# Computational Geometry • Lecture

## Linear Programming

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# Profit optimization

- You are the boss of a company, that produces two products $P_1$ und $P_2$ from three raw materials $R_1, R_2$ und $R_3$.
- Let's assume you produce $x_1$ items of the product $P_1$ and $x_2$ items of product $P_2$.
- Assume that items $P_1$, $P_2$ get profit of 300€ and 500€, respectively. Then the total profit is:

$$G(x_1, x_2) = 300x_1 + 500x_2$$

- Assume that the amout of raw material you need for $P_1$ and $P_2$ is:

$$\begin{aligned} P_1 &: \quad 4R_1 + R_2 \\ P_2 &: \quad 11R_1 + R_2 + R_3 \end{aligned}$$

- And in your warehouse there are $880R_1$, $150R_2$ and $60R_3$. So:

$$\begin{aligned} R_1 &: \quad 4x_1 + 11x_2 \leq 880 \\ R_2 &: \quad x_1 + x_2 \leq 150 \\ R_3 &: \quad x_2 \leq 60 \end{aligned}$$

- Which choice for $(x_1, x_2)$ maximizes your profit?

# Solution

*Linear constraints:*

$R_1: \quad 4x_1 + 11x_2 \leq 880$

$R_2: \quad x_1 + x_2 \leq 150$

$R_3: \quad x_2 \leq 60$

$x_1 \geq 0$

$x_2 \geq 0$

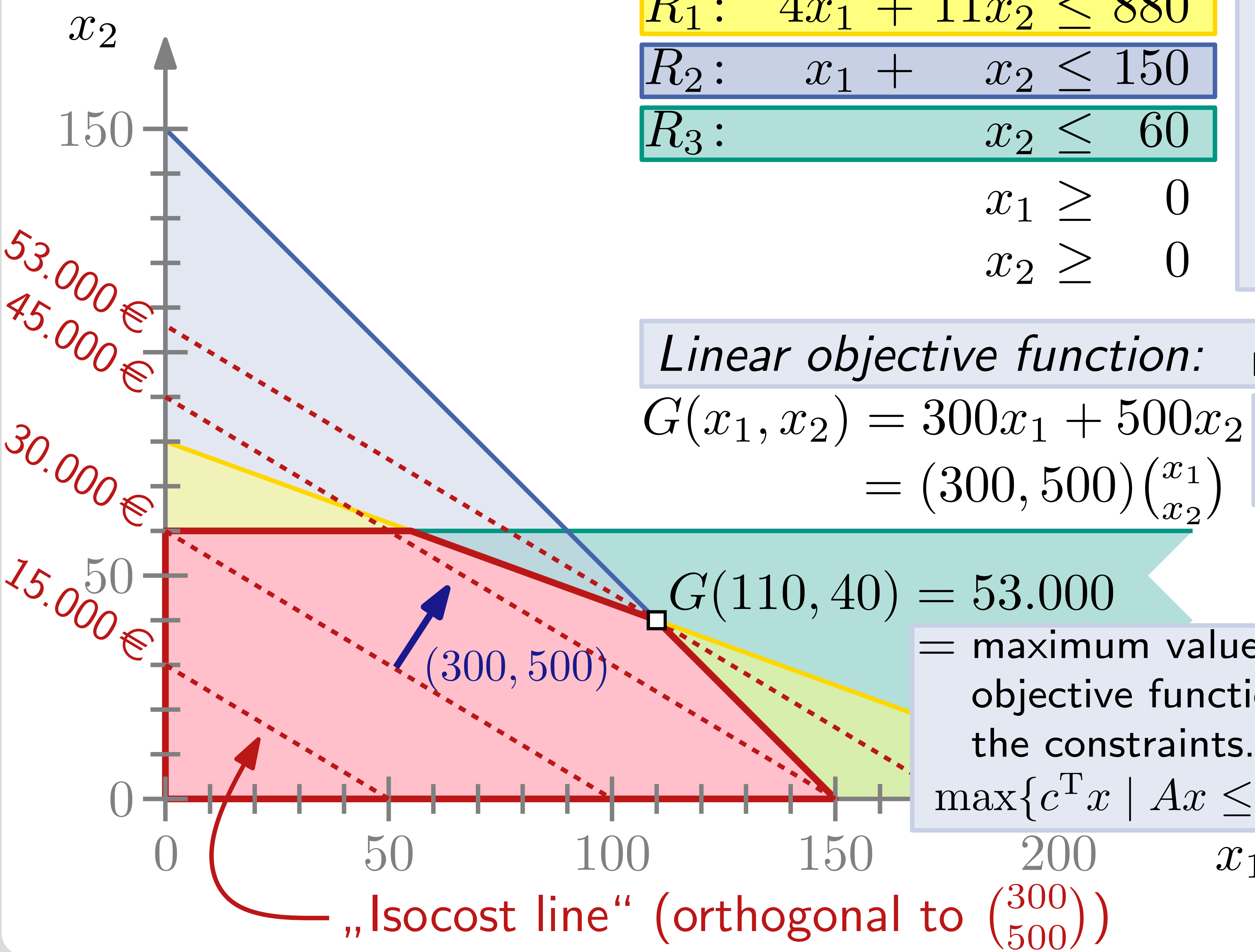
$Ax \leq b$

$x \geq 0$

*Linear objective function:* max $c^{\mathrm{T}}x$

$$G(x_1, x_2) = 300x_1 + 500x_2 = (300, 500)\binom{x_1}{x_2}$$

