.15 and can conclude that it is also u-cover resistant. However, concluding from union to global cover resistance needs extra attention, since it requires the host graphs to be connected. That some condition like that is necessary can be seen in Theorem 7.13.

Theorem 7.13
Let $\mathcal{K}$ denote the class of complete graphs and let $\mathcal{S}t$ denote the class of stars. Then $\mathcal{H} = \overline{\mathcal{K} \cup \mathcal{S}t}$ is a union-closed induced-hereditary host class that is l-cover resistant but not g-cover resistant.

Further, let $\mathcal{OS}$ denote the class of shift graphs of ordered graphs as introduced at the end of Chapter 4 Section 4.2. Let $\mathcal{F} = \overline{\mathcal{K} \cup \mathcal{S}t \cup \mathcal{OS}}$ be the union-closure of $\mathcal{H} \cup \mathcal{OS}$. Then $\mathcal{F}$ is u-cover resistant but neither g- nor l-cover resistant.

Proof. Since $\mathcal{K}$ and $\mathcal{S}t$ are induced-hereditary, so is $\mathcal{K} \cup \mathcal{S}t$ and $\mathcal{H}$. By its definition $\mathcal{H}$ is also union-closed.

Further, the class of complete graphs $\mathcal{K}$ is l-cover resistant by Theorem 7.2 and the class of stars $\mathcal{S}t$ is f-cover resistant by Theorem 7.3. This implies l-cover resistance of $\mathcal{K}$ and $\mathcal{S}t$ by Proposition 7.12. By Theorem 7.9 and Theorem 7.10 this proves that $\mathcal{H}$ is l-cover resistant.

Now consider the guest class $\mathcal{G} = \overline{\mathcal{K}} \cup \mathcal{OS}$. It is easy to verify that it is induced-hereditary. Let $H = H_c \uplus H_s \in \mathcal{H}$ be a host graph where $H_c \in \overline{\mathcal{K}}$ and $H_s \in \overline{\mathcal{S}t} \subseteq \mathcal{OS}$. Then H can be covered using H_c and H_s as two guests that cover themselves. Hence, we have $c_g^{\mathcal{G}}(\mathcal{H}) \leq 2$. On the other hand, consider the guest $H' = K_3 \uplus St_3$ where St_3 denotes the star with 3 leaves. Since $St_3 \notin \mathcal{K}$ and $K_3 \notin \mathcal{OS}$ and $K_3 \notin \mathcal{S}t$ we have $H' \notin \overline{\mathcal{K}}$ and $H' \notin \mathcal{OS}$ and $H' \notin \overline{\mathcal{S}t}$. Hence, we obtain $H' \notin \mathcal{G}$ which implies $\mathcal{H} \not\subseteq_e \mathcal{G}$, since H' contains no isolated vertices. With the definition of cover resistance we conclude that $\mathcal{H}$ is not g-cover resistant.

By Theorem 4.15 we know $\mathcal{OS}$ is u-cover resistant and with Theorem 7.9 and 7.10 we obtain that $\mathcal{F}$ is also u-cover resistant. Let $H = H_c \uplus H_s \uplus H_o \in \mathcal{H}$ be a host graph where $H_c \in \overline{\mathcal{K}}$ and $H_s \in \overline{\mathcal{S}t}$ and $H_o \in \mathcal{OS}$. Then H can be covered using H_c, H_s and H_o as three guests that cover themselves. Hence, we have $c_g^{\mathcal{G}}(\mathcal{F}) \leq 3$. However, we also have $c_g^{\mathcal{G}}(\mathcal{F}) \geq c_g^{\mathcal{G}}(\mathcal{H}) > 1$. Therefore $\mathcal{F}$ is not g-cover resistant.

Finally consider guest class $\mathcal{G}' = \overline{\mathcal{K}} \cup \mathcal{B}ip$ where $\mathcal{B}ip$ is the class of bipartite graphs. Again it is easy to verify that $\mathcal{G}'$ is induced-hereditary. By Corollary 4.13 we know $c_l^{\mathcal{B}ip}(\mathcal{OS}) \leq c_l^{\mathcal{B}ip}(\mathcal{S}) = 2$. With $\mathcal{S}t \subseteq \mathcal{OS}$ we obtain $c_l^{\mathcal{G}'}(\mathcal{F}) \leq 2$. On the other hand, we know that $C_5 \sqsubseteq \mathrm{S}(K_5^o)$ and $C_5 \notin \mathcal{K} \cup \mathcal{B}ip$. Thus $c_l^{\mathcal{G}'}(\mathcal{F}) \geq c_l^{\mathcal{G}'}(\mathcal{OS}) \geq c_l^{\mathcal{G}'}(\mathrm{S}(K_5^o)) \geq 2$. Hence, we have $c_l^{\mathcal{G}'}(\mathcal{F}) = 2$ and thus $\mathcal{F}$ is not l-cover resistant. □

The first construction of Theorem 7.13 is generally possible for two host classes that do not contain each other. This is because the g-cover resistance is defined different to the other cover resistances by demanding $\mathcal{H} \subseteq_e \mathcal{G}$ instead of $\mathcal{H} \subseteq_e \overline{\mathcal{G}}$, if the covering number is infinite. This supports the approach of only considering union-closed guest classes for the comparison of the local, folded and global covering number.

