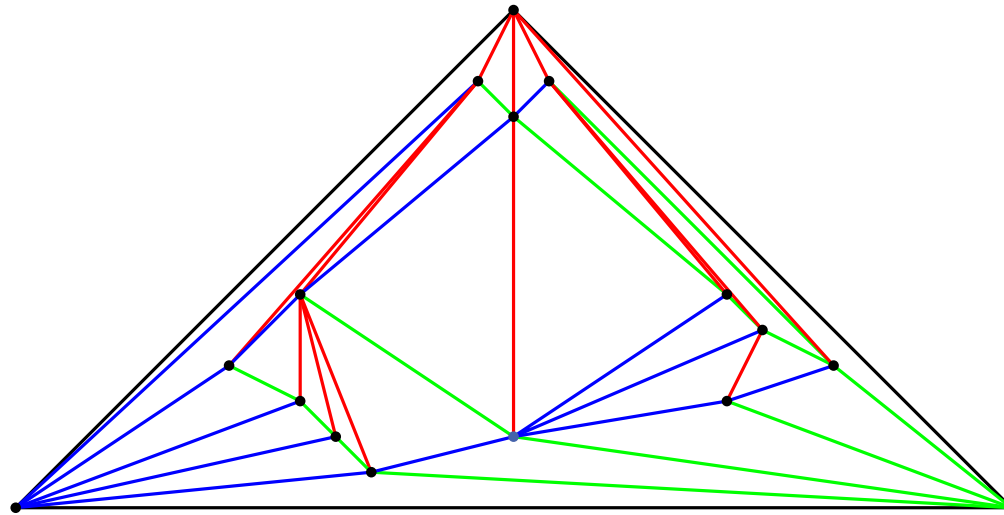


Algorithms for graph visualization

Layouts for planar graphs. Realizer method.

WINTER SEMESTER 2017/2018

Tamara Mchedlidze



Last lecture:

Theorem [De Fraysseix, Pach, Pollack '90]

Every n -vertex planar graph has a planar straight-line drawing of a size $(2n - 4) \times (n - 2)$.

This lecture:

Theorem [Schnyder '90]

Every n -vertex planar graph has a planar straight-line drawing of a size $(n - 2) \times (n - 2)$.

Overview

- Barycentric Coordinates and Barycentric Graph Representation.
- Schnyder Labelling. Existence.
- Schnyder Realizer.
- From Schnyder Realizer to Barycentric Representation.

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Barycentric Coordinates

Let $A, B, C, P \in \mathbb{R}^2$. A triple $(\alpha, \beta, \gamma) \in \mathbb{R}^3$ such that:

- $\alpha + \beta + \gamma = 1$
- $P = \alpha A + \beta B + \gamma C$

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Barycentric Representation

A **Barycentric Representation** of a graph $G = (V, E)$ is an assignment of barycentric coordinates to the vertices of G , i.e. it is an **injective** function

$v \in V \mapsto (v_a, v_b, v_c) \in \mathbb{R}^3$, such that:

- $v_a + v_b + v_c = 1$ for all $v \in V$
- for each $(x, y) \in E$ and each $z \in V \setminus \{x, y\}$, $\exists k \in \{a, b, c\}$ with $x_k < z_k$ and $y_k < z_k$.

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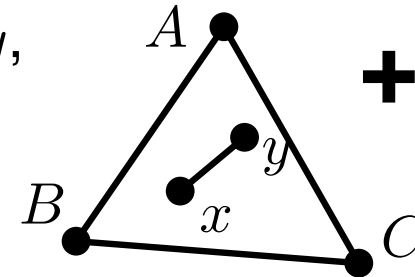
What does this condition mean?



Work alone, discuss with your neighbour(s) and then share

**3 min
+3 min**

- Consider the a-coordinates of points x and y , which is bigger?
- Interpret the condition below.



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Lemma [Schnyder '90]

Let $v \in V \mapsto (v_a, v_b, v_c) \in \mathbb{R}^3$ be a barycentric representation of a graph $G = (V, E)$ and let $A, B, C \in \mathbb{R}^2$. The function

$$f: v \in V \mapsto v_a A + v_b B + v_c C$$

gives a **planar** drawing of G inside triangle $\triangle ABC$.

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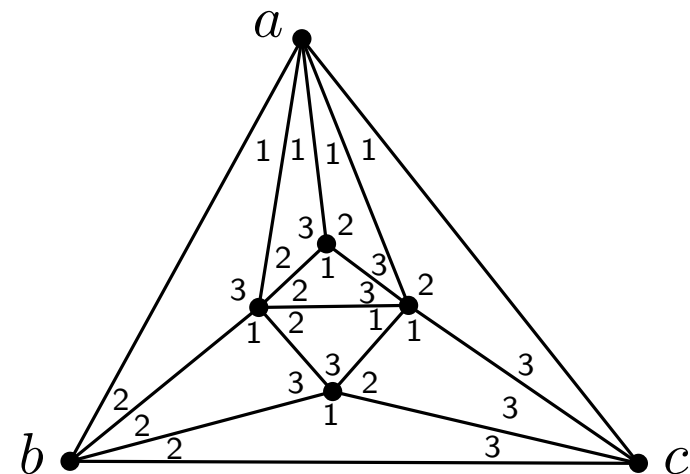
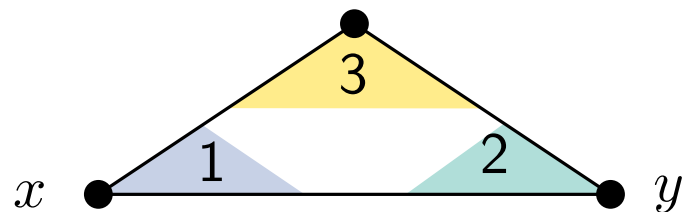
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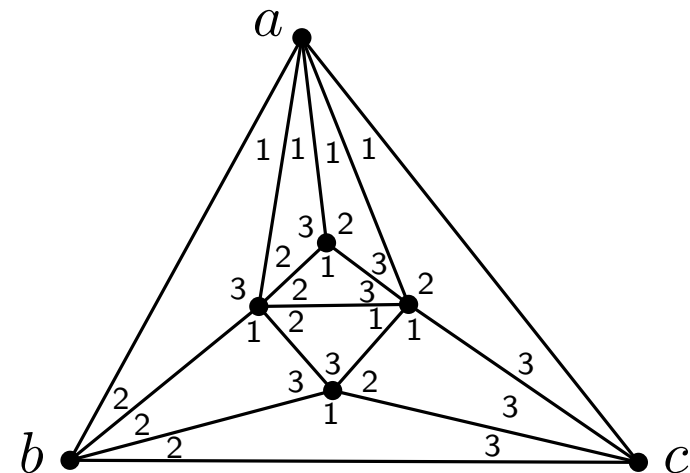
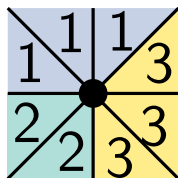
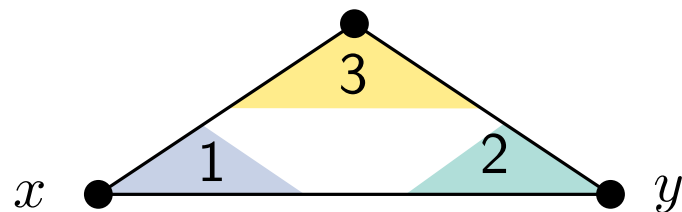


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Vertex The counterclockwise ordering of the labels around each vertex consists of a nonempty interval of 1's followed by a nonempty interval of 2's followed by a nonempty interval of 3's.



Schnyder Labeling

Theorem [Schnyder '90]

Every triangulated plane graph has a Schnyder labeling.

