

Algorithms for Graph Visualization

Flow Methods: Orthogonal Layout

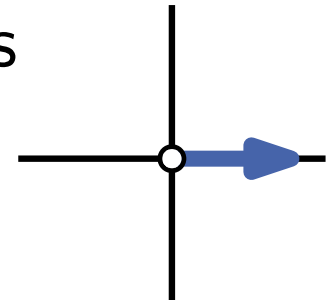
INSTITUT FÜR THEORETISCHE INFORMATIK · FAKULTÄT FÜR INFORMATIK

Tamara Mchedlidze
16.01.2017

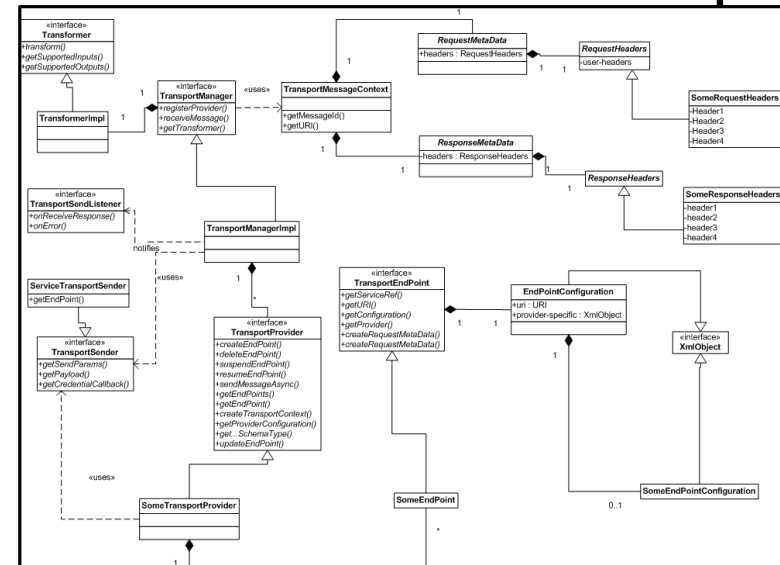
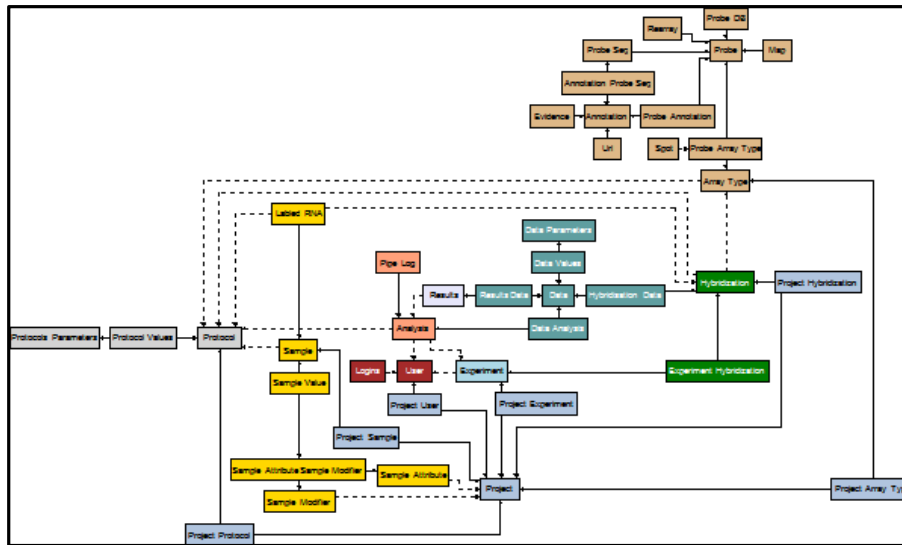


Orthogonal Layout

- Edges consist of vertical and horizontal segments
- Applied in many areas

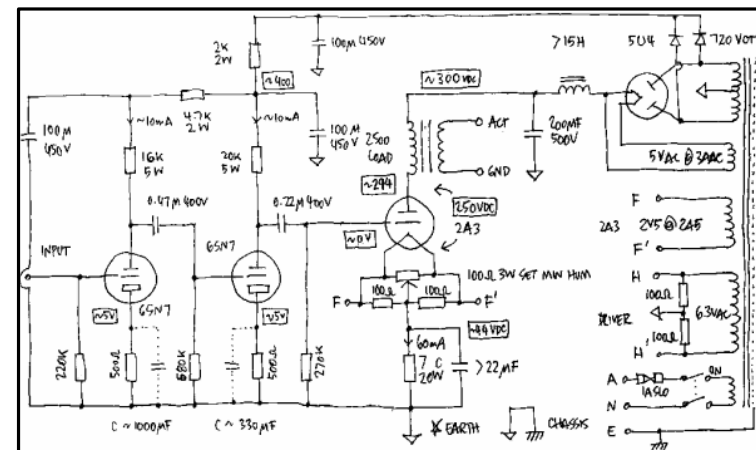
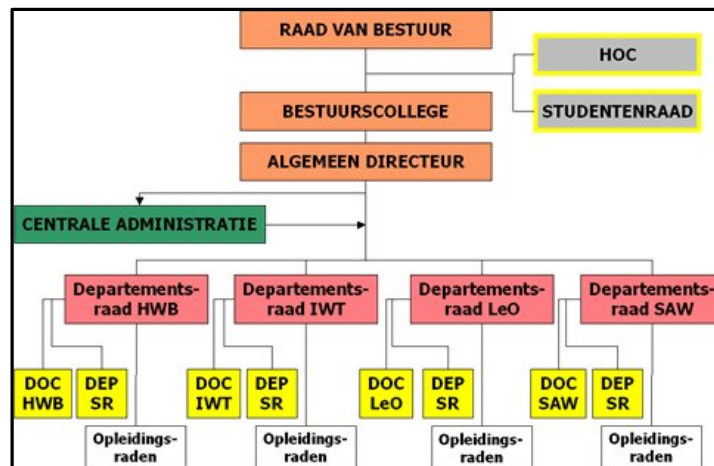


ER diagram in OGDF



UML diagram by Oracle

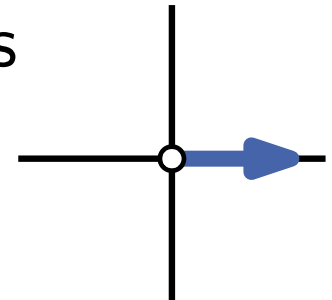
Organigram of HS Limburg



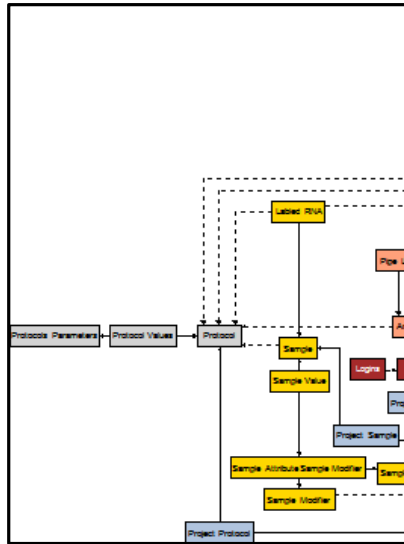
Circuit diagram by Jeff Atwood

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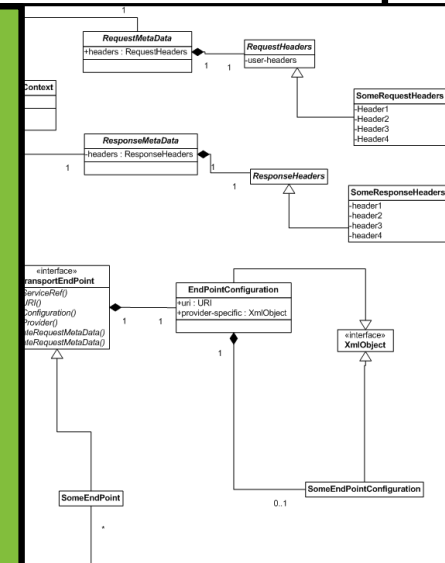
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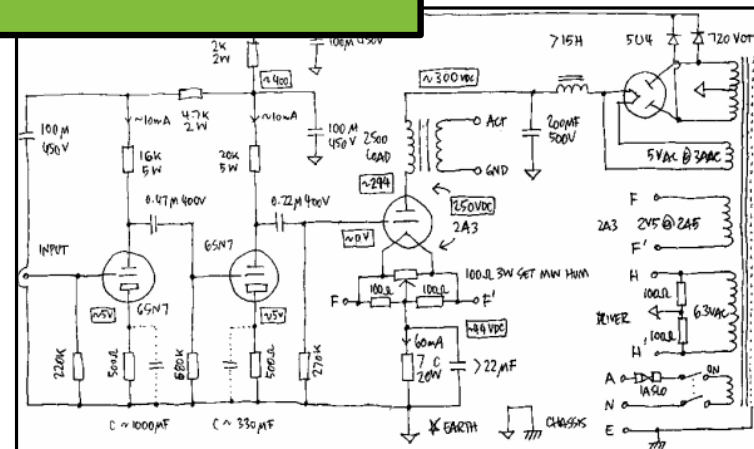
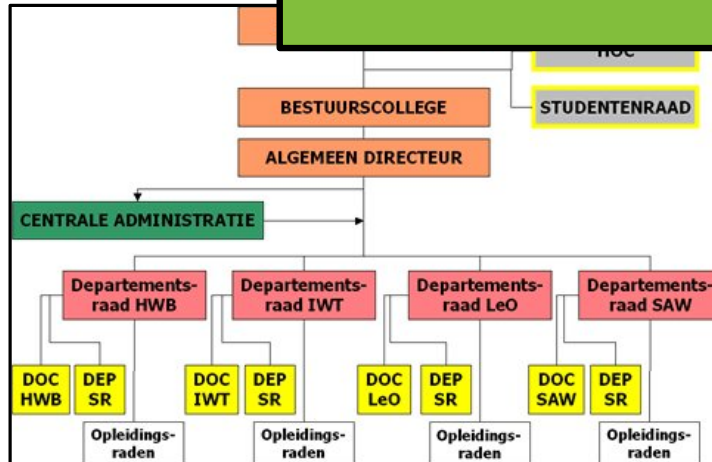
Aesthetic criteria:

- number of bends
- length of edges
- width, height, area
- monotonicity of edges
- ...

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(Planar) Orthogonal Drawings

Three-step approach: *Topology – Shape – Metrics*

[Tamassia SIAM J. Comput. 1987]

$$V = \{v_1, v_2, v_3, v_4\}$$

$$E = \{v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4\}$$

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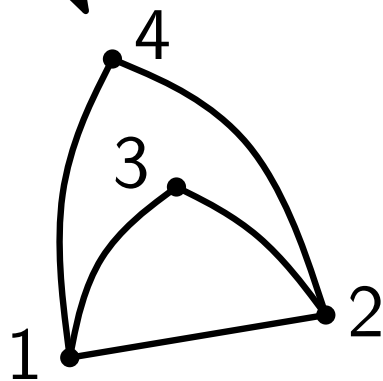
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Reduce Crossings

combinatorial
embedding/
planarization



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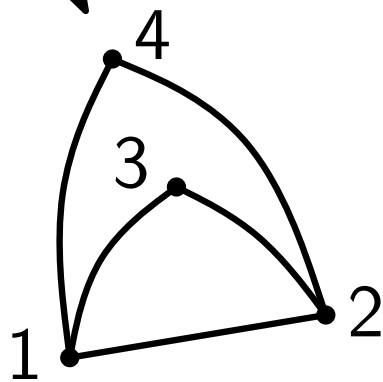
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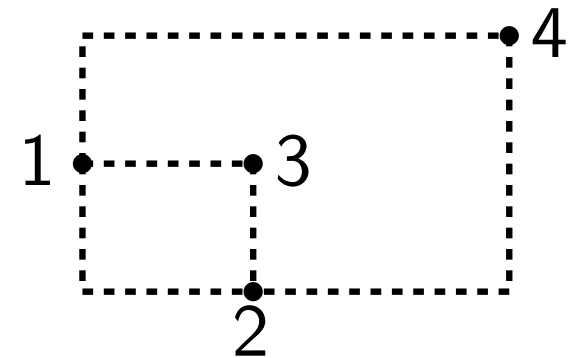
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Bend Minimization

orthogonal
representation



(Planar) Orthogonal Drawings

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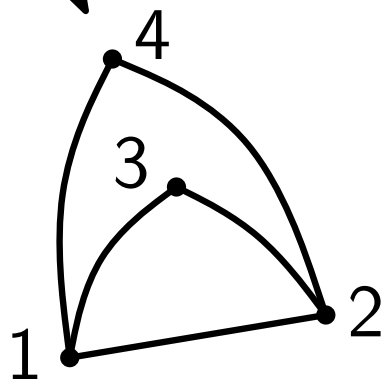
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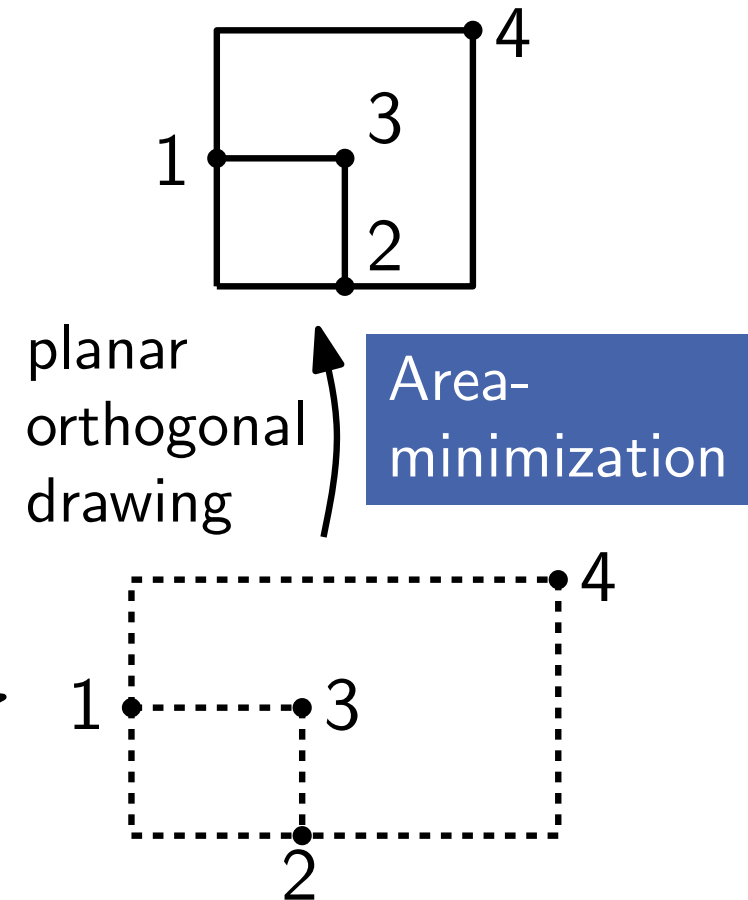
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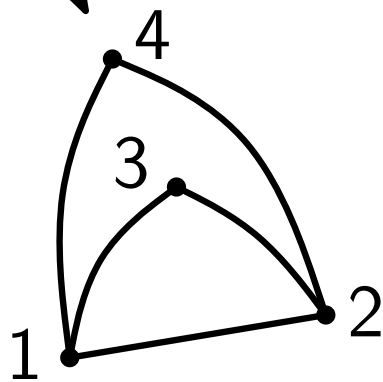
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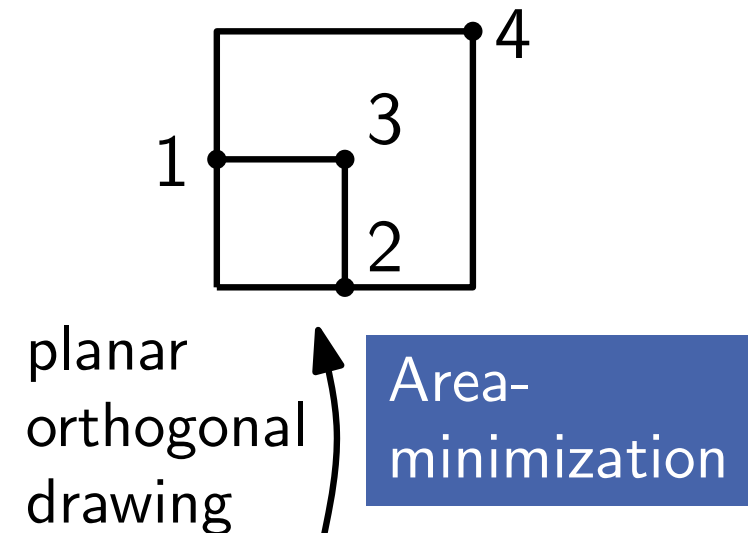
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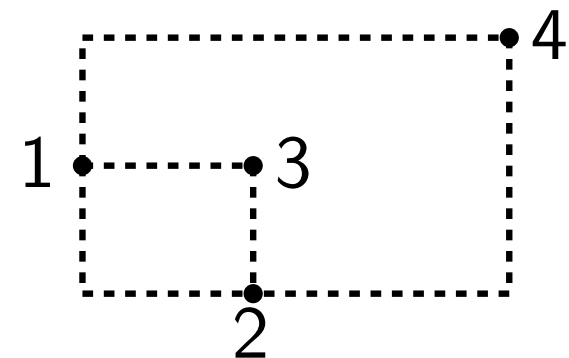
Bend Minimization

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planar
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Area-
minimization



Bend Minimization with Given Embedding

Problem: Geometric Bend Minimization

Given: ■ planar Graph $G = (V, E)$ with maximum degree 4
■ combinatorial embedding \mathcal{F} and outer face f_0

Find: orthogonal drawing with minimum number of bends that preserves the embedding

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compare with the following variation

Problem Combinatorial Bend Minimization

Given: ■ planar Graph $G = (V, E)$ with maximum degree 4
■ combinatorial embedding \mathcal{F} and outer face f_0

Find: **orthogonale representation** $H(G)$ with minimum number of bends that preserves the embedding

Orthogonal Representation

Given: planar Graph $G = (V, E)$, set of faces \mathcal{F} ,
outer face f_0

Find: orthogonale representation $H(G) = \{H(f) \mid f \in \mathcal{F}\}$

Face representation $H(f)$: of f is a clockwise* ordered
sequence of edge descriptions (e, δ, α) with

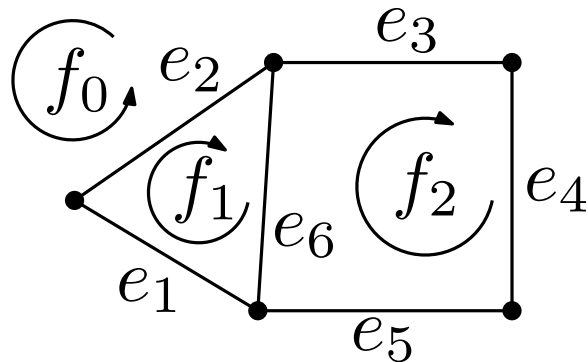
- e edge of f
- δ is sequence of $\{0, 1\}^*$ (0 = right bend, 1 = left bend)
- α is angle $\in \{\frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\}$ between e and next edge e'

Orthogonal Representation: Example

$$H(f_0) = ((e_1, 11, \frac{\pi}{2}), (e_5, 111, \frac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \frac{\pi}{2}))$$

$$H(f_1) = ((e_1, 00, \frac{3\pi}{2}), (e_2, \emptyset, \frac{\pi}{2}), (e_6, 00, \pi))$$

$$H(f_2) = ((e_5, 000, \frac{\pi}{2}), (e_6, 11, \frac{\pi}{2}), (e_3, \emptyset, \pi), (e_4, \emptyset, \frac{\pi}{2}))$$



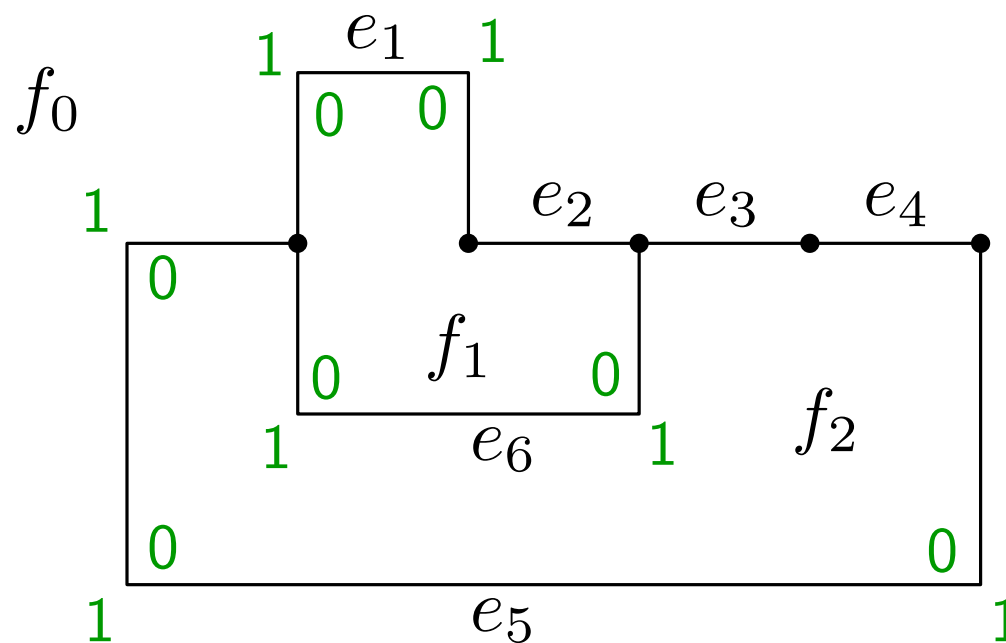
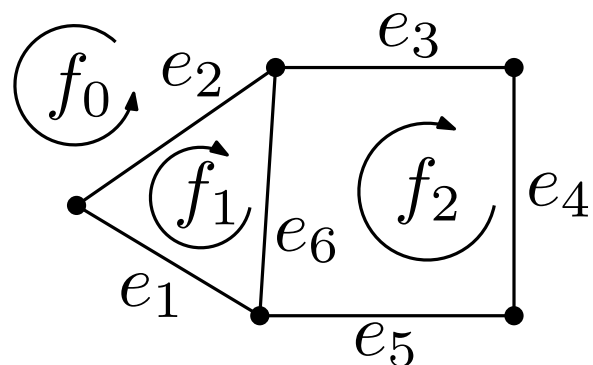
Combinatorial “drawing” of $H(G)$?

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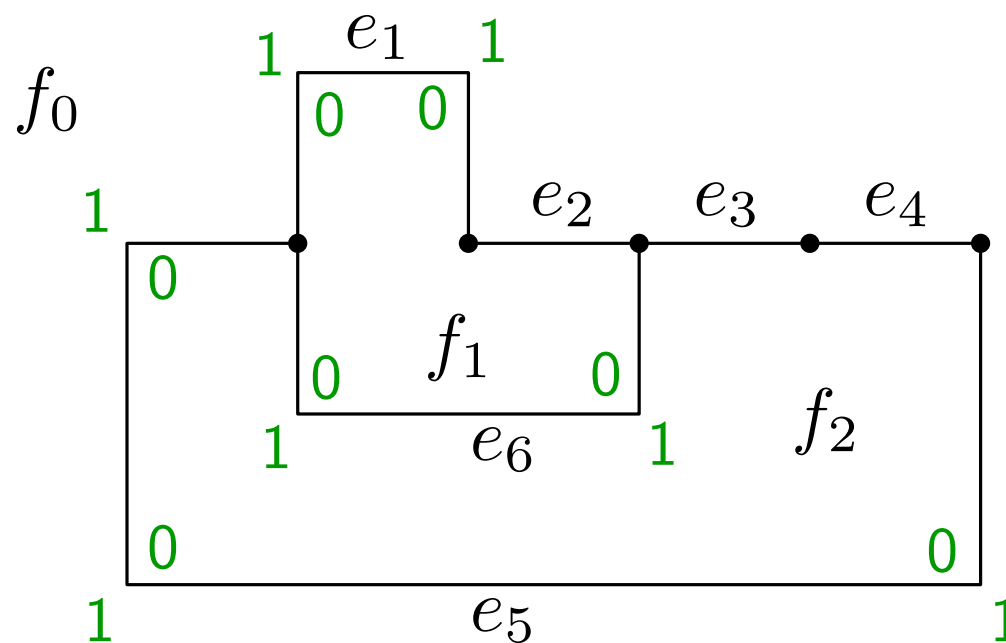
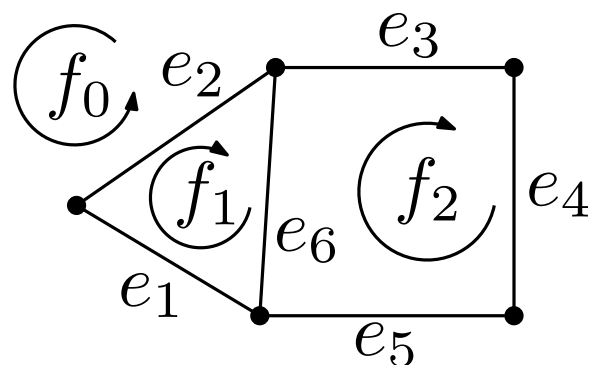


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$$H(f_1) = ((e_1, 00, \frac{3\pi}{2}), (e_2, \emptyset, \frac{\pi}{2}), (e_6, 00, \pi)) \quad \text{is } f_0 \text{ listed wrongly!?$$

