Computational Geometry · Lecture

Projects & Polygon Triangulation

INSTITUTE FOR THEORETICAL INFORMATICS · FACULTY OF INFORMATICS

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02.11.2015

Karlsruhe Institute of Technology

Projects

Project Specifications

Groups:

- 2 or 3 students, assigned to a supervisor

Tools:

- Use any programming language, should run with little effort on Linux

Due date:

- 08.02.2016

Visualization:

- Need to visualize output; ipe, svg, video? This is also helpful for debugging output
- Do not need to use graphics APIs, unless you really want to

Group Presentations:

- Each groups gives a 20-minute presentation (last 2 weeks of class)

Project—Next Steps

Project proposals: due in 2 weeks (16.11.2015):

- No more than 1 page
- Formalize your problem
- Describe geometric primitives
- State any simplifying assumptions
- Describe the algorithms / data structures you will use
- **Avoid brute force algorithms**

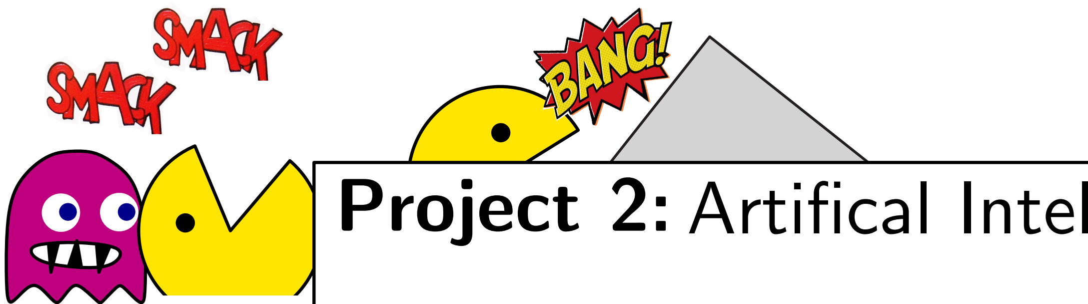
Supervisor:

- Assigned supervisor (Tamara, Darren, or Benjamin)
- Meet with supervisor in two weeks to discuss proposal.

Now, form groups and sit by each other!

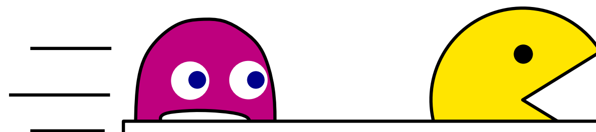
Time to present the projects...

Project 1: Collision Detection



moving obje
moving obje

Project 2: Artifical Intelligence I

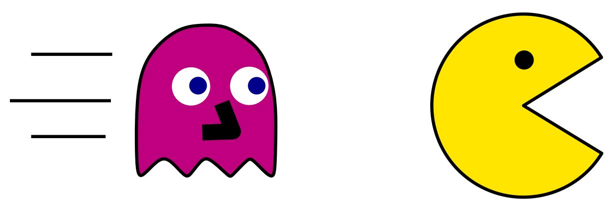


Ghosts se
until they

Fast visi

Is Pacm

Project 3: Artifical Intelligence II



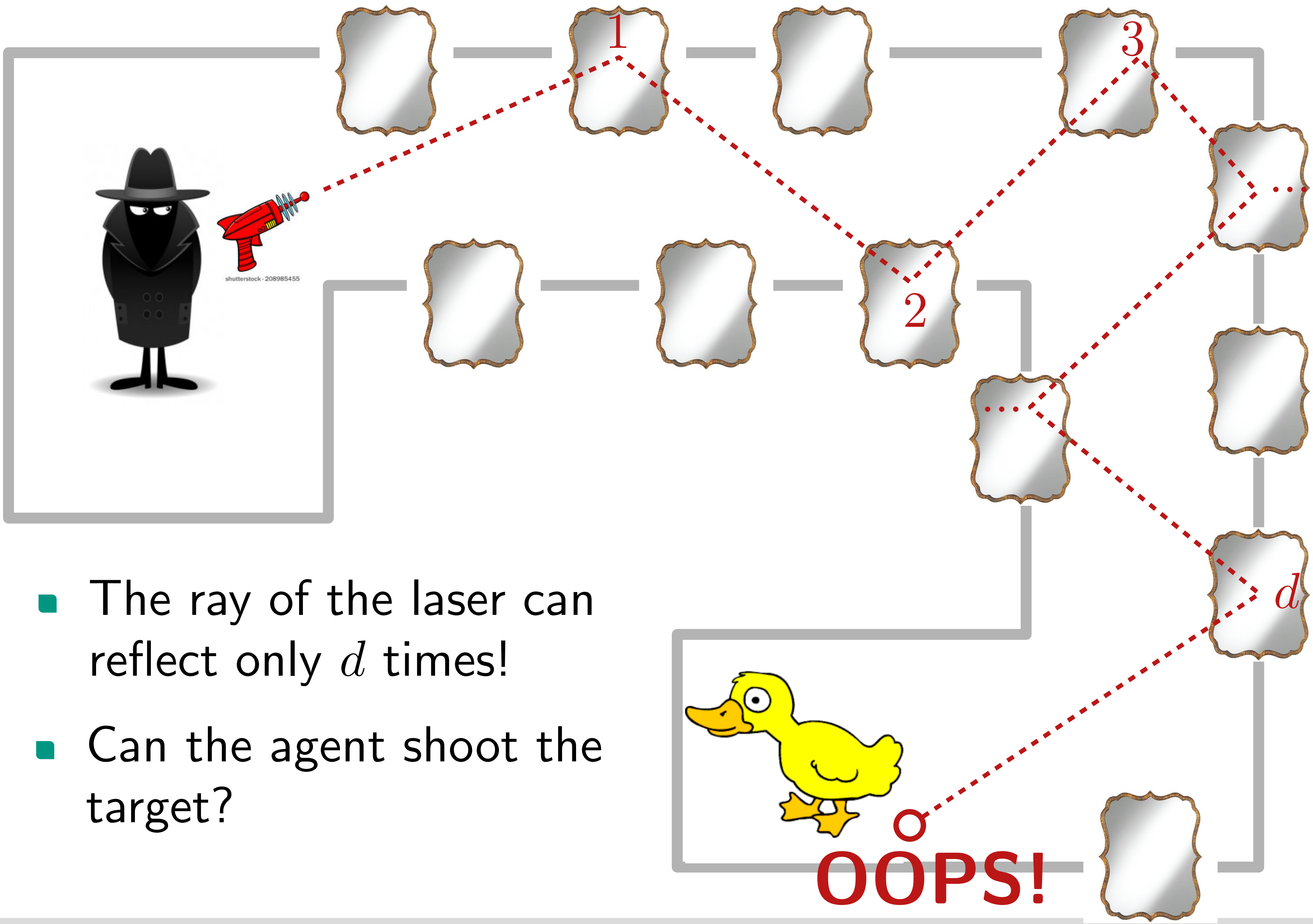
Ghosts smelling Pacman follow him until they lose the track.

Fast smell check:

Can Packman be smelled from point p?