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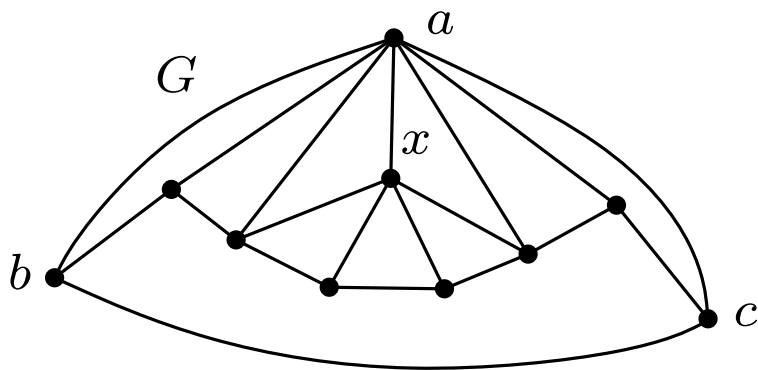
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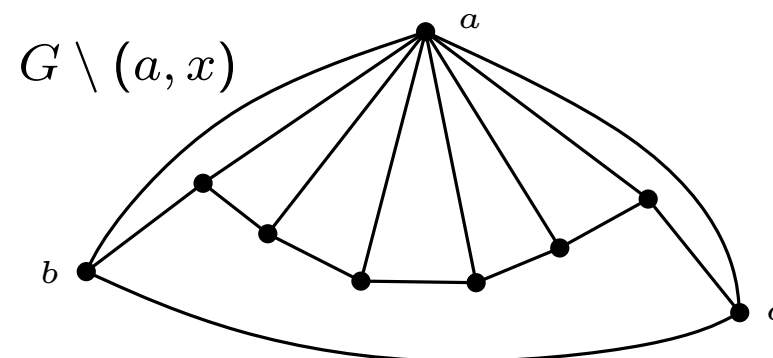
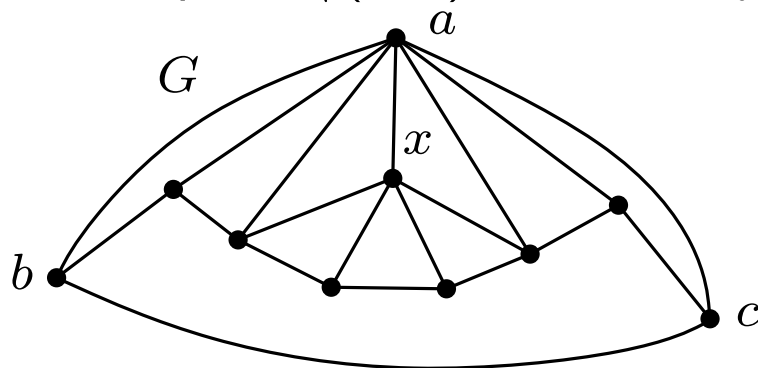


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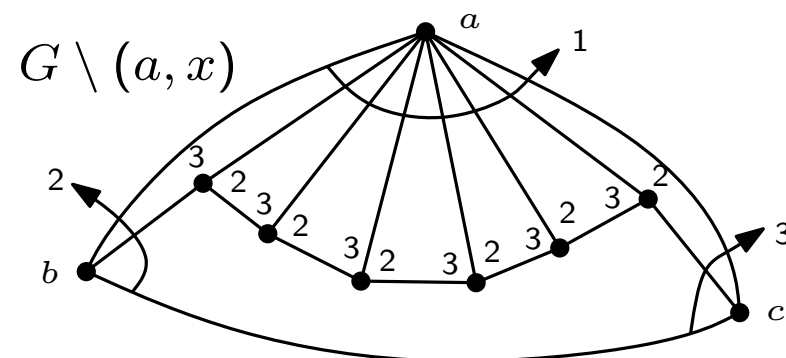
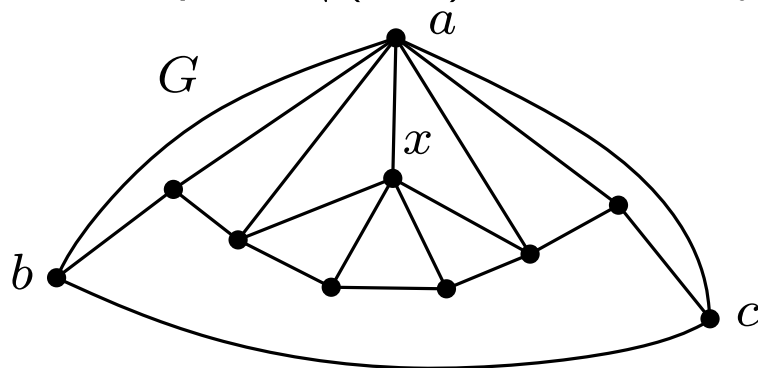


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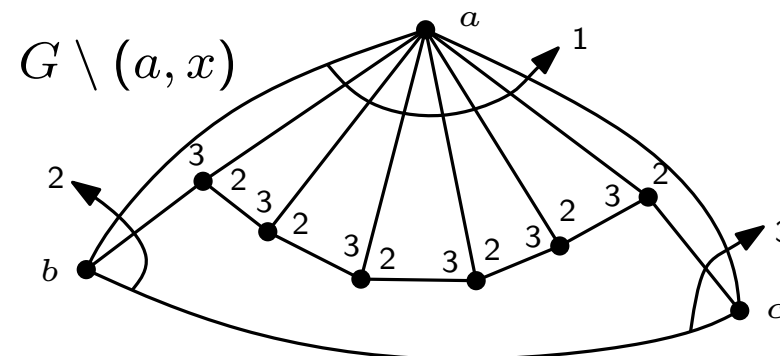
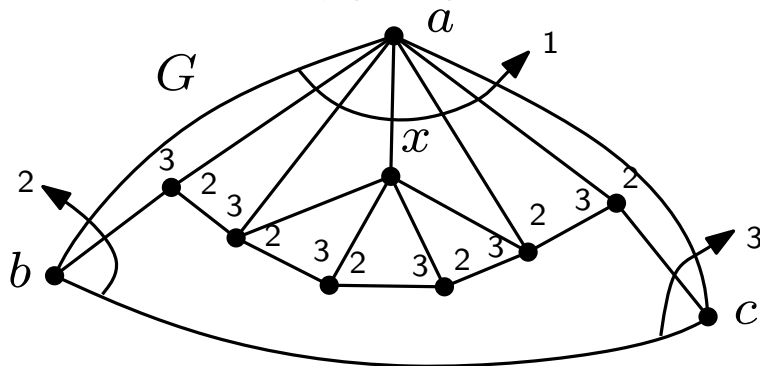


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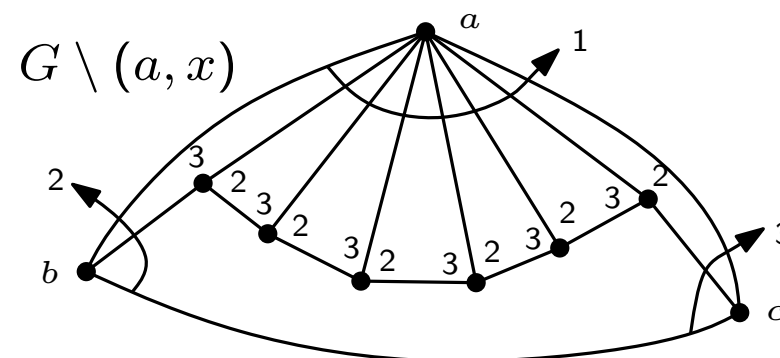
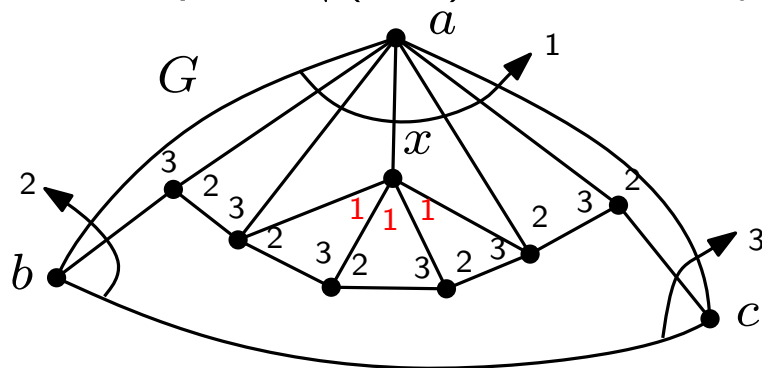