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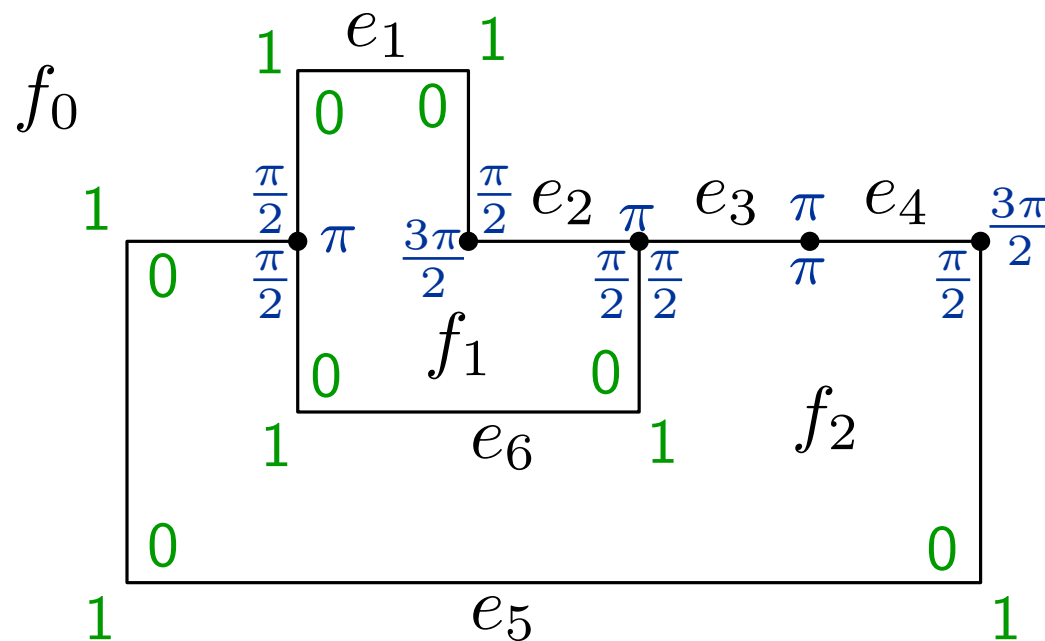
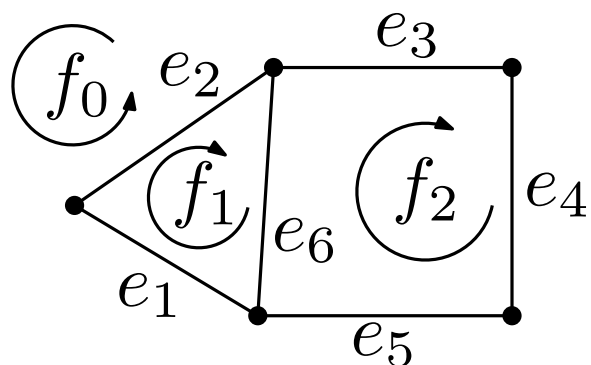


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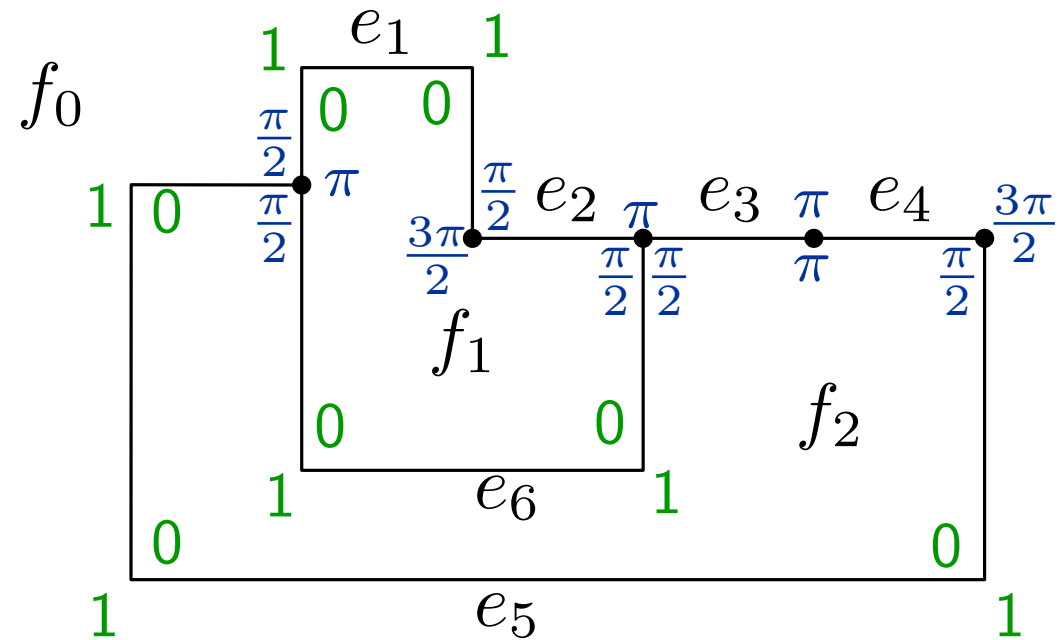
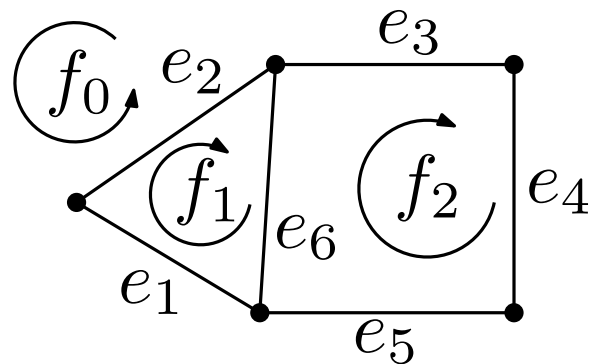


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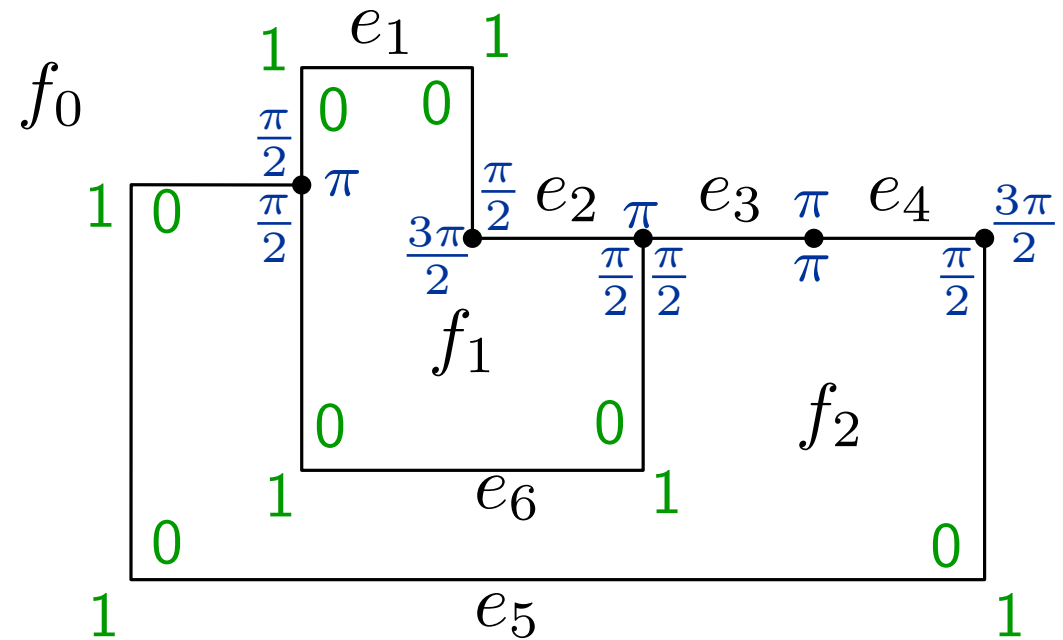
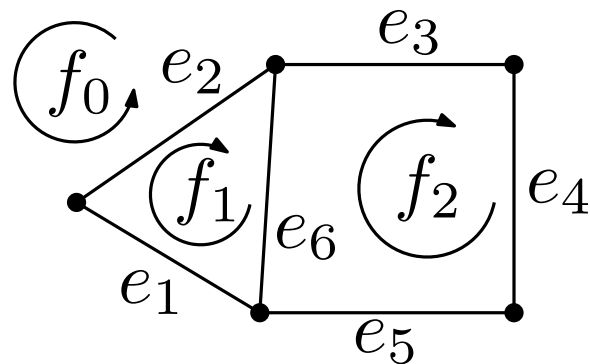


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concrete coordinates are not fixed yet!

Correctness of an Orthogonal Representation

(H1) $H(G)$ corresponds to \mathcal{F}, f_0

Correctness of an Orthogonal Representation

- (H1) $H(G)$ corresponds to \mathcal{F}, f_0
- (H2) for an edge $\{u, v\}$ shared by faces f and g with $((u, v), \delta_1, \alpha_1) \in H(f)$ and $((v, u), \delta_2, \alpha_2) \in H(g)$ sequence δ_1 is reversed and inverted δ_2

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- (H3) Let $|\delta|_0$ (resp. $|\delta|_1$) be the number of zeros (resp. ones) in δ and $r = (e, \delta, \alpha)$. For $C(r) := |\delta|_0 - |\delta|_1 + 2 - 2\alpha/\pi$ holds that: $\sum_{r \in H(f)} C(r) = 4$ for $f \neq f_0$ and $\sum_{r \in H(f_0)} C(r) = -4$

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- (H4) For each node v the summ of incident angles is 2π

Problem Combinatorial Bend Minimization

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Idea: formulate as a network flow problem

- a unit of flow represents an angle $\pi/2$
- flow from vertices to faces represents the angles at the vertices
- flow between adjacent faces represent the bends at the edges

Reminder: s - t Flow Network

Flow network $(D = (V, A); s, t; c)$ with

- directed graph $D = (V, A)$
- Edge capacity $c : A \rightarrow \mathbb{R}_0^+$
- Source $s \in V$, Sink $t \in V$

A function $X : A \rightarrow \mathbb{R}_0^+$ is called **s - t -flow**, if:

$$0 \leq X(u, v) \leq c(u, v) \quad \forall (u, v) \in A \quad (1)$$

$$\sum_{(u, v) \in A} X(u, v) - \sum_{(v, u) \in A} X(v, u) = 0 \quad \forall u \in V \setminus \{s, t\} \quad (2)$$

Reminder: General Flow Network

Flow network $(D = (V, A); \ell; u; b)$ with

- directed graph $D = (V, A)$
- edge **lower bound** $\ell : A \rightarrow \mathbb{R}_0^+$
- edge **capacity** $c : A \rightarrow \mathbb{R}_0^+$
- node **production/consumption** $b : V \rightarrow \mathbb{R}$ with $\sum_{i \in V} b(i) = 0$

An assignement $X : A \rightarrow \mathbb{R}_0^+$ is called **flow**, if:

$$\ell(u, v) \leq X(u, v) \leq c(u, v) \quad \forall (u, v) \in A \quad (3)$$

$$\sum_{(u, v) \in A} X(u, v) - \sum_{(v, u) \in A} X(v, u) = b(u) \quad \forall u \in V \quad (4)$$

(A) Valid Flow:

Find a Valid Flow $X : A \rightarrow \mathbb{R}_0^+$, such that.

- Lower bounds and capacities $\ell(e), u(e)$ are respected (inequalities (3))
- Consumption/production is $b(v)$ satisfied (inequality (4))

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(B) Minimum Cost Flow

Find a valid flow $X : A \rightarrow \mathbb{R}_0^+$, that minimizes cost function $\text{cost}(X)$ (over all valid flows)

Flow Network for Bend Minimization

Flow Network $N(G) = ((V \cup \mathcal{F}, A); \ell; c; b; \text{cost})$

- $A = \{(v, f) \in V \times \mathcal{F} \mid v \text{ incident to } f\} \cup \{(f, g) \in \mathcal{F} \times \mathcal{F} \mid f, g \text{ adjacent through edge } e\}$

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- $b(v) = 4 \quad \forall v \in V$
- $b(f) = -2(d_G(f) - 2) \quad \forall f \in \mathcal{F} \setminus \{f_0\}$
- $b(f_0) = -2(d_G(f_0) + 2)$

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- } $\Rightarrow \sum_w b(w) \stackrel{?}{=} 0$

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$$\forall (f, g) \in A, f, g \in \mathcal{F}$$

$$\ell(f, g) := 0 \leq X(f, g) \leq \infty =: c(f, g)$$

$$\text{cost}(f, g) = 1$$

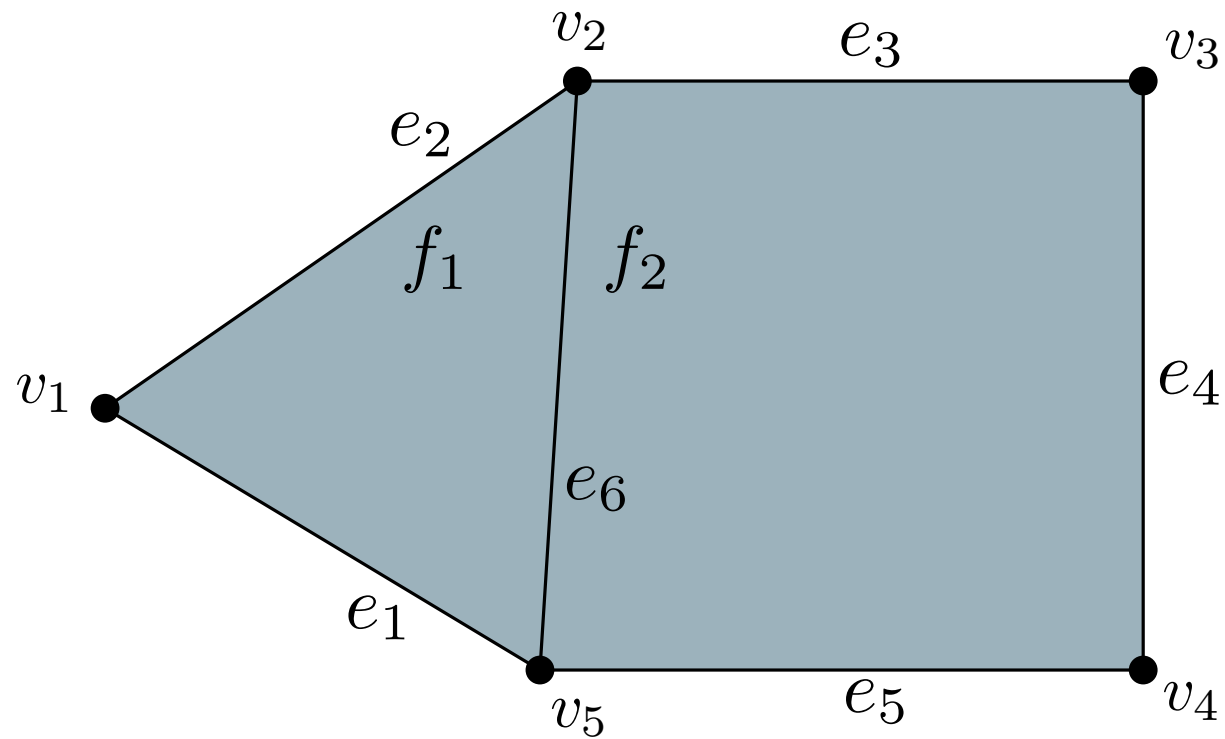
$$\forall (v, f) \in A, v \in V, f \in \mathcal{F}$$

$$\ell(v, f) := 1 \leq X(v, f) \leq 4 =: c(v, f)$$

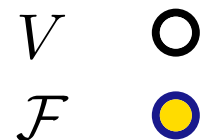
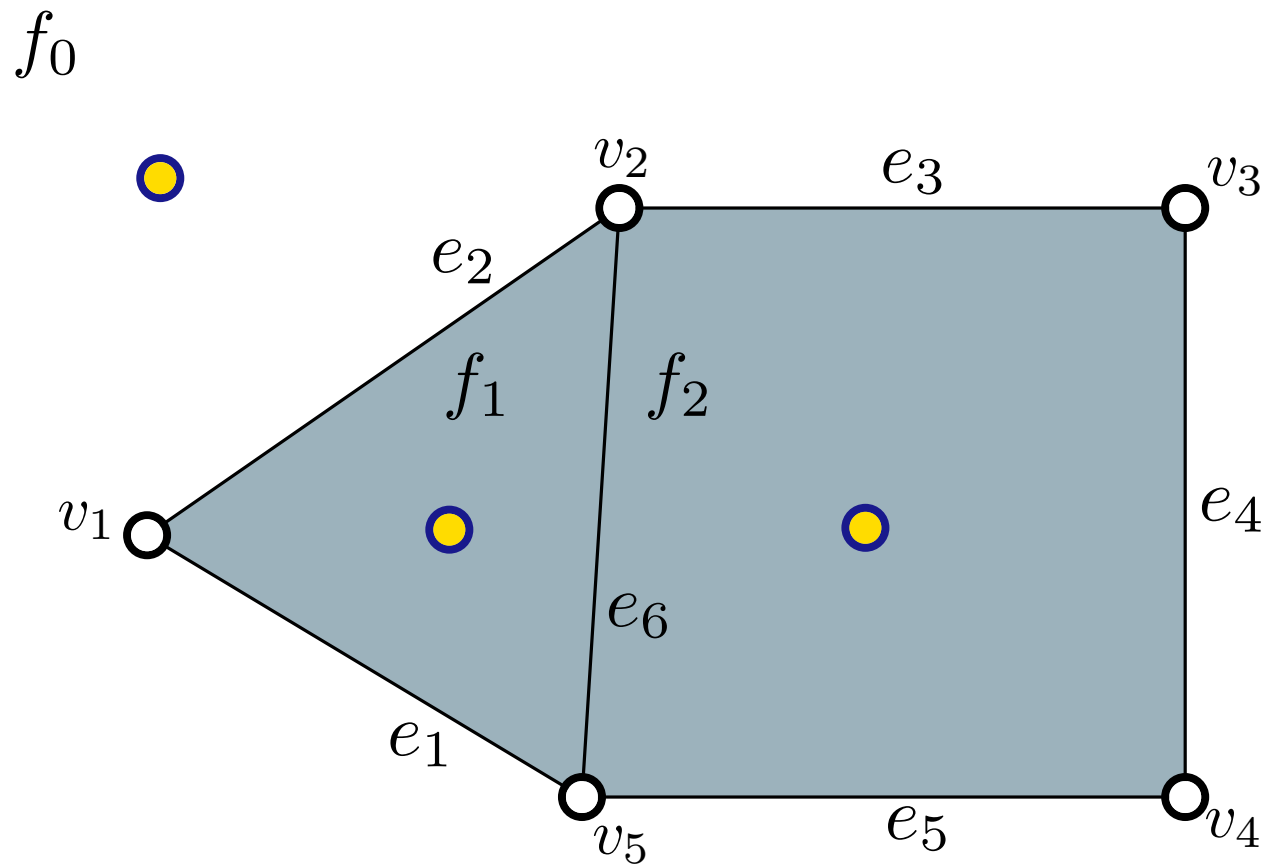
$$\text{cost}(v, f) = 0$$

Example Flow Network

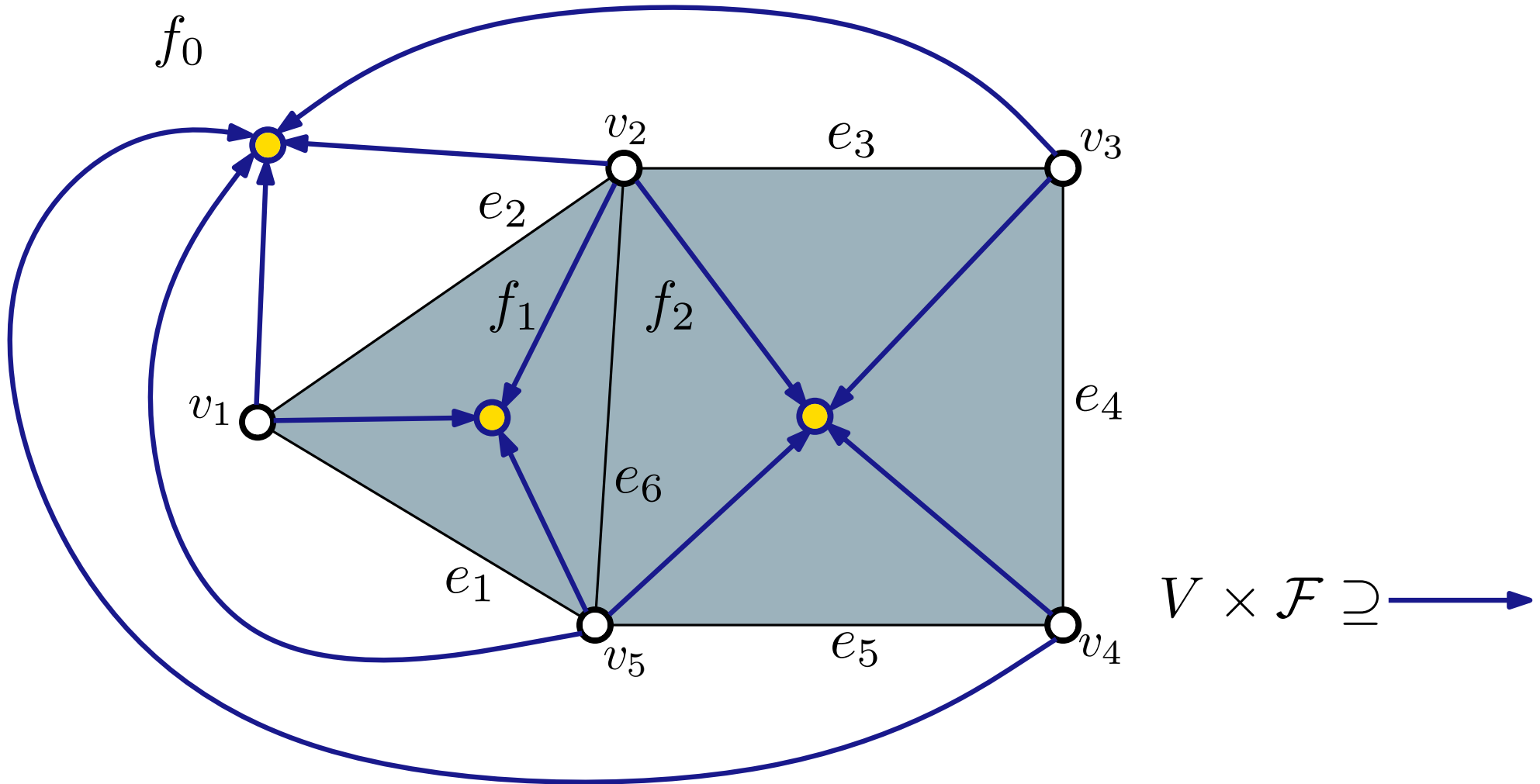
f_0



Example Flow Network



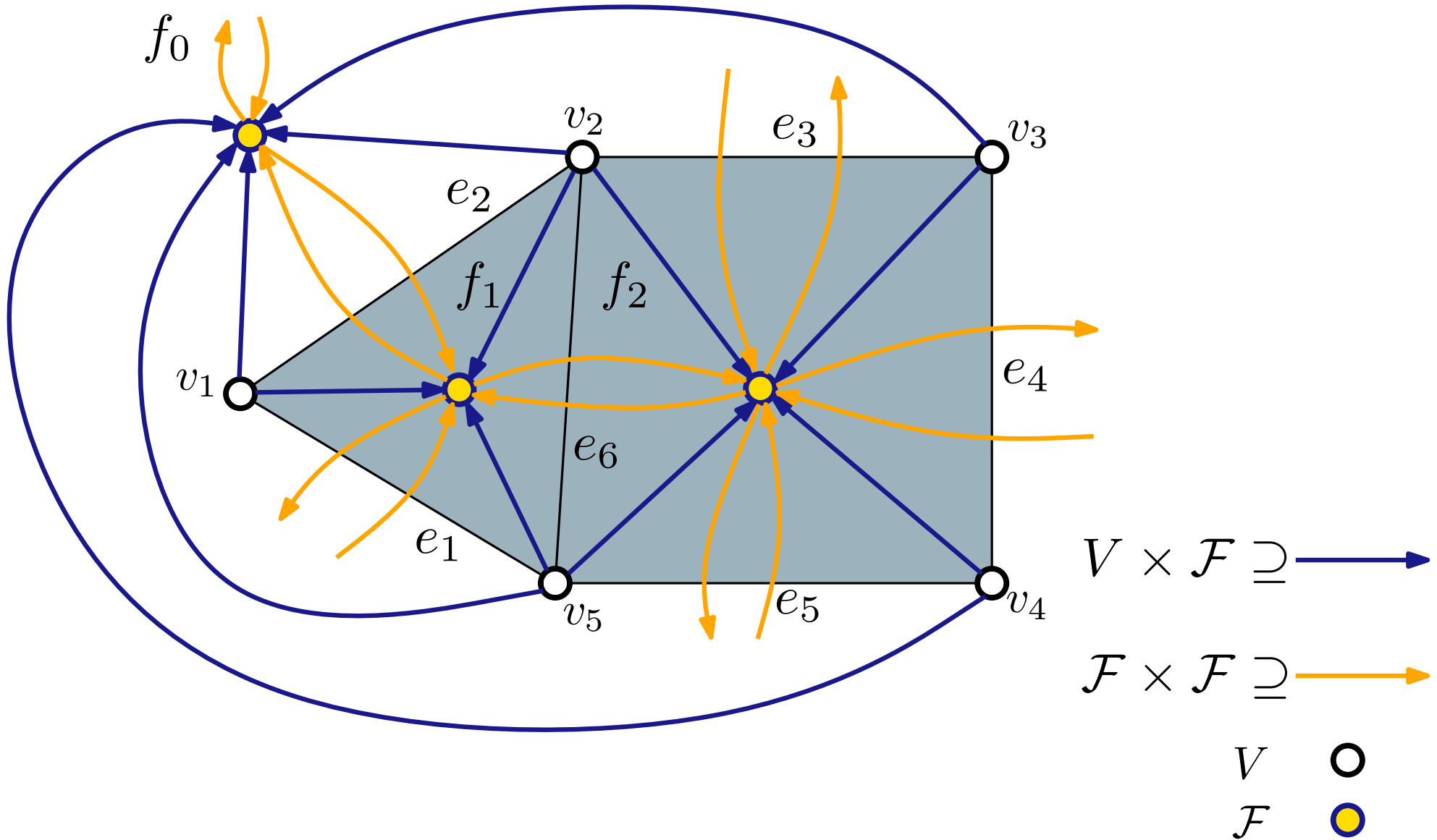
Example Flow Network



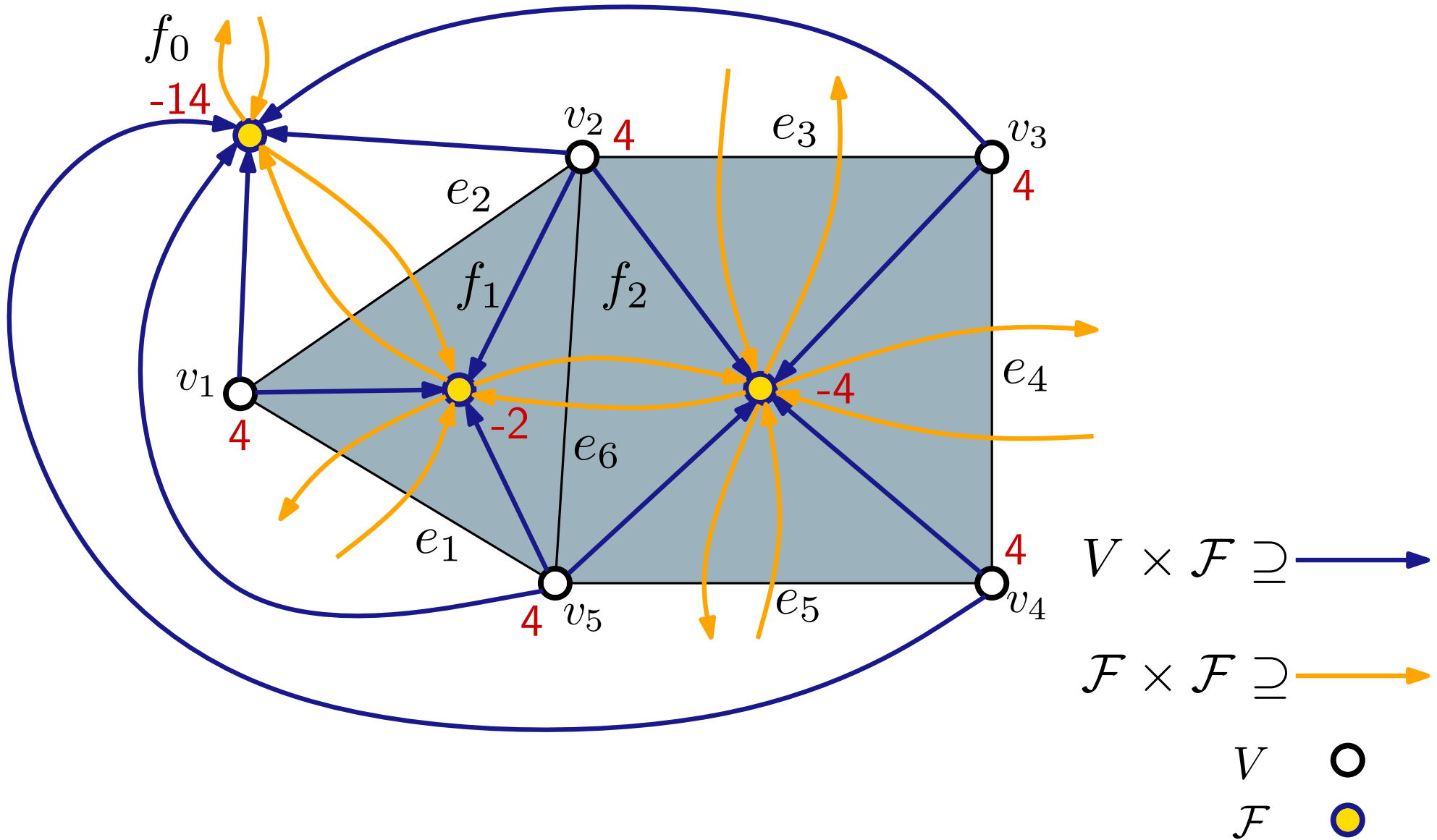
$V \times \mathcal{F} \supseteq \longrightarrow$