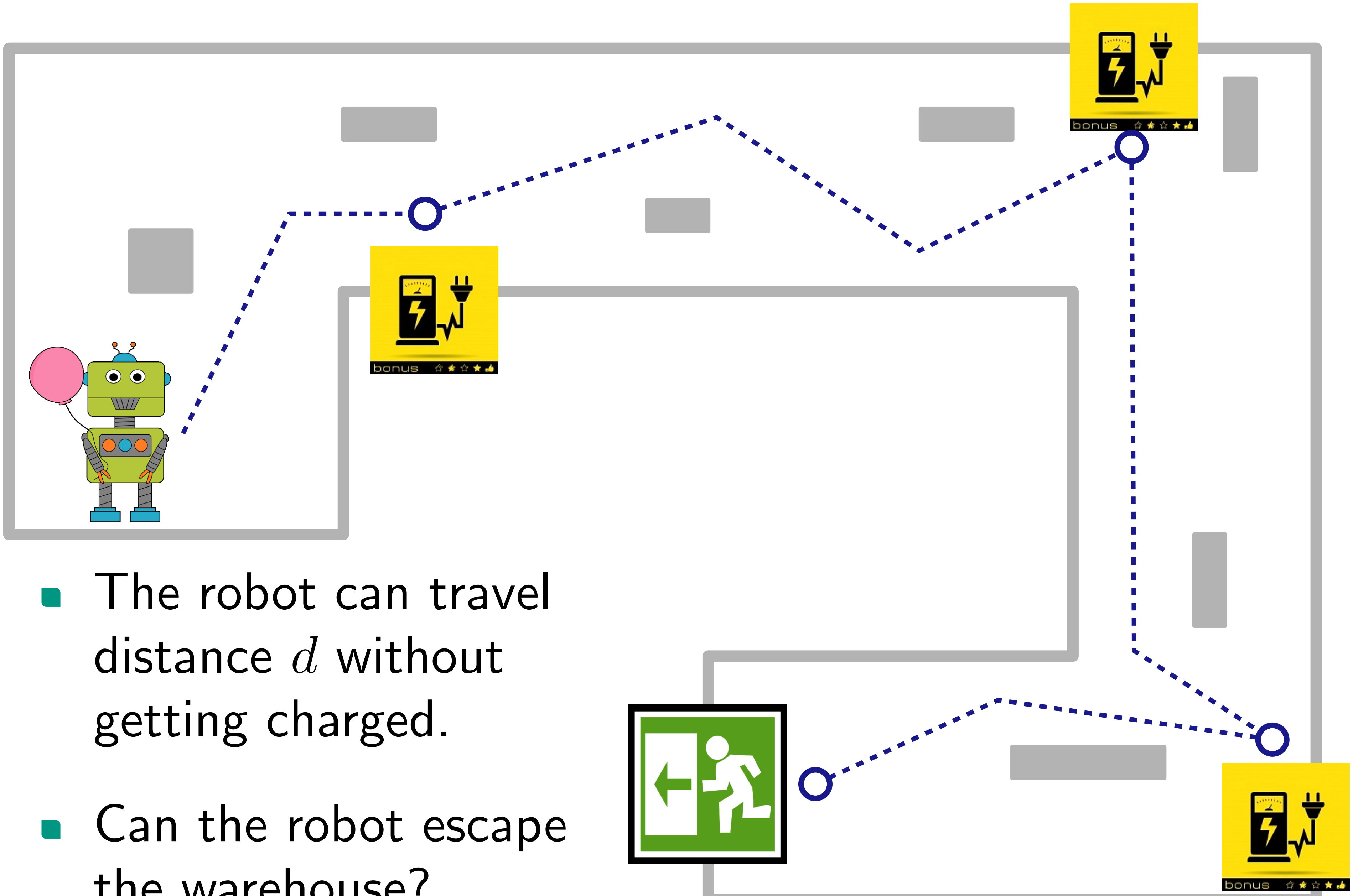
Secret Agent

Shooting Secret Agent



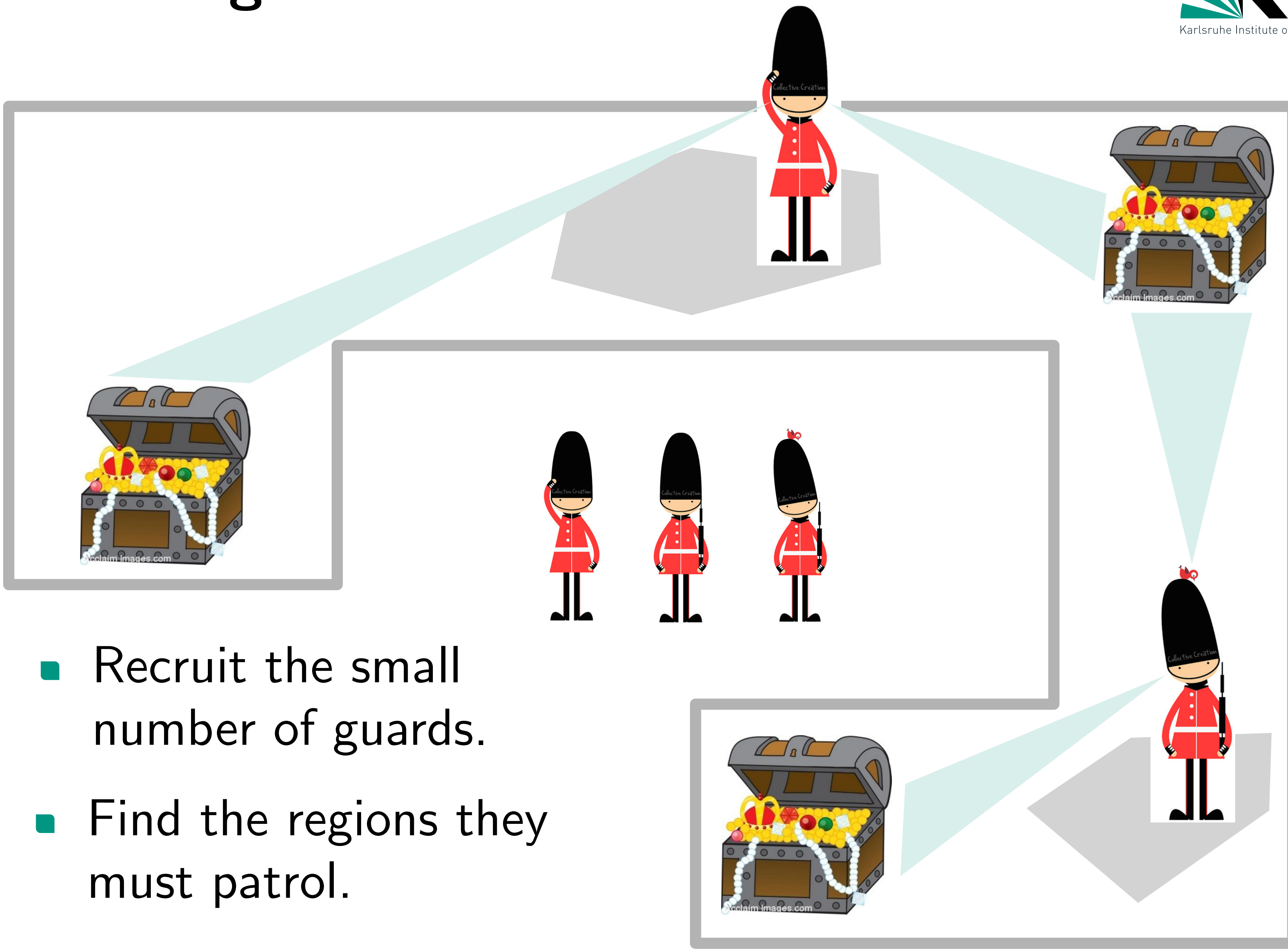
- The ray of the laser can reflect only d times!
- Can the agent shoot the target?

Secret Agent's Robot

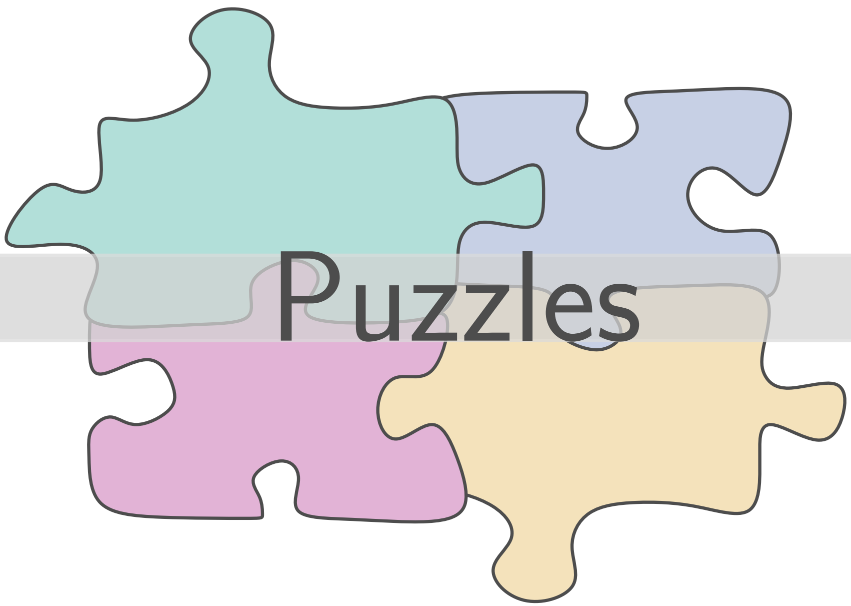


- The robot can travel distance d without getting charged.
- Can the robot escape the warehouse?

Secret Agent Protects Jewels

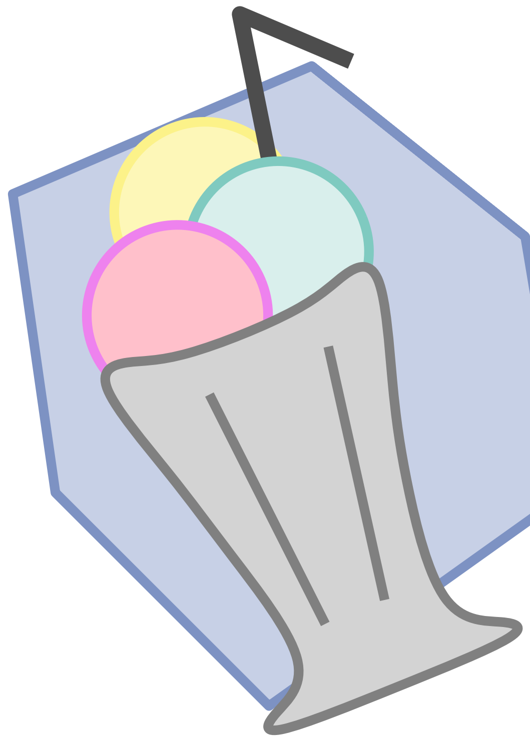


- Recruit the small number of guards.
- Find the regions they must patrol.