*c - normal vector*




= maximum value of the objective function under the constraints.

$\max\{c^{\mathrm{T}}x \mid Ax \leq b, x \geq 0\}$

# Linear programming

**Definition:** Given a set of linear constraints $H$ and a linear objective function $c$ in $\mathbb{R}^d$, a **linear program** (LP) is formulated as follows:

$$
\begin{array}{rrcl}
\text{maximize} & c_1x_1 + c_2x_2 + \cdots + c_dx_d & & \\
\text{under constr.} & a_{1,1}x_1 + \cdots + a_{1,d}x_d & \leq & b_1 \\
 & a_{2,1}x_1 + \cdots + a_{2,d}x_d & \leq & b_2 \\
 & & \vdots & \\
 & a_{n,1}x_1 + \cdots + a_{n,d}x_d & \leq & b_n
\end{array}
\left.\vphantom{\begin{array}{c}a\\a\\a\\a\end{array}}\right\} H
$$

- $H$ is a set of half-spaces in $\mathbb{R}^d$.
- We are searching for a point $x \in \bigcap_{h \in H} h$, that maximizes $c^T x$, i.e. $\max\{c^{\mathrm{T}}x \mid Ax \leq b, x \geq 0\}$.
- Linear programming is a central method in operations research.

# Algorithms for LPs

There are many algorithms to solve LPs:

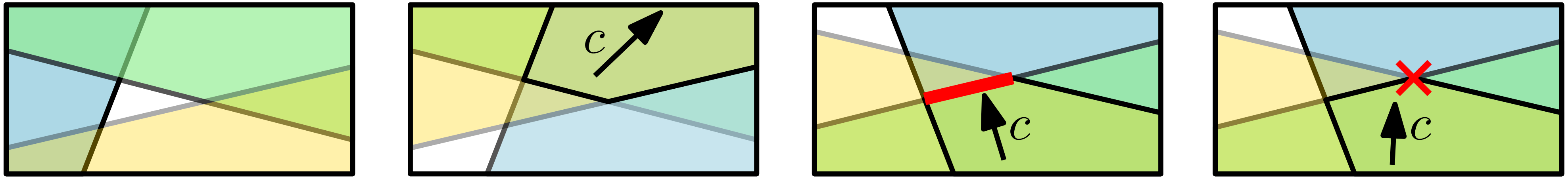
- Simplex-Algorithm [Dantzig, 1947]
- Ellipsoid-Method [Khatchiyan, 1979]
- Interior-Point-Method [Karmarkar, 1979]

They work well in practice, especially for large values of $n$ (number of constraints) and $d$ (number of variables).