V ○
 \mathcal{F} ●

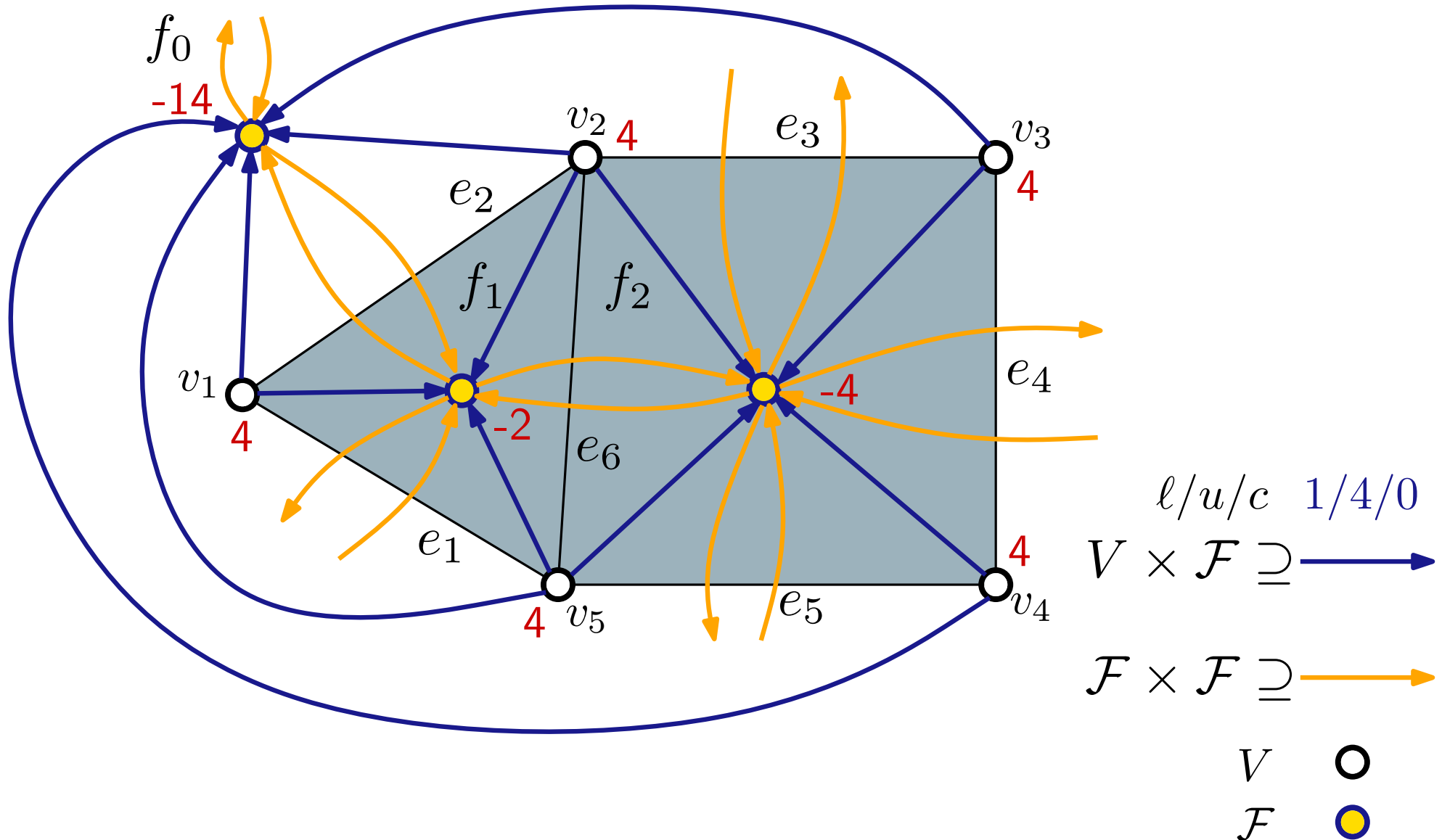
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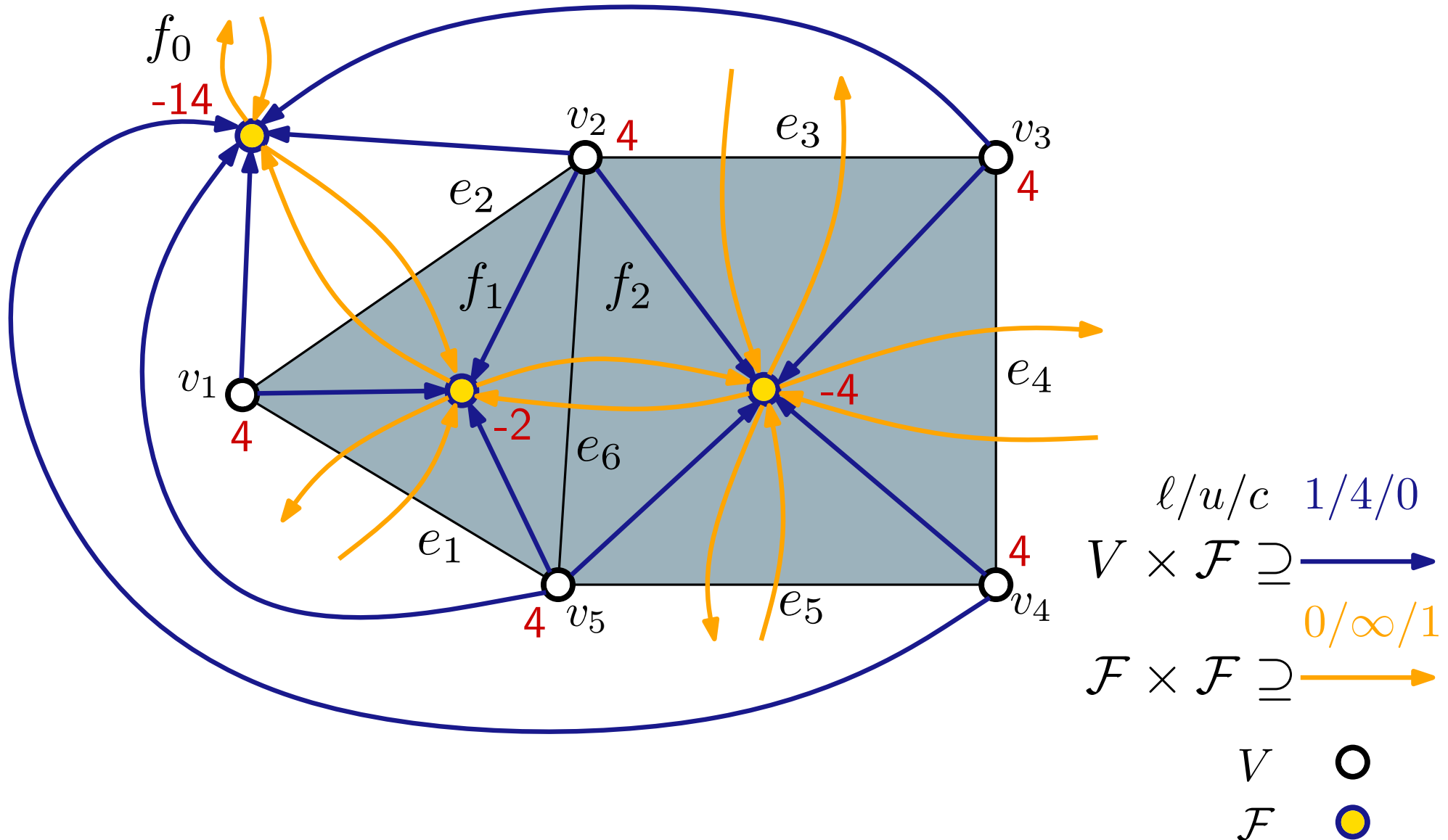
Example Flow Network



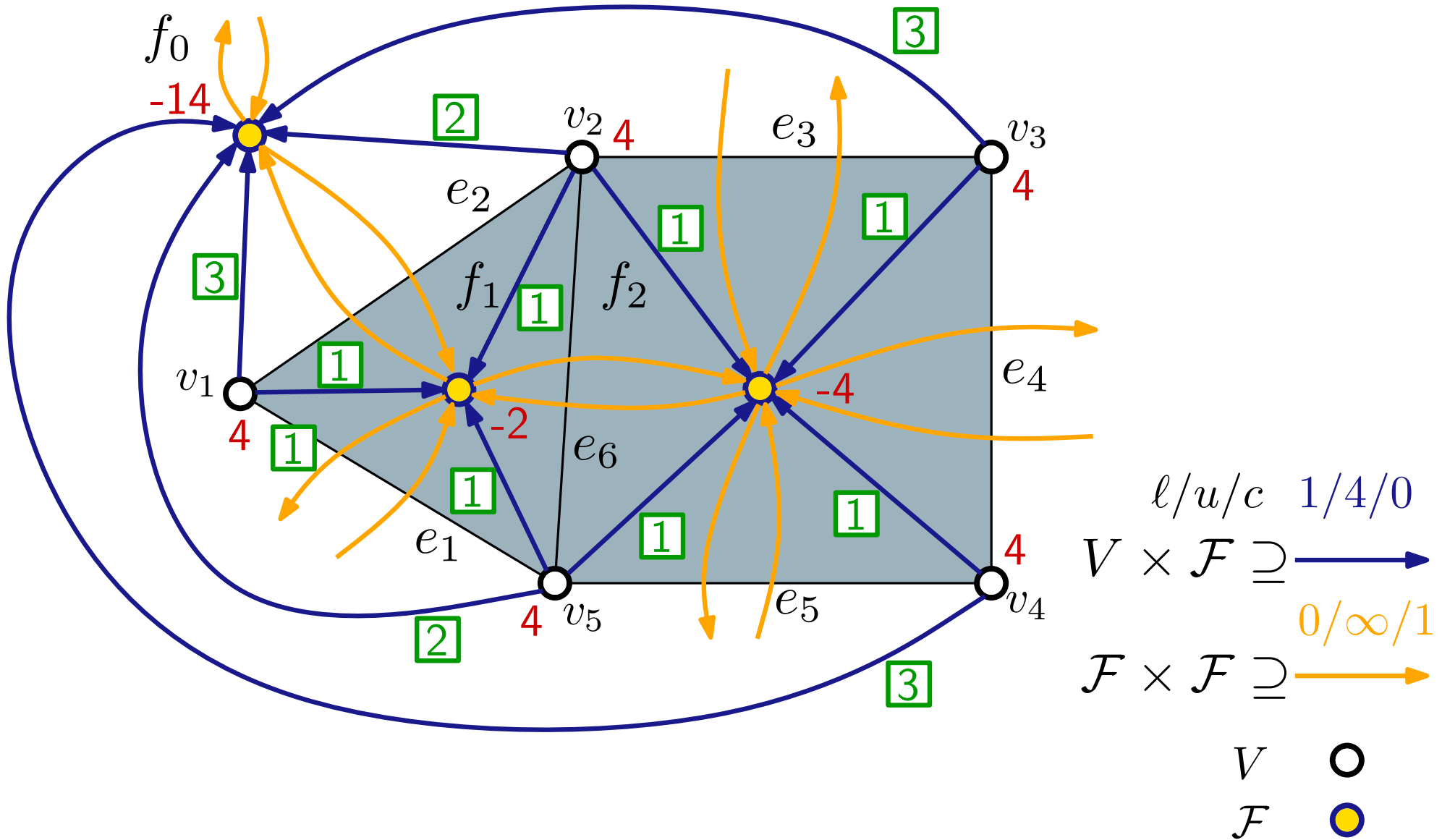
Example Flow Network



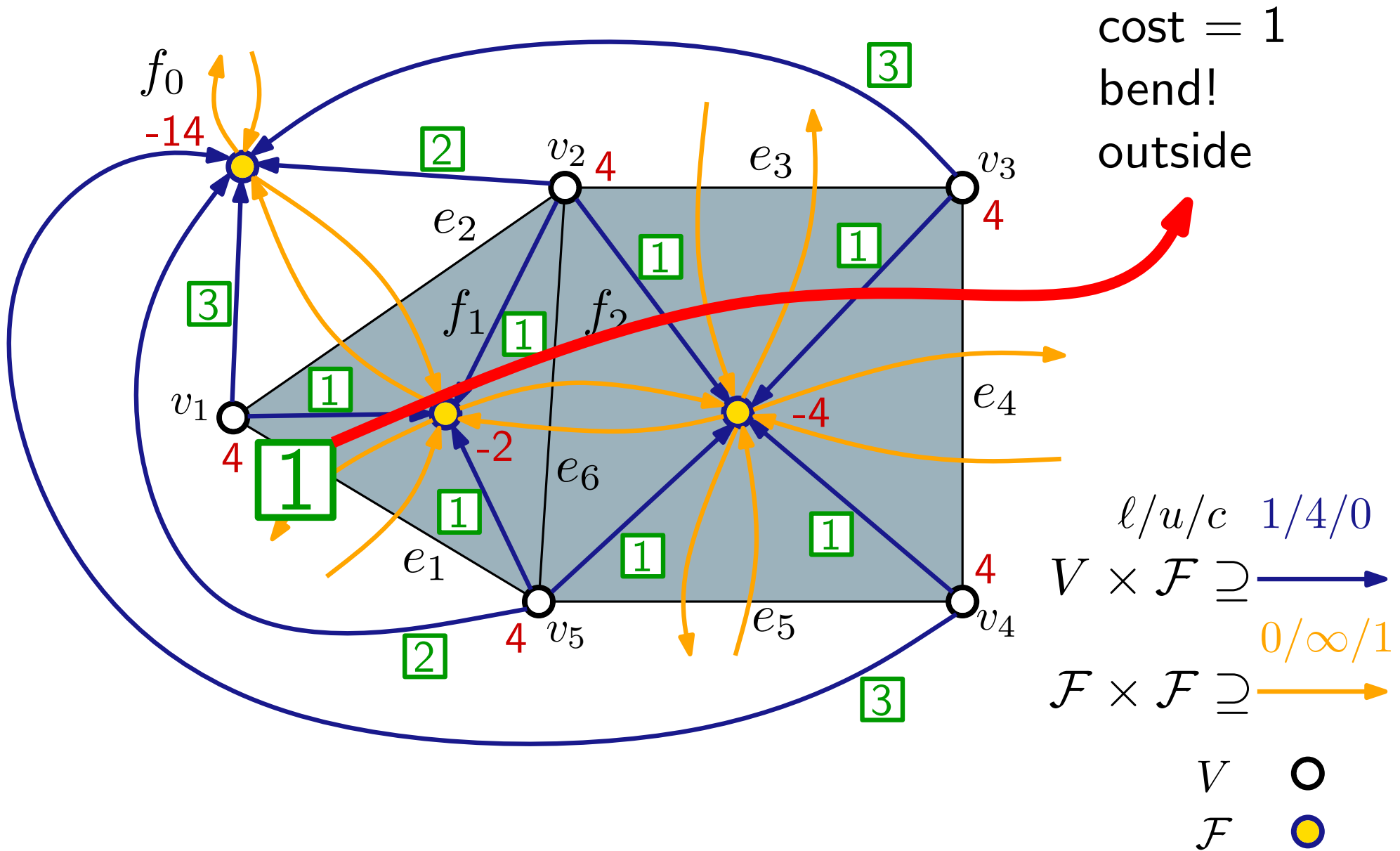
Example Flow Network



Example Flow Network



Example Flow Network



cost = 1
bend!
outside

$l/u/c$ 1/4/0
 $V \times \mathcal{F} \supseteq$ (blue arrow)
 0/ ∞ /1
 $\mathcal{F} \times \mathcal{F} \supseteq$ (orange arrow)

V ○
 \mathcal{F} ●

Main Statement

Thm 1: A planar embedded graph (G, \mathcal{F}, f_0) has a valid orthogonal description $H(G)$ with k bends iff the flow network $N(G)$ has a valid flow X with cost k .

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- Construct flow X in $N(G)$ with cost k
 - define assignement $X : A \rightarrow \mathbb{R}_0^+$
 - show that X is a valid flow and has cost k

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- From Theorem 1 it follows that a combinatorial orthogonal bend minimization problem for embedded planar graphs can be solved using an algorithm for Min-Cost-Flow Problem.

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- Bend minimization without a given combinatorial embedding is an NP-hard problem. [Garg, Tamassia SIAM J. Comput. 2001]
- The decision problem whether a graph has an orthogonal drawing with at most $f(e)$ bends per edge e is polynomially solvable when $f(e) \geq 1$ for all $e \in E$.

[Bläsius et al. GD 2010]

(Planare) Orthogonale Zeichnungen

Three-step approach: *Topology – Shape – Metrics*

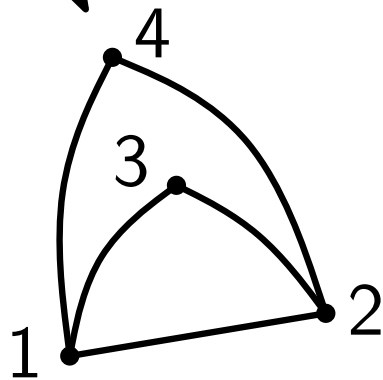
[Tamassia SIAM J. Comput. 1987]

$$V = \{v_1, v_2, v_3, v_4\}$$

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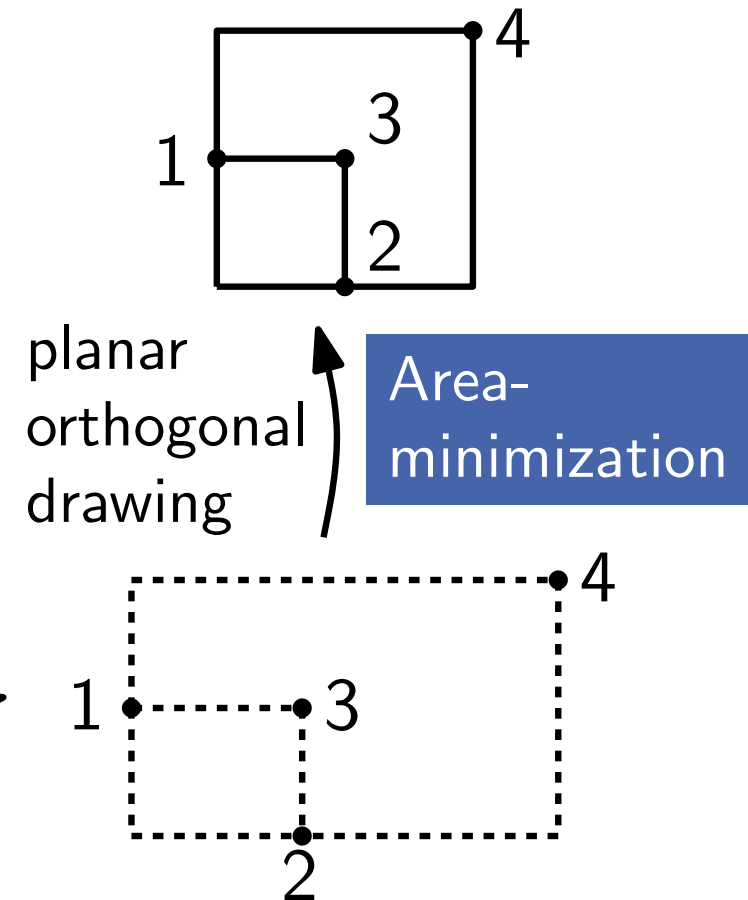
Reduce Crossings

combinatorial
embedding/
planarization



Bend Minimization

orthogonal
representation



Area-
minimization

planar
orthogonal
drawing

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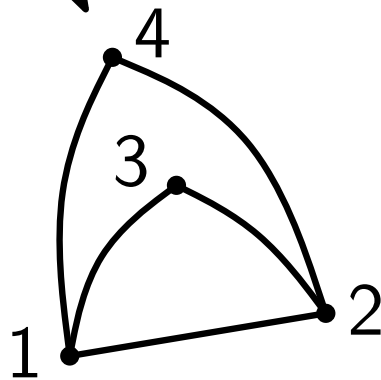
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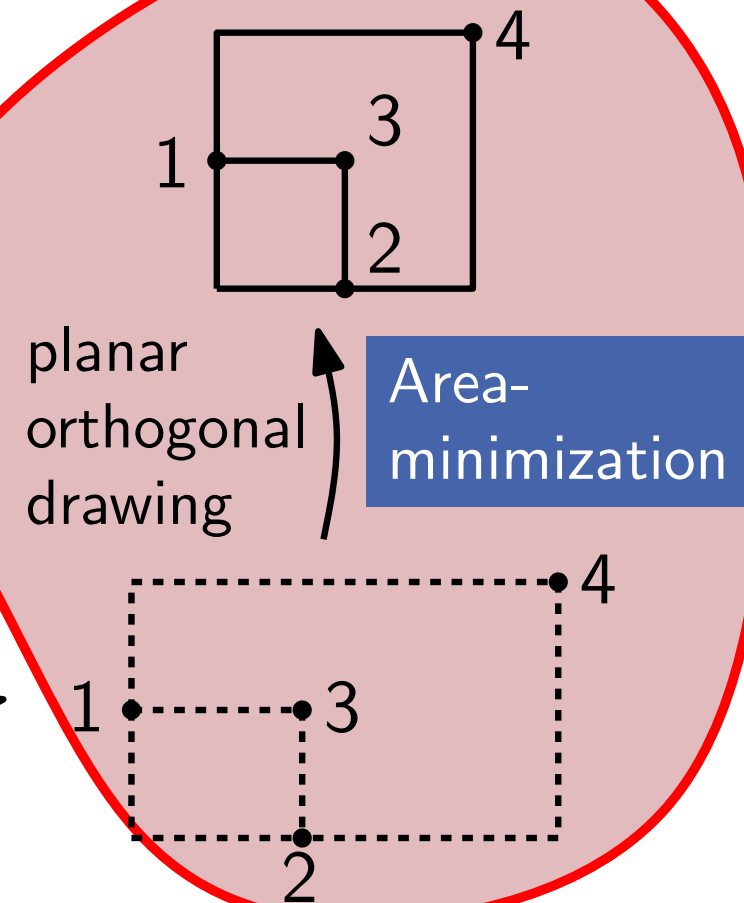
Reduce Crossings

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orthogonal
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planar
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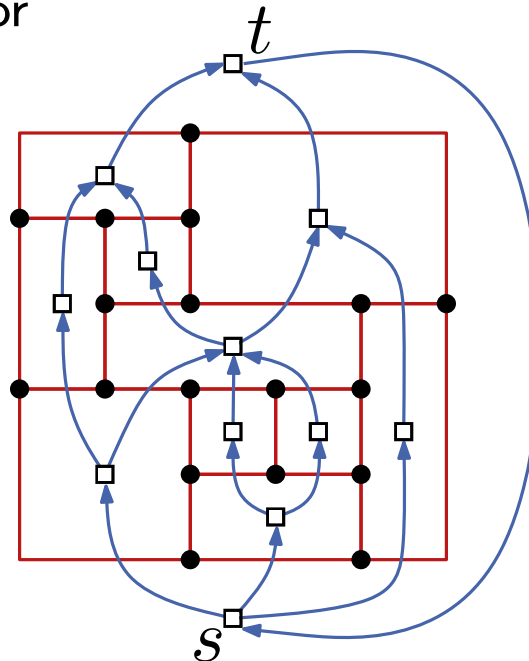
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We will formulate a flow network for
(horizontal) compaction

Flow Network for Edge Length Computation

Def: Flow Network $N_{\text{hor}} = ((W_{\text{hor}}, A_{\text{hor}}); \ell; u; b; \text{cost})$