By Theorem 4.15 and Corollary 4.13, the class of shift graphs is u-cover resistant but not l-cover resistant. By Theorem 7.2 the class of complete graphs is l-cover resistant. However, since $c_f^{\mathcal{B}ip}(\mathcal{K}) \leq 2$ by Theorem 4.3 and $K_3 \notin \mathcal{B}ip$, the class $\mathcal{K}$ is not f-cover resistant.

Hence, the f-cover resistance is a stronger property than the l-cover resistance, which in turn is a stronger property than the u-cover resistance.

However, there are many host classes for which these cover resistances are equivalent as Corollary 7.14 shows.

We recall Theorem 4.17:

Theorem 4.17
Let $\mathcal{G}$ be an induced-hereditary guest class and let H be a host graph. Let (S, ϕ) be a k-folded $\mathcal{G}$-cover of H with $S = \{G_1, \ldots, G_m\}$. Then there is a $(k^2\lceil\log(\chi(H))\rceil)$-union $\mathcal{G}$-cover (S', ϕ') of H, such that every guest in S' is induced subgraph of one of the guests in S. Especially we have $c_u^{\mathcal{G}}(H) \leq c_f^{\mathcal{G}}(H)^2\lceil\log(\chi(H))\rceil$.

Theorem 4.17 provides a non-separability of folded- and union-covering number on host graphs of bounded chromatic number. This non-separability implies an equivalence of the cover resistances.

Corollary 7.14
Let $r \in \mathbb{N}$ and let $\mathcal{H}$ be a host class with $\chi(\mathcal{H}) \leq r$, i.e., let for every host graph $H \in \mathcal{H}$ hold that $\chi(H) \leq r$.

Then $\mathcal{H}$ is f-cover resistant if and only if it is u-cover resistant.

Proof. By Proposition 7.12 we know $\mathcal{H}$ is u-cover resistant if it is f-cover resistant. Hence, let $\mathcal{H}$ be u-cover resistant.

Let $\mathcal{G}$ be an induced-hereditary guest class with $c_f^{\mathcal{G}}(\mathcal{H}) = k < \infty$. Let $H \in \mathcal{H}$. By Theorem 4.17 we have $c_u^{\mathcal{G}}(H) \leq c_f^{\mathcal{G}}(\mathcal{H})^2\lceil\log(\chi(H))\rceil \leq k^2 r < \infty$. With Proposition 3.1 and Proposition 3.4 this implies $c_f^{\mathcal{G}}(H) \leq c_u^{\mathcal{G}}(H) \leq 1$. Therefore, host class $\mathcal{H}$ is f-cover resistant. □

In another case we obtain the l-cover resistance from the g-cover resistance as stated in Theorem 7.16.

Lemma 7.15
Let $\mathcal{H}$ be a g-cover resistant host class and $\mathcal{A}$ be the class of all graphs. Then we have

$$\forall k \in \mathbb{N} \forall H \in \mathcal{H} : \exists H' \in \mathcal{H} : \forall k\text{-global } \mathcal{A}\text{-cover } (S, \phi) \text{ of } H' : \exists G \in S : H \sqsubseteq_e G$$

Proof. We prove the statement by induction on k. For $k = 1$ just consider $H' = H$. In every 1-global cover of H' the guest G must contain all edges of H' and we thus have $H = H' \sqsubseteq_e G$.

Let $k \geq 1$ and the statement be true for k. Then there is a graph H_k such that every k-global cover of H_k has a guest G_k with $H \sqsubseteq_e G_k$. By Theorem 7.5 there is a graph H_{2k}, such that in every 2-global cover of H_{2k} there is a guest G'_{2k} with $H_k \sqsubseteq_e G'_{2k}$.

Consider a $2k$-global cover (S, ϕ) of H_{2k}. Now partition the set S into two sets A and B of (at most) k guests each.

Consider the union G_A of all guests in A and the union G_B of all guests in B. We obtain a 2-global cover of H_{2k} by using G_A and G_B as guests with the mapping of ϕ. Hence, we have $H_k \sqsubseteq_e G_A$ or $H_k \sqsubseteq_e G_B$. Let without loss of generality $H_k \sqsubseteq_e G_A$. By definition of G_A, this copy of H_k is globally covered by k guests in $A \subseteq S$. By definition of H_k this implies there is a guest G_k in $A \subseteq S$ with $G \sqsubseteq_e G_k$. Since not all guests have to be used, also a $(k + 1)$-global cover of H_{2k} contains a guest G' with $H \sqsubseteq_e G'$. This concludes the induction. □

Let $\mathcal{H}$ be a graph class. We call $\mathcal{H}$ *universal-vertex-closed*, if for $H \in \mathcal{H}$ adding a *universal vertex* x to H that is adjacent to all vertices in H results in a graph $H' \in \mathcal{H}$. I.e., if for all $H \in \mathcal{H}$ with $x \notin H$ we have $(V(H) \cup \{x\}, E(H) \cup xV) \in \mathcal{H}$ where $xV = \{xv \in \binom{V \cup \{x\}}{2} \mid v \in V\}$.

