# Computational Geometry · Lecture
## Introduction & Convex Hulls

INSTITUT FÜR THEORETISCHE INFORMATIK · FAKULTÄT FÜR INFORMATIK

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# AlgoGeom-Team

## Lecturers

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## Exercise Leader

- Benjamin Niedermann
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- Office hours: by appointment

## Schedule

- Lecture: Mon. 15:45 – 17:15, SR 301
- Exercises: Wed. 15:45 – 17:15, SR 301 (starting Oct. 28)

# Organization

## Website

`http://i11www.iti.kit.edu/teaching/winter2015/compgeom/`

- Course Information
- Lecture Slides
- Exercises
- Additional Material

## Computational Geometry in Computer Science Master's Studies

Bachelor

Master

Algorithms 1+2

Theoretical Basics

Algorithms for Planar Graphs

Computational Geometry

⋮

Algorithm design, theoretical basics, computer graphics

# Organization

## Exercises

- Every second Wednesday starting 28.10
- Exercise problems posted at least one week before an exercise session.
- Rinforce lecture material, help prepare for exam.

## What will the exercises involve?

- Weekly work for about 45–60 minutes
- Active participation in exercises is expected
- Volunteers will work problems on the board
- Can hand in exercises for feedback
- Variations will be announced
  - Exercise on 16.12 instead of 23.12

# Computational Geometry

**Objectives:** At the end of the course you will be able to...

- explain concepts, structures, and problem definitions
- understand the discussed algorithms, and explain and analyze them
- select and adapt appropriate algorithms and data structures
- analyze new geometric problems and develop efficient solutions

**Prior Knowledge:** Algorithms and Elementary Geometry

**Course Time Breakdown:** 5LP = 150h

- Time in lectures and exercise sessions ca. 35h
- Preparation and review ca. 25h
- Working on exercises ca. 20h
- Project work ca. 40h
- Exam preparation ca. 30h

# Modules and Grading

Karlsruhe Institute of Technology

## Master's in Computer Science

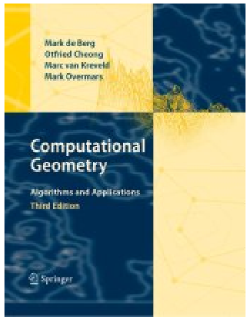
- Computational Geometry (IN4INAG) [5 LP]
- Algorithm Engineering & Applications (IN4INAEA) [5 LP]
- Design and Analysis of Algorithms (IN4INDAA) [5 LP]
- Computer Graphics Algorithms (IN4INACG) [3 LP]

## Test Modalities

More on this in 1–2 weeks

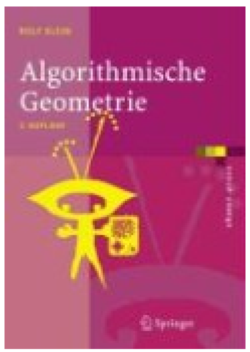
- Semester-long project in small teams (application-driven geometric algorithms)
  $\rightarrow$ 20% of grade
- One oral examination (about 20 minutes)
  $\rightarrow$ 80% of grade

# Class Materials

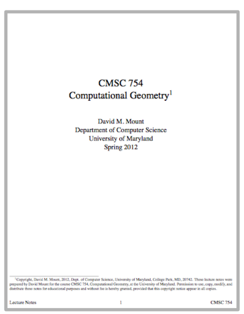



M. de Berg, O. Cheong, M. van Kreveld, M. Overmars:
Computational Geometry: Algorithms and Applications
Springer, 3rd Edition, 2008

**PDF available on springerlink.com!**




Rolf Klein:
Algorithmische Geometrie
Springer, 2nd Edition, 2005




David Mount:
Computational Geometry
Lecture Notes CMSC 754, U. Maryland, 2012

http://www.cs.umd.edu/class/spring2012/cmsc754/Lects/cmsc754-lects.pdf

Both books are available in the library!

# What is Computational Geometry?

Karlsruhe Institute of Technology

## Algorithmische Geometrie

Computational geometry is a branch of computer science that deals with algorithmic solutions to geometric problems. A central problem is the storage and processing of geometric data...such as points, lines, circles, polygons...

## Where is Computational Geometry Used?

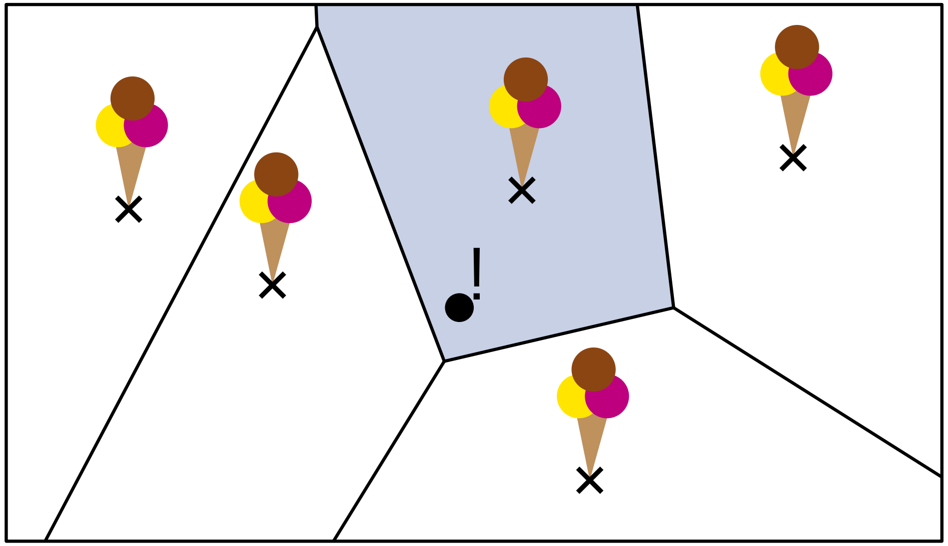
- Computer Graphics and Image Processing
- Visualization
- Geographic Information Systems (GIS)
- Robotics
- ...

## Central Themes

- Geometric algorithms and data structures
- Discrete and combinatorial geometric problems

# Example 1

It's a hot 42°C summer day in Karlsruhe. Suppose you know the location of every ice cream shop in the city. How can you determine the closest ice cream shop for any location on a map?