- $W_{\text{hor}} = \mathcal{F} \setminus \{f_0\} \cup \{s, t\}$
- $A_{\text{hor}} = \{(f, g) \mid f, g \text{ share a horizontal segment and } f \text{ lies below } g\} \cup \{(t, s)\}$
- $\ell(a) = 1 \quad \forall a \in A_{\text{hor}}$
- $u(a) = \infty \quad \forall a \in A_{\text{hor}}$
- $\text{cost}(a) = 1 \quad \forall a \in A_{\text{hor}}$
- $b(f) = 0 \quad \forall f \in W_{\text{hor}}$

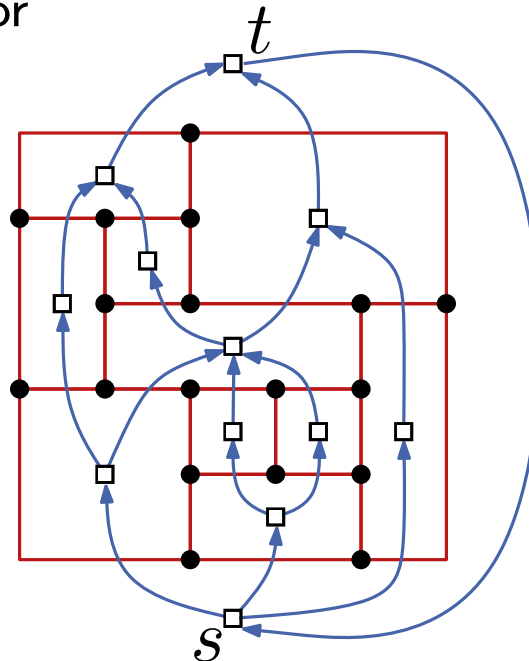


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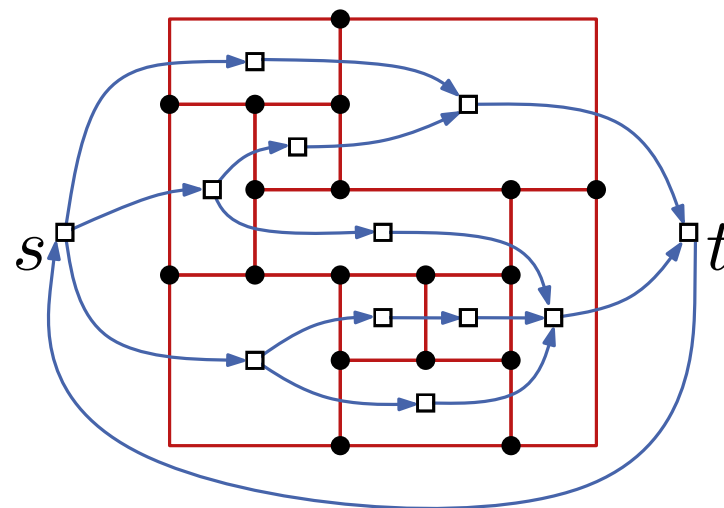
s and t represent lower and upper side of f_0

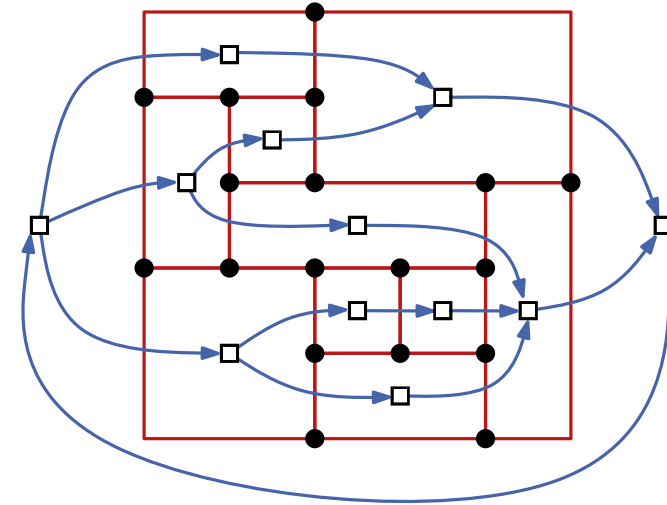
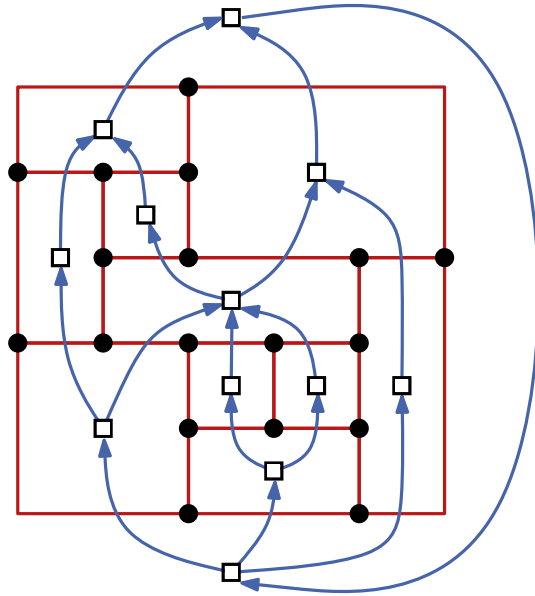


Flow Network for Edge Length Computation

Def: Flow Network $N_{\text{ver}} = ((W_{\text{ver}}, A_{\text{ver}}); \ell; u; b; \text{cost})$

- $W_{\text{ver}} = \mathcal{F} \setminus \{f_0\} \cup \{s, t\}$
- $A_{\text{ver}} = \{(f, g) \mid f, g \text{ share a vertical segment and } f \text{ lies to the left of } g\} \cup \{(t, s)\}$
- $\ell(a) = 1 \quad \forall a \in A_{\text{hor}}$
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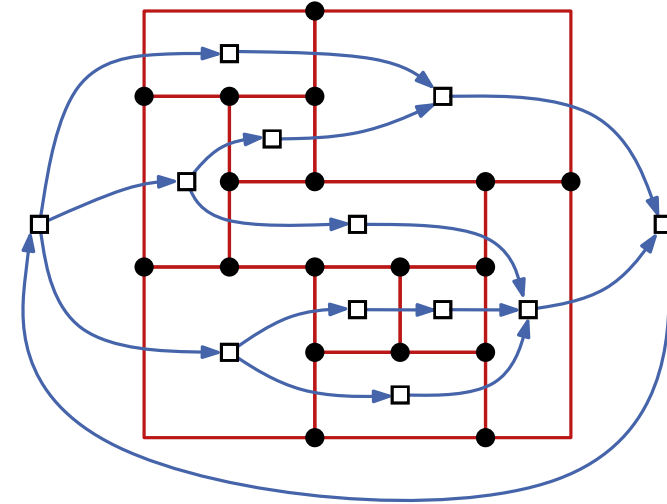
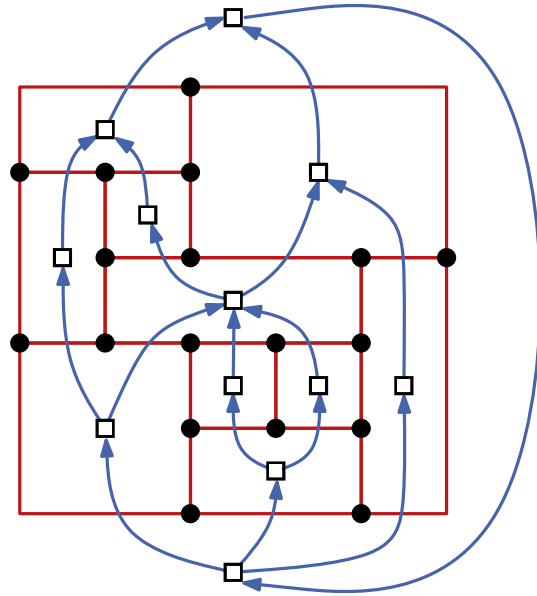




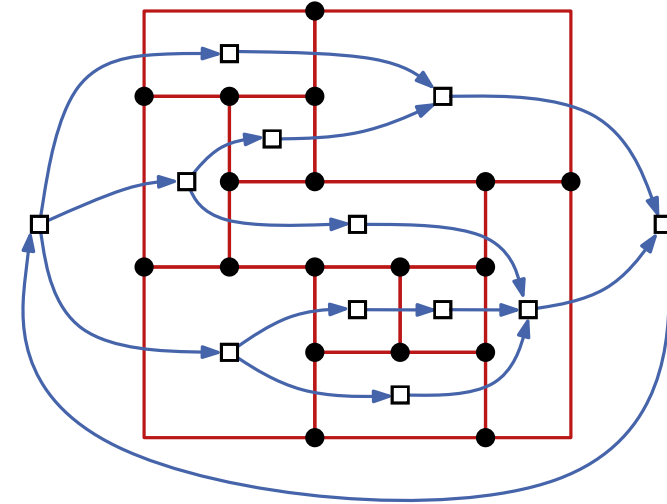
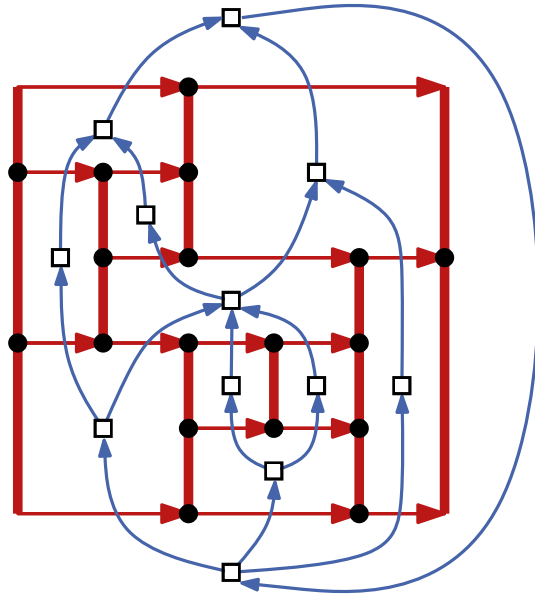
Thm 2: Integer flows x_{hor} and x_{ver} in N_{hor} and N_{ver} with minimum cost induce:

- valid orthogonal layout with respect to edge length
- $|x_{\text{hor}}(t, s)|$ and $|x_{\text{ver}}(t, s)|$ represent the width and height of the layout
- $\sum_{a \in A_{\text{hor}}} x_{\text{hor}}(a) + \sum_{a \in A_{\text{ver}}} x_{\text{ver}}(a)$ represent the total edge length

The layout can be computed in $O(n^{3/2})^*$ time

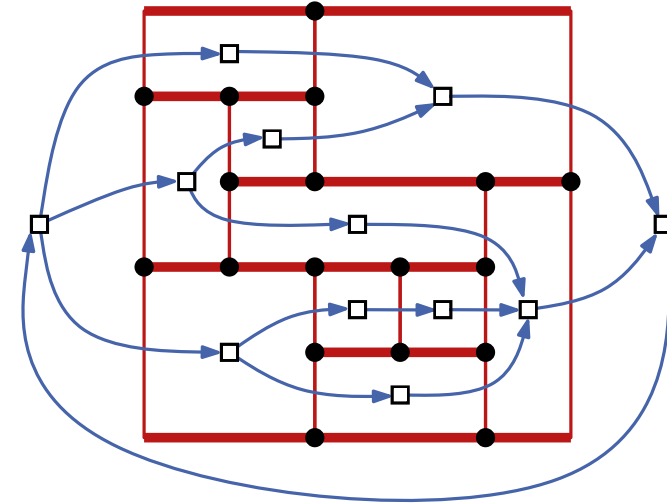
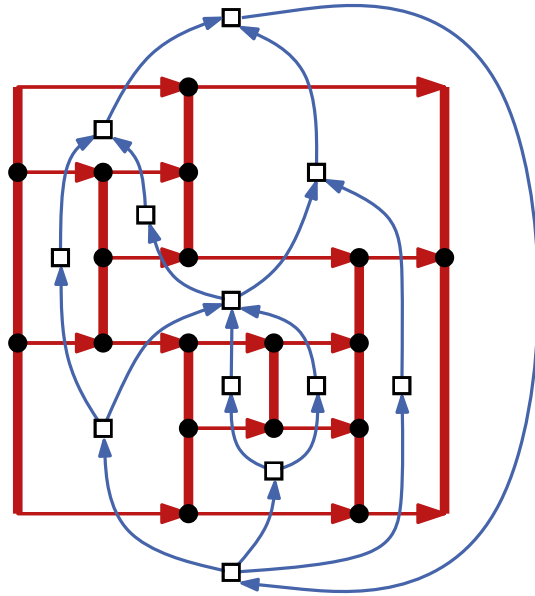


- construct the duals N_{hor}^* and N_{ver}^* of N_{hor} and N_{ver}
- topological numbering T_{hor} and T_{ver} of N_{hor}^* and N_{ver}^*
- for edge (f, g) of N_{hor} set flow
 $x_{hor}(f, g) = T_{hor}(b) - T_{hor}(a)$, where b is dual vertex on the left and b is dual vertex on the right of (f, g) , similar for x_{ver}
- easy to see that the constructed assignments x_{hor}, x_{ver} is a valid flow with smallest value (not minimum cost)



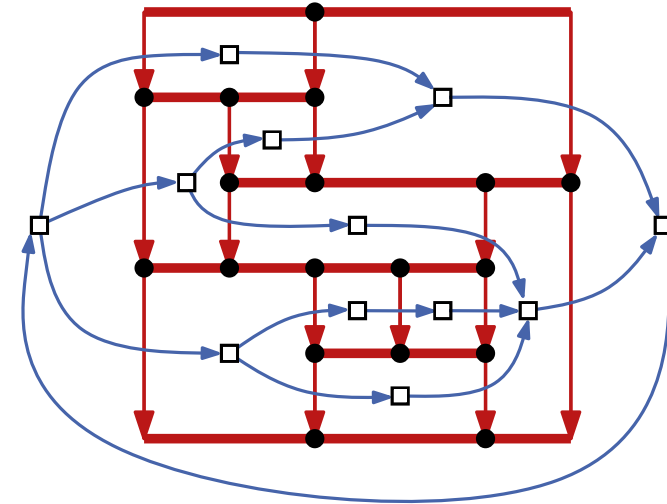
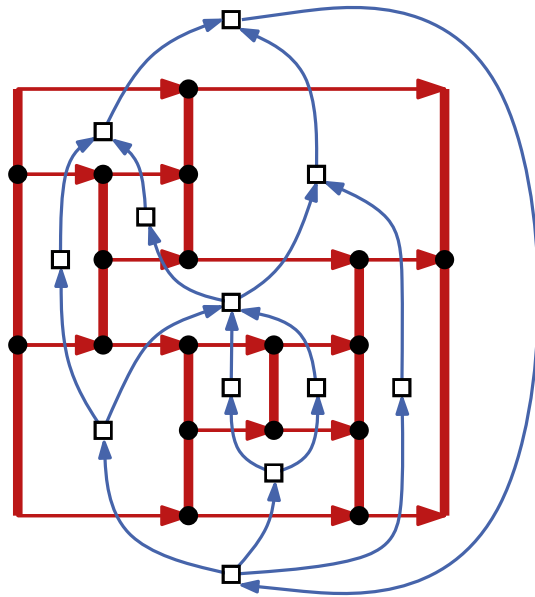
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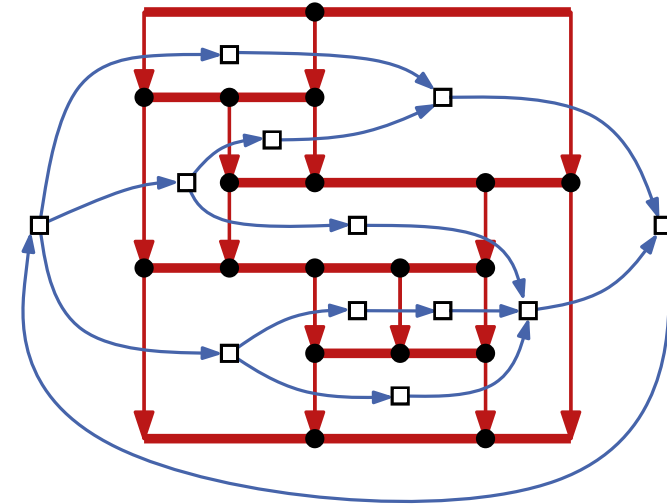
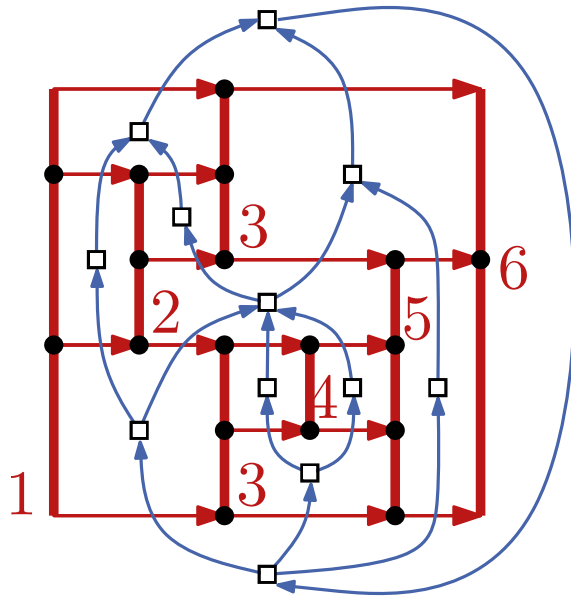


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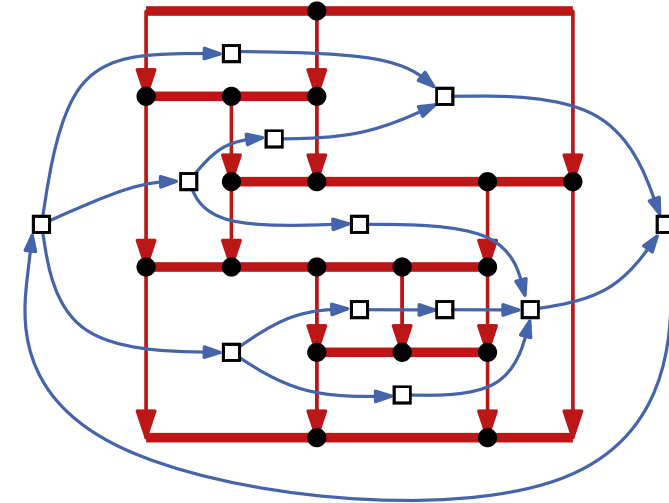
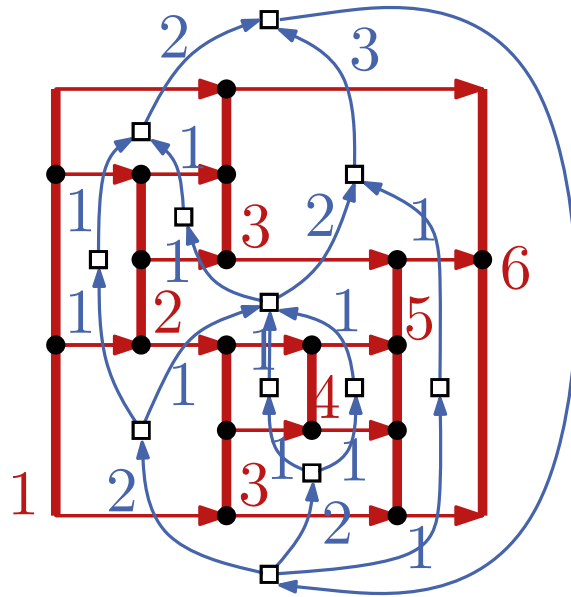
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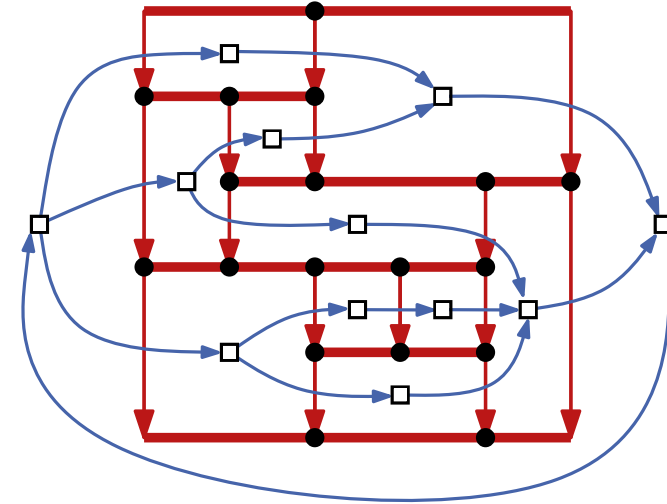
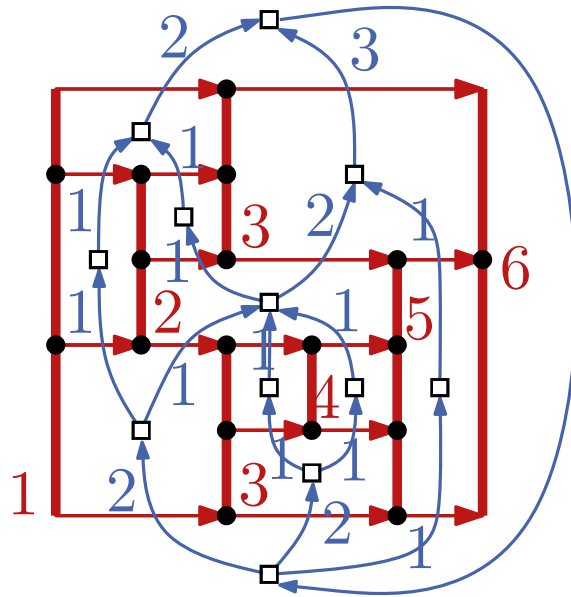
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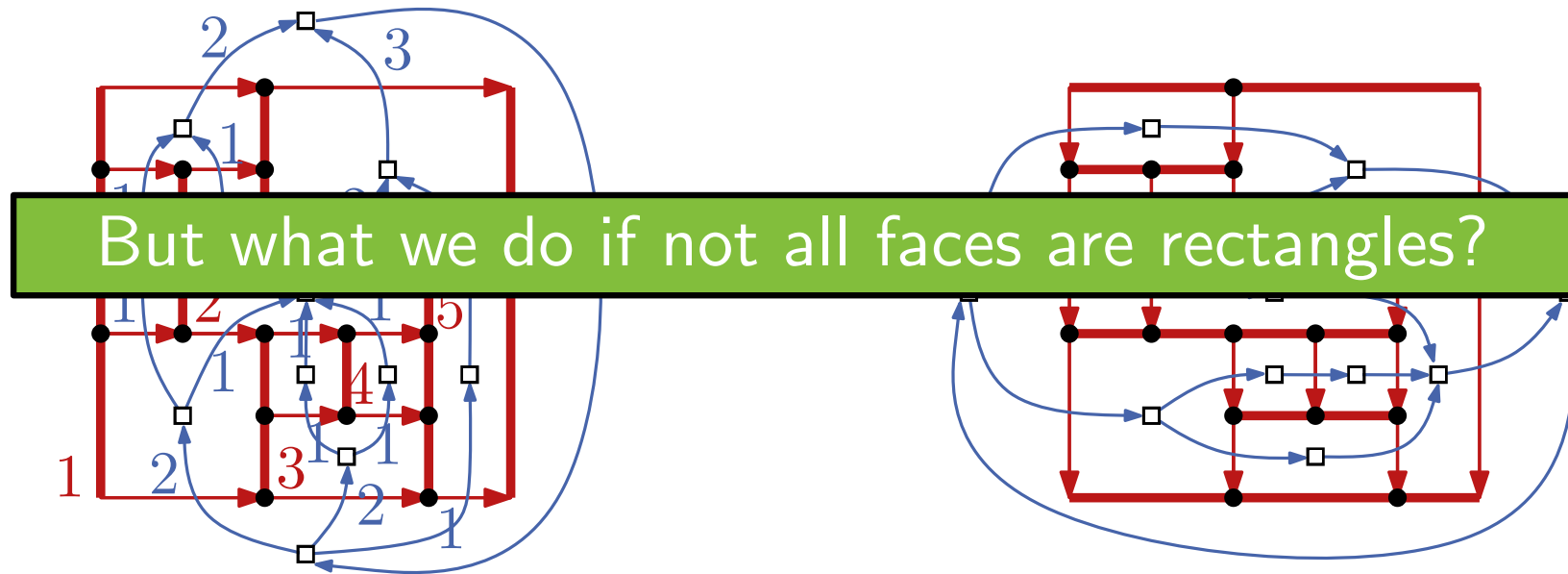
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Faster Flow Computation

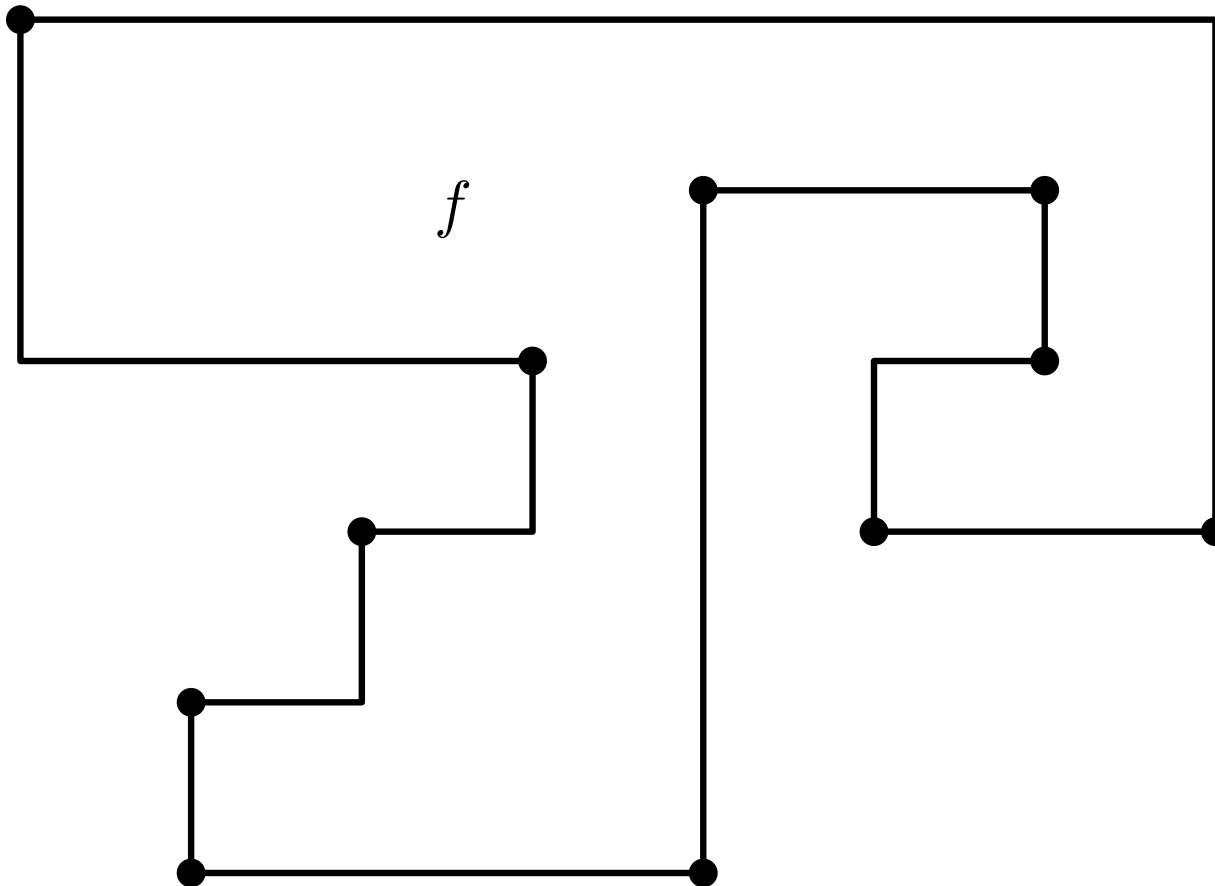


- This approach finds minimum width, height, area, but does not guarantee minimum total edge length
- Time complexity $O(n)$