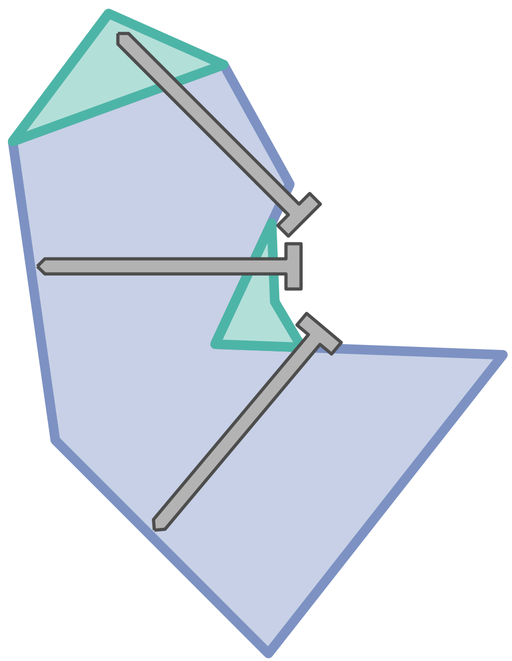
Puzzles

Puzzle # 1: Food fit



Does not fit!

Puzzle # 2: Protect the colony



Given:

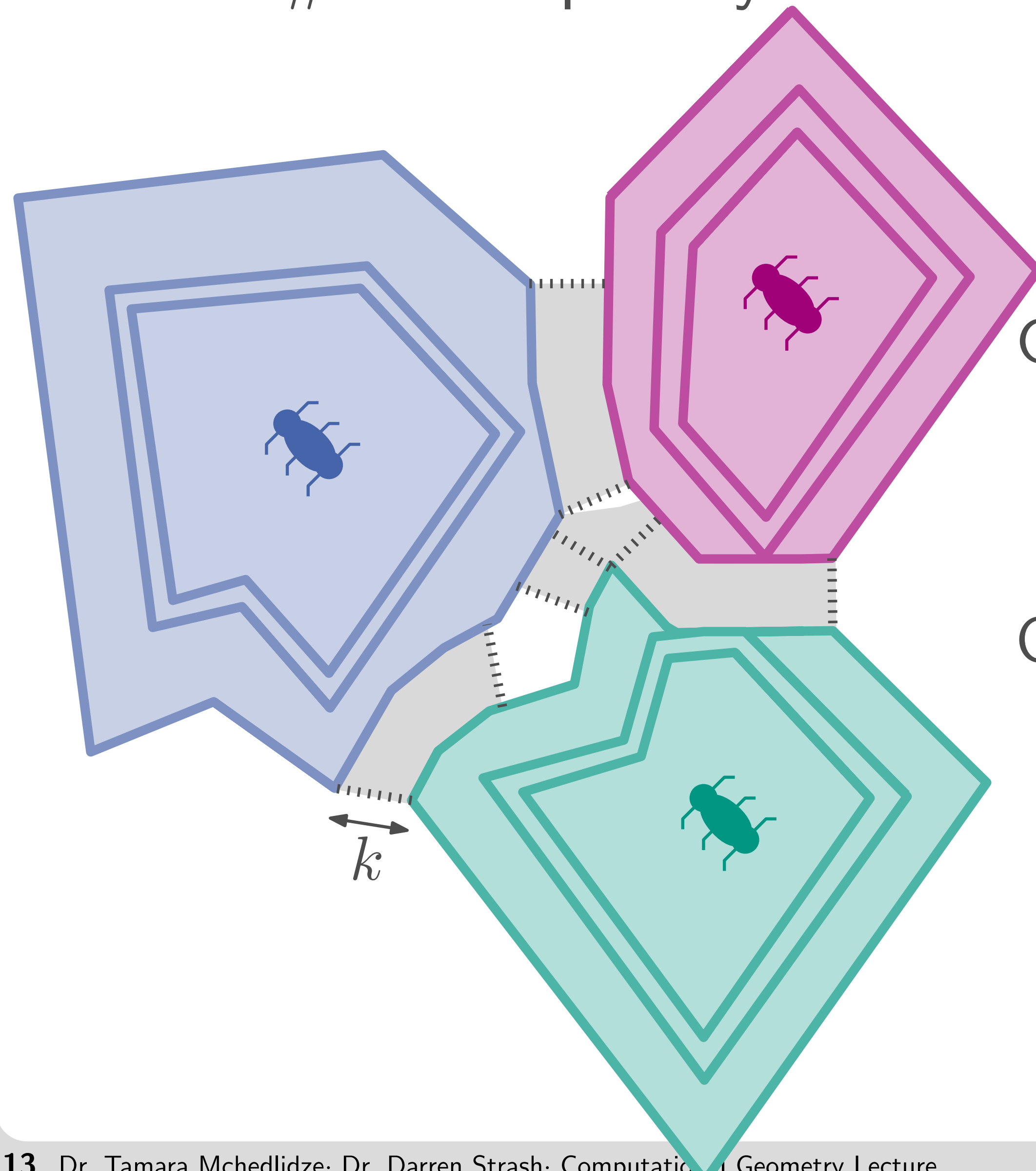
- A region representing an ant colony's home
- A nail of length l, to be hammered into the home

Output:

- Will the nail break the home apart?
- How can we make "small" changes to protect the home?

Puzzles

Puzzle # 3: Keep away



Given:

- A collection of ant colonies that grow over time
- A keep-away distance k

Output:

- The state of the colonies after t time steps

Project Selection

Projects

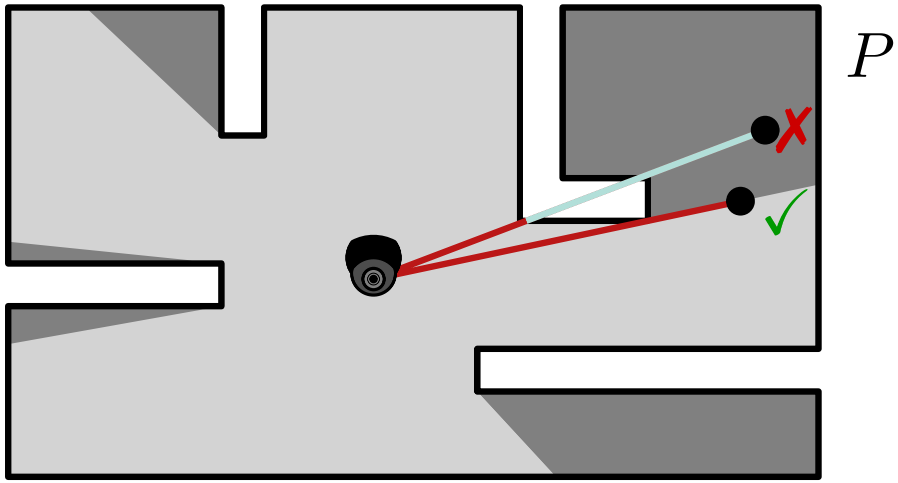
Karlsruhe Institute of Technology

1. **Pacman:** Collision detection
2. **Pacman:** Ghosts activate on sight
3. **Pacman:** Ghosts activate on smell
4. **Secret Agent:** Shoot the laser
5. **Secret Agent:** Save the robot
6. **Secret Agent:** Guard the jewels
7. **Puzzle:** Food fit
8. **Puzzle:** Protect the colony
9. **Puzzle:** Keep away

Polygon Triangulation

The Art-Gallery-Problem

Task: Install a number of cameras in an art gallery so that every part of the galery is visible to at least one of them.



Assumption: Art gallery is a *simple* polygon P with n corners (no self-intersections, no holes)

Observation: each camera observes a star-shaped region

Definition: Point $p \in P$ is *visible* from $c \in P$ if $\overline{cp} \in P$

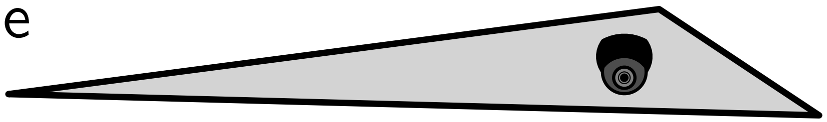
Goal: Use as few cameras as possible!

NP-hard!

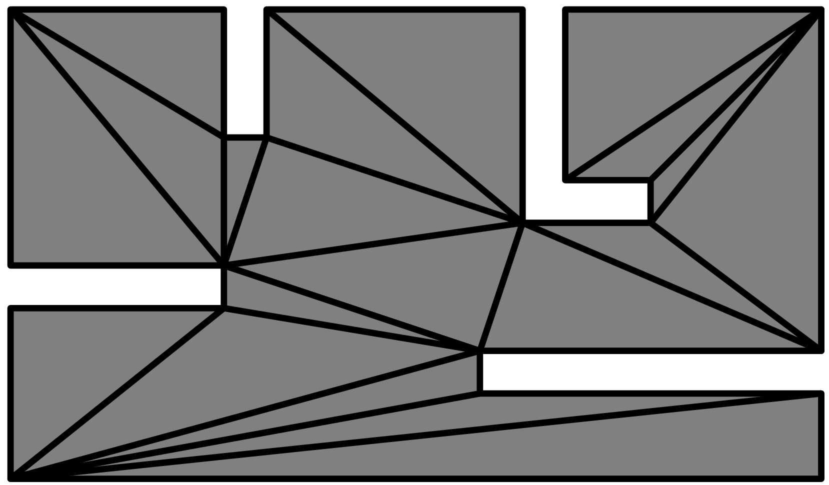
$\rightarrow$ The number depends on the number of corners n and on the shape of P

Problem Simplification

Observation: It is easy to guard a triangle



Idea: Decompose P into triangles and guard each of them

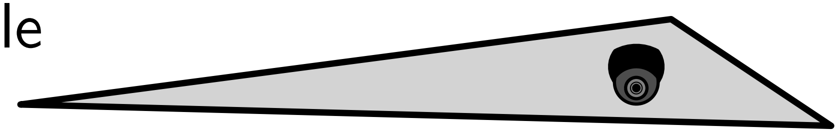


Theorem 1: Each simple polygon with n corners admits a triangulation; any such triangulation contains exactly $n-2$ triangles.

