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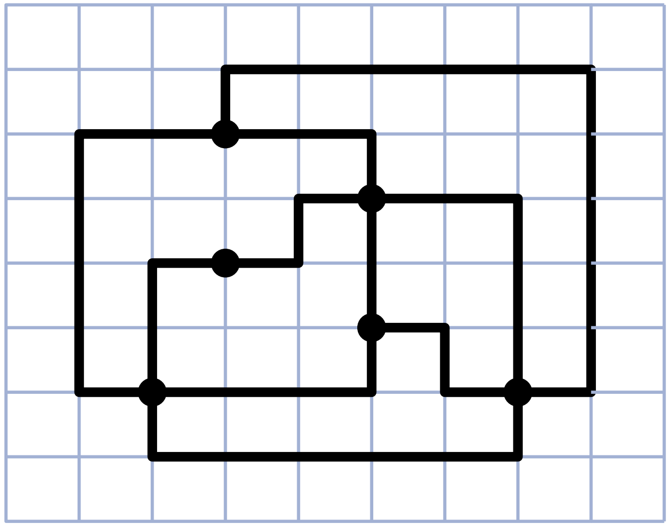
Algorithms for graph visualization

Incremental algorithms. Orthogonal drawing.

WINTER SEMESTER 2016/2017

Tamara Mchedlidze



Definition

Definition: Orthogonal Drawing

A drawing Γ of a graph $G = (V, E)$ is called orthogonal if its veritices are drawn as points and each edge is represented as a sequence of alternating horizontal and vertical segments.

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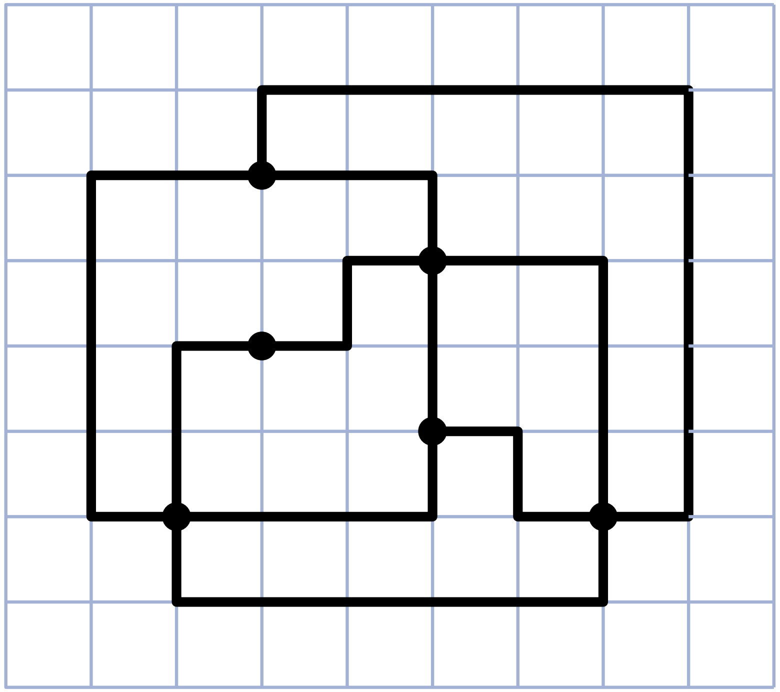

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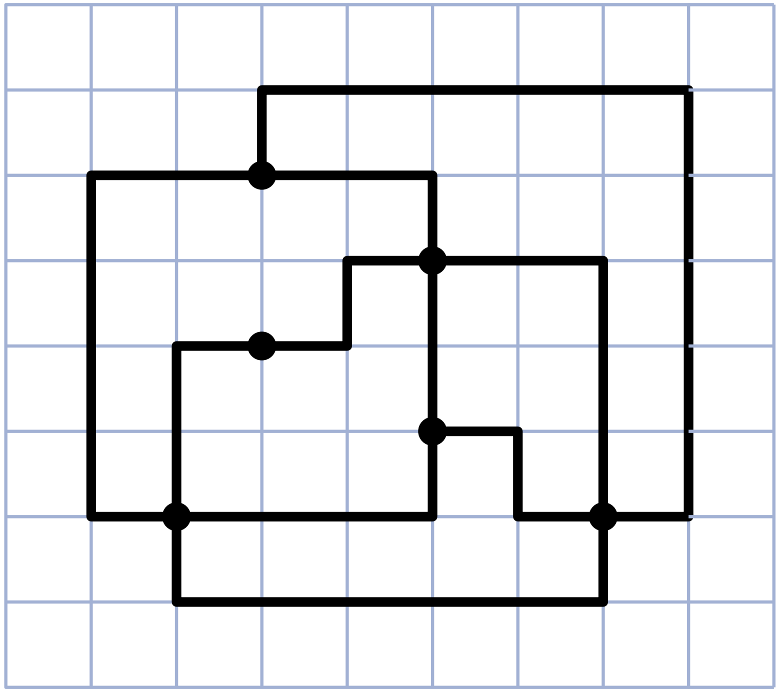

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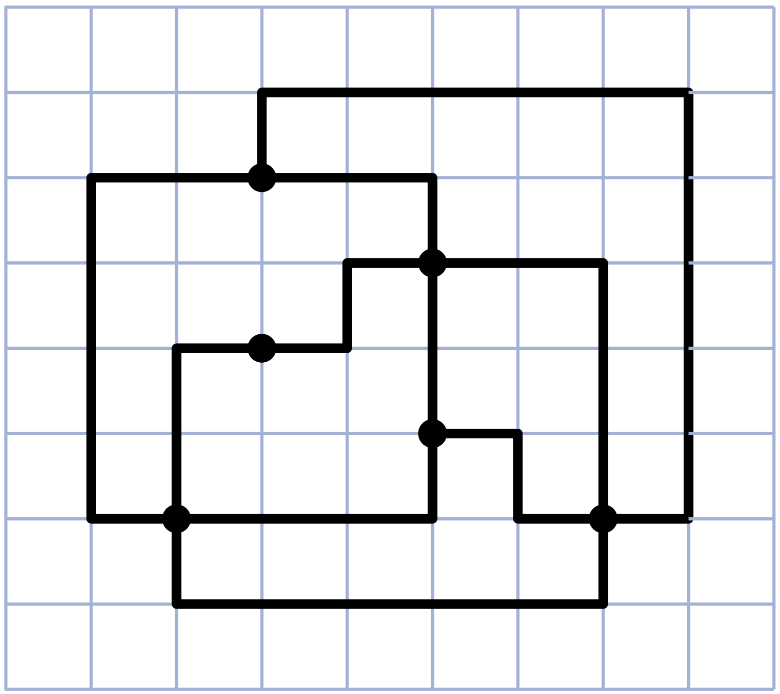

- Edges lie on the grid, i.e, bends lie on grid points

- degree of each vertex has to be at most 4

Applications

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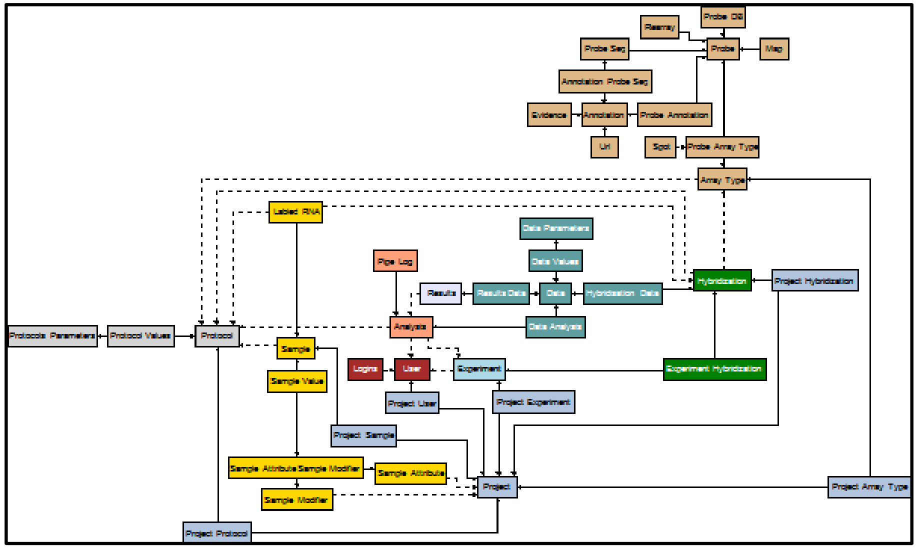
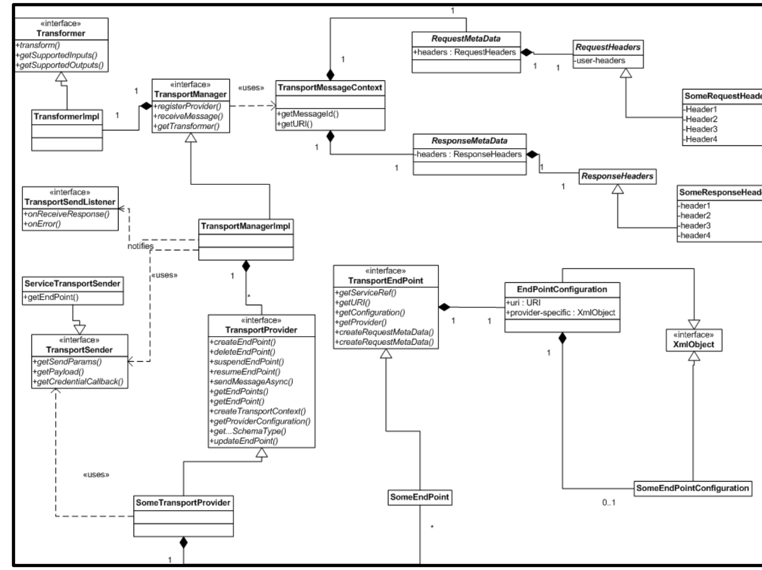
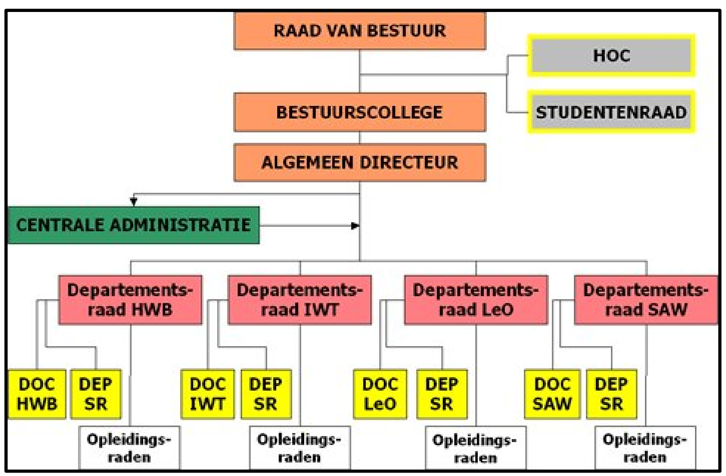
ER diagram in OGDF



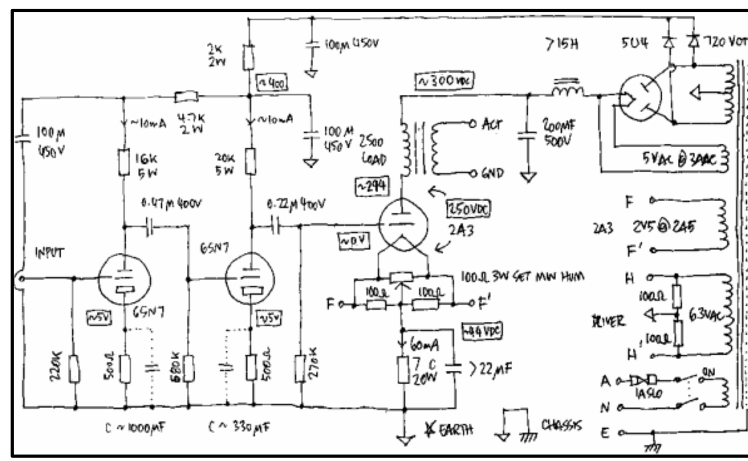
UML diagram by Oracle



Organigram of HS Limburg



Circuit diagram by Jeff Atwood



Applications

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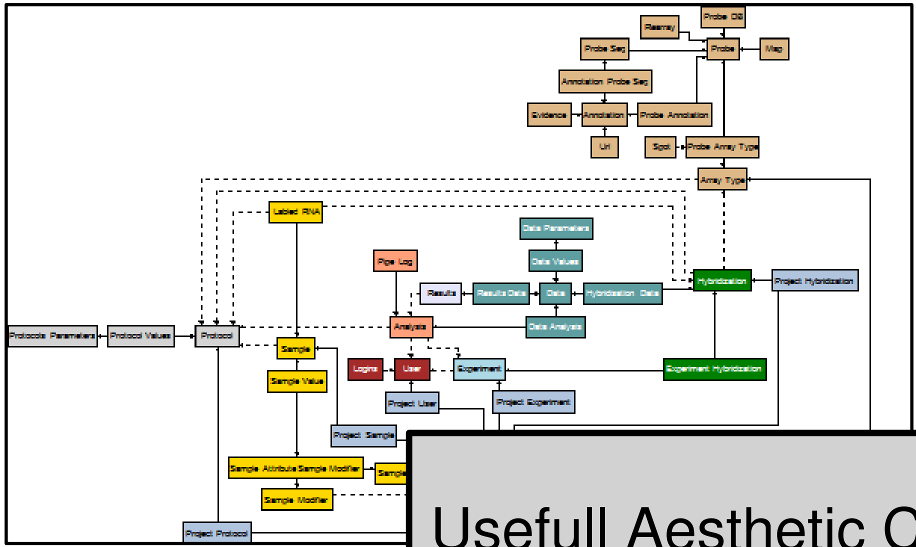

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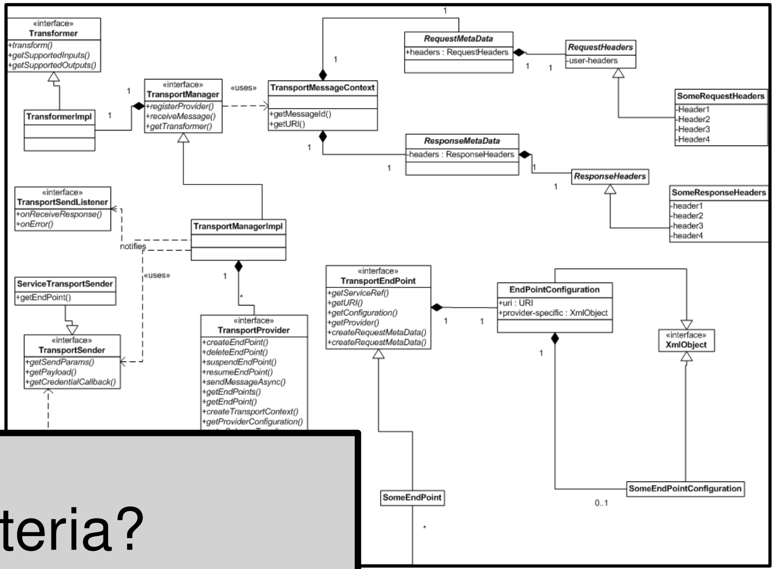
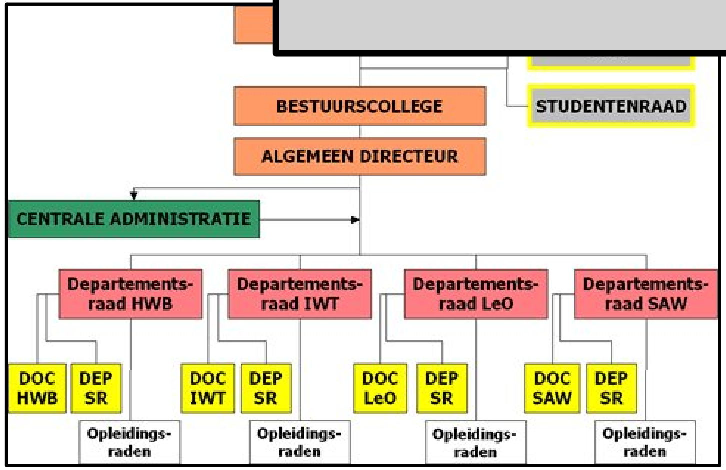


UML diagram by Oracle

Usefull Aesthetic Criteria?



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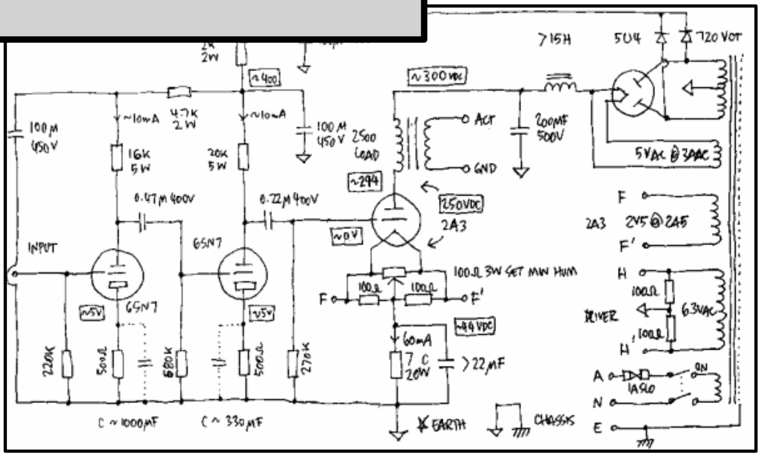

Circuit diagram by Jeff Atwood

st-ordering

Definition: st-ordering

An st-ordering of a graph $G = (V, E)$ is an ordering of the vertices $\{v_1, v_2, \ldots, v_n\}$, such that for each j, $2 \leq j \leq n - 1$, vertex v_j has at least one neighbour v_i with $i < j$, and at least one neighbour v_k with $k > j$.