**Today:** Special case $d = 2$

Possibilities for the solution space

feasible region $\bigcap H$ is bounded




$\bigcap H = \emptyset$ infeasible

$\bigcap H$ is unbounded in the direction $c$

solution is not unique

unique solution

# First approach

**Idea:** Compute the feasible region $\bigcap H$ and search for the angle $p$, that maximizes $c^T p$.

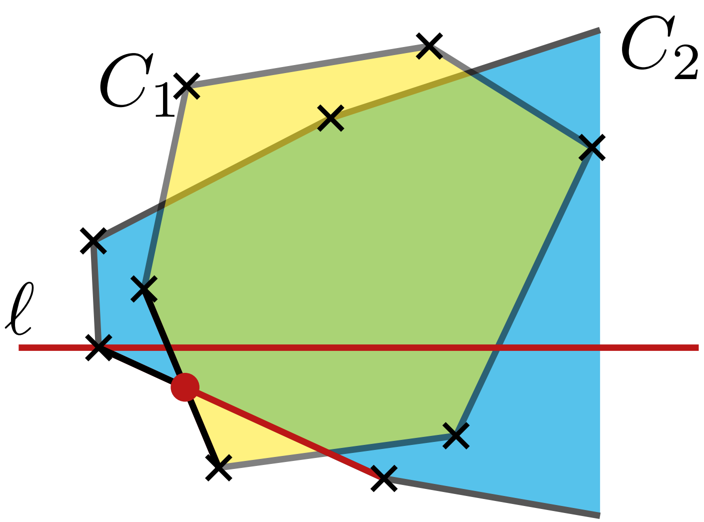
- The half-planes are convex
- Let's try a simple Divide-and-Conquer Algorithm

```
IntersectHalfplanes(H)
  if |H| = 1 then
  |  C ← H
  else
  |  (H_1, H_2) ← SplitInHalves(H)
  |  C_1 ← IntersectHalfplanes(H_1)
  |  C_2 ← IntersectHalfplanes(H_2)
  |  C ← IntersectConvexRegions(C_1, C_2)
  return C
```

# Intersect convex regions

IntersectConvexRegions$(C_1, C_2)$ can be implemented using a sweep line method:

- consider the left and the right boundaries of $C_1$ and $C_2$
- move the sweep line $\ell$ from top to bottom and save the crossed edges $(\leq 4)$
- The nodes of $C_1 \cup C_2$ define events. We process every event in $O(1)$ time, dependent on the type of the edges incident to the event vertex.




**Theorem 1:**
The intersection of two convex polygonal regions in the plane with $n_1 + n_2 = n$ nodes can be computed in $O(n)$ time.

# Running time of IntersectHalfplanes($H$)

IntersectHalfplanes($H$)

**if** $|H| = 1$ **then**
  $C \leftarrow H$
**else**
  $(H_1, H_2) \leftarrow$ SplitInHalves($H$)
  $C_1 \leftarrow$ IntersectHalfplanes($H_1$)
  $C_2 \leftarrow$ IntersectHalfplanes($H_2$)
  $C \leftarrow$ IntersectConvexRegions($C_1, C_2$)
**return** $C$

**Task:** What is the running time of IntersectHalfplanes($H$)?

Recursive formula

$$T(n) = \begin{cases} O(1) & \text{when } n = 1 \\ O(n) + 2T(n/2) & \text{when } n > 1 \end{cases}$$

Master Theorem $\Rightarrow$
$T(n) \in O(n \log n)$

# Running time of IntersectHalfplanes($H$)

IntersectHalfplanes($H$)

**if** $|H| = 1$ **then**

- feasible region $\bigcap H$ can be found in $O(n \log n)$ time
- $\bigcap H$ has $O(n)$ nodes
- the node $p$ that maximizes $c^T p$ can therefore be found in $O(n \log n)$ time