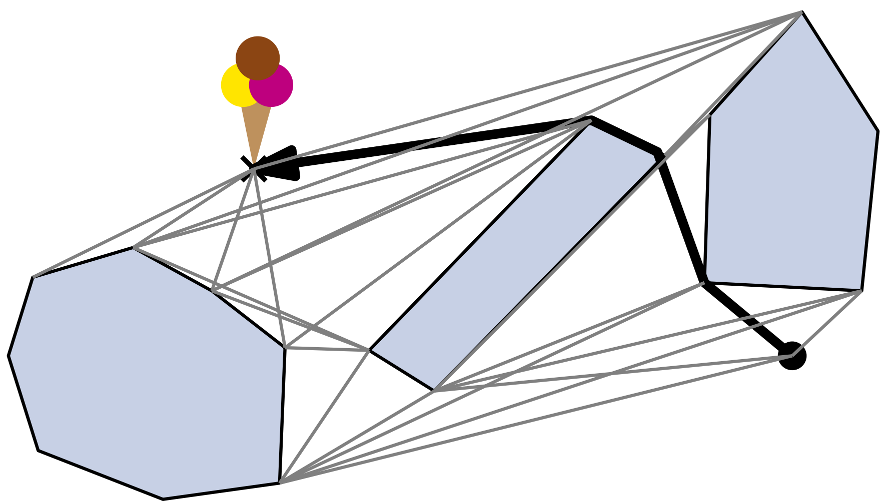



The solution is a *division* of $\mathbb{R}^2$, called a **Voronoi Diagram**.

Many applications, including: site planning, nearest-neighbor finding, robot motion planning, radio cells . . .

# Example 2

Now it is 50°C in Karlsruhe. We want to send a robot to buy an ice cream cone. How can the robot reach the destination without passing through houses, park benches, and trees?

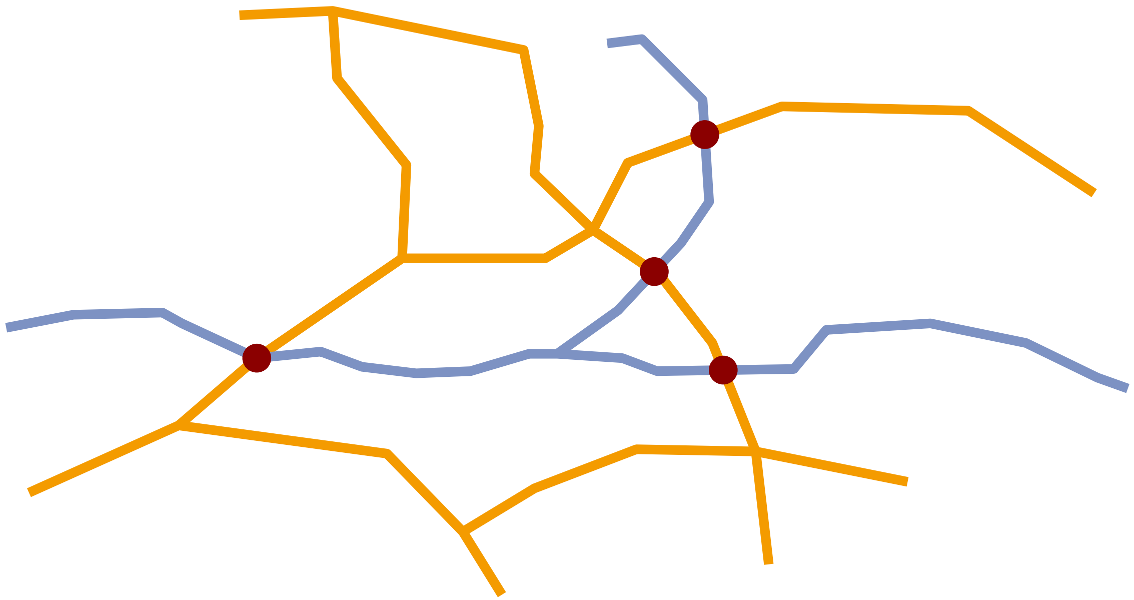


Motion planning problem in robotics:
Given a set of obstacles with a start and destination point, find a collision-free shortest route (e.g., using the **visibility graph).**

# Example 3

Maps in geographic information systems consist of several levels (e.g., roads, water, borders, etc.). When superimposing several layers, what are the intersection points?

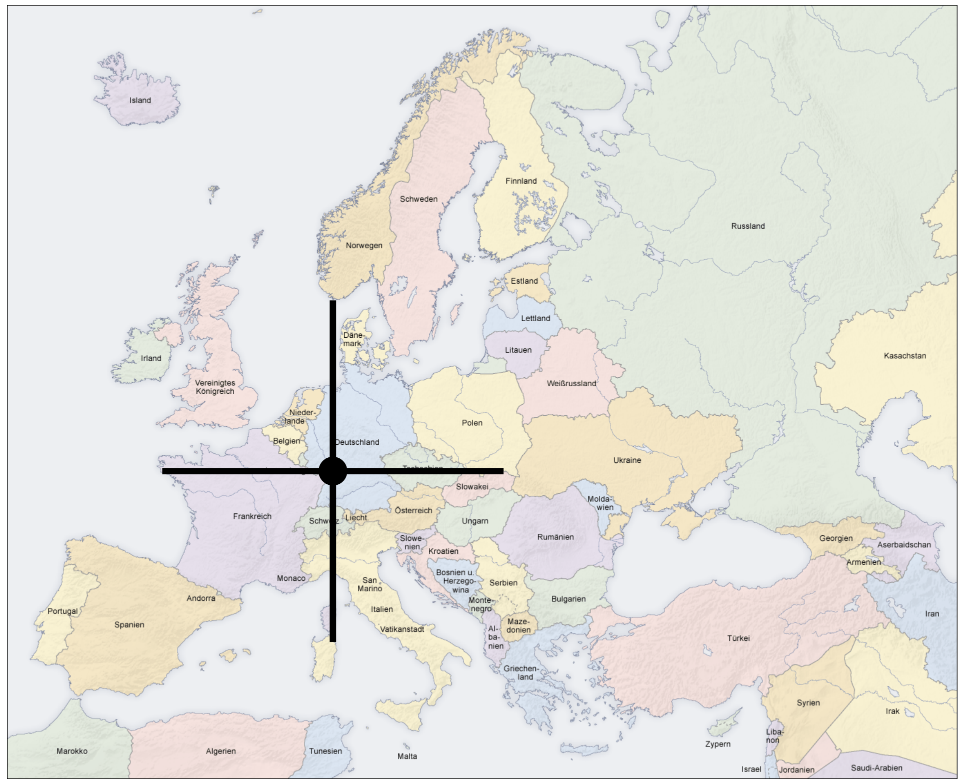
One example is to to view all roads and rivers as a set of links and ask for the bridges. For these, you have to find all intersections between the two layers.



Testing all edge pairs is slow. How can you quickly find all intersections?

