Computational Geometry · Lecture
Line Segment Intersection

INSTITUT FÜR THEORETISCHE INFORMATIK · FAKULTÄT FÜR INFORMATIK

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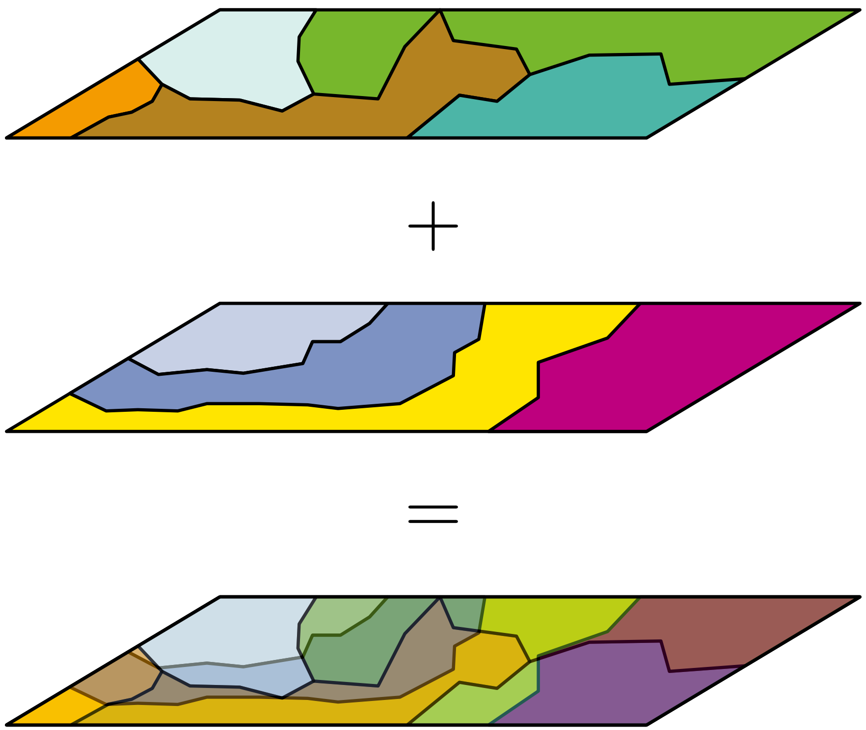
Aside: Organizational Items

Overlaying Map Layers

Example: Given two different map layers whose intersection is of interest.



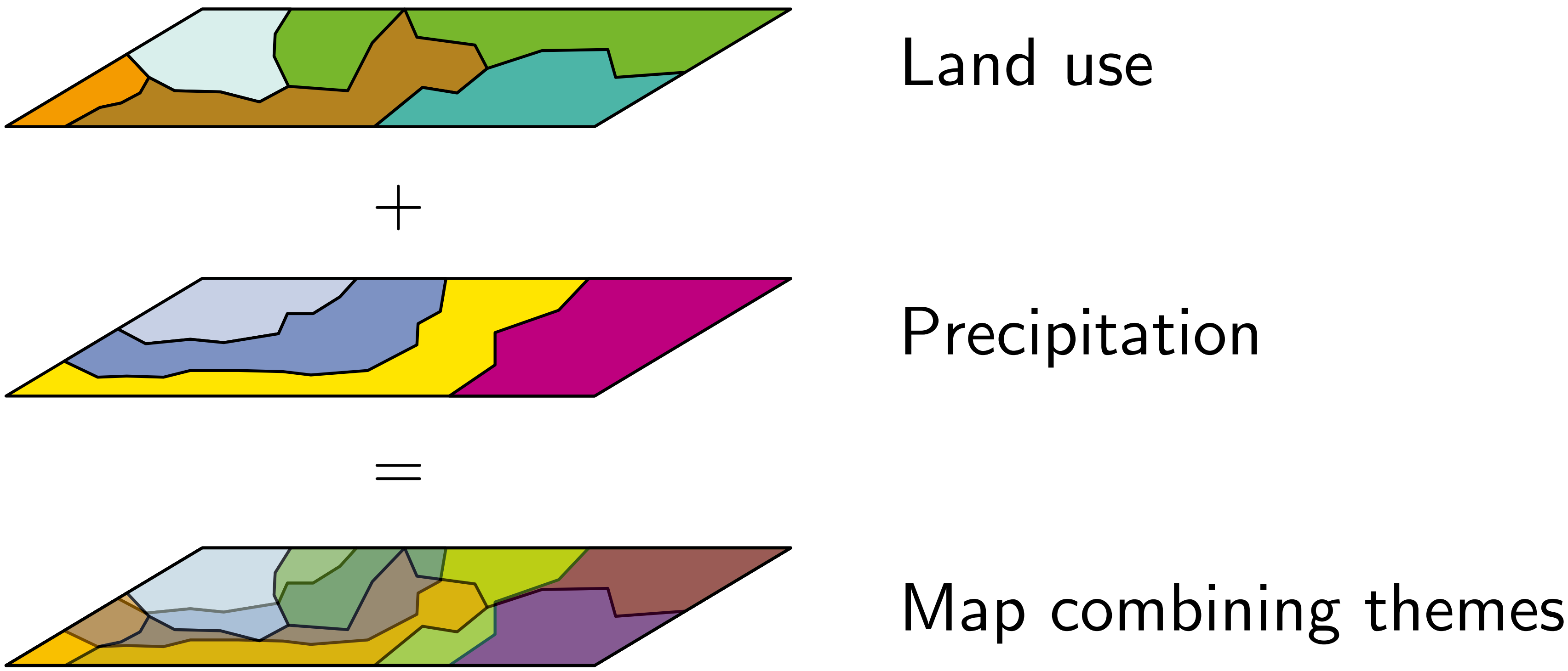
Land use

Precipitation

Map combining themes

Overlaying Map Layers

Example: Given two different map layers whose intersection is of interest.



- Regions are polygons
- Polygons are line segments
- **Calculate all line segment intersections**
- Compute regions

Problem Formulation

Given: Set $S = \{s_1, \ldots, s_n\}$ of line segments in the plane

Output:
- all intersections of two or more line segments
- for each intersection, the line segments involved.

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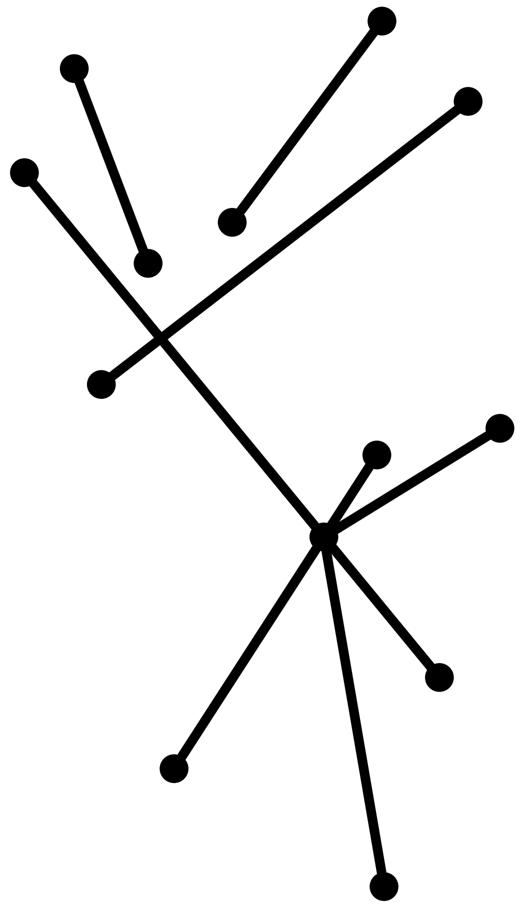
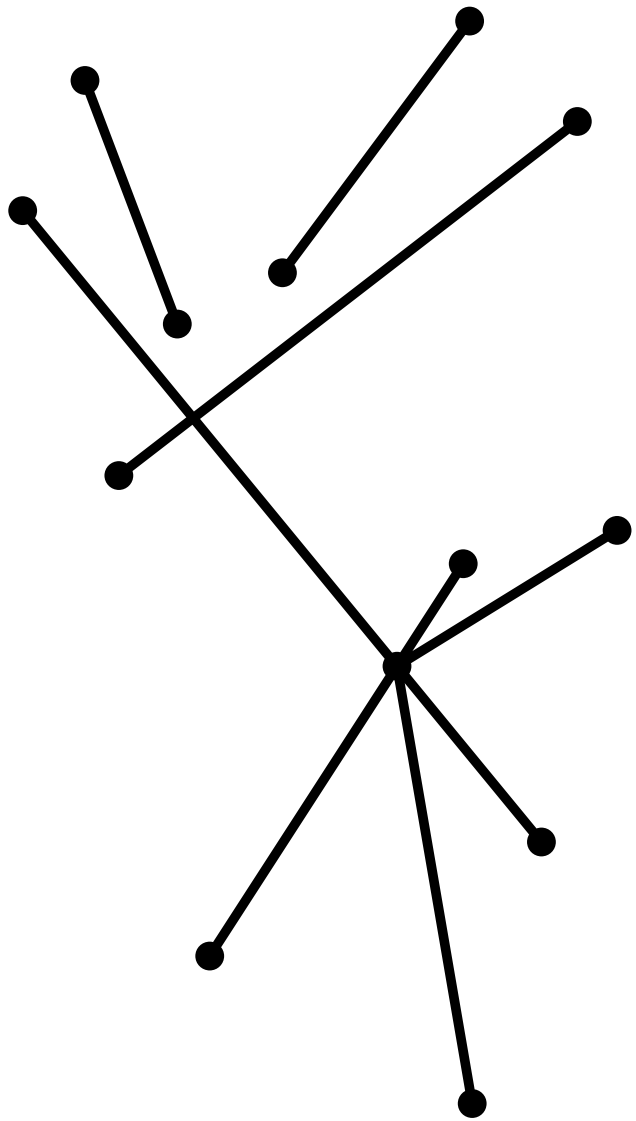


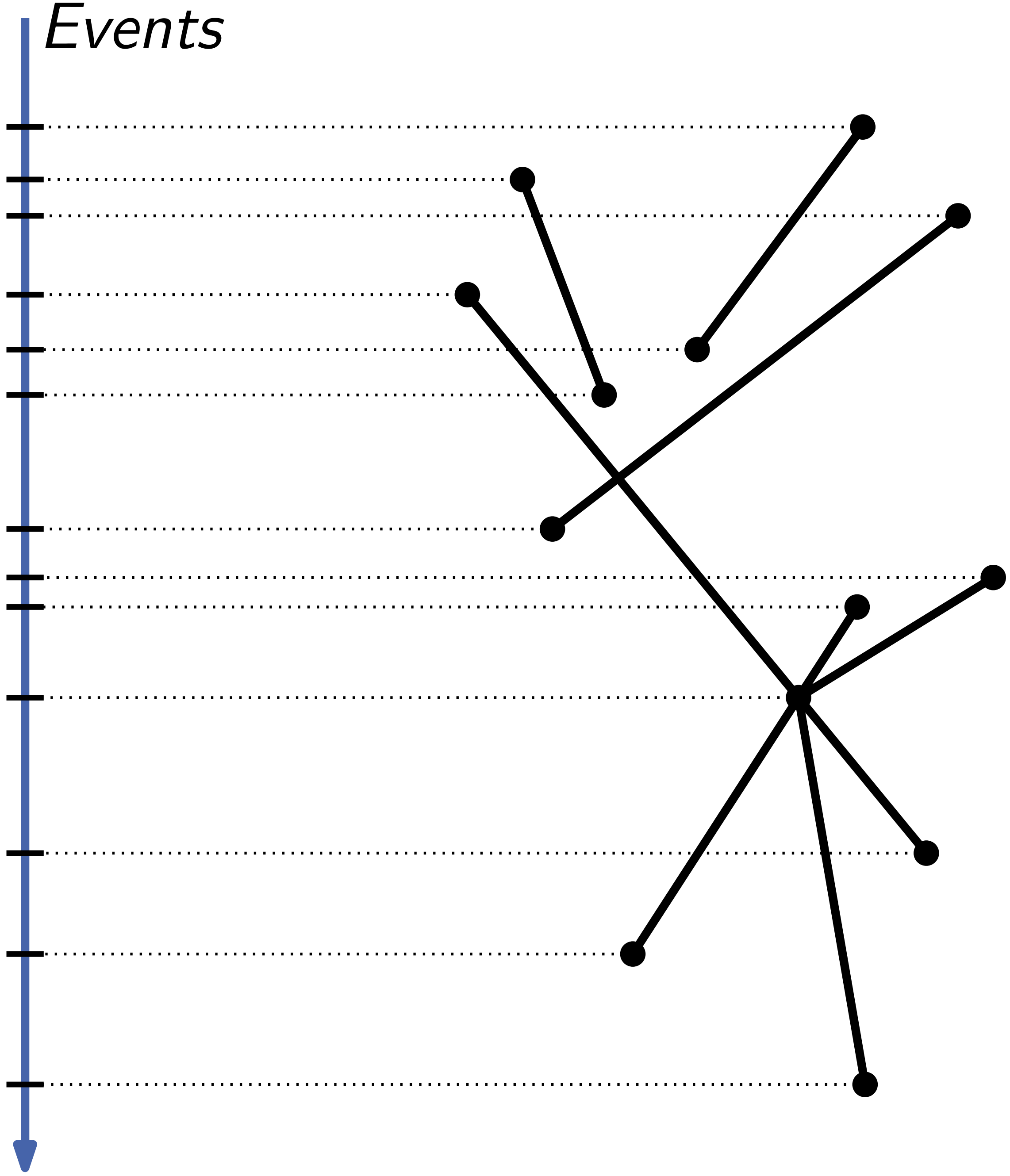
Discussion:
– How can you solve this problem naively?
– Is this already optimal?
– Are their better approaches?

The Sweep-Line Method: An Example



The Sweep-Line Method: An Example

Events



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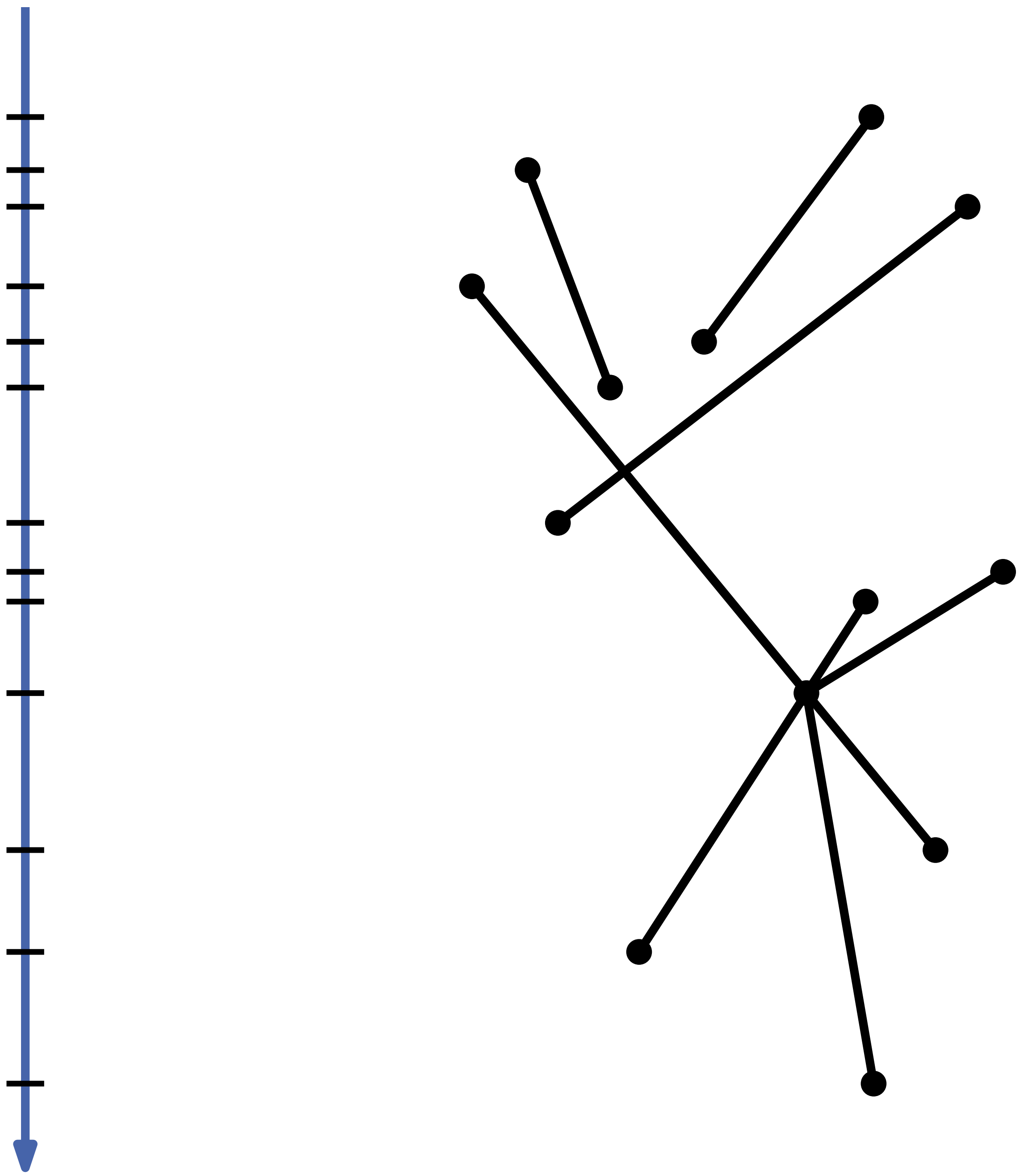

