

Algorithms for Graph Visualization

Force-Directed Algorithms

INSTITUT FÜR THEORETISCHE INFORMATIK · FAKULTÄT FÜR INFORMATIK

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21.12.2016



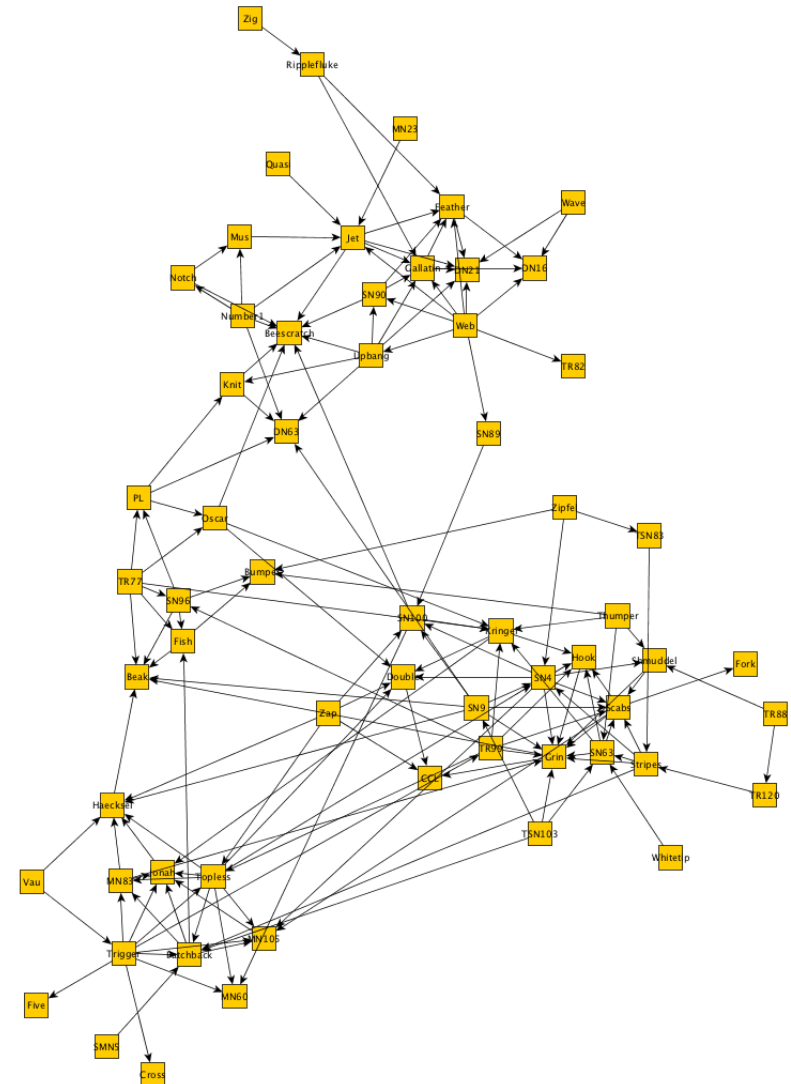
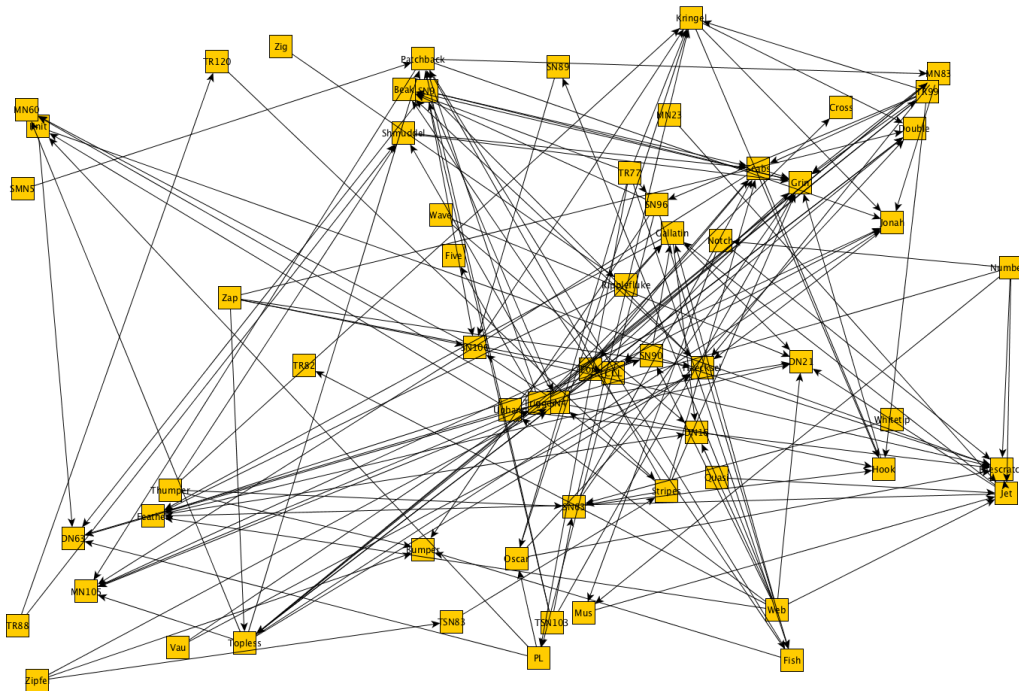
Introduction

- Before: always based on some properties: tree, series-parallel graph, planar graph
- and on some additional information: ordering of the vertices, decompositions into SP-components
- Today: more direct and intuitive method based on physical analogies
- The methods are very popular: intuitiveness, easy to program, generality, fairly satisfactory results,...

General Layout Problem

Given: Graph $G = (V, E)$

Find: Clear and readable drawing of G



Which aesthetic criteria would you optimize?

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Criteria:

- adjacent nodes are close
- non-adjacent far apart
- edges short, straight-line, similar length
- densely connected parts (clusters) form communities
- as few crossings as possible
- nodes distributed evenly

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Optimization criteria partially contradict each other

Example: Fixed edge-length

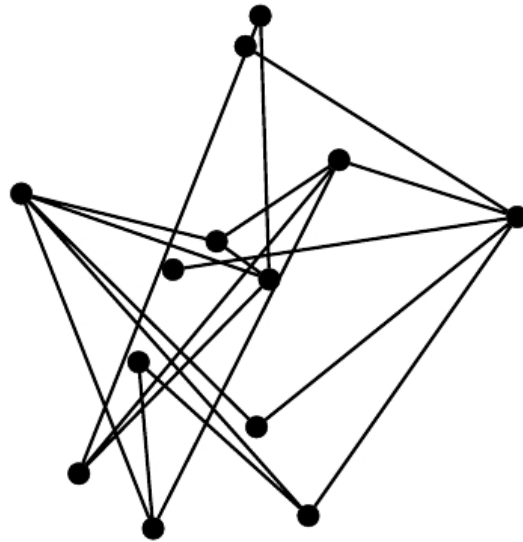
Given: Graph $G = (V, E)$, required edge length $\ell(e)$, $\forall e \in E$

Find: Drawing of G which realizes all the edge lengths

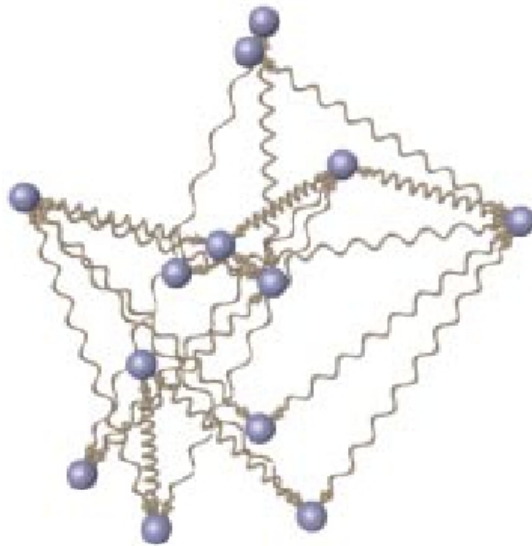
NP-hard for

- edge lengths $\{1, 2\}$ [Saxe, '80]
- planar drawing with unit edge length [Eades, Wormald, '90]

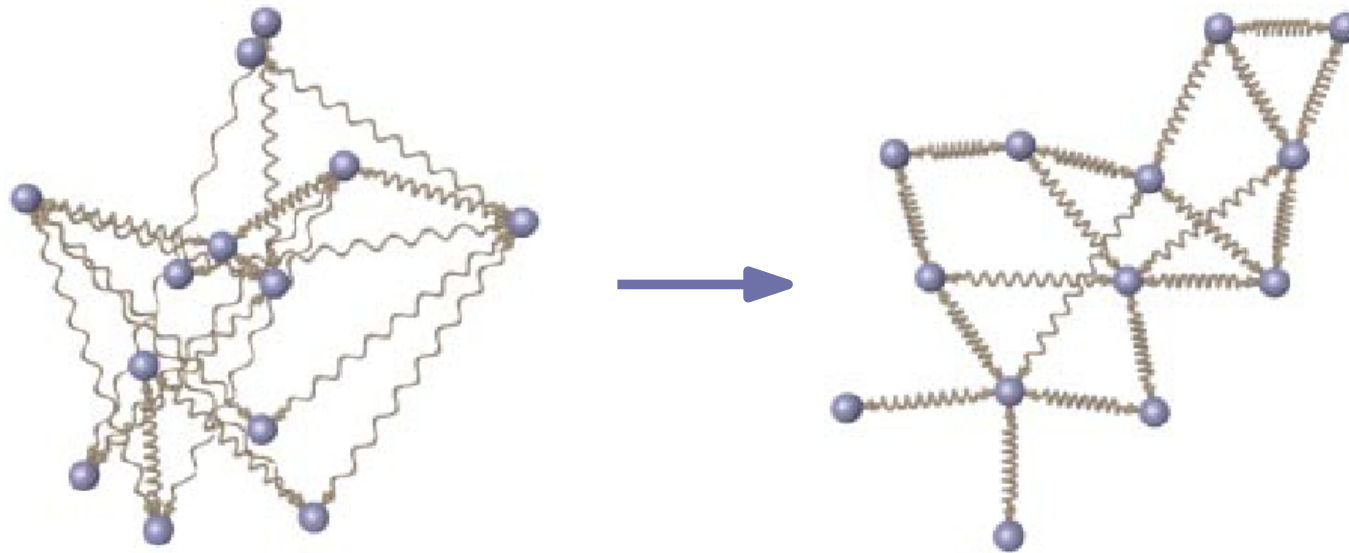
Physical Model



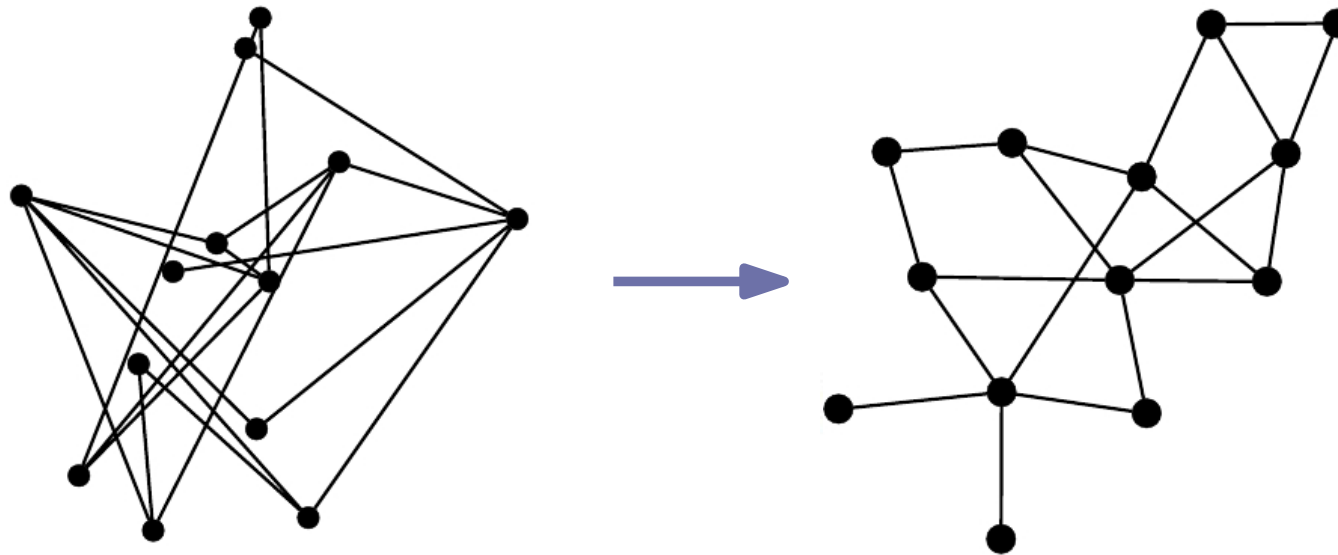
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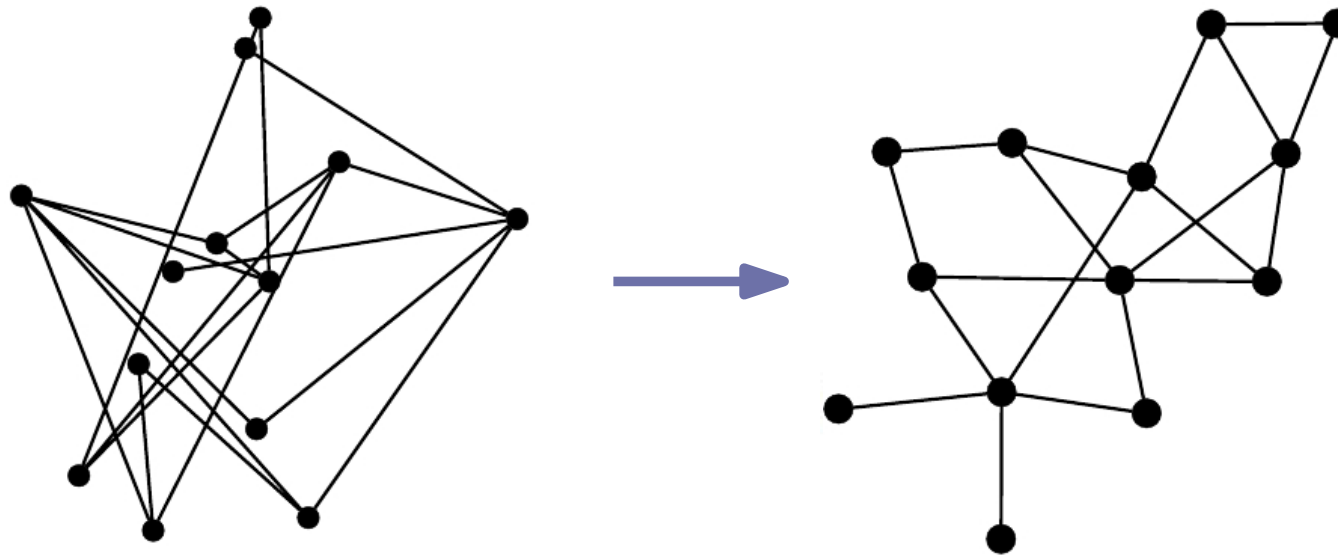
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So-called **spring-embedder** algorithms that work according to this or similar principles are among the most frequently used graph-drawing methods in practice.