Theorem 7.16
Let $\mathcal{H}$ be a universal-vertex-closed g-cover resistant host class. Then $\mathcal{H}$ is l-cover resistant.

Proof. We aim to use Theorem 7.5. We prove the Property $(\star\star\star)$ by induction on k. For $k = 1$ just consider $H' = H$ and the statement holds, since all edges must be covered while the guests must be vertex disjoint.

Let $k \geq 1$ and for k let hold that

$$\forall H \in \mathcal{H} : \exists H' \in \mathcal{H} : \forall k\text{-local } \mathcal{A}\text{-cover } (S, \phi) \text{ of } H' : H \sqsubseteq_e \dot{\bigcup} S.$$

Let $H \in \mathcal{H}$. Then there is a graph H_k such that every k-local cover of H_k has a guest ${G'}_k$ with $H \sqsubseteq {G'}_k$. By Lemma 7.15 there is a graph H'_{k+1}, such that in every $(k+2)$-global cover of H'_{k+1}, there is a guest G'_{k+1} with $H_k \sqsubseteq G'_{k+1}$. Now let H_{k+1} be the graph H'_{k+1} to which a (universal) vertex x is added that is adjacent to all other vertices. I.e., we define $H_{k+1} = (V(H_{k+1}), E(H_{k+1}))$ where $V(H_{k+1}) = V(H'_{k+1}) \dot{\cup} \{x\}$ and $E(H_{k+1}) = E(H'_{k+1}) \dot{\cup} \{vx \mid x \in V(H'_{k+1})\}$.

Consider a $(k+1)$-local cover (S, ϕ) of H_{k+1}. There are at most $(k+1)$ guests $G_1, \ldots, G_s$ in S covering x. Consider the union G_R of the remaining guests. We obtain a $(k+2)$-global cover of H'_{k+1} by using $G_1, \ldots, G_s, G_R$ as guests and the mapping of ϕ. By definition of H'_{k+1} there is a guest G'_{k+1} with $H_k \sqsubseteq G'_{k+1}$. Assume G'_{k+1} is one of the guests $G_1, \ldots, G_s$ covering H'_{k+1}. Then $H \sqsubseteq H_k \sqsubseteq G'_{k+1} = G_i$ for some $i = 1, \ldots, s$. Hence, assume $H_k \sqsubseteq G_R$. Observe that every vertex v of H'_{k+1} is adjacent to x in H_{k+1}. As the corresponding edge vx is covered by a guest in $\{G_1, \ldots, G_s\}$, vertex v is covered by that guest and at most k other guests. Therefore (S, ϕ) induces a k-local cover on G_R. This cover also covers a copy of H_k, since $H_k \sqsubseteq G_R$. By the definition of H_k this implies there is a guest G'_k in S with $H \sqsubseteq G'_k$. This concludes the induction. Therefore $\mathcal{H}$ is l-cover resistant by Theorem 7.5. ☐

7.6 Negative Results

In the preceding sections we focused on cover resistant host classes. In this section we provide sufficient conditions for host classes to be not cover resistant and give examples of such classes.

Plenty of host classes are not cover resistant. Those host classes include all classes whose graphs have bounded chromatic number, but are not all bipartite, as Proposition 7.17 shows. The perhaps most notable examples for such classes are the planar and the outer-planar graphs. Another example are (a, b)-sparse graphs which include forests.

Proposition 7.17
Let $\mathcal{H}$ be a host class with $\max\{\chi(H) \mid H \in \mathcal{H}\} = r \geq 3$. *Then $\mathcal{H}$ is not g-cover resistant and not u-cover resistant.*

Proof. We consider the guest class $\mathcal{B}ip$ of bipartite graphs. By Lemma 4.12 we know for every $H \in \mathcal{H}$ that $c_u^{\mathcal{B}ip}(H) \leq c_g^{\mathcal{B}ip}(H) = \lceil \log(\chi(H)) \rceil \leq \lceil \log(r) \rceil$. Hence, we have $c_u^{\mathcal{B}ip}(\mathcal{H}) \leq c_g^{\mathcal{B}ip}(\mathcal{H}) < \infty$.

However, there is a host graph H' with $\chi(H') = r \geq 3$. This host graph H' is not bipartite and therefore not contained in the guest class $\mathcal{B}ip = \overline{\mathcal{B}ip}$. Especially we have $\mathcal{H} \not\sqsubseteq_e \mathcal{B}ip = \overline{\mathcal{B}ip}$. ☐

Another negative result is that most finite classes are not cover resistant as shown in Proposition 7.18.

Proposition 7.18
Let $\mathcal{H}$ be a host class containing only a finite number of elements. Then the following holds:

(i) Host class $\mathcal{H}$ is g-cover resistant if and only if every host graph in $\mathcal{H}$ has at most one edge.