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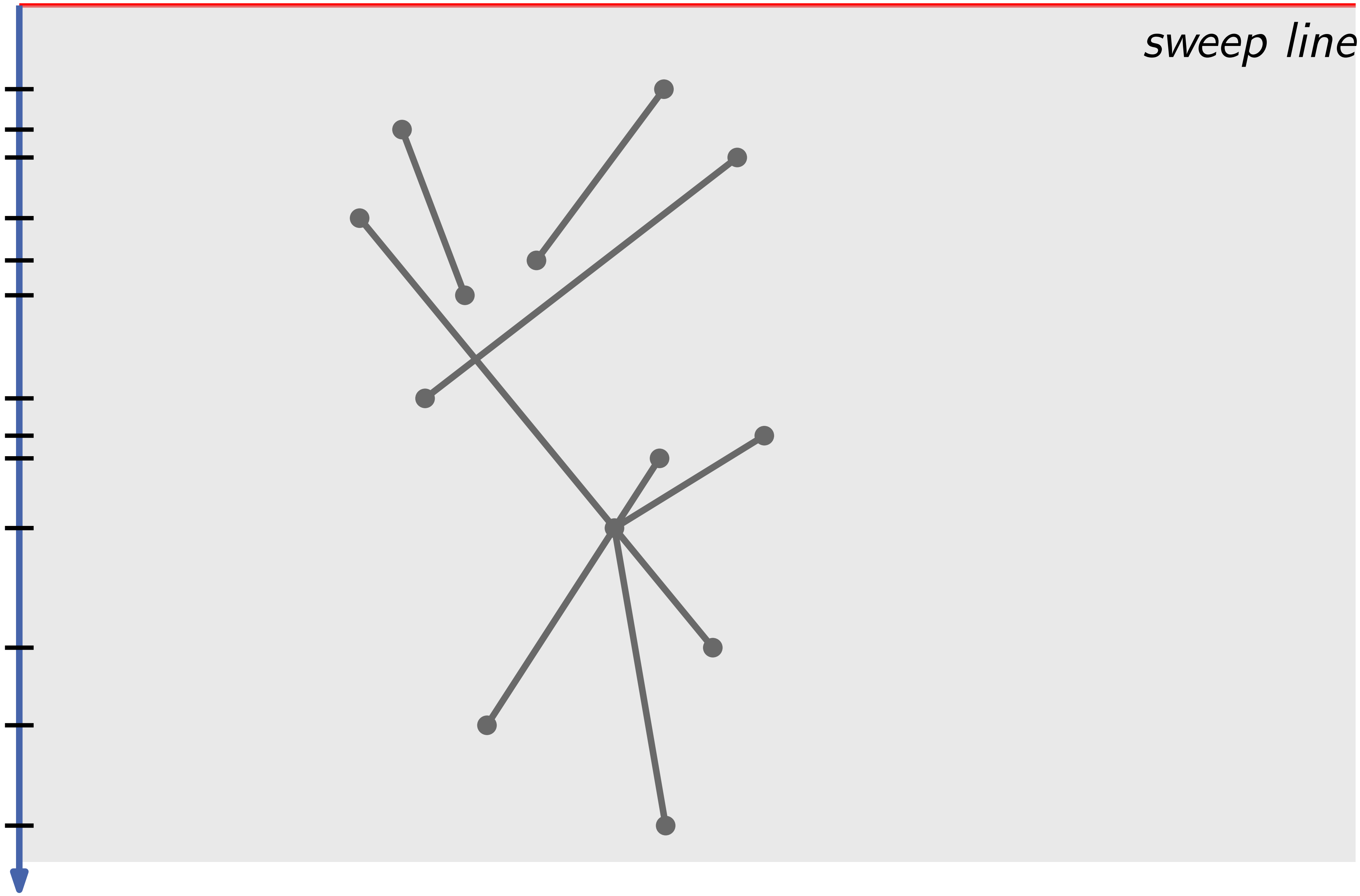

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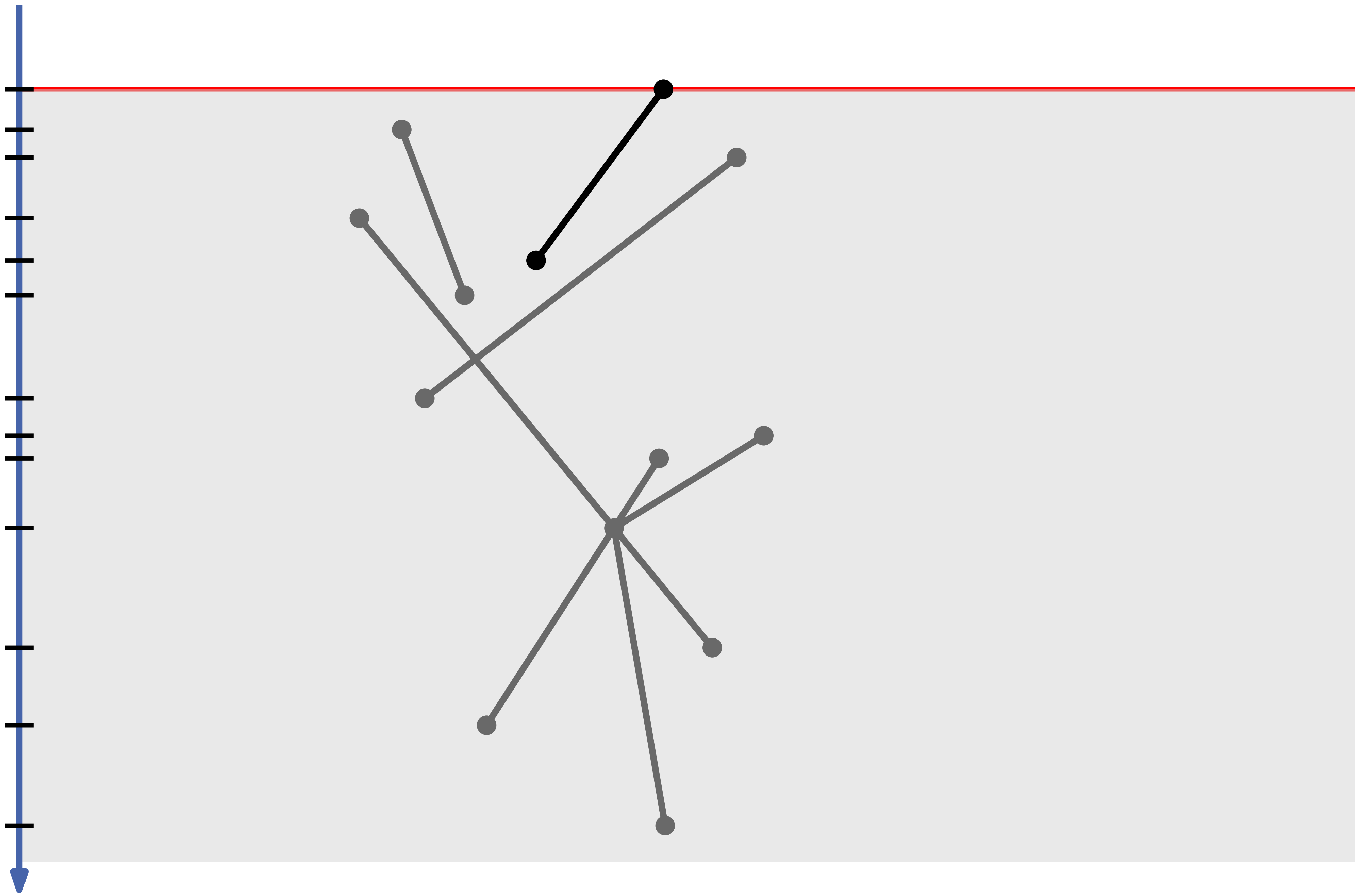

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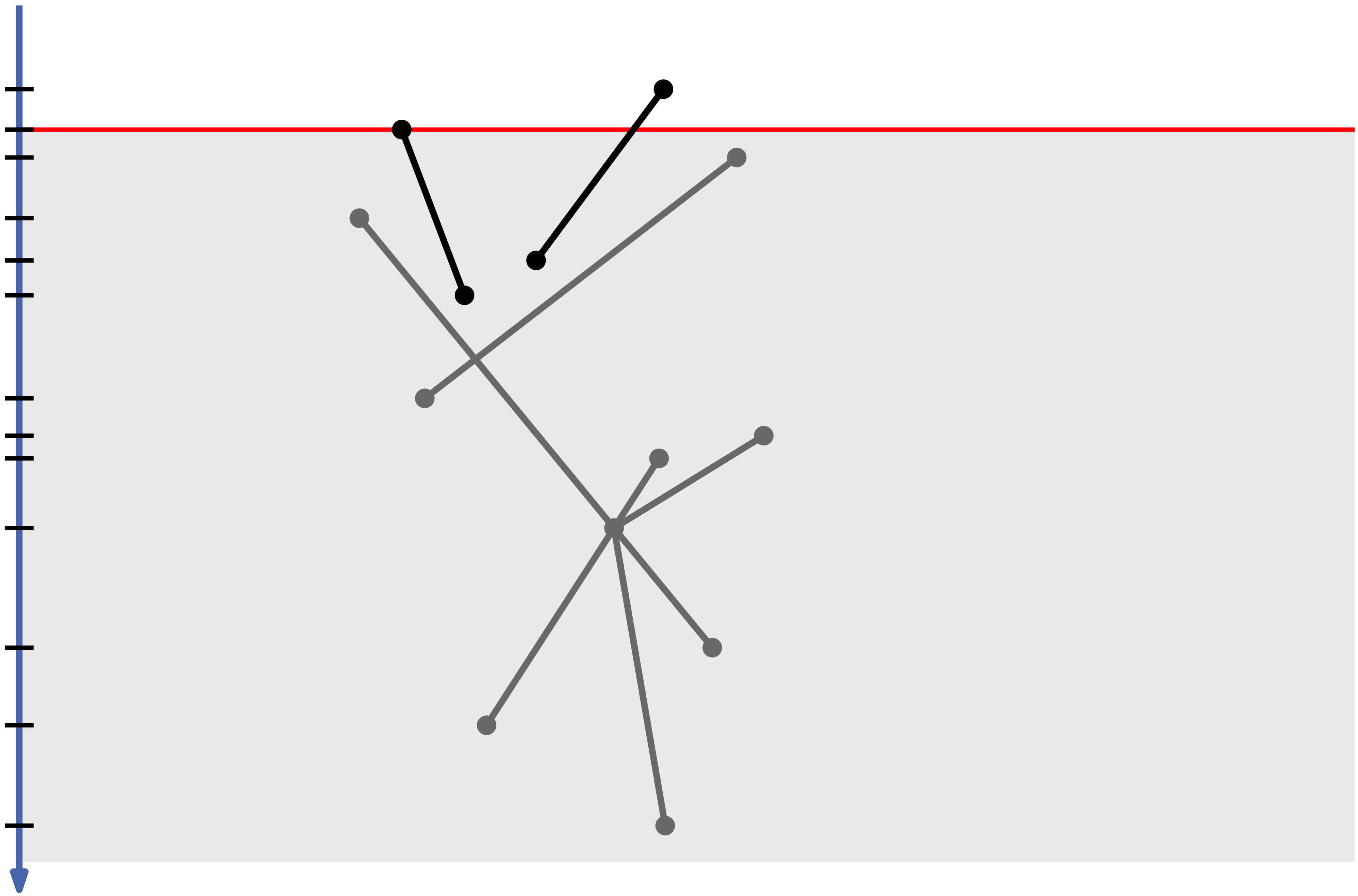

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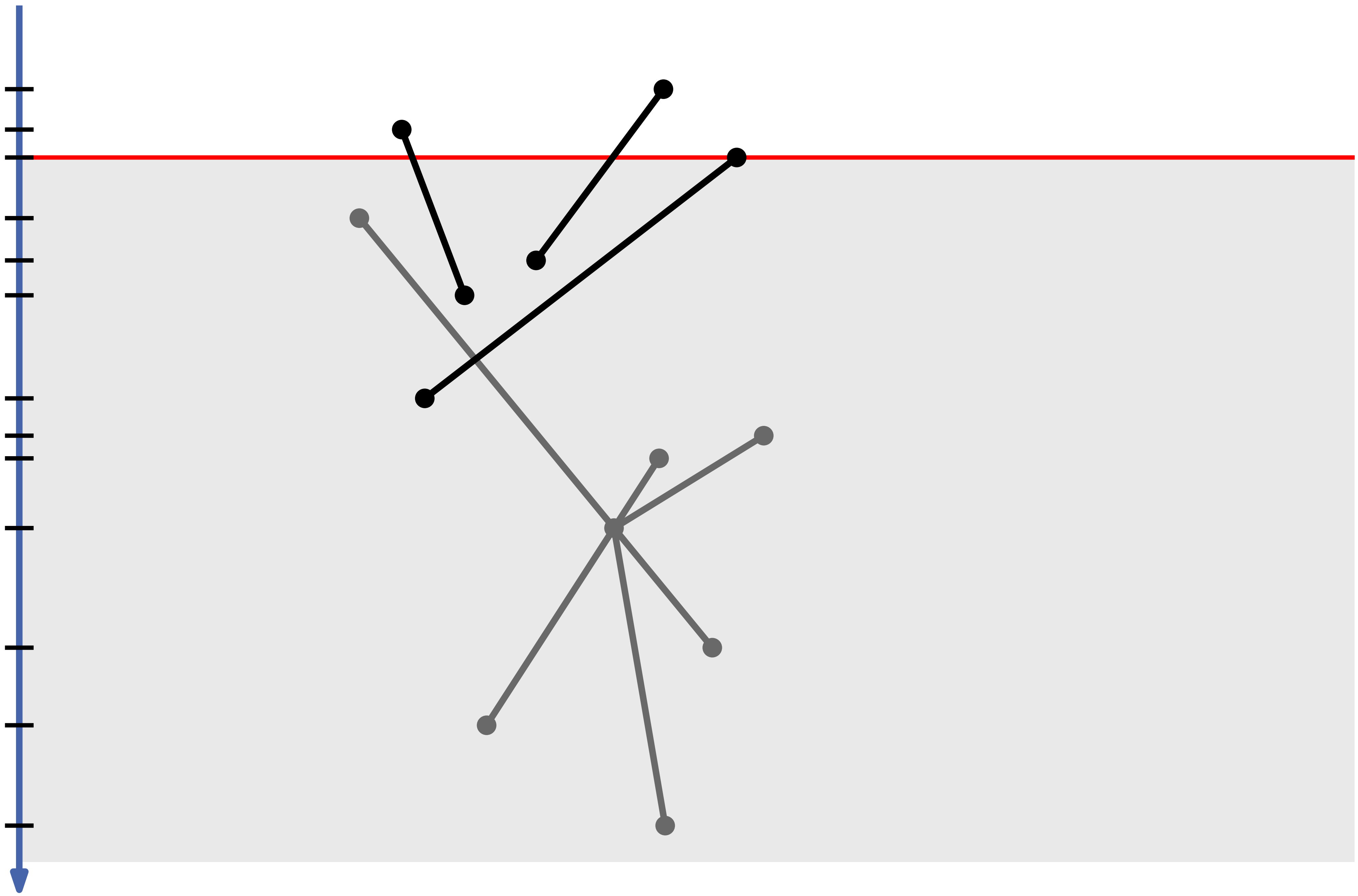