# Example 4

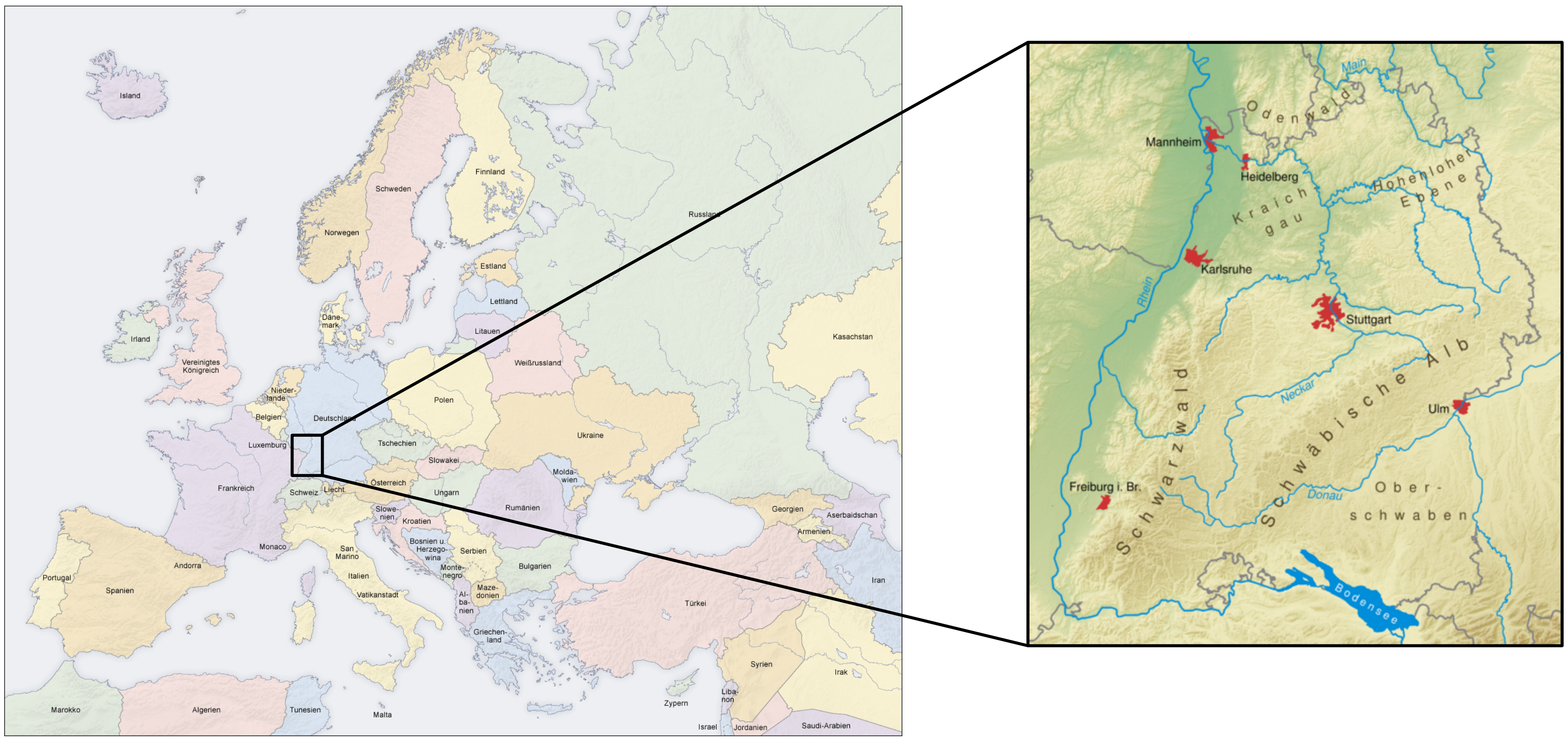
Given a map and a query point $q$ (e.g., a mouse click), determine the country containing $q$.




**We want** a fast data structure for answering point queries.

# Example 5

A navigation system should display a current map. How can we effectively choose the data to display?



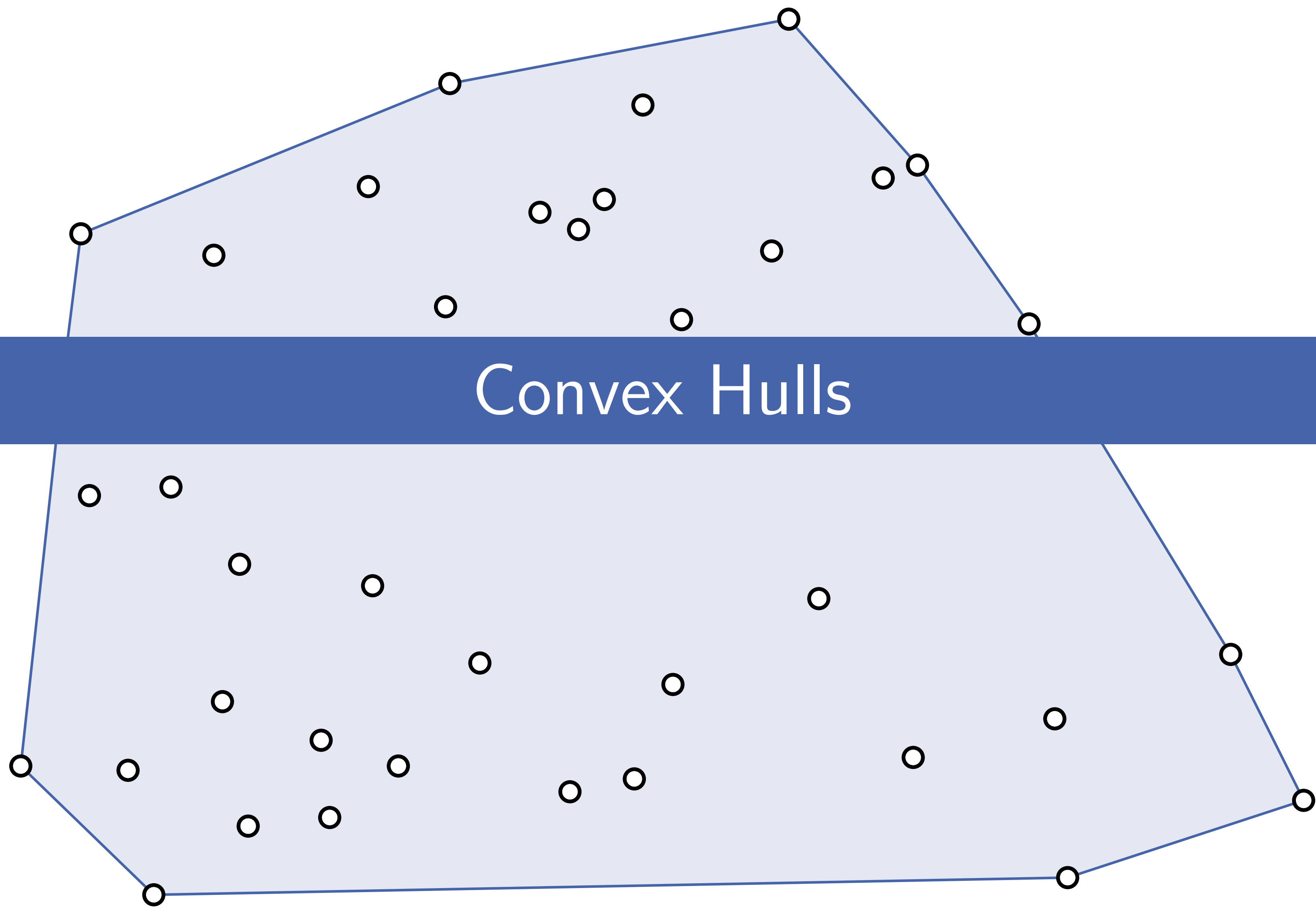

Evaluating each map feature is unrealistic.