(ii) Host class $\mathcal{H}$ is f-cover resistant if and only if every host graph in $\mathcal{H}$ has maximum degree at most 1.

Proof. First we prove (i).

"$\Leftarrow$": Let every host graph in $\mathcal{H}$ contain at most one edge. Let $\mathcal{G}$ be an induced-hereditary guest class. If $\mathcal{H}$ contains only independent sets, then we have $c_g^{\mathcal{G}}(\mathcal{H}) = 0$ by Proposition 3.5. Hence, assume $\mathcal{H}$ contains a host graph H_1 with an edge. In the first case, if there is also a guest $G \in \mathcal{G}$ containing an edge, then $K_2 \sqsubseteq G$ and by induced-heredity $\mathcal{G}$ contains K_2 which can be used as singe guest (without folding) to cover the single edge of any host graph in $\mathcal{H}$. Hence, we have $c_g^{\mathcal{G}}(\mathcal{H}) = 1$. In the second case, no guest in $\mathcal{G}$ contains an edge. Hence the edge of H_1 can not be covered using guests in $\mathcal{G}$. Therefore, we get $c_g^{\mathcal{G}}(\mathcal{H}) = \infty$. This proves $\mathcal{H}$ is g-cover resistant.

"$\Rightarrow$": Now let $\mathcal{H}$ contain a graph H_2 of size $||H_2|| \geq 2$. We consider the guest class $\mathcal{M}_1$ containing only K_2. Since there is only a finite number of different host graphs in $\mathcal{H}$, there exists a positive integer $m > 1$ which is the maximum size a host graph in $\mathcal{H}$ can have, while there actually is a host graph H_m with this number $||H_m|| = m > 1$.

By using another copy of K_2 as guest for every edge of a host graph $H \in \mathcal{H}$ and mapping it to the corresponding edge, we can cover every host graph with at most m guests. We therefore have $c_g^{\mathcal{M}_1}(\mathcal{H}) \leq m$. On the other hand, we actually need m guests to cover all edges of H_m since every guest can only cover one edge. We therefore have $1 < m = c_g^{\mathcal{G}}(\mathcal{H}) < \infty$. Thus, host class $\mathcal{H}$ is not g-cover resistant.

Now we prove (ii).

"$\Leftarrow$": Let every host graph in $\mathcal{H}$ have maximum degree at most 1. Then let $\mathcal{G}$ be an induced-hereditary guest class. If $\mathcal{H}$ contains only independent sets, then we have $c_f^{\mathcal{G}}(\mathcal{H}) = 0$ by Proposition 3.5. Hence, assume $\mathcal{H}$ contains a host graph H_1 with an edge. In the first case, if there is also a guest $G \in \mathcal{G}$ containing an edge, then $K_2 \sqsubseteq G$ and by induced-heredity $\mathcal{G}$ contains K_2. Let $H \in \mathcal{H}$. By covering every edge of H by another copy of K_2 no vertex is covered more than once (as it has degree at most 1). Hence, we have $c_f^{\mathcal{G}}(\mathcal{H}) = 1$. In the second case, no guest in $\mathcal{G}$ contains an edge. Hence the edge of H_1 can not be covered using guests in $\mathcal{G}$. Therefore, we get $c_f^{\mathcal{G}}(\mathcal{H}) = \infty$. This proves $\mathcal{H}$ is f-cover resistant.

"$\Rightarrow$":Now let $\mathcal{H}$ contain a graph H_2 with a vertex v of degree at least $\deg(v) \geq 2$. We consider the guest class $\mathcal{M}_1$ containing only K_2. Since there is only a finite number of different host graphs in $\mathcal{H}$, there exists a positive integer $m > 1$ which is the maximum maximum degree $\Delta(H)$ a host graph $H \in \mathcal{H}$ can have, while there actually is a host graph H_m with $\Delta(H_m) = m$.

By using another copy of K_2 as guest for every edge of a host graph $H \in \mathcal{H}$ and mapping it to the corresponding edge, we can cover every host graph $H \in \mathcal{H}$ such that every vertex $v \in H$ is covered at most $\deg(v) \leq \Delta(H) = m$ times. We therefore have $c_g^{\mathcal{M}_1}(\mathcal{H}) \leq m$. On the other hand, there is a vertex v_m in H_m with $\deg(v_m) = \Delta(H_m) = m$. To cover all edges adjacent to v_m, that vertex must be covered m times, since every edge must be covered by another guest. We therefore have $1 < m = c_f^{\mathcal{G}}(\mathcal{H}) < \infty$. Thus, host class $\mathcal{H}$ is not f-cover resistant. □

7.7 f-Cover Resistance

In this section we give examples of f-cover resistant classes. Recall that this is the strongest cover resistance, since every host class that is f-cover resistant is also l- and u-cover resistant by Proposition 7.12.

Note that an f-cover resistant class $\mathcal{H}$ can only contain bipartite graphs: By Theorem 4.3 for every graph H we have $c_f^{\mathcal{B}ip}(H) \leq 2$, thus we have $c_f^{\mathcal{B}ip}(\mathcal{H}) \leq 2 < \infty$ for every host class $\mathcal{H}$.