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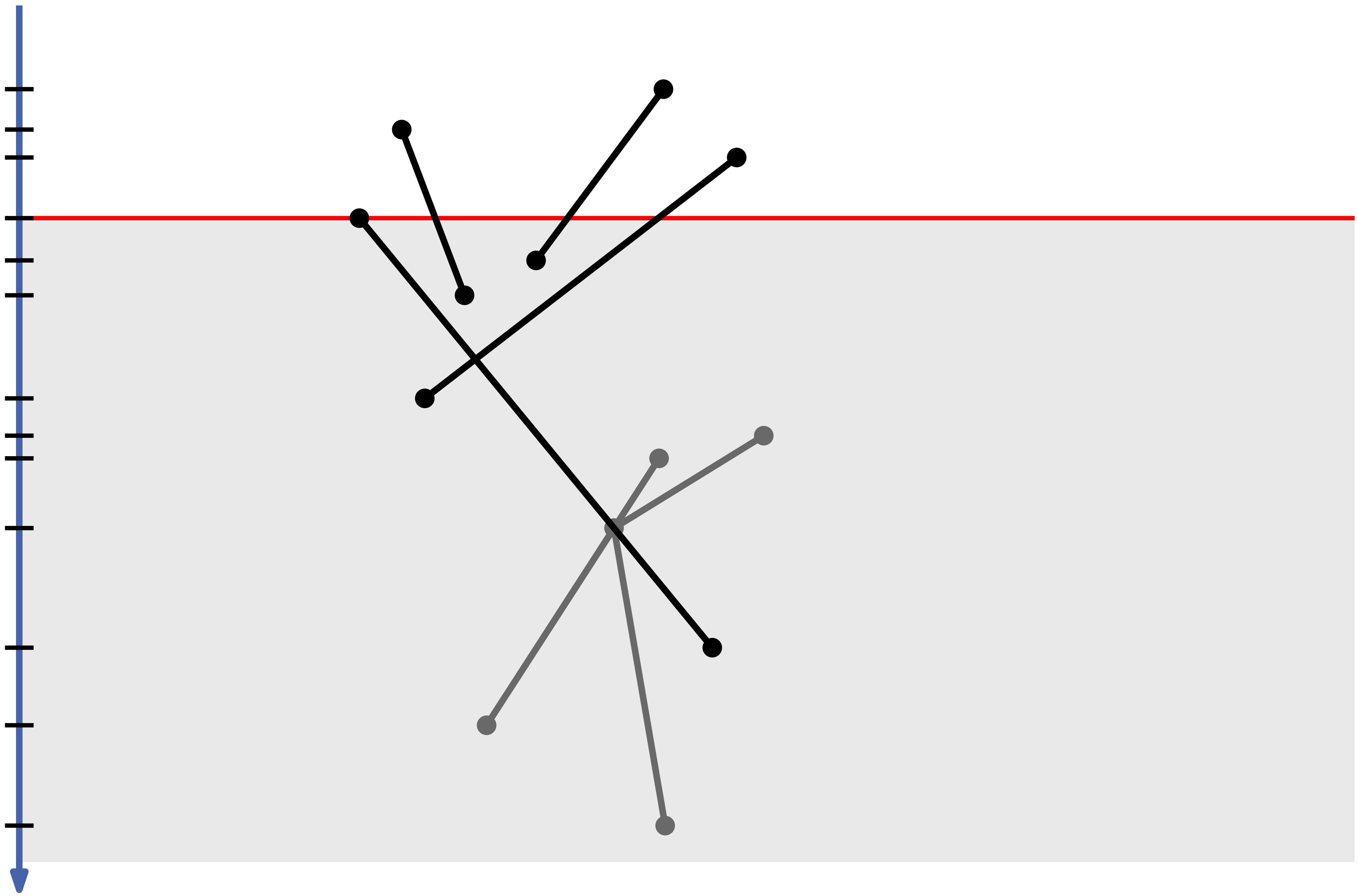

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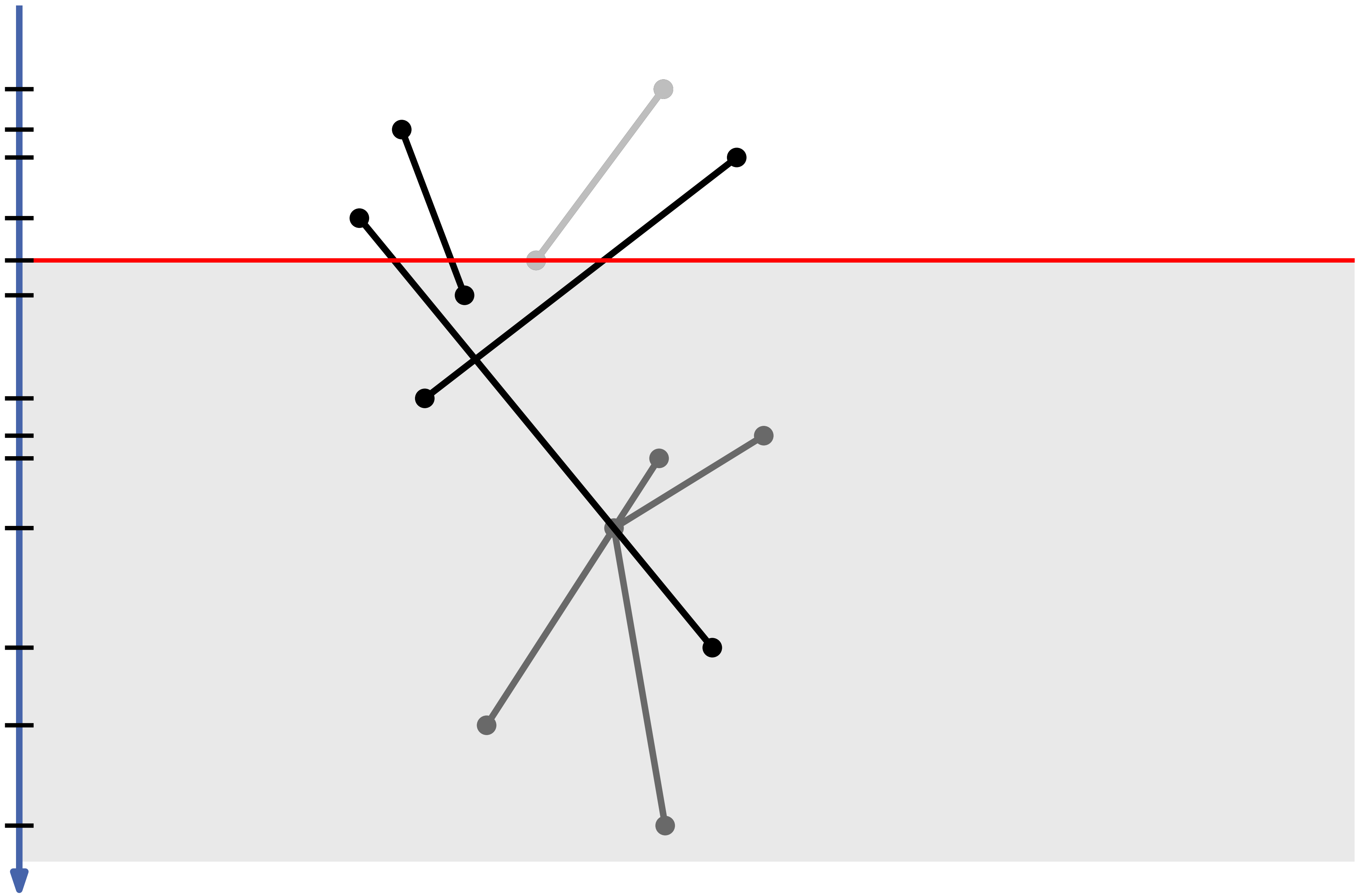

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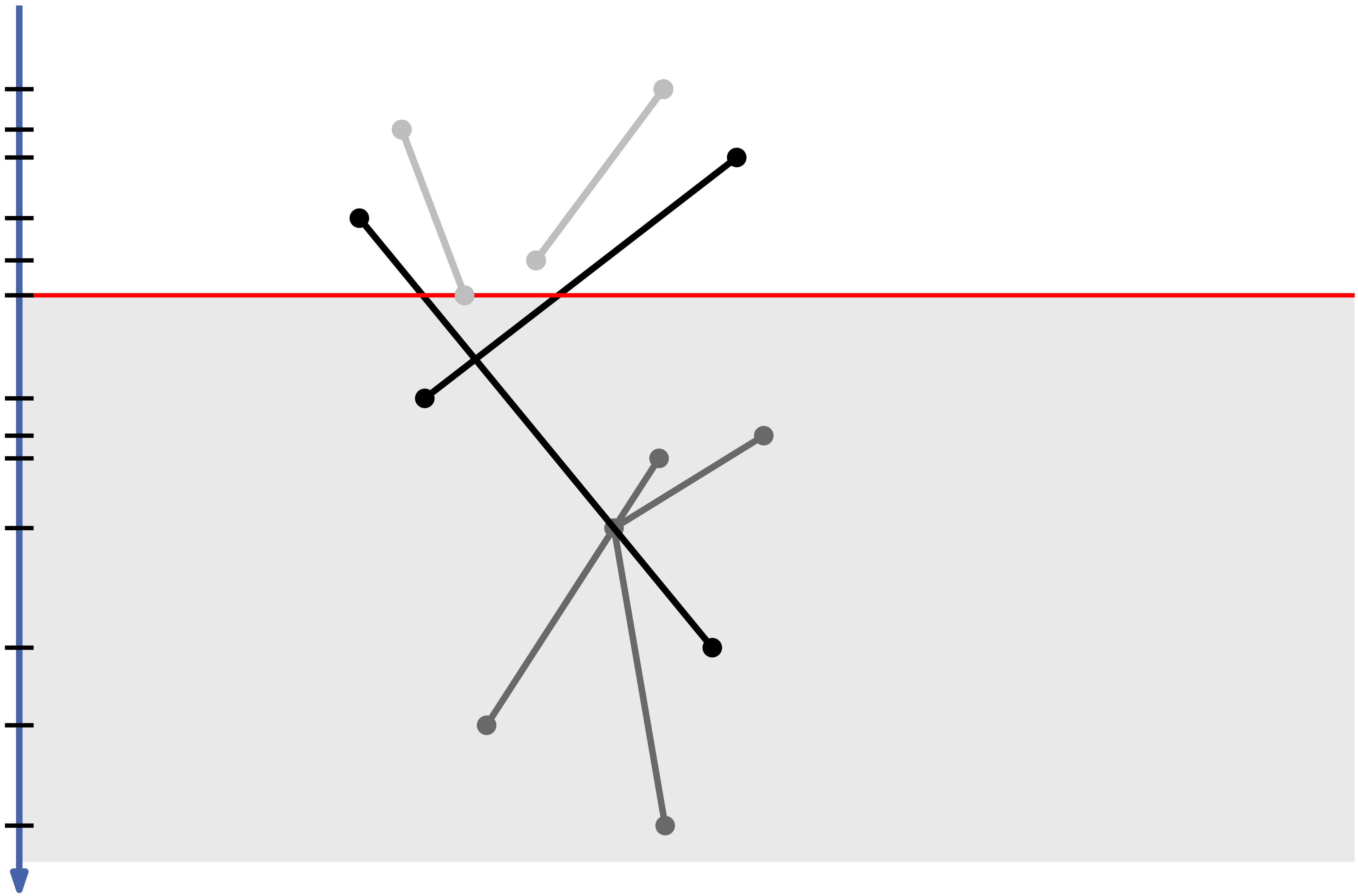

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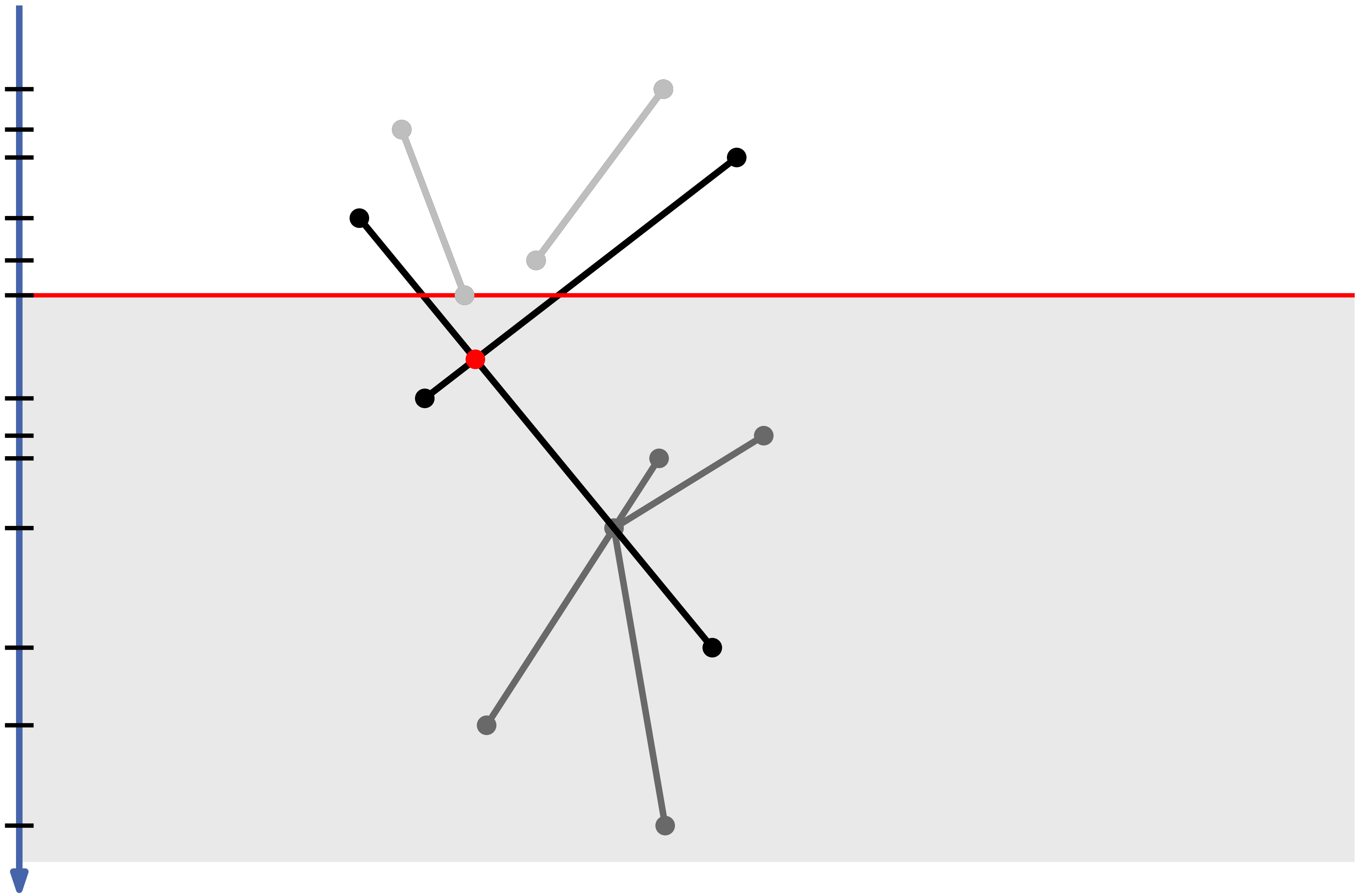