**We want** a fast data structure for answering range queries

# Topics

**We will cover the following topics:**

- Convex Hulls
- Line Segment Intersection
- Polygon Triangulation
- Geometric Linear Programming
- Data Structures for Range Queries
- Data Structure for Point Location Queries
- Voronoi Diagrams and Delaunay Triangulation
- Duality of Points and Lines
- Quadtrees
- Well-Separated Pair Decompositions
- Visibility Graphs
- ...



# Convex Hulls

# Mixing Ratios

Given...

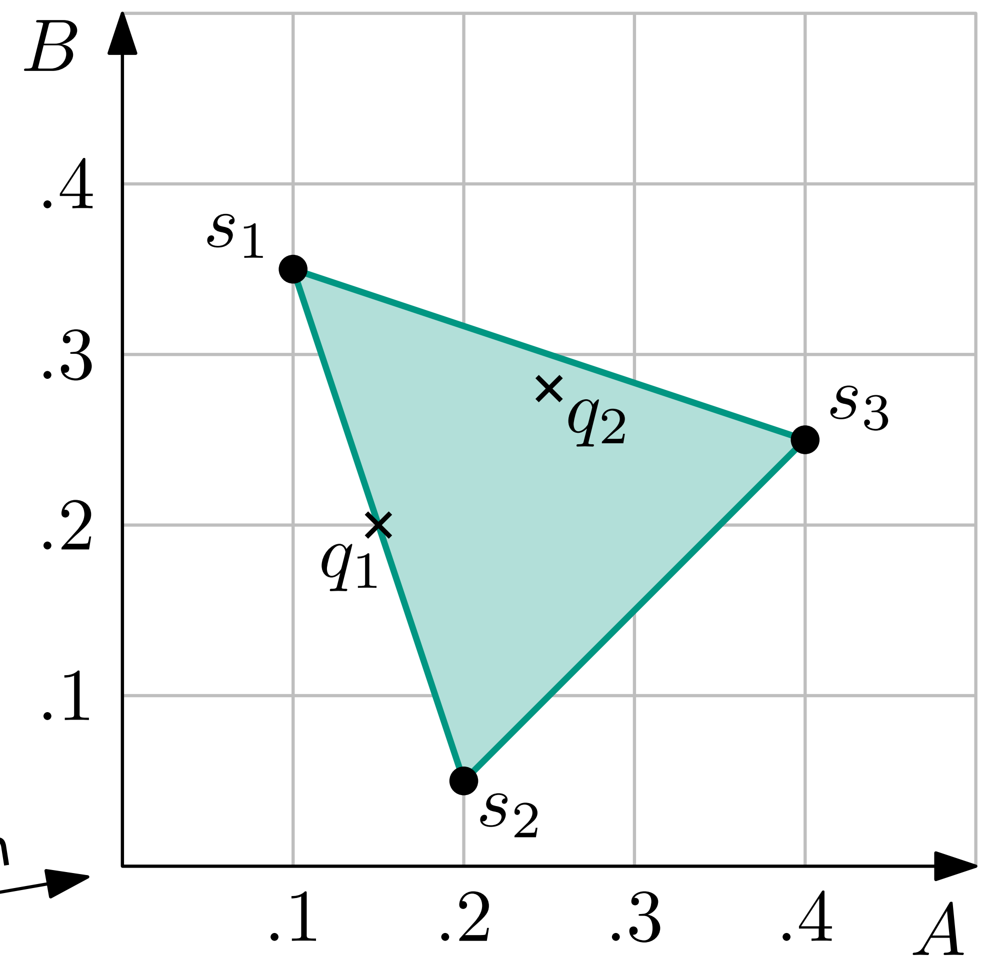
| Mixture | fraction $A$ | fraction $B$ |
|---|---|---|
| $s_1$ | 10 % | 35 % |
| $s_2$ | 20 % | 5 % |
| $s_3$ | 40 % | 25 % |

can we mix

| | | |
|---|---|---|
| $q_1$ | 15 % | 20 % |
| $q_2$ | 25 % | 28 % |

using $s_1$, $s_2$, $s_3$?

geom. interpretation




**Obs:** Given a set $S \subset \mathbb{R}^{\cancel{2}d}$ of mixtures, we can make another mixture $q \in \mathbb{R}^{\cancel{2}d}$ out of $S$ $\Leftrightarrow$ $q \in$ convex hull $CH(S)$.

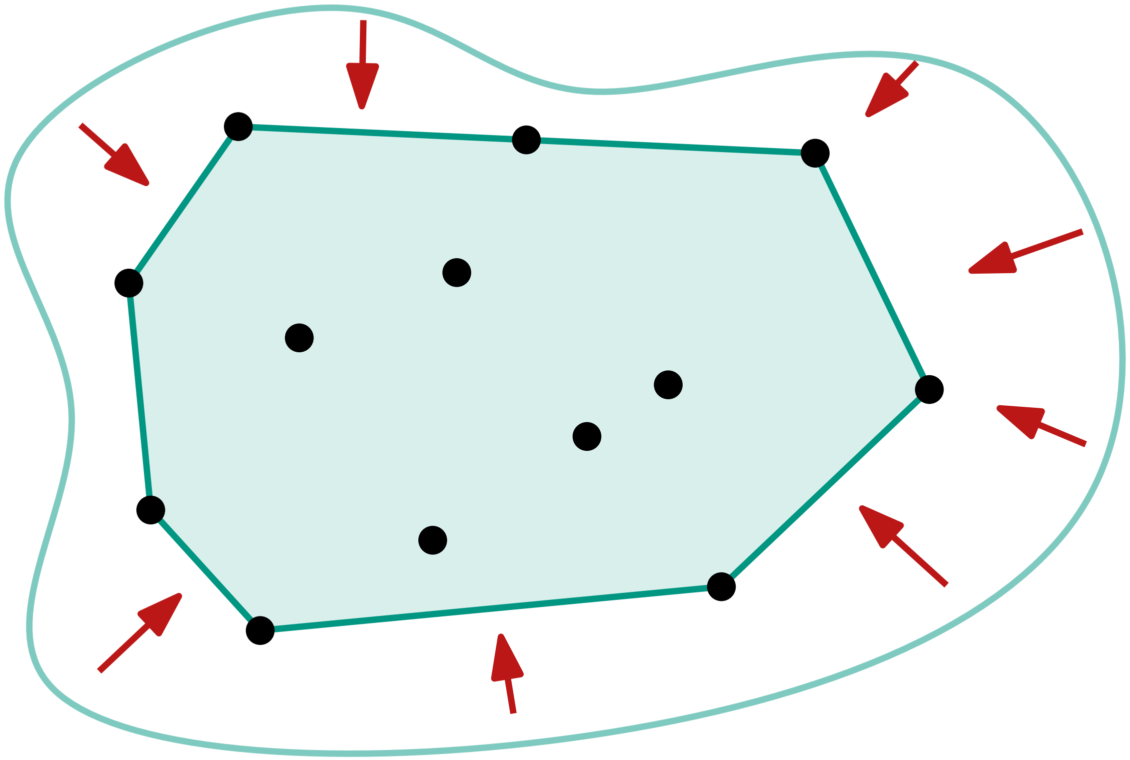
$$q = \sum_i \lambda_i s_i \text{ with } \sum_i \lambda_i = 1.$$

# Definition of Convex Hull

**Def:** A region $S \subseteq \mathbb{R}^2$ is called **convex**, when for two points $p, q \in S$ then line $\overline{pq} \in S$.
The **convex hull** $CH(S)$ of $S$ is the smallest convex region containing $S$.