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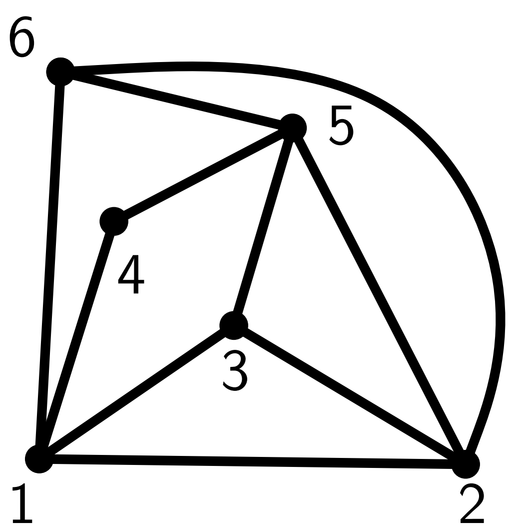

Example of an st-ordering

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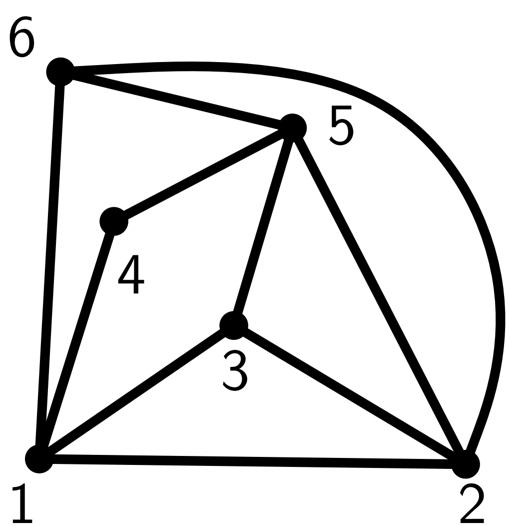
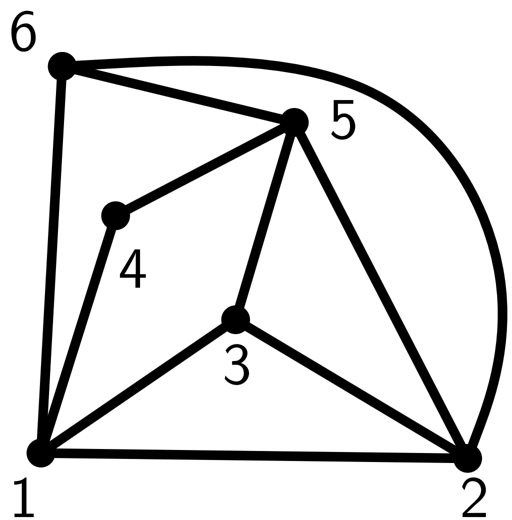


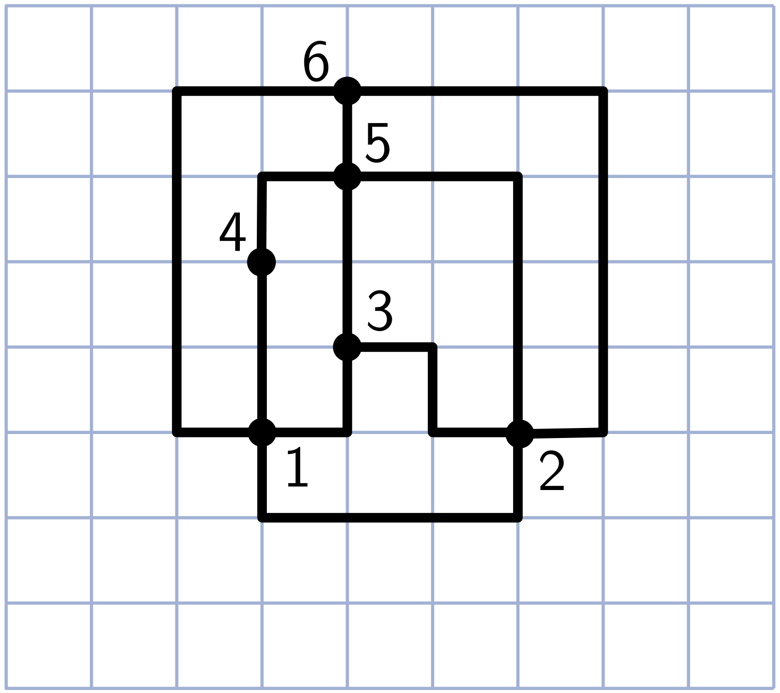
Example of an st-ordering

Theorem [Lempel, Even, Cederbaum, 66]

Let G be a biconnected graph G and let s, t be vetices of G. G has an st-ordering such that s appears as the first and t as the last vertex in this ordering.

Biedl & Kant Orthogonal Drawing Algorithm





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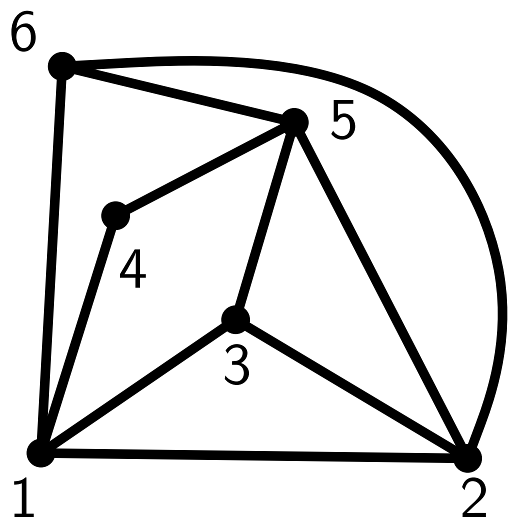
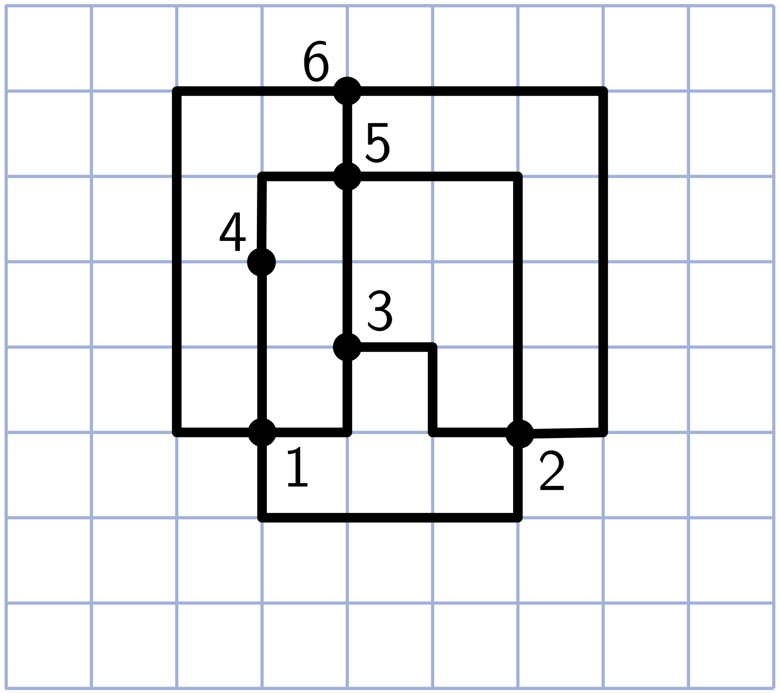
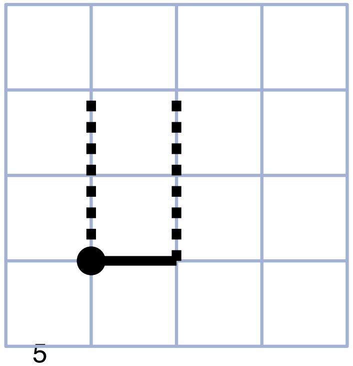




first vertex



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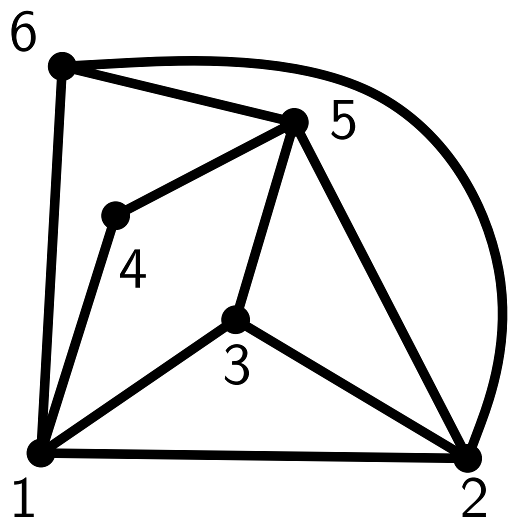
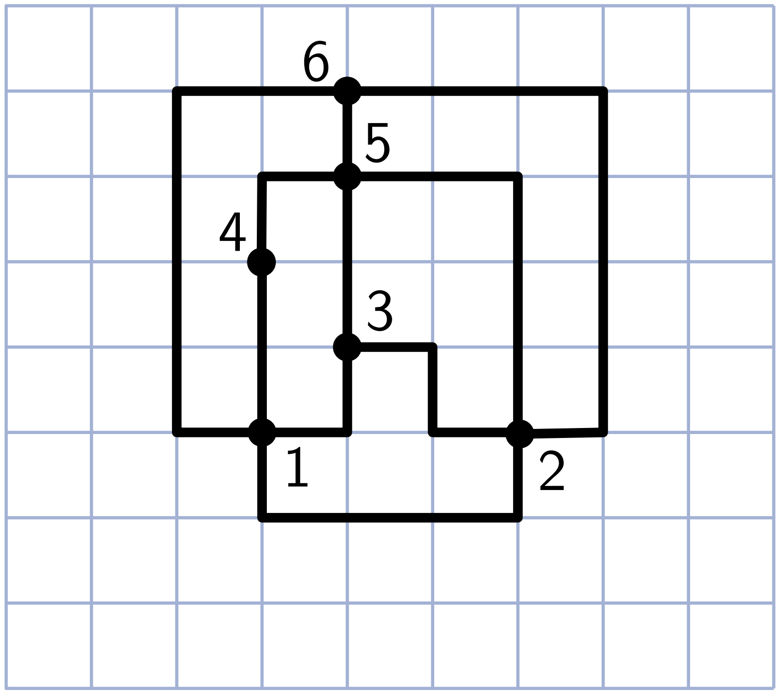
Biedl & Kant Orthogonal Drawing Algorithm





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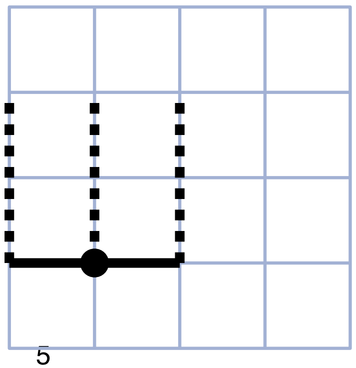

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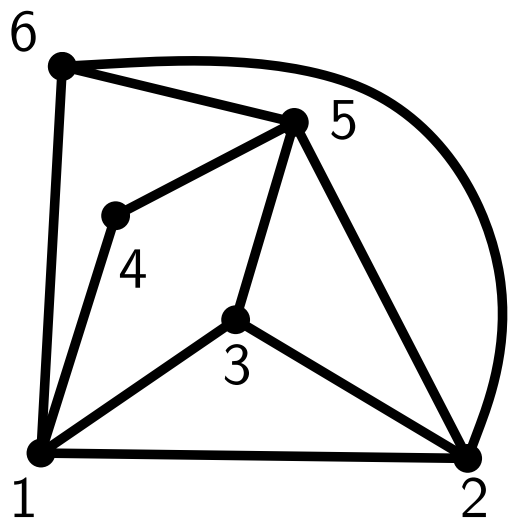

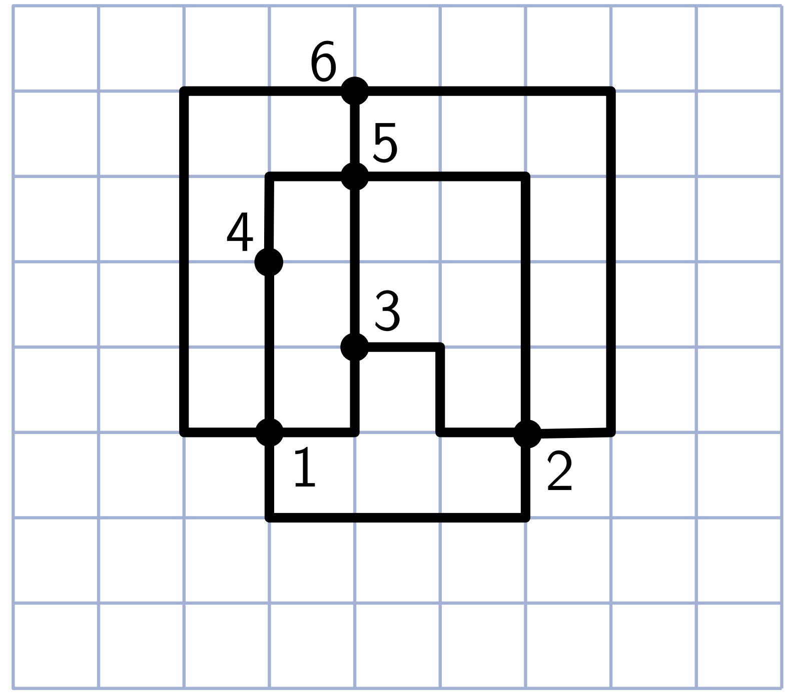

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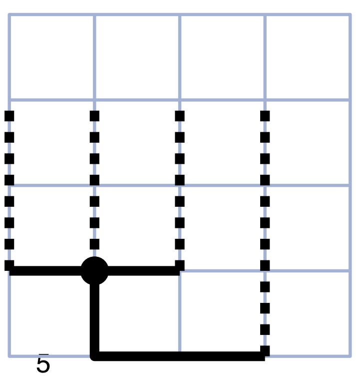

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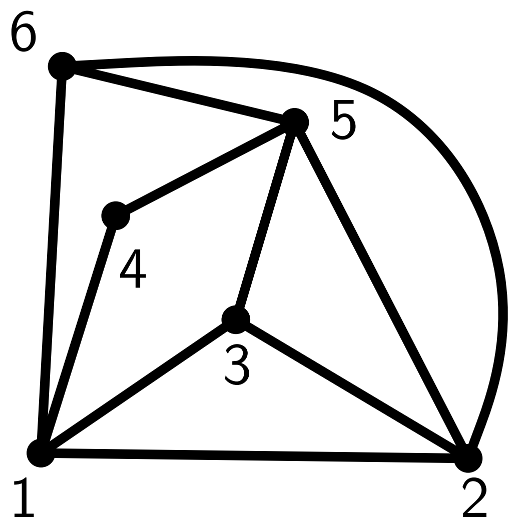
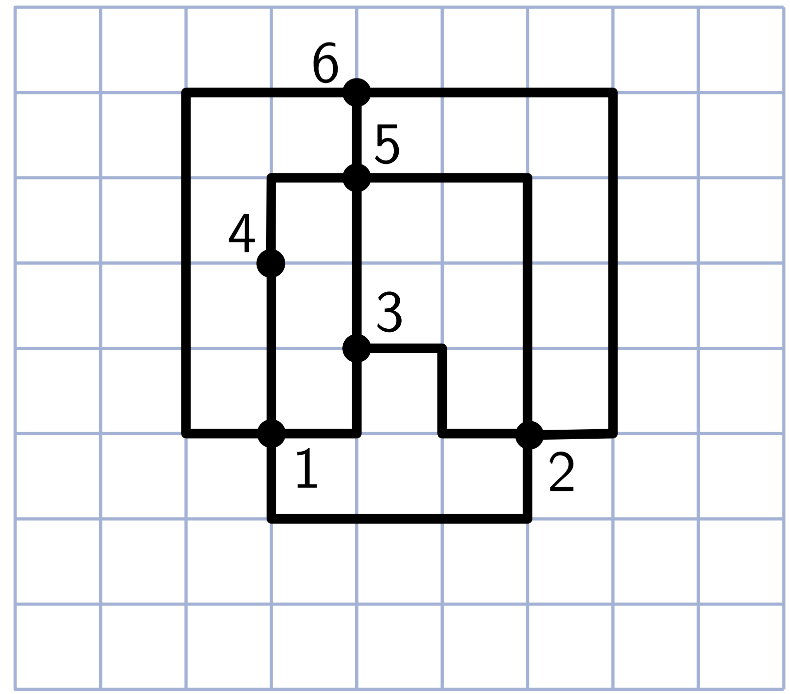
Biedl & Kant Orthogonal Drawing Algorithm





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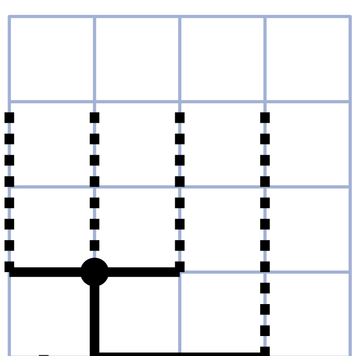
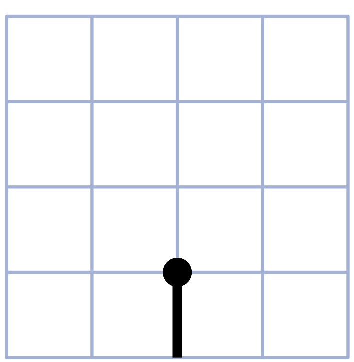


indegree = 1



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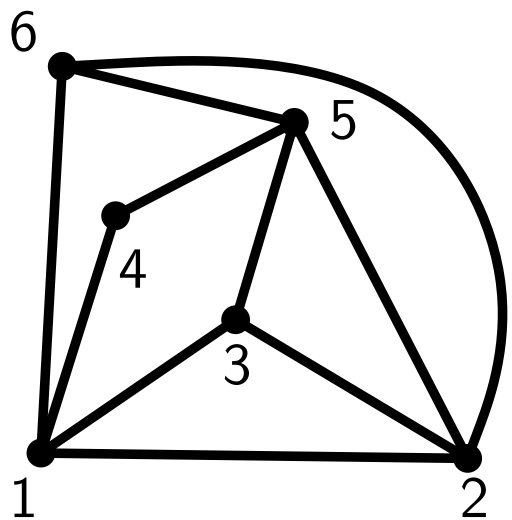

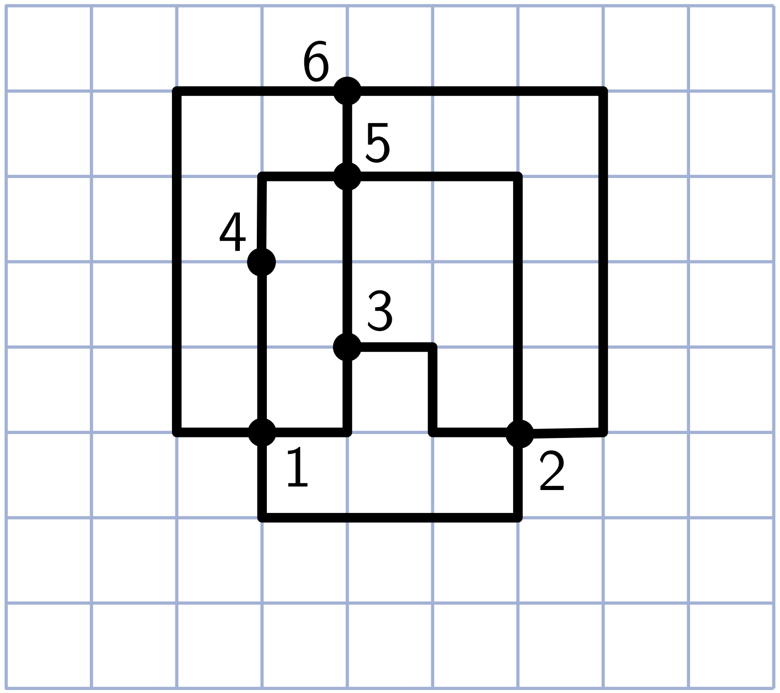

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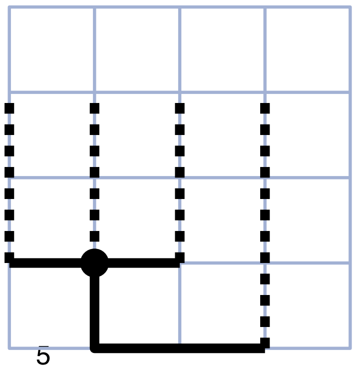

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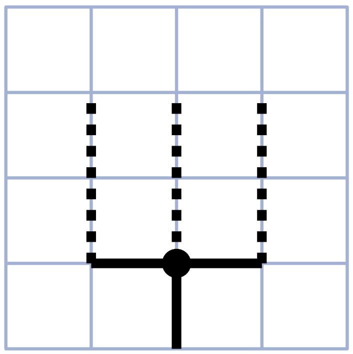