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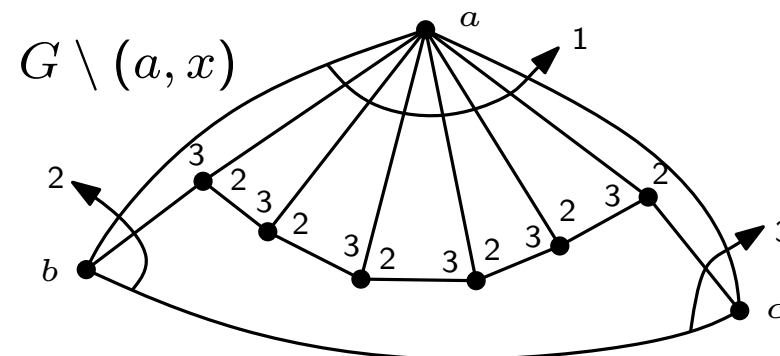
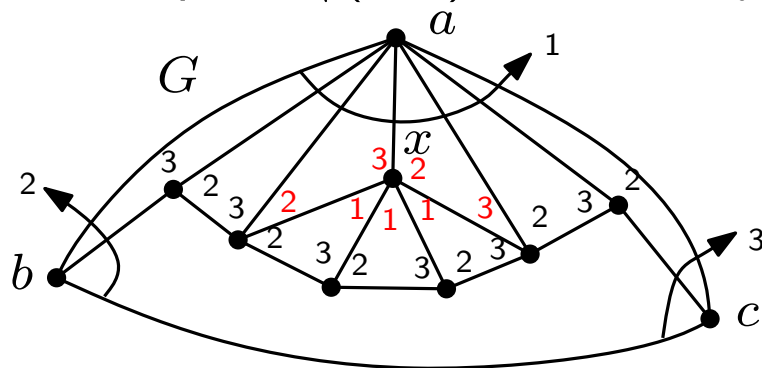


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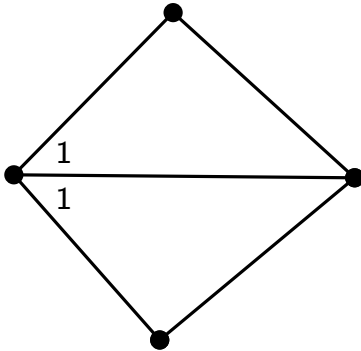