In physics:

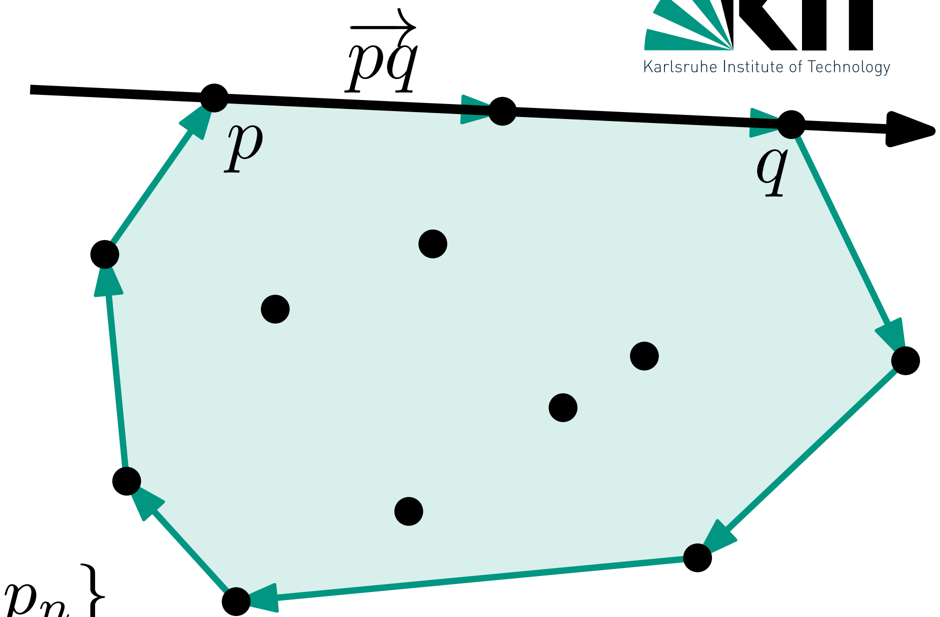
- put a large rubber band around all points
- and let it go!
- unfortunately, does not help algorithmically



In mathematics:

- define $CH(S) = \bigcap_{C \supseteq S:\ C \text{ convex}} C$
- does not help :-(

# Algorithmic Approach




**Lemma:**

For a set of points $P \subseteq \mathbb{R}^2$, $CH(P)$ is a convex polygon that contains $P$ and whose vertices are in $P$.

**Input:** A set of points $P = \{p_1, \ldots, p_n\}$

**Output:** List of vertices of $CH(P)$ in clockwise order

**Observation:**

$(p, q)$ is an edge of $CH(P) \Leftrightarrow$ each point $r \in P \setminus \{p, q\}$

- strictly right of the oriented line $\overrightarrow{pq}$ or
- on the line segment $\overline{pq}$

# A First Algorithm

FirstConvexHull($P$)

$E \leftarrow \emptyset$

**foreach** $(p, q) \in P \times P$ with $p \neq q$ **do**

Check all possible edges $(p, q)$

    *valid* $\leftarrow$ *true*

    **foreach** $r \in P$ **do**

        **if not** ($r$ strictly right of $\overrightarrow{pq}$ **or** $r \in \overline{pq}$) **then**

            *valid* $\leftarrow$ *false*

Test in $O(1)$ time with

$$\begin{vmatrix} x_r & y_r & 1 \\ x_p & y_p & 1 \\ x_q & y_q & 1 \end{vmatrix} < 0$$

$\rightarrow$ Exercise!

    **if** *valid* **then**

        $E \leftarrow E \cup \{(p, q)\}$

construct sorted node list $L$ of $CH(P)$ from $E$

**return** $L$

# Running Time Analysis

FirstConvexHull($P$)

$E \leftarrow \emptyset$

**foreach** $(p, q) \in P \times P$ with $p \neq q$ **do** $(n^2 - n)\cdot$

  *valid* $\leftarrow$ *true*

  **foreach** $r \in P$ **do**

    **if not** ($r$ strictly right of $\overrightarrow{pq}$ **or** $r \in \overline{pq}$) **then**

      *valid* $\leftarrow$ *false*

  **if** *valid* **then**

    $E \leftarrow E \cup \{(p, q)\}$

construct sorted node list $L$ of $CH(P)$ from $E$

**return** $L$

$\Theta(1)$ $\Theta(n)$ $\Theta(n^3)$ $O(n^2)$

**Lemma:** The convex hull of $n$ points in the plane can be computed in $O(n^3)$ time.

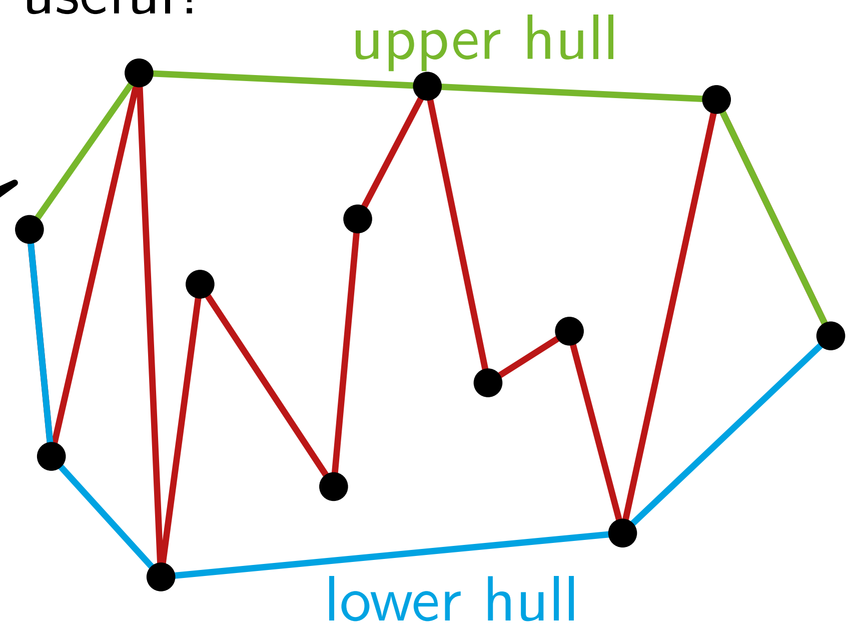
# Incremental Approach

**Idea:** For $i = 1, \ldots, n$ compute $CH(P_i)$ where $P_i = \{p_1, \ldots, p_i\}$

**Question:** Which ordering of the points is useful?

**Answer:** From left to right!

Consider the upper and lower hull separately




UpperConvexHull($P$)

$\langle p_1, p_2, \ldots, p_n \rangle \leftarrow$ sort $P$ from left to right
$L \leftarrow \langle p_1, p_2 \rangle$
**for** $i \leftarrow 3$ **to** $n$ **do**
    $L$.append$(p_i)$
    **while** $|L| > 2$ **and** the last 3 points in $L$ do not form right turn **do**
        remove the second-to-last point in $L$
**return** $L$

lower hull is handled similarly!