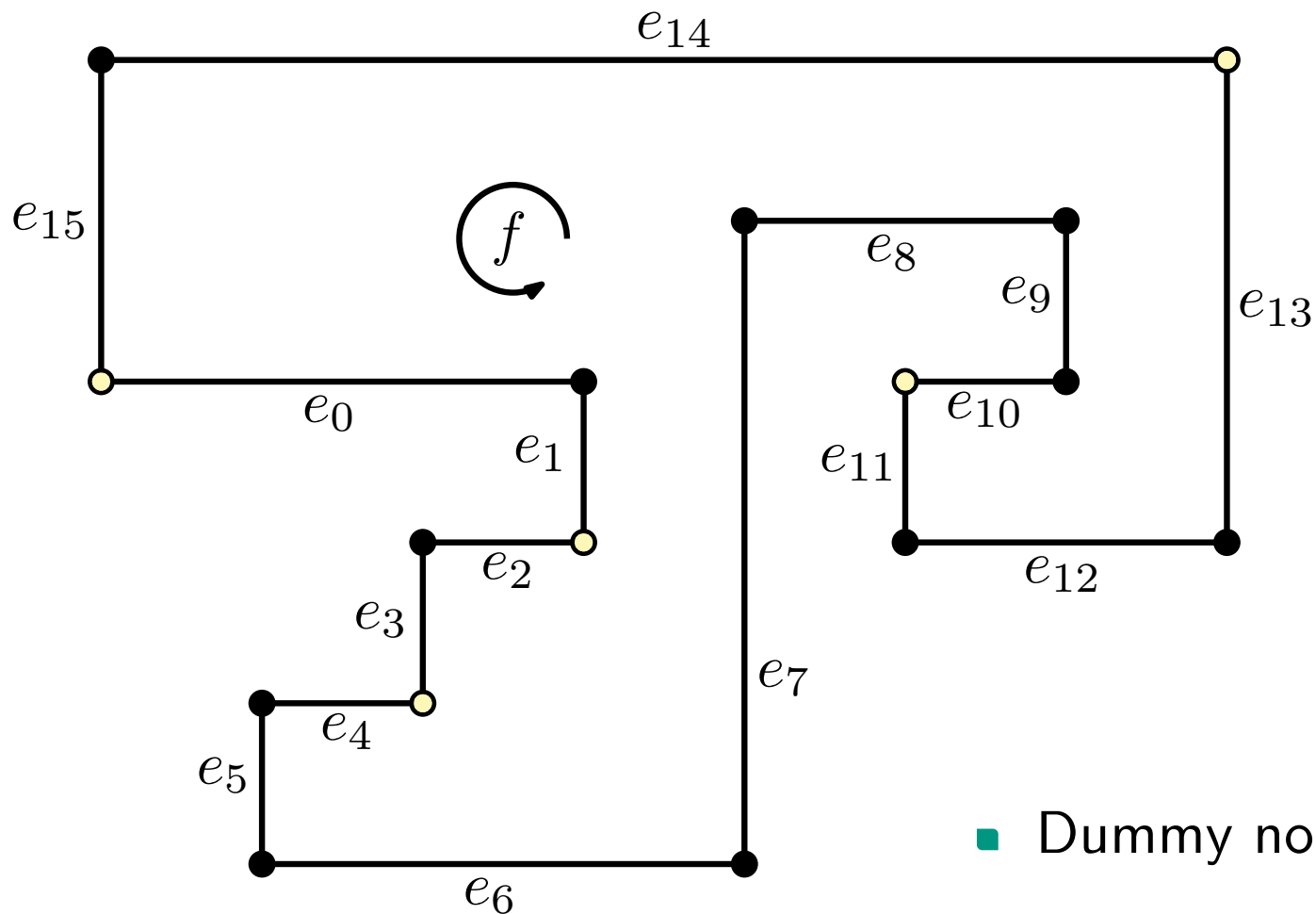


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Refinement of (G, H) – Inner Face

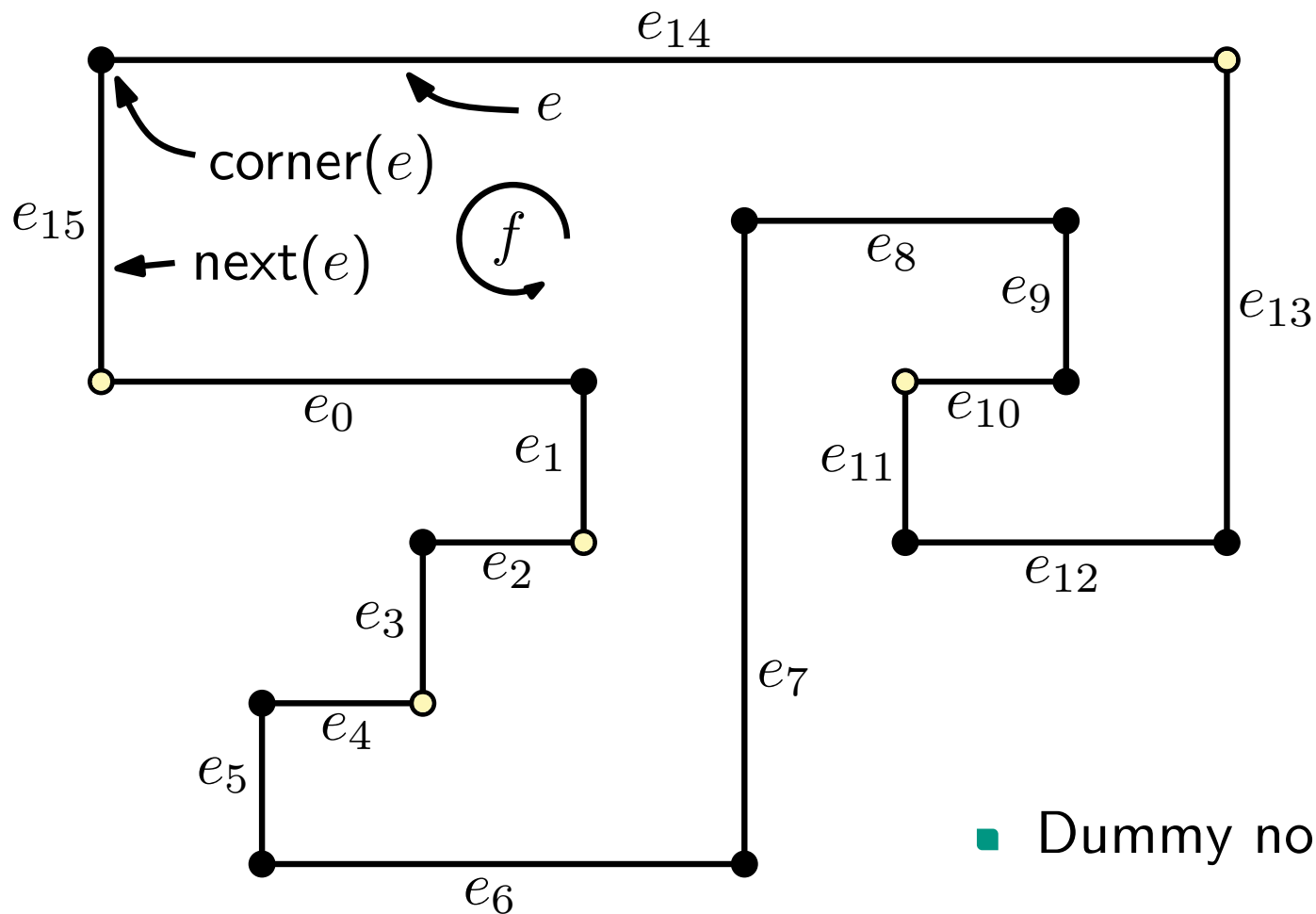


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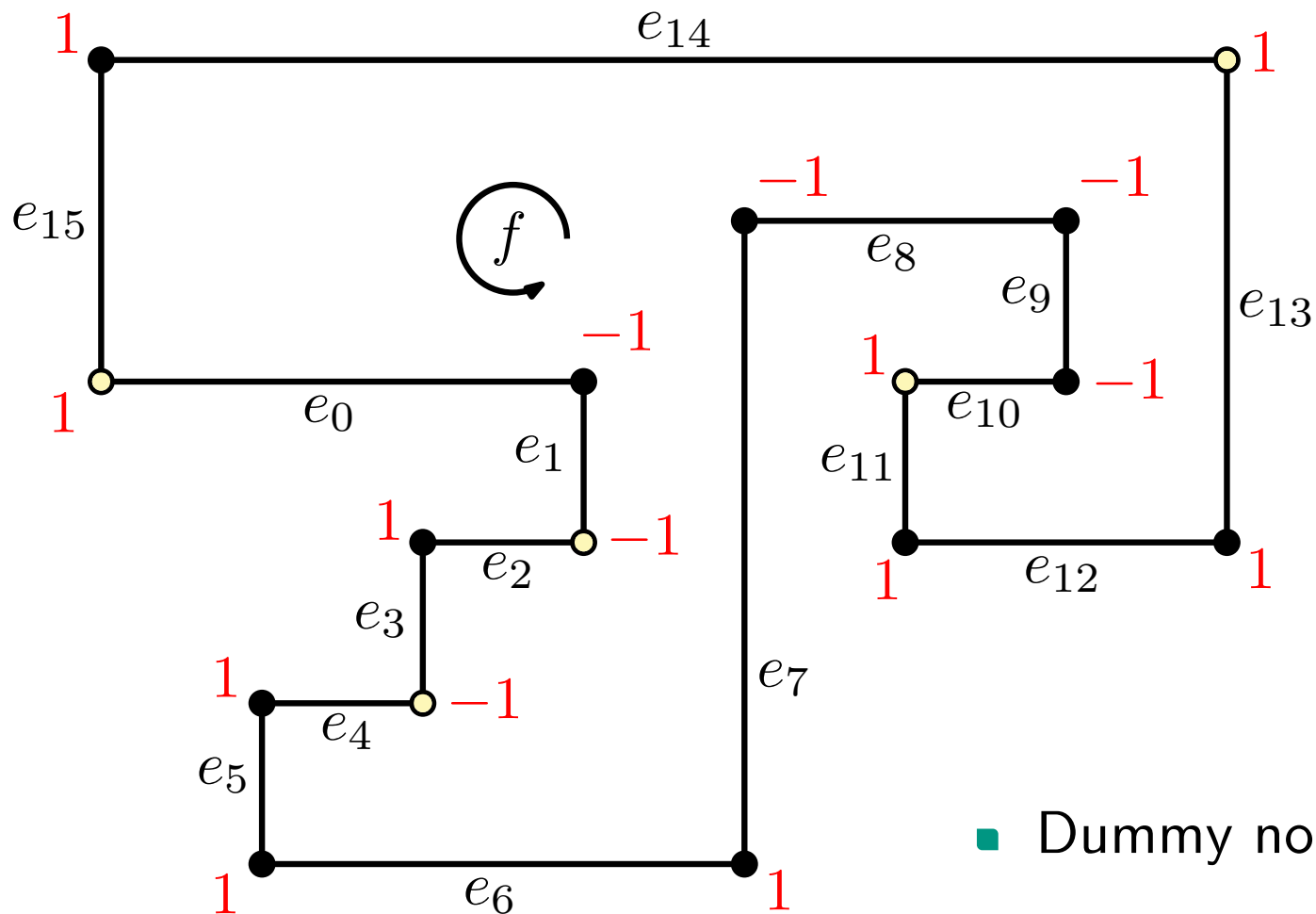
- Dummy nodes for bends

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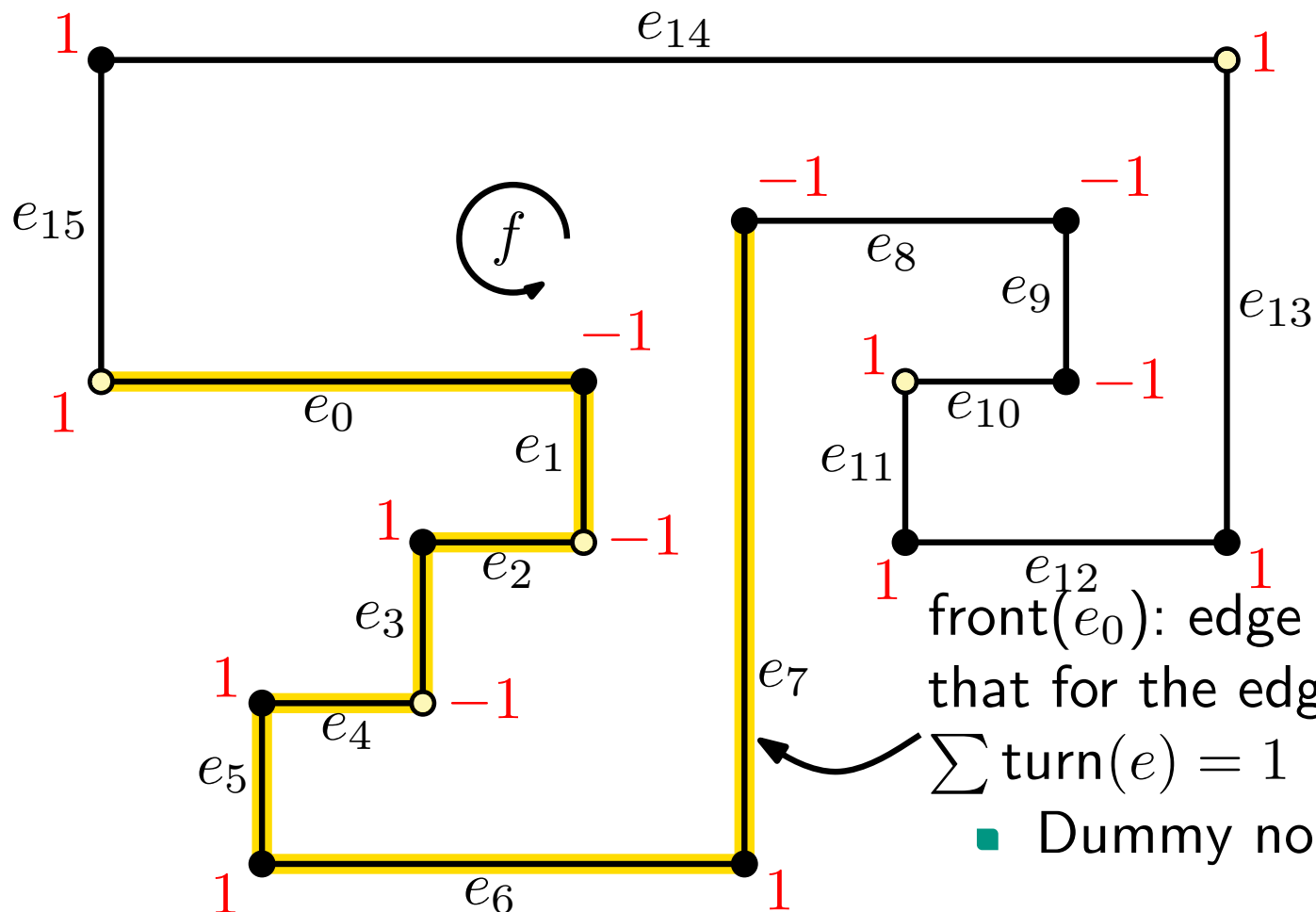
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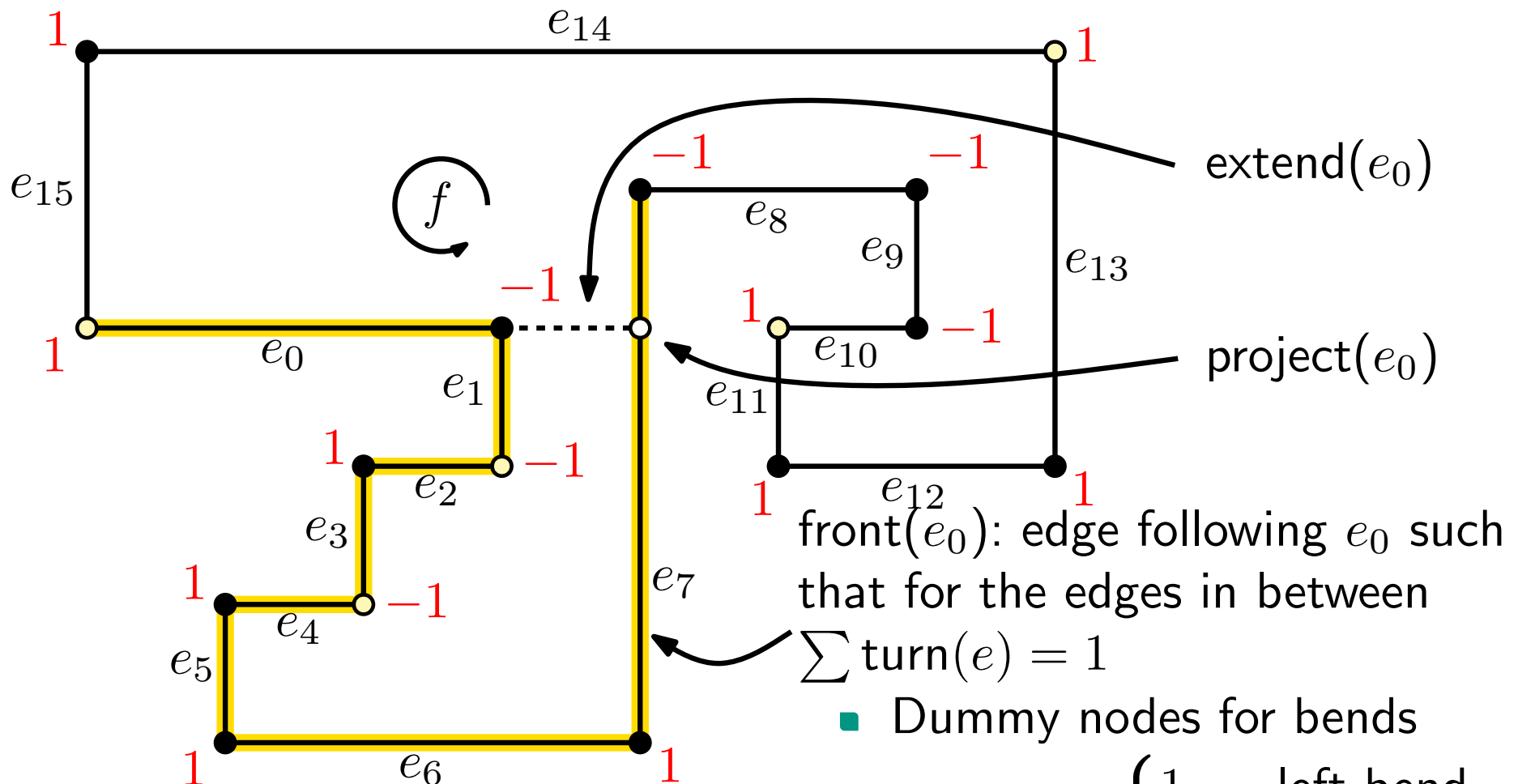