Hence, only subclasses of $\mathcal{B}ip$ can be f-cover resistant. For the simplest subclasses of $\mathcal{B}ip$ we have positive results. We start with the class of a single edge and the class of matchings as the smallest f-cover resistant classes.

Theorem 7.19
The class $\{K_2\}$ and the class $\mathcal{M}$ of matchings are f-cover resistant and g-cover resistant.

Proof. By Proposition 7.18 the class $\{K_2\}$ is f- and g-cover resistant. With Theorem 7.10 we know $\mathcal{M} = \overline{\{K_2\}}$ is also f-cover resistant.

Now let $\mathcal{G}$ be an induced-hereditary guest class with $\mathcal{M} \notin \mathcal{G}$ and let $k \geq 2$. Then there is a smallest matching $M_t \notin \mathcal{G}$ which is the disjoint union of t copies of K_2. Then consider matching $M_{(k-1)t+1}$, the disjoint union of $(k-1)t+1$ copies of K_2. The only components of $M_{(k-1)t+1}$ are copies of K_2 and thus only matchings can be guests in global covers of $M_{(k-1)t+1}$. Since every matching in $\mathcal{G}$ contains at most t edges and all edges of $M_{(k-1)t+1}$ must be covered, a global $\mathcal{G}$-cover of $M_{(k-1)t+1}$ contains at least k guests. Therefore, we have $c_g^{\mathcal{G}}(\mathcal{M}) = \infty$ and the class $\mathcal{M}$ is g-cover resistant. $\square$

The next simple class is the class $\mathcal{S}t$ of stars, which is also f-cover resistant as stated in Theorem 7.3.

Theorem 7.20 shows that the class $\mathcal{S}t$ is the next larger induced-hereditary f-cover resistant class to $\{K_2\}$.

Theorem 7.20
Let $\mathcal{H} \subseteq \mathcal{B}ip$ be an induced-hereditary host class with a graph $H' \in \mathcal{H}$ with $\Delta(H') > 1$ and $\mathcal{S}t \nsubseteq \mathcal{H}$. Then $\mathcal{H}$ is not f-cover resistant.

Proof. Since $\mathcal{S}t \nsubseteq \mathcal{H}$, there is a star S_D that is not contained in $\mathcal{H}$. Assume for the sake of contradiction that there is a host graph $H \in \mathcal{H}$ with $\Delta(H) \geq D$. Then there is a vertex $v \in H$ with $\deg(v) \geq D$. But vertex v and D of its neighbours induce S_D in H, since H must be bipartite and the neighbours of v can not be adjacent. The host class $\mathcal{H}$ is induced-hereditary, and thus, we have $S_D \in \mathcal{H}$. This contradicts the definition of S_D.

Therefore, every host graph $H \in \mathcal{H}$ has maximum degree at most $D-1$. Then, by Vizing's Theorem [Viz64], every host graph $H \in \mathcal{H}$ has a D-global $\mathcal{M}$-cover. Hence, we have $c_f^{\mathcal{M}}(\mathcal{H}) \leq c_g^{\mathcal{M}}(\mathcal{H}) \leq D$. On the other hand, we have $\mathcal{H} \nsubseteq_e \mathcal{M}$, since $\Delta(H') > 1$. Hence, host class $\mathcal{H}$ is not f-cover resistant. $\square$

Another f-cover resistant host class is the class $\mathcal{C}$-$\mathcal{B}ip$ of complete bipartite graphs. To prove this, we consider a result by Irving which is related to the *bipartite Ramsey number* [Irv78].

Lemma 7.21 (Irving [Irv78])
The class $\mathcal{C}$-$\mathcal{B}ip$ of complete bipartite graphs has the Induced Ramsey Property.

We already have everything we need to show $\mathcal{C}$-$\mathcal{B}ip$ is f-cover resistant.

Corollary 7.22
The class $\mathcal{C}$-$\mathcal{B}ip$ of complete bipartite graphs is g- and f-cover resistant.

Proof. By Lemma 7.21 the class $\mathcal{C}$-$\mathcal{B}ip$ has the Induced Ramsey Property. By Corollary 7.6 it is thus g-cover resistant. With Proposition 7.12 we obtain that $\mathcal{C}$-$\mathcal{B}ip$ is also u-cover resistant. Finally, by Corollary 7.14 class $\mathcal{C}$-$\mathcal{B}ip$ is also f-cover resistant. □

While Theorem 7.20 verifies there is no f-cover resistant host class between the class of matchings $\mathcal{M}$ and the class of stars $\mathcal{S}t$, we do not provide such a statement for classes between $\mathcal{S}t$ and $\mathcal{C}$-$\mathcal{B}ip$. Indeed this is a much larger gap and it is at least not obvious whether such a class can be f-cover resistant.