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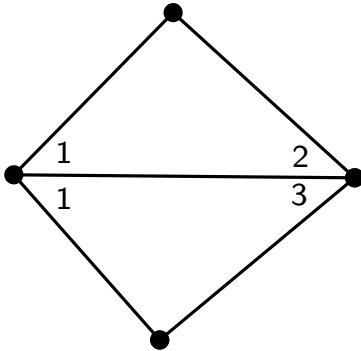
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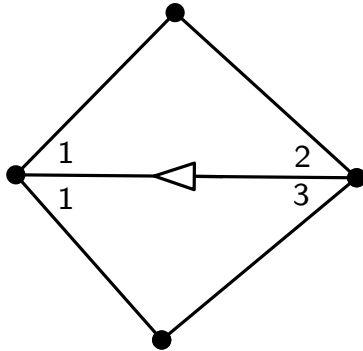
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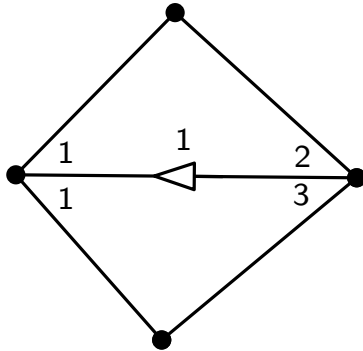
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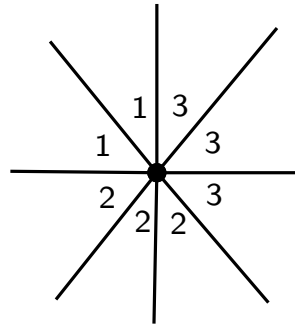
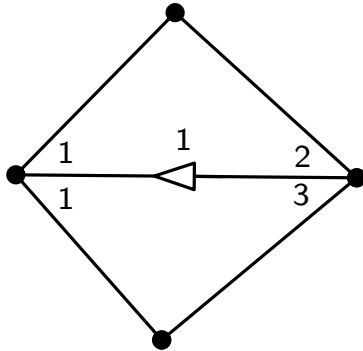
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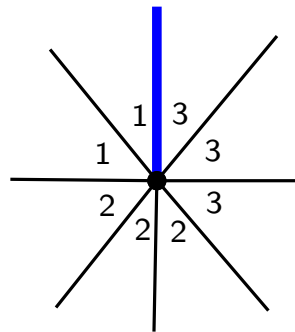
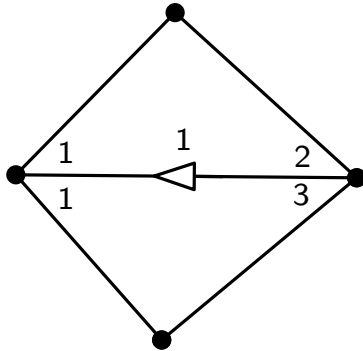
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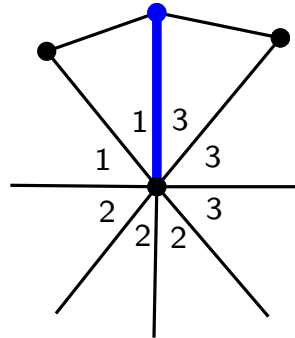
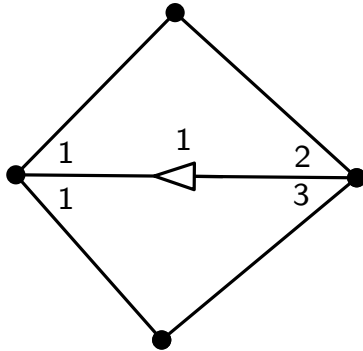
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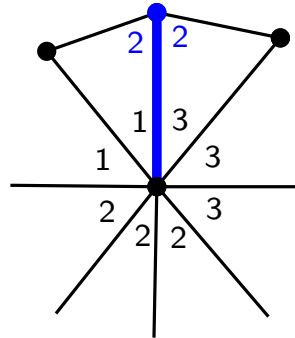
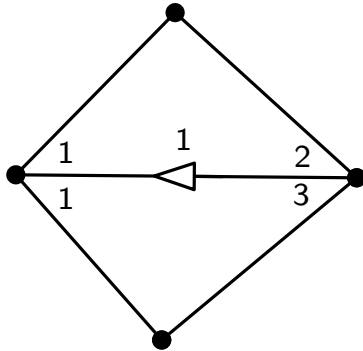
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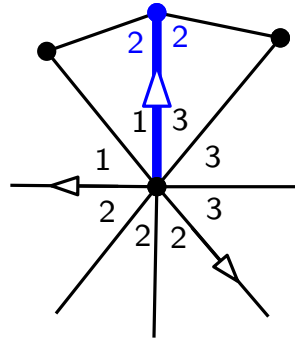
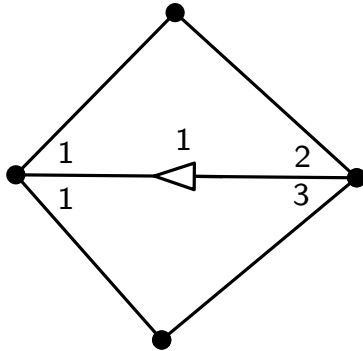
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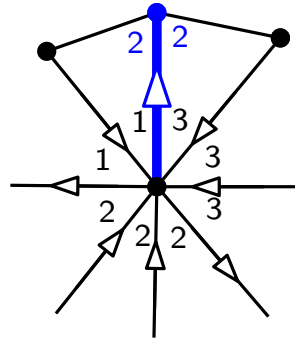
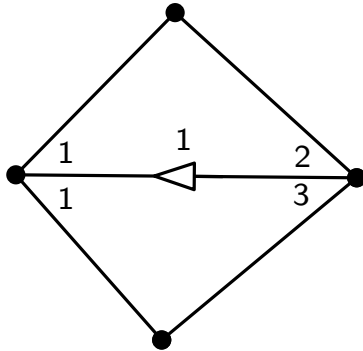
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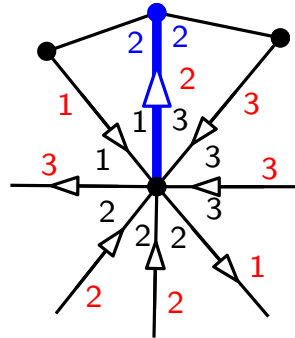
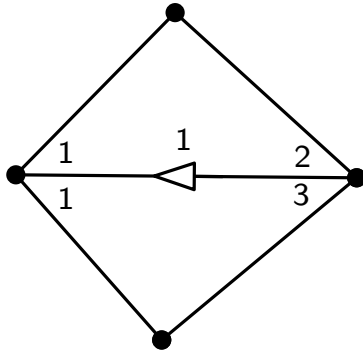
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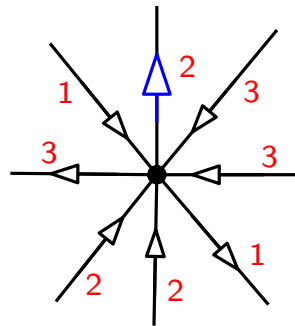
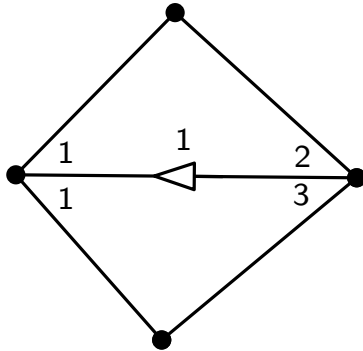
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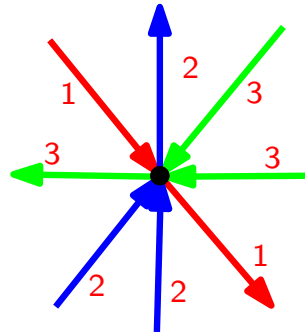
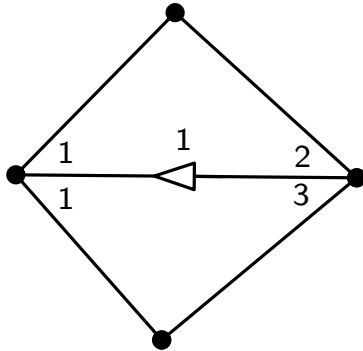
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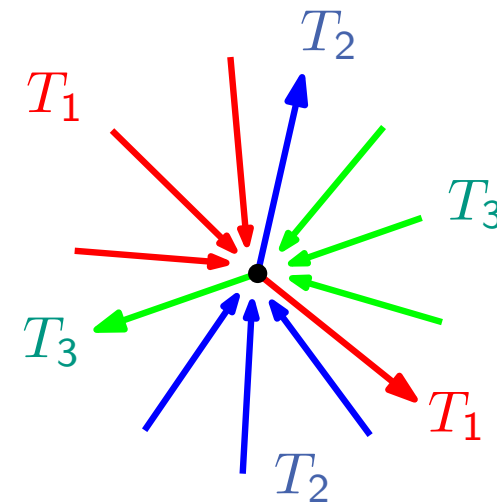
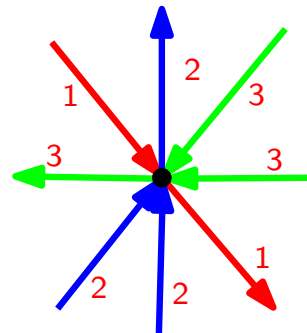
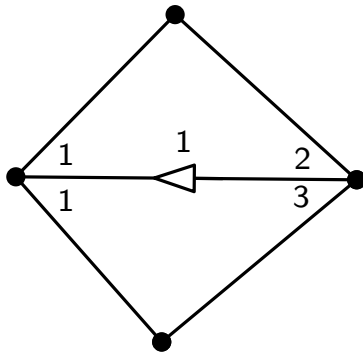


Schnyder Realizer

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Definition: Schnyder Forest

A **Schnyder Forest** or a **Realizer** of a planar triangulated graph $G = (V, E)$ is a partition of the inner edges of E into three sets of oriented edges T_1, T_2, T_3 such that for each inner vertex $v \in V$ hold:

- v has an outgoing edge in each of T_1, T_2, T_3
- The counterclockwise order of the edges around v is as follows: edges leaving in T_1 , entering in T_3 , leaving in T_2 , entering in T_1 , leaving in T_3 , entering in T_2 .

Recall that:

Theorem [Schnyder '90]

Every triangulated plane graph has a Schnyder labeling.