**Task:** What is the running time of IntersectHalfplanes($H$)?

Recursive formula

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# Bounded LPs

**Idea:** Instead of computing the feasible region and then searching for the optimal angle, do this incrementally.

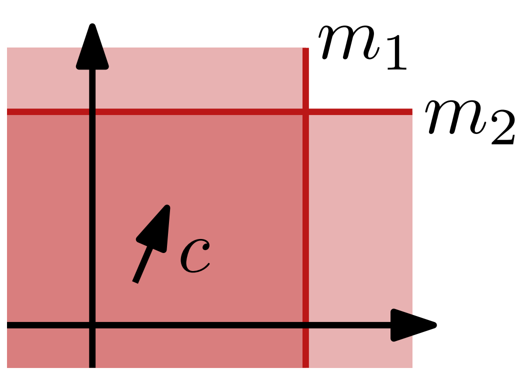
**Invariant:** Current best solution is a unique corner of the current feasible polygon

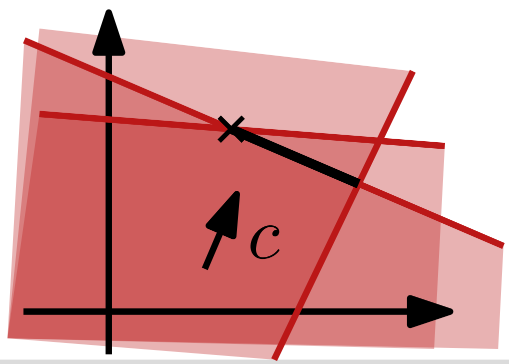
How to deal with the unbounded feasible regions?

When the optimal point is not unique, select lexicographically smallest one!

Define two half-planes for a big enough value $M$

$$m_1 = \begin{cases} x \leq M & \text{if } c_x > 0 \\ -x \leq M & \text{otherwise} \end{cases} \qquad m_2 = \begin{cases} y \leq M & \text{if } c_y > 0 \\ -y \leq M & \text{otherwise} \end{cases}$$



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Consider a LP $(H, c)$ with $H = \{h_1, \ldots, h_n\}$, $c = (c_x, c_y)$. We denote the first $i$ constraints by $H_i = \{m_1, m_2, h_1, \ldots, h_i\}$, and the feasible polygon defineed by them by $C_i = m_1 \cap m_2 \cap h_1 \cap \cdots \cap h_i$

# Properties

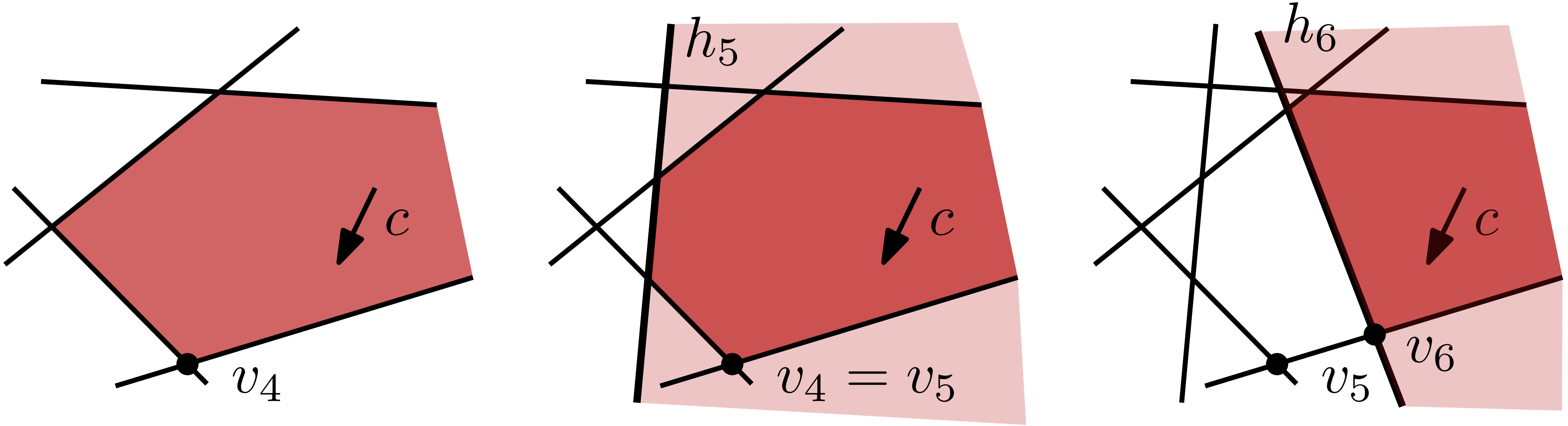
- each region $C_i$ has a single optimal angle $v_i$
- it holds that: $C_0 \supseteq C_1 \supseteq \cdots \supseteq C_n = C$