“To each node in the graph, a force is applied, and the nodes move the system to a minimal energy state. [Lades, 04]”

$$\ell = \ell(e)$$

ideal spring length for edge e

$$p_v = (x_v, y_v)$$

position of node v

$$\|p_u - p_v\|$$

Euclidean distance between u and v

$$\overrightarrow{p_u p_v}$$

unit vector pointing from u to v

Model:

- repulsive force between two non-adjacent nodes u and v

$$f_{\text{rep}}(p_u, p_v) = \frac{c_{\text{rep}}}{\|p_v - p_u\|^2} \cdot \overrightarrow{p_u p_v}$$

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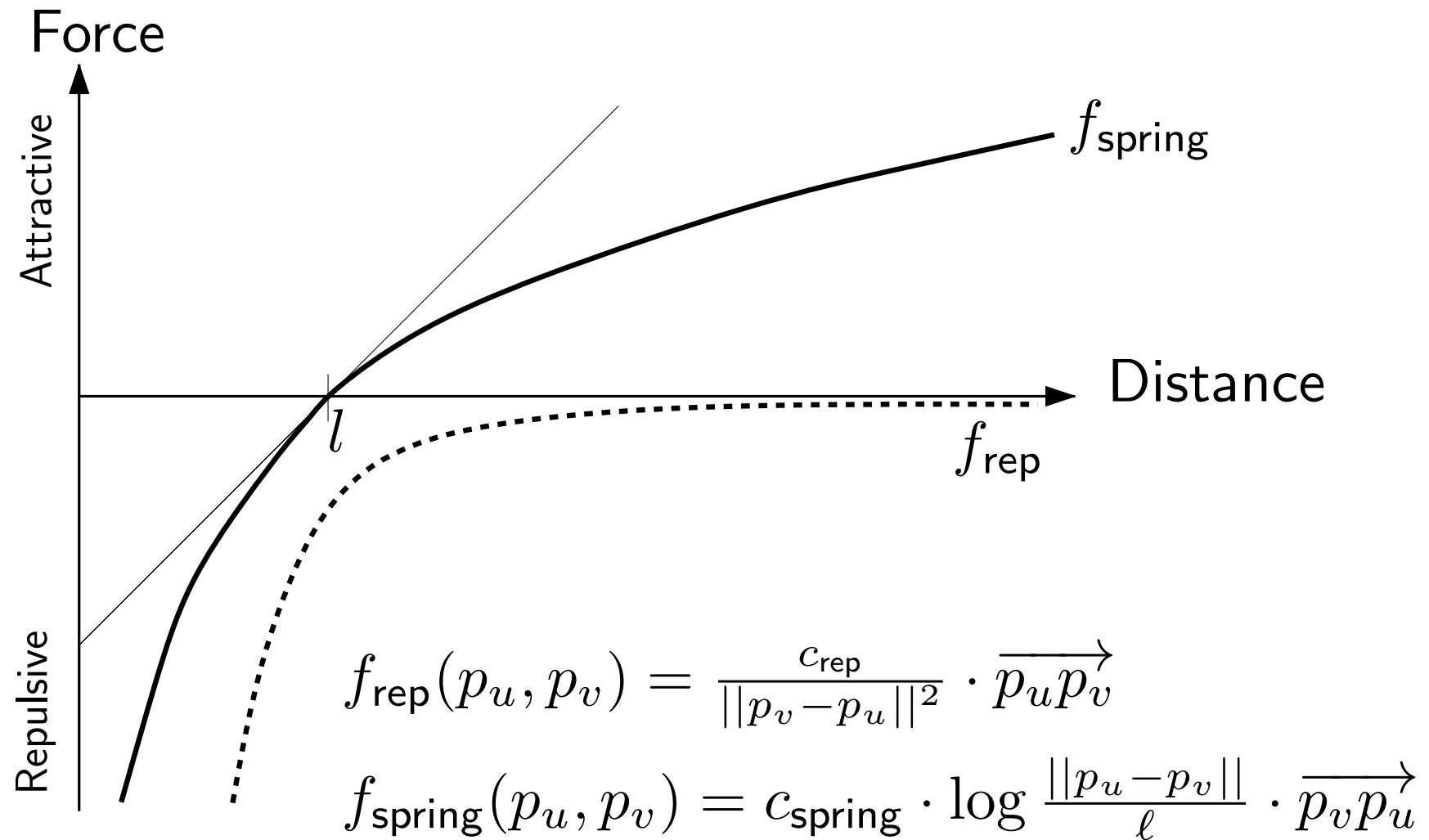
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- resulting displacement vector for node v

$$F_v = \sum_{u: \{u, v\} \notin E} f_{\text{rep}}(p_u, p_v) + \sum_{u: \{u, v\} \in E} f_{\text{spring}}(p_u, p_v)$$

Diagram of Spring-Embedder Forces (Eades, 1984)



Input: $G = (V, E)$ connected undirected graph with initial placement $p = (p_v)_{v \in V}$, number of iterations $K \in \mathbb{N}$, threshold $\varepsilon > 0$, constant $\delta > 0$

Output: Layout p with "low internal stress"

$t \leftarrow 1$

while $t < K$ **and** $\max_{v \in V} \|F_v(t)\| > \varepsilon$ **do**

foreach $v \in V$ **do**

$$F_v(t) \leftarrow \sum_{u: \{u,v\} \notin E} f_{\text{rep}}(p_u, p_v) + \sum_{u: \{u,v\} \in E} f_{\text{spring}}(p_u, p_v)$$

foreach $v \in V$ **do**

$$p_v \leftarrow p_v + \delta \cdot F_v(t)$$

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Demo

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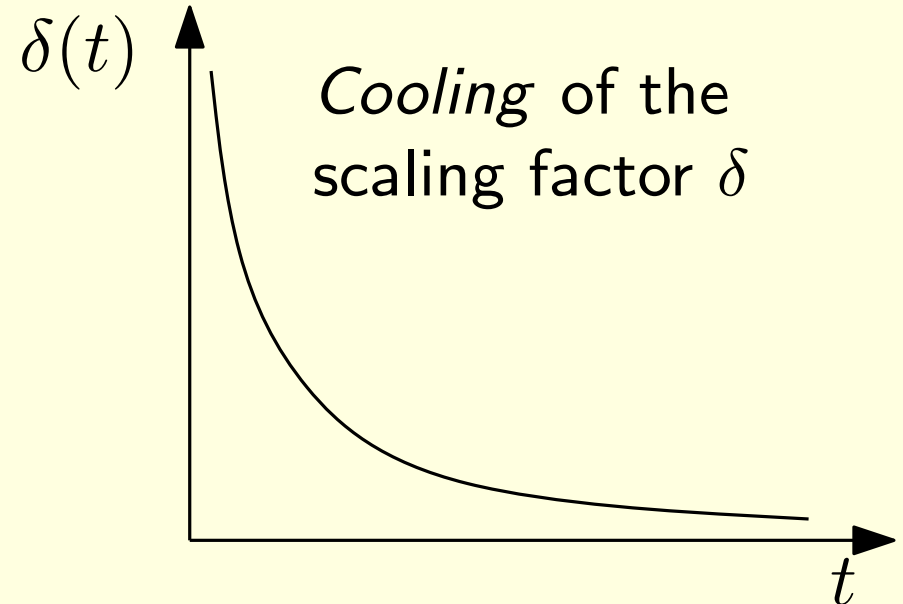
$$F_v(t) \leftarrow \sum_{u: \{u,v\} \notin E} \frac{1}{|u-v|} \cdot (p_u - p_v)$$

$$\sum_{u: \{u,v\} \in E} f_{uv} \cdot (p_u - p_v)$$

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- very simple Algorithm
- good results for small and medium-sized graphs
- empirically good representation of symmetry and structure

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Influence

- Original paper by Peter Eades got 1600 citations (400 in the past two years)
- Basis for many further ideas

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- repulsive force between **all** node pairs u and v

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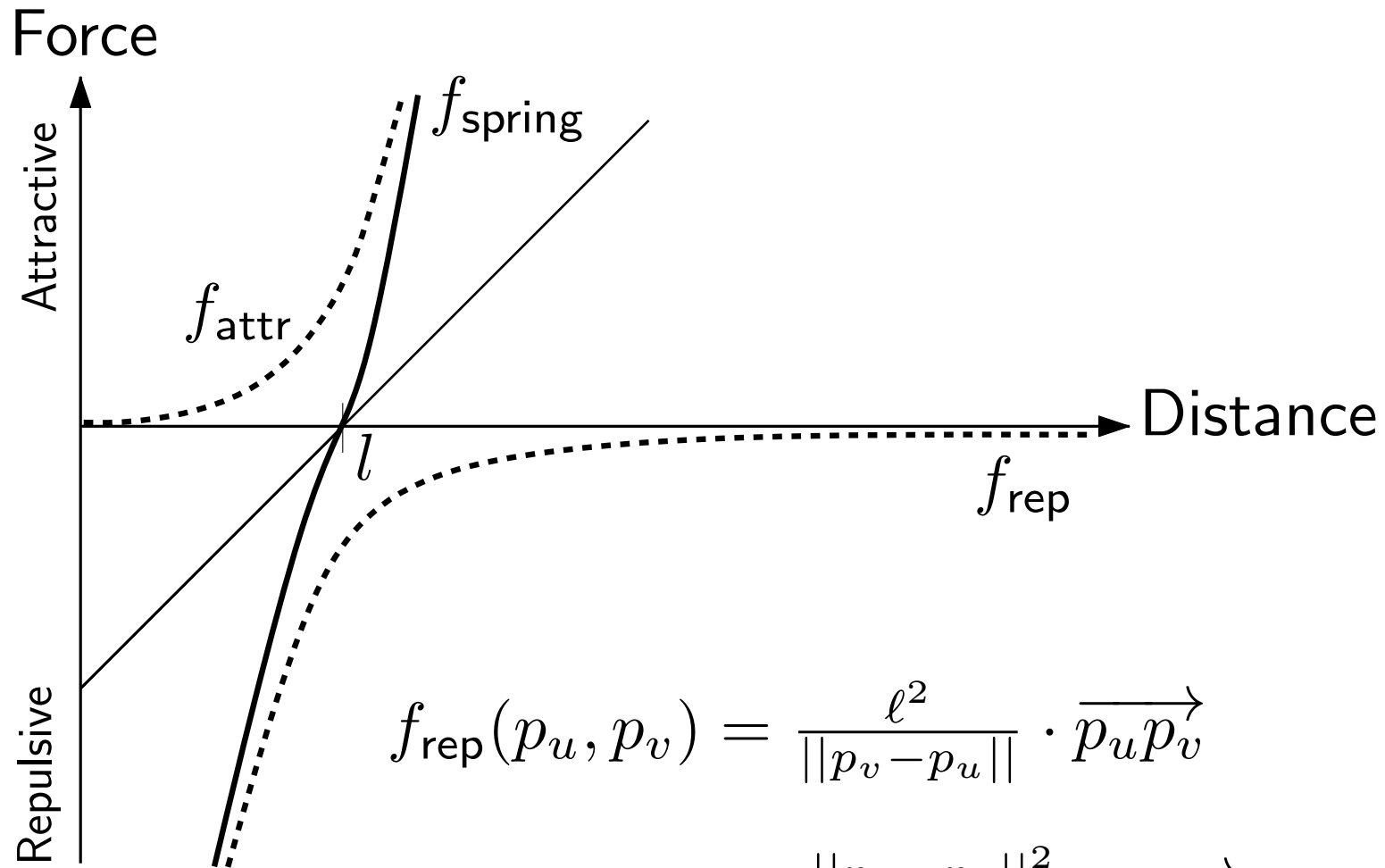
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Diagramm of Fruchterman & Reingold Forces

KIT
Karlsruhe Institute of Technology

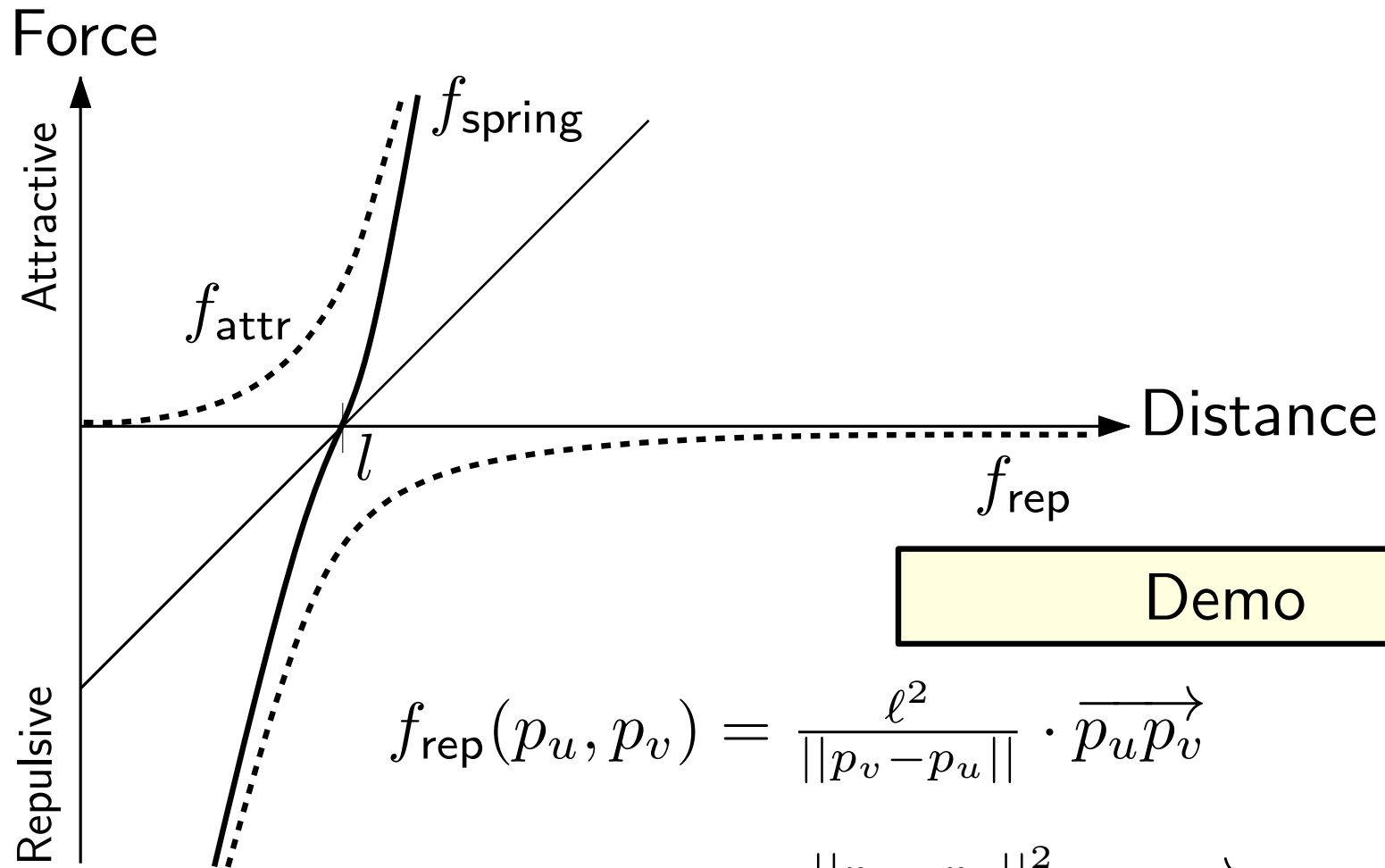


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- **Inertia**
- **Gravitation**
- **Magnetic forces**

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set $f_{\text{attr}}(p_u, p_v) \leftarrow f_{\text{attr}}(p_u, p_v) \cdot 1/\Phi(v)$

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■ Gravitation

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$f_{\text{grav}}(p_v) = c_{\text{grav}} \cdot \Phi(v) \cdot \overrightarrow{p_v p_{\text{bary}}}$

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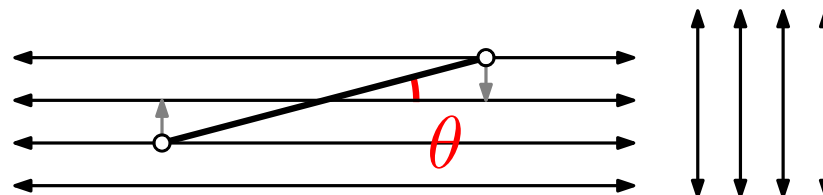
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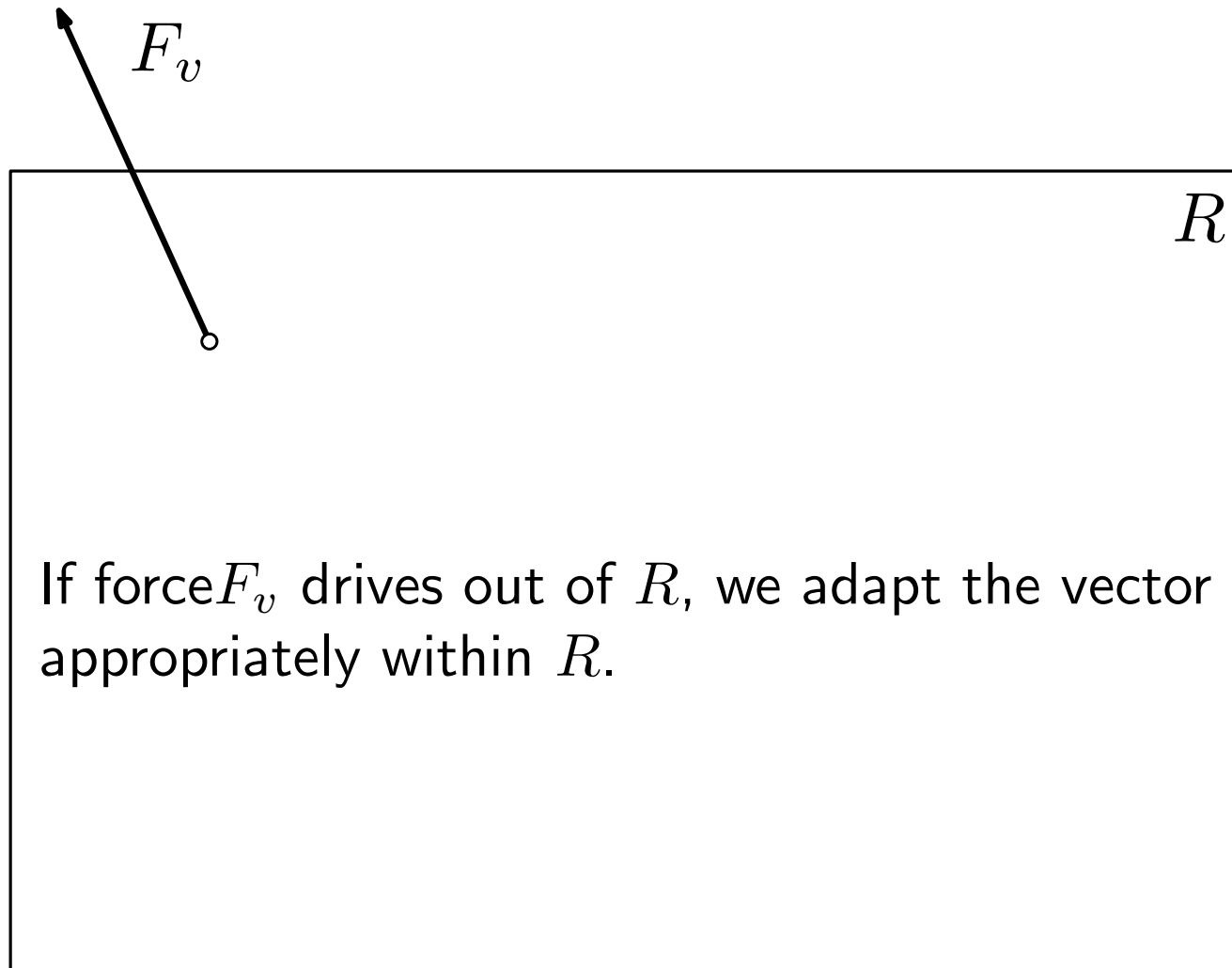
– define magnetic fields (e.g. vertical, horizontal)