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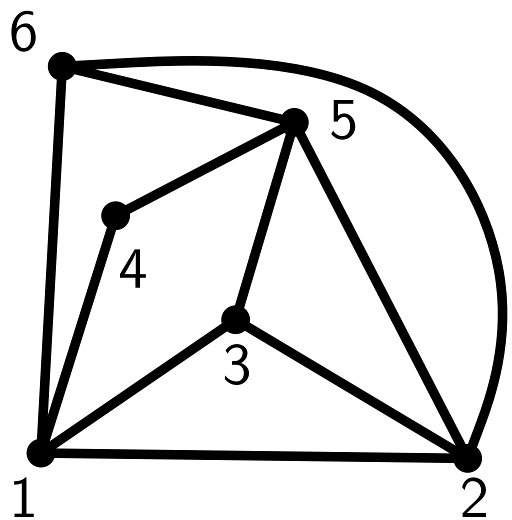
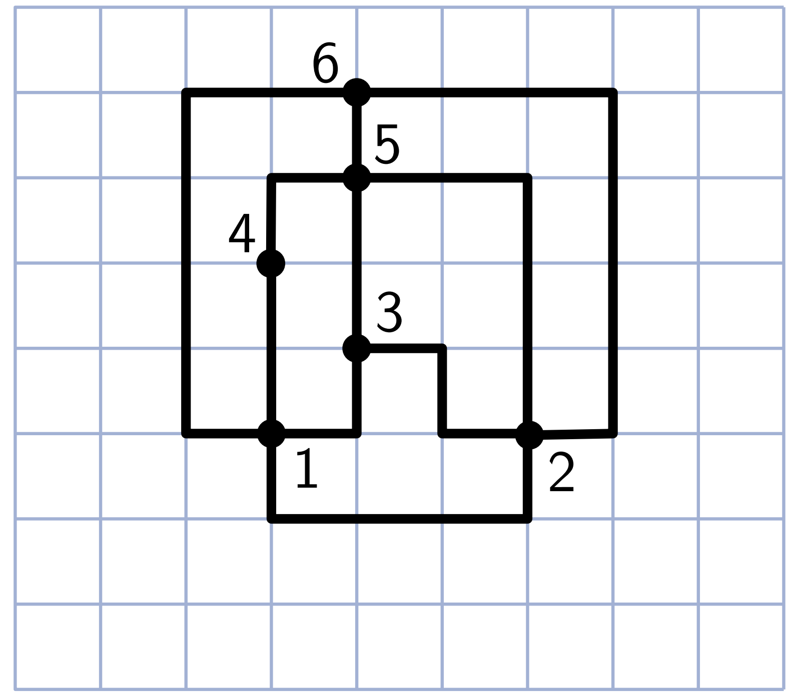
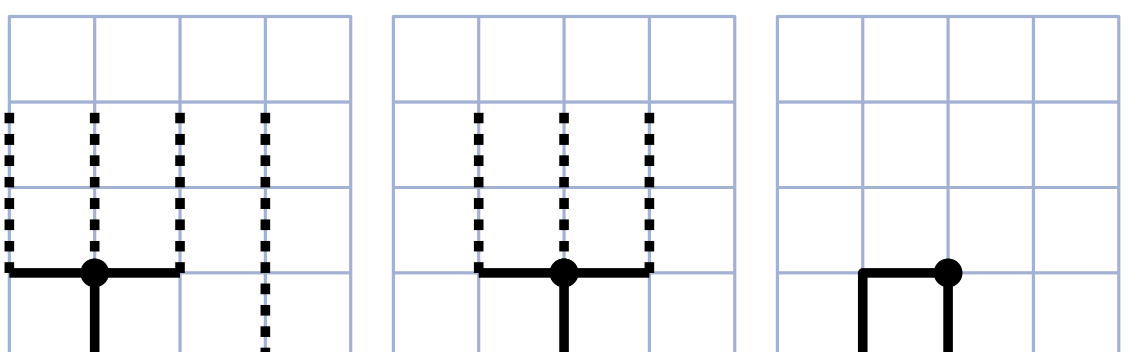




first vertex indegree = 1 indegree = 2



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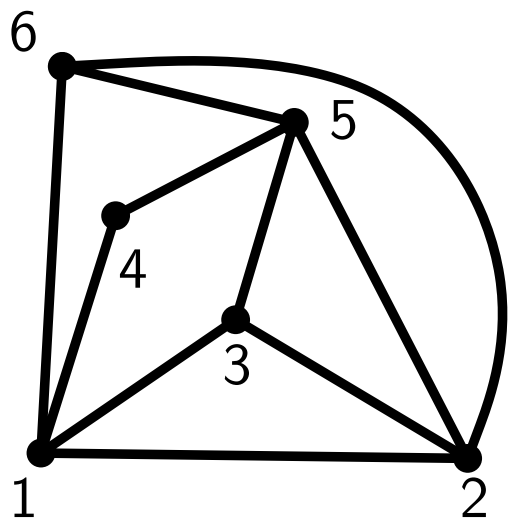

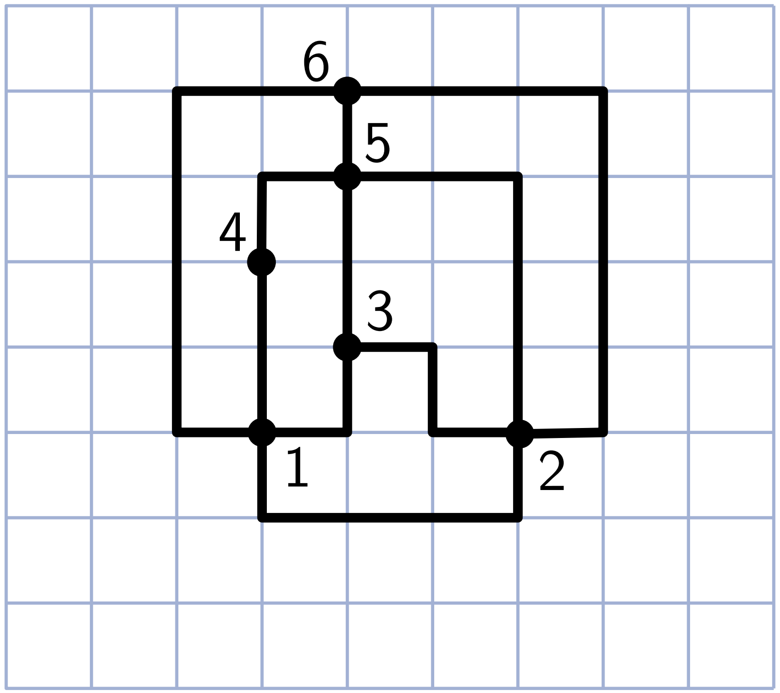

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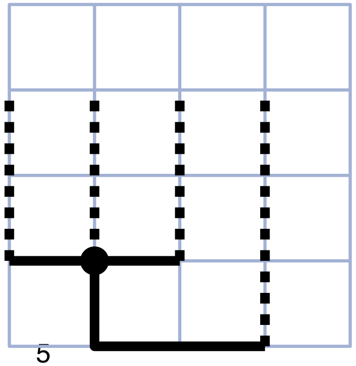

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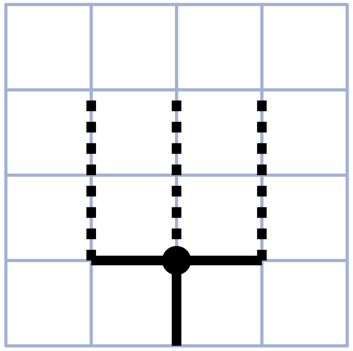
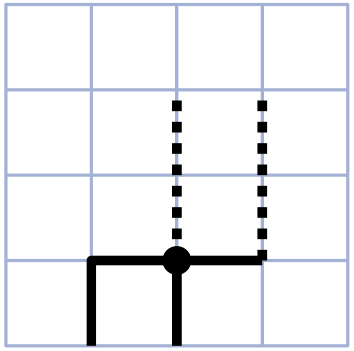


indegree = 2



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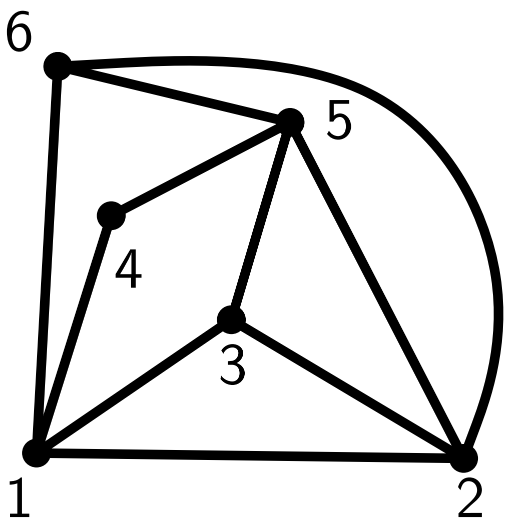
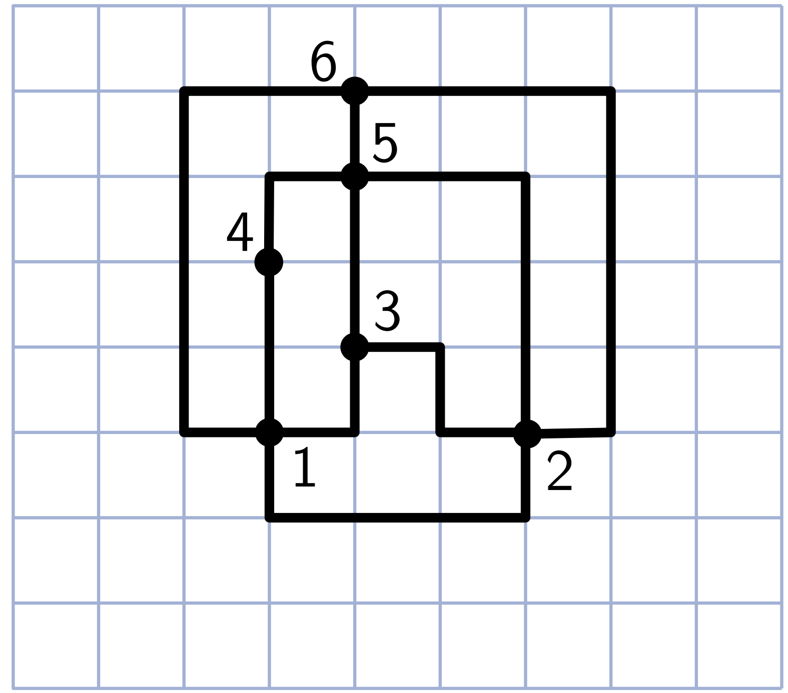
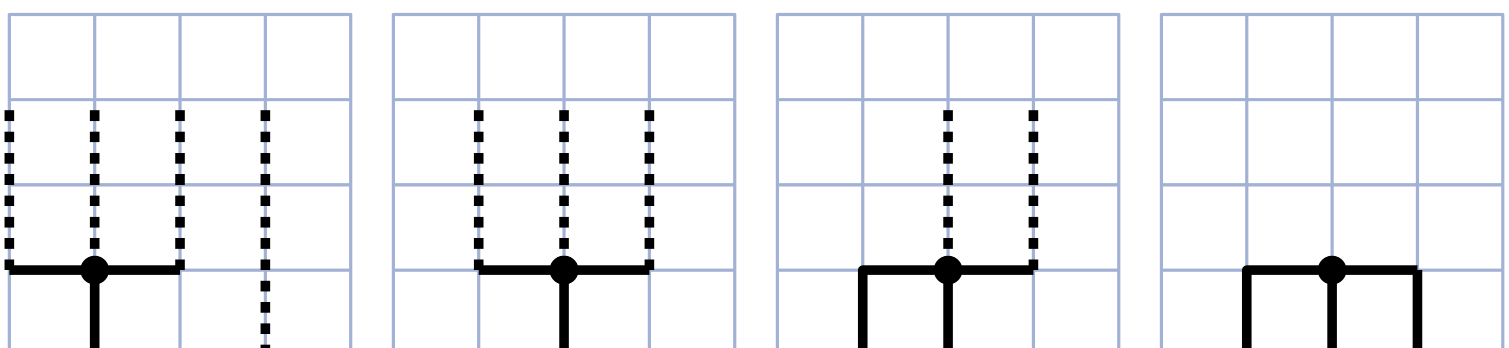
Biedl & Kant Orthogonal Drawing Algorithm





first vertex   indegree = 1   indegree = 2   indegree = 3



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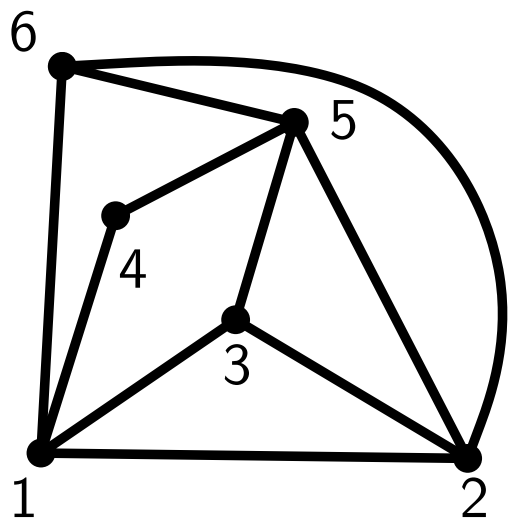

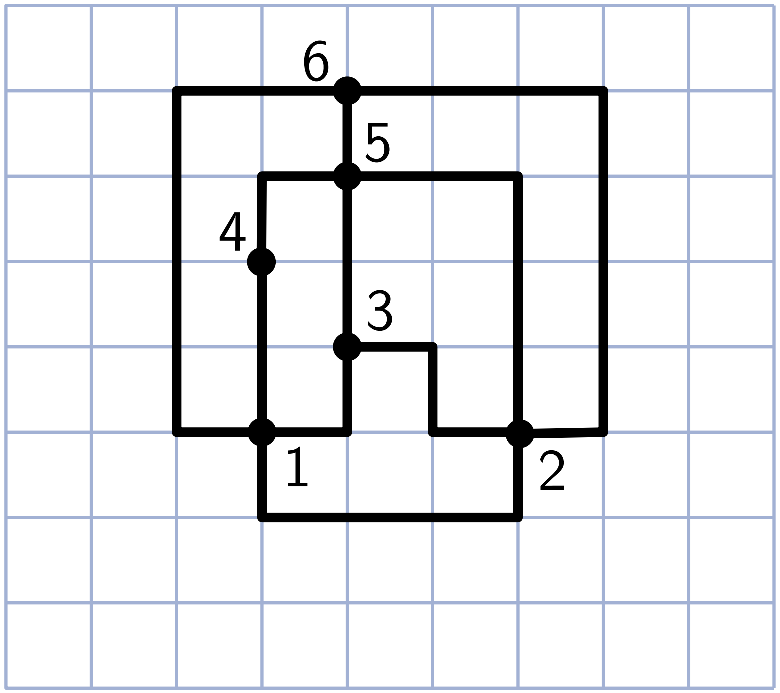

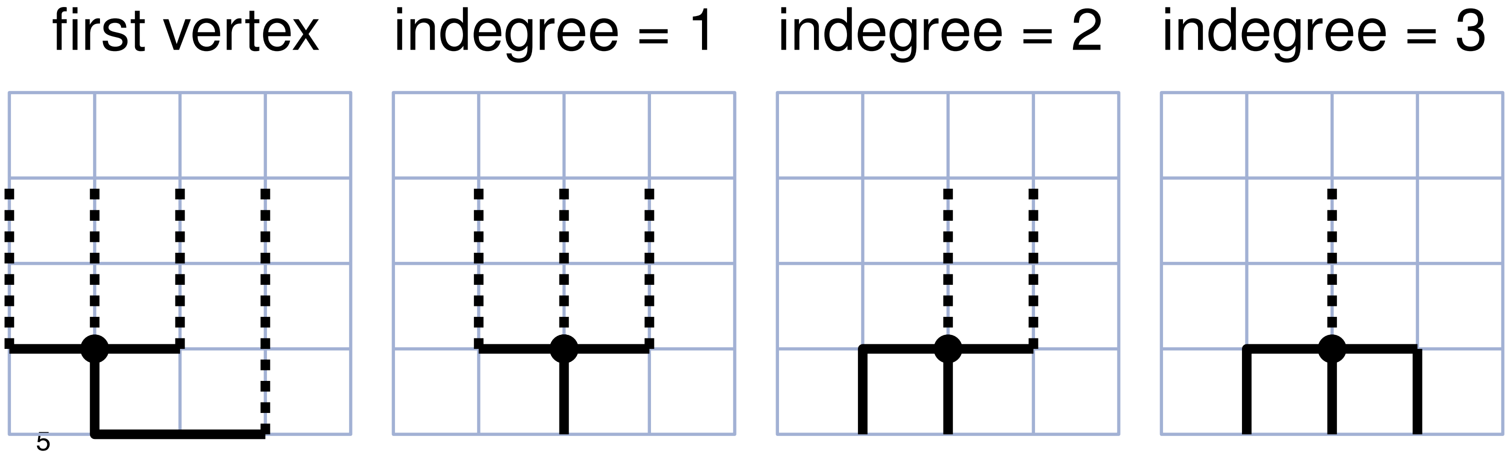

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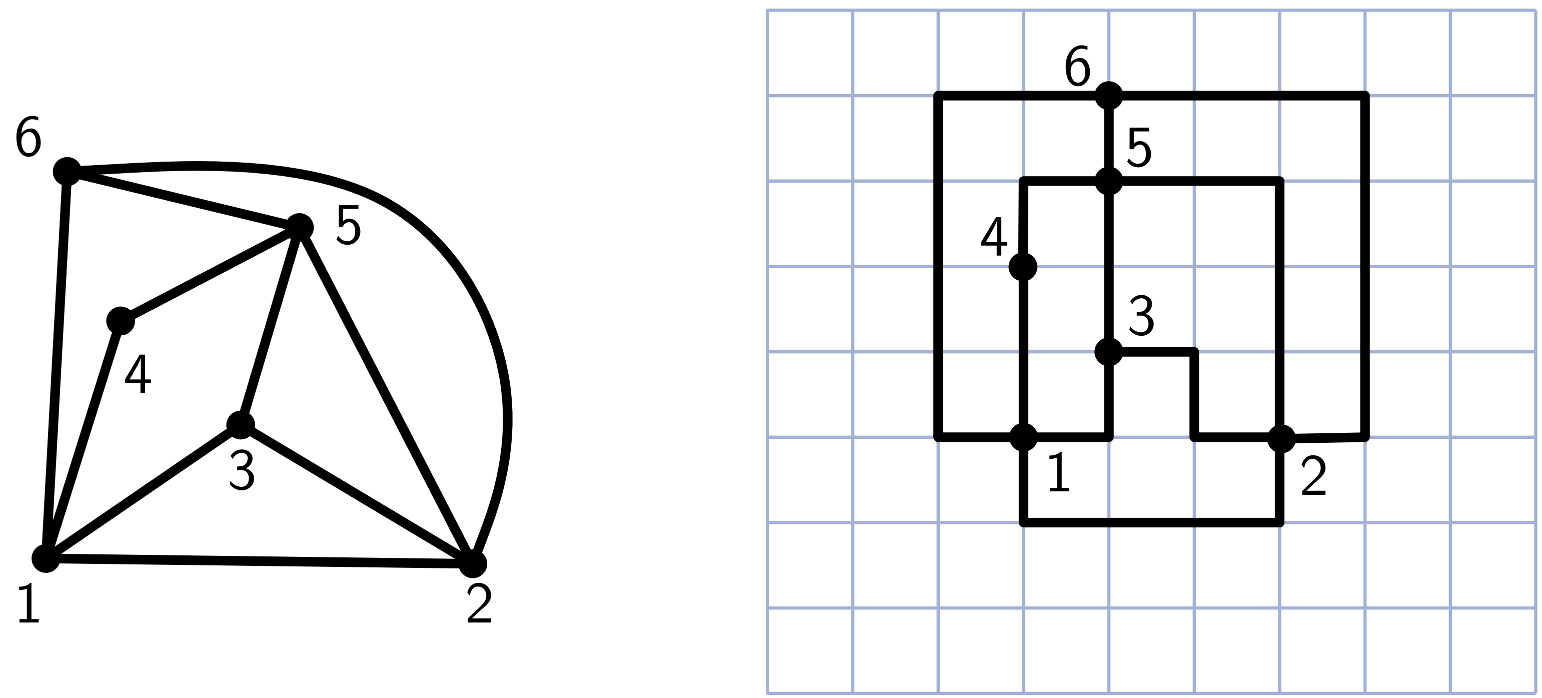

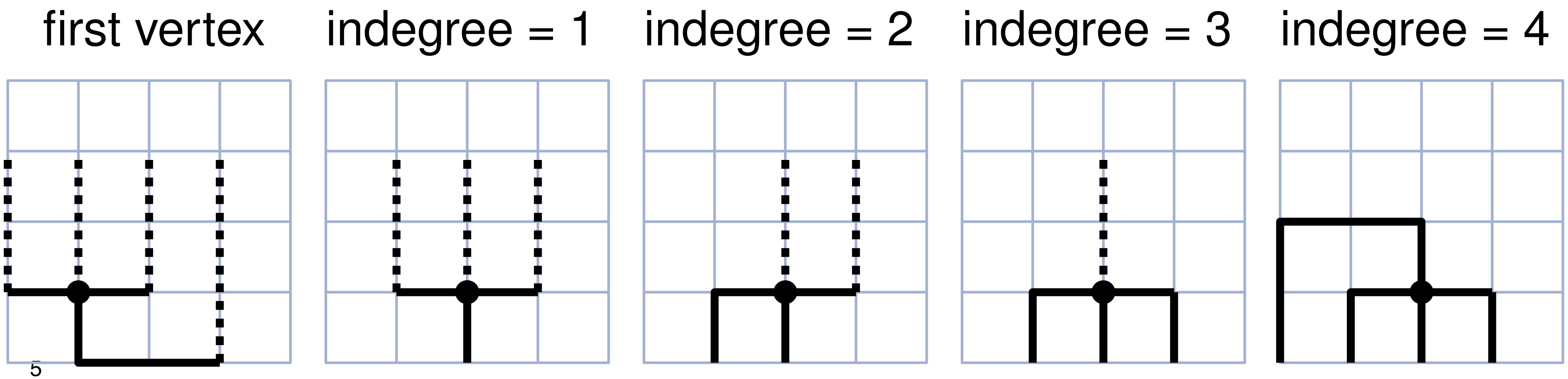

Biedl & Kant Orthogonal Drawing Algorithm

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Lemma (Area of Biedl & Kant drawing)

The width is $m - n + 1$ and the height at most $n + 1$.