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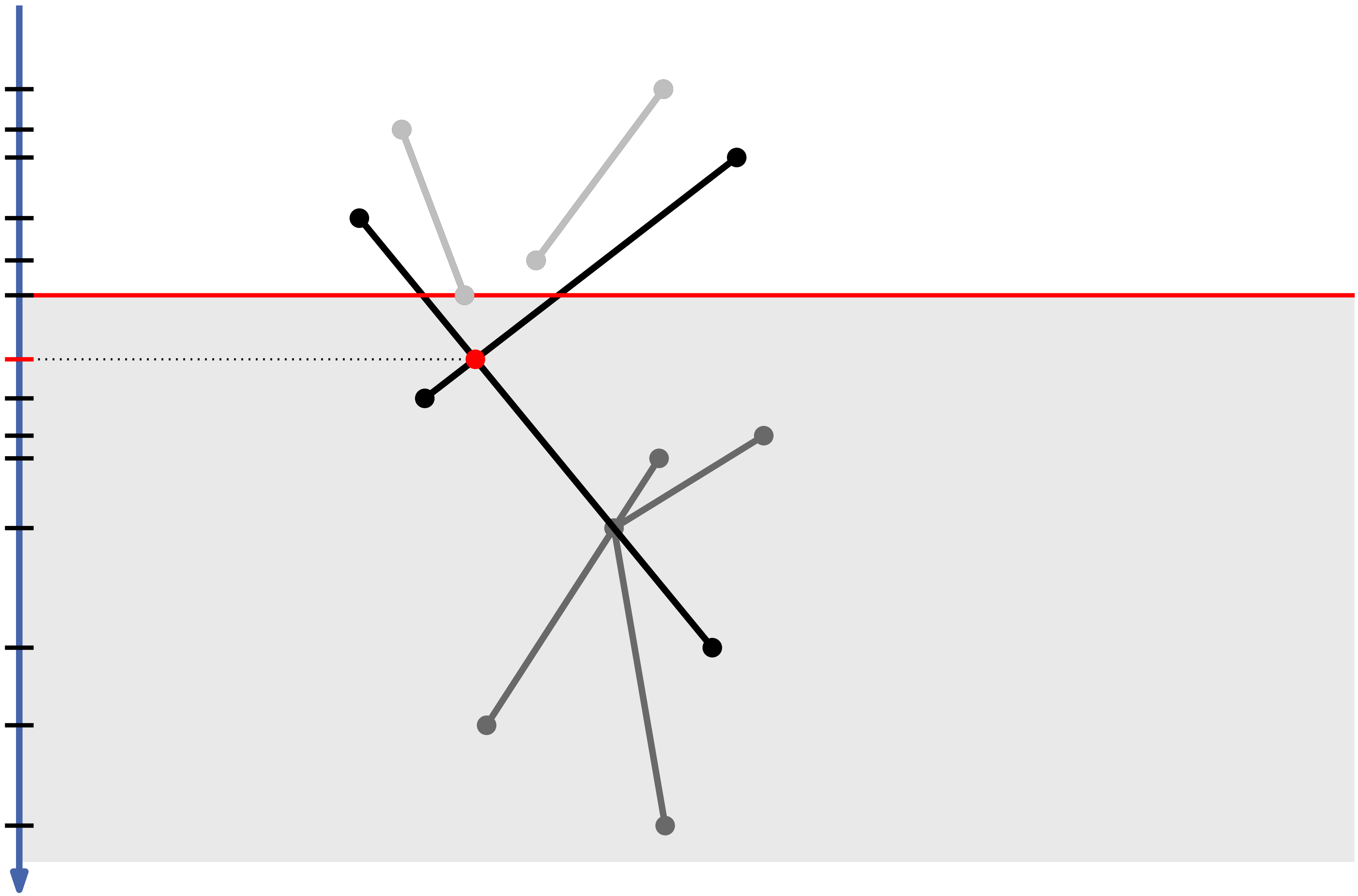

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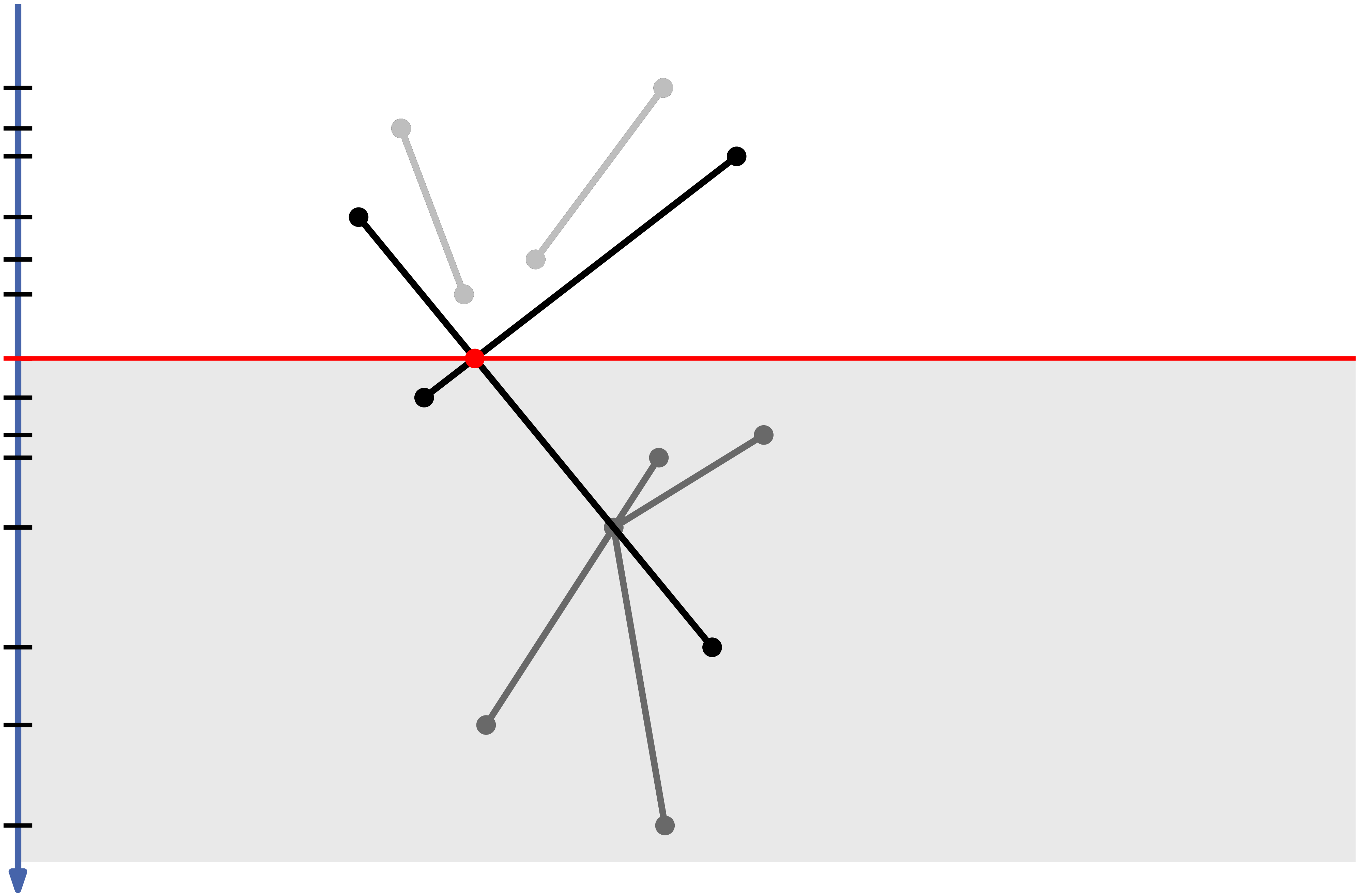

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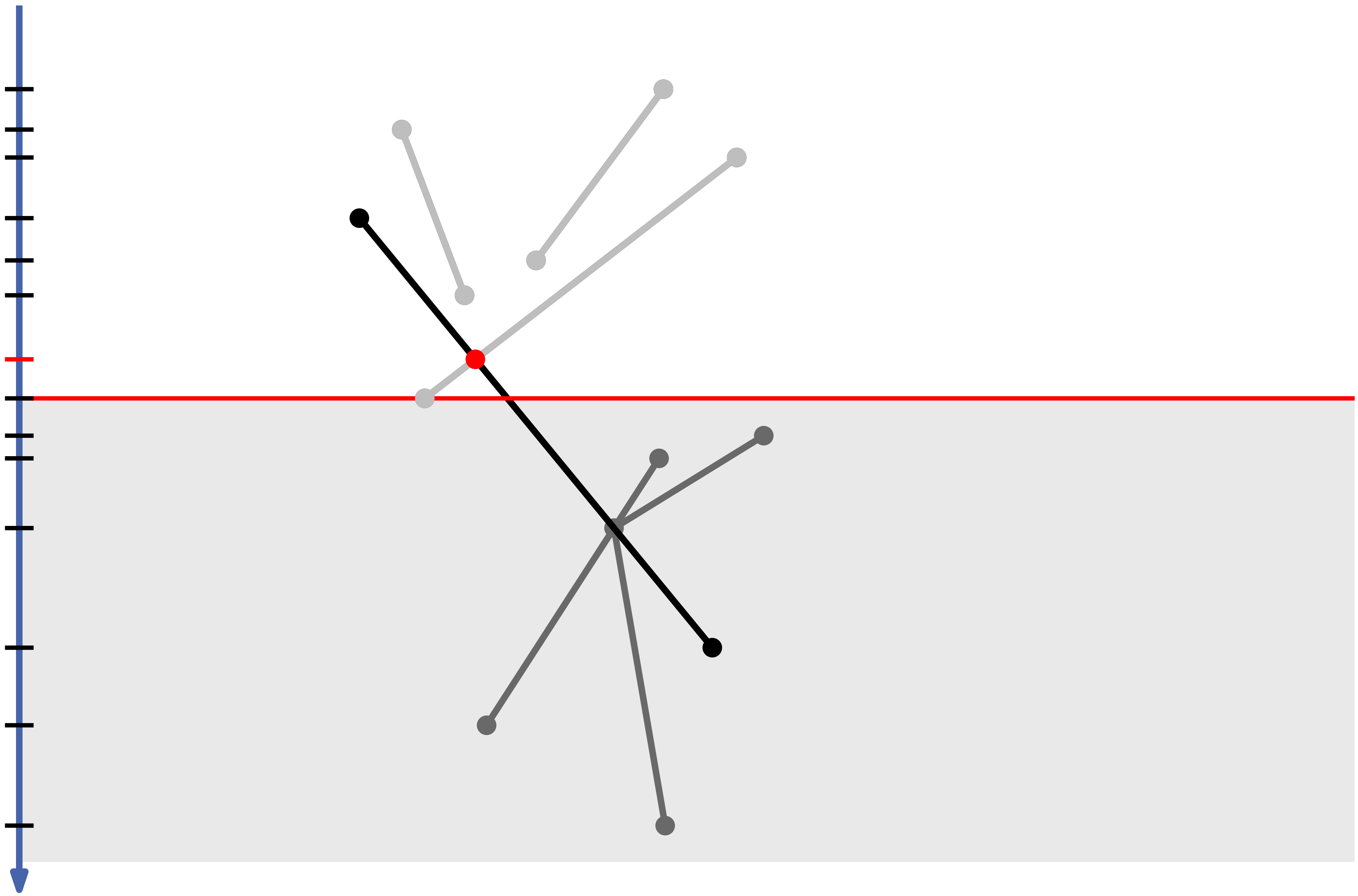

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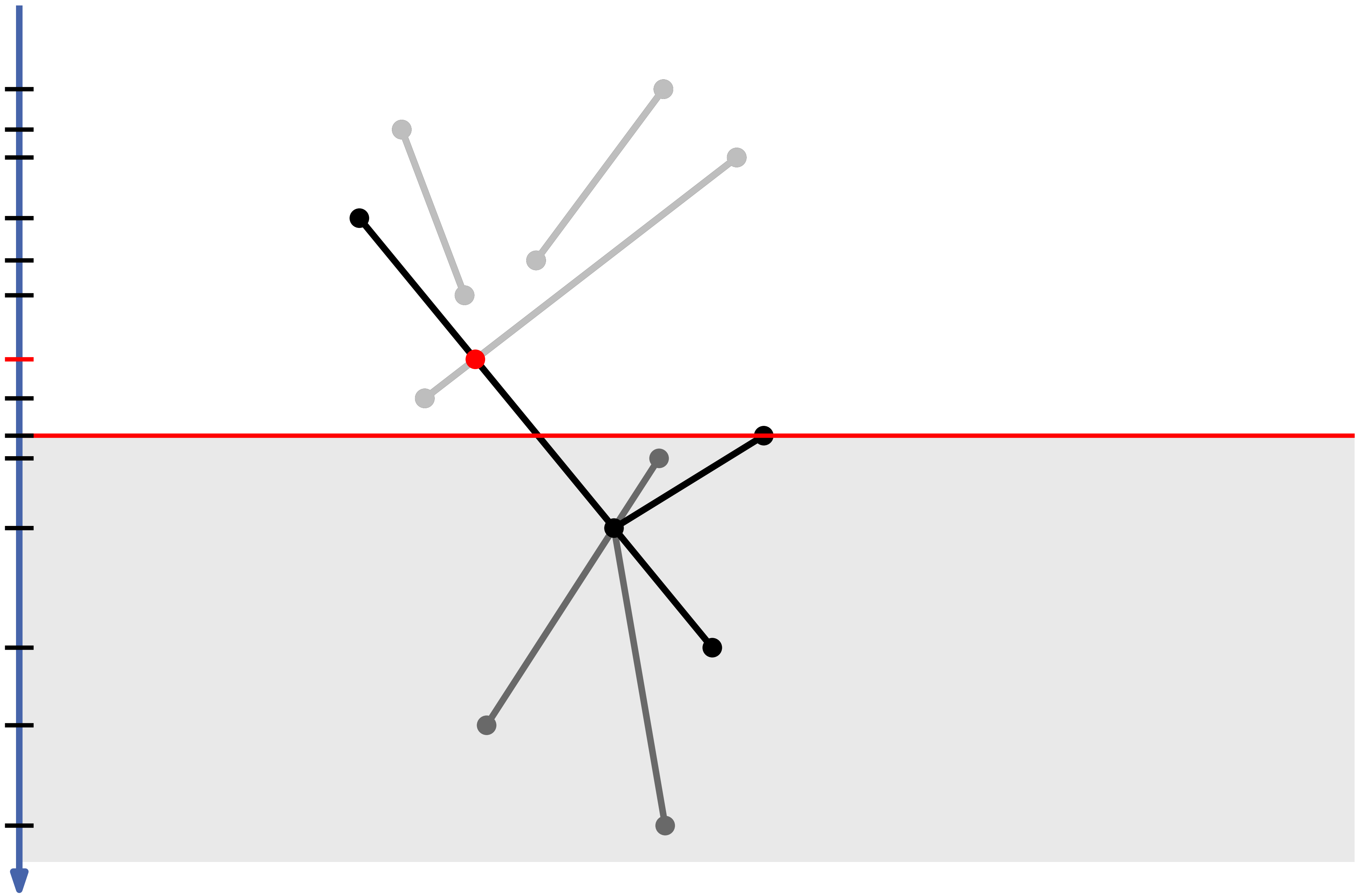

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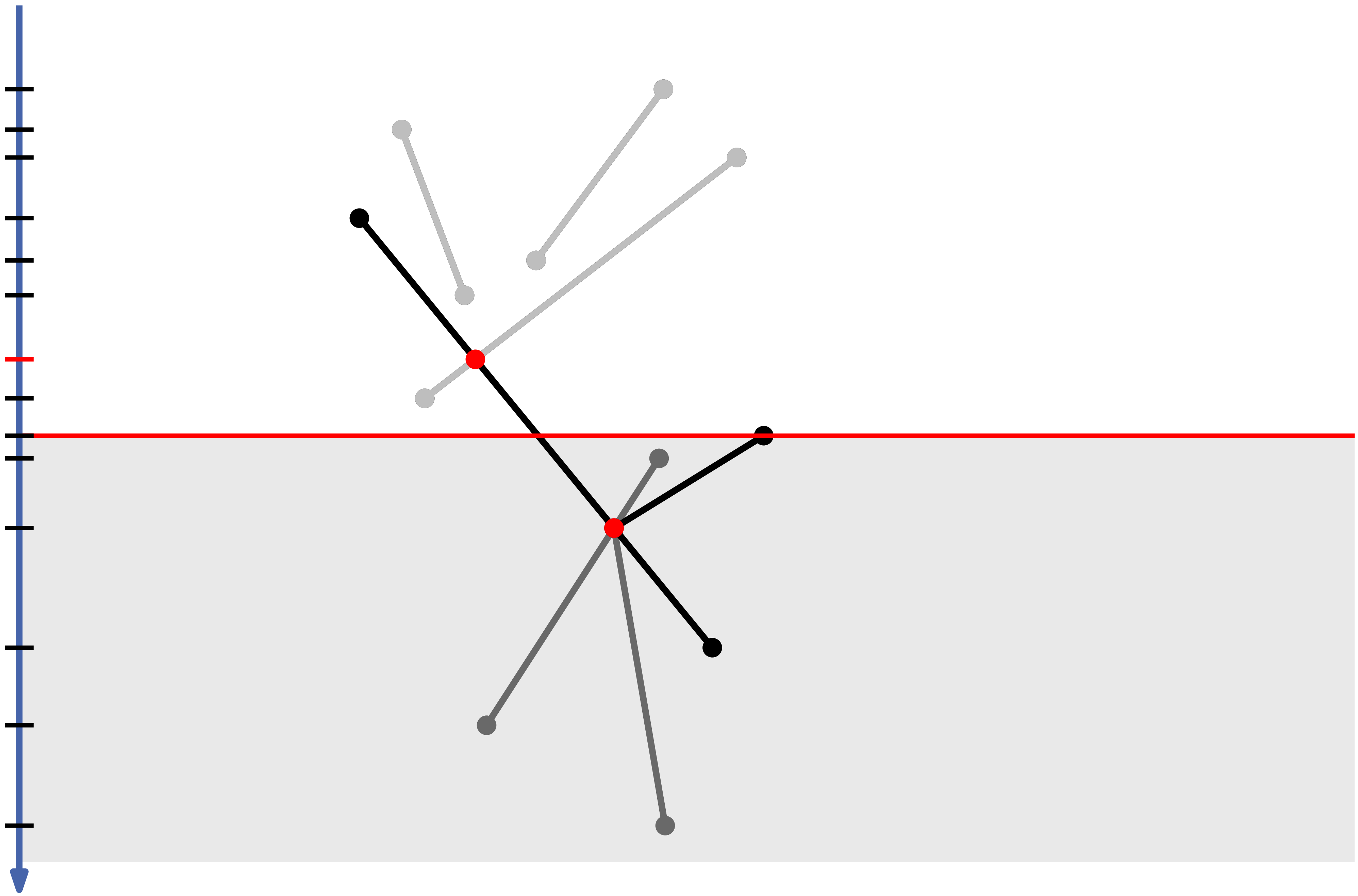