– angle θ between edge and the direction of the field

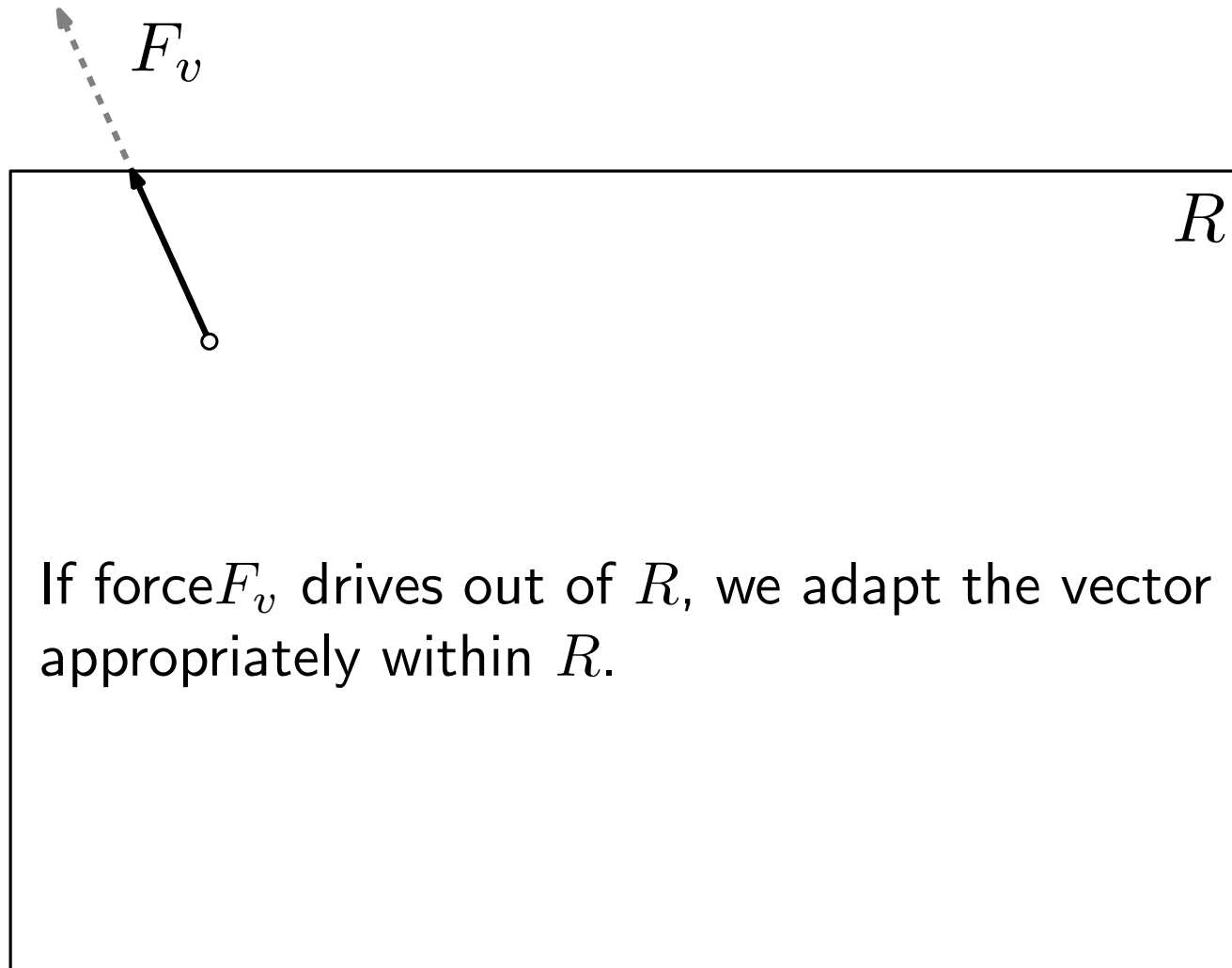
– define force that reduces this angle



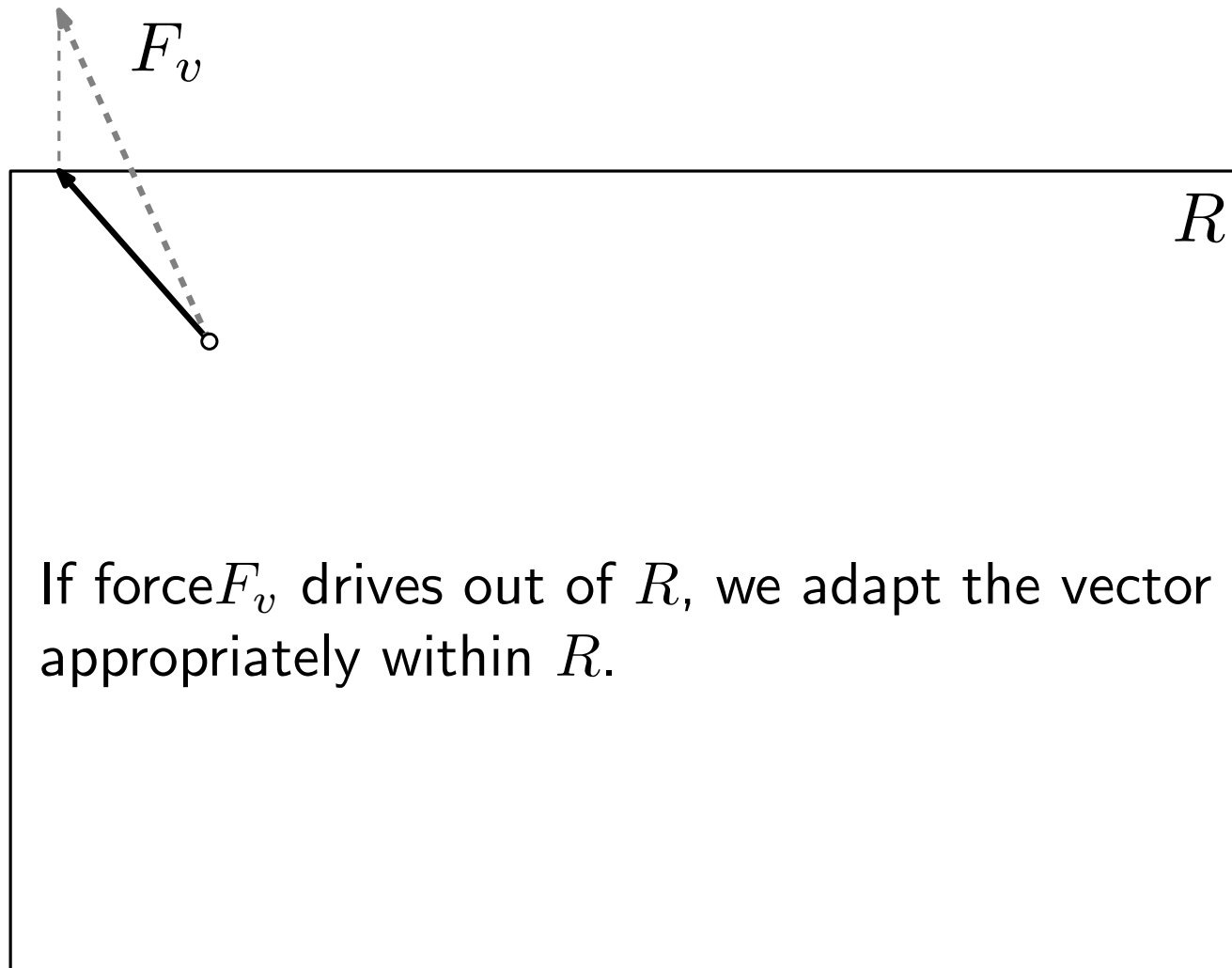
Bounded Drawing Area



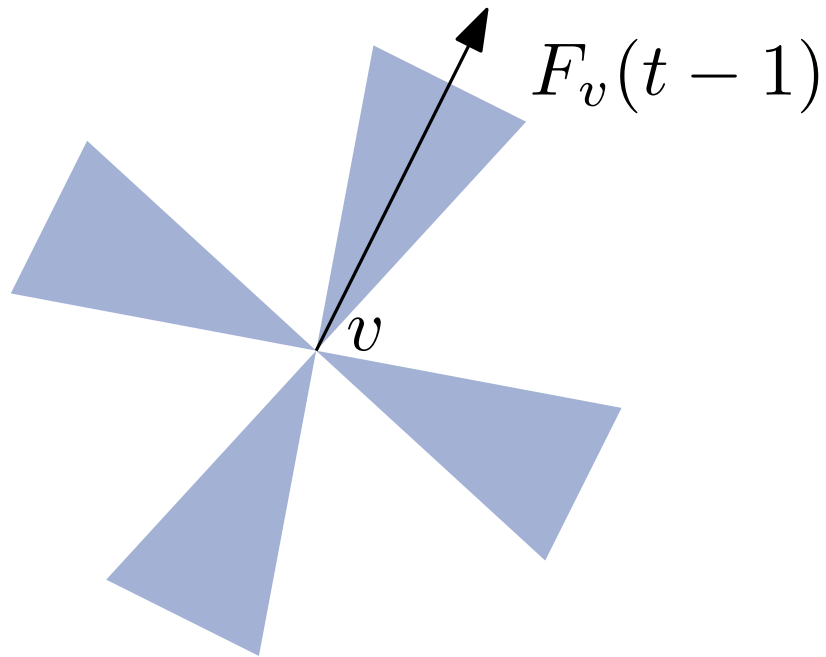
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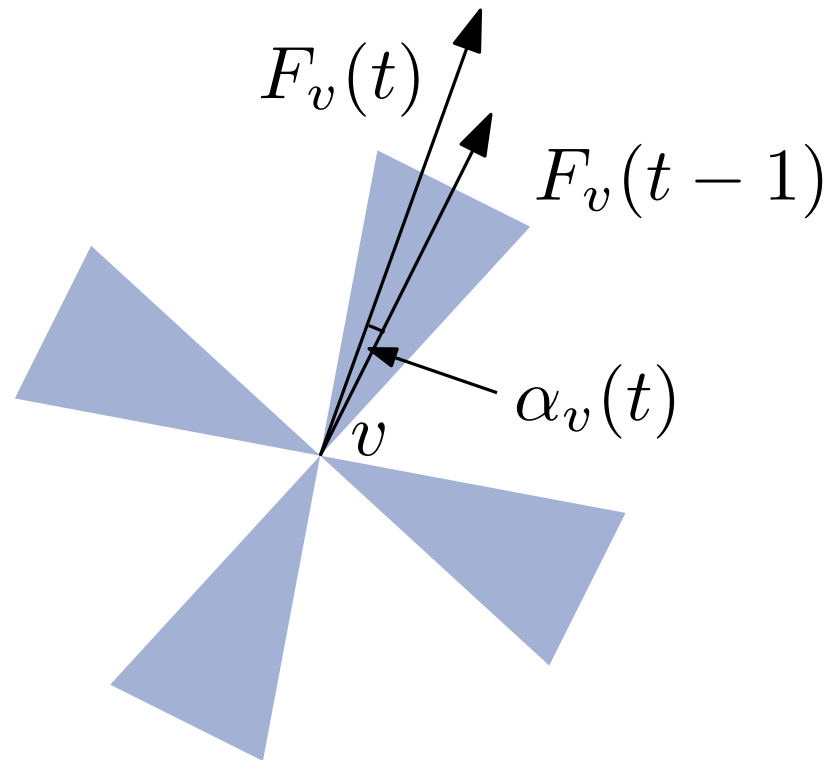


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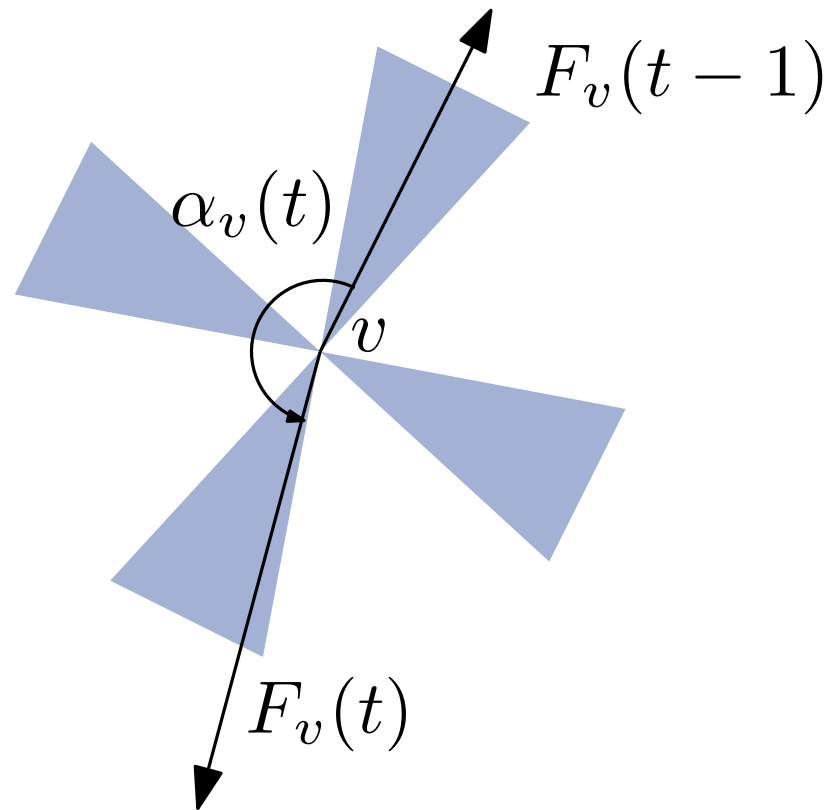


- store previous displacement vector $F_v(t - 1)$





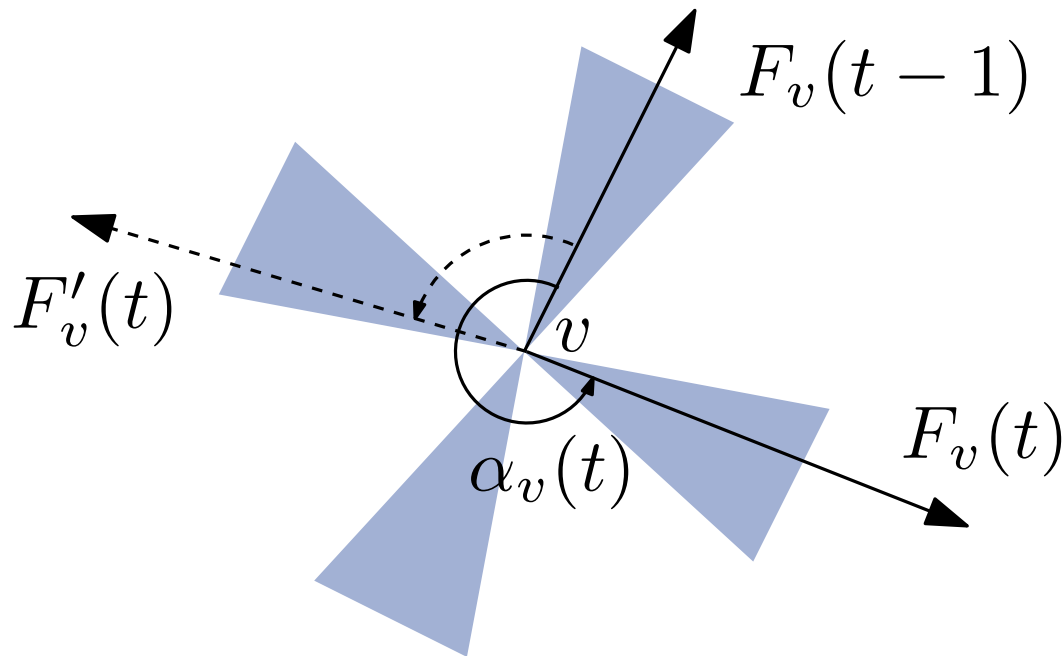
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- local temperature**
 - $\cos(\alpha_v(t)) \approx 1$: similar direction
→ increase the temperature



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oscillation
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- $\cos(\alpha_v(t)) \approx 0$:
Rotation
→ update rotation counter and decrease temperature if necessary

Advantages

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- further modifications improve layout quality and lead to faster convergence

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- Variants of Fruchterman and Reingold algorithms are probably the most popular force-based methods (original paper cited 3500 times (1000 in the past two years))

Discussion

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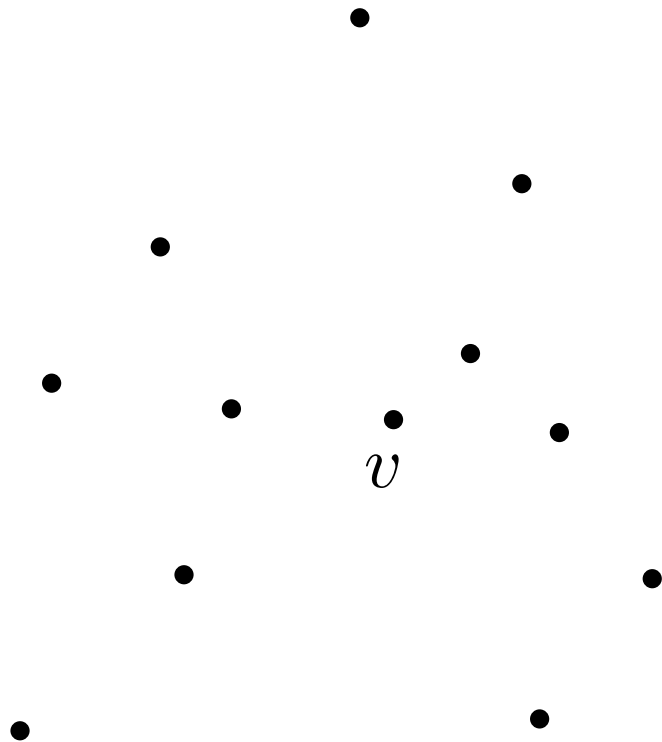
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Could we reduce this?