Biedl & Kant Orthogonal Drawing Algorithm

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The width is $m - n + 1$ and the height at most $n + 1$.

Proof

- Width: At each step we increase the number of columns by $outdeg(v_i) - 1$, if $i > 1$ and $outdeg(v_1)$ for v_1.

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There are at most $2m - 2n + 4$ bends.

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Lemma (Number of bends in Biedl & Kant drawing)

There are at most $2m - 2n + 4$ bends.

Proof

- Each vertex v_i, $i \neq 1, n$, introduces $indeg(v_i) - 1$ and $outdeg(v_i) - 1$ new bends.

Biedl & Kant Orthogonal Drawing Algorithm

Lemma (Number of bends per edge in Biedl & Kant drawing)

All edges but one bent at most twice. The exceptional edge bents at most three times.

Biedl & Kant Orthogonal Drawing Algorithm

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Proof

- Let (v_i, v_j), $i < j$, $i, j \neq 1, n$. Then $outdeg(v_i)$, $indeg(v_j) \leq 3$. I.e (v_i, v_j) gets at most one bend after placement of v_i and at most one before placement of v_j. Edges outgoing from v_1 can me made 2-bend by using the column below v_1 for the edge (v_1, v_2).

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Lemma (planarity)

For planar embedded graphs, with v_1 and v_n on the outer face, the algorithm produces a planar drawing.

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Lemma (planarity)

For planar embedded graphs, with v_1 and v_n on the outer face, the algorithm produces a planar drawing.

Proof

- Consider a planar embedding of G. Let $v_1, \ldots, v_n$ be an st-ordering of G. Let G_i be the graph induced by $v_1, \ldots, v_i$. We now prove that if G is planar, vertex v_{i+1} lies on the outer face of G_i.

Biedl & Kant Orthogonal Drawing Algorithm

Lemma (planarity)

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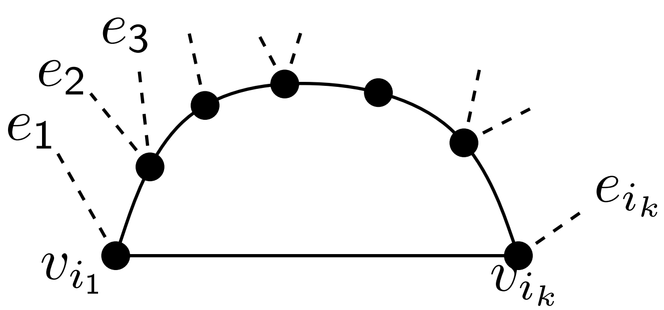
Proof (Continuation)

- Let E_i be the edges outgoing from the vertices of G_i in the order they appear in the embedded G.



Biedl & Kant Orthogonal Drawing Algorithm

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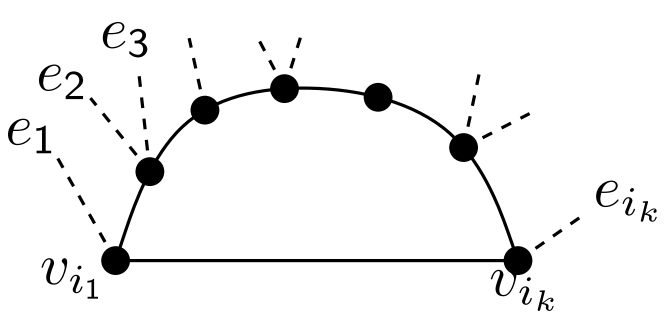
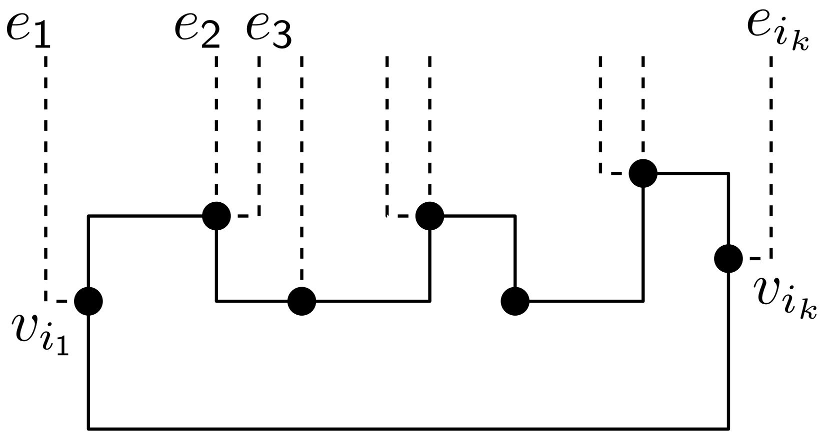
Proof (Continuation)

- Let E_i be the edges outgoing from the vertices of G_i in the order they appear in the embedded G.
- During the execution of the algorithm we make sure that edges of E_i appear in the same order in the orthogonal drawing of G_i.





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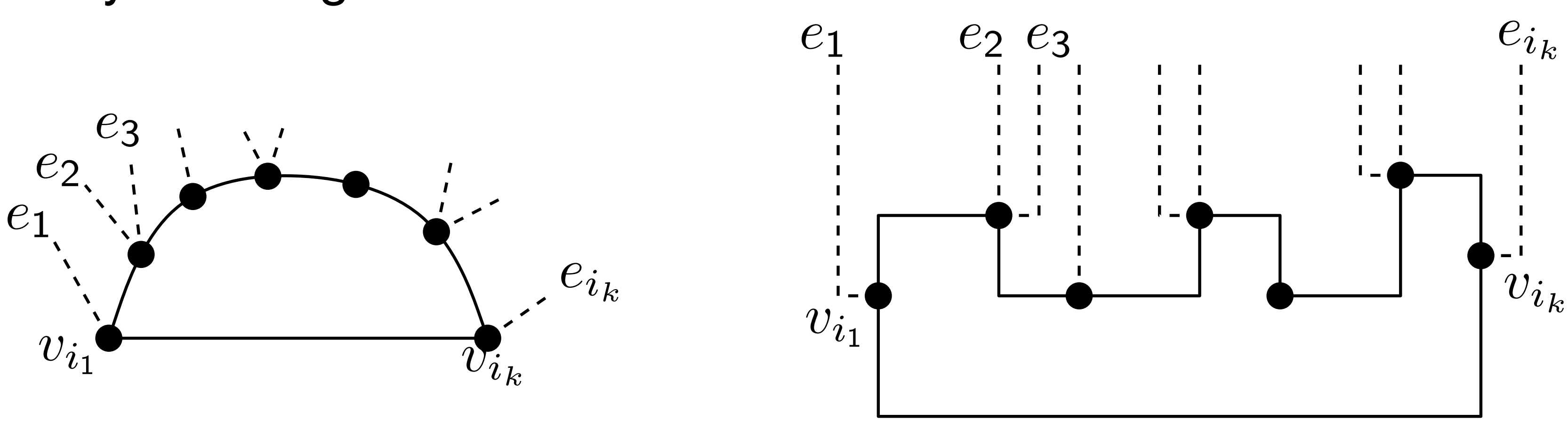

Biedl & Kant Orthogonal Drawing Algorithm

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Lemma (planarity)

For planar embedded graphs, with v_1 and v_n on the outer face, the algorithm produces a planar drawing.

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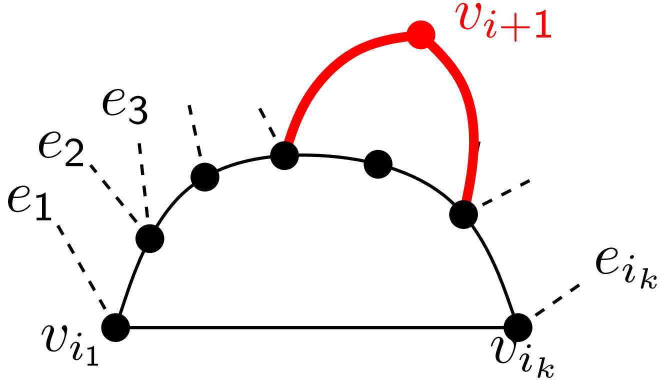

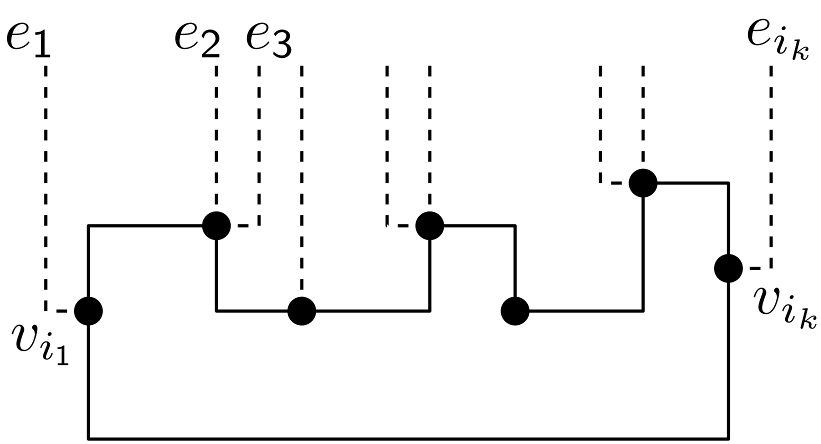

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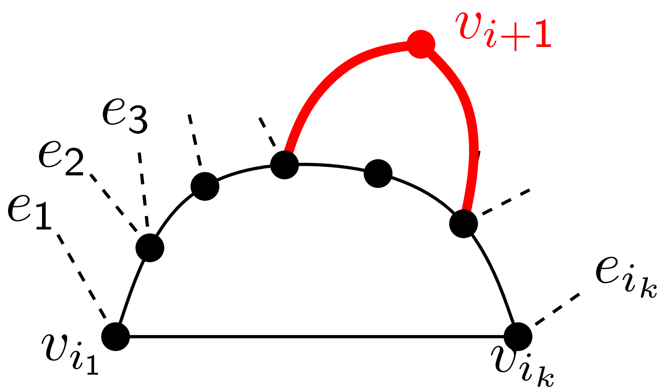

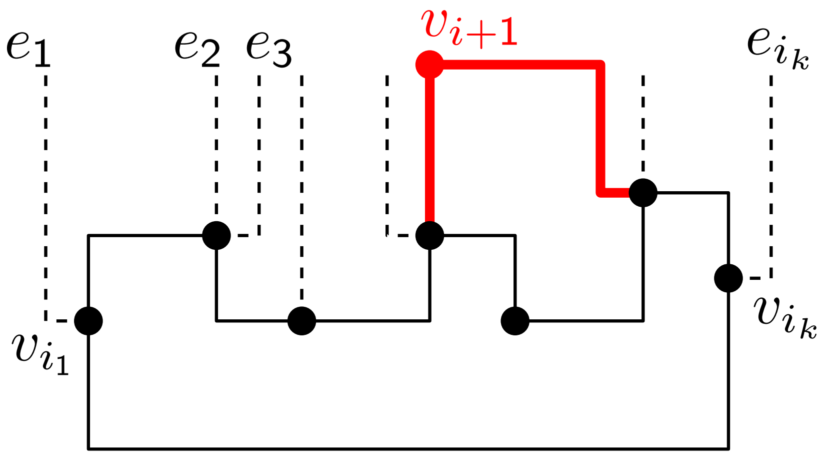

Biedl & Kant Orthogonal Drawing Algorithm

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Theorem (Biedl & Kant 98)

A biconnected graph G with vertex-degree at most 4 admits an orthogonal drawing such that:

- Area is $(m - n + 1) \times n + 1$
- Each edge (except maybe for one) has at most 2 bends
- The exceptional edge has at most 3 bends
- The total number if bends is at most $2m - 2n + 4$
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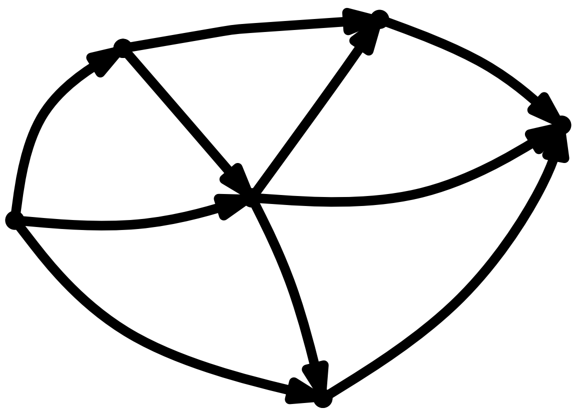
- For the construction we have used an st-ordering of G!

st-digraph, topological ordering

Definition: st-digraph

Let G be a directed graph. A vertex s (resp. t) is called **source** (resp. **sink**) of G if it cas only outgoing (resp. incomming edges). A directed acyclic graph with one source and one sink is called st**-digraph**.



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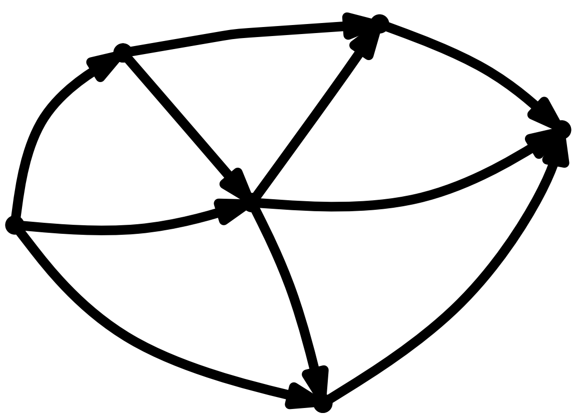

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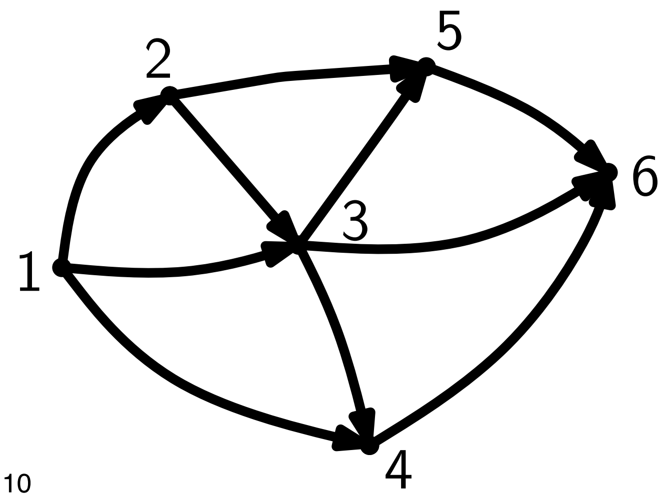

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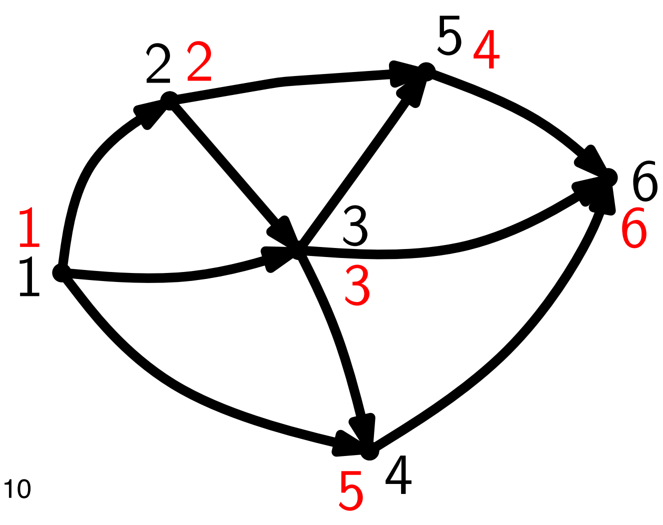

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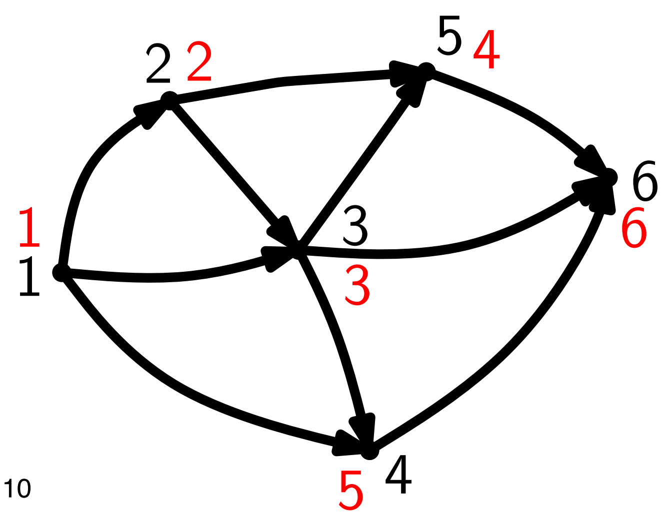

How to construct a topological ordering?