# Running Time Analysis

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UpperConvexHull($P$)

$\langle p_1, p_2, \dots, p_n \rangle \leftarrow$ sort $P$ from right to left — $O(n \log n)$

$L \leftarrow \langle p_1, p_2 \rangle$

**for** $i \leftarrow 3$ **to** $n$ **do** — $\cancel{(n-2)\cdot}$ — $O(n)$

  $L$.append($p_i$)

  **while** $|L| > 2$ **and** last 3 points in $L$ do not form right turn **do**

    remove the second-to-last point from $L$ — $\cancel{?}$

**return** $L$

## Amortized Analysis

- Each point is inserted into $L$ exactly once
- A point in $L$ is removed at most once from $L$
- $\Rightarrow$ Running time of the **for** loop including the **while** loop is $O(n)$

**Theorem 1:** The convex hull of $n$ points in the plane can be computed in $O(n \log n)$ time. $\rightarrow$ *Graham's Scan*.

# Alternative Approach: Gift Wrapping

**Idea:** Begin with a point $p_1$ of $CH(P)$, then find the next edge of $CH(P)$ in clockwise order.

GiftWrapping($P$)

| | | |
|---|---|---|
| $p_1 = (x_1, y_1) \leftarrow$ rightmost point in $P$; $p_0 \leftarrow (x_1, \infty)$; $j \leftarrow 1$ | | $O(n)$ |
| **while** true **do** | $O(h)$ | $O(n \cdot h)$ |
| $\quad p_{j+1} \leftarrow \arg\max\{\angle p_{j-1}, p_j, q \mid q \in P \setminus \{p_{j-1}, p_j\}\}$<br>$\quad$ **if** $p_{j+1} = p_1$ **then** break **else** $j \leftarrow j+1$ | $O(n)$ | |
| **return** $(p_1, \ldots, p_{j+1})$ | | |

**Theorem 2:** The convex hull $CH(P)$ of $n$ points $P$ in $\mathbb{R}^2$ can be computed in $O(n \cdot h)$ time using *Gift Wrapping* (also called *Jarvis' March*), where $h = |CH(P)|$.

$\rightarrow$ more on that in the exercises!

# Comparison

**Which algorithm is better?**

- Graham's Scan: $O(n \log n)$ time
- Jarvis' March: $O(n \cdot h)$ time

It depends on how large $CH(P)$ is!

**Idea:** Combine the two approaches into an optimal algorithm!

# Chan's Algorithm

Suppose we know $h$:

ChanHull($P$,$h$)

Divide $P$ into sets $P_i$ with $\leq h$ nodes

**for** $i$ from 1 to $\lceil n/h \rceil$ **do**
  Compute with GrahamScan $CH(P_i)$

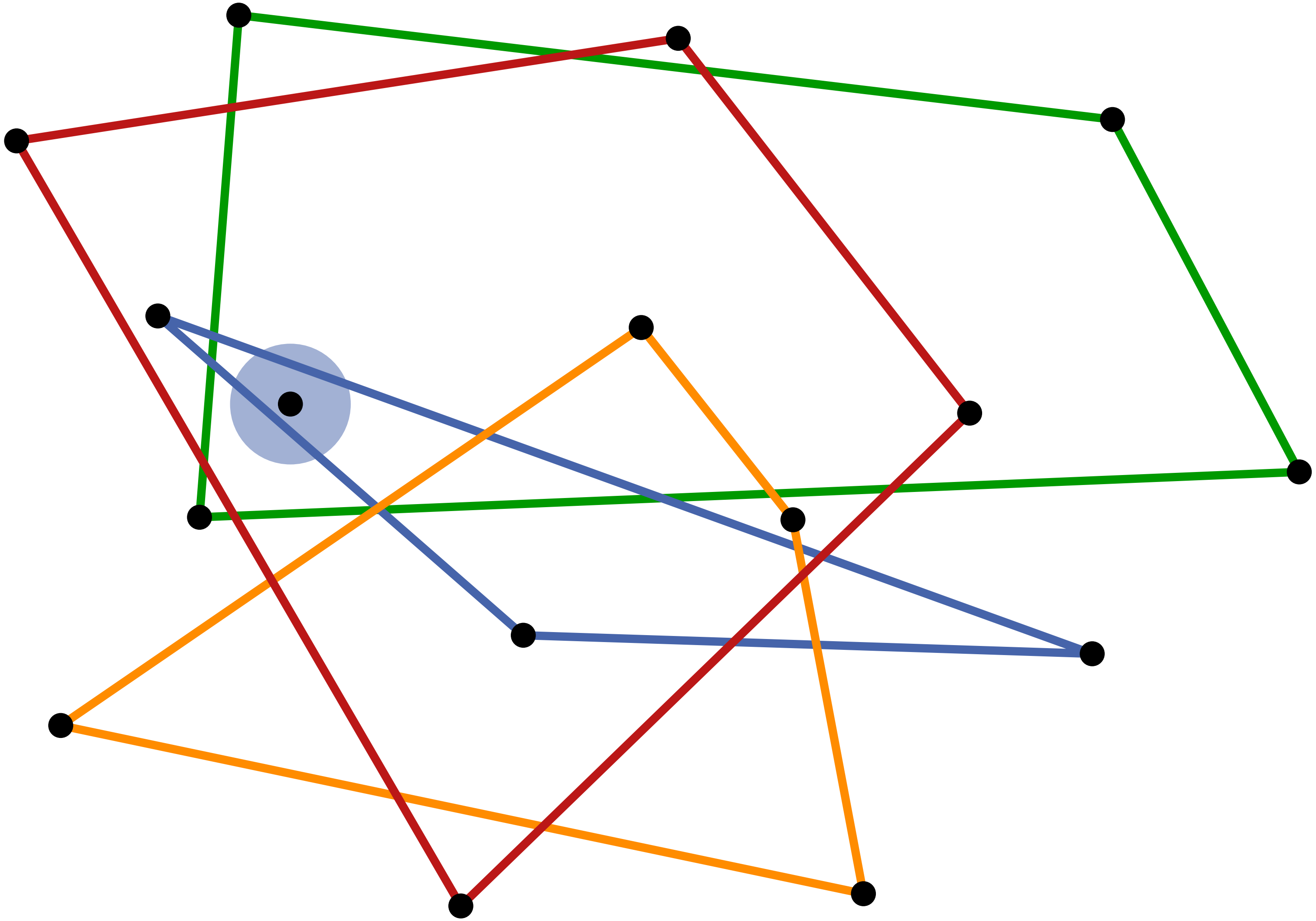
GrahamScan

$p_1 = (x_1, y_1) \leftarrow$ rightmost point in $P$
$p_0 \leftarrow (x_1, \infty)$
**for** $j = 1$ **to** $h$ **do**
  **for** $i = 1$ **to** $\lceil n/h \rceil$ **do**
    $q_i \leftarrow \arg\max\{\angle p_{j-1} p_j q \mid q \in P_i \setminus \{p_{j-1}, p_j\}\}$
  $p_{j+1} \leftarrow \arg\max\{\angle p_{j-1} p_j q \mid q \in \{q_1, \ldots, q_{\lceil n/h \rceil}\}\}$
**return** $(p_1, \ldots, p_h)$