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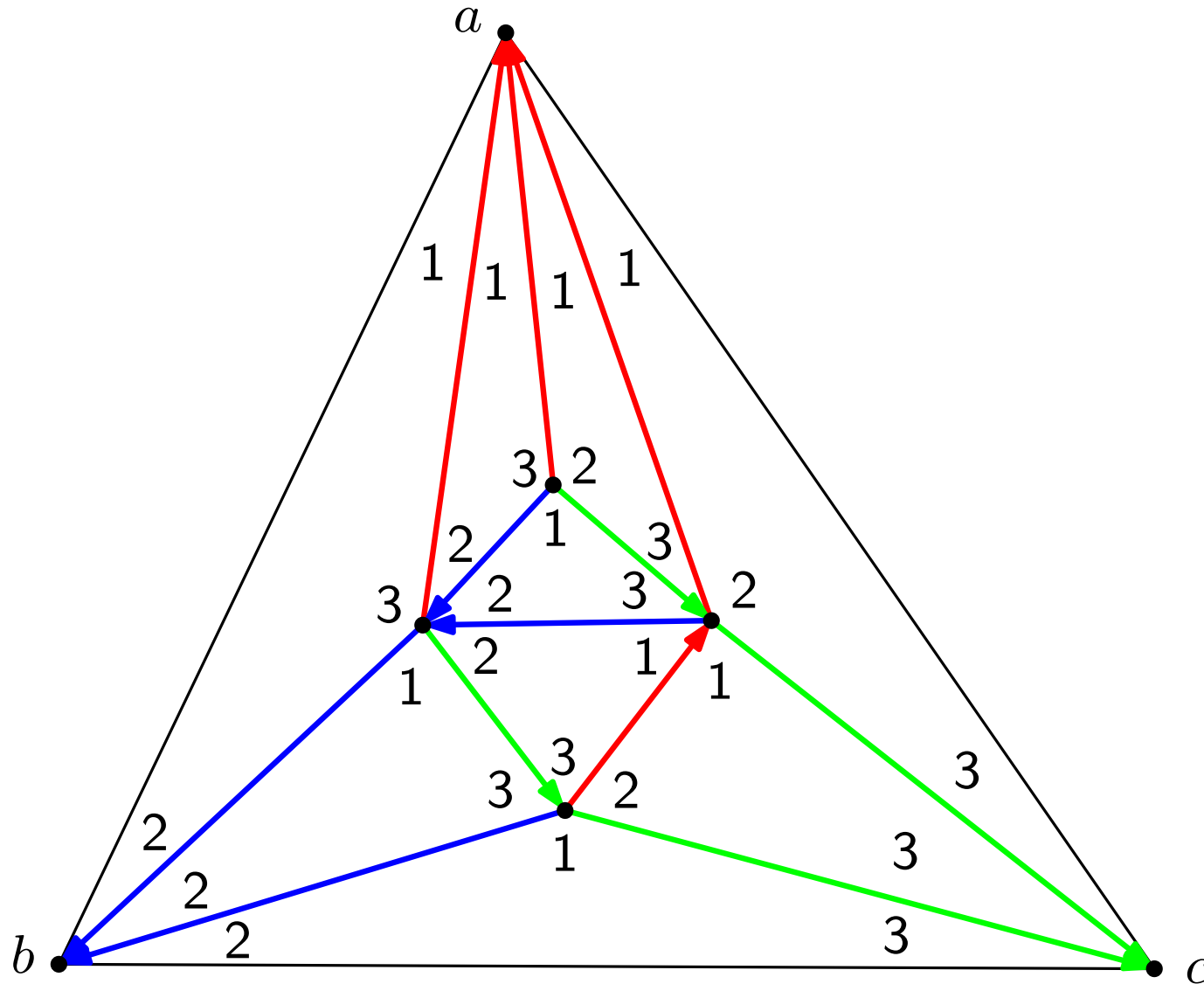
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By this theorem and by previous construction:

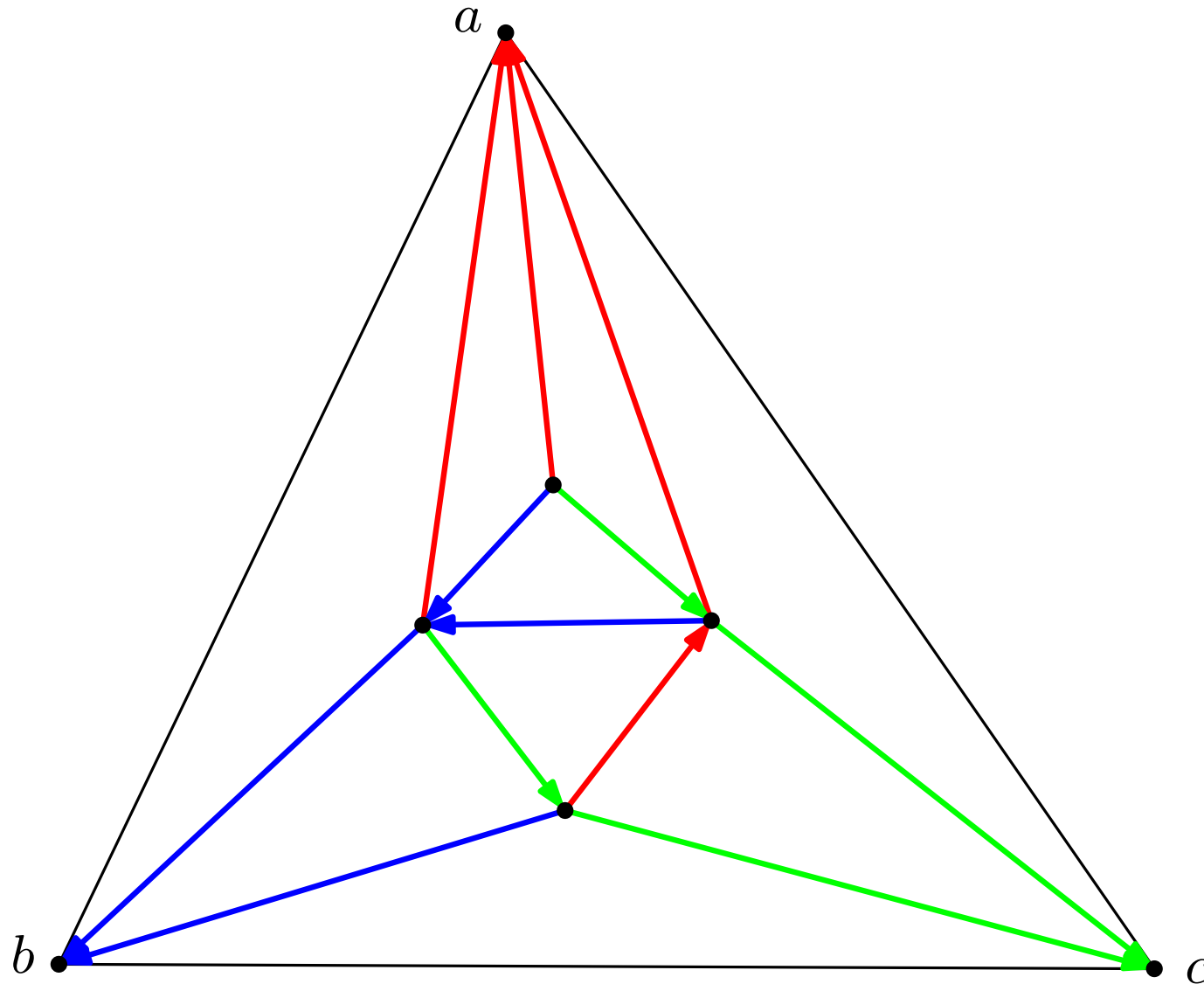
Theorem [Schnyder '90]