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st-ordering

Construction of an st-ordering:

G is undirected biconnected graph

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st-ordering

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st-ordering

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st-ordering

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G is undirected biconnected graph → (Orient edges of G) → **G' is an st-digraph** → **Let $v_1, \dots, v_n$ be a topological ordering of G'**

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st-ordering

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st-ordering

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EXAMPLE

st-ordering

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EXAMPLE

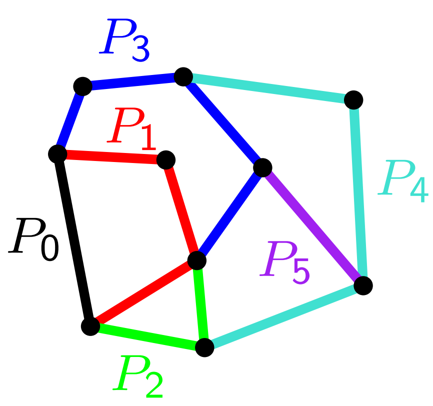
st-ordering

Definition: Ear decomposition

An ear decomposition $D = (P_0, \ldots, P_r)$ of an undirected graph $G = (V, E)$ is a partition of E into an ordered collection of edge disjoint paths $P_0, \ldots, P_r$, such that:

- P_0 is an edge
- $P_0 \cup P_1$ is a simple cycle
- both end-vertices of P_i belong to $P_0 \cup \cdots \cup P_{i-1}$
- no internal vertex of P_i belong to $P_0 \cup \cdots \cup P_{i-1}$

An ear decomposition of open if $P_0, \ldots, P_r$ are simple paths.



st-ordering

Lemma (Ear decomposition)

Let $G = (V, E)$ be a biconnected graph G and let $(s, t) \in E$. G has an open ear decomposition $(P_0, \dots, P_r)$, where $P_0 = (s, t)$.

st-ordering

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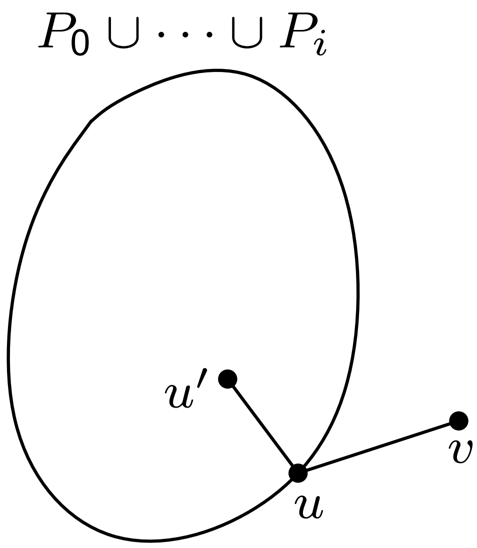

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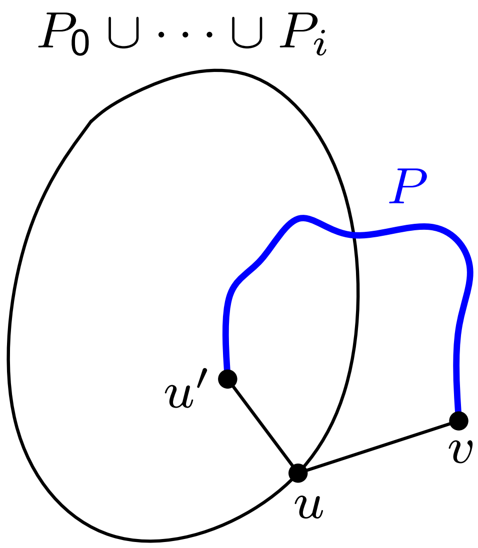

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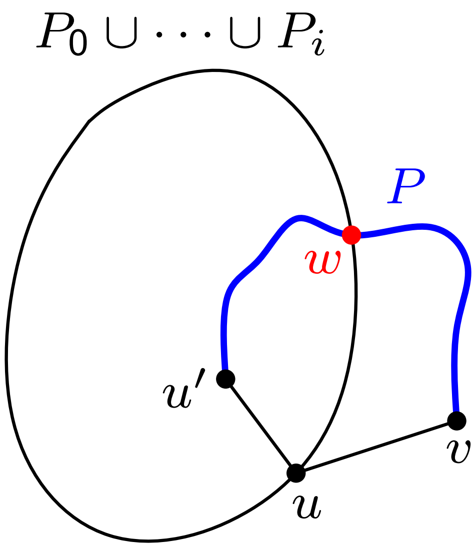

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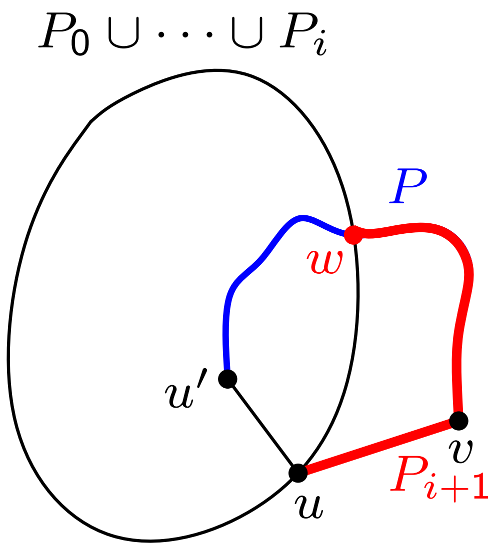

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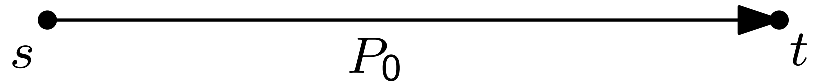

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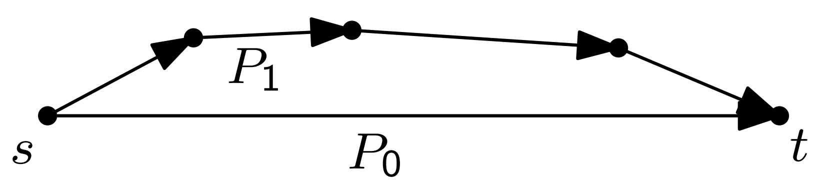

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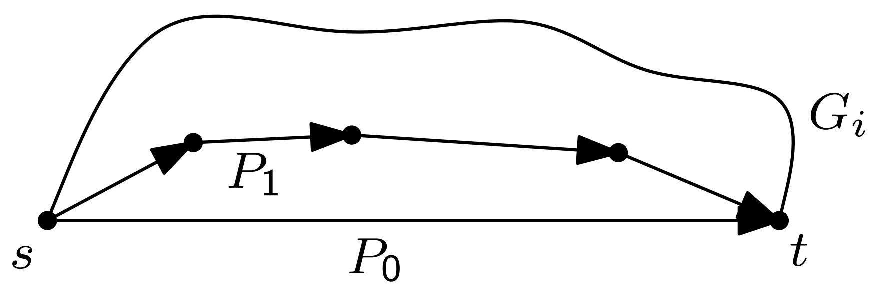

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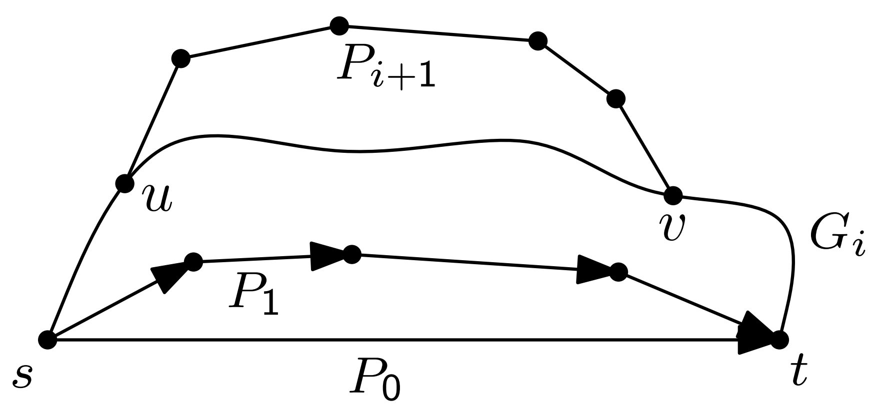

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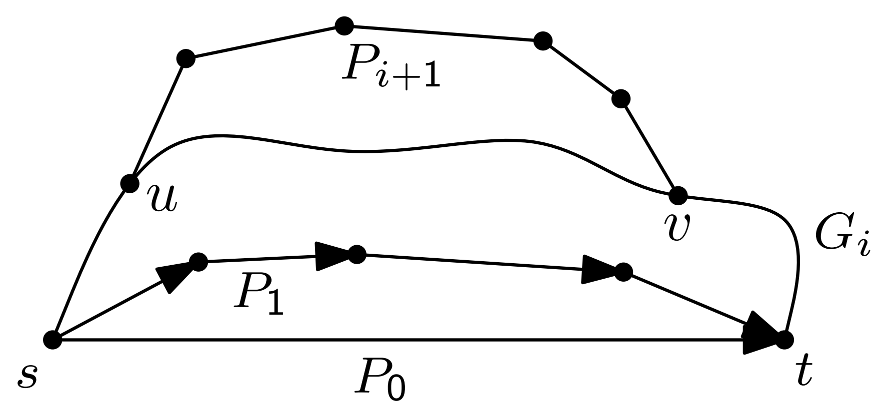

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st-ordering

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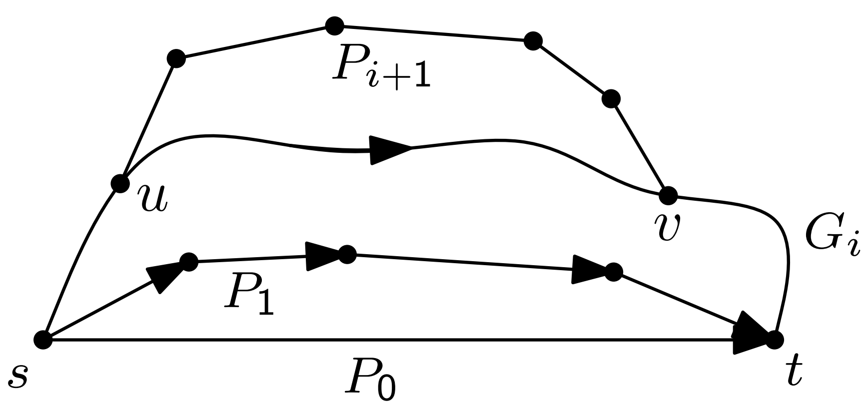

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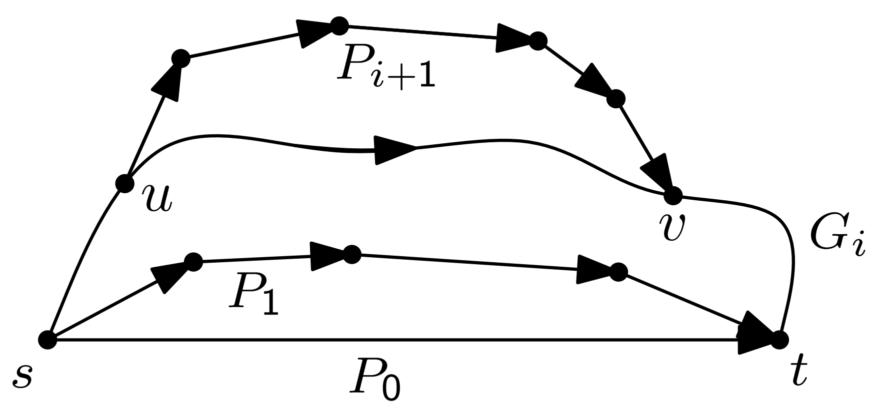

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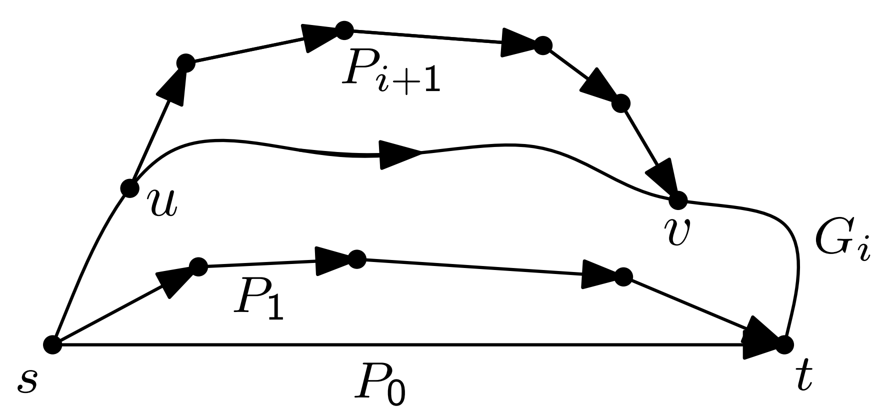

- Distinguish two cases based on whether u and v are connected by a directed path or not.
- Result is an st-digraph

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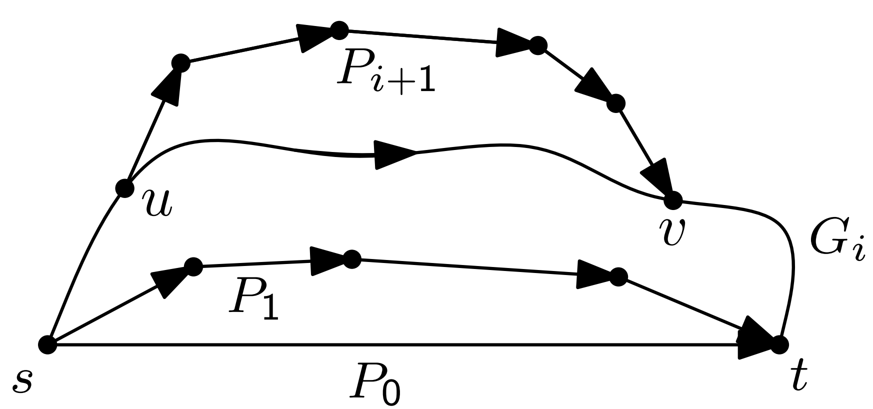
EXAMPLE



- Distinguish two cases based on whether u and v are connected by a directed path or not.
- Result is an st-digraph

Construction of an st-ordering:

G is undirected biconnected graph

→ Orient edges of G ✓ →

G' is an st-digraph

→

Let $v_1, \ldots, v_n$ be a topological ordering of G'

↓ Since G' is an st-digraph, for v_i $(i \neq 1, n)$ $\exists$ (v_j, v_i) and (v_i, v_k). By the property of topological ordering $j < i$ and $i < k$.

$v_1, \ldots, v_n$ is an st-ordering of G

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Ear decomposition of G

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Ear decomposition of G

Orient ears

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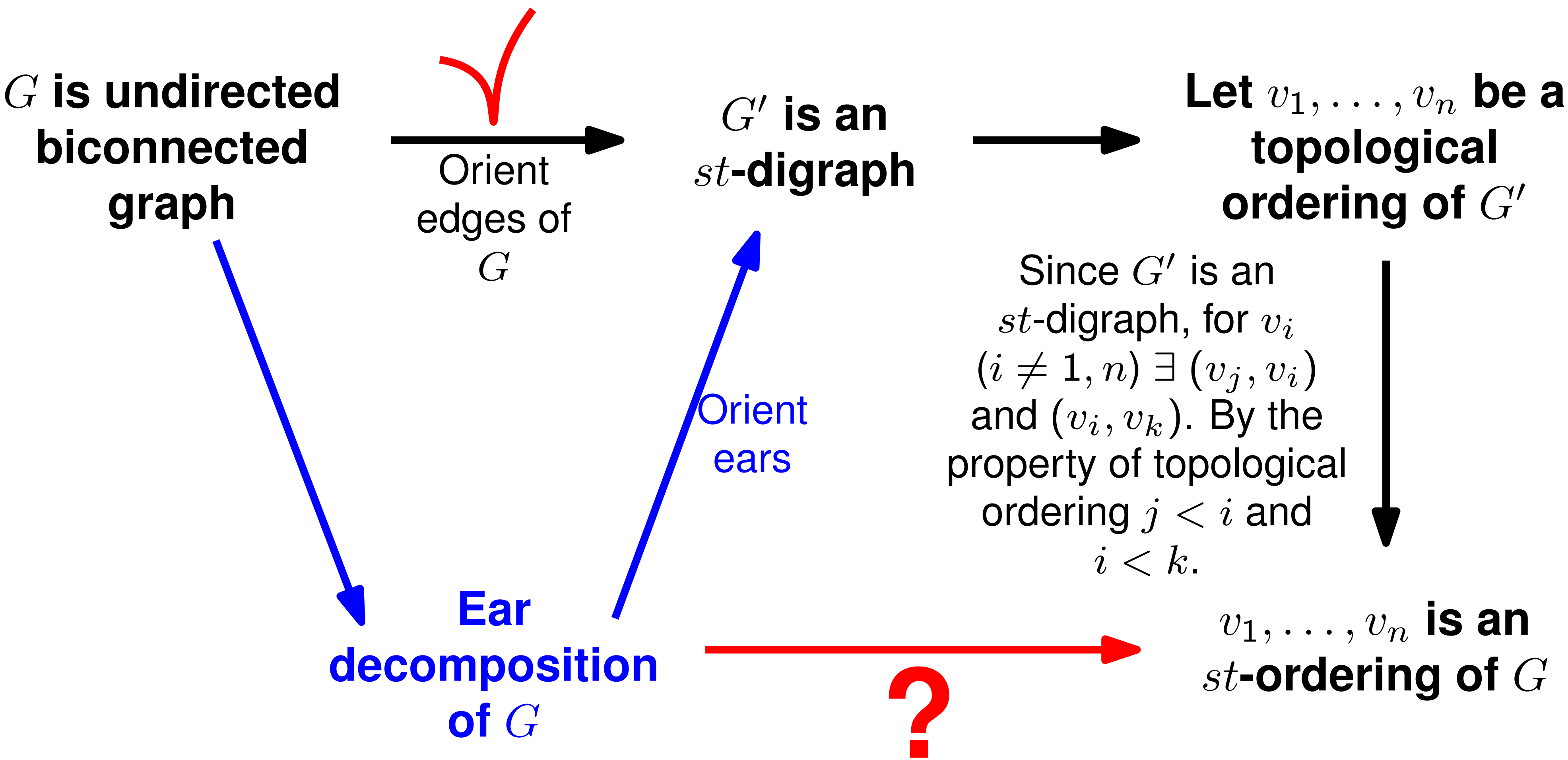
Construction of an st-ordering:



Direct construction of st-ordering from ear decomposition

st-ordering

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- We construct it incrementally, considering $G_i = P_0 \cup \cdots \cup P_i$, $i = 0, \ldots, r$.

st-ordering

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- For G_1, let $P_1 = \{u_1, \dots, u_p\}$, here $u_1 = s$ and $u_p = t$. The sequence $L = \{u_1, \dots, u_p\}$ is an st-ordering of G_1.

st-ordering

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- Assume that L contains an st-ordering of G_i and let ear $P_{i+1} = \{v_1, \ldots, v_q\}$. We insert vertices $v_1, \ldots, v_q$ to L after vertex v_1.