front(e_0): edge following e_0 such that for the edges in between

$$\sum \text{turn}(e) = 1$$

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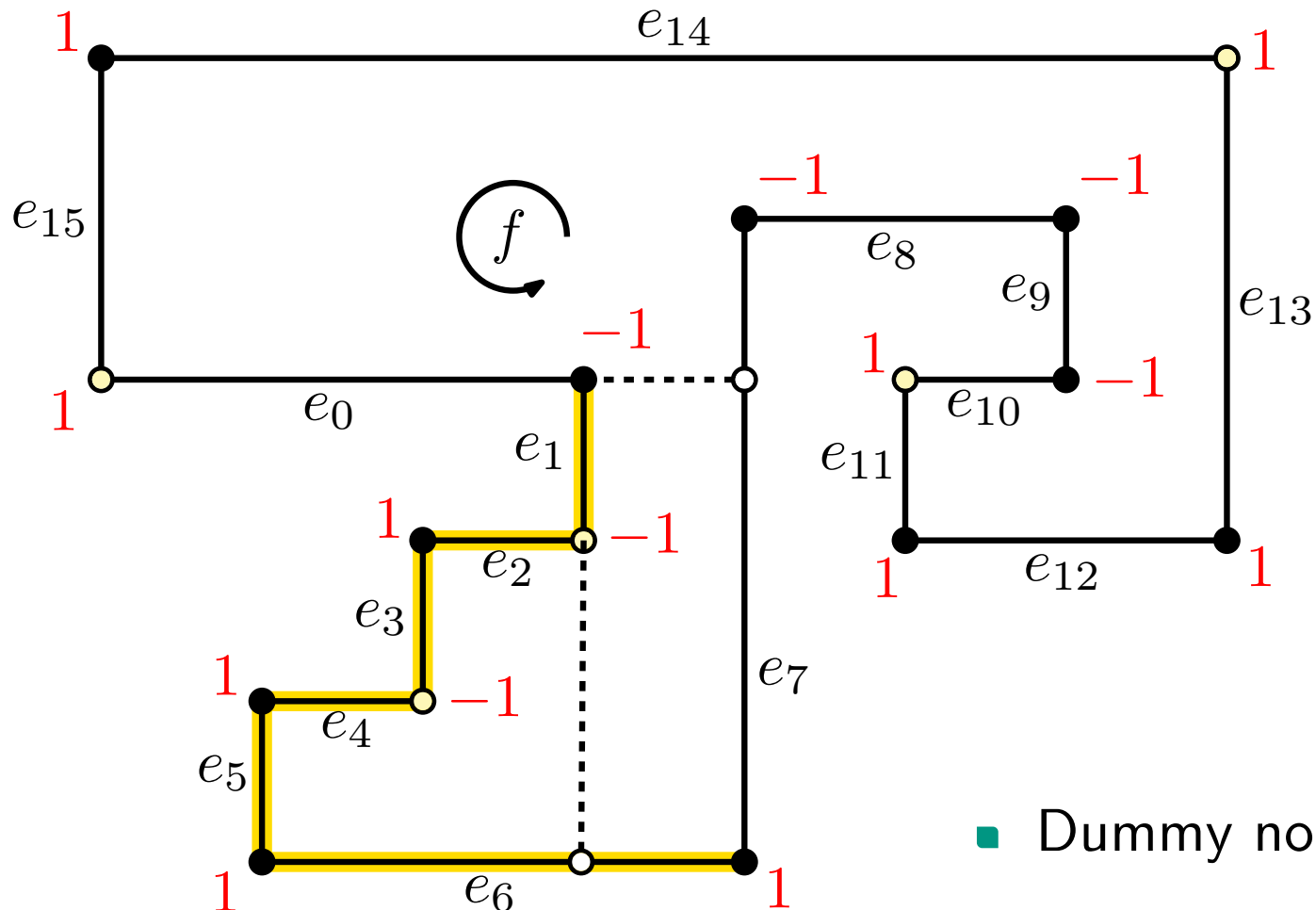
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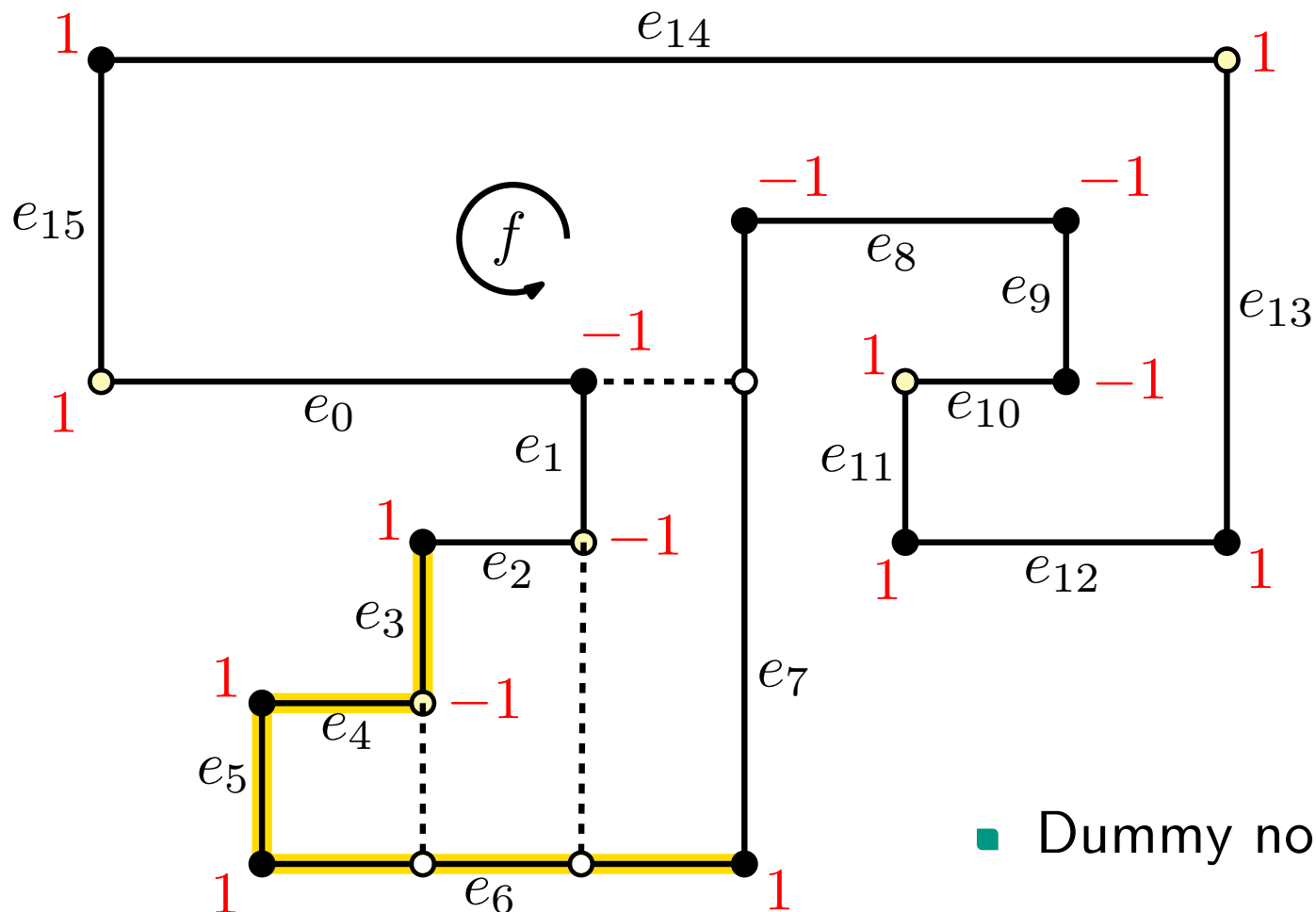
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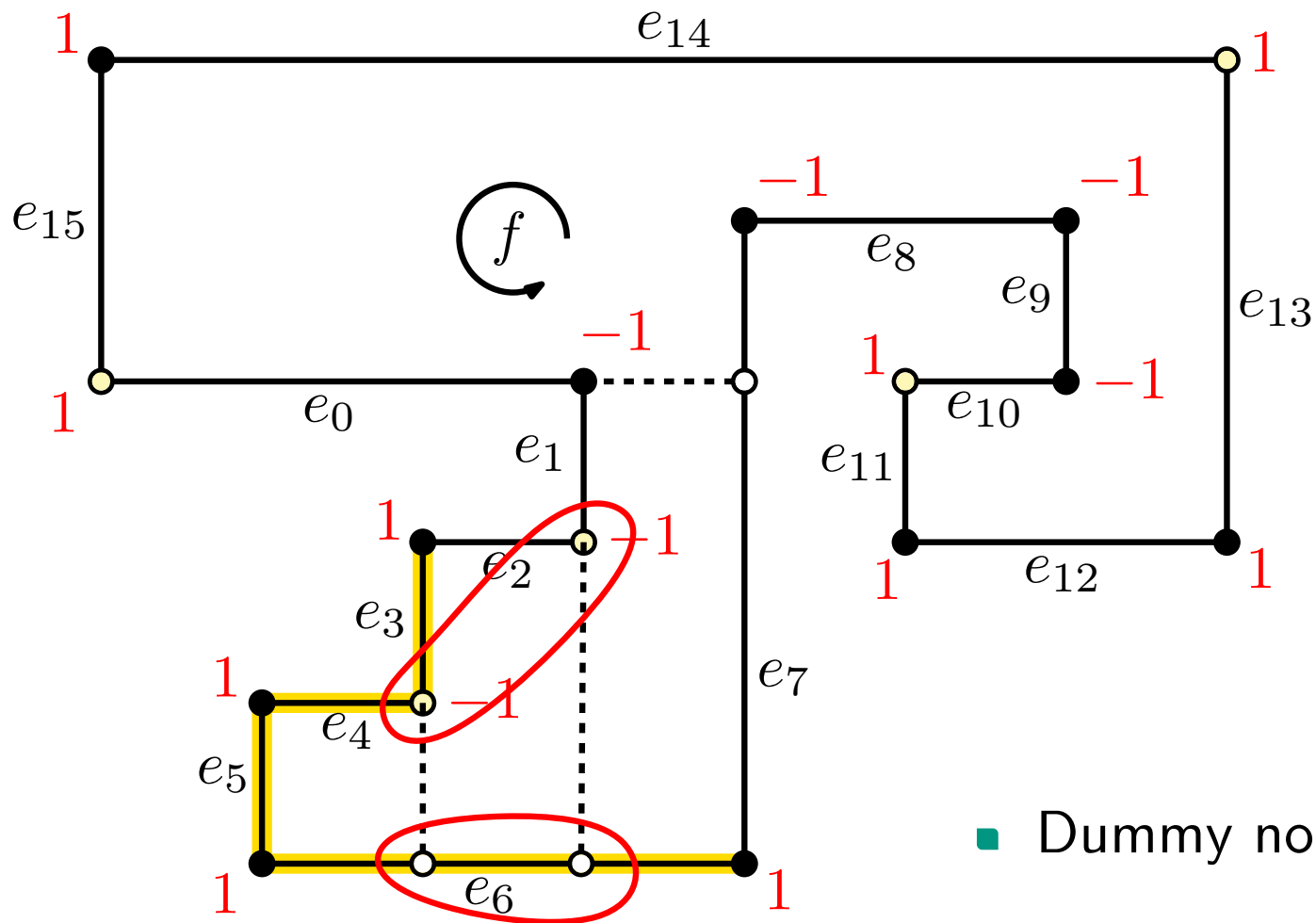
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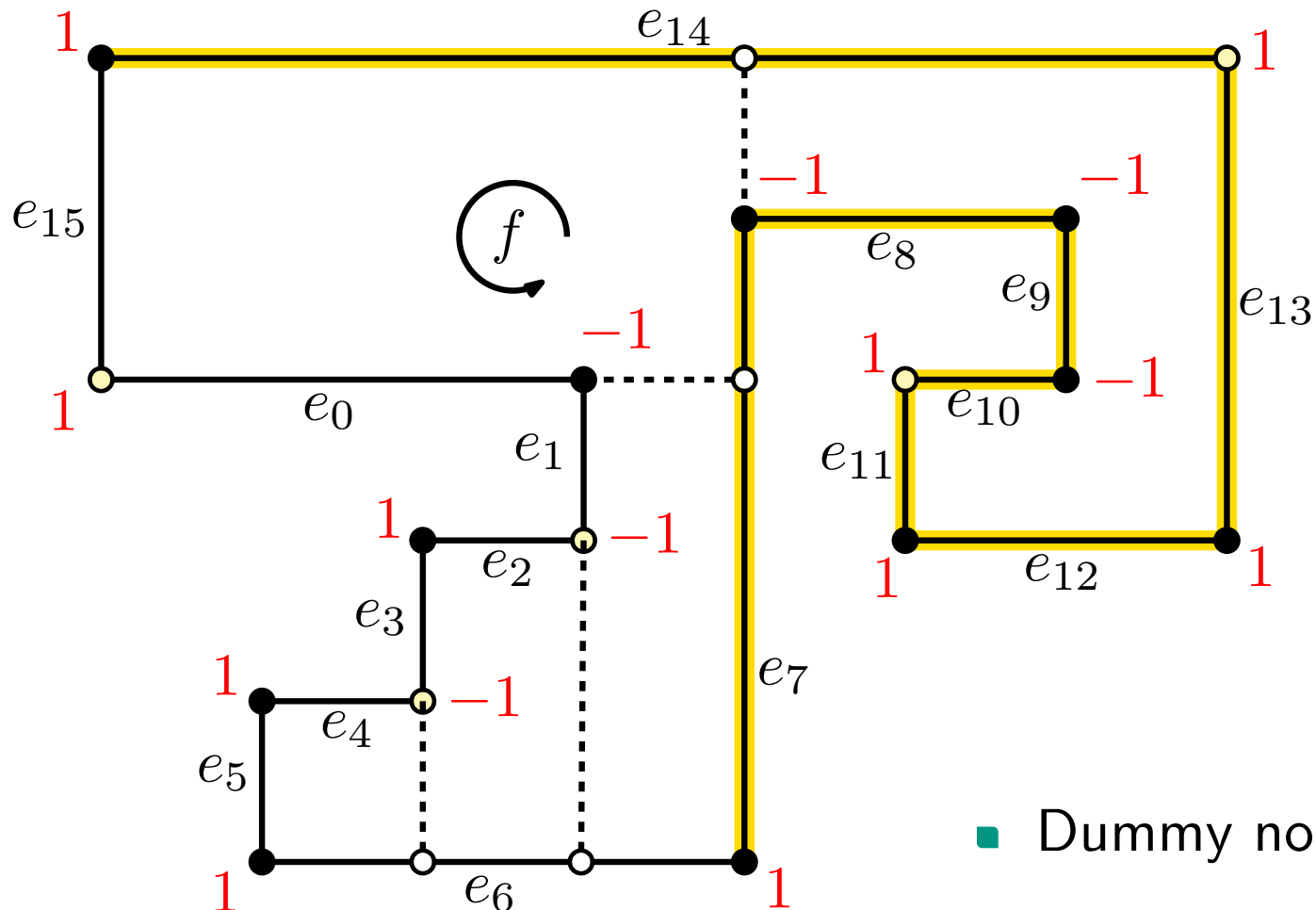
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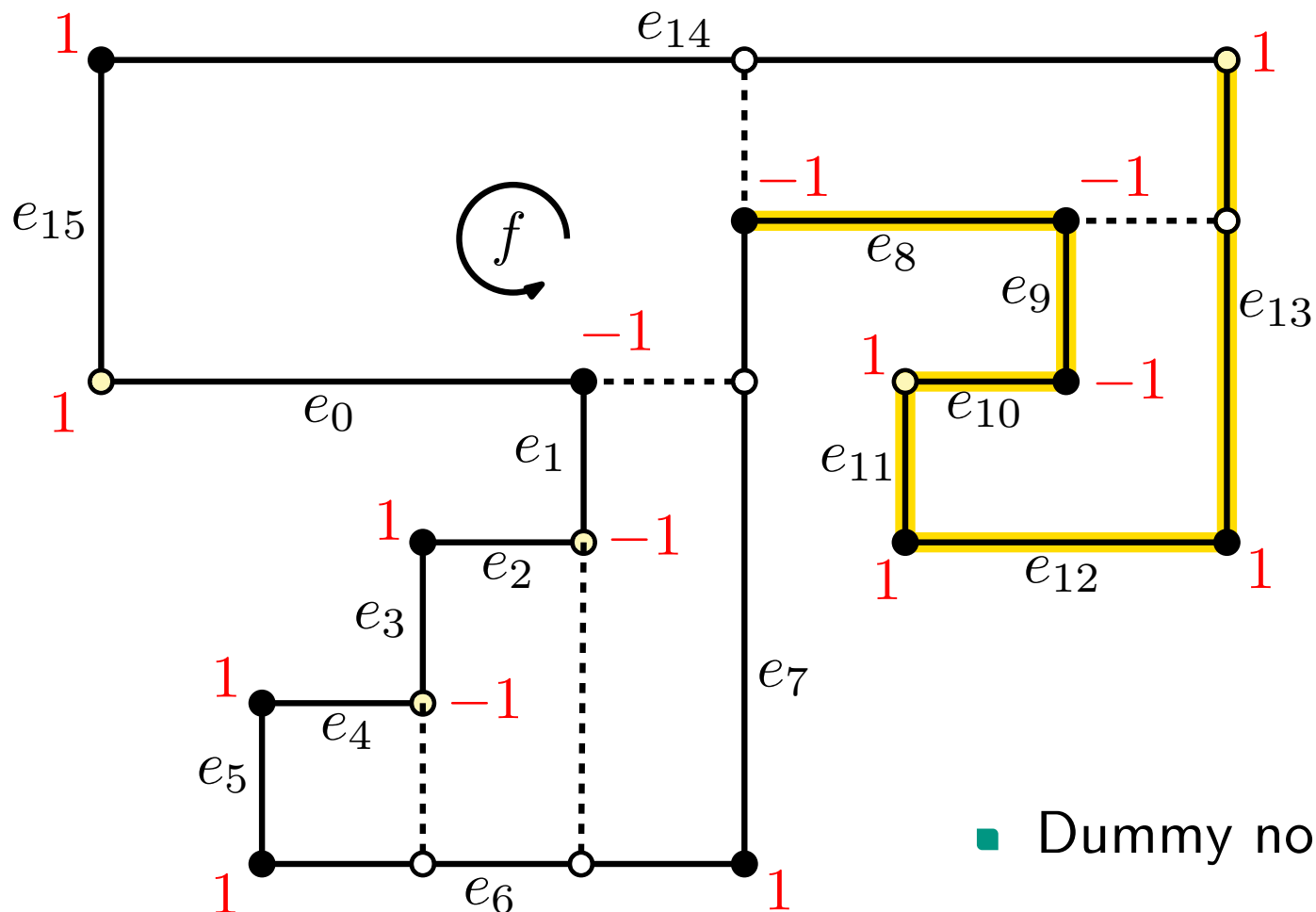
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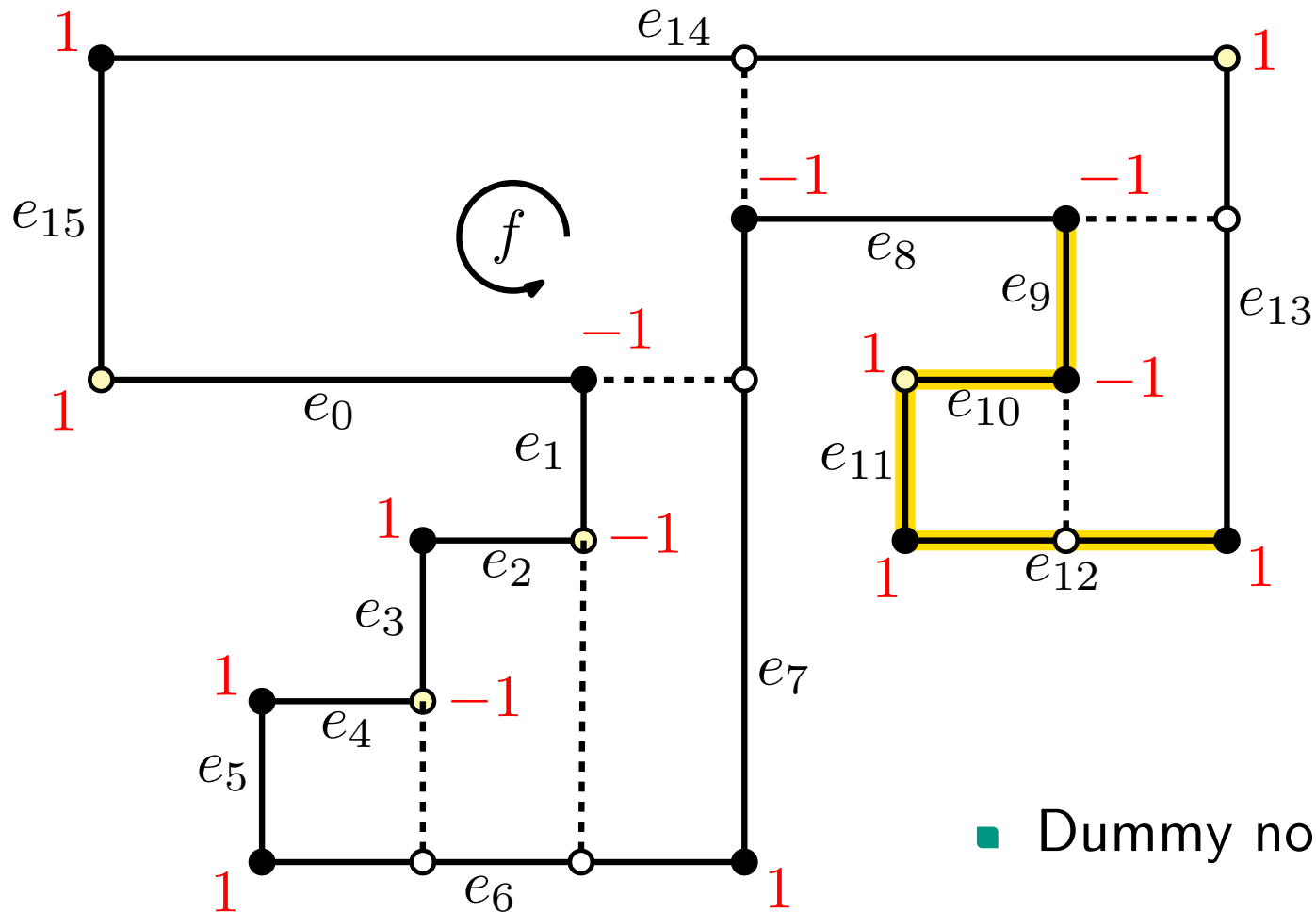
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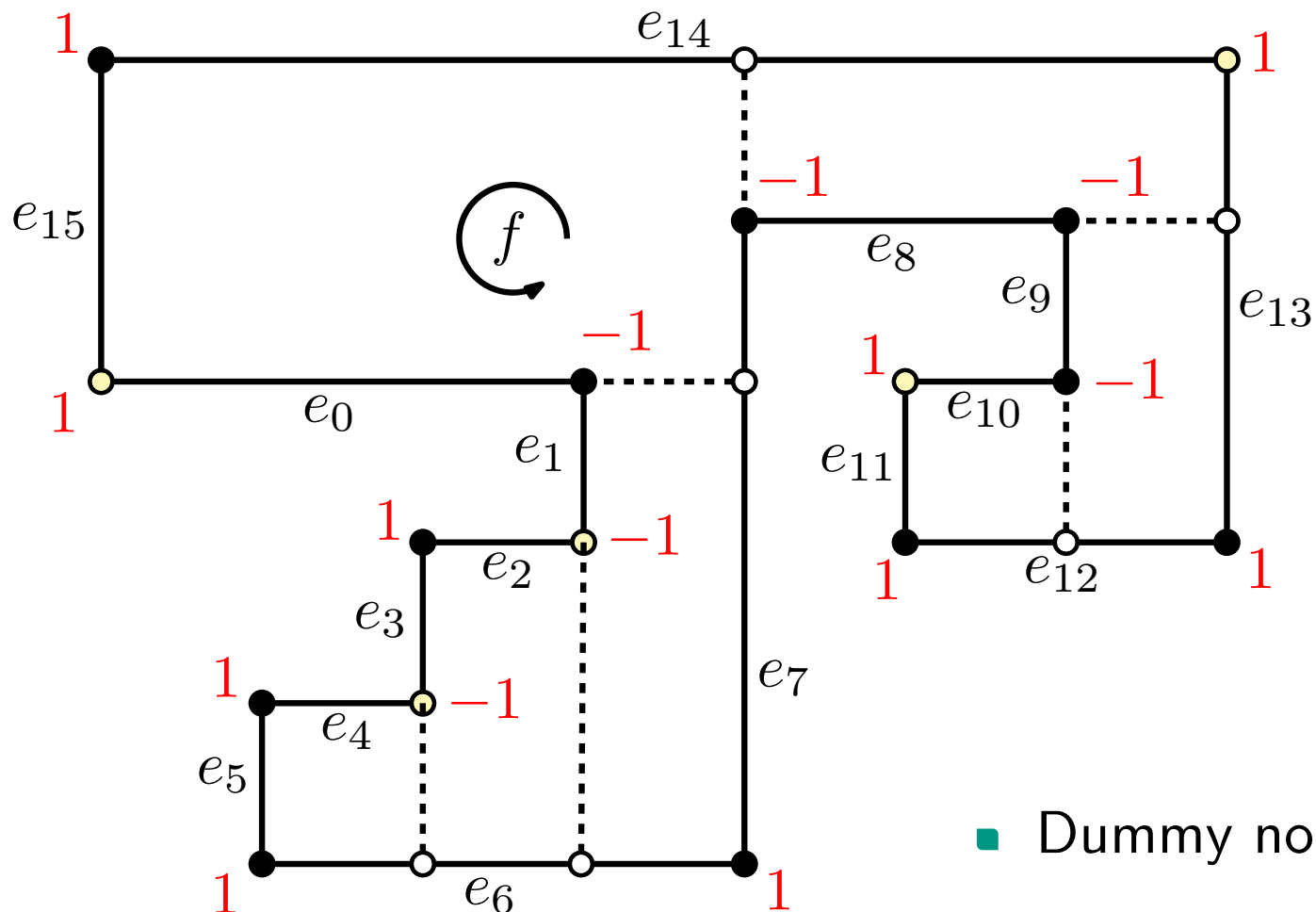
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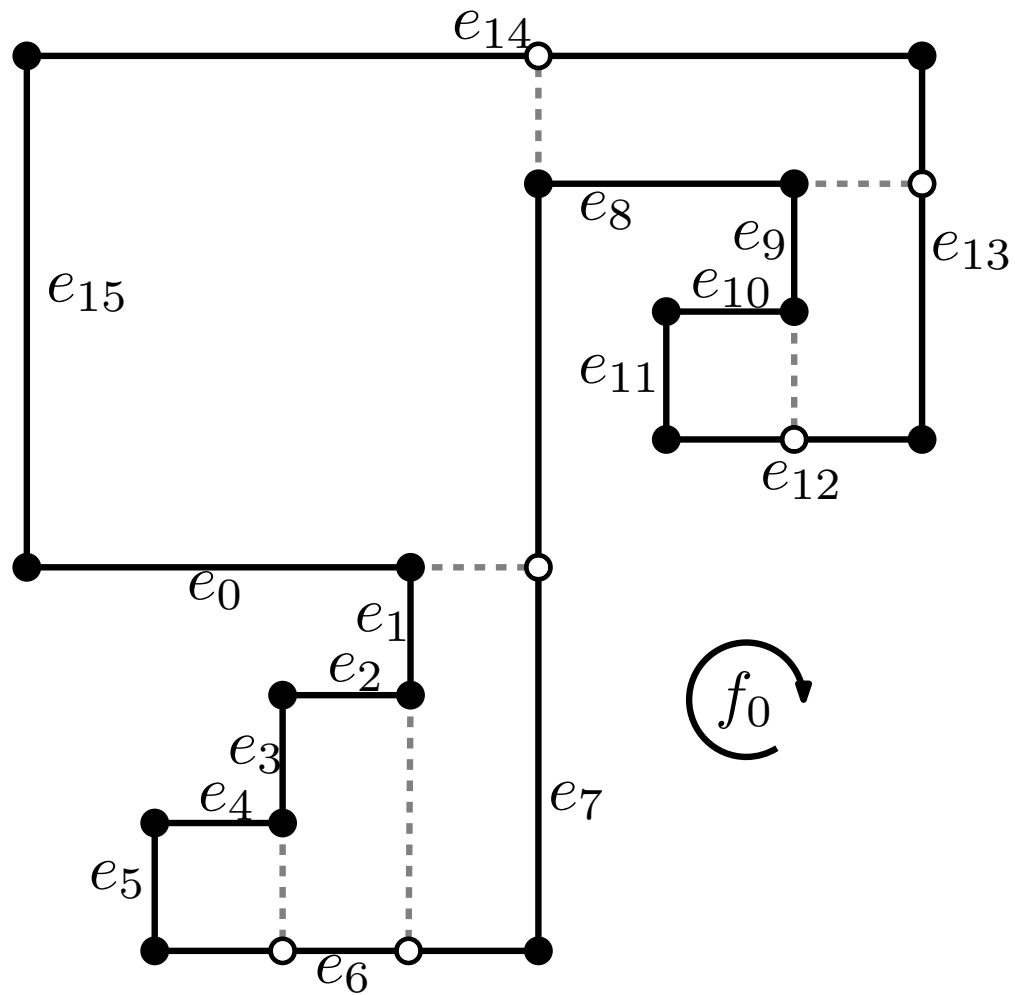
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Refinement of (G, H) – Inner Face

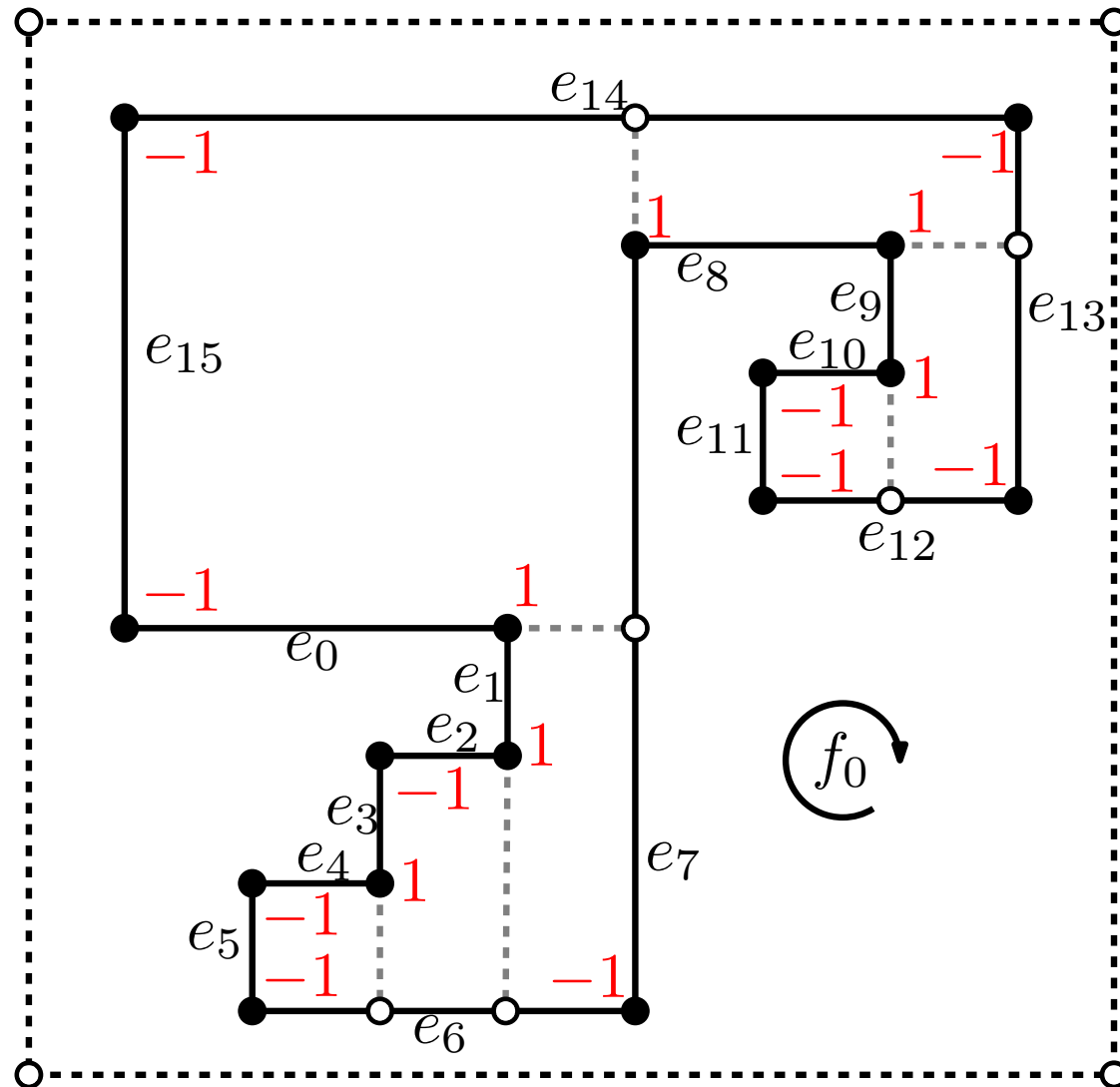


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Refinement of (G, H) – Outer Face