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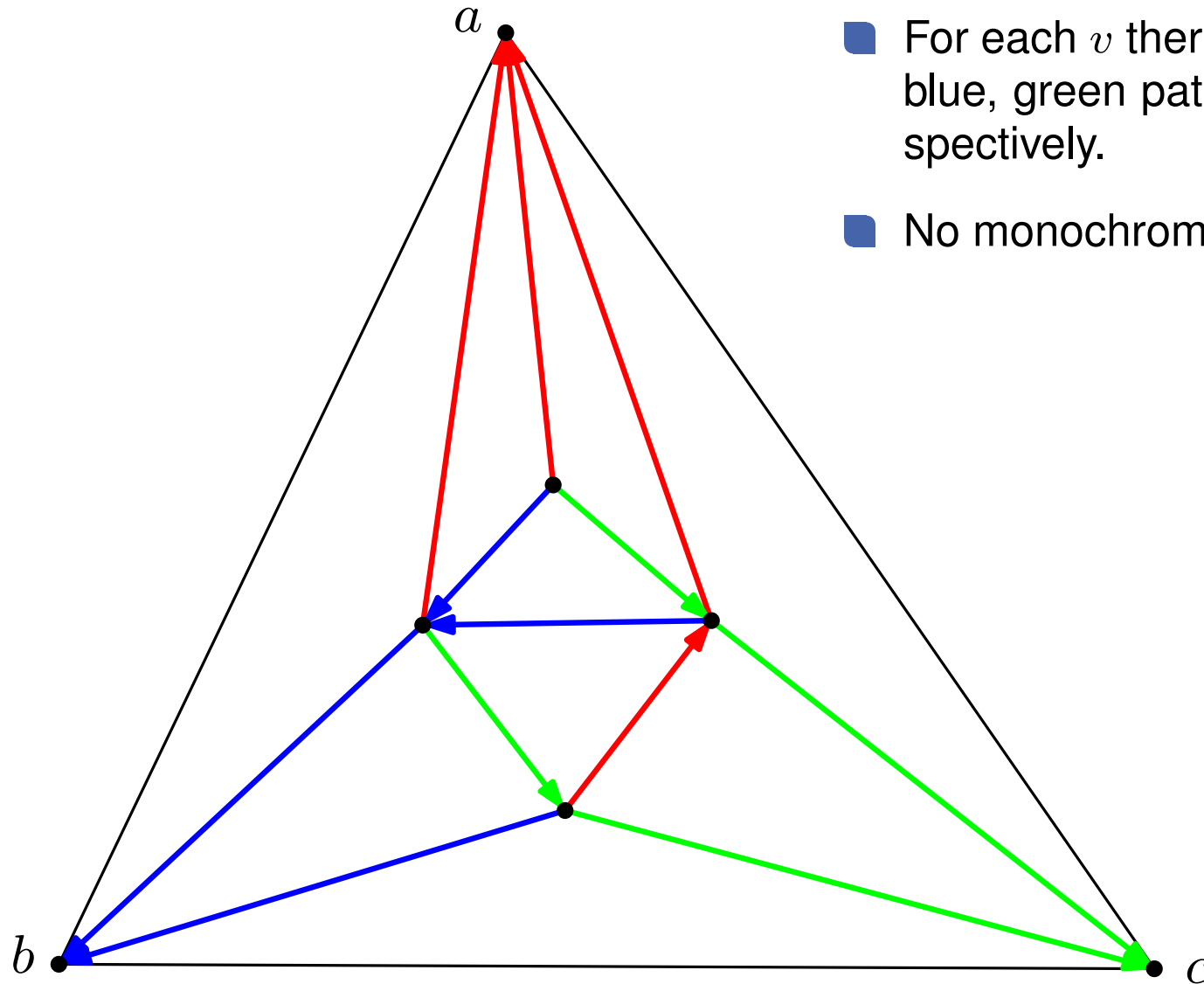
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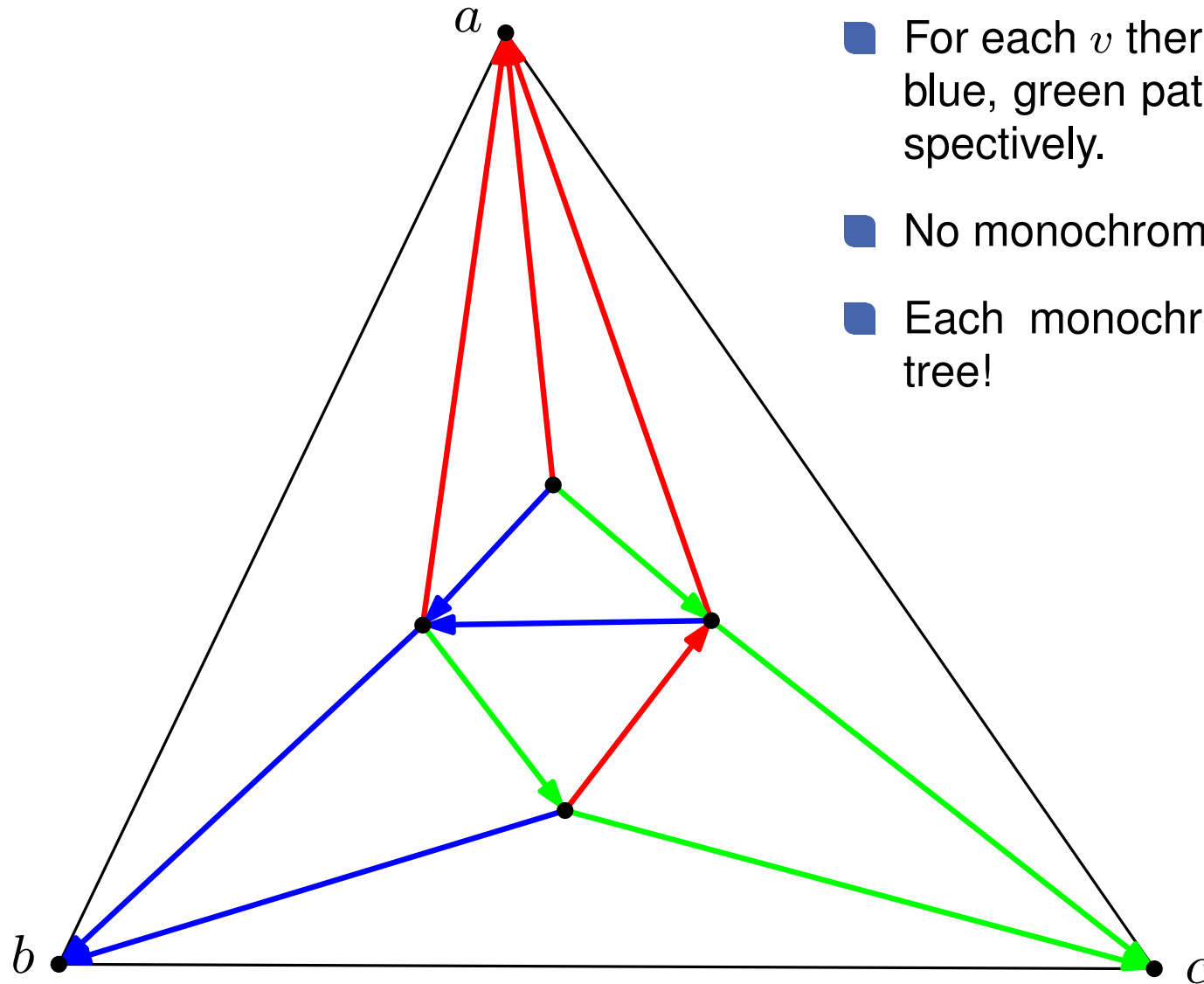