We finally consider the class $\mathcal{B}ip$ of all bipartite graphs itself. Since only subclasses of $\mathcal{B}ip$ can be f-cover resistant, this is the largest f-cover resistant class. Our proof of the f-cover resistance of $\mathcal{B}ip$ is based on Lemma 7.23, a well known result of Induced Ramsey Theory.

Lemma 7.23 (e.g. Diestel [Die05][Lemma 9.3.3])
The class $\mathcal{B}ip$ of bipartite graphs has the Induced Ramsey Property.

Corollary 7.24
The class $\mathcal{B}ip$ of bipartite graphs is g- and f-cover resistant.

Proof. This statement can be proven analogously to Corollary 7.22 using Lemma 7.23. □

Corollary 7.24 presents $\mathcal{B}ip$ as a very special class in terms of the folded covering number. With Theorem 4.3 it implies $\mathcal{B}ip$ is a necessary subclass for every induced-hereditary guest class with bounded folded covering number for *any* host graph H. This is shown in Corollary 7.26.

Lemma 7.25
Let $B \in \mathcal{B}ip$. Then there is a connected bipartite graph B' with $B \sqsubseteq B' \in \mathcal{B}ip$.

Proof. Let the partition sets of $B \in \mathcal{B}ip$ be the sets X and Y. Then we define $B' = (V, E)$ where $V = V(B) \mathbin{\dot\cup} \{x, y\}$ and $E = E(B) \cup \{xz \mid z \in Y\} \cup \{zy \mid z \in X\} \cup \{xy\}$. I.e., let B' be the bipartite graph obtained by adding a vertex x adjacent to all vertices in Y and a vertex y adjacent to all vertices in X and also adjacent to the new vertex x. Then bipartite graph B' is connected and we have $B \sqsubseteq B'$. □

Corollary 7.26
Let $\mathcal{H}$ be the host class of all graphs and let $\mathcal{G}$ be an induced-hereditary guest class. Then we have

$$c_f^{\mathcal{G}}(\mathcal{H}) < \infty \Leftrightarrow \mathcal{B}ip \subseteq \mathcal{G} \Leftrightarrow c_f^{\mathcal{G}}(\mathcal{H}) \leq 2.$$

Proof. By Theorem 4.3 and Proposition 3.1, we have $\mathcal{B}ip \subseteq \mathcal{G} \Rightarrow c_f^{\mathcal{G}}(\mathcal{H}) \leq c_f^{\mathcal{B}ip}(\mathcal{H}) = 2 < \infty$.

On the other hand, let $c_f^{\mathcal{G}}(\mathcal{H}) < \infty$. Especially we have $c_f^{\mathcal{G}}(\mathcal{B}ip) < \infty$. By Corollary 7.24 the host class $\mathcal{B}ip$ is f-cover resistant and this implies $\mathcal{B}ip \subseteq_e \overline{\mathcal{G}}$. Since $\mathcal{B}ip$ is closed under adding isolated vertices, so is $\overline{\mathcal{G}}$. Therefore we have $\mathcal{B}ip \subseteq \overline{\mathcal{G}}$. Since $\mathcal{G}$ is induced-hereditary, this means $\mathcal{G}$ contains all connected graphs in $\mathcal{B}ip$ and with Lemma 7.25 we obtain that $\mathcal{B}ip \subseteq \mathcal{G}$. This concluded the proof. □

Since $\mathcal{B}ip$ is the class of all graphs with chromatic number at most 2, Corollary 7.26 highlights the relevance of the chromatic number for the folded covering number.

Note that we successfully applied cover resistance results to find a characterization of all induced-hereditary guest classes with bounded folded covering number.

7.8 The Class of All Graphs is l-Cover Resistant

The goal of this section is to find a similar characterization to Corollary 7.26 for the local and union covering number. However, it is a kind of more negative result, since the class of all graphs $\mathcal{H}$ is itself l-cover resistant. Hence, we find for every other induced-hereditary guest class $\mathcal{G}$ host graphs with arbitrarily large local covering numbers.

We use the Induced Ramsey Property which was proven by Deuber et al. [Deu75].

Theorem 7.27 (Deuber et al. [Deu75]; Diestel [Die05][Theorem 9.3.1])
The class $\mathcal{A}$ of all graphs has the Induced Ramsey Property.

As an immediate consequence of Theorem 7.27, we obtain the g- and l-cover resistance of all graphs.

Corollary 7.28
The host class $\mathcal{A}$ of all graphs is g- and l-cover resistant.

Proof. As a direct consequence of Theorem 7.27 and Corollary 7.6 we obtain that $\mathcal{A}$ is g-cover resistant. We note that $\mathcal{A}$ is obviously universal-vertex-closed and with Theorem 7.16 we also have that $\mathcal{A}$ is l-cover resistant. □

As in the section on f-cover resistance, we obtain a characterisation of all induced-hereditary guest classes with bounded local covering number for all host graphs: Only classes containing all graphs have a bounded l-cover resistance.