How the optimal angle $v_{i-1}$ changes when the half plane $h_i$ is added?

**Lemma 1:** For $1 \leq i \leq n$ and bounding line $\ell_i$ of $h_i$ holds that:

(i) If $v_{i-1} \in h_i$ then $v_i = v_{i-1}$,

(ii) otherwise, either $C_i = \emptyset$ or $v_i \in \ell_i$.



# One-dimentional LP

In case(ii) of Lemma 1, we search for the best point on the segment $\ell_i \cap C_{i-1}$:

- we parametrize $\ell_i : y = ax + b$
- define new objective function $f_c^i(x) = c^T \binom{x}{ax+b}$
- for $j \leq i-1$ let $\sigma_x(\ell_j, \ell_i)$ denote the $x$-coordinate of $\ell_j \cap \ell_i$

This gives us the following one-dimentional LP:

$$\text{maximize} \quad f_c^i(x) = c_x x + c_y(ax+b)$$

$$\text{with constr.} \quad \begin{array}{lll} x & \leq & \sigma_x(\ell_j, \ell_i) \quad \text{if } \ell_i \cap h_j \text{ is limited to the right} \\ x & \geq & \sigma_x(\ell_j, \ell_i) \quad \text{if } \ell_i \cap h_j \text{ is limited to the left} \end{array}$$

**Lemma 2:** A one-dimentional LP can be solved in linear time. In particular, in case (ii), one can compute the new angle $v_i$ or decide whether $C_i = \emptyset$ in $O(i)$ time.

# Incremental Algorithm

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2dBoundedLP$(H, c, m_1, m_2)$

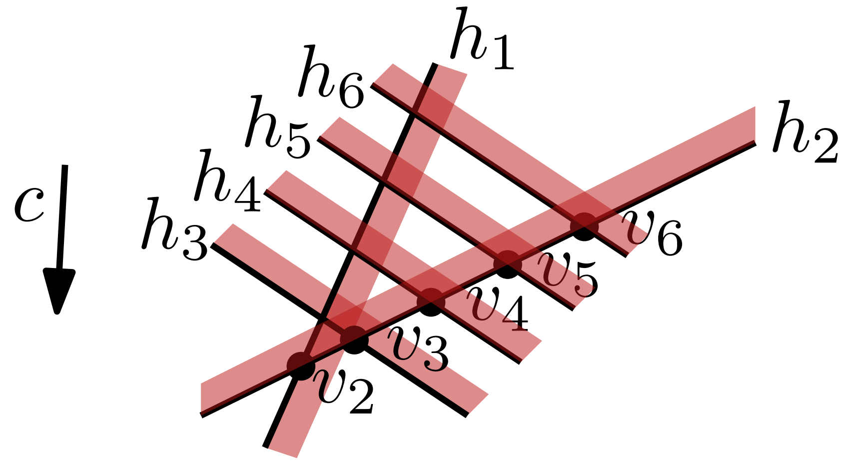
$C_0 \leftarrow m_1 \cap m_2$
$v_0 \leftarrow$ unique angle of $C_0$
**for** $i \leftarrow 1$ **to** $n$ **do**
  **if** $v_{i-1} \in h_i$ **then**
    $v_i \leftarrow v_{i-1}$   $O(1)$
  **else**
    $v_i \leftarrow$ 1dBoundedLP$(\sigma(H_{i-1}), f_c^i)$
    **if** $v_i =$ nil **then**   $O(i)$
      **return** infeasible
**return** $v_n$