Schnyder Realizer



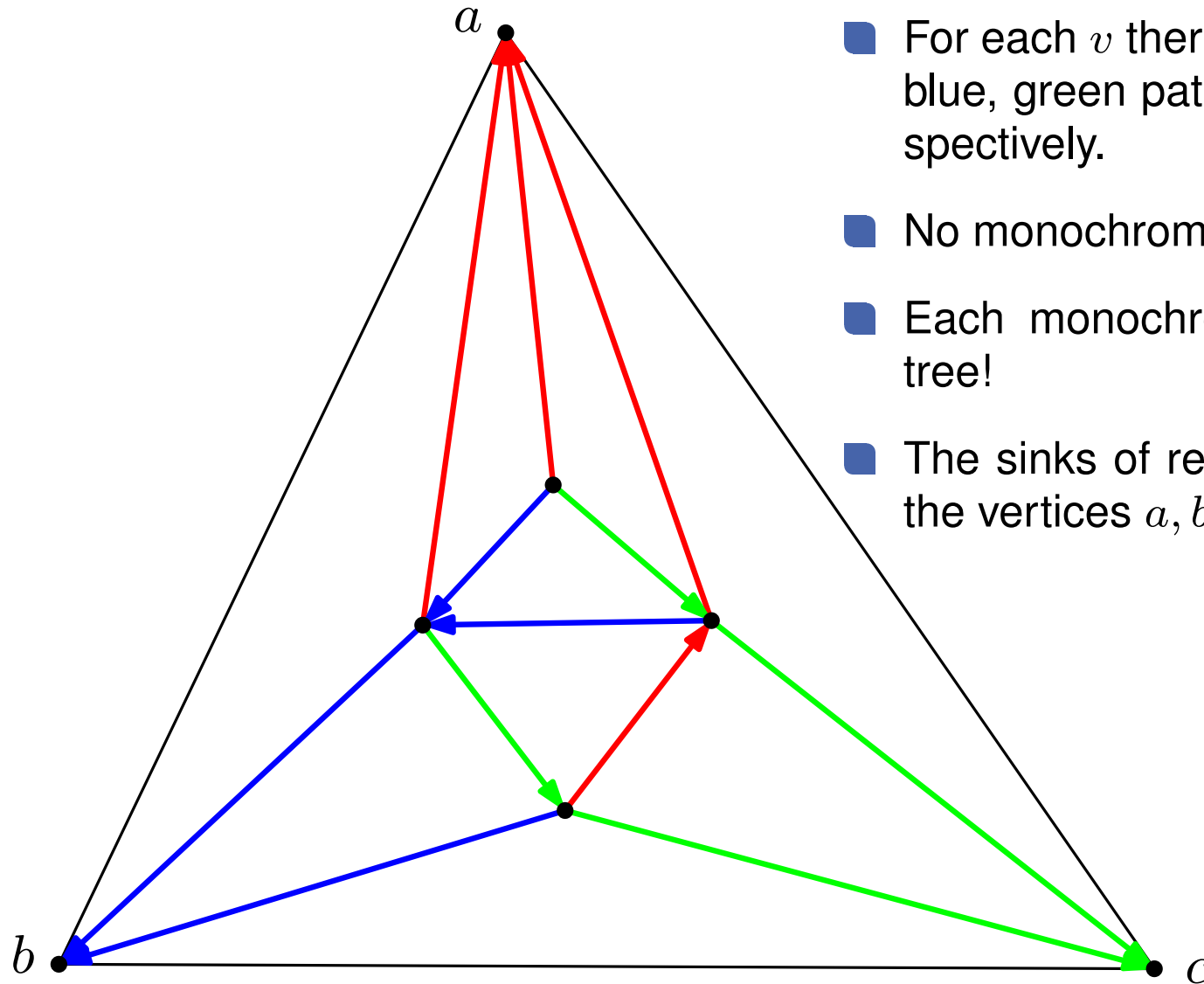
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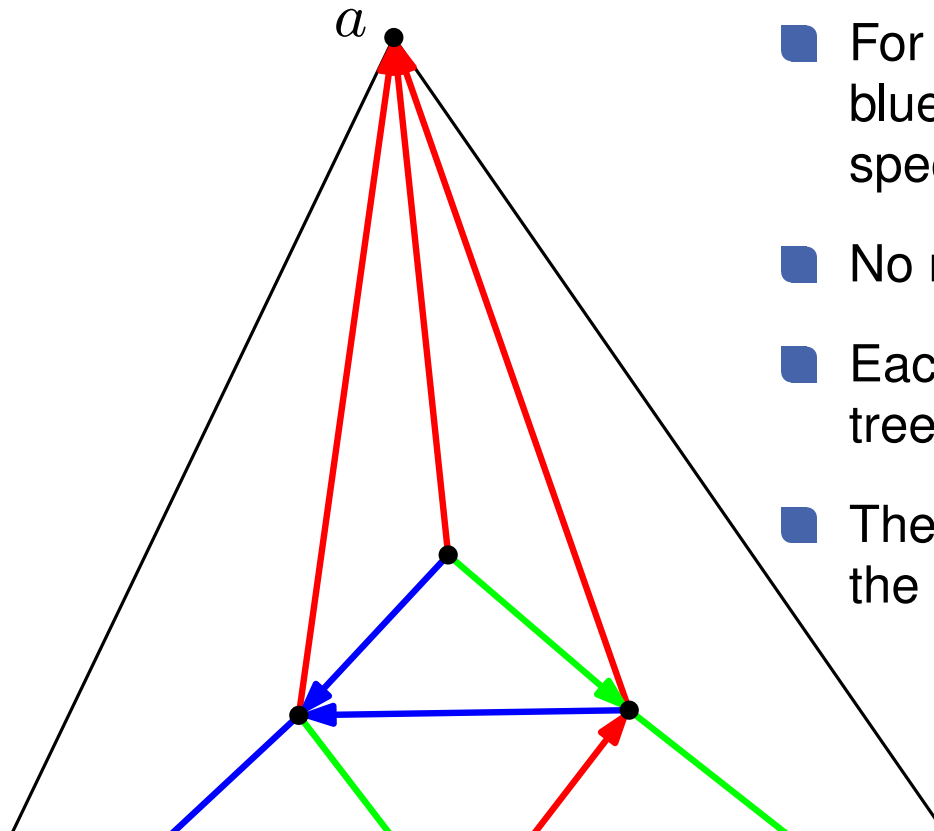


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Work with your neighbour(s) and then share

- Prove that each monochromatic graph is a tree.

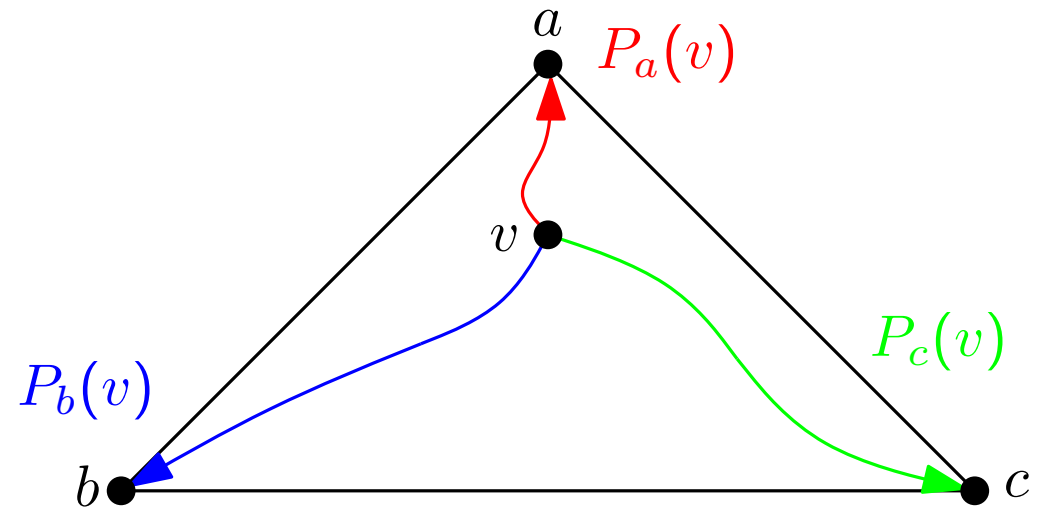
5 min

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- From Schnyder Realizer to Barycentric Representation.