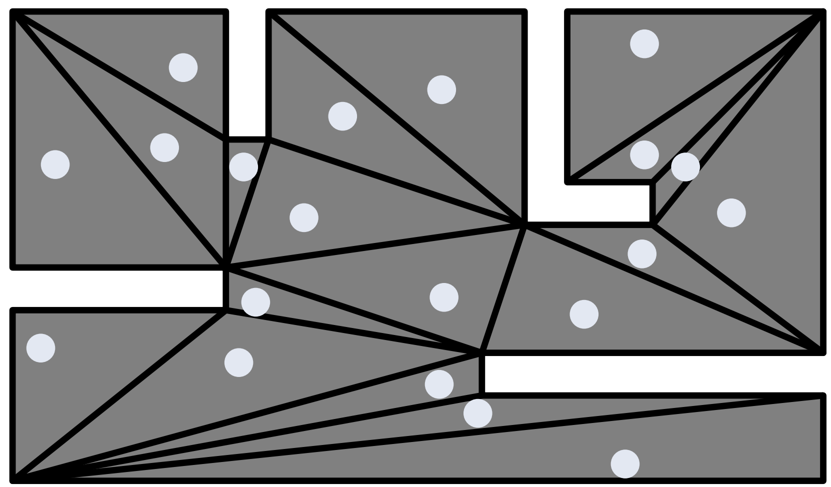
The proof implies a recursive $O(n^2)$-Algorithm!

Problem Simplification

Observation: It is easy to guard a triangle



Idea: Decompose P into triangles and guard each of them

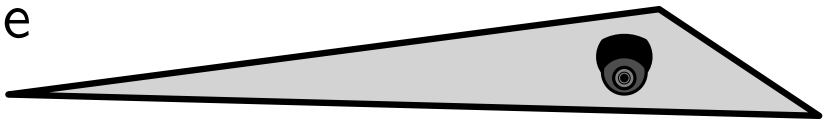


Theorem 1: Each simple polygon with n corners admits a triangulation; any such triangulation contains exactly $n-2$ triangles.

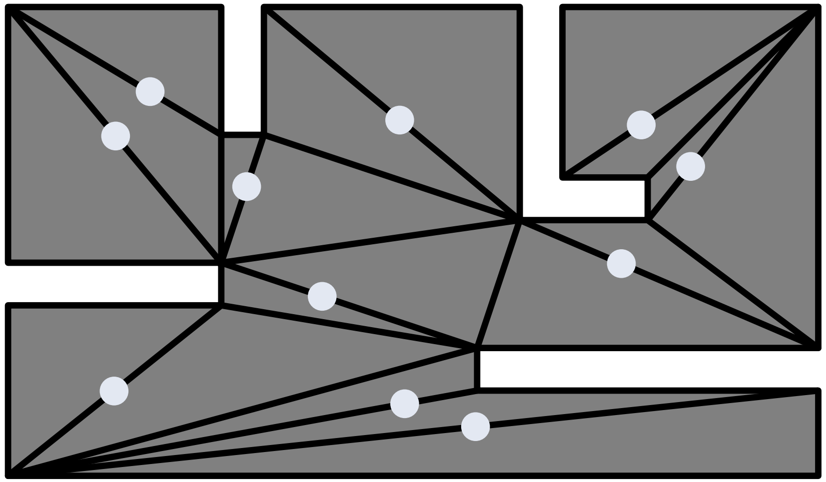
- P could be guarded by $n-2$ cameras placed in the triangles

Problem Simplification

Observation: It is easy to guard a triangle



Idea: Decompose P into triangles and guard each of them

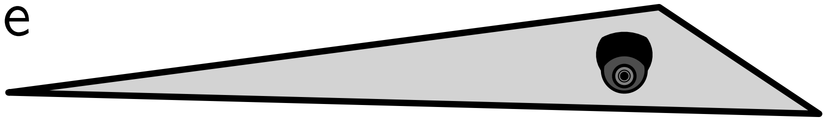


Theorem 1: Each simple polygon with n corners admits a triangulation; any such triangulation contains exactly $n-2$ triangles.

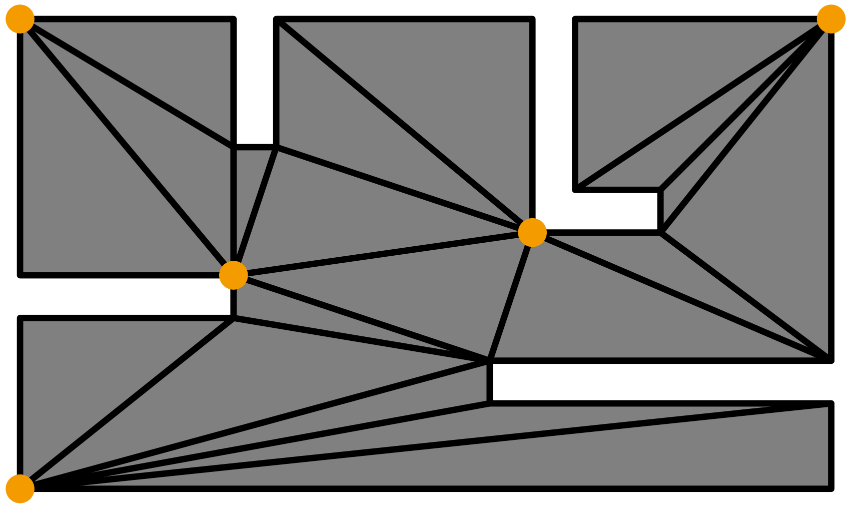
- P could be guarded by $n-2$ cameras placed in the triangles
- P can be guarded by $\approx n/2$ cameras placed on the diagonals

Problem Simplification

Observation: It is easy to guard a triangle



Idea: Decompose P into triangles and guard each of them



Theorem 1: Each simple polygon with n corners admits a triangulation; any such triangulation contains exactly $n-2$ triangles.

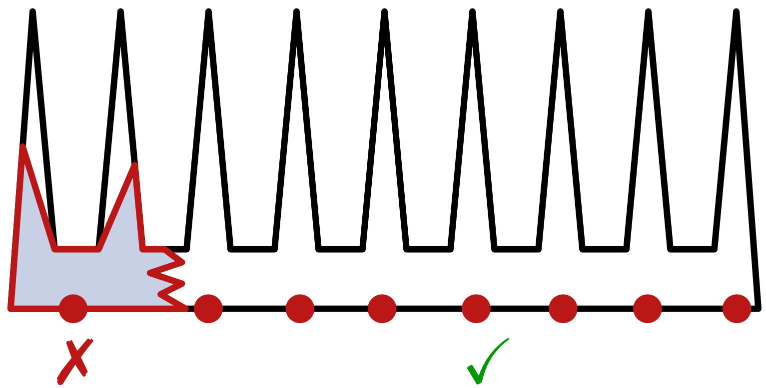
- P could be guarded by $n-2$ cameras placed in the triangles
- P can be guarded by $\approx n/2$ cameras placed on the diagonals
- P can be observed by even smaller number of cameras placed on the corners

The Art-Gallery-Theorem [Chvátal '75]

Theorem 2: For a simple polygon with n vertices, $\lfloor n/3 \rfloor$ cameras are sometimes necessary and always sufficient to guard it.

Proof:

- Find a simple polygon with n corners that requires $\approx n/3$ cameras!



- Sufficiency on the board

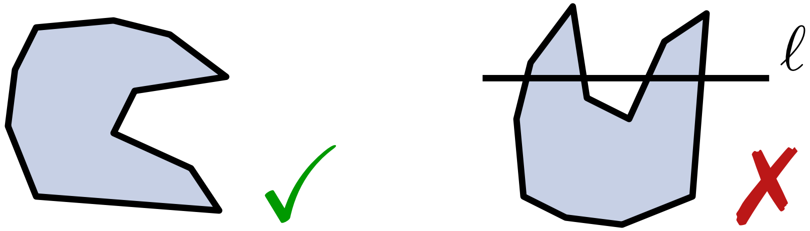
Conclusion: Given a triangulation, the $\lfloor n/3 \rfloor$ cameras that guard the polygon can be placed in $O(n)$ time.