worst-case running time:
$T(n) = \sum_{i=1}^{n} O(i) = O(n^2)$

**Lemma 3:** Algorithmus 2dBoundedLP needs $\Theta(n^2)$ time to solve an LP with $n$ contraints and 2 variables.

# What else can we do?

**Obs.:** It is not the half-planes $H$ that force the high running time, but the order in which we consider them.




How to find (quickly) a good ordering?

# Randomized incremental algorithm

2dRandomizedBoundedLP$(H, c, m_1, m_2)$

```
C_0 ← m_1 ∩ m_2
v_0 ← unique angle of C_0
H ← RandomPermutation(H)
for i ← 1 to n do
    if v_{i-1} ∈ h_i then
        v_i ← v_{i-1}
    else
        v_i ← 1dBoundedLP(σ(H_{i-1}), f_c^i)
        if v_i = nil then
            return infeasible
return v_n
```

# Random permutation

RandomPermutation($A$)

**Input**: Array $A[1 \dots n]$
**Output**: Array $A$, rearranged into a random permutation

**for** $k \leftarrow n$ **to** 2 **do**
  $r \leftarrow$ Random($k$)    Random num. between 1 and $k$
  exchange $A[r]$ and $A[k]$

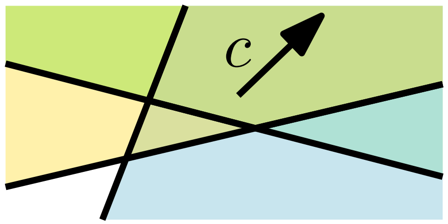
**Obs.:** The running time of 2dRandomizedBoundedLP depends on the random permutation computed by the procedure RandomPermutation. In the following we compute the **expected running time**.

**Theorem 2:** A two-dimentional LP with $n$ constraints can be solved in $O(n)$ randomized expected time.

# Unbounded LPs

**Till now:** Artificial contraints to bound $C$ by $m_1$ and $m_2$

**Next:** recongnize and deal with an unbonded LP




$\bigcap H$ unbounded in the direction $c$

**Def.:** A LP $(H, c)$ is called **unbounded**, if there exists a ray $\rho = \{p + \lambda d \mid \lambda > 0\}$ in $C = \bigcap H$, such that the value of the objective function $f_c$ becomes arbitrarily large along $\rho$.

It must be that:

- $\langle d, c \rangle > 0$
- $\langle d, \eta(h) \rangle \geq 0$ for all $h \in H$ where $\eta(h)$ is the **normal** vector of $h$ oriented towards the feasible side of $h$

# Characterization

**Lemma 4:** A LP $(H, c)$ is unbounded iff there is a vector $d \in \mathbb{R}^2$ such that

- $\langle d, c \rangle > 0$
- $\langle d, \eta(h) \rangle \geq 0$ for all $h \in H$
- LP $(H', c)$ with $H' = \{h \in H \mid \langle d, \eta(h) \rangle = 0\}$ is feasible.

Test whether $(H, c)$ is unbounded with a one-dimentional LP:

**Step 1:**

- rotate coordinate system till $c = (0, 1)$
- normalize vector $d$ with $\langle d, c \rangle > 0$ as $d = (d_x, 1)$
- For normal vector $\eta(h) = (\eta_x, \eta_y)$ it should hold that $\langle d, \eta(h) \rangle = d_x \eta_x + \eta_y \geq 0$
- Let $\bar{H} = \{d_x \eta_x + \eta_y \geq 0 | h \in H\}$
- Check whether this one-dim. LP $\bar{H}$ is feasible