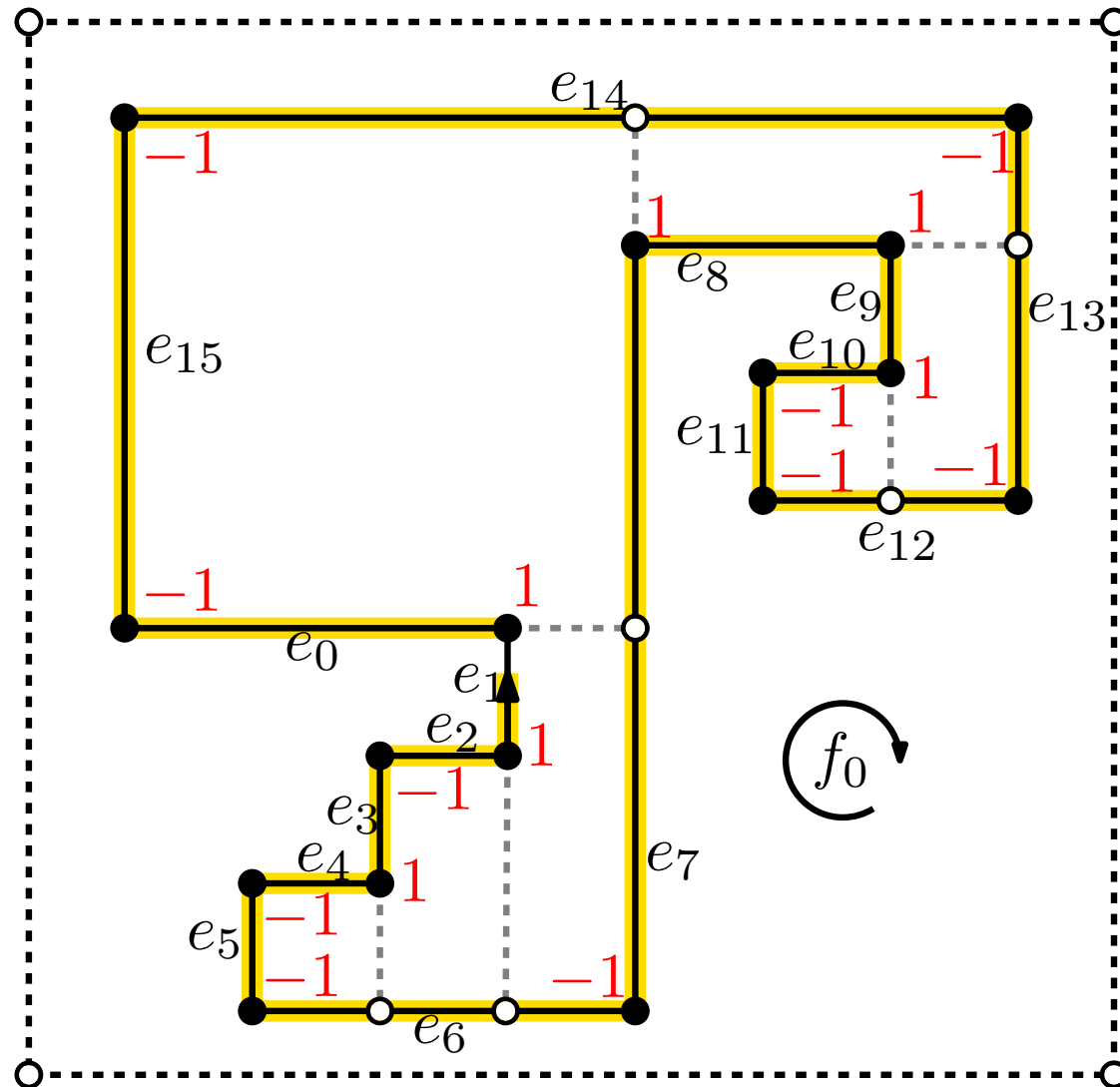
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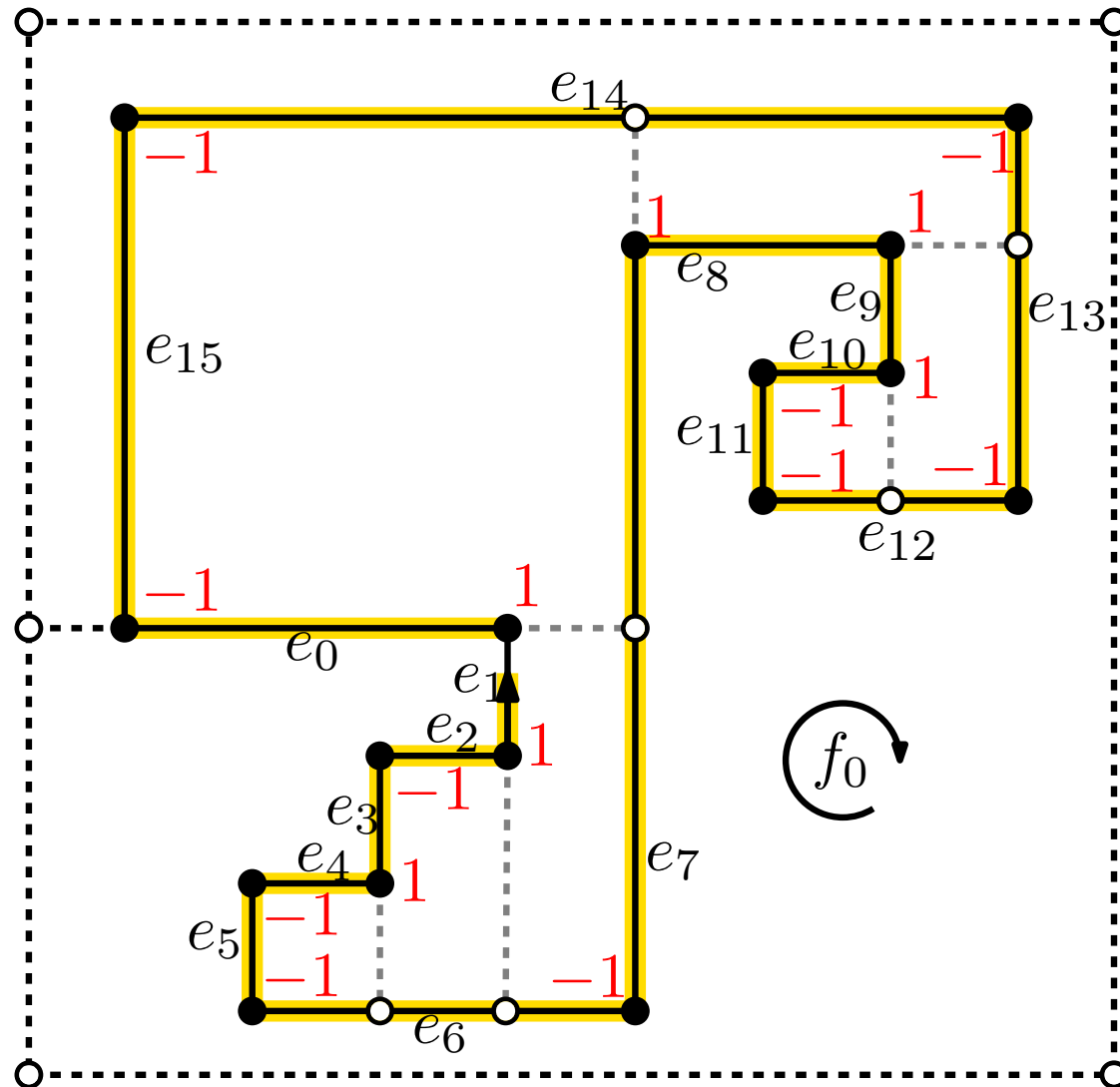
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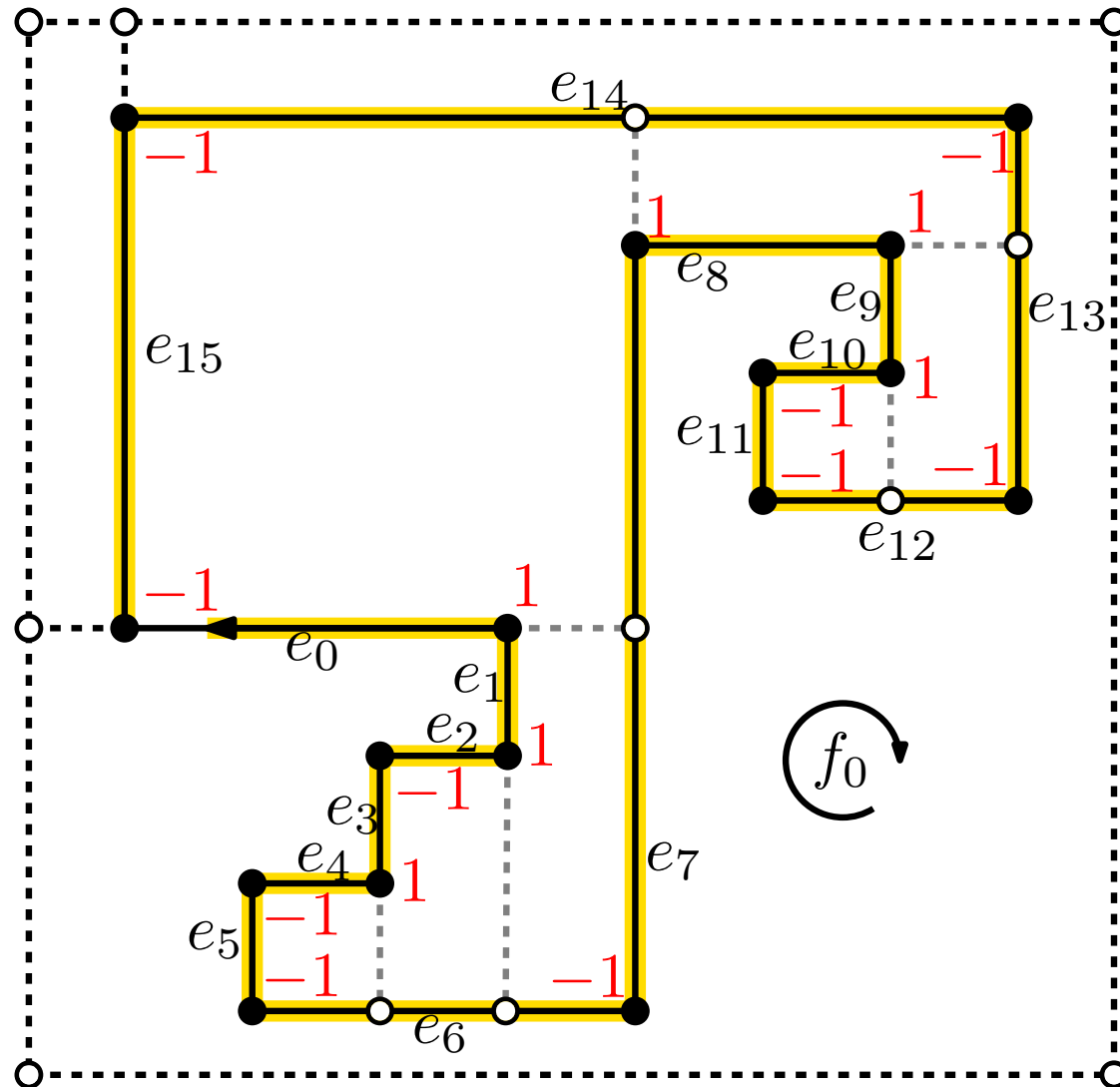
Refinement of (G, H) – Outer Face



- $\text{front}(e)$ may be undefined
- when $\sum \text{turn}(e) < 1$ for the complete turn around f_0 , project on R