Proof of Art-Gallery-Theorem: Overview

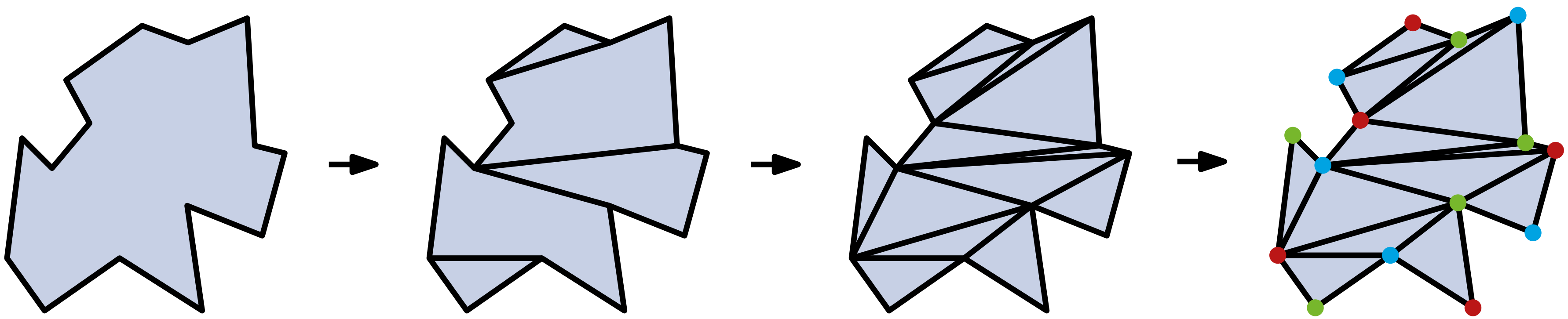
3-step process:

- Step 1: Decompose P into y-monotone polygons

  Definition: A polygon is y-monotone, if for any horizontal line ℓ, the interection $\ell \cap P$ is connected.

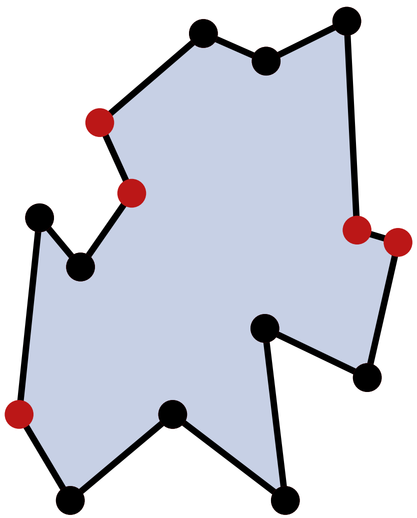


- Step 2: Triangulate the resulting y-monotone polygons
- Step 3: use DFS to color the vertices of the polygon



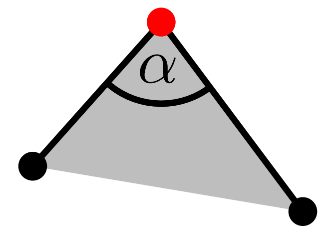
Partition into y-monotone Polygons

Idea: Five different types of vertices



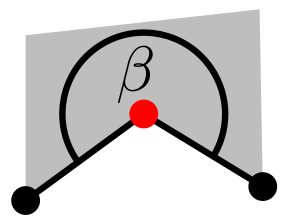
– *Turn vertices*:
vertical change in direction

- *start* vertices



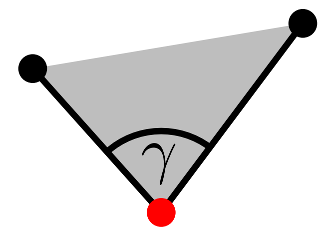
if $\alpha < 180°$

- *split* vertices



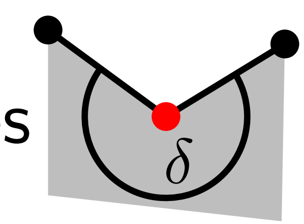
if $\beta > 180°$

- *end* vertices



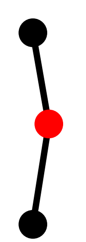
if $\gamma < 180°$

- *merge* vertices



if $\delta > 180°$

– *regular* vertices

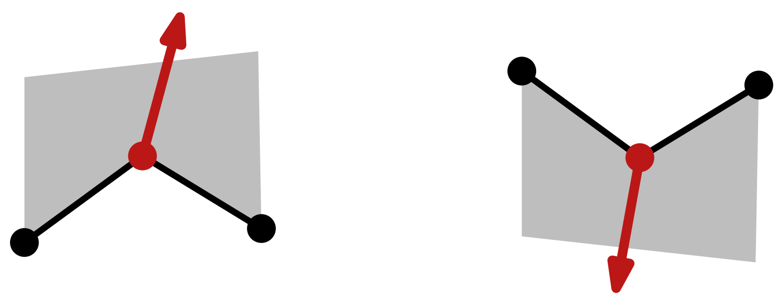


Characterization

Lemma 1: A polygon is y-monotone if it has no split vertices or merge vertices.

Proof: On the blackboard

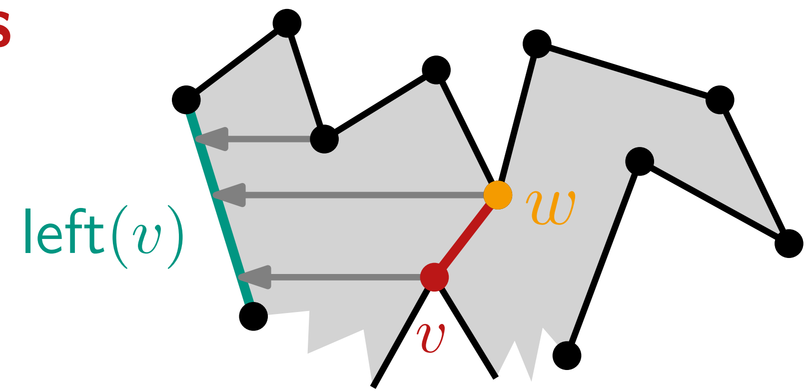
$\Rightarrow$ We need to eliminate all split and merge vertices by using diagonals



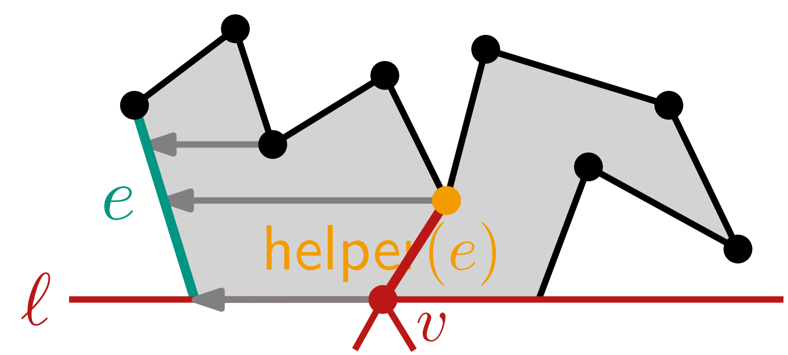
Observation: The diagonals should neither cross the edges of P nor the other diagonals

Ideas for Sweep-Line-Algorithm

1) Diagonals for the split vertices



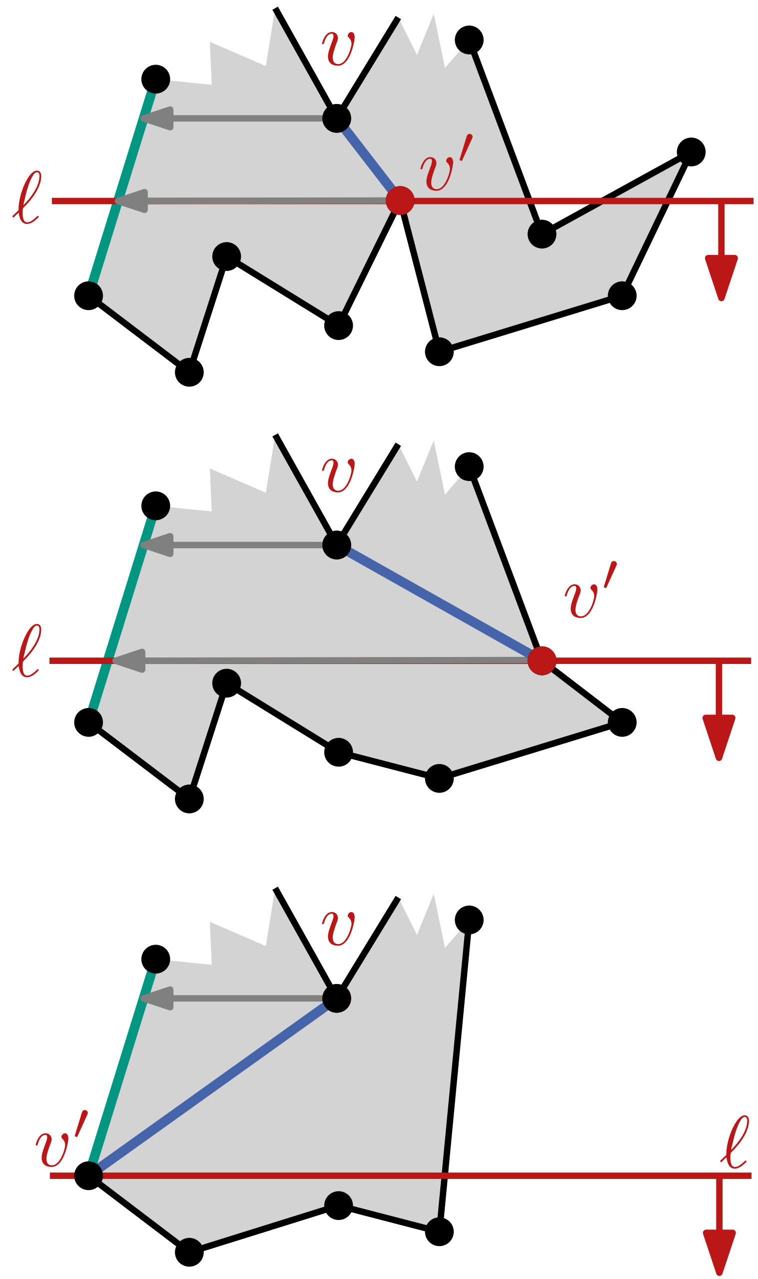
- compute for each vertex v its left adjacent edge left(v) with respect to the horizontal sweep line ℓ
- connect split vertex v to the nearest vertex w above v, such that left$(w) = $ left(v)
- for each edge e save the botommost vertex w such that left$(w) = e$; notation helper$(e) :=$ w
- when ℓ passes through a split vertex v, we connect v with helper(left(v))