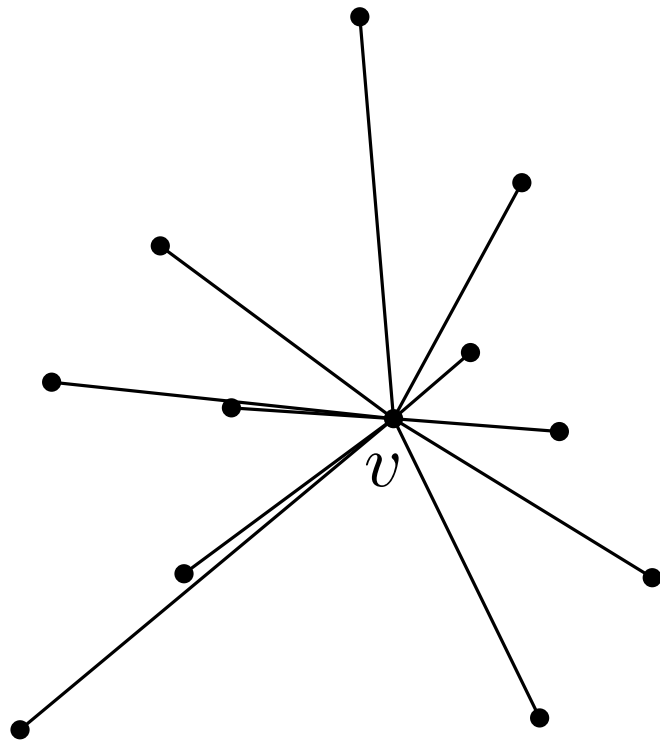
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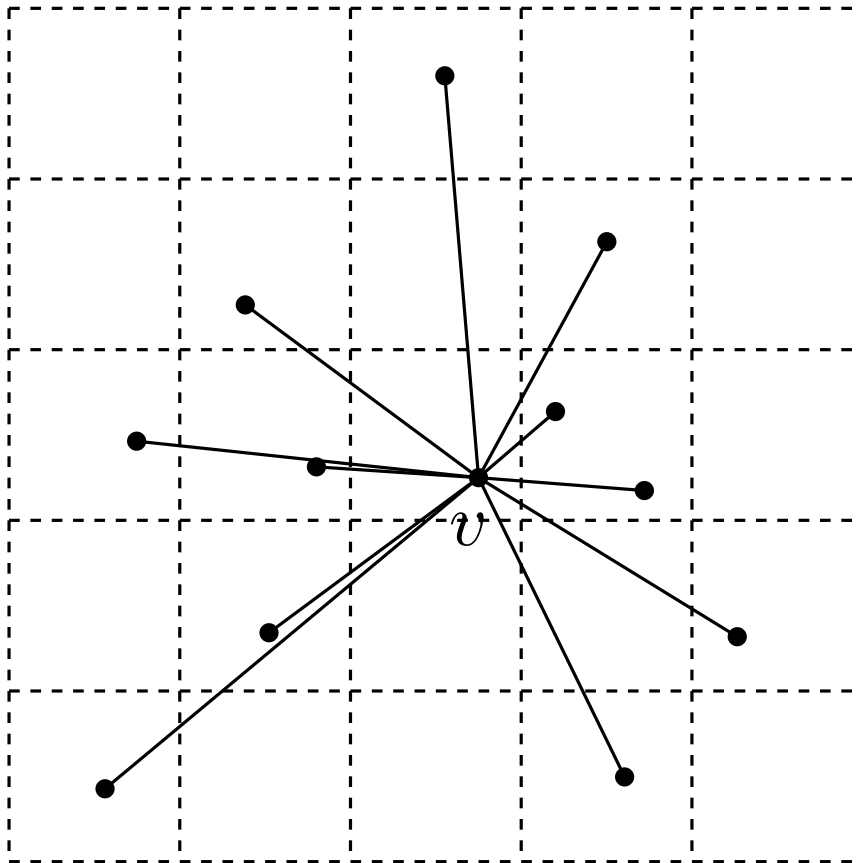
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Grid Version (Fruchterman, Reingold, 1990)

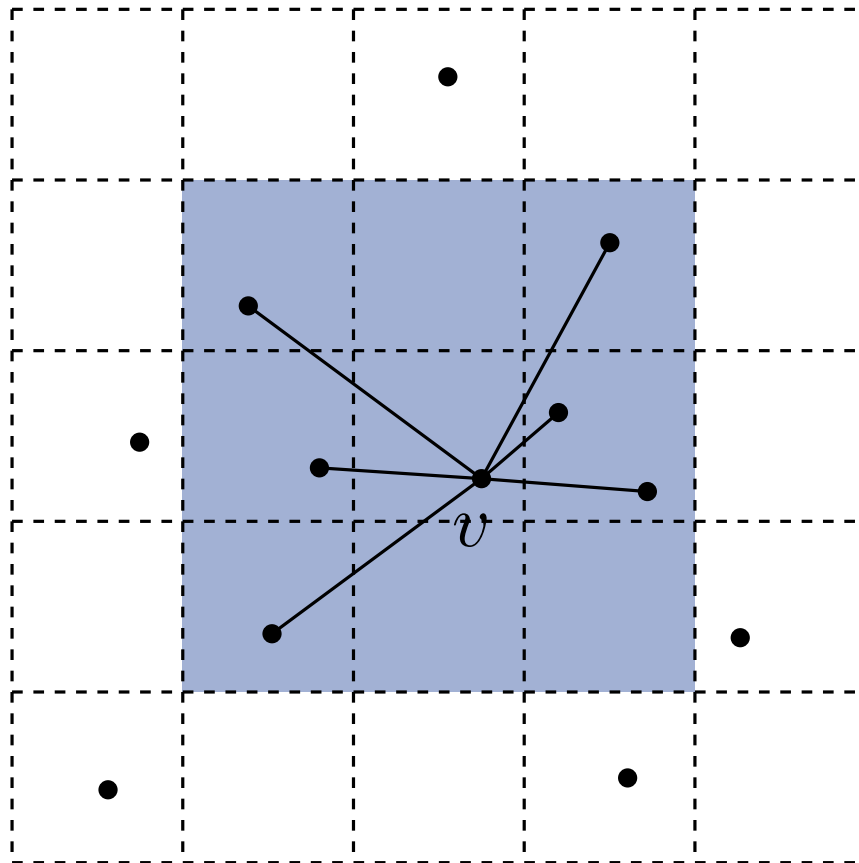


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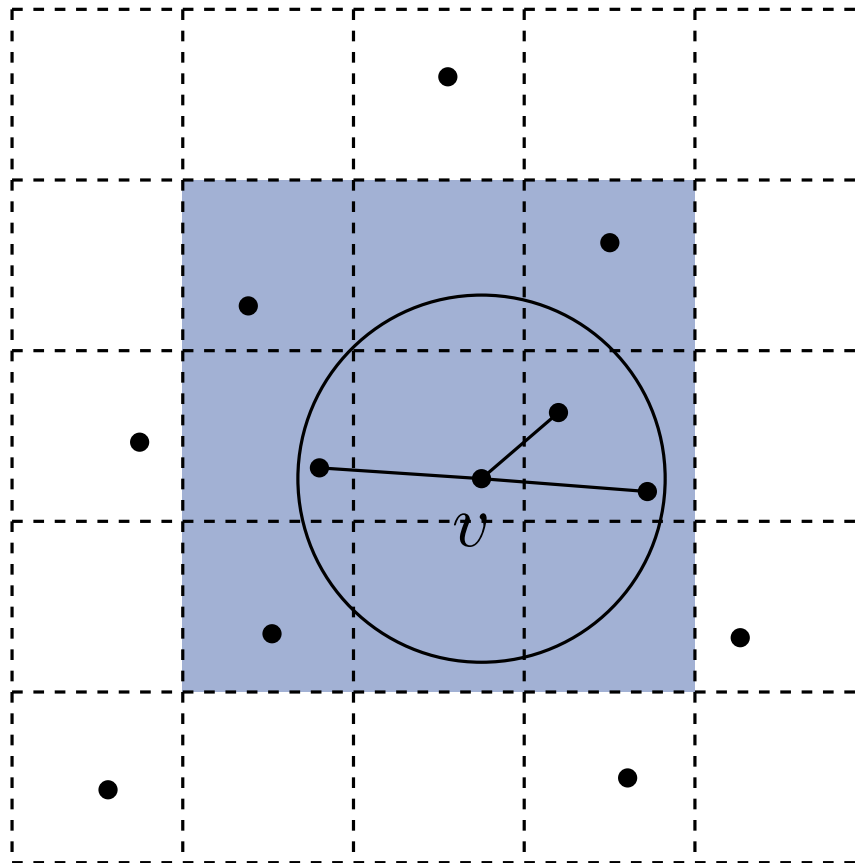




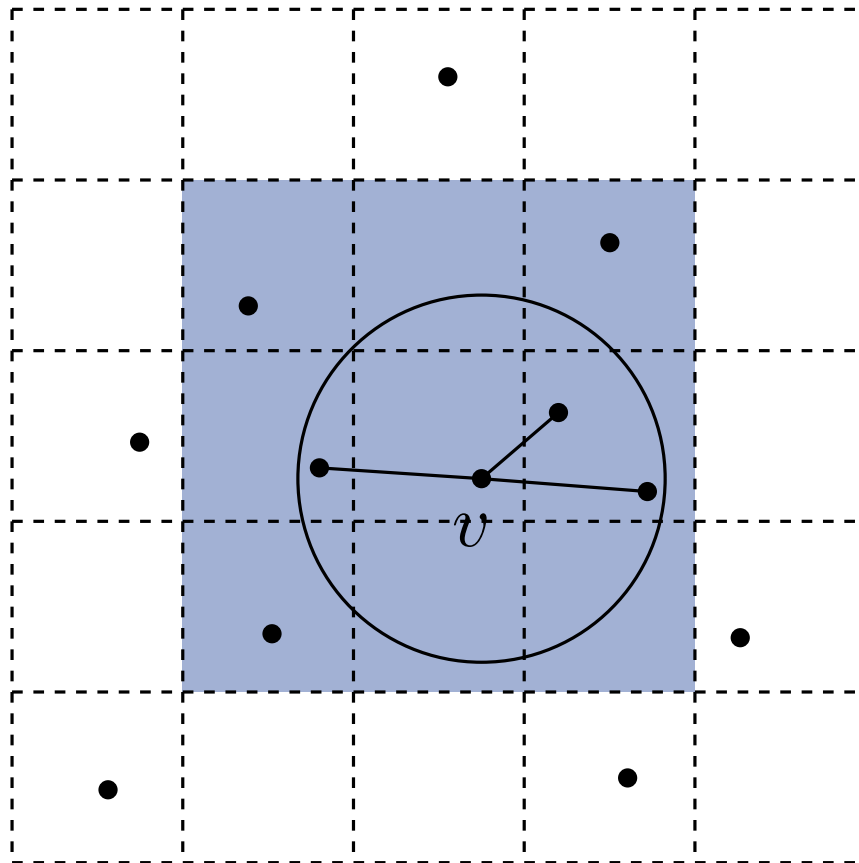
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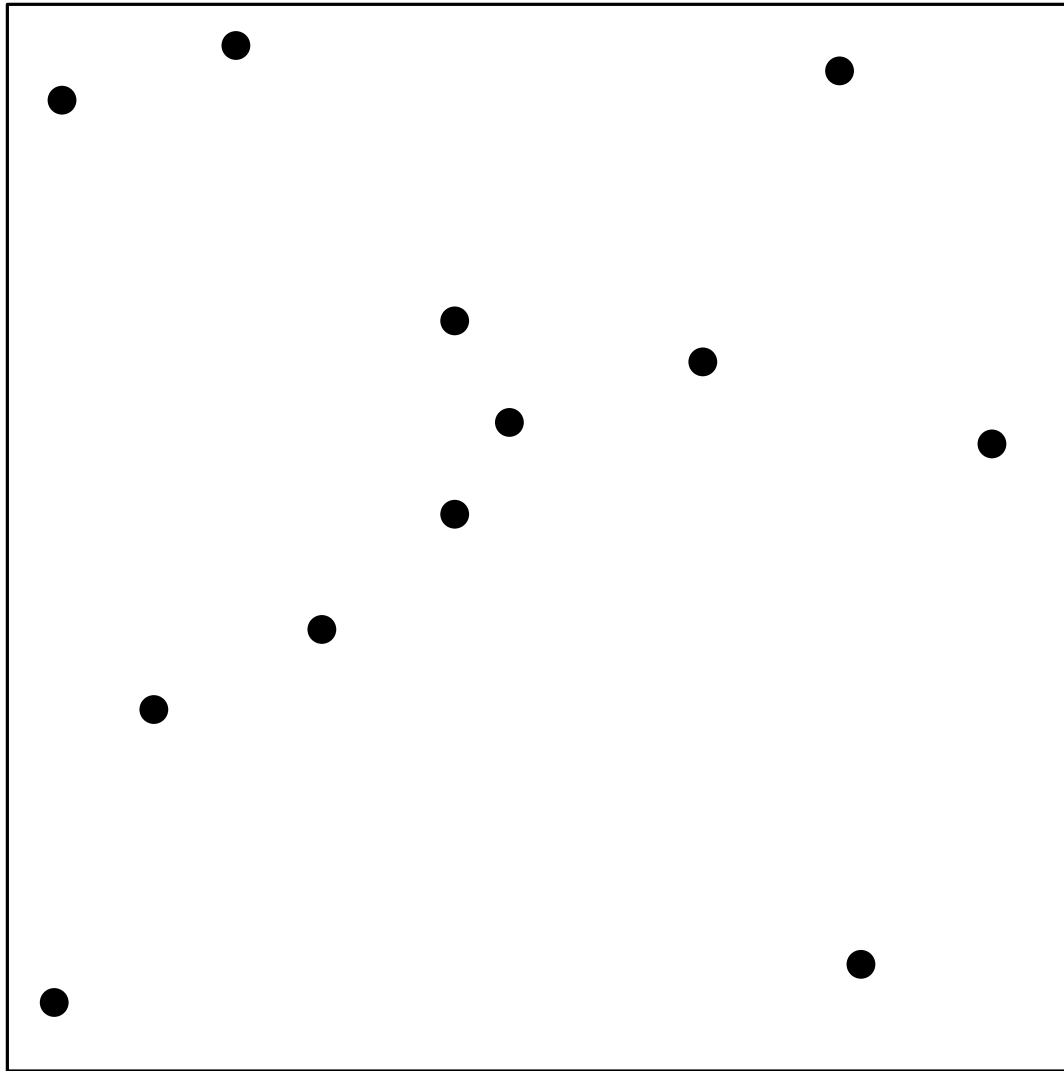


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Discussion

- meaningful idea to improve runtime
- worst-case no advantage
- Quality loss (e.g., oscillation d_{\max})