Corollary 7.29
Let $\mathcal{A}$ be the class of all graphs and let $\mathcal{G}$ be an induced-hereditary guest class. Then we have

$$c_l^{\mathcal{G}}(\mathcal{A}) < \infty \Leftrightarrow \mathcal{G} = \mathcal{A} \Leftrightarrow c_l^{\mathcal{G}}(\mathcal{A}) = 1.$$

Proof. If $\mathcal{G} = \mathcal{A}$, then we have $c_l^{\mathcal{G}}(\mathcal{A}) = 1 < \infty$.

Hence, assume $c_l^{\mathcal{G}}(\mathcal{A}) < \infty$. Since $\mathcal{A}$ is by Corollary 7.28 an l-cover resistant class, we obtain $\mathcal{A} \subseteq_e \overline{\mathcal{G}} \subseteq \mathcal{A}$. Since $\overline{\mathcal{G}}$ is closed under adding isolated vertices this implies $\overline{\mathcal{G}} = \mathcal{A}$.

Let $G \in \mathcal{A}$ be a graph. Then consider the graph G' obtained by adding a vertex x that is adjacent to every vertex in G, i.e., let $G' = (V(G) \mathbin{\dot\cup} x, E(G) \cup \{xv \mid v \in G\})$. Graph G' is by definition connected. Since we have $G' \in \overline{\mathcal{G}}$ this implies $G' \in \mathcal{G}$. Hence, we have $\mathcal{A} \subseteq \mathcal{G} \subseteq \mathcal{A}$. This concludes the proof. □

As a consequence, we find for any induced-hereditary guest class $\mathcal{G}$ host graphs with arbitrarily large local covering number, unless $\mathcal{G}$ is the class of all graphs.

8. Conclusion

In this thesis, we investigated the global, the union, the local and the folded covering number of host graphs (and classes) with regards to different guest classes. Specifically, we investigated the host class of shift graphs paired with the guest class of bipartite graphs. For the guest class of (a, b)-sparse graphs we additionally provided an algorithm which computes an optimal global $\mathcal{G}(a, b)$-cover for a given graph H. Finally, we introduced and investigated the cover resistances for several host classes.

8.1 Separability

We provided the host class of shift graphs with the guest class of (complete) bipartite graphs as separation of the local and the global covering number where the guest class is subgraph-hereditary. With Theorem 4.4 we thus have in some sense separations with strongest possible restrictions on the guest classes.

However, Stumpf asked in his Bachelor Thesis about weaker versions of separations with minor-closed guest classes that may still be possible [Stu15]. On the one hand, Knauer and Ueckerdt provided a family of host graphs such that for $\mathcal{S}t$, the guest class of stars, the union-covering number is about twice the local-covering number [KU16]. This raises the question whether an arbitrarily large factor is possible.

Question 8.1 (Stumpf [Stu15])
Is there for every $r > 0$ a minor-closed guest class $\mathcal{G}$ and a host graph H such that $c_u^{\mathcal{G}}(H) \geq r \cdot c_l^{\mathcal{G}}(H)$?

On the other hand, we have only one example where the folded and the local covering number differ at all for a minor-closed guest class. Namely, for the guest class of linear forests (i.e., disjoint unions of paths), these parameters can differ by one by a result of Stumpf [Stu15]. Hence, even a difference by two with a minor-closed guest class would be a new result.

Question 8.2 (Stumpf [Stu15])
Let $\mathcal{G}$ be a minor-closed union-closed guest class and H be a host graph. By how much can $c_f^{\mathcal{G}}(H)$ and $c_l^{\mathcal{G}}(H)$ differ?

More questions arise in relation to cover resistances.

8.2 The Guest Class of (a,b)-Sparse Graphs

Let $a, b \in \mathbb{N}_0$ and let $b < 2a$. We showed that for a host graph H we have $c_g^{\mathcal{G}(a,b)}(H) = c_u^{\mathcal{G}(a,b)}(H) = c_l^{\mathcal{G}(a,b)}(H)$ and we have $c_g^{\mathcal{G}(a,b)}(H) \leq k$ if and only if H is (ka, kb)-sparse. To this end we made use of the (a,b)-sparsity matroid structure of H. By the restriction $b < 2a$ we ensure that the subgraph induced by a single edge is allowed. For a planar graph P with $|V(P)| \geq 3$ we have $|E(Q)| \leq 3|V(P)| - 6$. Hence, a natural generalization of the (a,b)-sparsity is the $(a,b)_d$*-sparsity* which can be defined as follows. Let $a, b, d \in \mathbb{N}_0$ and let $b < da$. A graph H is $(a,b)_d$*-sparse* if for every $G \subseteq H$ with $|V(G)| \geq d$ we have $|E(G)| \leq a|V(G)| - b$. Note that b must be further restricted for $d > 2$ to allow more graphs than just matchings. We obtain a corresponding lower bound. For this lower bound we obtain an equivalence to $(ka, kb)_d$-sparsity corresponding to the one for the known case for $d = 2$.

Question 8.3
Do we have for $d \in \mathbb{N}$ that a graph H is $(ka, kb)_d$-sparse if and only if its global covering number with regards to $(a,b)_d$-sparse graphs is at most k?