Gift Wrapping

# Example

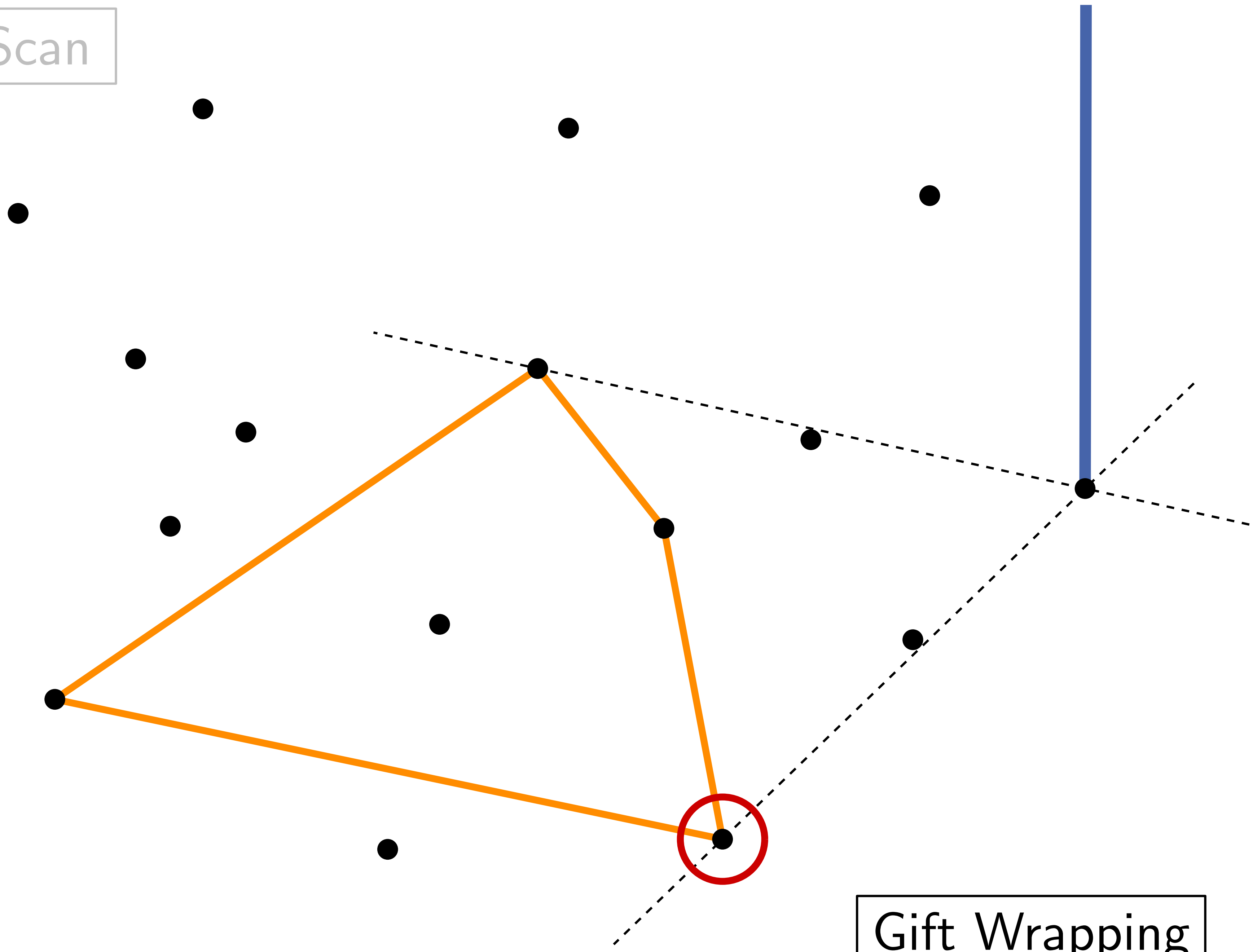
GrahamScan



n = 16

# Example

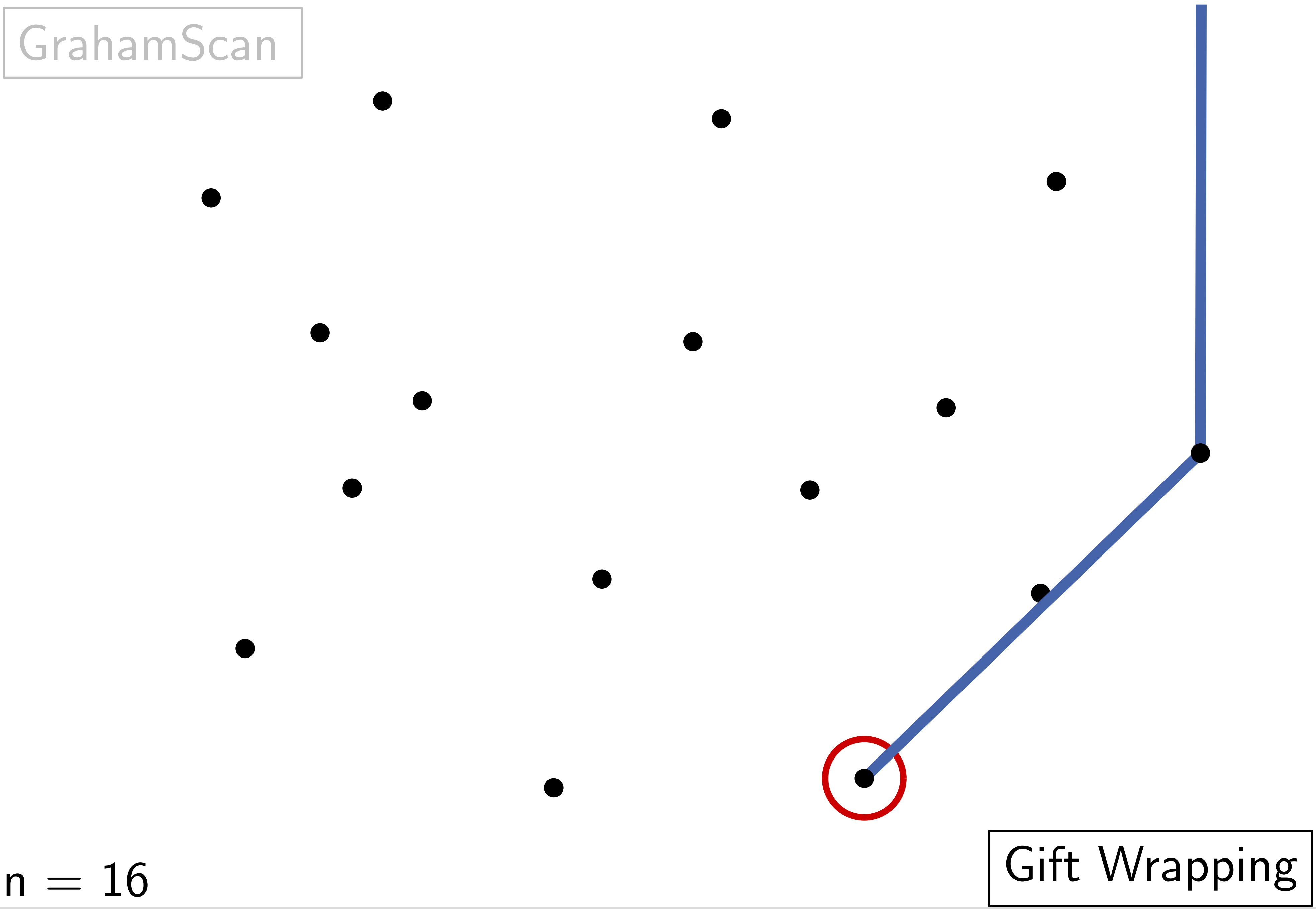
GrahamScan



$n = 16$

Gift Wrapping

# Example



# Chan's Algorithm

Suppose we know $h$:

ChanHull($P$,$h$)

Divide $P$ into sets $P_i$ with $\leq h$ nodes

**for** $i$ from 1 to $\lceil n/h \rceil$ **do** $\mathcal{O}(n/h)$

  Compute with GrahamScan $CH(P_i)$ $\mathcal{O}(h \log h)$

$\mathcal{O}(n \log h)$

$p_1 = (x_1, y_1) \leftarrow$ rightmost point in $P$

$p_0 \leftarrow (x_1, \infty)$

**for** $j = 1$ **to** $h$ **do** $\mathcal{O}(h) \cdot \mathcal{O}(n/h) = \mathcal{O}(n)$

  **for** $i = 1$ **to** $\lceil n/h \rceil$ **do** $\mathcal{O}(\log h) \rightarrow$ Exercise!

    $q_i \leftarrow \arg\max\{\angle p_{j-1} p_j q \mid q \in P_i \setminus \{p_{j-1}, p_j\}\}$

  $p_{j+1} \leftarrow \arg\max\{\angle p_{j-1} p_j q \mid q \in \{q_1, \ldots, q_{\lceil n/h \rceil}\}\}$

**return** $(p_1, \ldots, p_h)$

$\mathcal{O}(n \log h)$

Total: $\mathcal{O}(n \log h)$

# Chan's Algorithm

**But in general $h$ is unknown!**

ChanHull($P$,$m$)

Divide $P$ into sets $P_i$ with $\leq m$ nodes

**for** $i$ from 1 to $\lceil n/m \rceil$ **do** $\mathcal{O}(n/m)$

  Compute with GrahamScan $CH(P_i)$ $\mathcal{O}(m \log m)$

$\mathcal{O}(n \log m)$

$p_1 = (x_1, y_1) \leftarrow$ rightmost point in $P$

$p_0 \leftarrow (x_1, \infty)$

**for** $j = 1$ **to** $m$ **do** $\mathcal{O}(m) \cdot \mathcal{O}(n/m) = \mathcal{O}(n)$

  **for** $i = 1$ **to** $\lceil n/m \rceil$ **do** $\mathcal{O}(\log m)$

    $q_i \leftarrow \arg\max\{\angle p_{j-1} p_j q \mid q \in P_i \setminus \{p_{j-1}, p_j\}\}$

  $p_{j+1} \leftarrow \arg\max\{\angle p_{j-1} p_j q \mid q \in \{q_1, \ldots, q_{\lceil n/m \rceil}\}\}$

  **if** $p_{j+1} = p_1$ **then return** $(p_1, \ldots, p_{j+1})$

**return** failure

$\mathcal{O}(n \log m)$

Total: $\mathcal{O}(n \log m)$

# What to do with $m$?

FullChanHull($P$)

**for** $t = 0, 1, 2, \ldots$ **do**

  $m = \leftarrow \min\{n, 2^{2^t}\}$