Quad-Tree

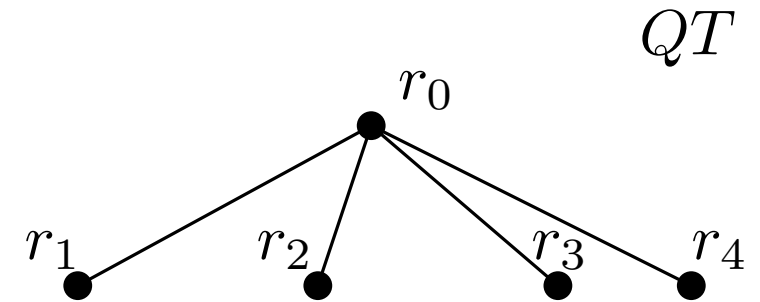
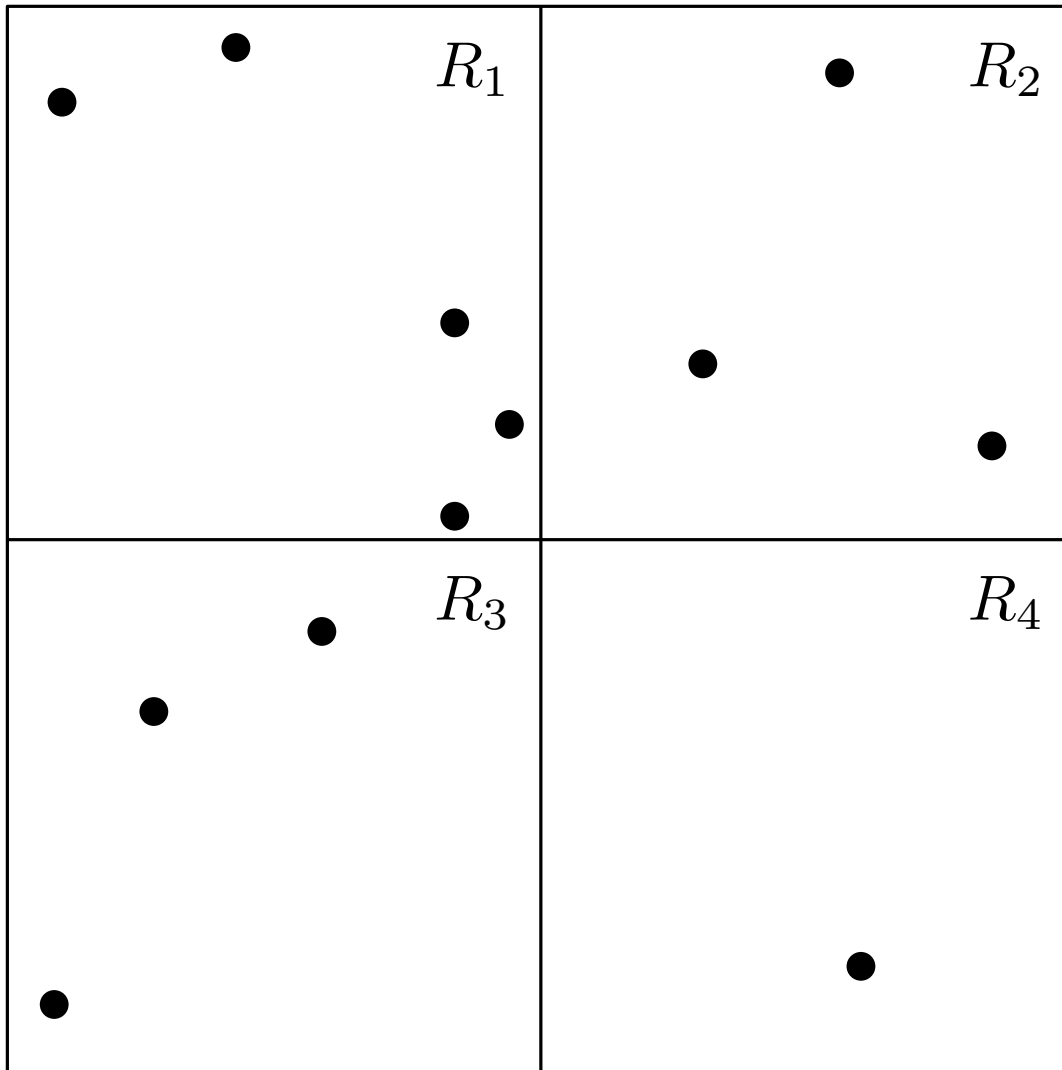


R_0

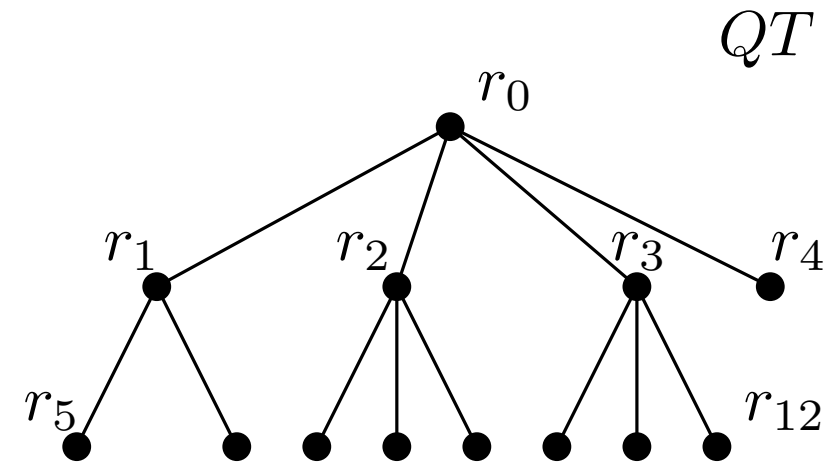
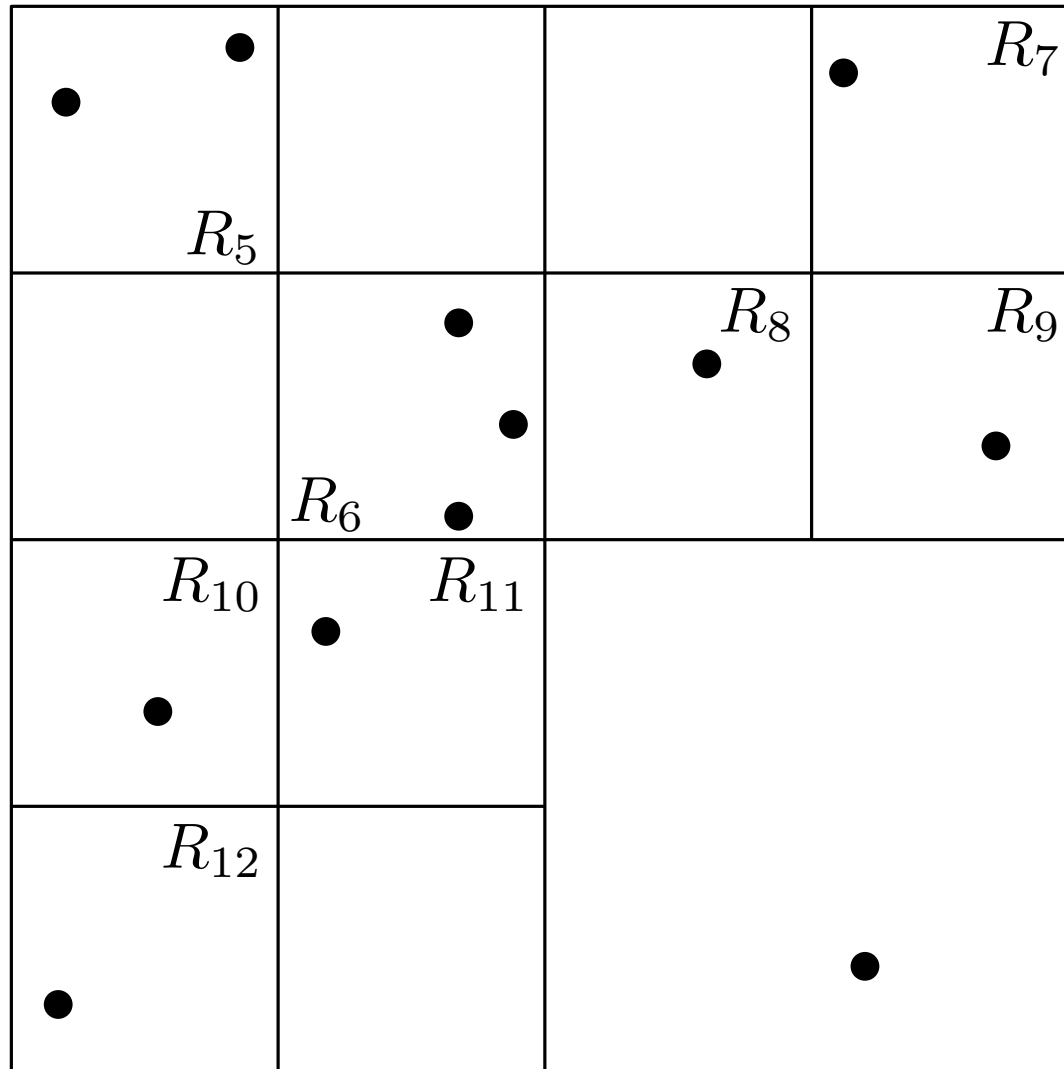


QT

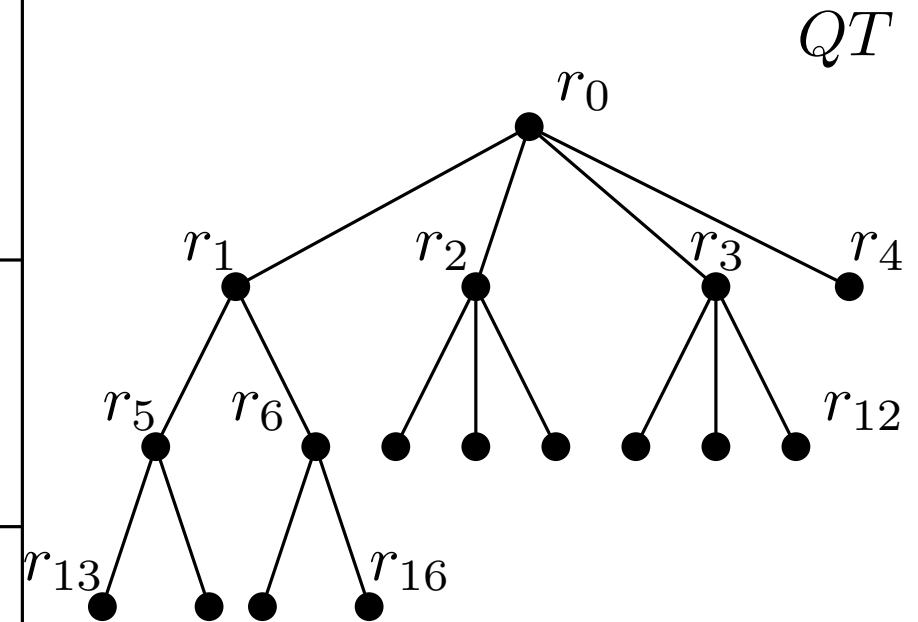
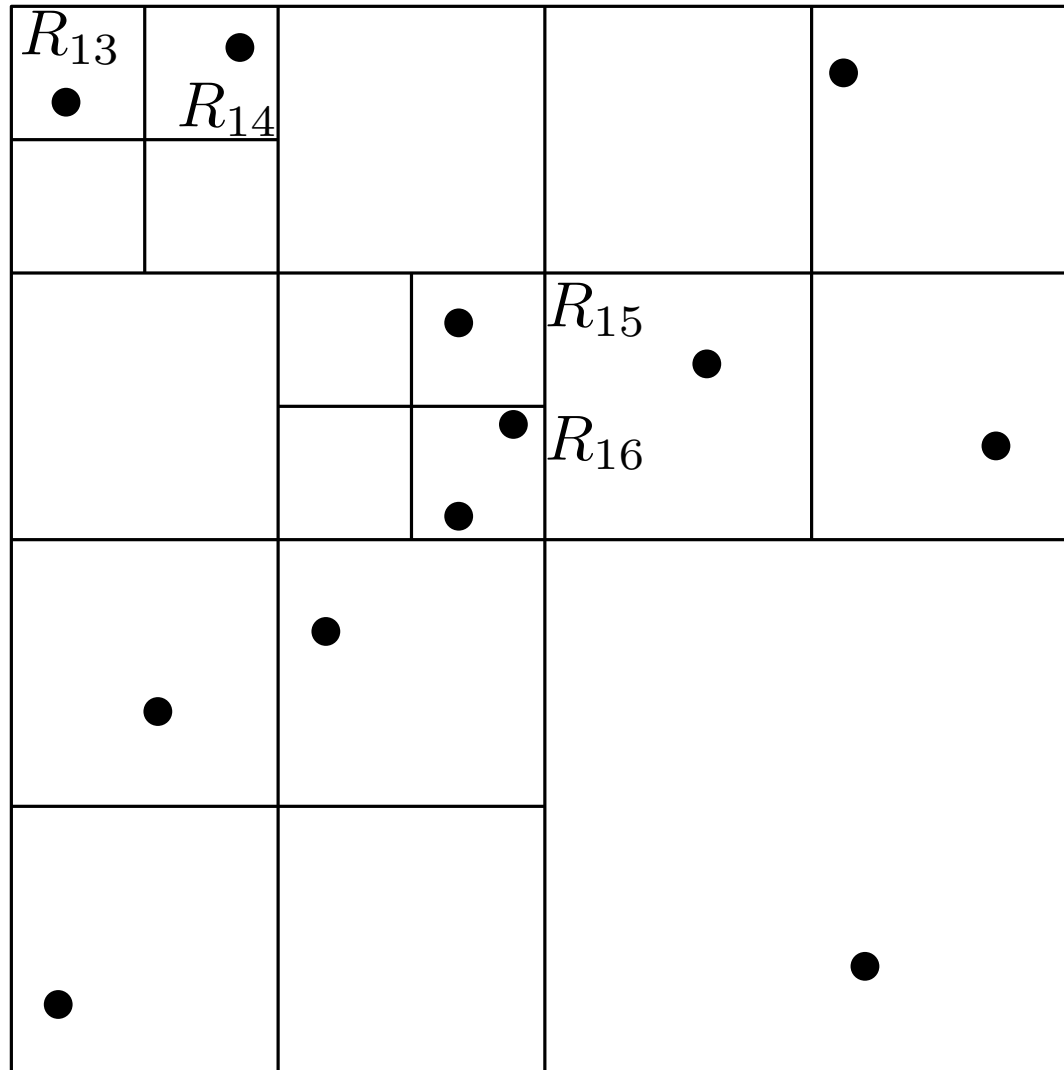
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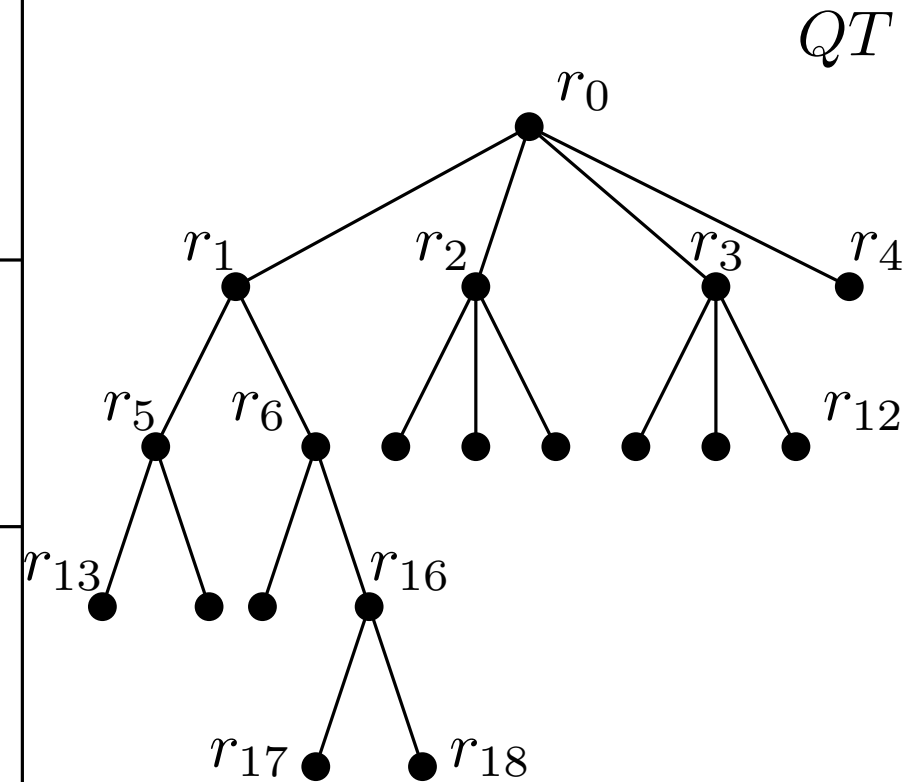
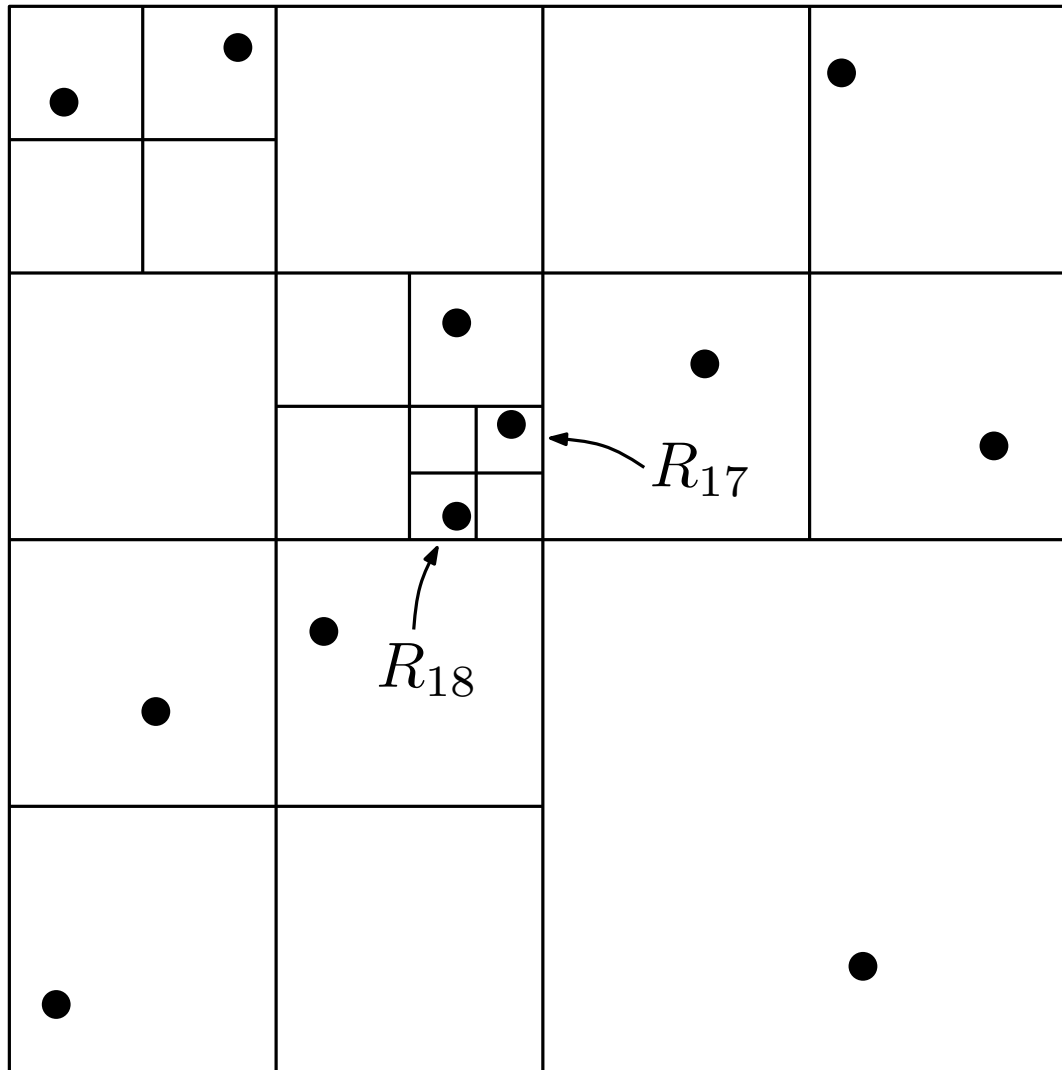
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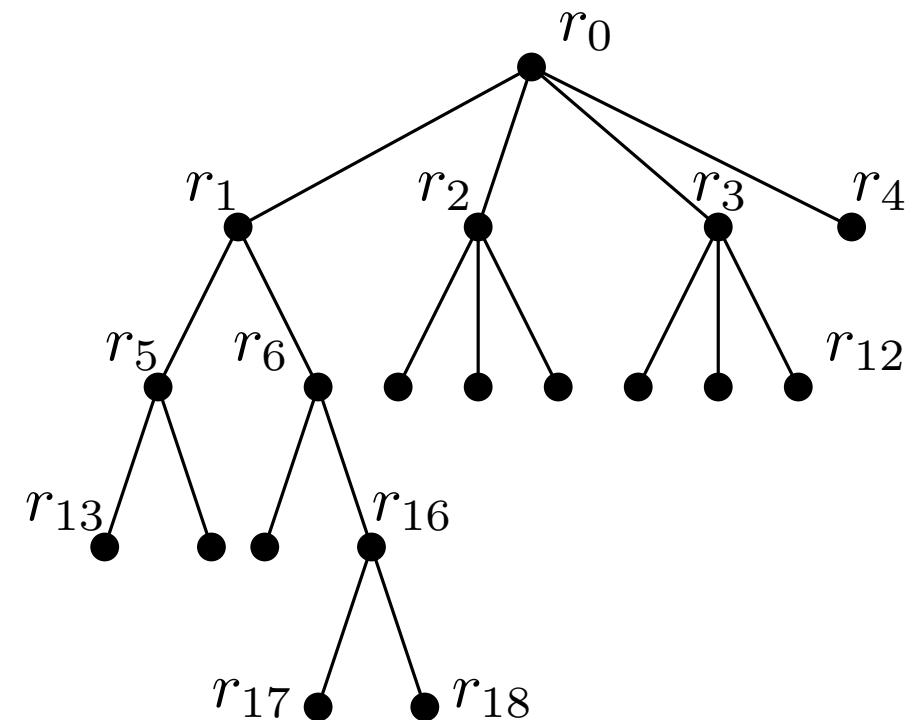
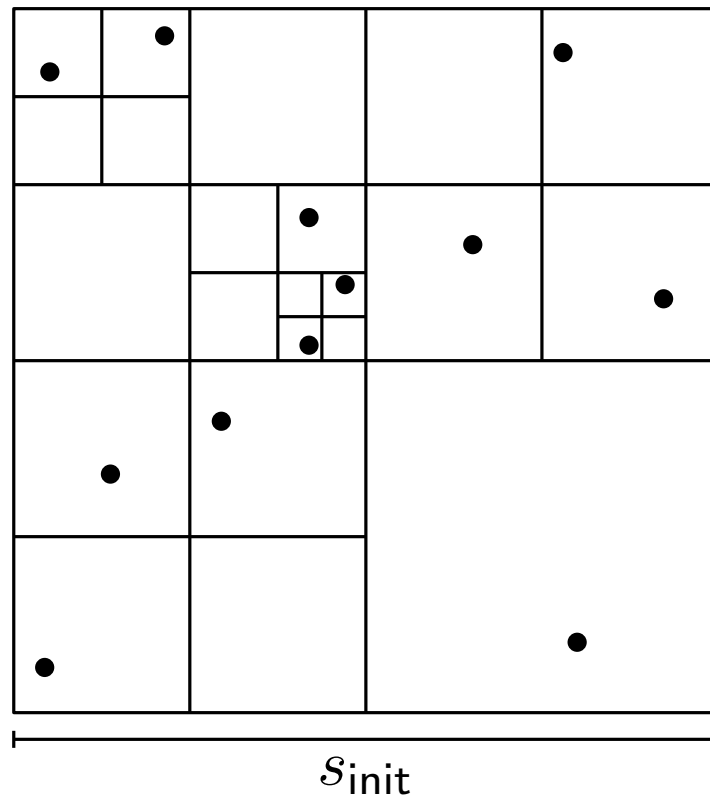