st-ordering

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EXAMPLE

st-ordering

Direct construction of st-ordering from ear decomposition

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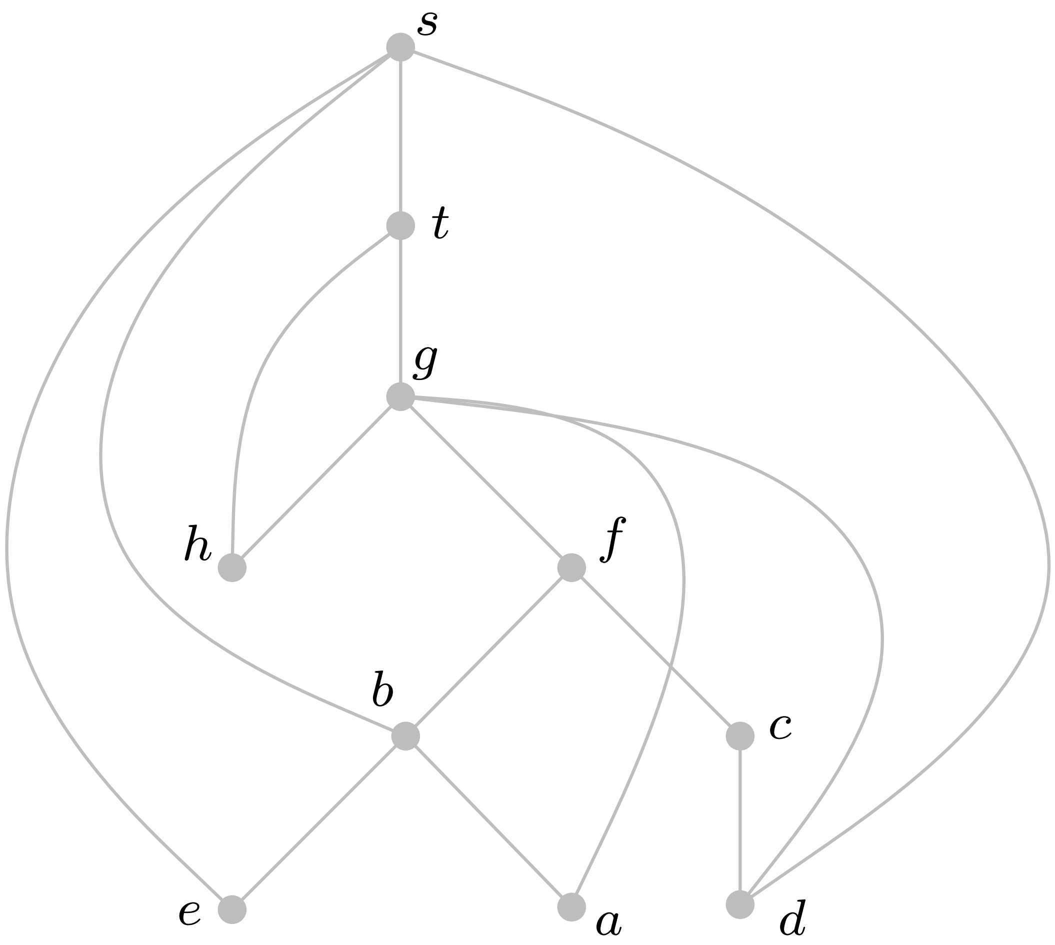
- For G_1, let $P_1 = \{u_1, \ldots, u_p\}$, here $u_1 = s$ and $u_p = t$. The sequence $L = \{u_1, \ldots, u_p\}$ is an st-ordering of G_1.
- Assume that L contains an st-ordering of G_i and let ear $P_{i+1} = \{v_1, \ldots, v_q\}$. We insert vertices $v_1, \ldots, v_q$ to L after vertex v_1.
- **Why this is an st-ordering?** Let G'_{i+1} be an st-orientation of G_i as constructed in the previous proof. L is a topological ordering of G'_{i+1} and therefore an st-ordering of G_i

EXAMPLE

st-ordering: implementation

Algorithm: st-ordering (example)

(Implementation details - Based on DFS)



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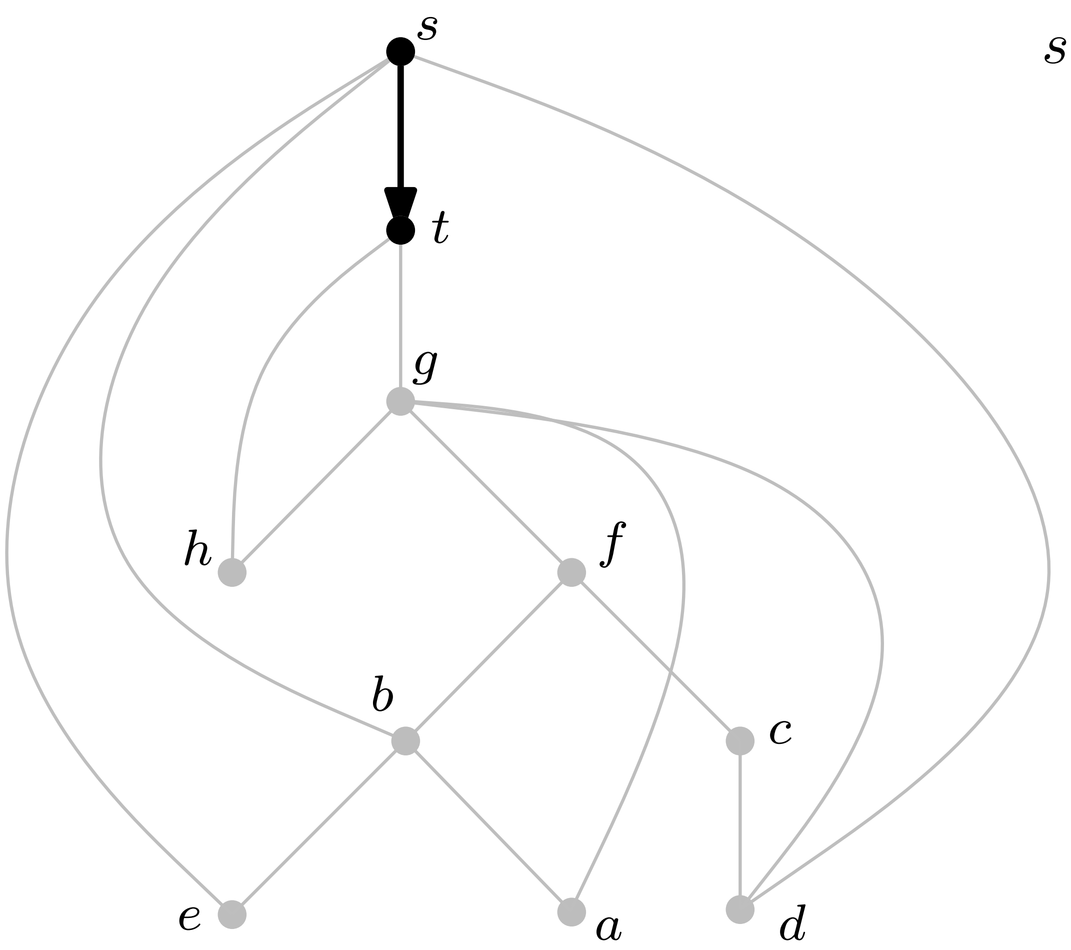

s, t

st-ordering: implementation

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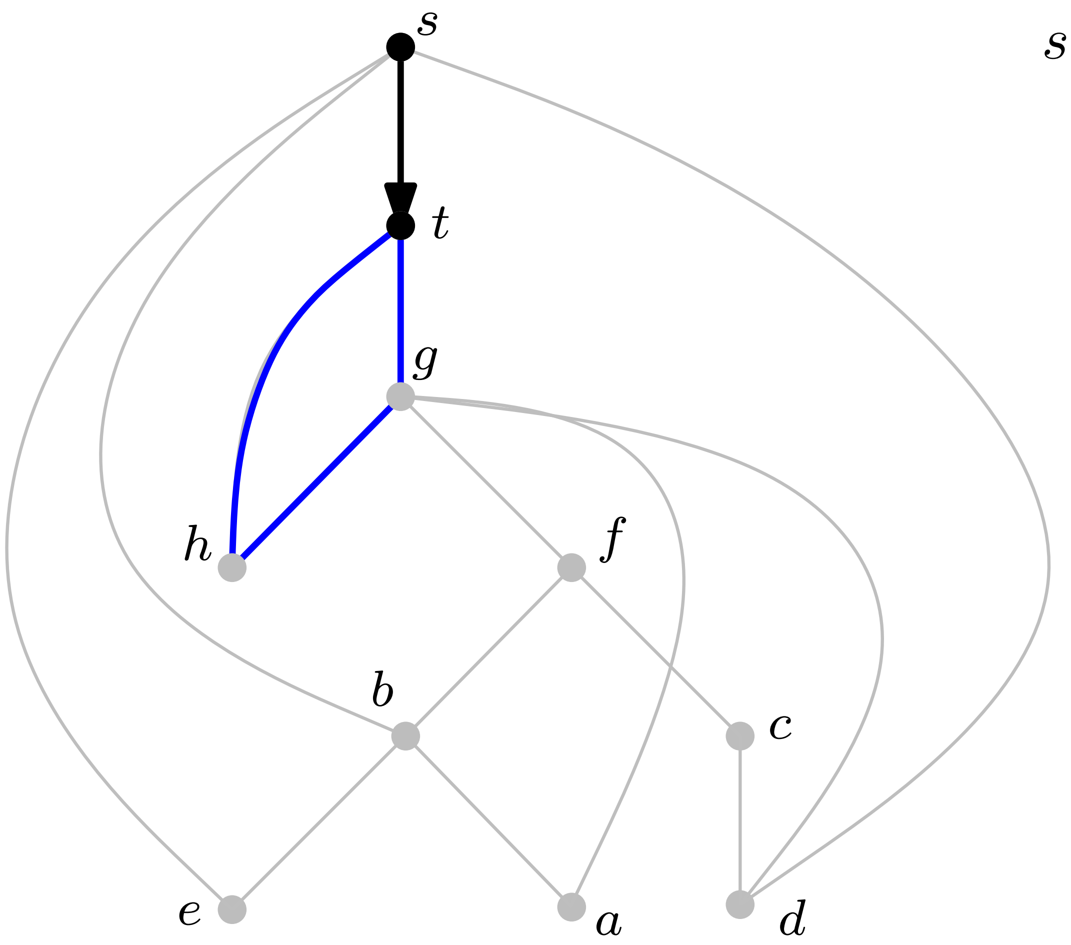

s, t

st-ordering: implementation

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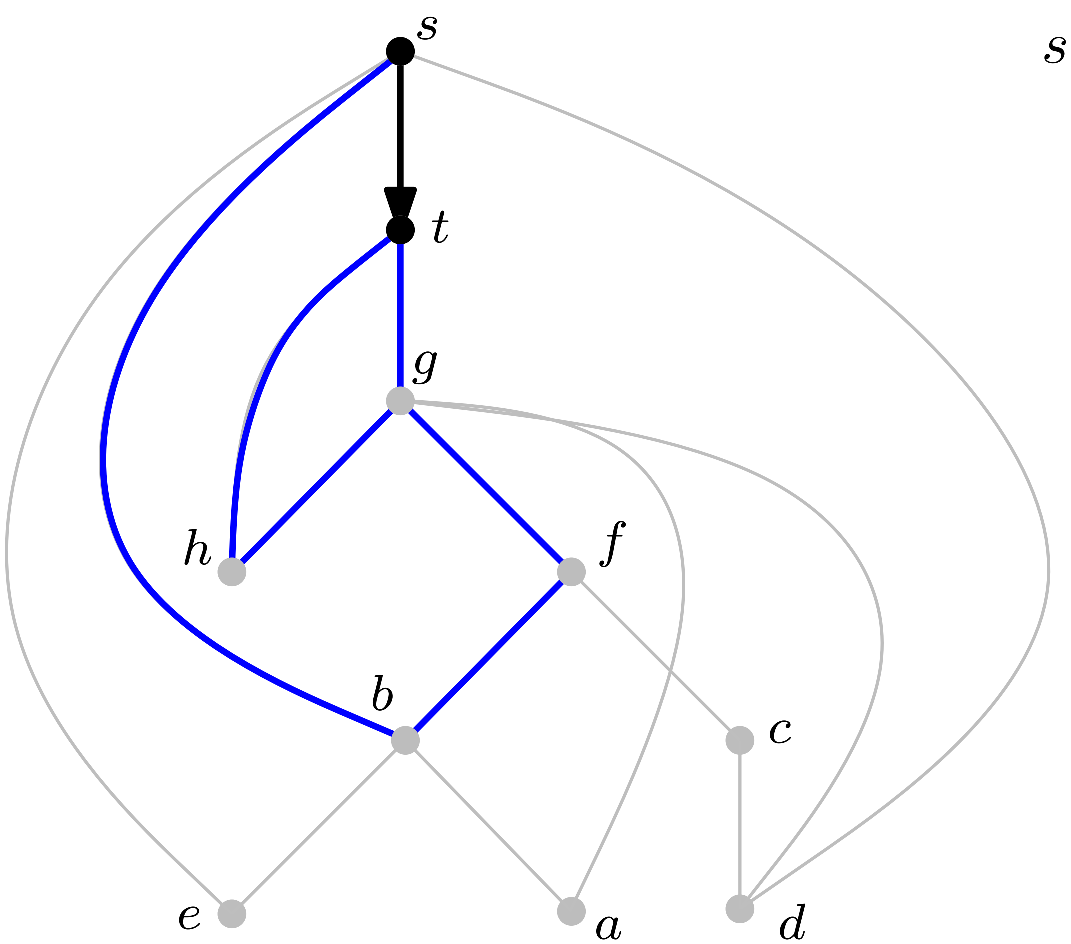

s, t

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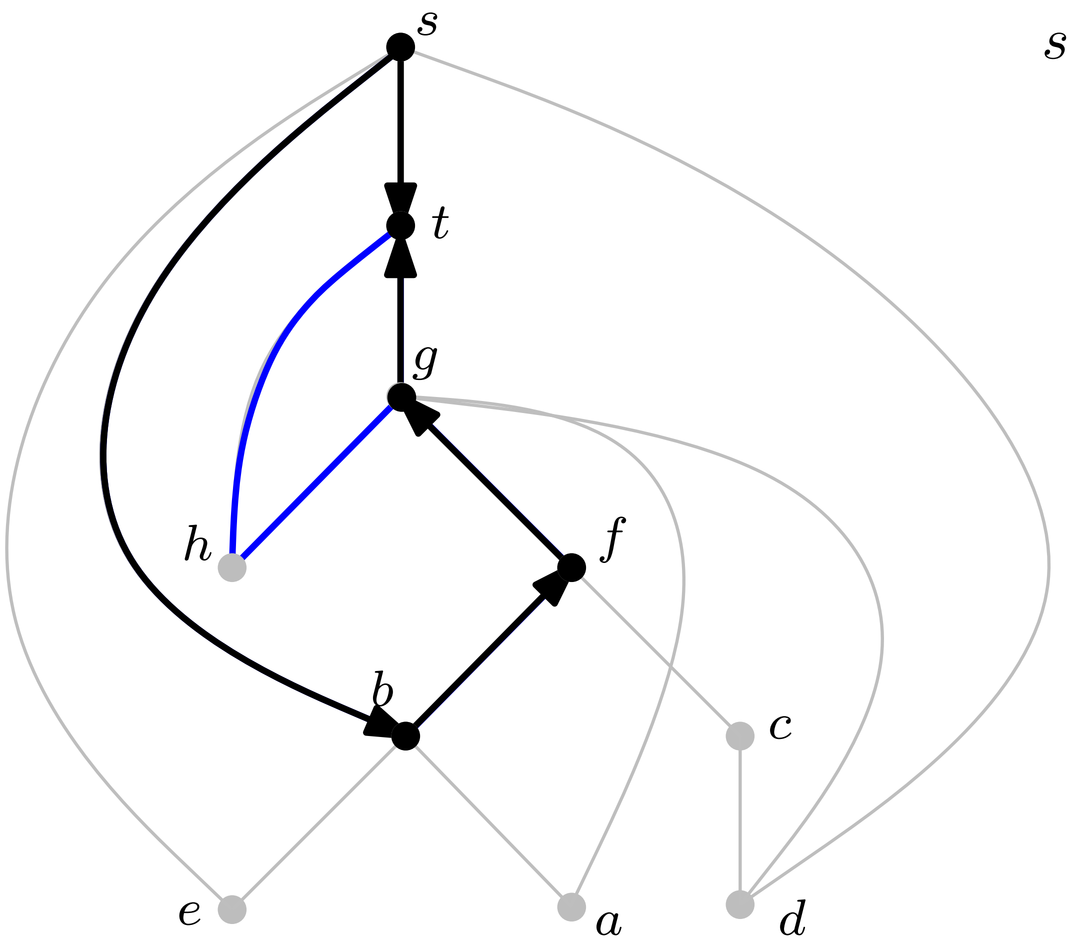
st-ordering: implementation

Algorithm: st-ordering (example)

(Implementation details - Based on DFS)

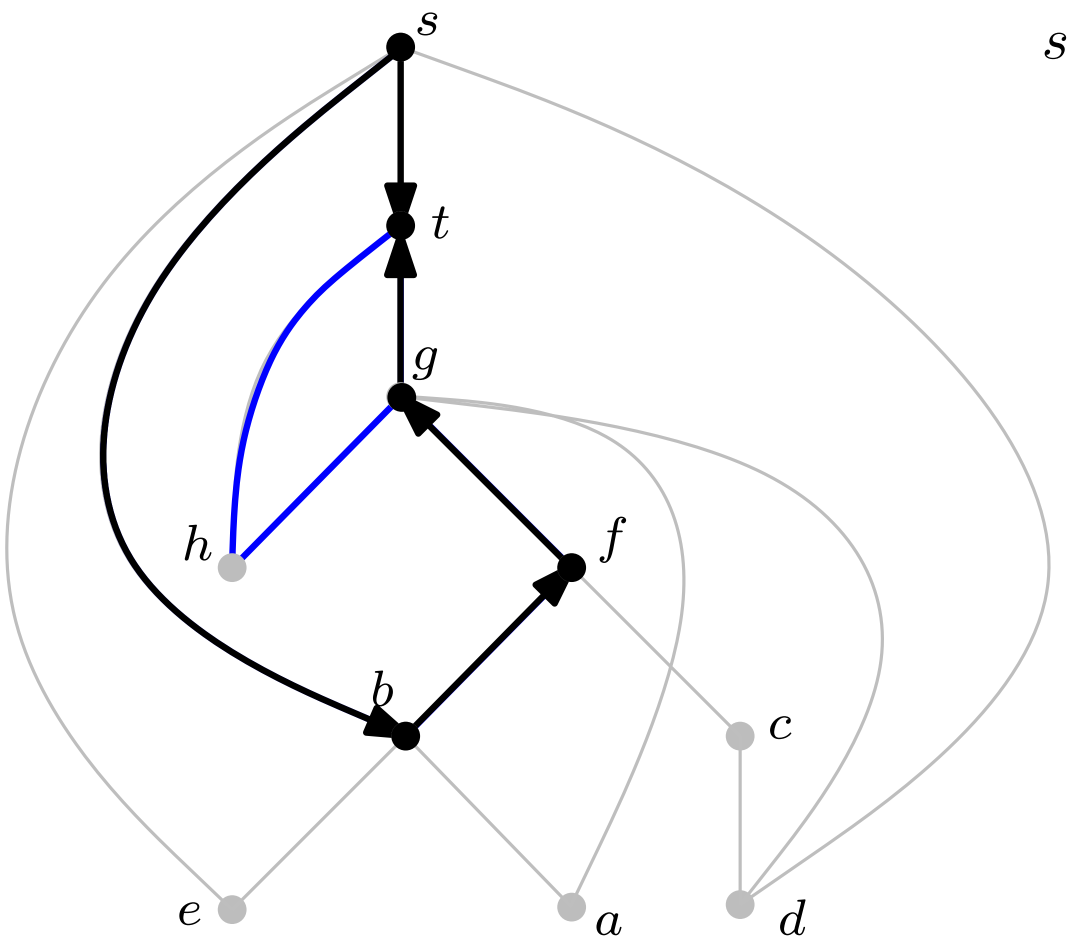


s, t

st-ordering: implementation

Algorithm: st-ordering (example)

(Implementation details - Based on DFS)