Ideas for Sweep-Line-Algorithm

2) Diagonals for merge vertices

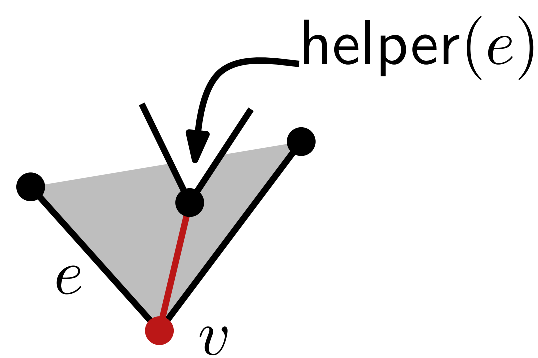
- when the vertex v is reached, we set helper(left(v)) $= v$
- when we reach a split vertex v' such that left(v') $=$ left(v) the diagonal (v, v') is introduced
- in case we reach a regular vertex v' such that helper(left(v')) is v the diagonal (v, v') is introduced
- if the end of v' of left(v) is reached, then the diagonal (v, v') is introduced



Algorithm MakeMonotone(P)

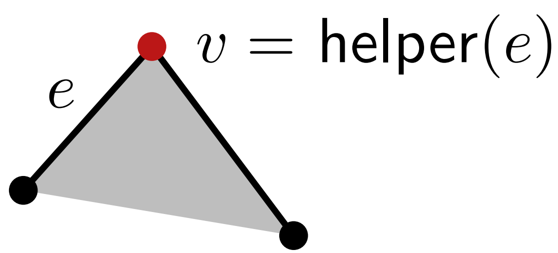
MakeMonotone(Polygon P)

$\mathcal{D} \leftarrow$ doubly-connected edge list for $(V(P), E(P))$
$\mathcal{Q} \leftarrow$ priority queue for $V(P)$ sorted lexicographically; $\mathcal{T} \leftarrow \emptyset$ (binary search tree for sweep-line status)
while $\mathcal{Q} \neq \emptyset$ **do**
  $v \leftarrow \mathcal{Q}$.nextVertex()
  $\mathcal{Q}$.deleteVertex(v)
  handleVertex(v)
return $\mathcal{D}$



handleStartVertex(vertex v)

$\mathcal{T} \leftarrow$ add the left edge e
helper(e) $\leftarrow v$



handleEndVertex(vertex v)

$e \leftarrow$ left edge
if isMergeVertex(helper(e)) **then**
  $\mathcal{D} \leftarrow$ add edge (helper(e), v)
remove e from $\mathcal{T}$

Algorithm MakeMonotone(P)

MakeMonotone(Polygon P)

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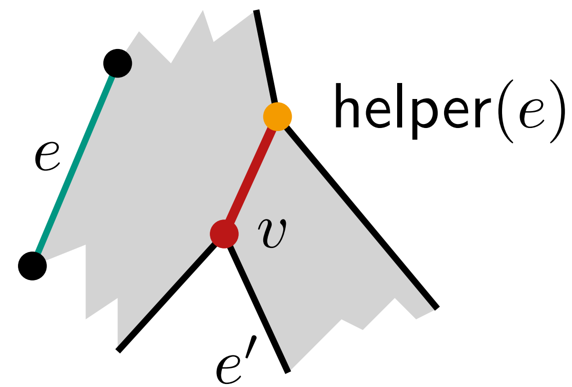
while $\mathcal{Q} \neq \emptyset$ **do**
    $v \leftarrow \mathcal{Q}$.nextVertex()
    $\mathcal{Q}$.deleteVertex(v)
    handleVertex(v)
return $\mathcal{D}$

handleSplitVertex(vertex v)

$e \leftarrow$ Edge to the left of v in $\mathcal{T}$
$\mathcal{D} \leftarrow$ add edge (helper(e), v)
helper(e) $\leftarrow v$
$\mathcal{T} \leftarrow$ add the right edge e' of v
helper(e') $\leftarrow v$



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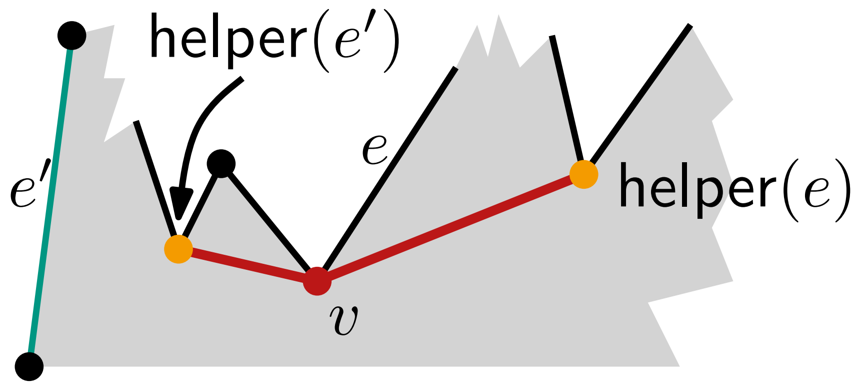


handleMergeVertex(vertex v)

$e \leftarrow$ right edge
if isMergeVertex(helper(e)) **then**
    $\mathcal{D} \leftarrow$ add edge (helper(e), v)
remove e from $\mathcal{T}$
$e' \leftarrow$ edge to the left of v in $\mathcal{T}$
if isMergeVertex(helper(e')) **then**
    $\mathcal{D} \leftarrow$ add edge (helper(e'), v)
helper(e') $\leftarrow v$

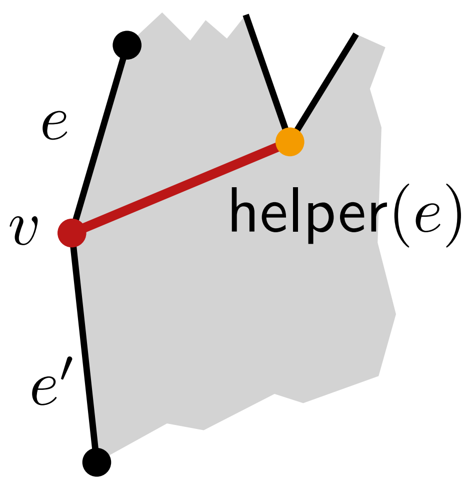
Algorithm MakeMonotone(P)

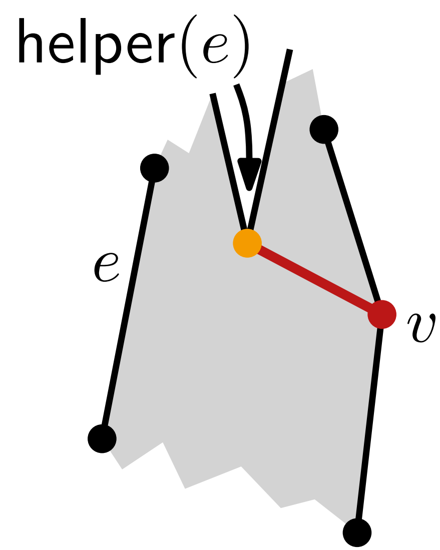
MakeMonotone(Polygon P)

$\mathcal{D} \leftarrow$ doubly-connected edge list for $(V(P), E(P))$
$\mathcal{Q} \leftarrow$ priority queue for $V(P)$ sorted lexicographically; $\mathcal{T} \leftarrow \emptyset$ (binary search tree for sweep-line status)
while $\mathcal{Q} \neq \emptyset$ **do**
  $v \leftarrow \mathcal{Q}$.nextVertex()
  $\mathcal{Q}$.deleteVertex(v)
  handleVertex(v)
return $\mathcal{D}$

handleRegularVertex(vertex v)

if P lies locally to the left of v **then**
  $e, e' \leftarrow$ above, below edge
  if isMergeVertex(helper(e)) **then**
    $\mathcal{D} \leftarrow$ add edge (helper(e), v)
  remove e from $\mathcal{T}$
  $\mathcal{T} \leftarrow$ add e'; helper(e') $\leftarrow v$
else
  $e \leftarrow$ edge to the left of v
  add e to $\mathcal{T}$
  if isMergeVertex(helper(e)) **then**
    $\mathcal{D} \leftarrow$ add (helper(e), v)
  helper(e) $\leftarrow v$





Analysis

Lemma 2: The algorithm MakeMonotone computes a set of crossing-free diagonals of P, which partitions P into y-monotone polygons.