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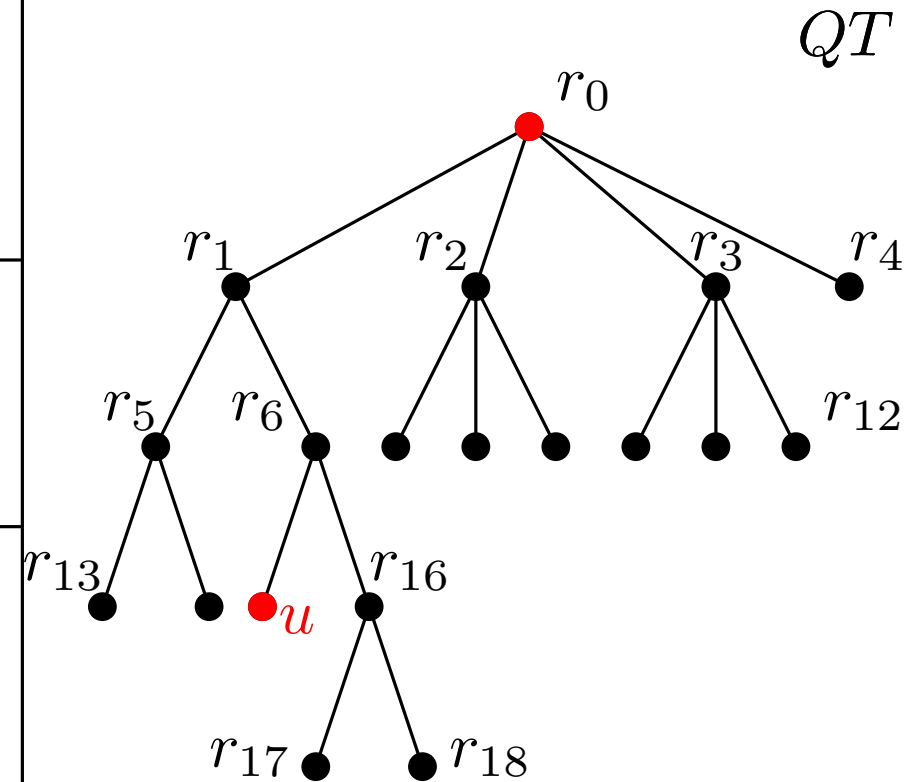
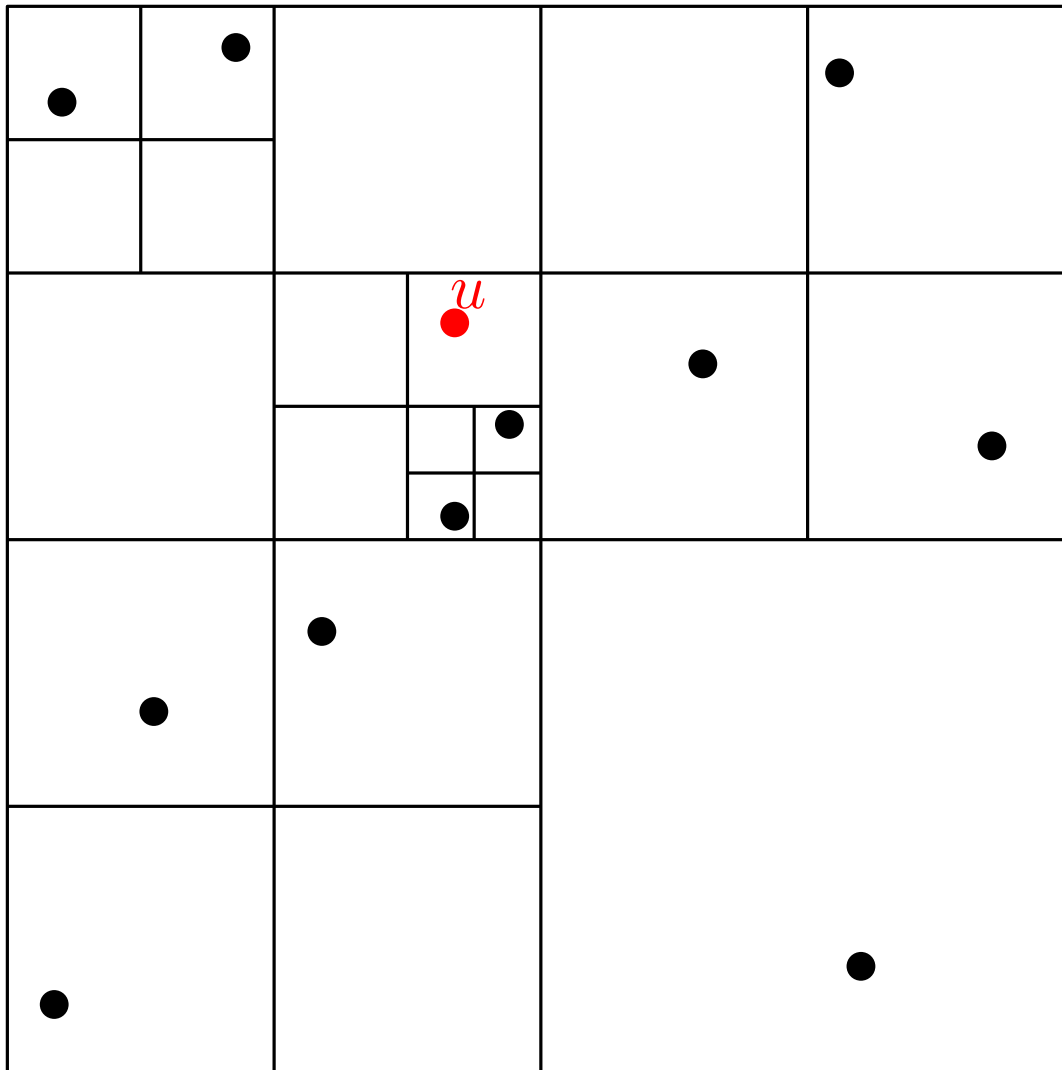


Properties of Quad-Tree

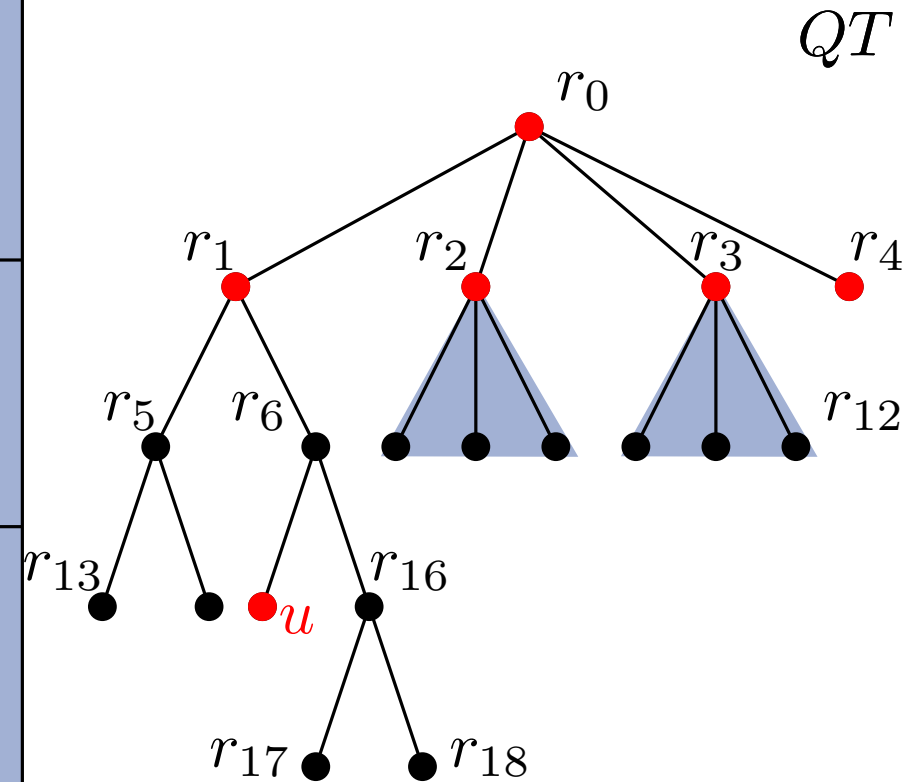
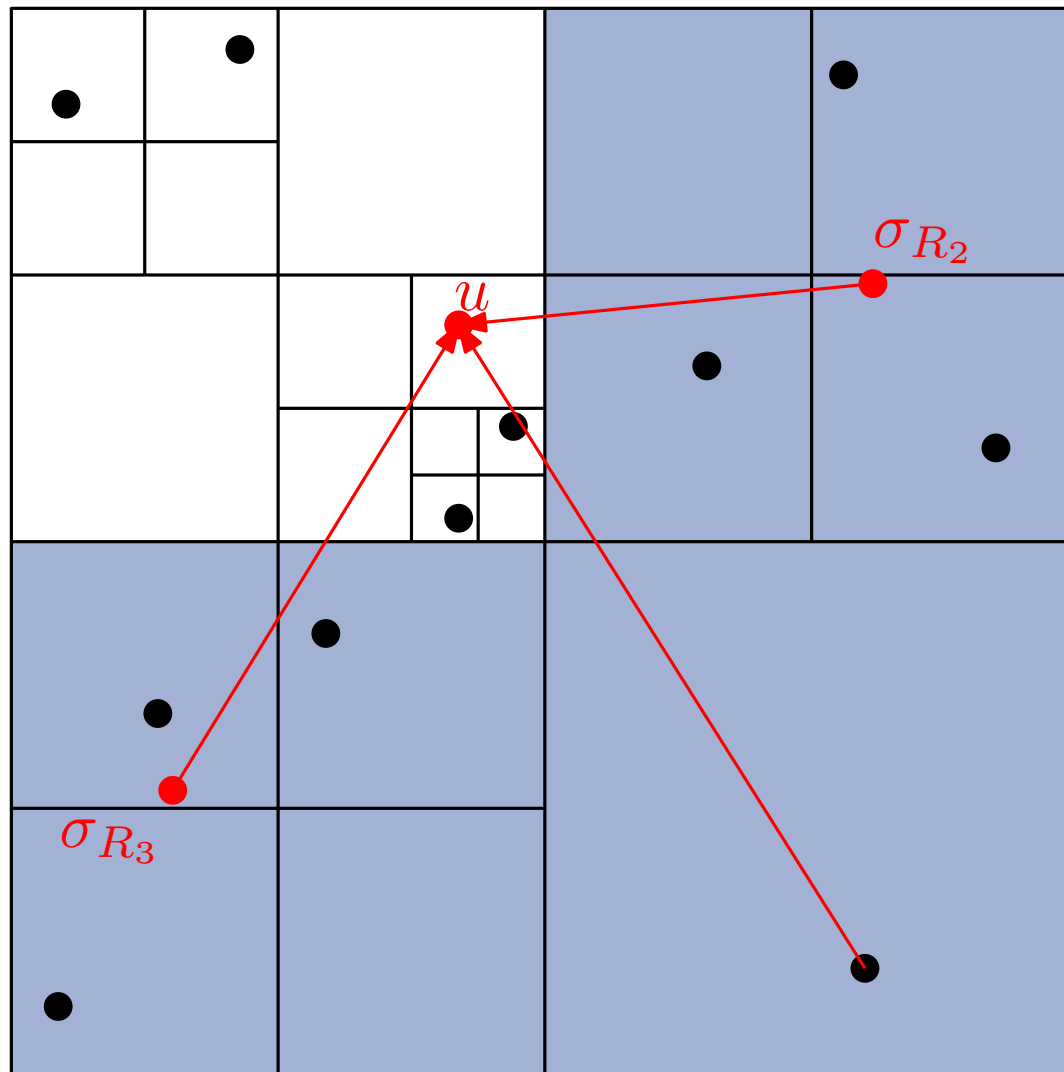
- height $h \leq \log \frac{s_{init}}{d_{min}} + \frac{3}{2}$, here d_{min} -smallest distance
- time and space $O(hn)$
- *compressed* quad-tree in $O(n \log n)$ time