$s, \underline{b}, \underline{f}, \underline{g}, t$

st-ordering: implementation

Algorithm: st-ordering (example)
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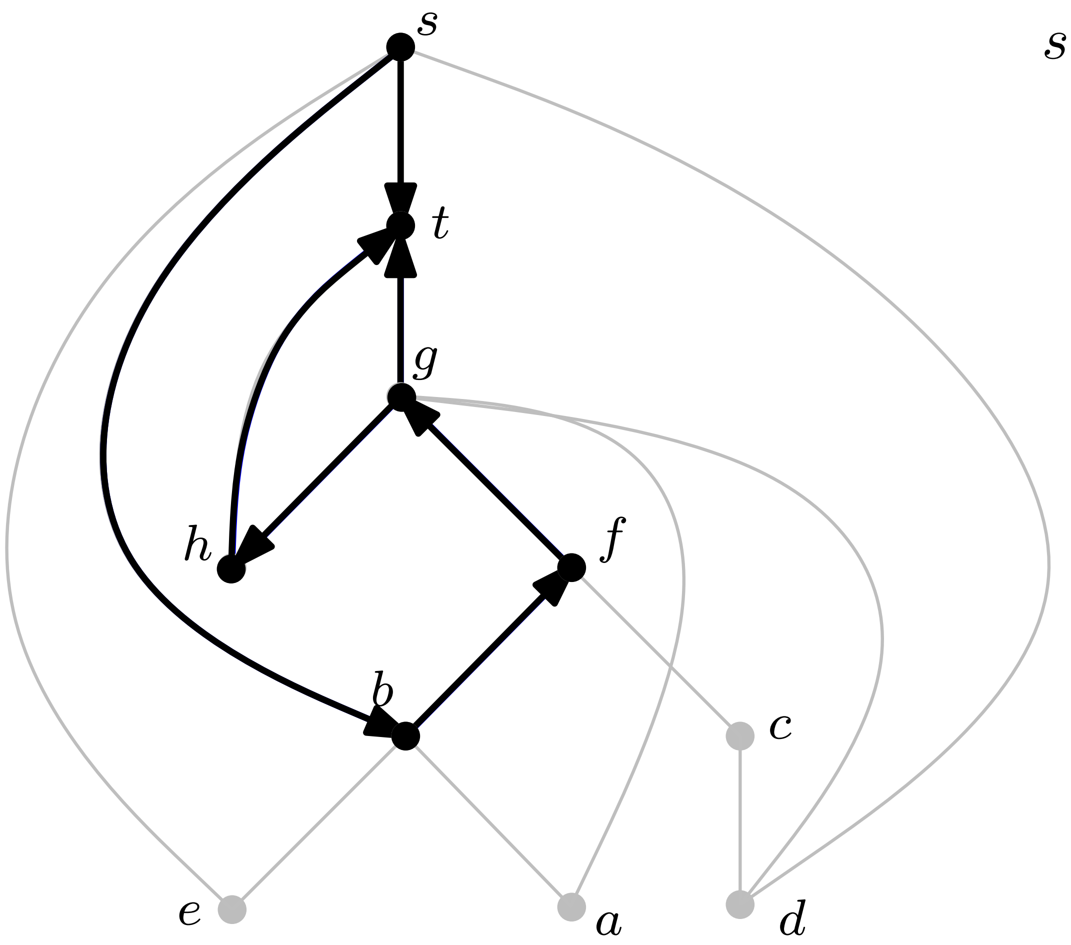

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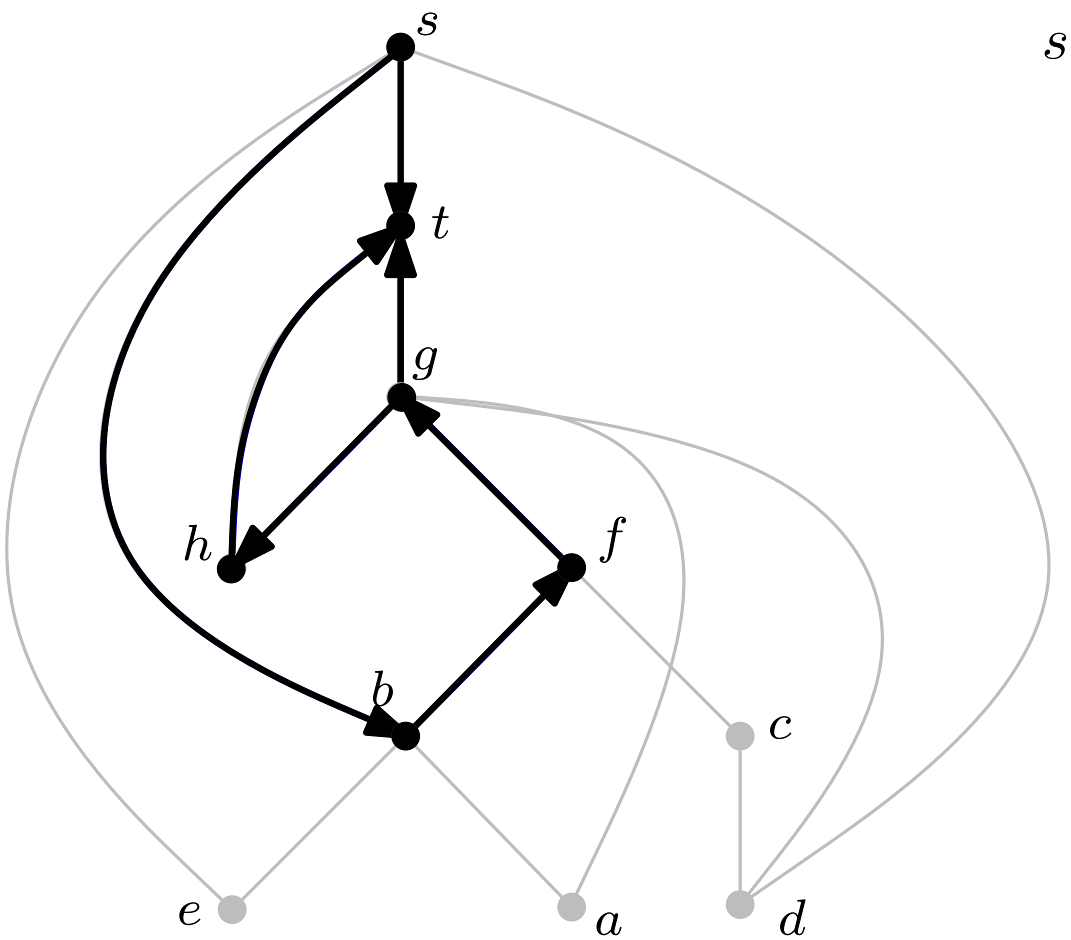
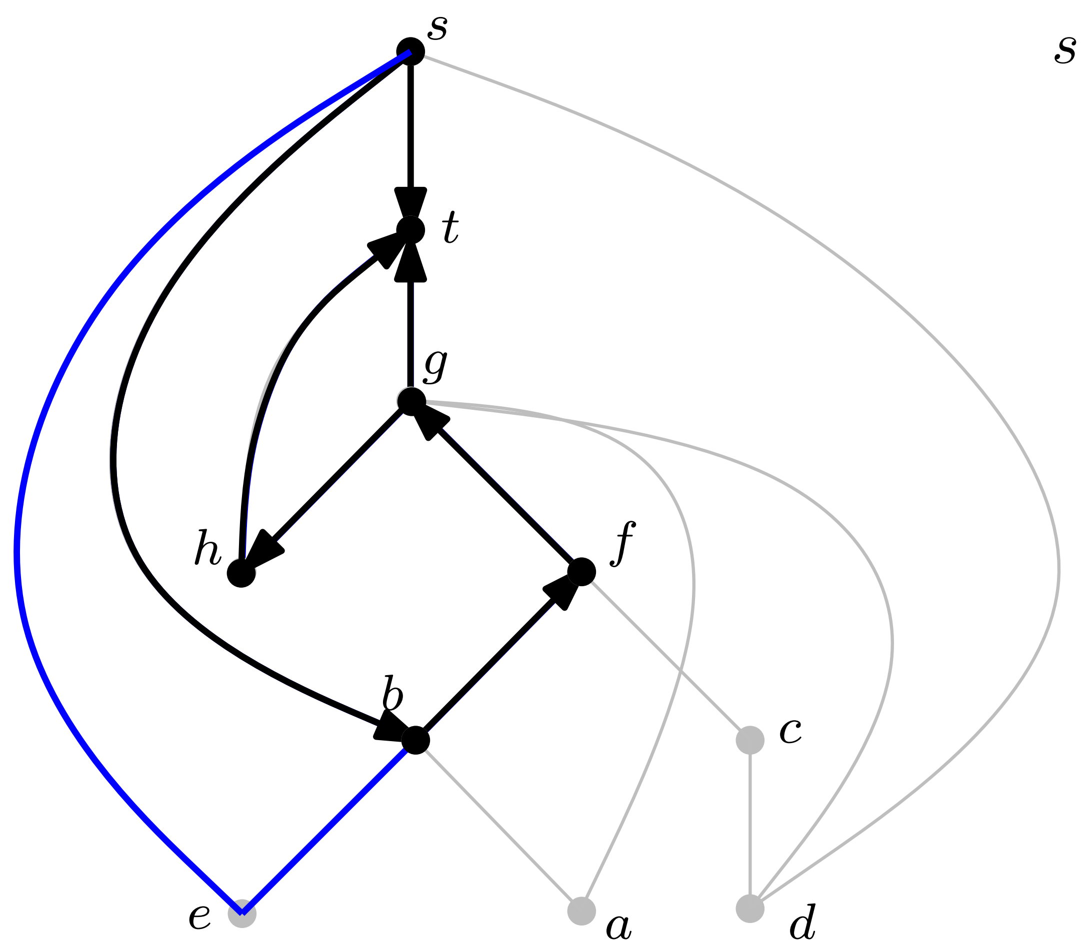


$s, b, f, g, \underline{h}, t$

st-ordering: implementation

Algorithm: st-ordering (example)

(Implementation details - Based on DFS)



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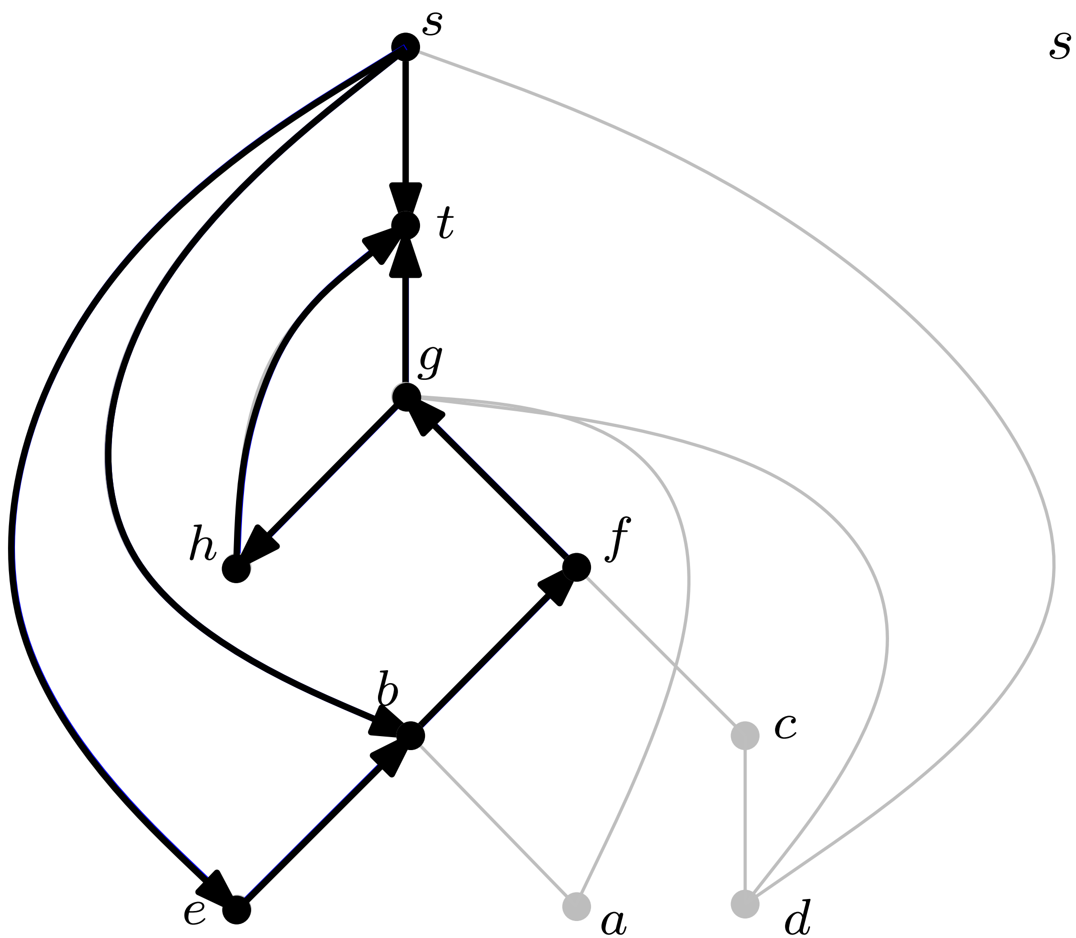

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st-ordering: implementation

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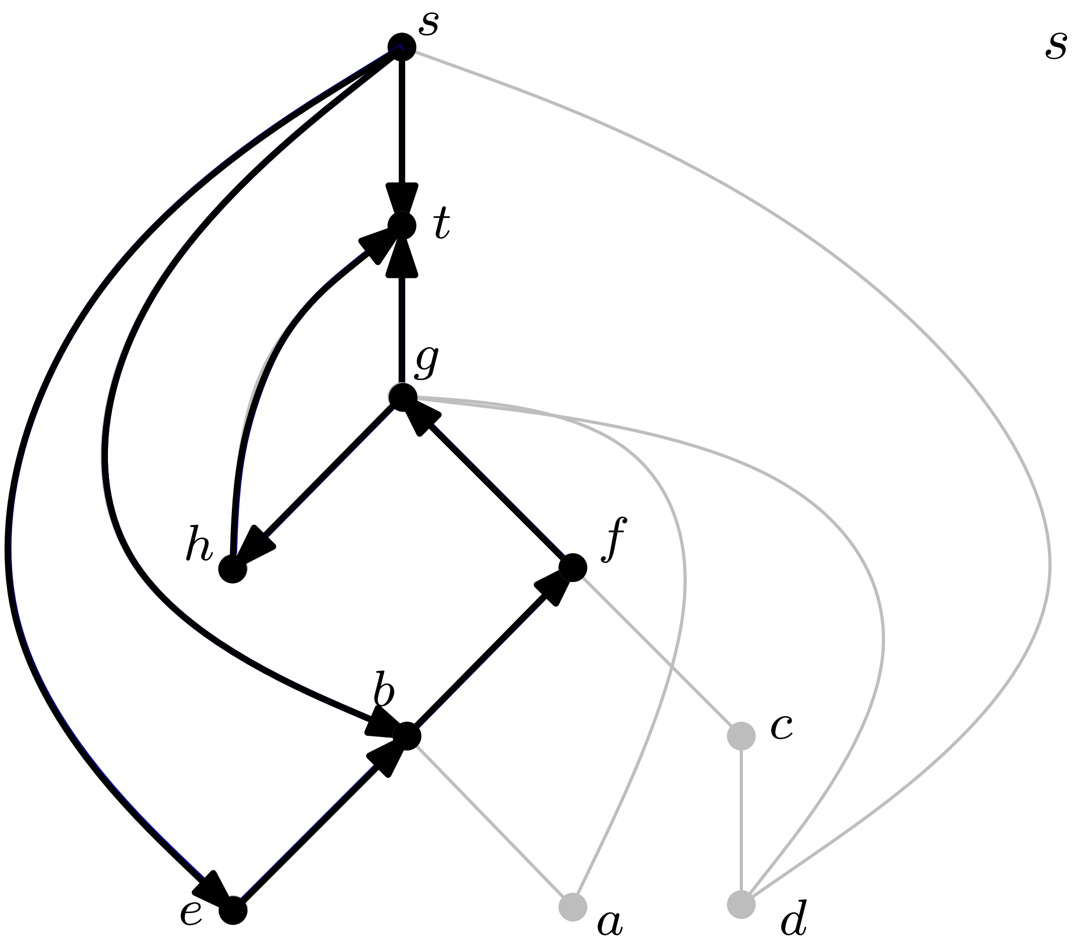
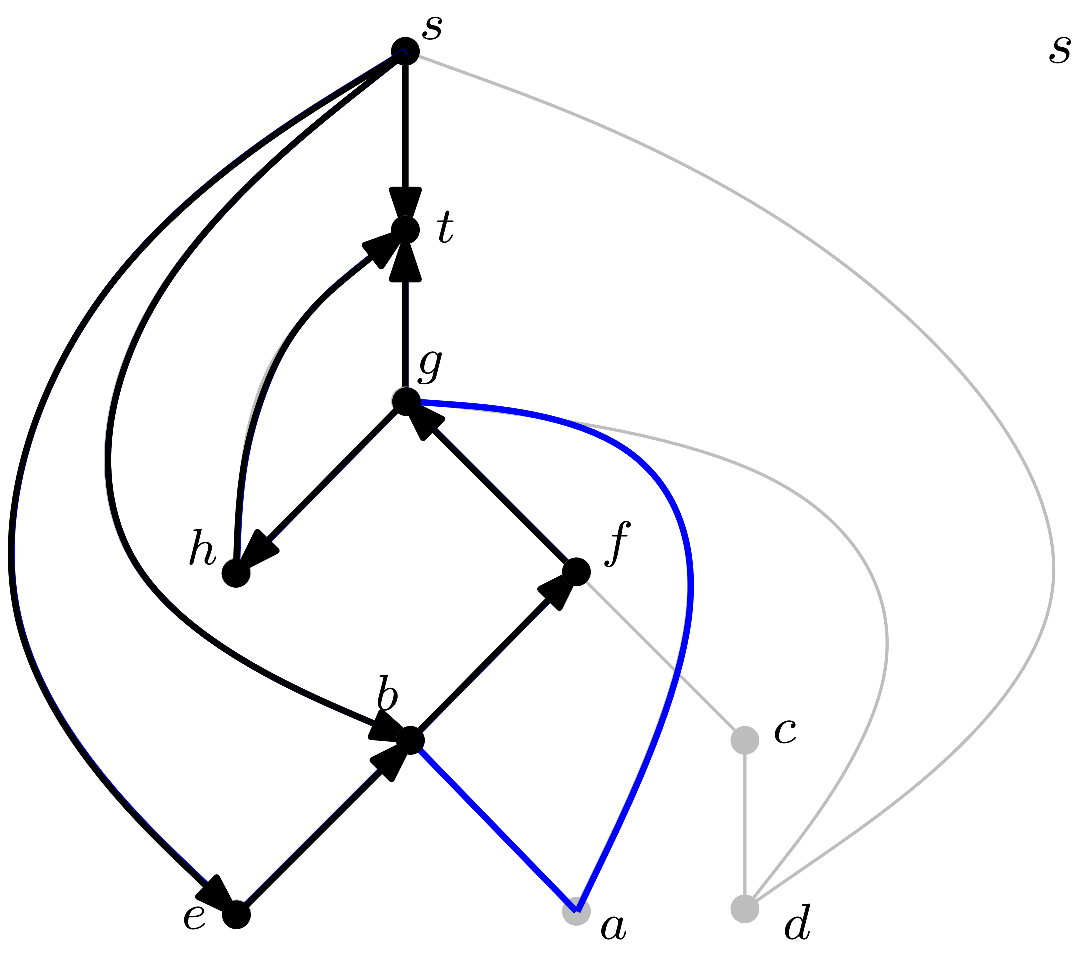


$s, \underline{e}, b, f, g, h, t$

st-ordering: implementation

Algorithm: st-ordering (example)
(Implementation details - Based on DFS)