Theorem 3: A simple polygon with n vertices can be partitioned into y-monotone polygons in $O(n \log n)$ time and $O(n)$ space.

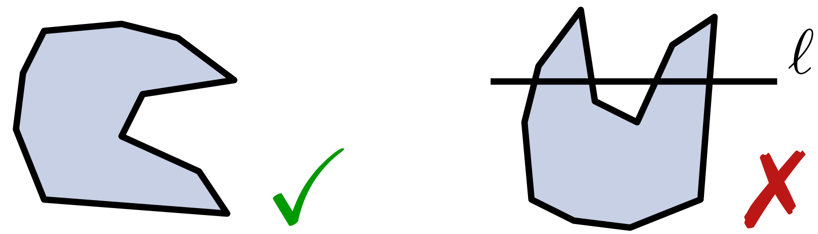
- Construct priority queue $\mathcal{Q}$: $O(n)$
- Initialize sweep-line status $\mathcal{T}$: $O(1)$
- Handle a single event: $O(\log n)$
  - $\mathcal{Q}$.deleteMax: $O(\log n)$
  - Find, remove, add element in $\mathcal{T}$: $O(\log n)$
  - Add diagonals to $\mathcal{D}$: $O(1)$
- Space: obviously $O(n)$

Proof of Art-Gallery-Theorem: Overview

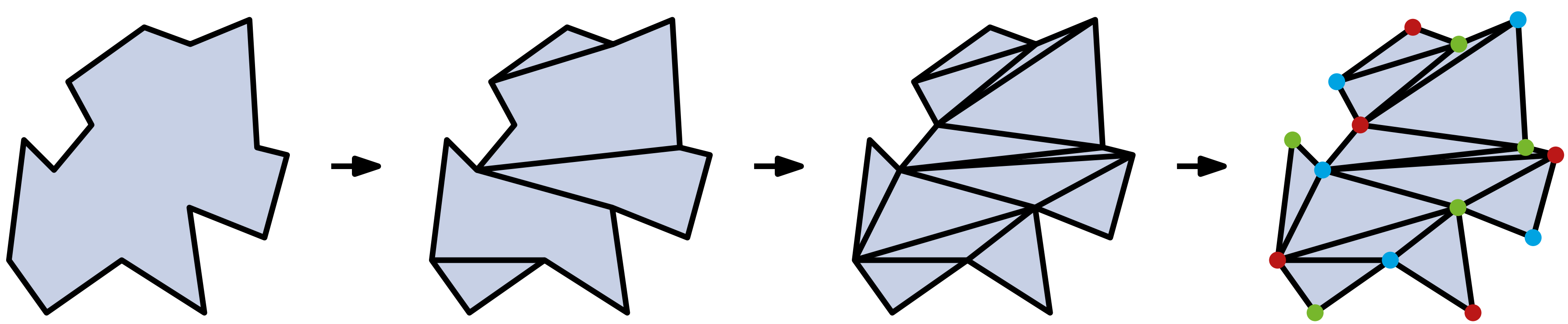
Three-step procedure:

- Step 1: Decompose P in y-monotone polygons ✓

  Definition: A polygon P is y-monotone, if for each horizontal line ℓ the intersection $\ell \cap P$ is connected.



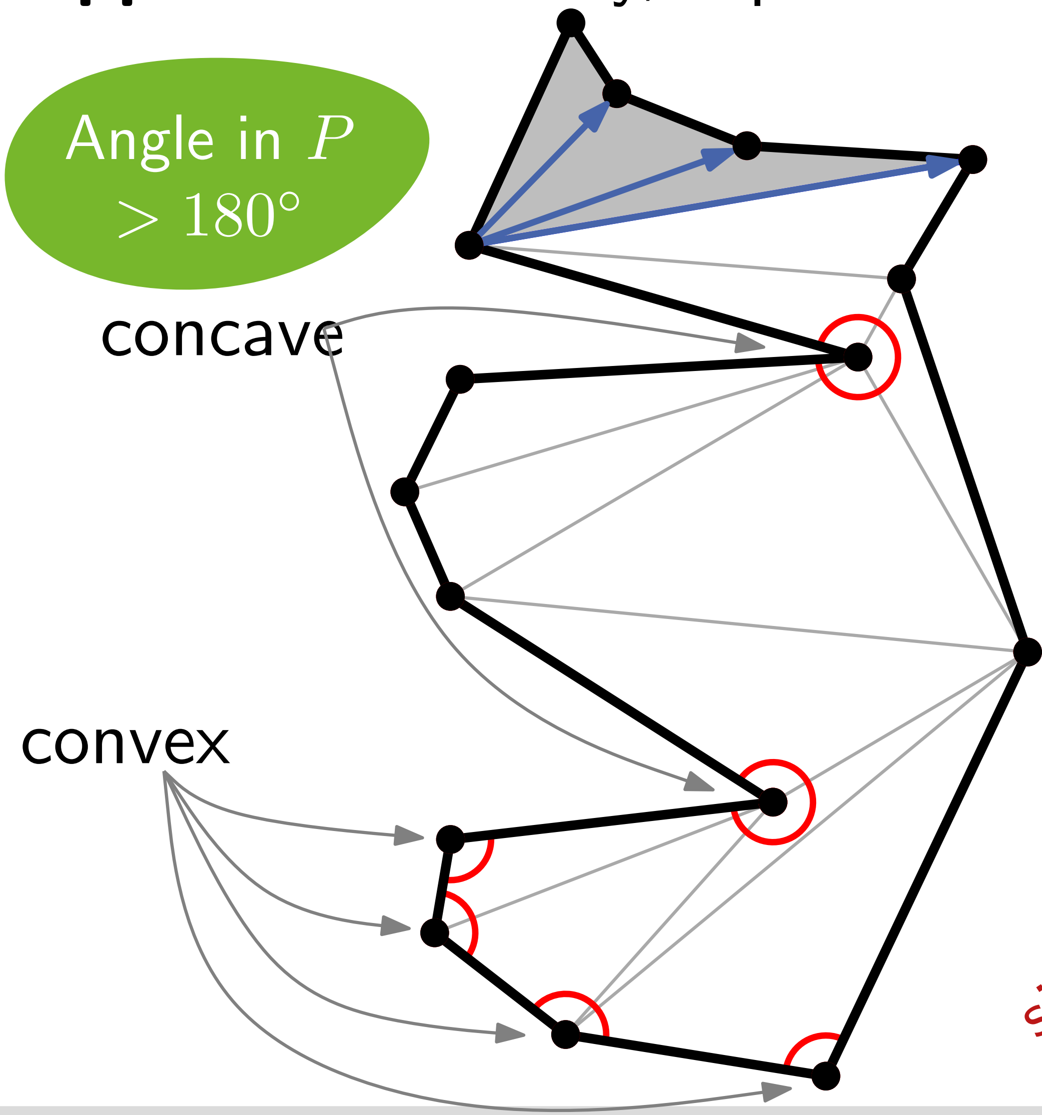
- Step 2: Triangulate y-monotone polygons **ToDo!**
- Step 3: use DFS to color the triangulated polygon ✓



Triangulate y-monotone Polygon

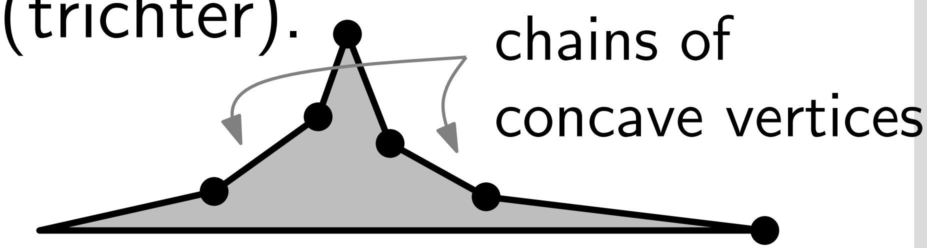
Reminder: The left and the right boundary of the polygon have decreasing y-coordinates

Approach: Greedy, top down traversal of both sides

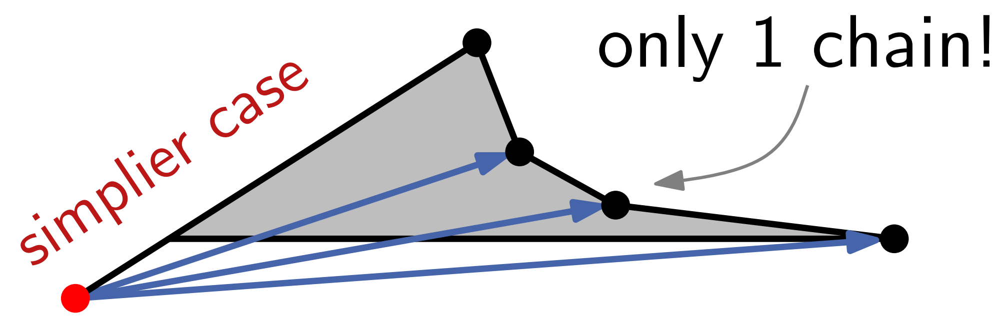


Invariant?