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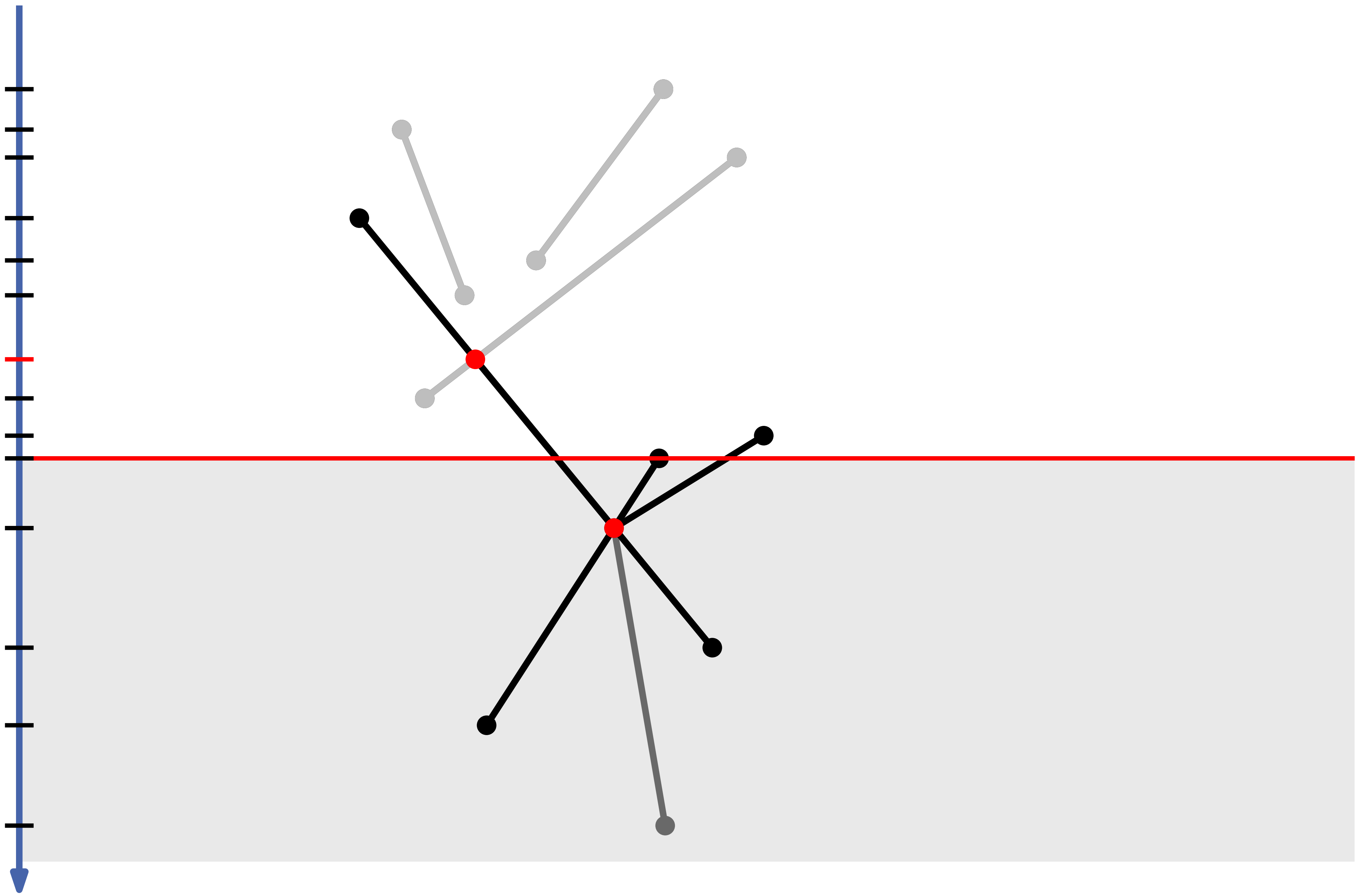

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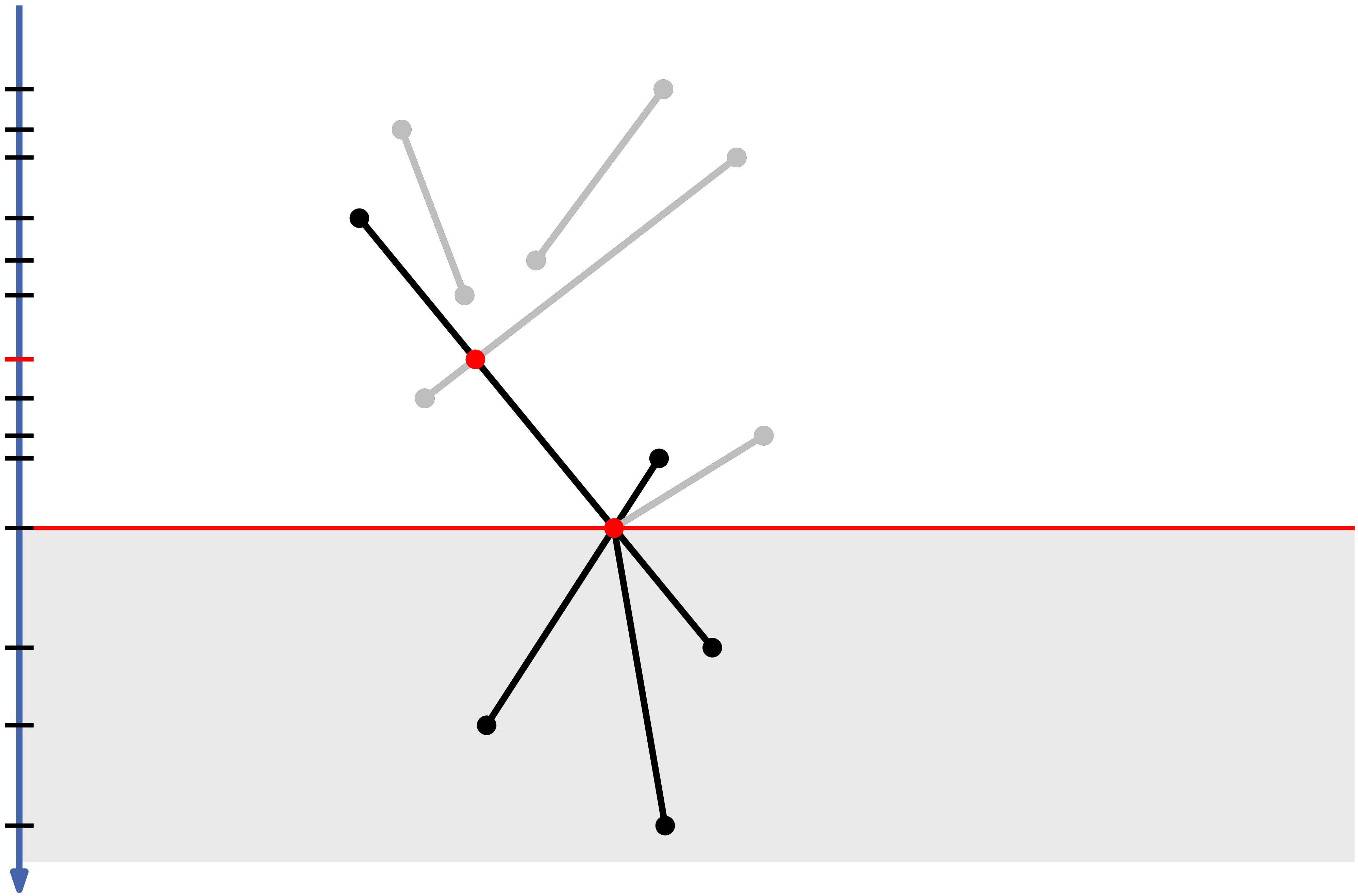

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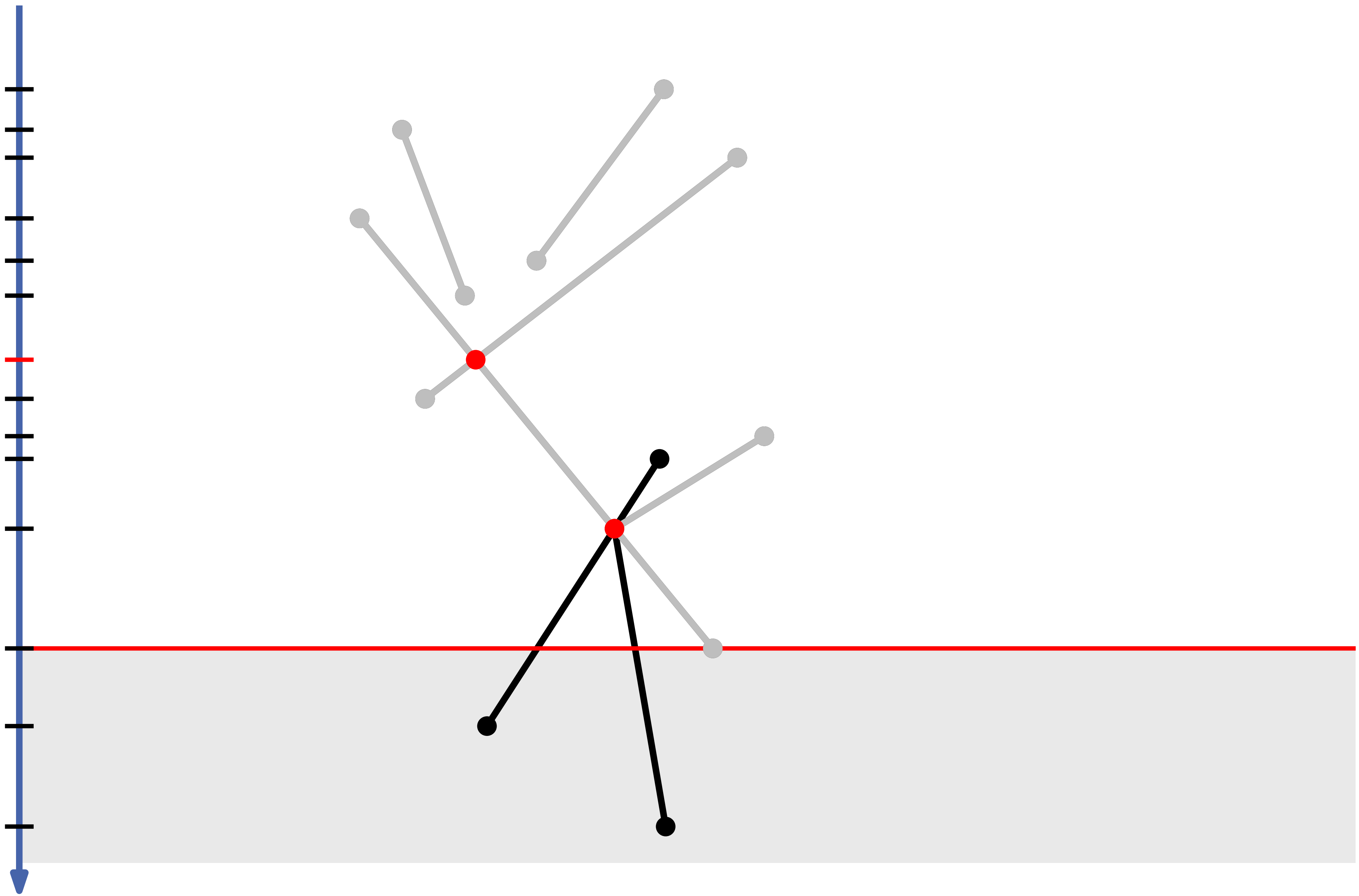

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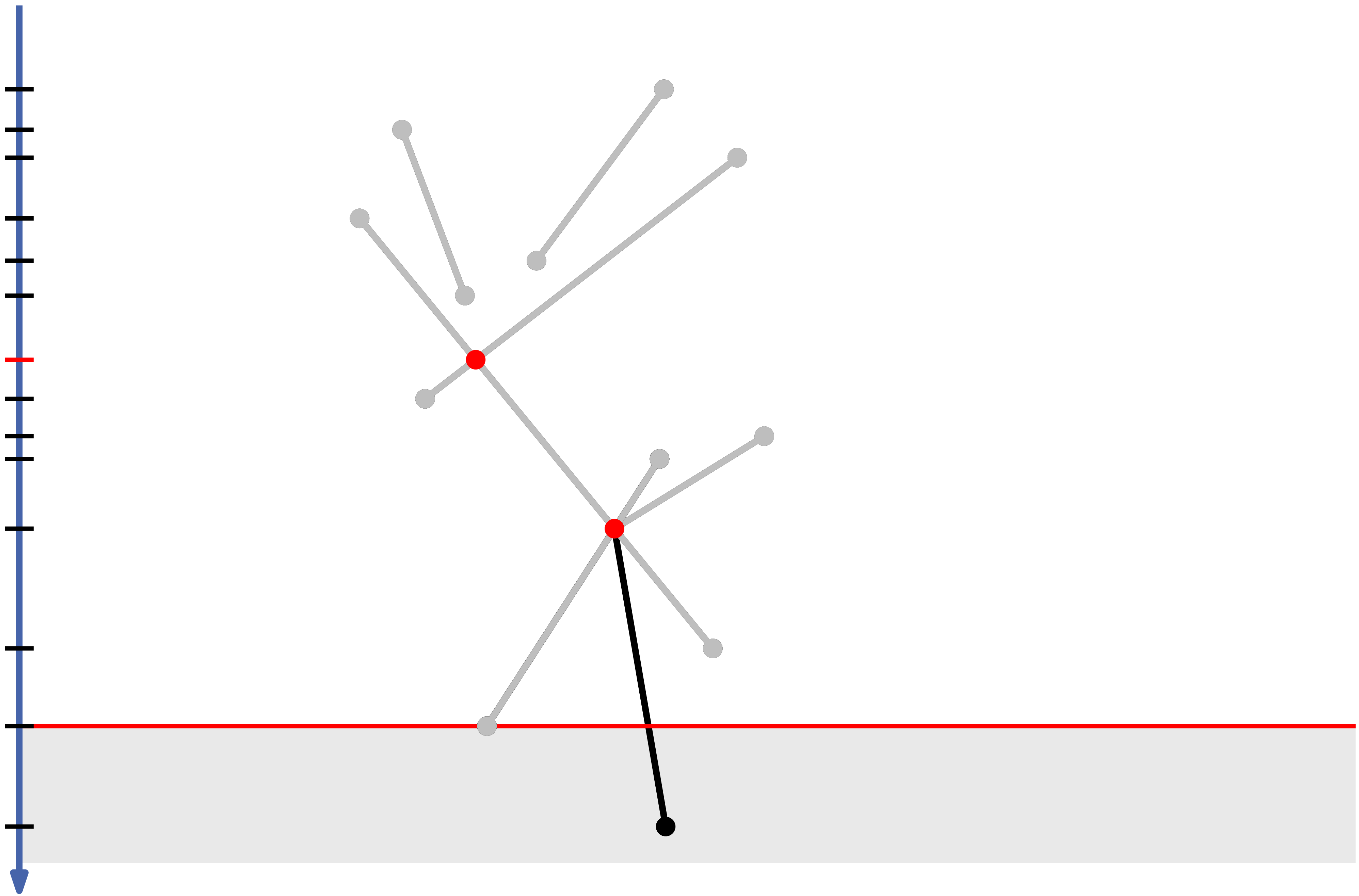