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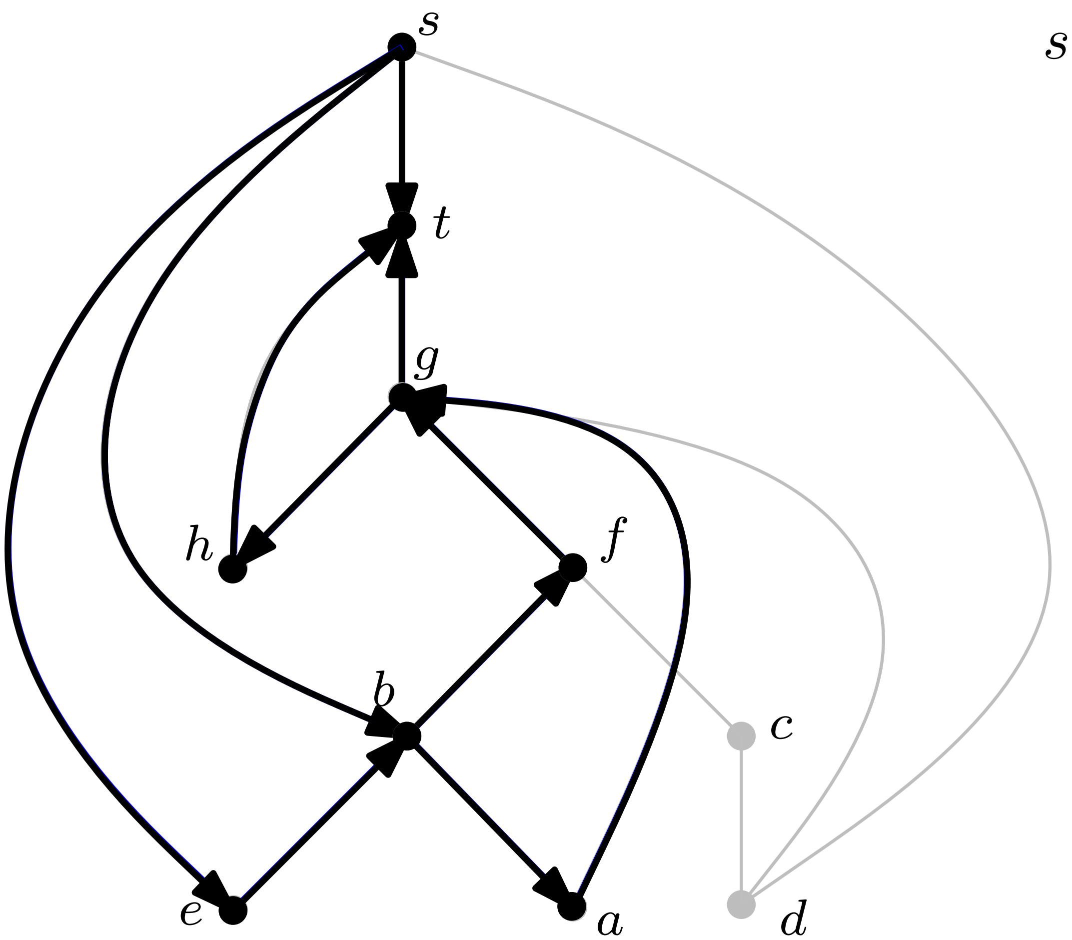

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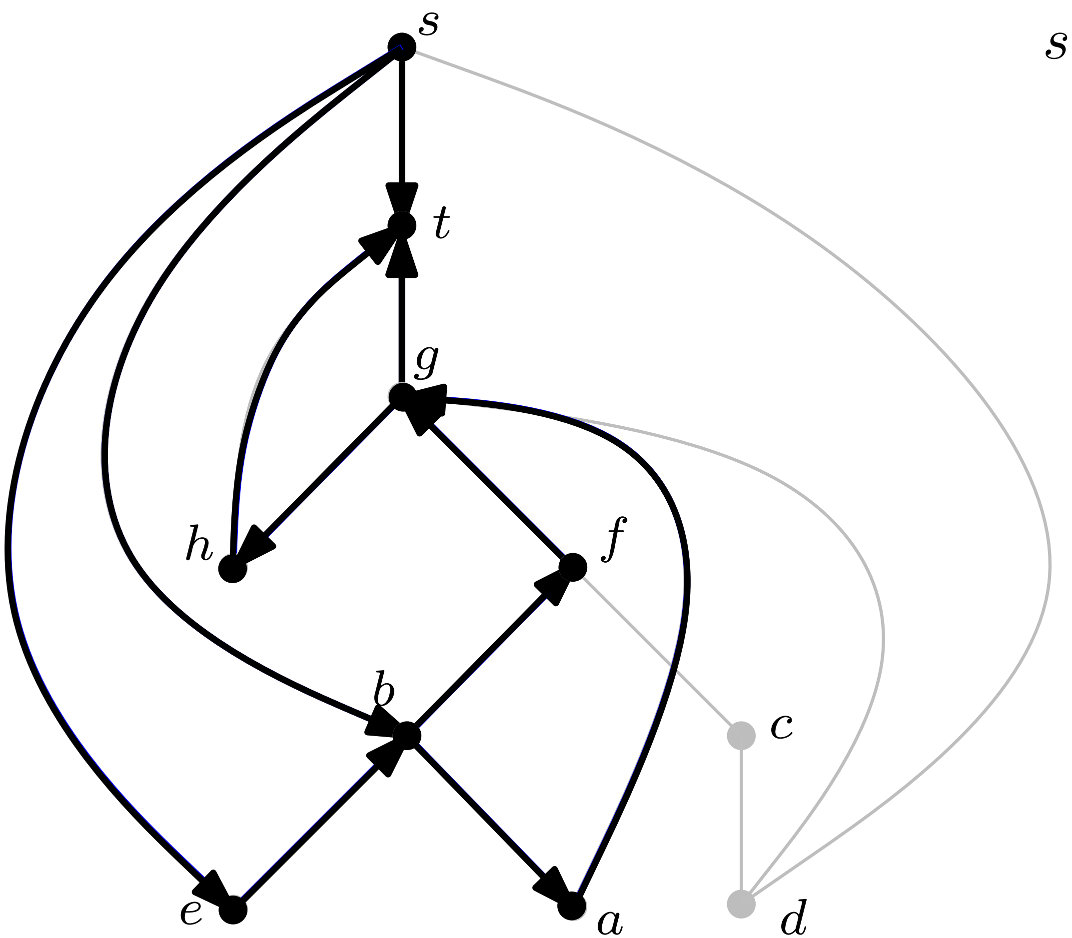

$s, e, b, \underline{a}, f, g, h, t$

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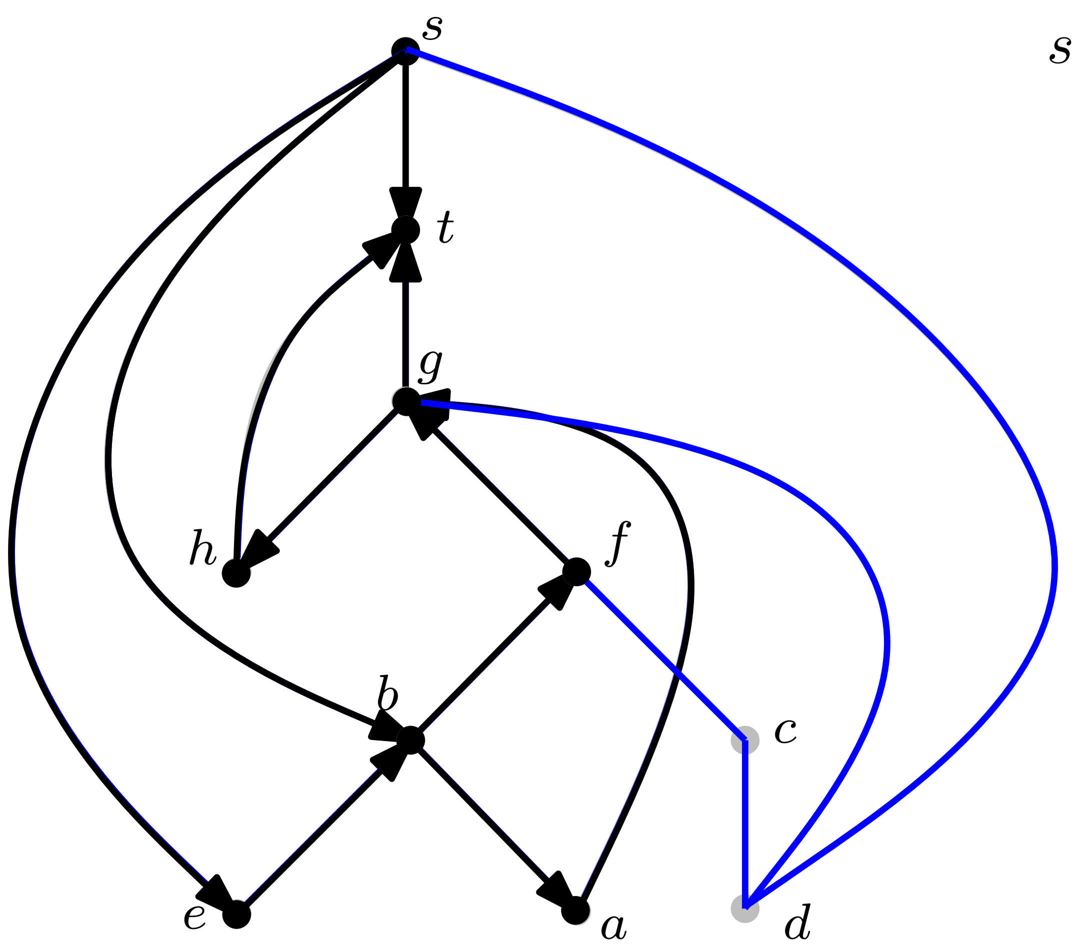

$s, e, b, \underline{a}, f, g, h, t$

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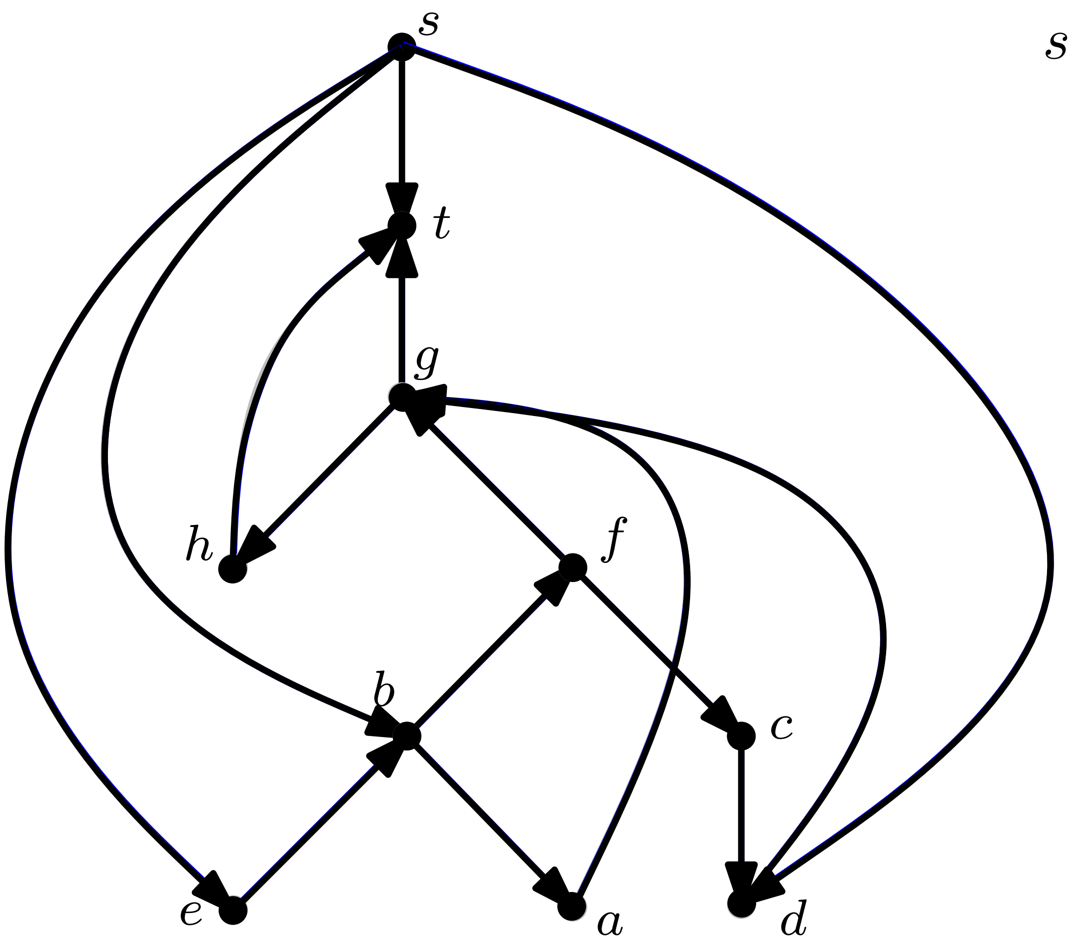
Lehrstuhl Algorithmik I

st-ordering: implementation

Algorithm: st-ordering (example)

(Implementation details - Based on DFS)



$s, e, b, \underline{a}, f, g, h, t$

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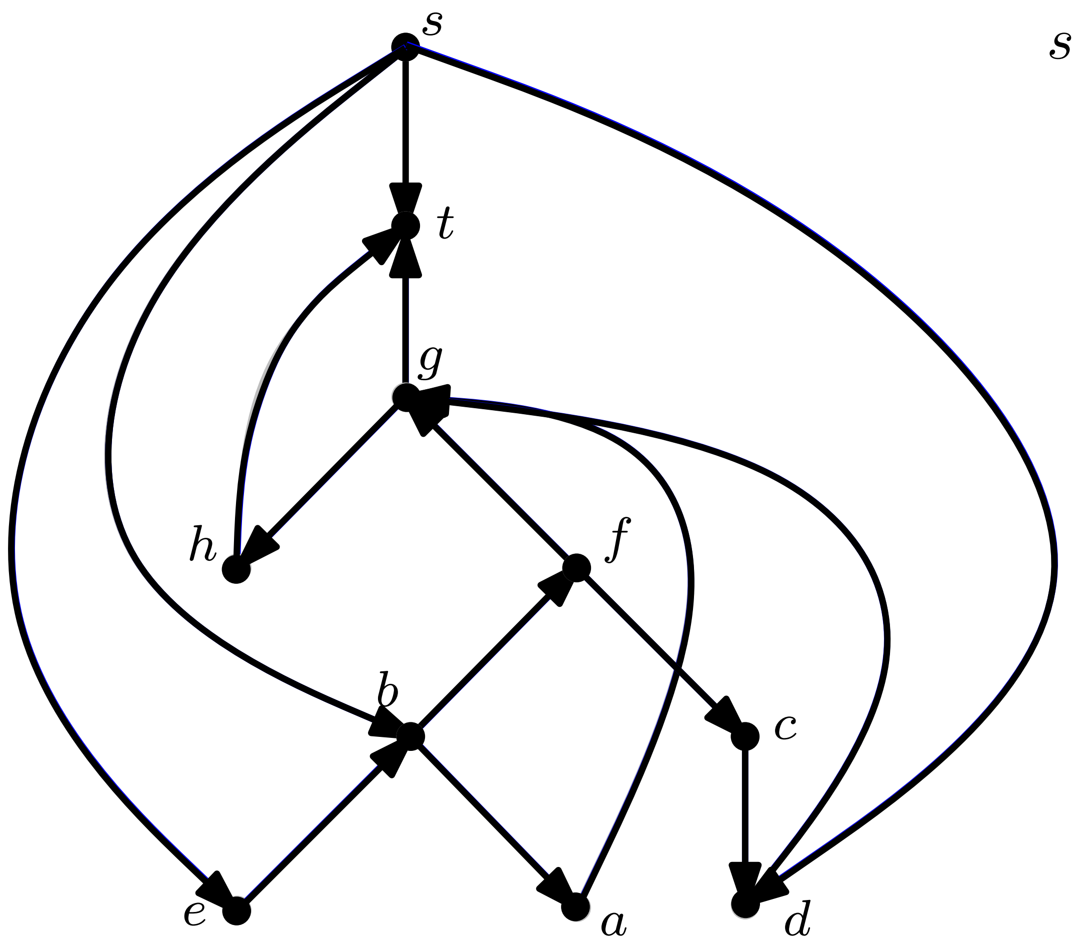


$s, e, b, a, f, \underline{c}, \underline{d}, g, h, t$

st-ordering: implementation

Algorithm st-ordering

Data: Undirected biconnected graph $G = (V, E)$, edge $\{s, t\} \in E$
Result: List L of nodes representing an st-ordering of G)

```
dfs(vertex v) begin
    i ← i + 1;  DFS[v] ← i;
    while there exists non-enumerated e = {v, w} do
        DFS[e] ← DFS[v];
        if w not enumerated then
            CHILDEDGE[v] ← e; PARENT[w] ← v;
            dfs(w);
        else
            {w, x} ← CHILDEDGE[w]; D[{w, x}] ← D[{w, x}] ∪ {e};
            if x ∈ L then process_ears(w → x);
            ;

begin
    initialize L as {s, t};
    DFS[s] ← 1;  i ← 1; DFS[{s, t}] ← 1; CHILDEDGE[s] ← {s, t};
    dfs(t);
```

st-ordering: implementation

Function *process_ears*

```
process_ears(tree edge w → x) begin
    foreach v ↪ w ∈ D[w → x] do
        u ← v;
        while u ∉ L do u ← PARENT[u];
        ;
        P ← (u →* v ↪ w);
        if w → x is oriented from w to x (resp.from x to w) then
            orient P from w to u (resp. from u to w);
            paste the inner nodes of P to L
            before (resp. after) u ;
        foreach tree edge w′ → x′ of P do process_ears(w′ → x′); ;
    D[{w, x}] ← ∅;
```

st-ordering: implementation

Function *process_ears*

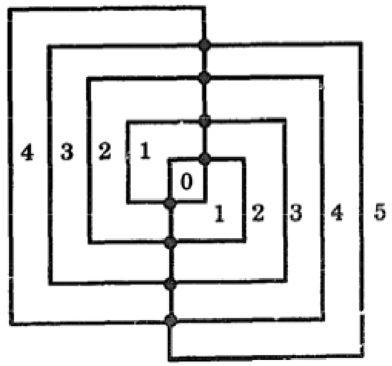
process_ears(tree edge $w \to x$) begin
- **foreach** $v \hookrightarrow w \in D[w \to x]$ **do**
  - $u \leftarrow v$;
  - **while** $u \notin L$ **do** $u \leftarrow PARENT[u]$;
  - ;
  - $P \leftarrow (u \xrightarrow{*} v \hookrightarrow w)$;
  - **if** $w \to x$ *is oriented from* w *to* x *(resp.from* x *to* w*)* **then**
    - orient P from w to u (resp. from u to w);
    - paste the inner nodes of P to L before (resp. after) u ;
  - **foreach** *tree edge* $w' \to x'$ *of* P **do** $process\_ears(w' \to x')$; ;
- $D[\{w, x\}] \leftarrow \emptyset$;

Theorem

The described algorithm produces an st-ordering of a given biconnected graph $G = (V, E)$ in $O(E)$ time.

Discussion

- Today: incremental algorithm for orthogonal drawings, with worst-case guarantees. $2n + 4$ bends in total, which is almost optimal. Lower bound: $2n - 2$.
- Algorithms is simple, linear-time, works for non-planar graphs
- For planar graphs produces planar drawing
- Uses st-ordering
- Construction of st-ordering using ear-decomposition



lower bound

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- Algorithms is simple, linear-time, works for non-planar graphs
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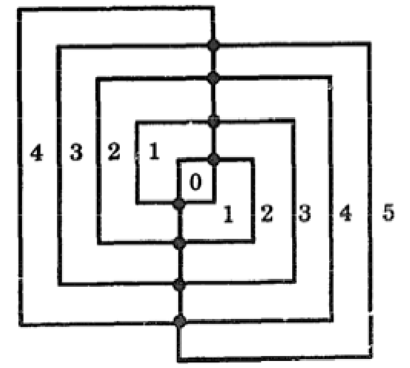
- Construction of st-ordering using ear-decomposition
- Next: algorithm based on network flow, that achieves minimum number of bends



lower bound