Unfortunately, the $(a,b)_d$-sparse subgraphs of a graph do not induce a matroid. Thus, a new way appears necessary for a positive answer. One may also consider a guest class $\mathcal{G}(a,b)$ of (a,b)-sparse graphs for negative b. However, in that case $\mathcal{G}(a,b)$ is not union-closed and small guests are beneficial. This probably requires a quite different approach.

We provided an algorithm to compute an optimal global $\mathcal{G}(a,b)$-cover for a given graph $H = (V, E)$ with a runtime in $\mathrm{O}(|V| \cdot |E|^2)$. If H is rather sparse, this is significantly worse than a runtime in $\mathrm{O}(k \log(k)|E|^2)$ where $k = c_g^{\mathcal{G}(a,b)}(H)$, since k can be considered bounded. This runtime, however, suffices to compute k. Hence, it is open by how much the runtime of our algorithm can be improved.

While we did not proof that the folded covering number with regards to $\mathcal{G}(a,b)$ always meets the given lower bound, it appears at least very likely. Indeed, an approach of translating our algorithm for optimal global covers into the folded setting by using corresponding definitions of augmenting paths, range graphs and global pebble paths, with additional steps appears very promising. If this approach works, we can find a k-folded $\mathcal{G}(a,b)$ cover for every (ka, b)-sparse graph H. As a consequence we would obtain that the folded covering number indeed always meets the given lower bound.

Finally, for investigation of the relation between local and union-covering number it would also be interesting whether there are other guest classes for which the local and the union covering number coincide. Especially, it would be interesting to know for which subclasses of $\mathcal{G}(a,b)$ this applies. Here we have outer-planar graphs in mind, but also the class of planar graphs for $(3,6)_3$-sparsity.

8.3 Cover Resistance

We introduced the cover resistances and characterized them similar to the Induced Ramsey Property. As a consequence, we obtained that in terms of strength the g-cover resistance neatly fits between the Induced Ramsey Property and the Weak Induced Ramsey Property. This raises the question whether it coincides with one of these properties.

Question 8.4
Is the g-cover resistance equivalent to the Induced Ramsey Property or the Weak Induced Ramsey Property (or both)?

For the characterization of the cover resistances, we established for $i = f, u, g$, that for a host class $\mathcal{H}$ that is not i-cover resistant there exists an induced-hereditary guest class $\mathcal{G}$ with $c_i^{\mathcal{G}}(\mathcal{H}) = 2$. However, we gave the host class of line-graphs of 3-uniform hypergraphs as example where our argumentation does not work for $i = l$. Indeed, it is questionably whether the corresponding statement for $i = l$ is true at all.

Question 8.5
Let $\mathcal{L}^3$ denote the class of line-graphs of 3-uniform hypergraphs. Is there an induced-hereditary guest class $\mathcal{G}$ with $c_l^{\mathcal{G}}(\mathcal{L}^3) = 2$?

In most of the known separations the host class is actually i-cover resistant, where i indicates the kind of the covering number which becomes arbitrarily large. The host class of shift graphs is probably not u-cover resistant. However, we have the class $\mathcal{OS} \subseteq \mathcal{S}$ of shift graphs of ordered graphs which is u-cover resistant and also provides a separation of the local and the union covering number together with the guest class of bipartite graphs.

The host class of line-graphs $\mathcal{L}$ is also not u-cover resistant, since every line-graph can be covered by a corresponding ordered shift graph and two disjoint unions of complete graphs (one union for all incoming edges and one for all outgoing edges). However, we claim that we have for every induced-hereditary guest class $\mathcal{G}$ that $c_u^{\mathcal{G}}(\mathcal{L}) \in \{1, 2, 3, \infty\}$ (since $c_u^{\mathcal{G}}(\mathcal{L}) < \infty$ implies that $\mathcal{G}$ contains all complete graphs and some kind of graphs class similar to $\mathcal{OS}$). This property is quite similar to the u-cover resistant. For $i = f, l, u, g$, we call a host class $\mathcal{H}$ *weakly i-cover resistant*, if there is an $l \in \mathbb{N}$ such that for every induced-hereditary guest class $\mathcal{G}$ we have $c_i^{\mathcal{G}}(\mathcal{H}) \in [l] \cup \{0, \infty\}$. Many of our results for cover resistance can be translated to results for weak cover resistance. With this definition all known separations are in some sense connected to the property of (weak) cover resistance.

Question 8.6
Is there an induced-hereditary host class $\mathcal{H}$, an induced-hereditary guest class $\mathcal{G}$ and $i, j = f, l, u$ such that

1. *we have $c_i^{\mathcal{G}}(\mathcal{H}) = 2$ and $c_j^{\mathcal{G}}(\mathcal{H}) = \infty$*
2. *there is no weakly j-cover resistant host class $\mathcal{H}' \subseteq \mathcal{H}$ with $\mathcal{H}' \not\subseteq \overline{\mathcal{G}}$?*

However note that, since it is quite natural to use results from Ramsey Theory to obtain host classes with arbitrarily large covering numbers, our previous results are no strong indication for a negative answer to Question 8.6.

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