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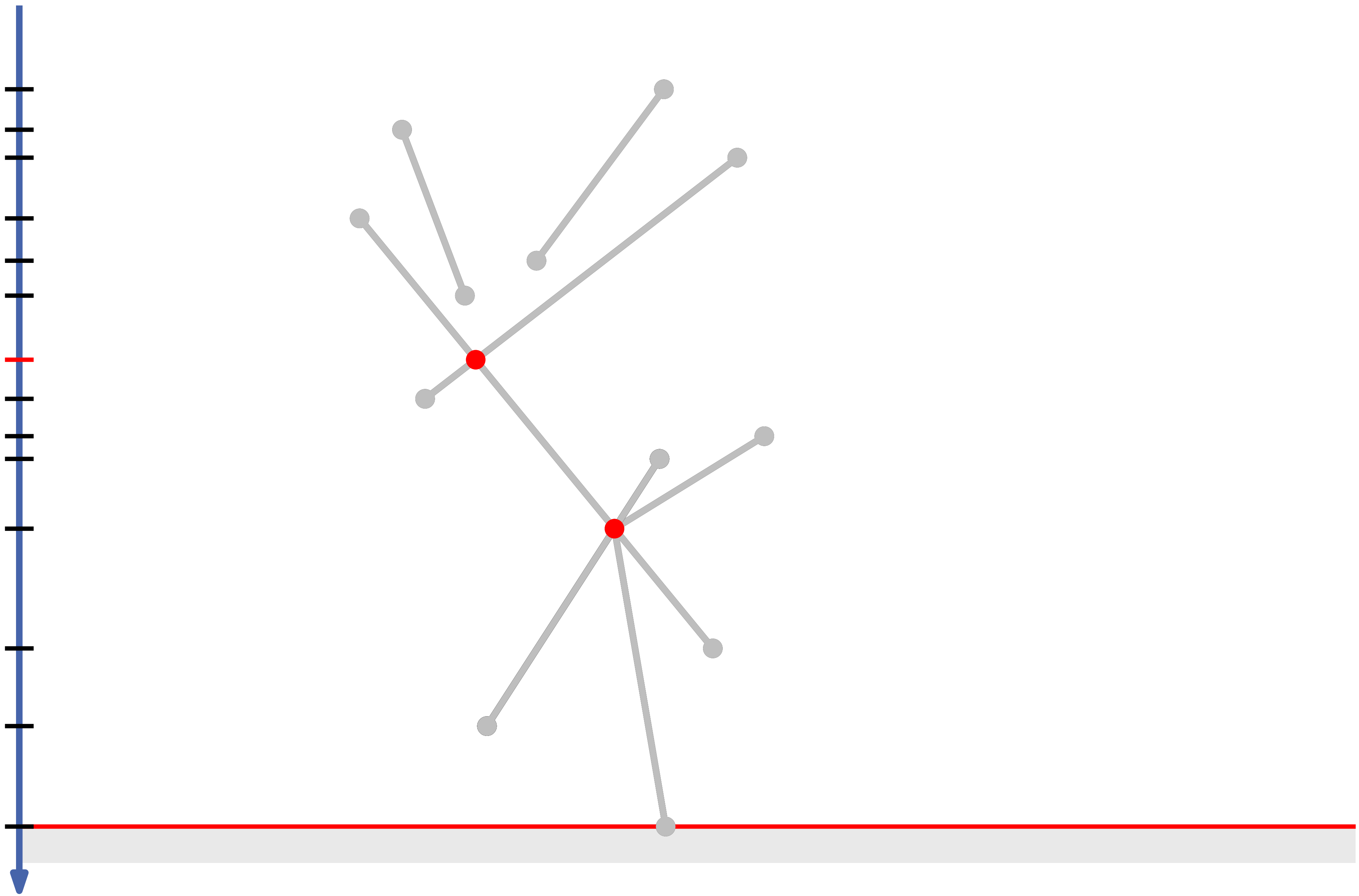

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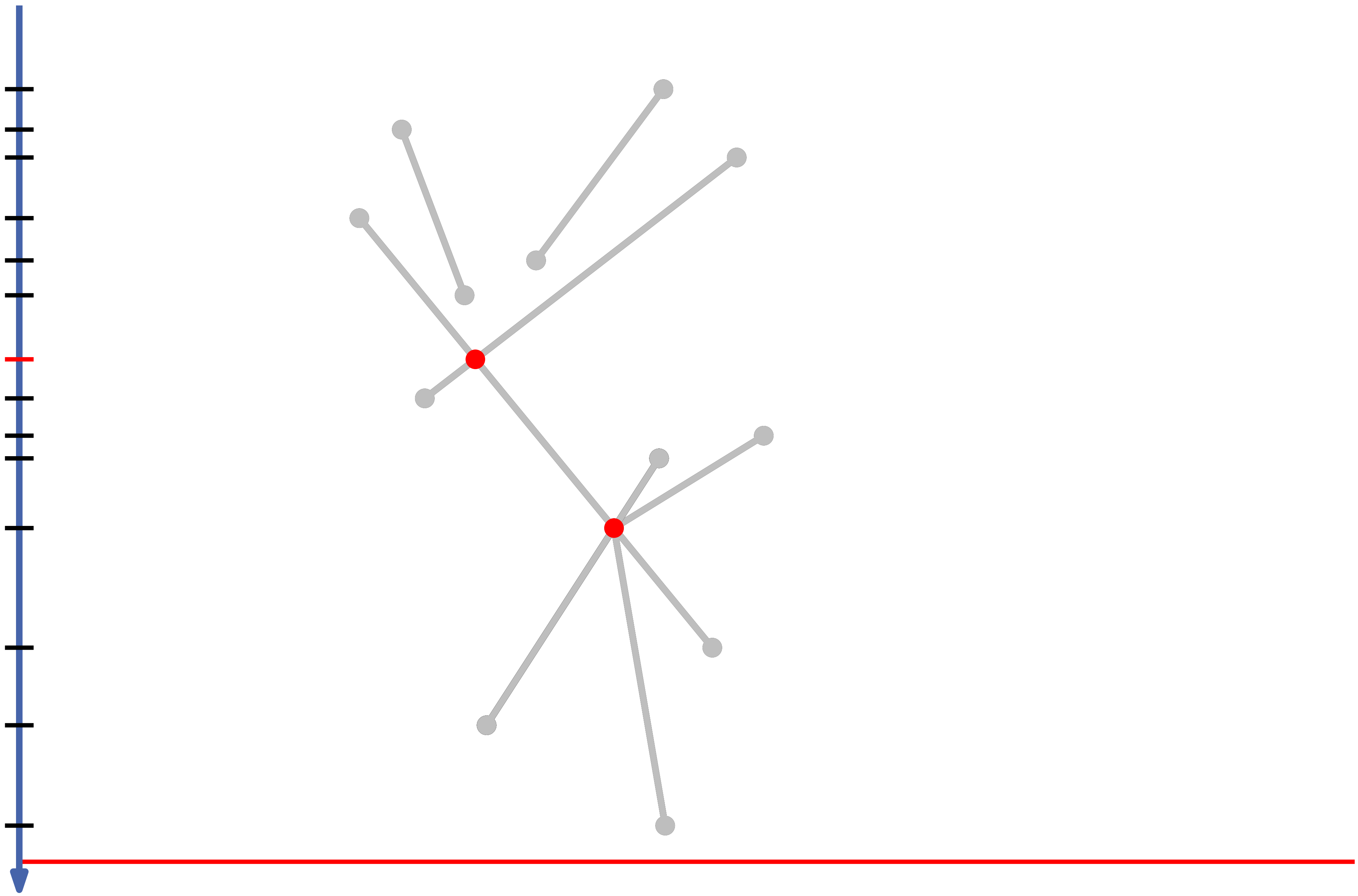
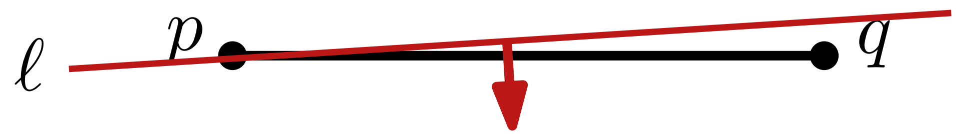


Data Structures

1.) Event Queue $\mathcal{Q}$

- define $p \prec q \quad \Leftrightarrow_{\text{def.}} \quad y_p > y_q \vee (y_p = y_q \wedge x_p < x_q)$



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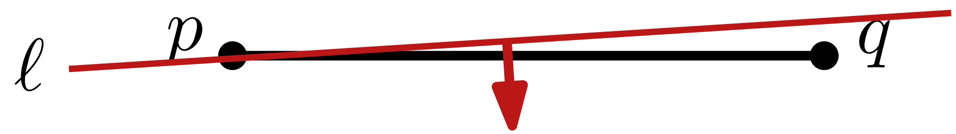

- Store events by $\prec$ in a **balanced binary search tree**
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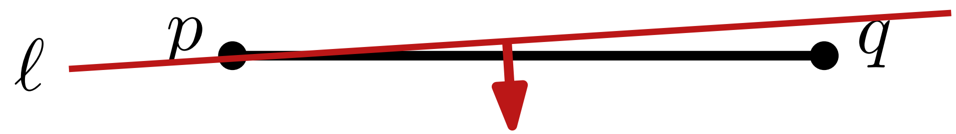

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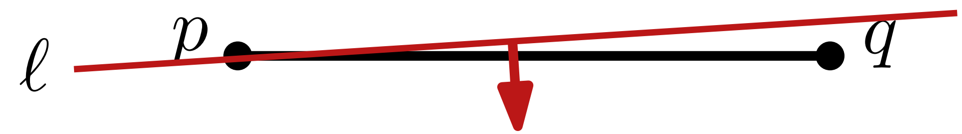

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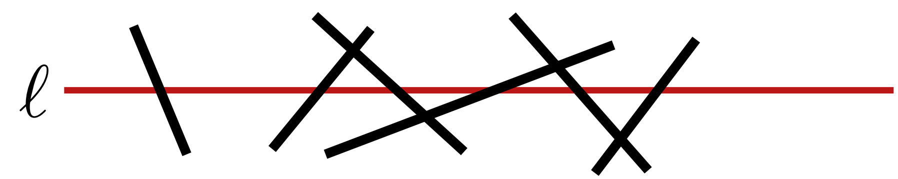

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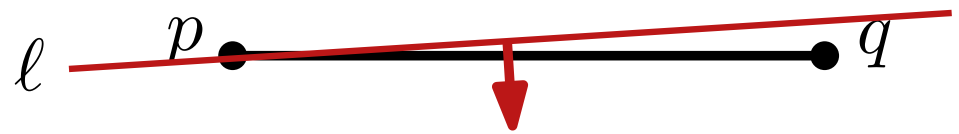

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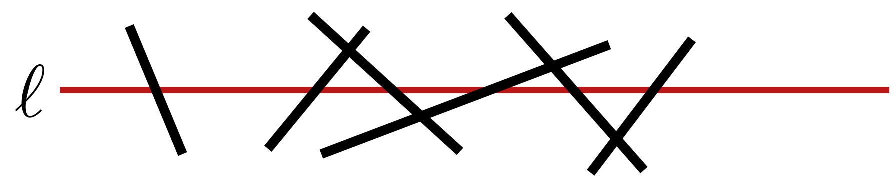

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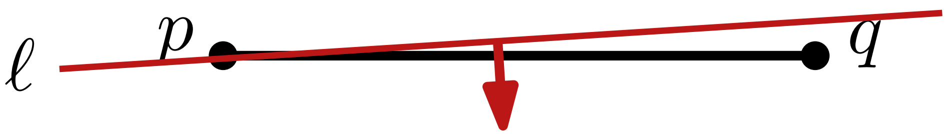

- Store events by $\prec$ in a **balanced binary search tree**
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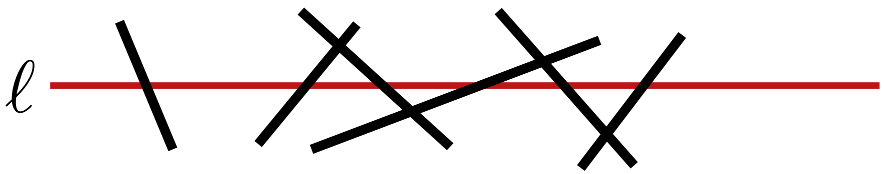

- Stores ℓ cut lines ordered from left to right
- Required operations `insert`, `delete`, `findNeighbor`
- This is also a balanced binary search tree with line segments stored in the leaves!

Algorithm

FindIntersections(S)

Input: Set S of line segments

Output: Set of all intersection points and the line segments involved

$\mathcal{Q} \leftarrow \emptyset; \quad \mathcal{T} \leftarrow \emptyset$

foreach $s \in S$ **do**

    $\mathcal{Q}$.insert(upperEndPoint(s))

    $\mathcal{Q}$.insert(lowerEndPoint(s))

while $\mathcal{Q} \neq \emptyset$ **do**

    $p \leftarrow \mathcal{Q}$.nextEvent()

    $\mathcal{Q}$.deleteEvent(p)

    handleEvent(p)

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while $\mathcal{Q} \neq \emptyset$ **do**

  $p \leftarrow \mathcal{Q}$.nextEvent()

  $\mathcal{Q}$.deleteEvent(p)

  handleEvent(p)

This is the core of the algorithm!