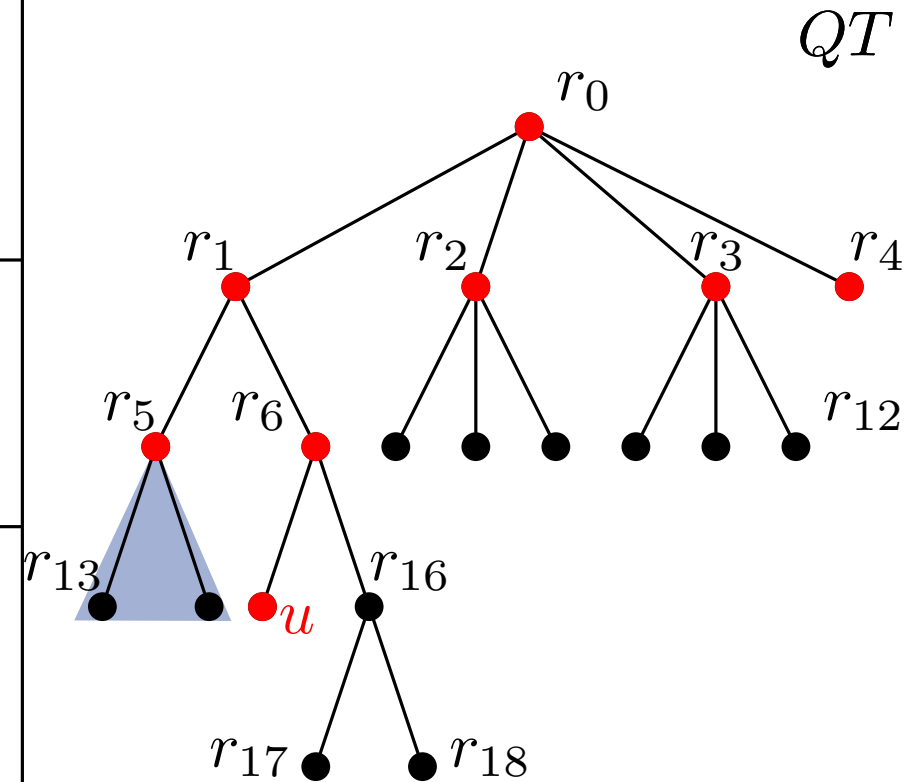
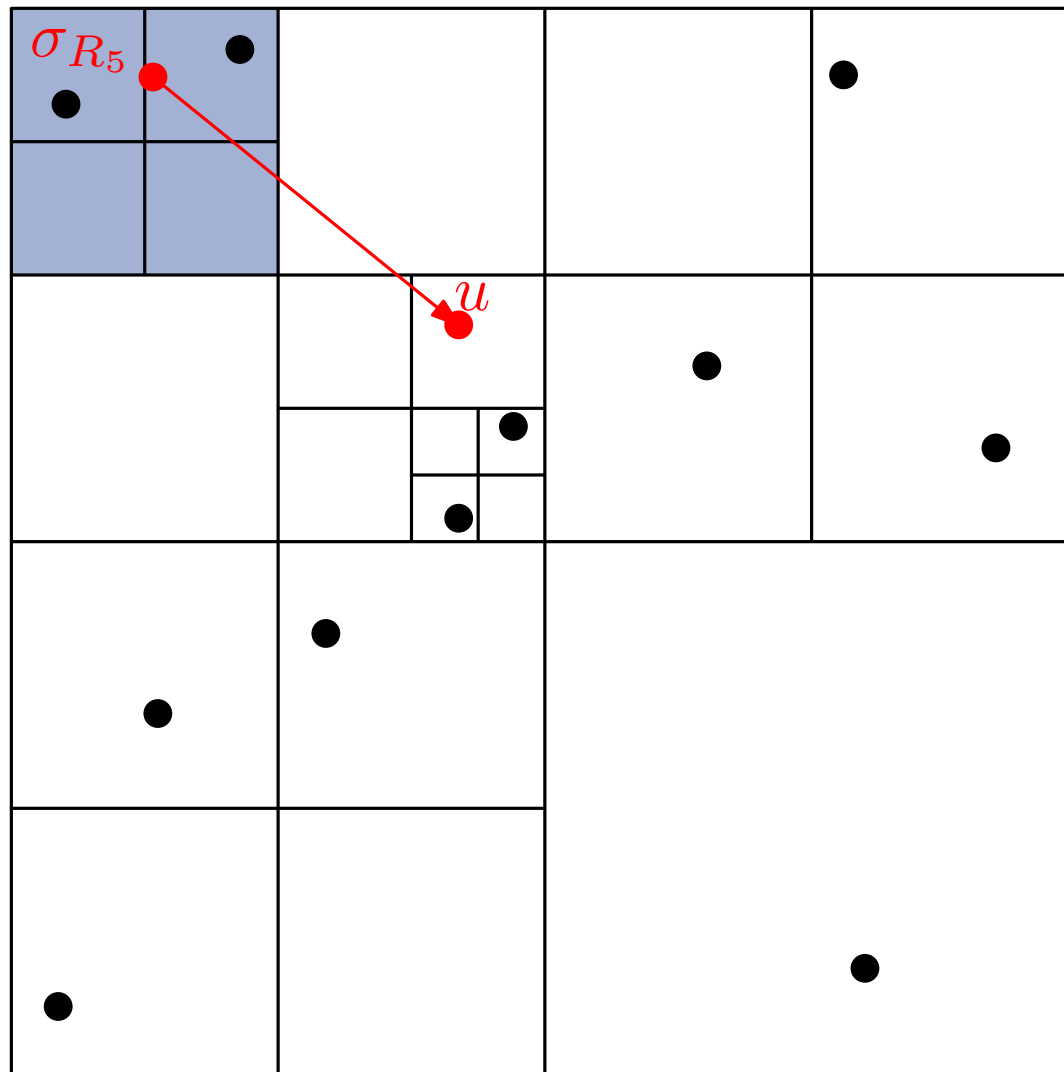
Forces with Quad-Trees (Barnes, Hut, 1986)



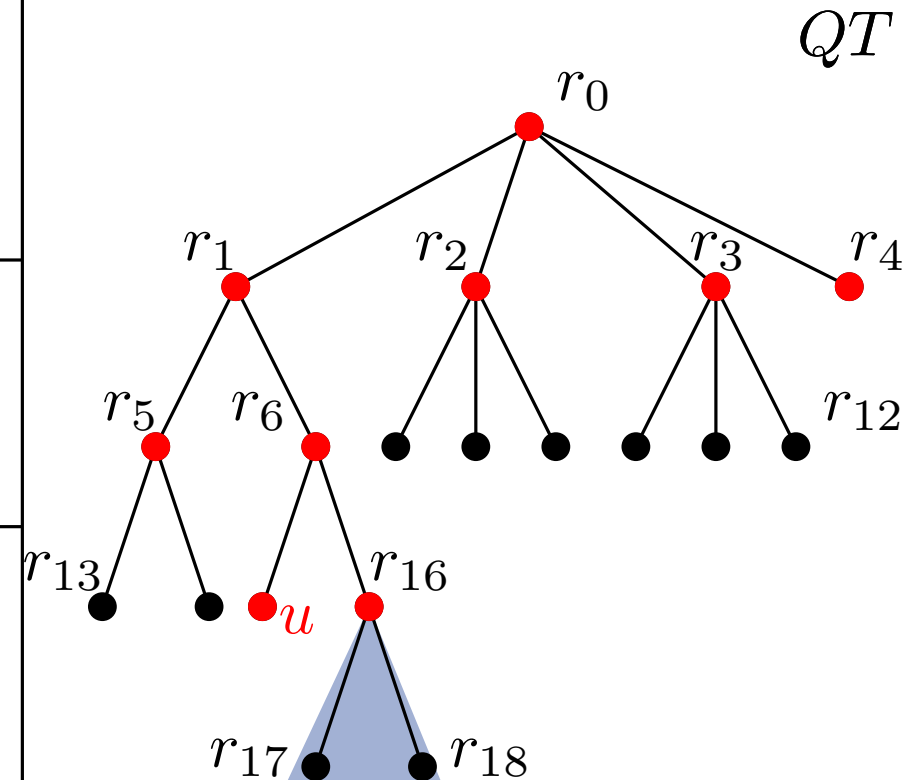
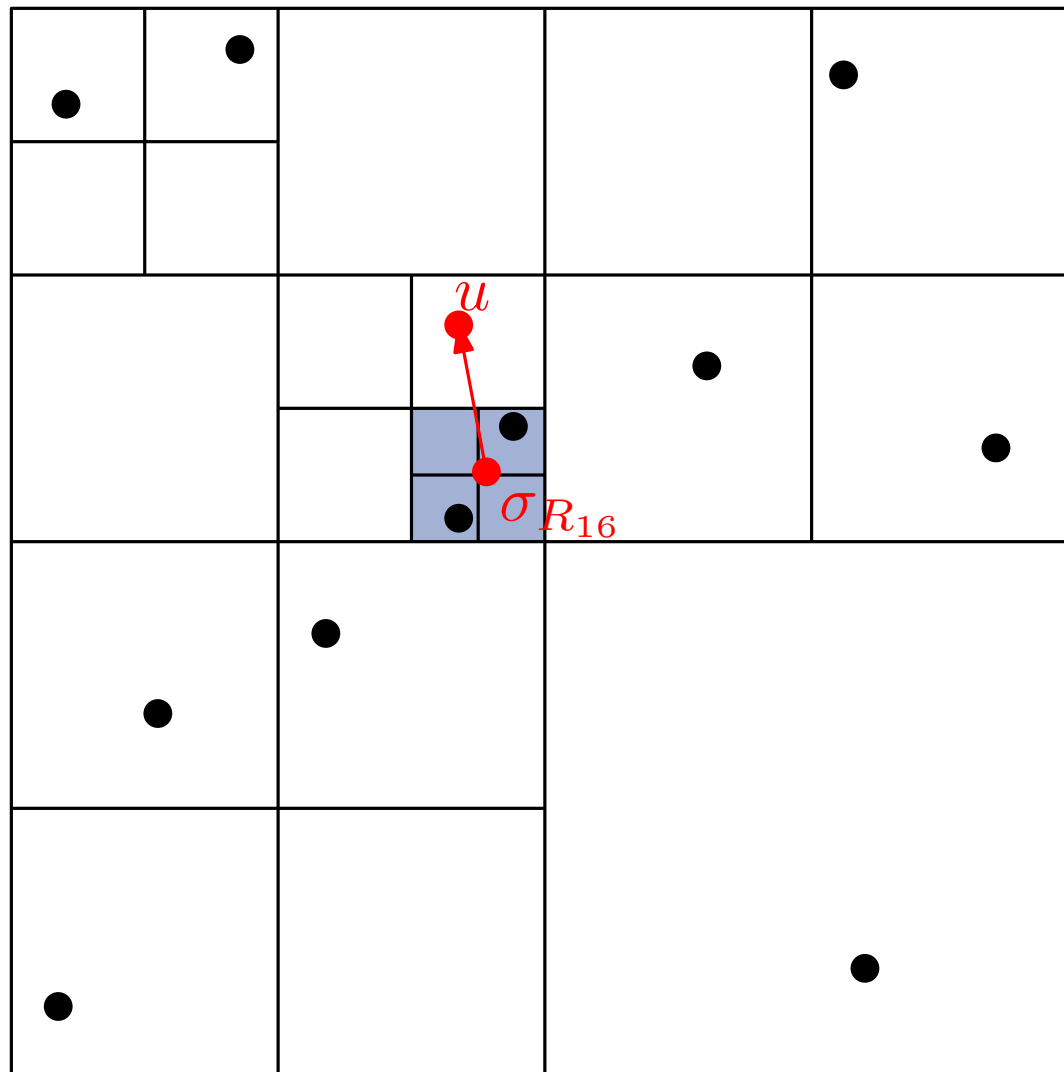
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GRIP – Graph dRawing with Intelligent Placement

(Gajer, Kobourov, 2004)

Approach

- top-down graph coarsening/filtration
- bottom-up calculation of the layout:
- clever placement of new nodes
- force-based refinement of their positions

GRIP Algorithm

Input: Graph $G = (V, E)$

$\mathcal{V} \leftarrow$ Filtering $V = V_0 \supset V_1 \supset \dots \supset V_k$

for $i = k$ **to** 0 **do**

foreach $v \in V_i \setminus V_{i+1}$ **do**

 compute neighbours of v

 compute initial position of v

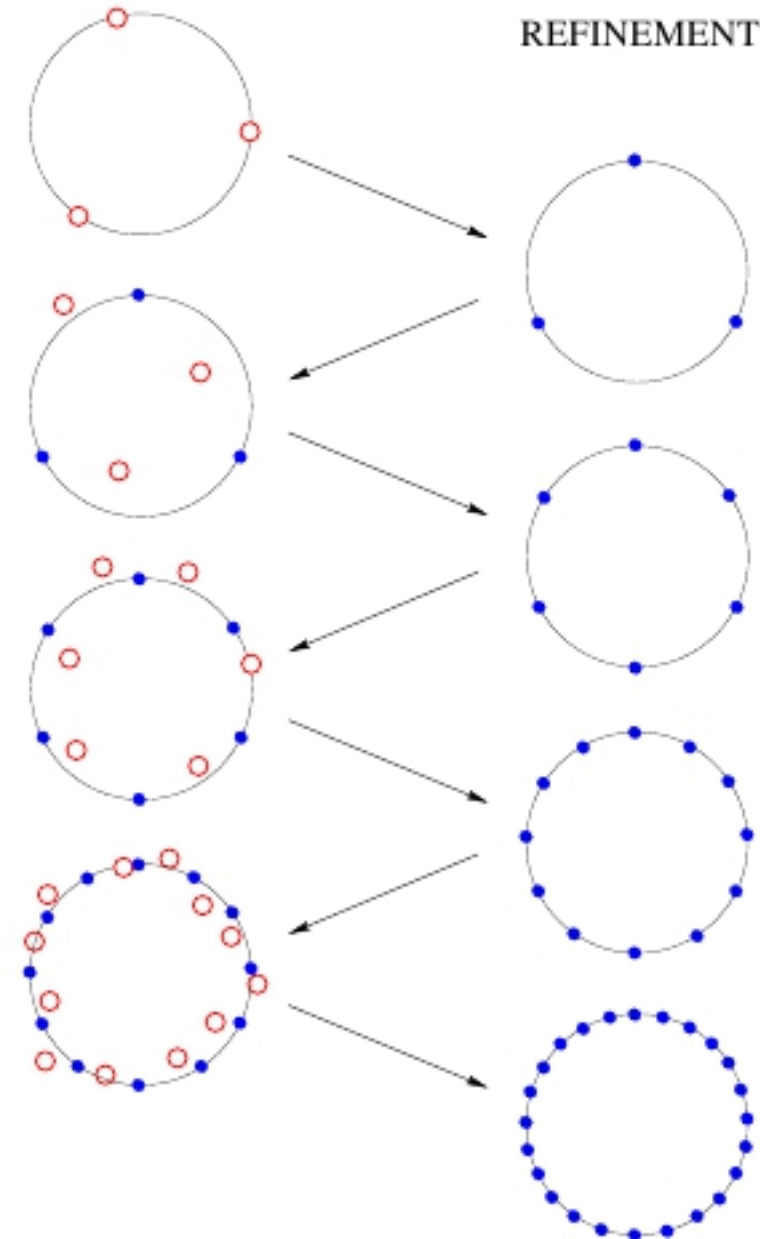
for $j = 1$ **to** rounds **do**

foreach $v \in V_i$ **do**

 force-based relaxation

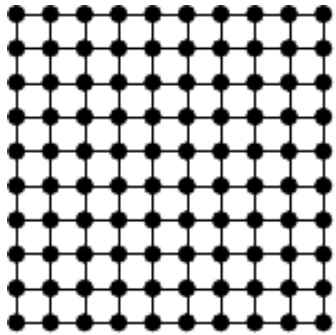
INITIAL PLACEMENT

REFINEMENT

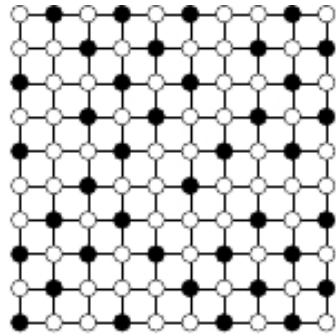


Maximal Independent Set (MIS) filtering

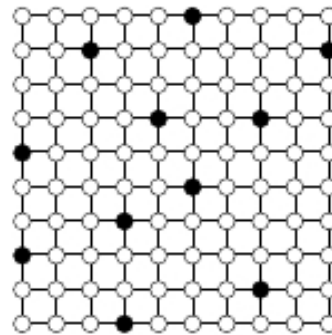
- sequence of node sets $V = V_0 \supset V_1 \supset \dots \supset V_k \supset \emptyset$
- V_i is (inclusion-) maximal set of nodes, so that
- distance in G between nodes in V_i is $\geq 2^{i-1} + 1$
- good balance between size of a level and depth of decomposition



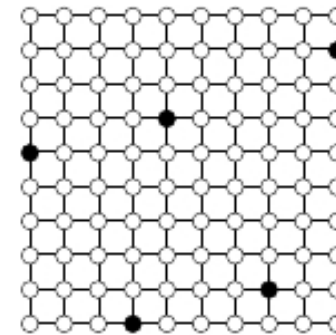
V_0



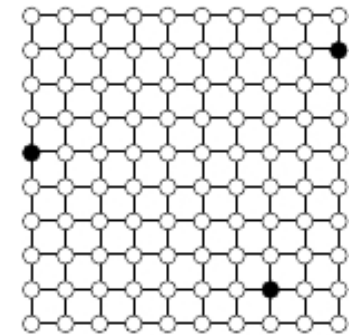
V_1



V_2



V_3



V_4

Algorithm

- incremental procedure: given V_i compute V_{i+1}
- select random element v in V_i
- remove all elements from V_i with distance $\leq 2^i$ to v
- use BFS from v with search depth 2^i
- until no further node remains in V_i

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Depth of Filtration

- for the last level k it holds that $2^k > \text{diam}(G)$
- therefore the depth is $O(\log \text{diam}(G))$