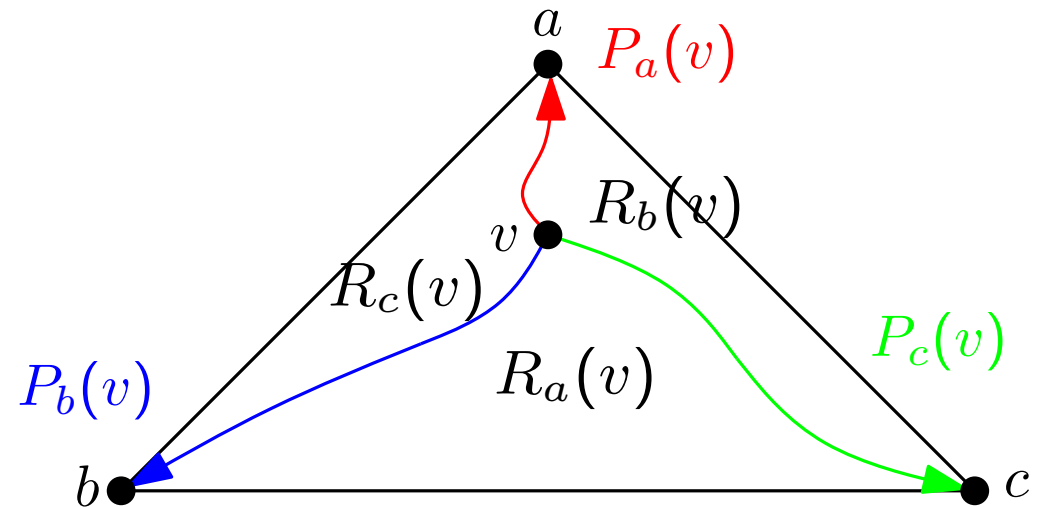
Face Regions

- Paths $P_a(v)$, $P_b(v)$, $P_c(v)$ cross only at vertex v .
- $R_a(v)$, $R_b(v)$, $R_c(v)$ are sets of faces.



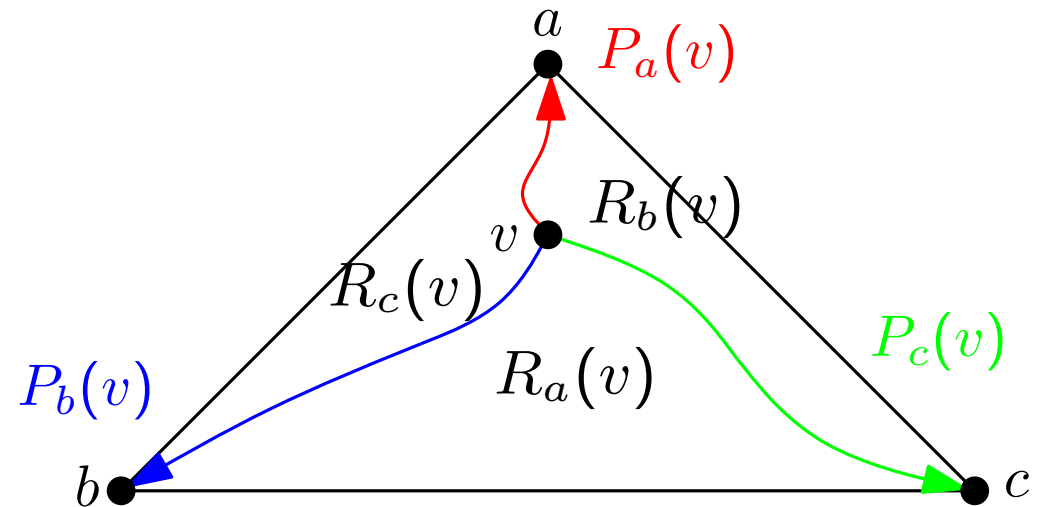
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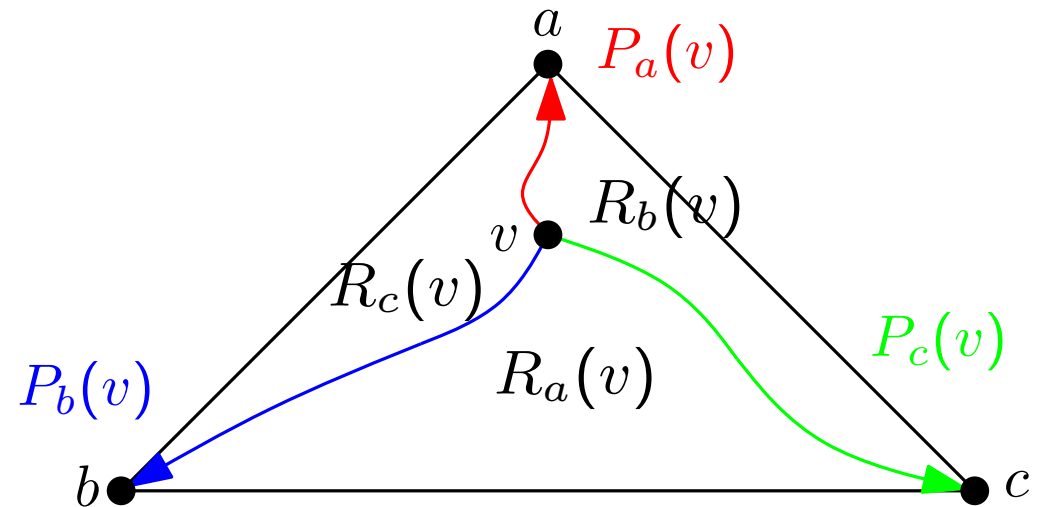


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For inner vertices $u \neq v$ it holds that $u \in R_i(v) \Rightarrow R_i(u) \subsetneq R_i(v)$.

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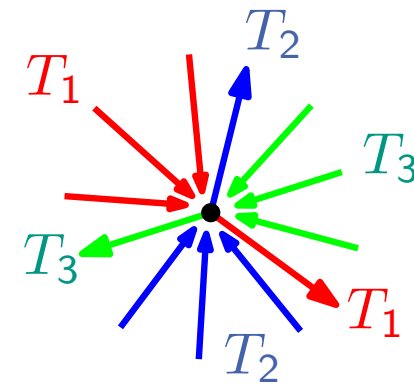
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Proof ...



Barycentric Representation

- Let barycentric coordinates of $v \in G \setminus a, b, c$ be (v_a, v_b, v_c) , where $v_a = |R_a(v)|/(2n - 5)$, $v_b = |R_b(v)|/(2n - 5)$ and $v_c = |R_c(v)|/(2n - 5)$.
- We set: $A = (2n - 5, 0)$, $B = (0, 2n - 5)$, $C = (0, 0)$.

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Satz [Schnyder '90]

The function

$$f: v \mapsto (v_a, v_b, v_c) = \frac{1}{2n - 5} (|R_a(v)|, |R_b(v)|, |R_c(v)|)$$

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Proof

- Condition 1 : $v_a + v_b + v_c = 1$.
- Condition 2: For each edge (u, v) and vertex $w \neq u, v$ at least one of three is true: $w_a > u_a, v_a$, $w_b > u_b, v_b$, $w_c > u_c, v_c$.

Final Remarks

- The resulting drawing is a grid drawing.
- It is bounded by the triangle $\triangle ABC$ with $A = (2n - 5, 0)$, $B = (0, 2n - 5)$, $C = (0, 0)$.
- It has area $2n - 5 \times 2n - 5$.

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How to obtain area $n - 2 \times n - 2$?

- Use weak barycentric coordinates $\frac{1}{n-1}(n_1(v), n_2(v), n_3(v))$,
 $n_i(v) = |\text{vertices in } R_i(v)| - |P_{i-1}(v)|$ with respect to $A = (n - 1, 0)$,
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 $B = (0, n - 1)$, $C = (0, 0)$.
- **Weak barycentric coordinates:** Triple (v_a, v_b, v_c) such that
 - $v_a + v_b + v_c = 1$
 - For each edge (u, v) and vertex $w \neq u, v$, $\exists k \in \{a, b, c\}$, such that $(u_k, u_{k+1}) <_{lex} (w_k, w_{k+1})$, and $(v_k, v_{k+1}) <_{lex} (w_k, w_{k+1})$.
 - Here we say that $(a, b) <_{lex} (c, d)$ iff $a < c$ or $a = c$ and $b < d$.



Schnyder Realizer Method

- [NR04] Book T. Nishizeki, Md. S. Rahman “Planar Graph Drawing. Chapter 4.3.