Algorithm

handleEvent(p)

$U(p) \leftarrow$ Line segments with p as upper endpoint
$L(p) \leftarrow$ Line segments with p as lower endpoint
$C(p) \leftarrow$ Line segments with p as interior point
if $|U(p) \cup L(p) \cup C(p)| \geq 2$ **then**
  return p and $U(p) \cup L(p) \cup C(p)$
remove $L(p) \cup C(p)$ from $\mathcal{T}$
add $U(p) \cup C(p)$ to $\mathcal{T}$
if $U(p) \cup C(p) = \emptyset$ **then** $//s_l$ and s_r, neighbors of p in $\mathcal{T}$
  $\mathcal{Q} \leftarrow$ check if s_l and s_r intersect below p
else $//s'$ and s'' leftmost and rightmost line segment in $U(p) \cup C(p)$
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Algorithm

handleEvent(p)

$U(p) \leftarrow$ Line segments with p as upper endpoint — Stored with p in $\mathcal{Q}$

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Neighbors in $\mathcal{T}$

if $|U(p) \cup L(p) \cup C(p)| \geq 2$ **then**

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if $|U(p) \cup L(p) \cup C(p)| \geq 2$ **then**

  return p and $U(p) \cup L(p) \cup C(p)$

remove $L(p) \cup C(p)$ from $\mathcal{T}$

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if $U(p) \cup C(p) = \emptyset$ **then** $//s_l$ and s_r, neighbors of p in $\mathcal{T}$

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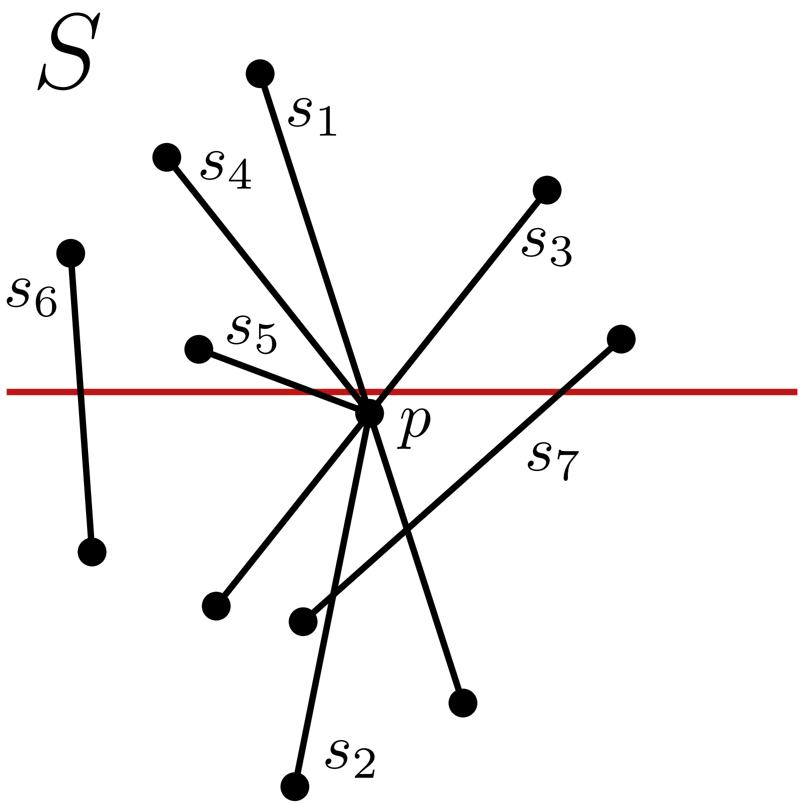
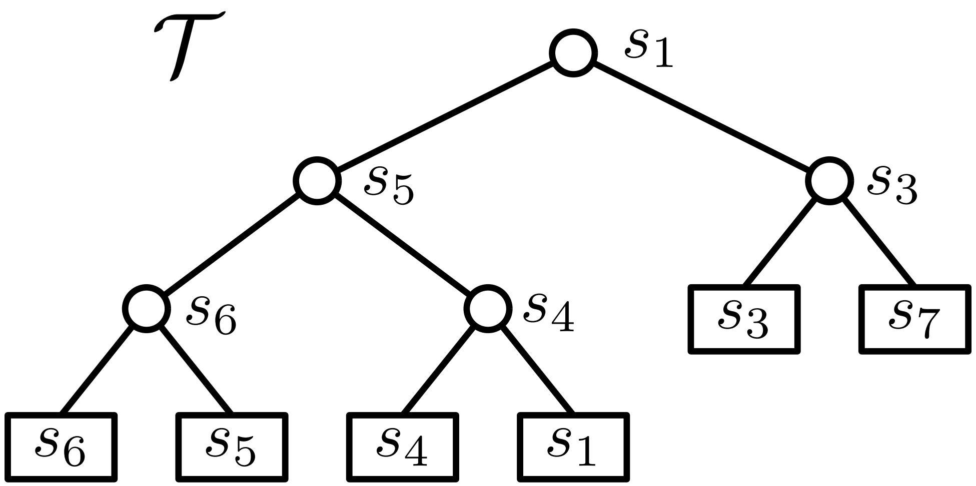
Stored with p in $\mathcal{Q}$

Neighbors in $\mathcal{T}$

Remove and insert reverses order in $C(p)$

What Happens Exactly?





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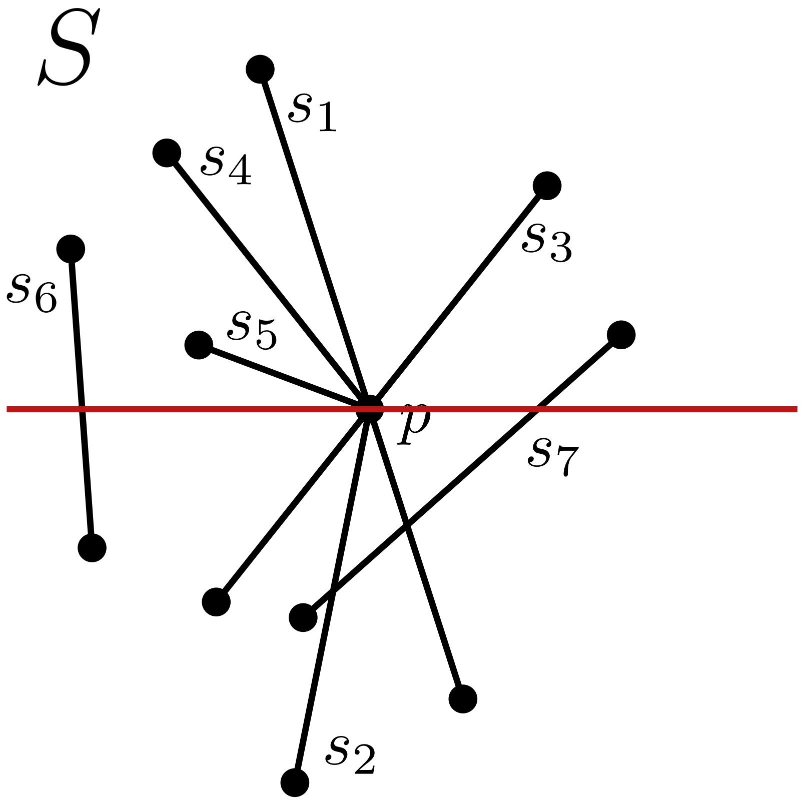
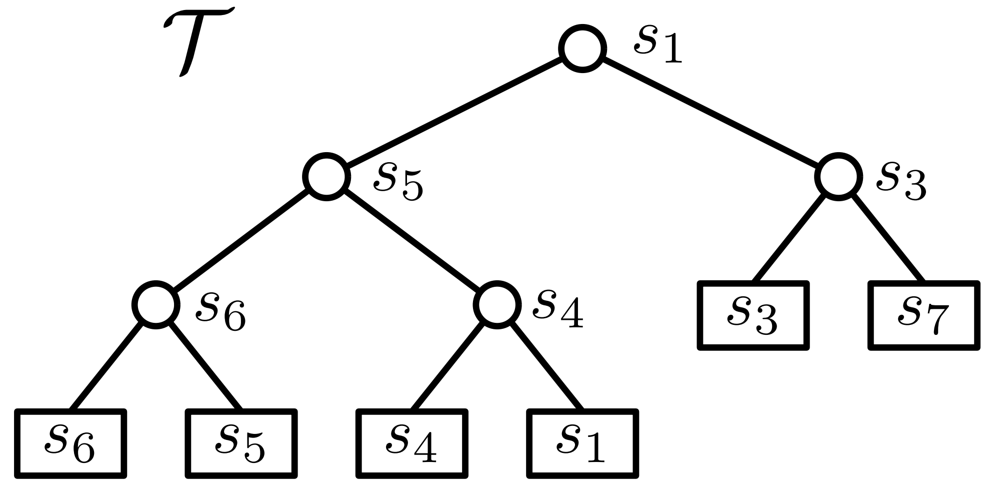
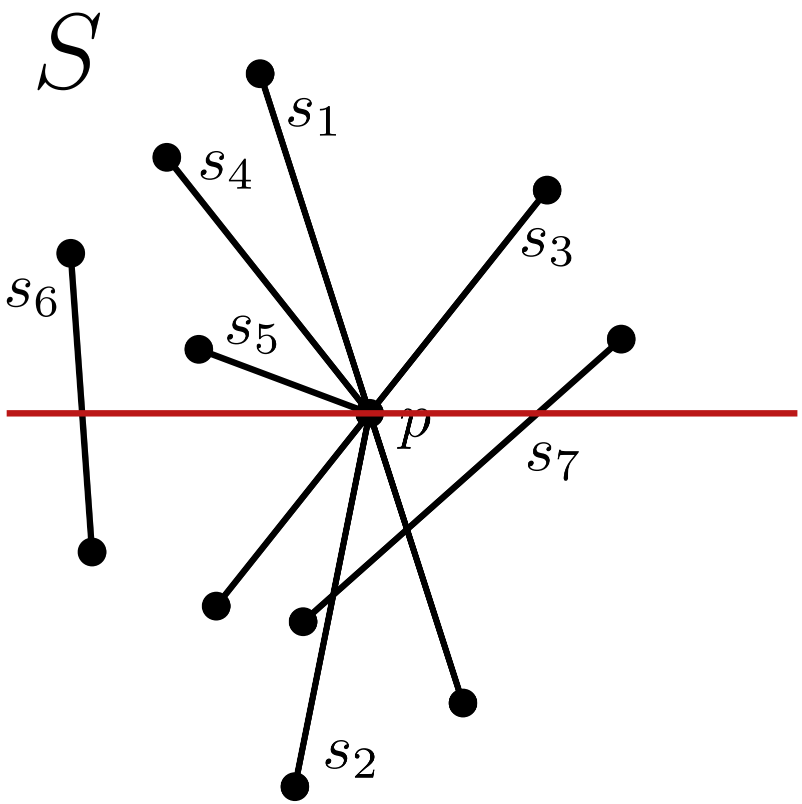
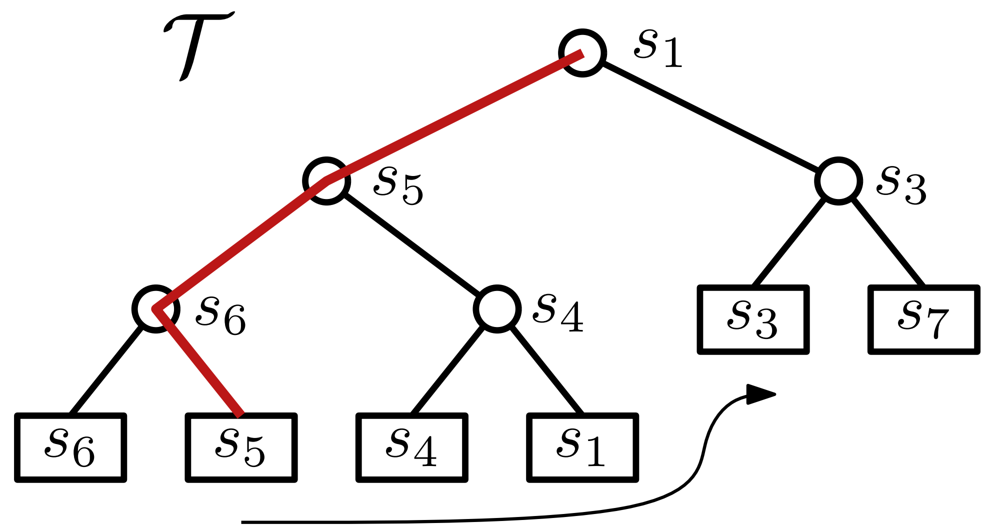




$U(p) = \{s_2\}$
$L(p) =$
$C(p) =$

What Happens Exactly?





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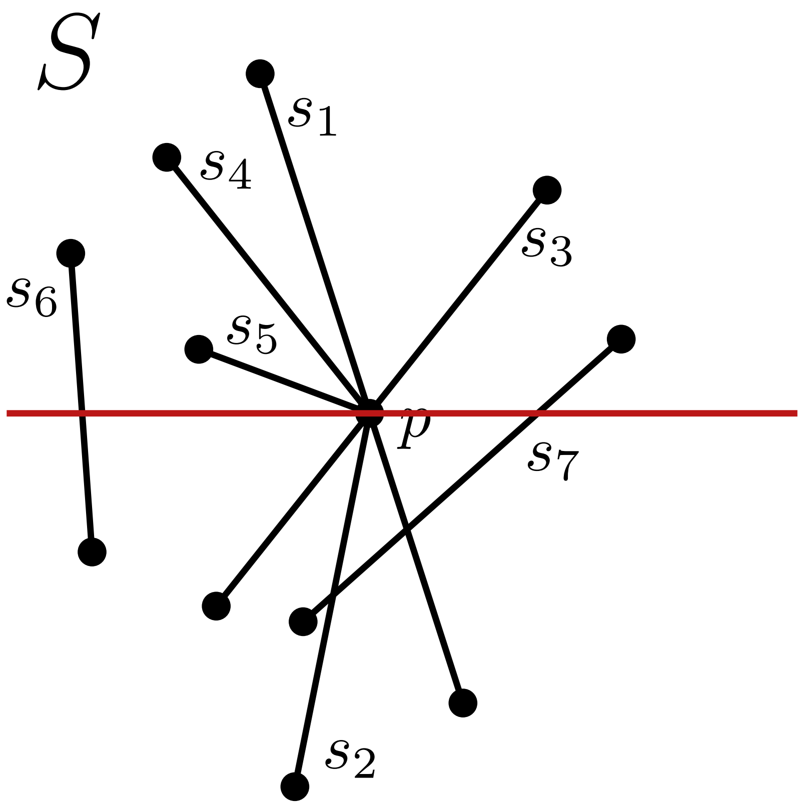