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Alternative

- coarsening with contracting matchings: end vertices of each matching edge are contracted

(Walshaw, JGAA 2003)

Step 1

- for each node $v \in V_i \setminus V_{i+1}$ find optimal position with respect to three adjacent nodes V_{i+1}

Step 2

- perform force-based refinement, where forces are computed locally only to a constant number of nearest neighbors in V_i

Step 1

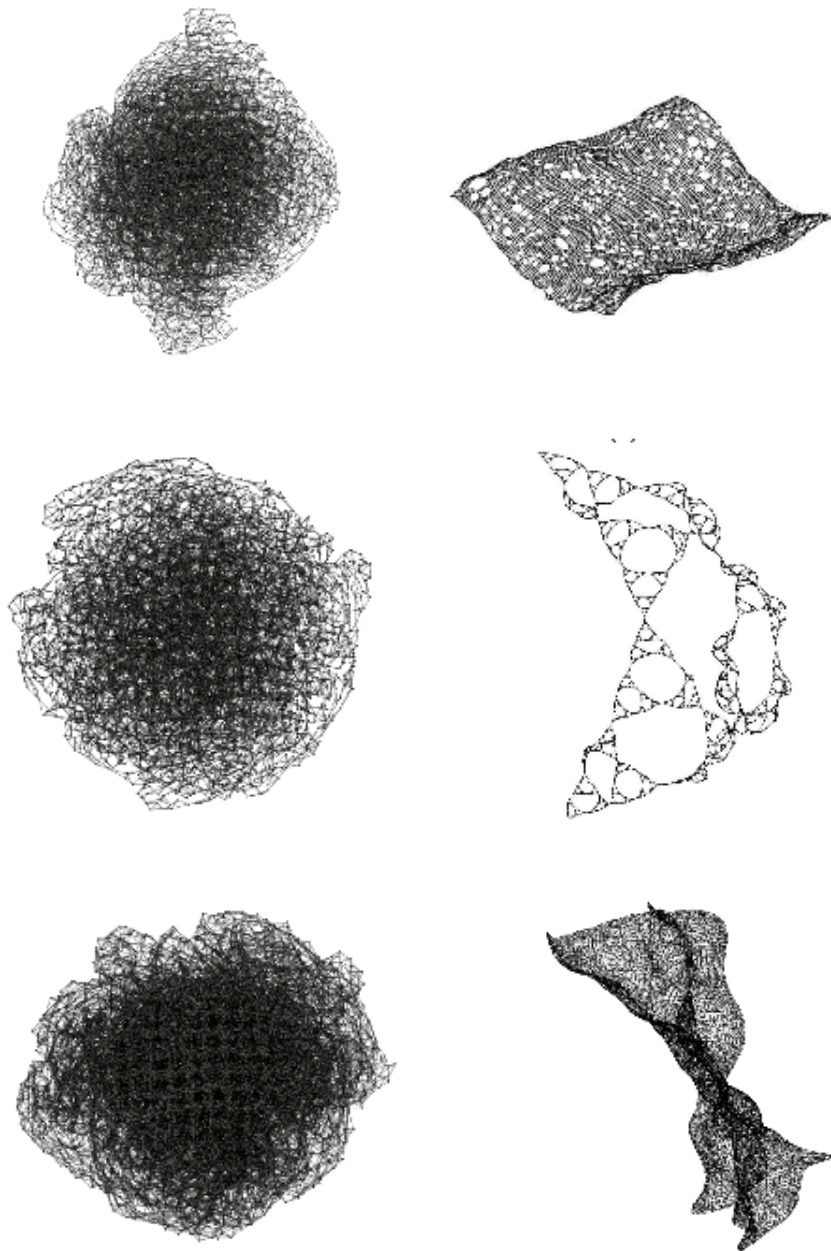
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Step 2

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Properties of GRIP

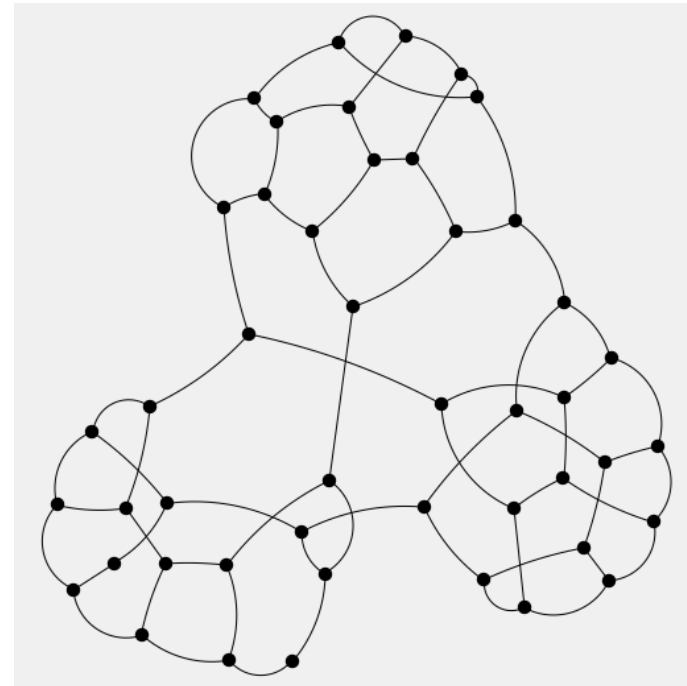
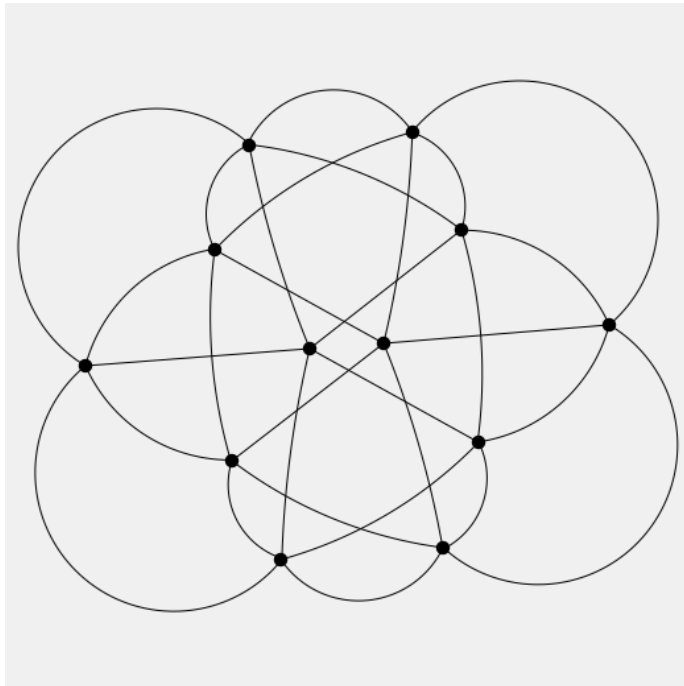
- intelligent step-by-step calculation starting from a good layout and using MIS-filtering
- significantly faster convergence
- graphs with $> 10,000$ nodes in seconds (2004)



Left: Grid version of
Fruchterman Reingold.
Right: GRIP

Lombardi-Spring-Embedder (Chernobelskiy et al. 2012)

- edges are circular arcs
- goal: optimal angular resolution $2\pi / \deg(v)$ at each node v
- additional rotational forces

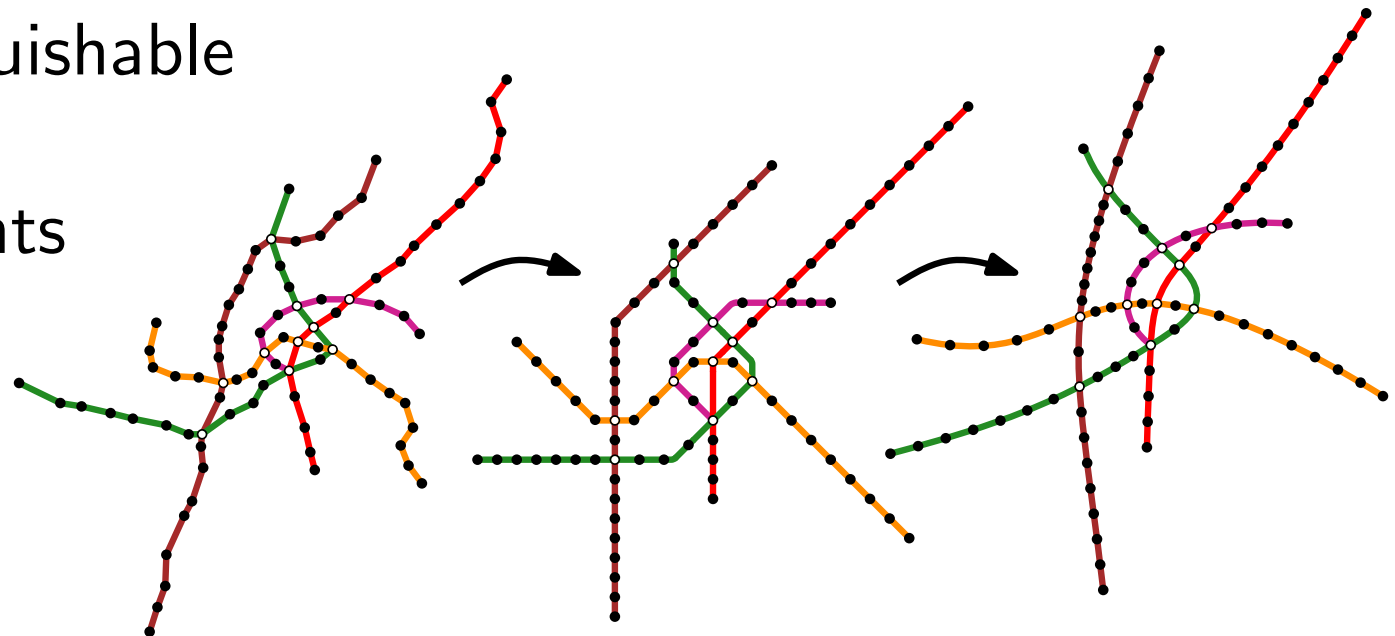


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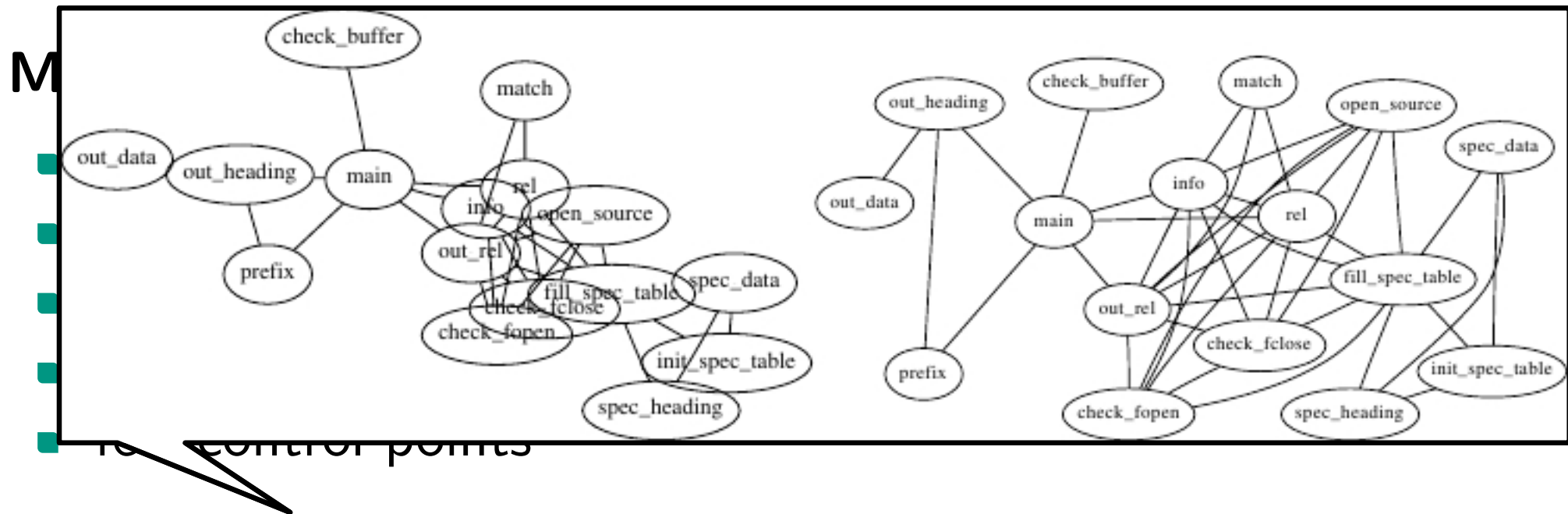
Metro Maps with Bézier curves (Fink et al. 2013)

- model paths as Bézier curves
- forces on nodes and control points:
- lines are distinguishable
- few bend points
- few control points



Lombardi-Spring-Embedder (Chernobelskiy et al. 2012)

- edges are circular arcs
- goal: optimal angular resolution $2\pi / \deg(v)$ at each node v
- additional rotational forces



Realistic Node Sizes (Gansner, North 1998)

- node positions are adjusted to avoid overlaps

Force-based Approaches are

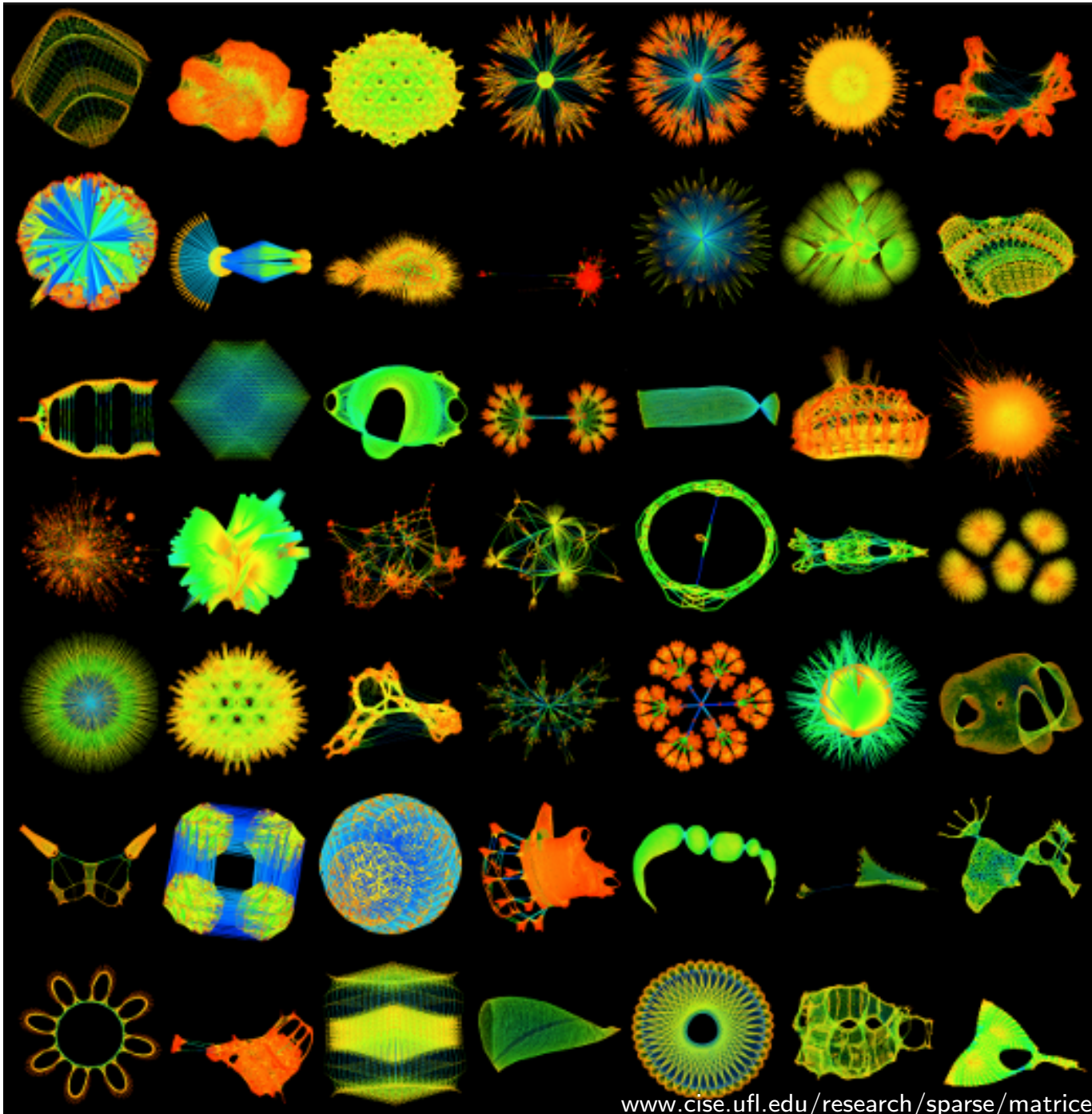
- easily understandable and implementable
- no special requirements on the input graph
- depending on the graphs (small and sparse) amazingly good layouts (Symmetries, Clustering, ...)
- easily adaptable and configurable
- robust
- scalable

Force-based Approaches are

- easily understandable and implementable
- no special requirements on the input graph
- depending on the graphs (small and sparse) amazingly good layouts (Symmetries, Clustering, ...)
- easily adaptable and configurable
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- scalable

But...

- usually no quality and running time guarantees
- bad choice of starting layout → slow convergence
- possibly slow for large graphs
- fine-tuning can be done by experts



www.cise.ufl.edu/research/sparse/matrices

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