  result $\leftarrow$ ChanHull($P$, $m$)   $\mathcal{O}(n \log m) =$

  **if** result $\neq$ failure **then** break   $\mathcal{O}(n \log 2^{2^t})$

**return** result

Running time:

$$\sum_{t=0}^{\lceil \log \log h \rceil} \mathcal{O}(n \log 2^{2^t}) = \mathcal{O}(n) \sum_{t=0}^{\lceil \log \log h \rceil} \mathcal{O}(2^t)$$

$$\leq \mathcal{O}(n) \cdot \mathcal{O}(2^{\log \log h}) = \mathcal{O}(n) \cdot \mathcal{O}(\log h)$$

$$= \mathcal{O}(n \log h)$$

# What to do with $m$?

FullChanHull($P$)

**for** $t = 0, 1, 2, \ldots$ **do**

$m =\leftarrow \min\{n, 2^{2^t}\}$

result $\leftarrow$ ChanHull($P$, $m$)

**if** result $\neq$ failure **then** break

**return** result

**Theorem 3:** The convex hull $CH(P)$ of $n$ points $P$ in $\mathbb{R}^2$ can be computed in $O(n \log h)$ time with Chan's Algorithm, where $h = |CH(P)|$.

# Discussion

### Is it possible to compute faster than $O(n \log n)$ or $O(n \log h)$ time?

Generally not! An algorithm to compute the convex hull can also sort (exercise!) $\Rightarrow$ lower bound $\Omega(n \log n)$.

### What happens in Graham's Scan when sorting $P$ is not unique?

Use lexicographic order!

### What happens with collinear points in $CH(P)$?

Graham: Forms no right turn, so an interior point is deleted.
Jarvis: Choose farthest point

### What about the robustness of the algorithms?

- Regarding robustness: imprecision of floating-point arithmetic
- FirstConvexHull possibly produces a valid polygon
- Graham and Jarvis always provide a polygon, but it may have minor defects

# Designing Geometric Algorithms–Guidelines

1.) Eliminate degenerate cases ($\rightarrow$ *general position*)

- unique $x$-coordinates
- no three collinear points
- ...

2.) Adjust degenerate inputs

- integrate into existing solutions
  (e.g., compute lexicographic order if $x$-coordinates are not unique)
- may require special treament

3.) Implementation

- primitive operations (available in libraries?)
- robustness