# Test auf Unbeschränktheit

**Step 2:** If Step 1 returns a feasible solution $d_x^\star$

- compute $H' = \{h \in H \mid d_x^\star \eta_x(h) + \eta_y(h) = 0\}$
- Normals to $H'$ are orthogonal to $d = (d_x, 1) \Rightarrow$ lines bounding half-planes of $H'$ are parallel to $d$
- intersect the bounding lines of $H'$ with $x$-axis $\rightarrow$ 1d-LP

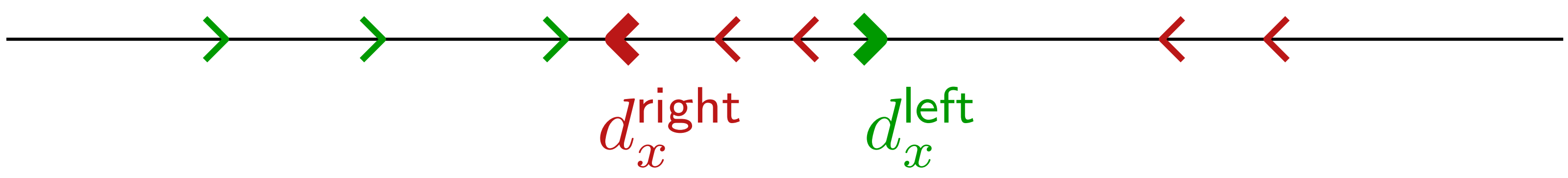
If the two steps result in a feasible solution, the LP $(H, c)$ is unbounded and we can construct the ray $\rho$.

If the LP $H'$ in Step 2 is infeasible, then then so is the initial LP $(H, c)$.

If the LP $\bar{H}$ of the Step 1 is infeasible, then by Lemma 4, $(H, c)$ is bounded.

# Certificates of boundness

**Obs.:** When the LP $\bar{H} = \{d_x\eta_x + \eta_y \geq 0 | h \in H\}$ of the Step 1 is infeasible, we can use this information further!




1d-LP $\bar{H}$ is infeasible $\Leftrightarrow$ the interval $[d_x^{\text{left}}, d_x^{\text{right}}] = \emptyset$

- let $h_1$ and $h_2$ be the half planes corresponding to $d_x^{\text{left}}$ and $d_x^{\text{right}}$
- There is no vector $d$ that would “satisfy” $h_1$ and $h_2$, thus
- the LP $(\{h_1, h_2\}, c)$ is already bounded
- $h_1$ and $h_2$ are **certificates** of the boundness
- use $h_1$ and $h_2$ in 2dRandomizedBoundedLP as $m_1$ and $m_2$

# Algorithms

2dRandomizedLP$(H, c)$

$\exists$? Vector $d$ with $\langle d, c\rangle > 0$ and $\langle d, \eta(h)\rangle \geq 0$ for all $h \in H$

**if** $d$ exists **then**

$H' \leftarrow \{h \in H \mid \langle d, \eta(h)\rangle = 0\}$

**if** $H'$ feasible **then**

**return** (ray $\rho$, unbounded)

**else**

**return** infeasible

**else**

$(h_1, h_2) \leftarrow$ Certificates for the boundness of $(H, c)$

$\overline{H} \leftarrow H \setminus \{h_1, h_2\}$

**return** 2dRandomizedBoundedLP$(\overline{H}, c, h_1, h_2)$

**Theorem 3:** A two-dimentional LP with $n$ constraints can be solved in $O(n)$ randomized expected time.

# Discussion

**Can the two-dimentional algorithms be generalized to more dimentions?**

Yes! The same way as the two-dimentional LP is solved incrementally with reduction to a one-dimentional LP, a $d$-dimentional LP can be solved by a randomized incremental algorithm with a reduction to $(d-1)$-dimentional LP. The expected running time is then $O(c^d d!\, n)$ for a constant $c$. The algorithm is therefore usefull only for small values on $d$.

The simple randomized incremental algorithm for two and more dimentions given in this lecture is due to Seidel (1991).