The already visited but not triangulated polygon has the shape of a *funnel* (trichter).



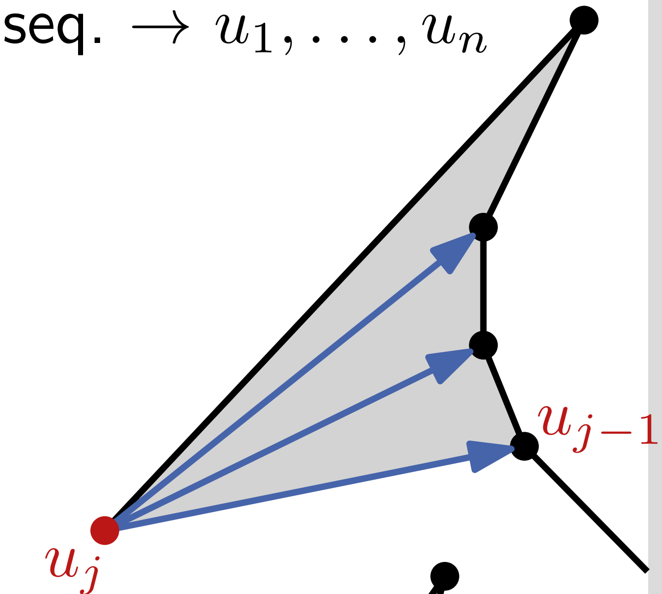
In our case:

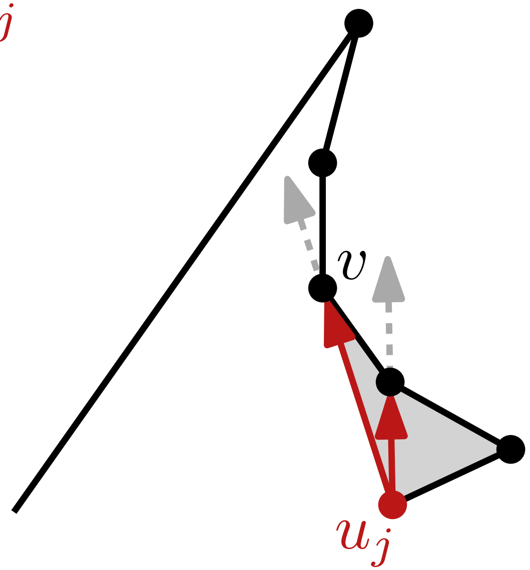


Algorithm TriangulateMonotonePolygon

TriangulateMonotonePolygon(Polygon P as doubly-connected list of edges)

```
Merge vertices on left and right chains into desc. seq. → u_1, ..., u_n
Stack S ← ∅; S.push(u_1); S.push(u_2)
for j ← 3 to n − 1 do
    if u_j and S.top() from different paths then
        while not S.empty() do
            v ← S.pop()
            if not S.empty() then draw (u_j, v)
        S.push(u_{j−1}); S.push(u_j)
    else
        v ← S.pop()
        while not S.empty() and u_j sees S.top() do
            v ← S.pop()
            draw diagonal (u_j, v)
        S.push(v); S.push(u_j)
Connect u_n to all the vertices in S (except for the first and the last)
```





Summary

Theorem 4: A y-monotone polygon with n vertices can be triangulated in $O(n)$ time.

Theorem 3: A simple polygon with n vertices can be partitioned into y-monotone polygons in $O(n \log n)$ time and $O(n)$ space.

recall

$\Downarrow$

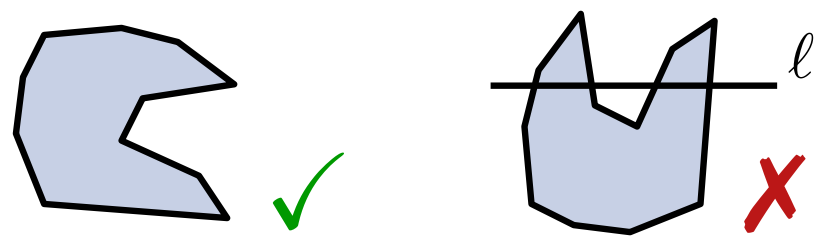
Theorem 5: A simple polygon with n vertices can be triangulated in $O(n \log n)$ time and $O(n)$ space.

Proof of Art-Gallery-Theorem: Overview

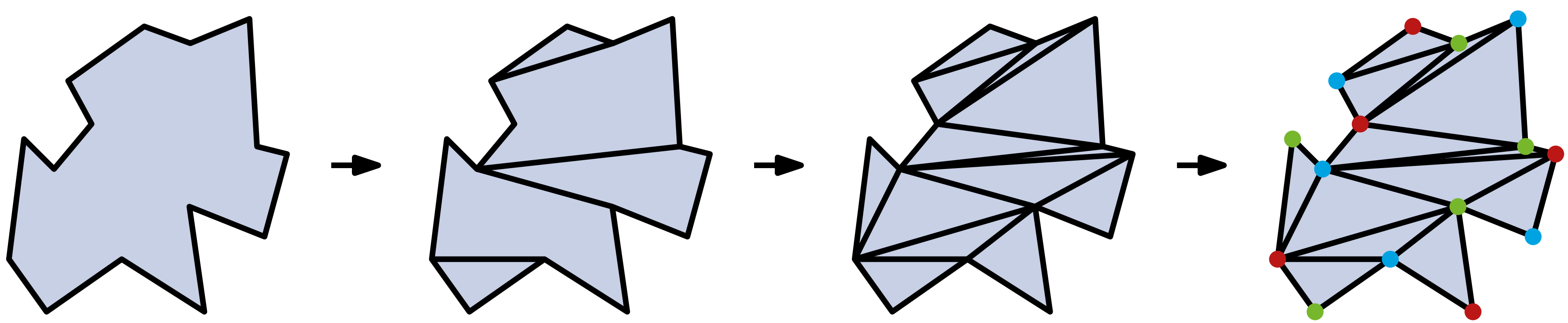
Three-step procedure:

- Step 1: Decompose P in y-monotone polygons ✓

  Definition: A polygon P is y-monotone, if for each horizontal line ℓ the intersection $\ell \cap P$ is connected.



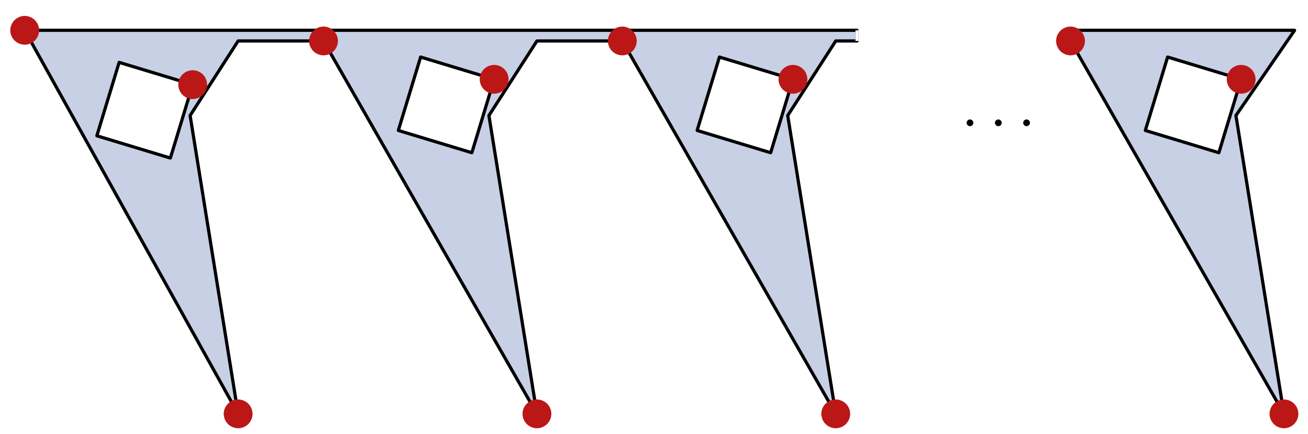
- Step 2: Triangulate y-monotone polygons ✓
- Step 3: use DFS to color the triangulated polygon ✓



Discussion

Can the triangulation algorithm be expanded to work with polygons with holes?

- Triangulation: yes
- But are $\lfloor n/3 \rfloor$ cameras still sufficient to guard it?
  No, a generalization of Art-Gallery-Theorems says that $\lfloor (n+h)/3 \rfloor$ cameras are sometimes necessary, and always sufficient, where h is the number of holes. [Hoffmann et al., 1991]



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Can we solve the triangulation problem faster for simple polygons?

Yes. The question whether it is possible was open for more than a decade. In the end of 80's a faster randomized algorithm was given, and in 1990 Chazelle presented a deterministic linear-time algorithm (complicated).