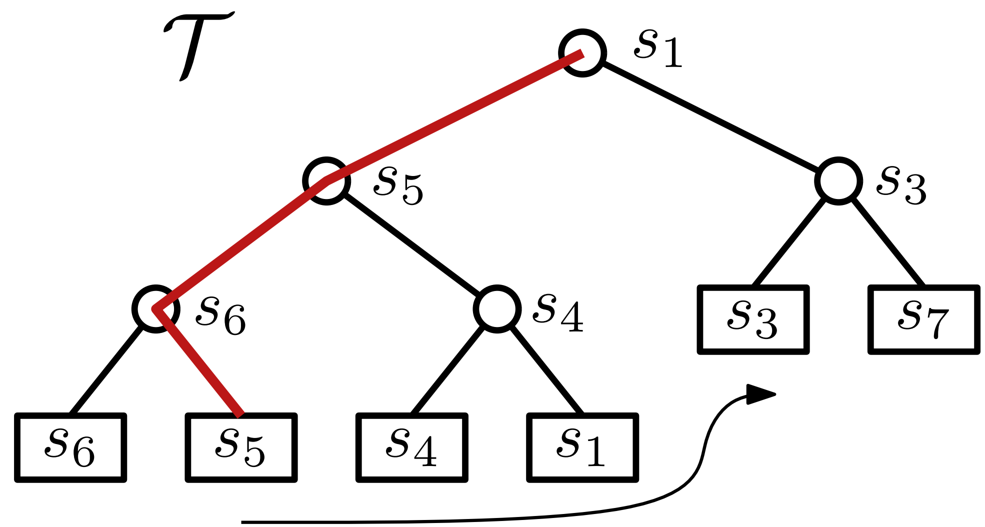

$U(p) = \{s_2\}$
$L(p) = \{s_4, s_5\}$
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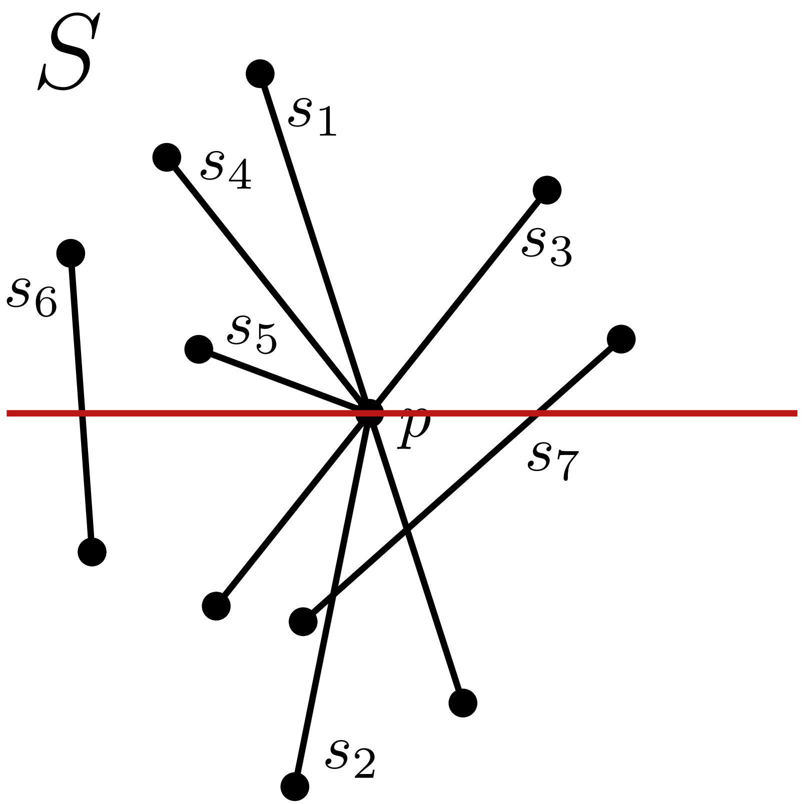
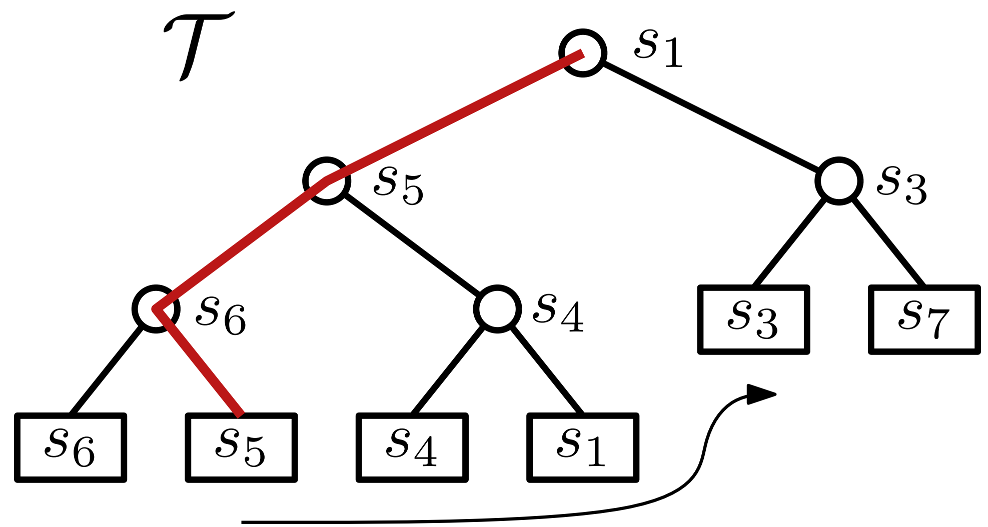
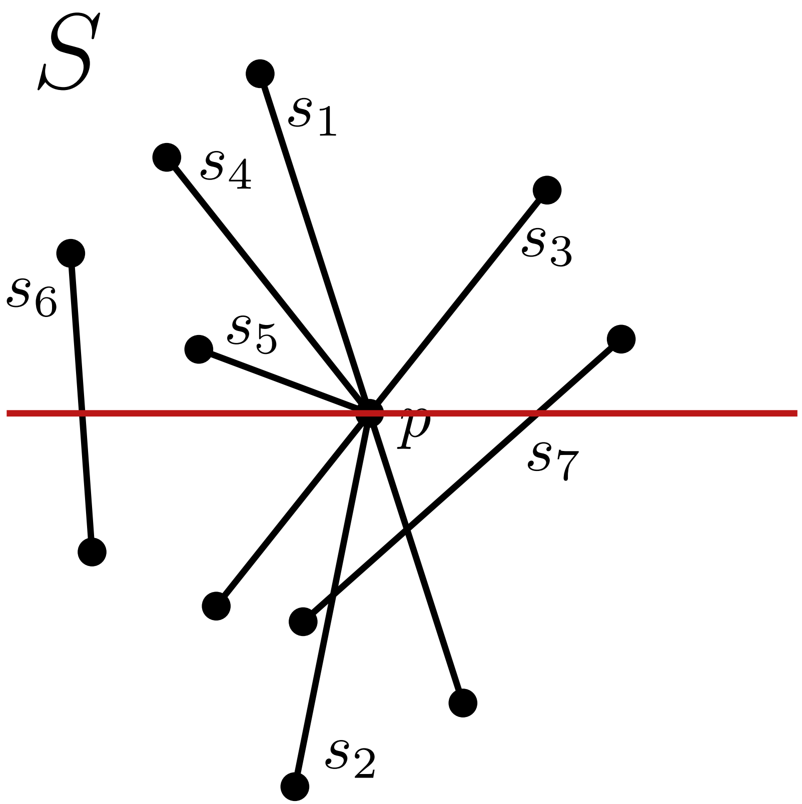


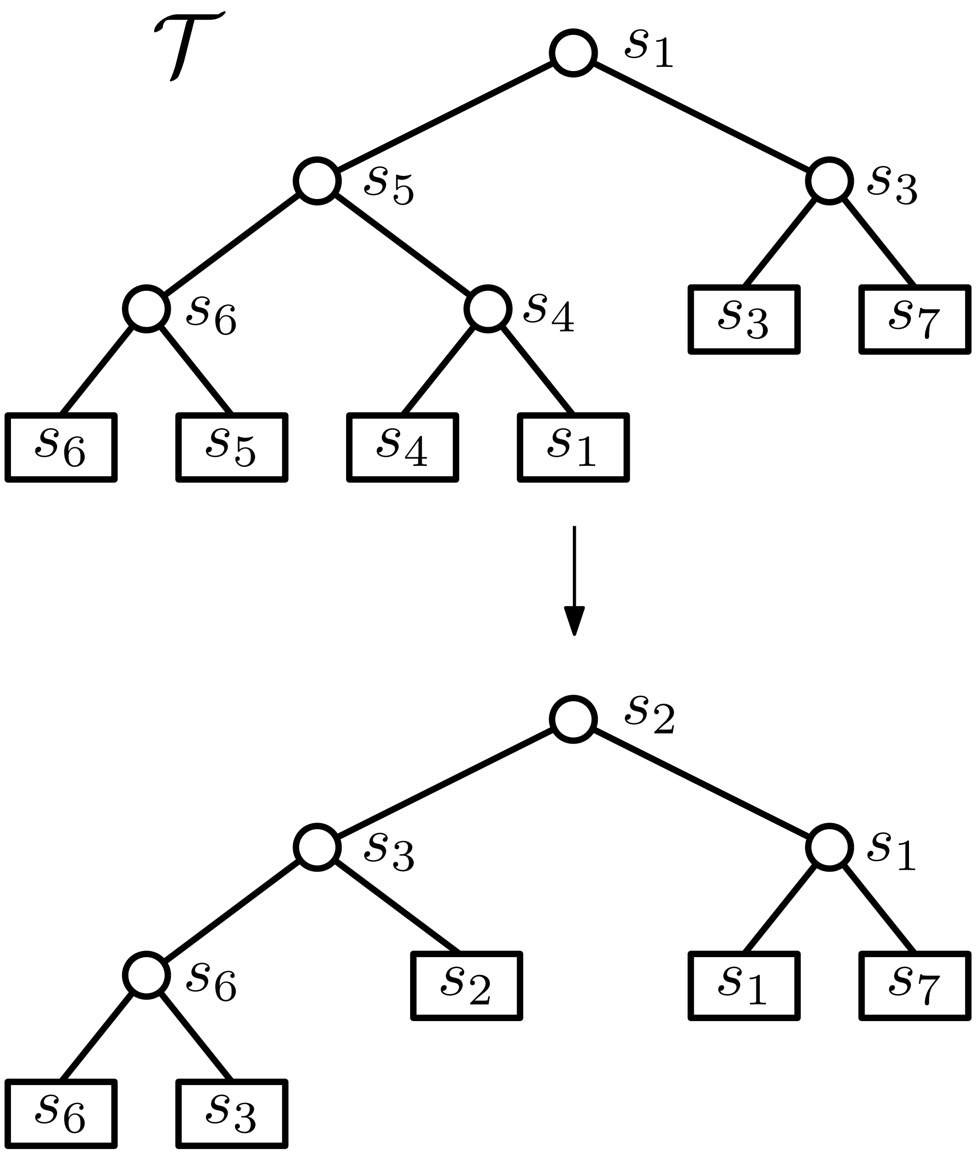


$U(p) = \{s_2\}$
$L(p) = \{s_4, s_5\}$
$C(p) = \{s_1, s_3\}$

Report $(p, \{s_1, s_2, s_3, s_4, s_5\})$

What Happens Exactly?

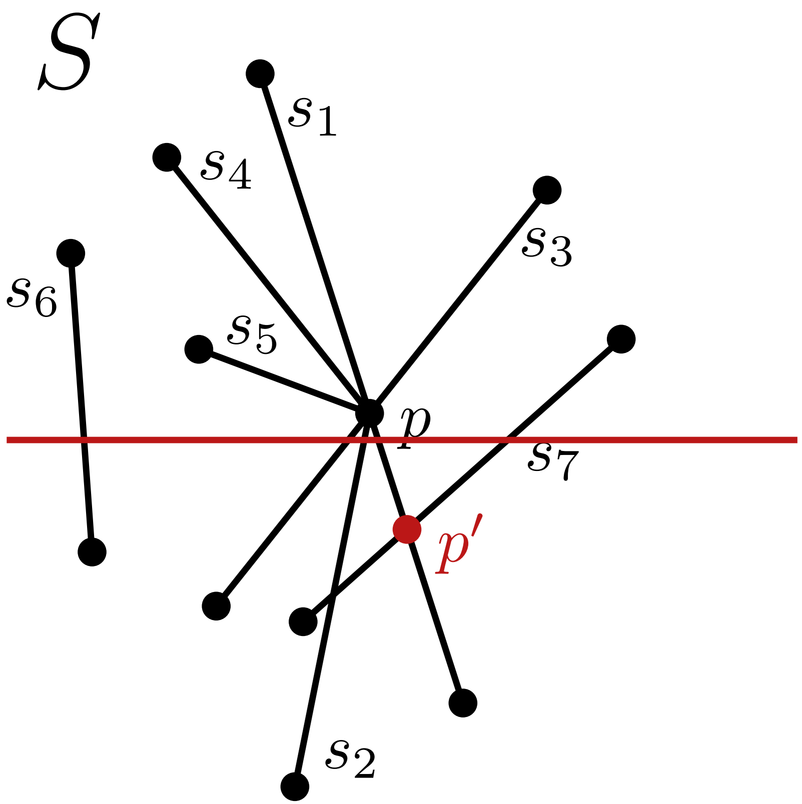


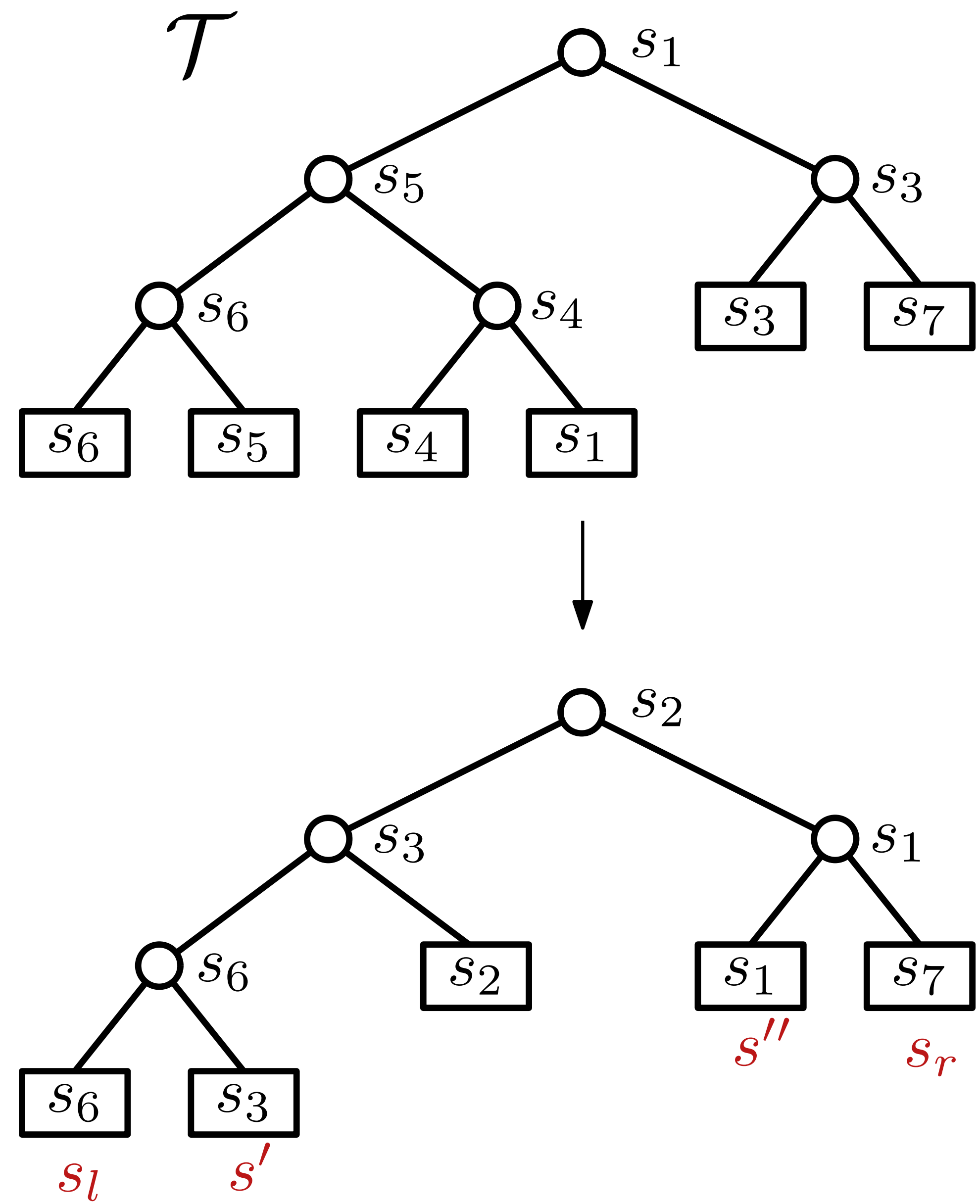


$U(p) = \{s_2\}$
$L(p) = \{s_4, s_5\}$
$C(p) = \{s_1, s_3\}$

Delete $L(p) \cup C(p)$; add $U(p) \cup C(p)$

What Happens Exactly?





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Add event $p' = s_1 \times s_7$ in $\mathcal{Q}$

Correctness

Lemma 1: Algorithm FindIntersections finds all intersection points and the line segments involved

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Proof:

Induction on the number of events processed.

Let p be an intersection point and all intersection points $q \prec p$ are already correctly computed.

Case 1: p is a line segment endpoint

- p was inserted in $\mathcal{Q}$
- $U(p)$ stores p
- $L(p)$ and $C(p)$ are in $\mathcal{T}$

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Case 2: p is not a line segment endpoint

Consider why p must be in $\mathcal{Q}$!

Running-Time Analysis

FindIntersections(S)

Input: Set S of line segments

Output: Set of all intersections with their line segments

$\mathcal{Q} \leftarrow \emptyset; \quad \mathcal{T} \leftarrow \emptyset$

foreach $s \in S$ **do**

  $\mathcal{Q}$.insert(upperEndPoint(s))

  $\mathcal{Q}$.insert(lowerEndPoint(s))

while $\mathcal{Q} \neq \emptyset$ **do**

  $p \leftarrow \mathcal{Q}$.nextEvent()

  $\mathcal{Q}$.deleteEvent(p)

  handleEvent(p)

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foreach $s \in S$ **do**

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- handleEvent(p)

?

Running-Time Analysis

handleEvent(p)

$U(p) \leftarrow$ Line segments with p as upper endpoint
$L(p) \leftarrow$ Line segments with p as lower endpoint
$C(p) \leftarrow$ Line segments with p as interior point
if $|U(p) \cup L(p) \cup C(p)| \geq 2$ **then**
  return p and $U(p) \cup L(p) \cup C(p)$

remove $L(p) \cup C(p)$ from $\mathcal{T}$
add $U(p) \cup C(p)$ to $\mathcal{T}$
if $U(p) \cup C(p) = \emptyset$ **then** $//s_l$ and s_r, neighbors of p in $\mathcal{T}$
  $\mathcal{Q} \leftarrow$ check if s_l and s_r intersect below p
else $//s'$ and s'' leftmost and rightmost line segment in $U(p) \cup C(p)$
  $\mathcal{Q} \leftarrow$ check if s_l and s' intersect below p
  $\mathcal{Q} \leftarrow$ check if s_r and s'' intersect below p

Lemma 2: Algorithm FindIntersections has running time $O(n \log n + I \log n)$, where I is the number of intersection points.

Summary

Thm 1: Let S be a set of n line segments in the plane. Then we can compute intersections in S together with the involved line segments in $O((n+I)\log n)$ time and $O(?)$ space.

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Proof:

- Correctness ✓
- Running time ✓
- Space
  - $\mathcal{T}$ has at most n elements
  - $\mathcal{Q}$ has at most $O(n + I)$ elements
  - reduction of $\mathcal{Q}$ to $O(n)$ space: an exercise

Consider how much space the data structures need!

Discussion

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How does this solve the map overlay problem?

Using an appropriate data structure (**doubly-connected edgelist**) for planar graphs we can compute in $O((n + I) \log n)$ time the overlay of two maps.
(Details in Ch. 2.3 of the book)