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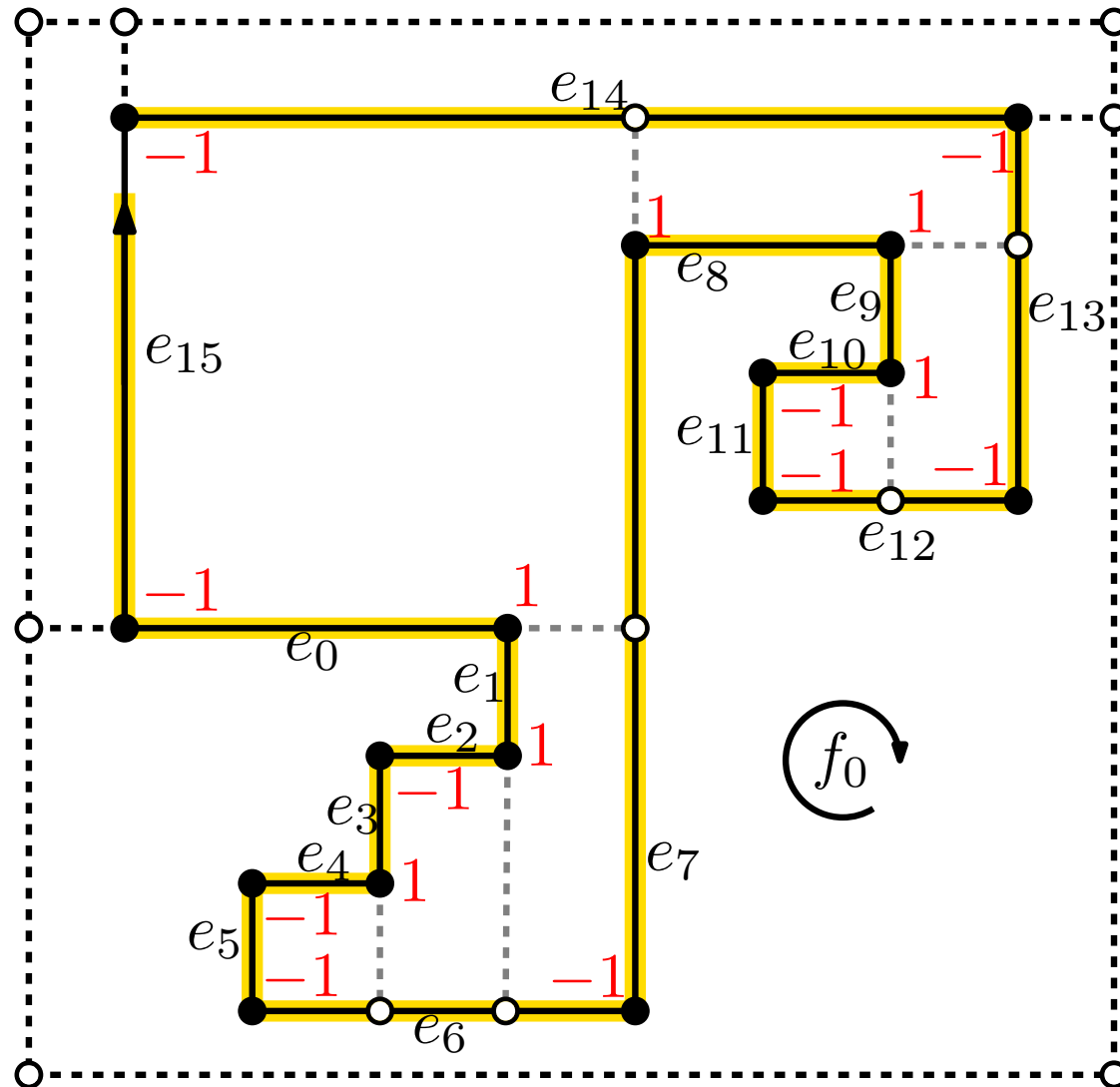
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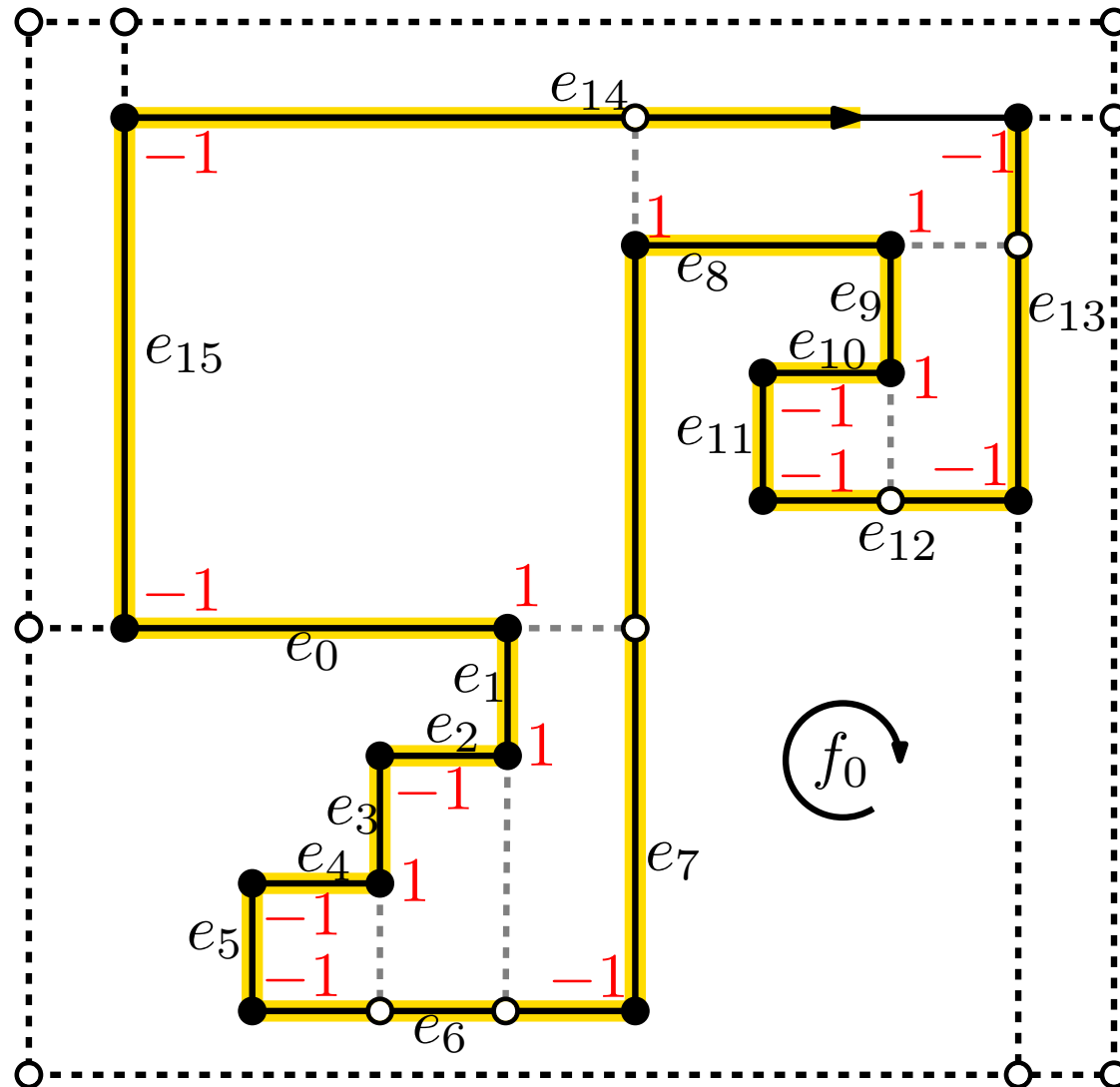
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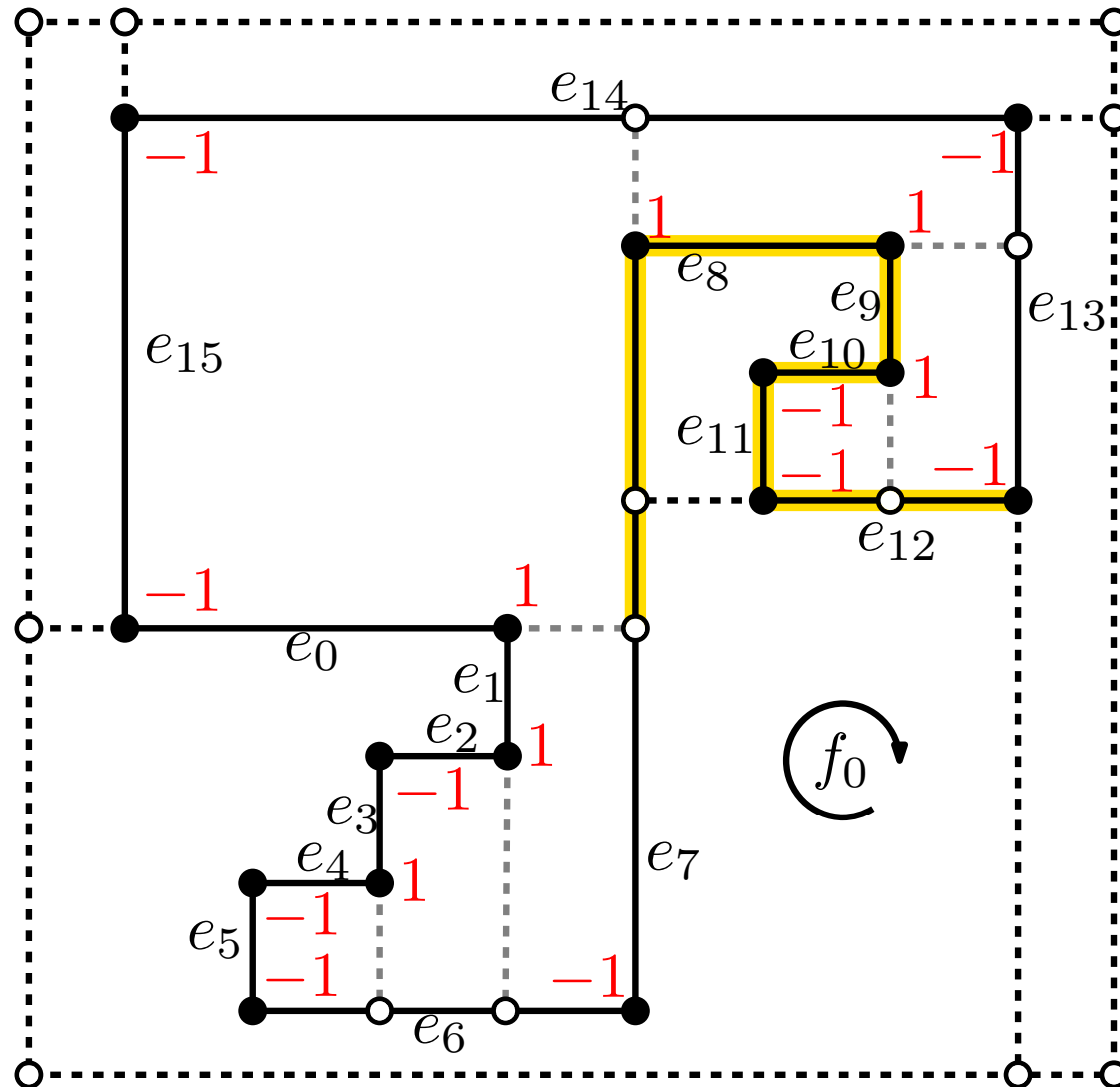
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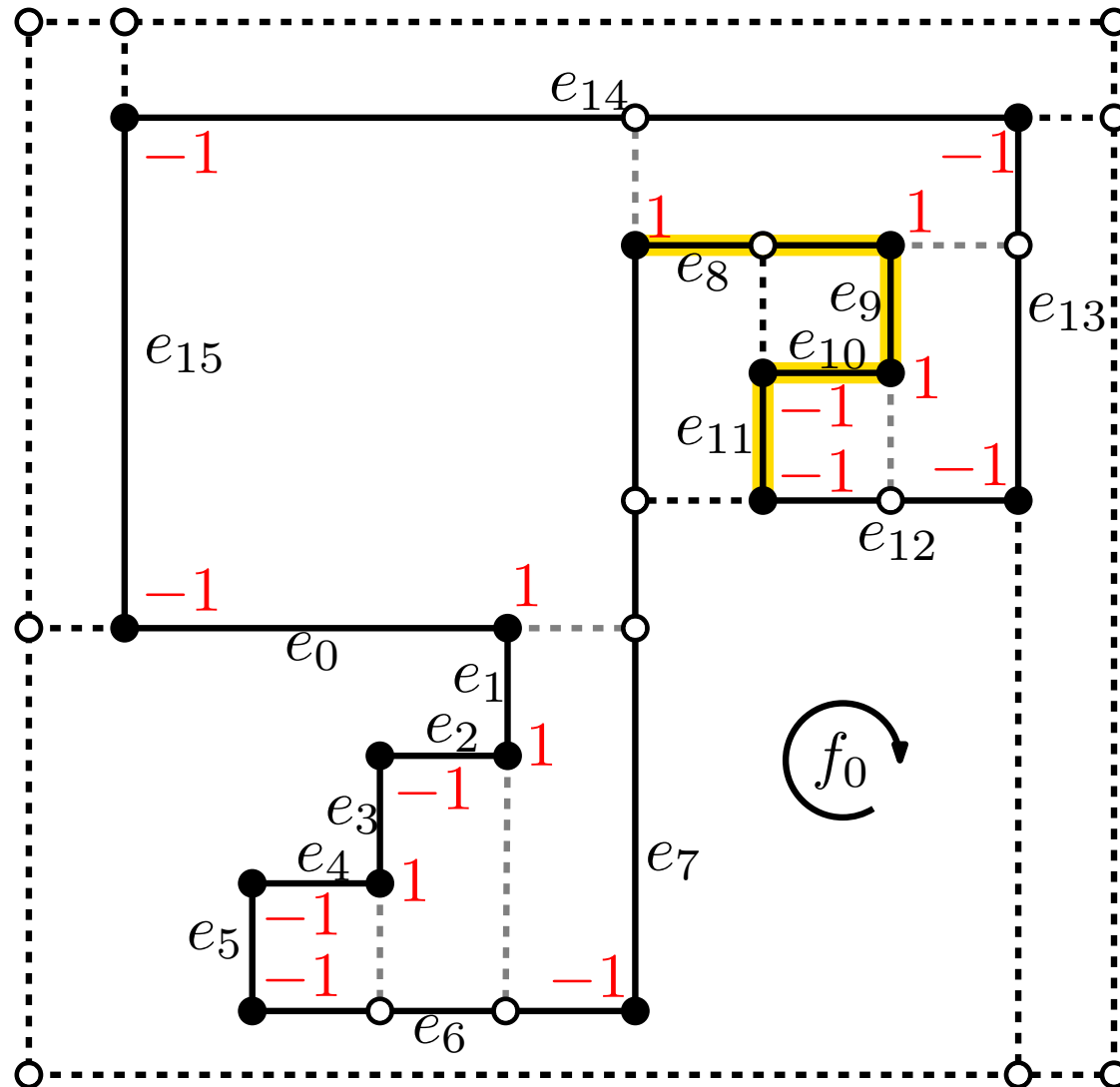
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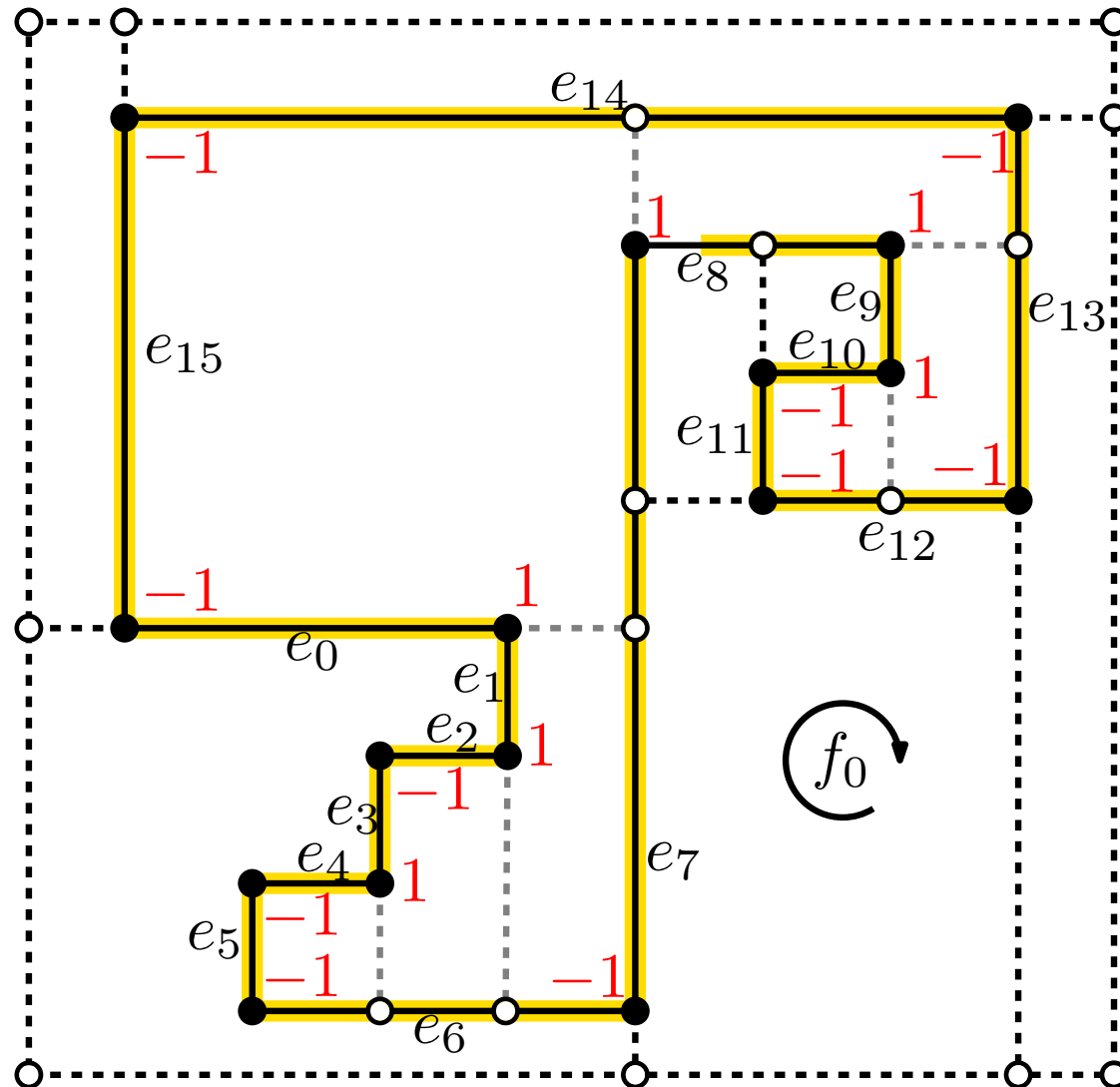
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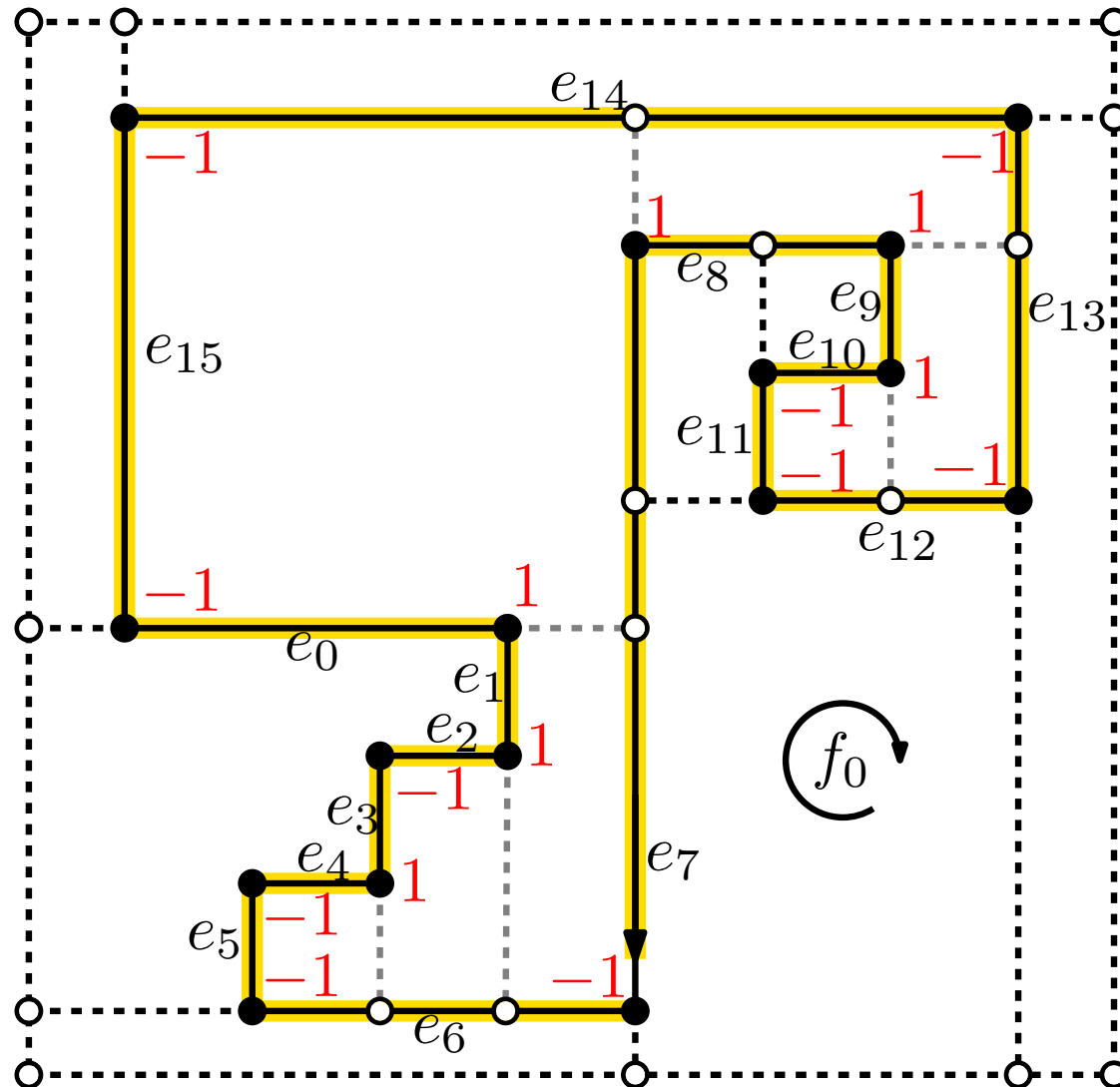
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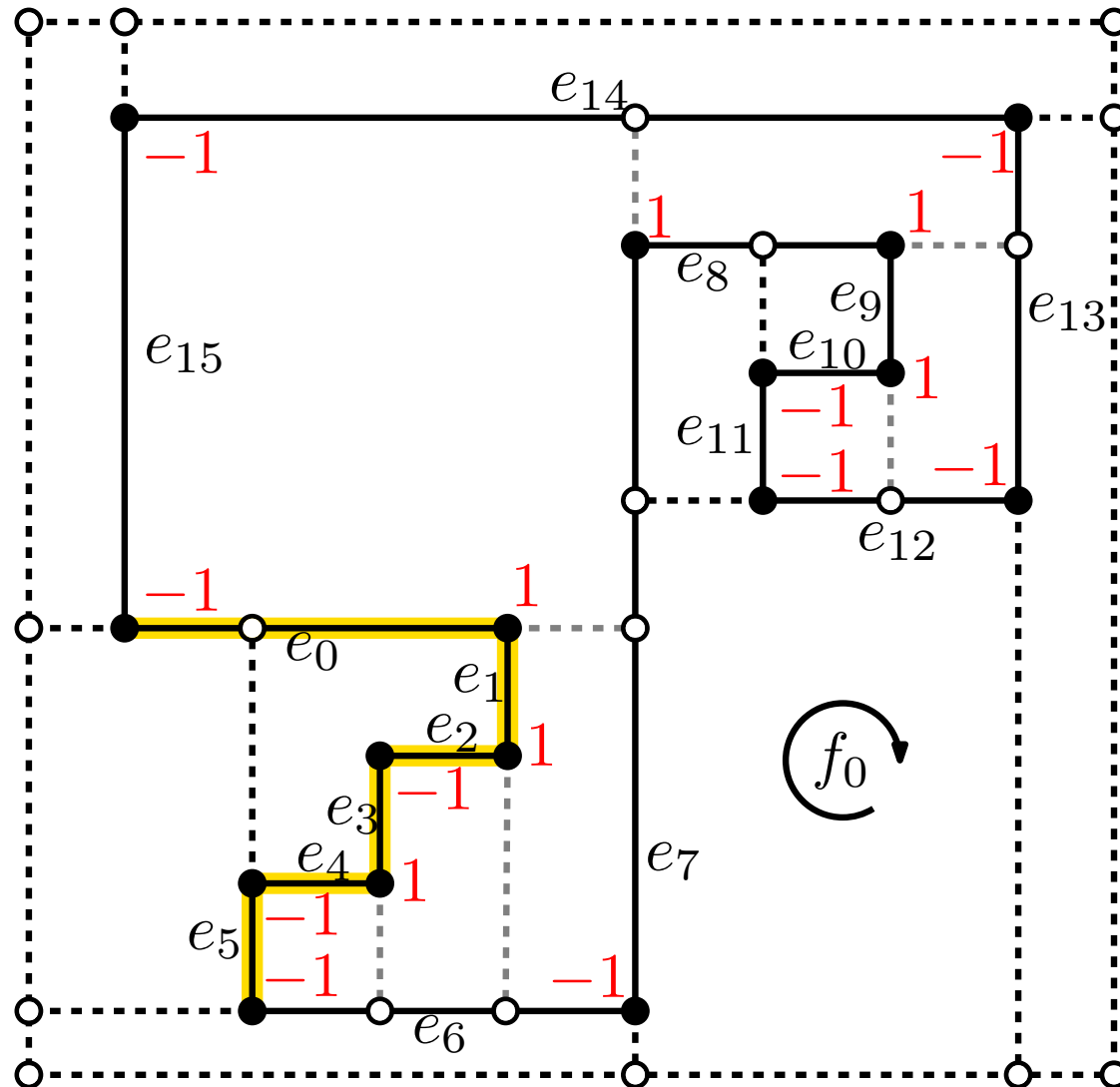
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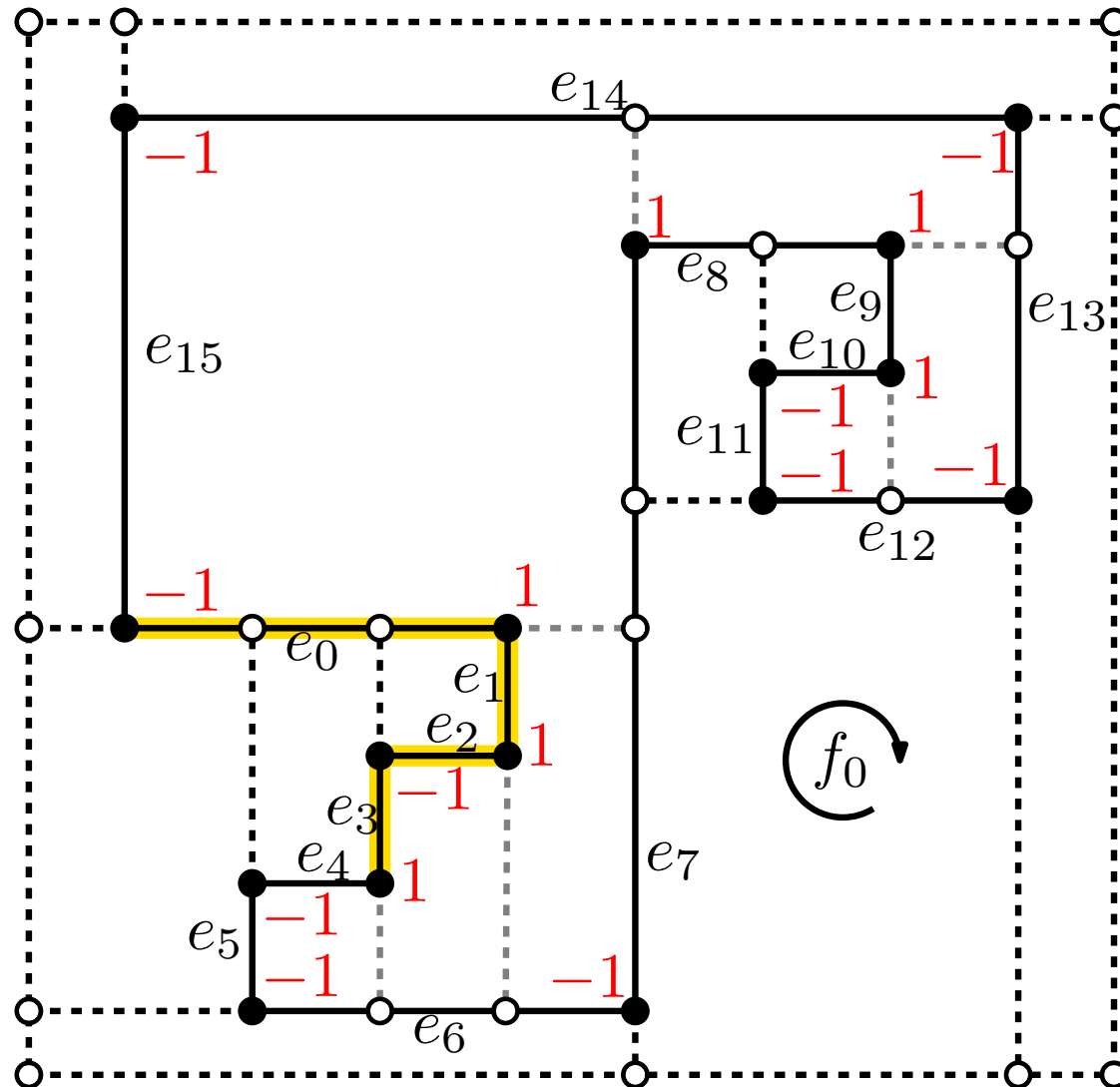
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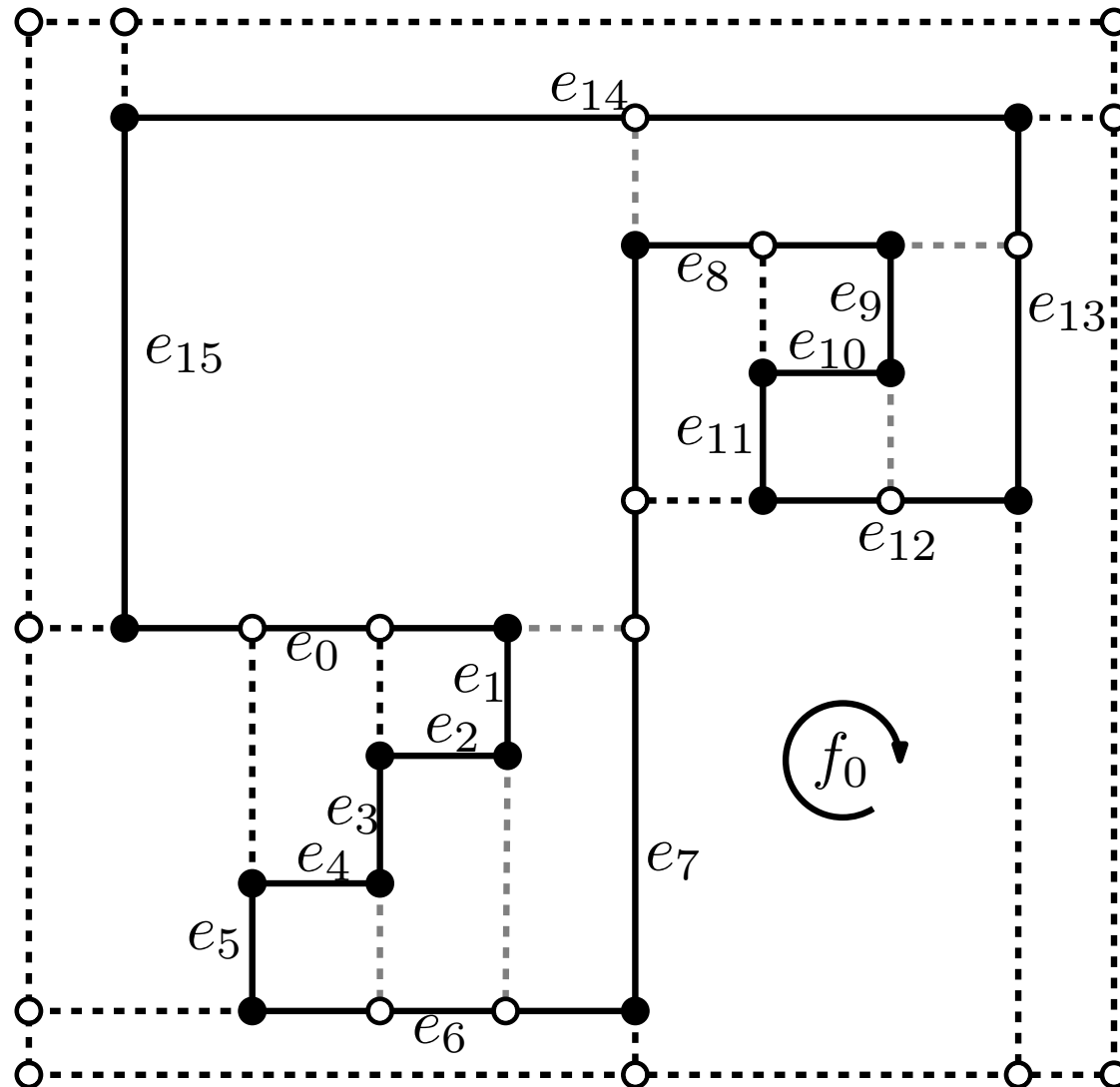
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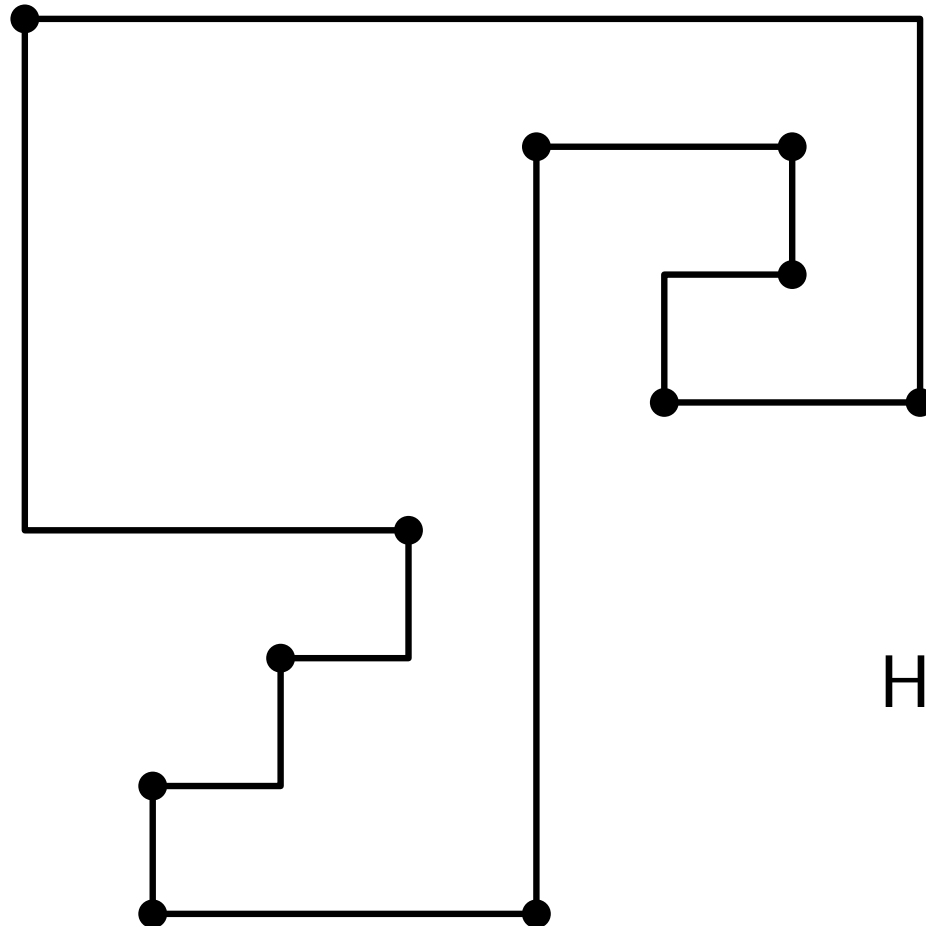
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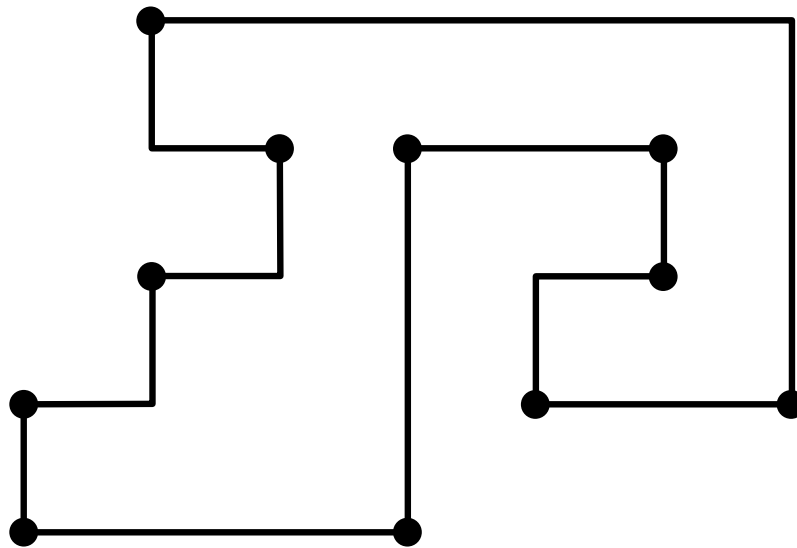
all faces are rectangles \rightarrow
apply flow network

Refinement of (G, H) – Outer Face



Has minimum area?

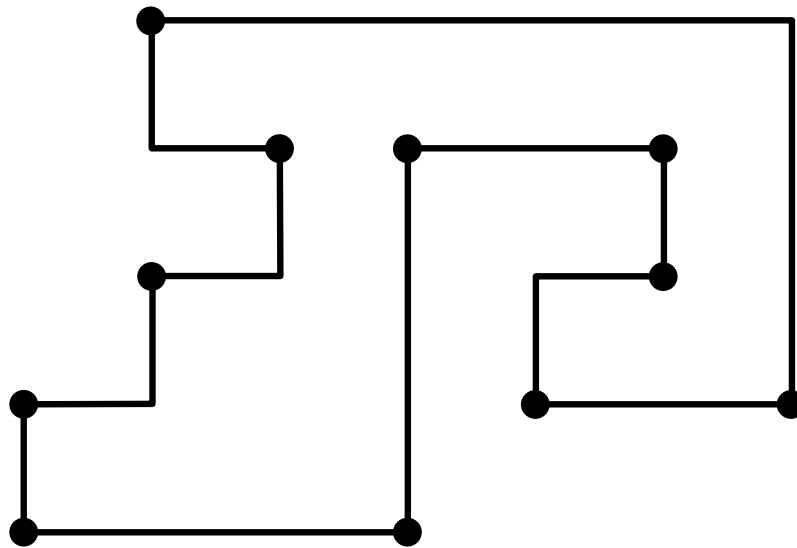
Refinement of (G, H) – Outer Face



Has minimum area?

NO!

Refinement of (G, H) – Outer Face



Has minimum area?

NO!

Area Minimization with a given orthogonal representation is an NP-hard problem!

Summary

- An orthogonal representation with minimum number of bends can be found in $O(n^{3/2})$ time
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[Patrignany CGTA 2001]

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- for non-planar graphs the area minimization is hard to approximate [Bannister, Eppstein, Simons JGAA 2012]