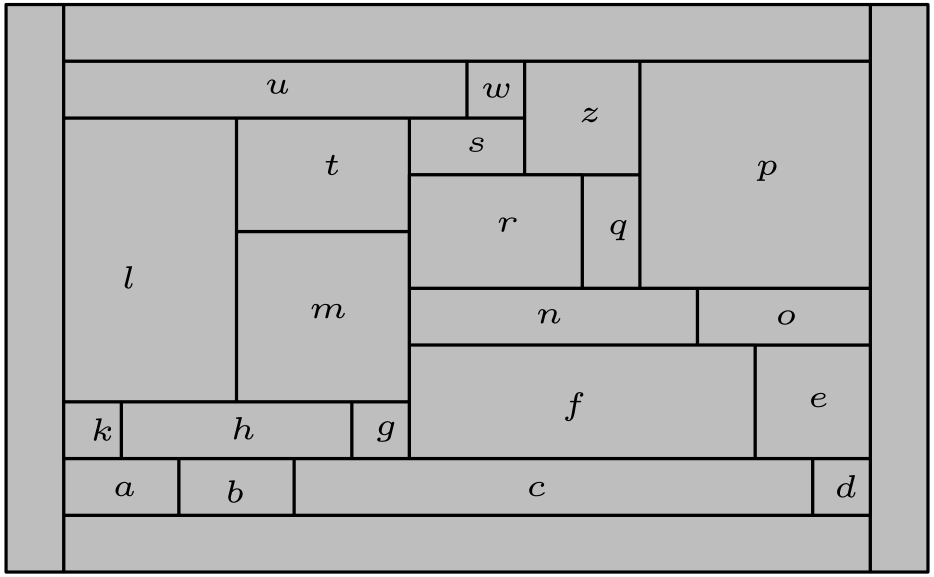
Karlsruhe Institute of Technology

Algorithms for graph visualization

Contact representations of planar graphs.

Winter semester 2018/2019

Tamara Mchedlidze



Contact representation

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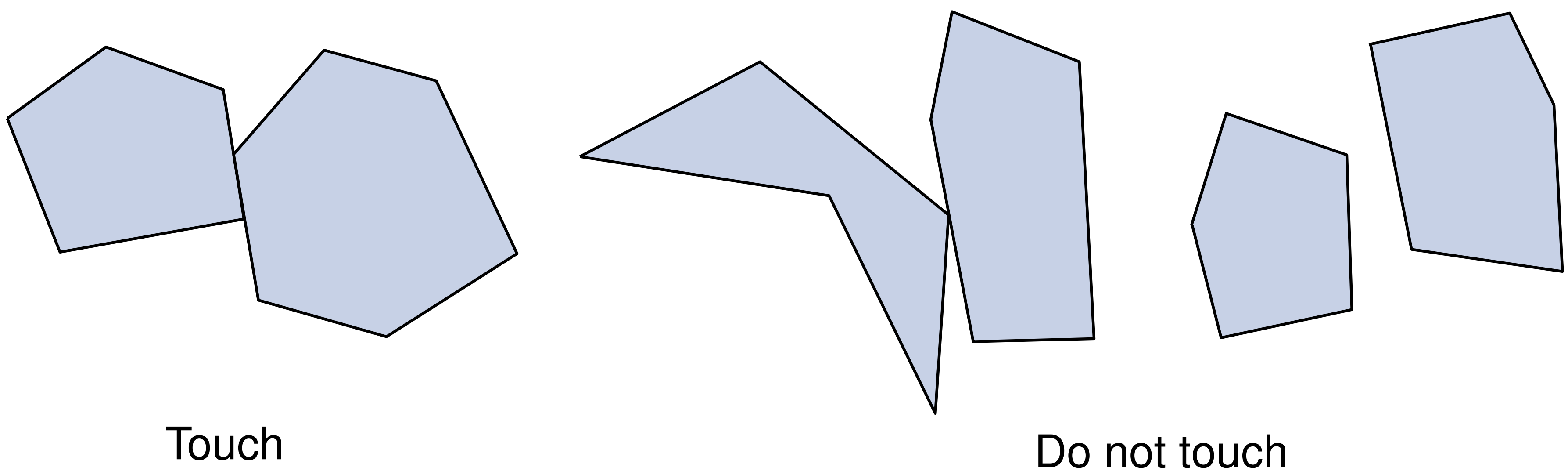
In a contact representation of a planar graph each vertex is represented as a geometrical object such that two object thouch if and only if the corresponding vertices are connected by an edge.



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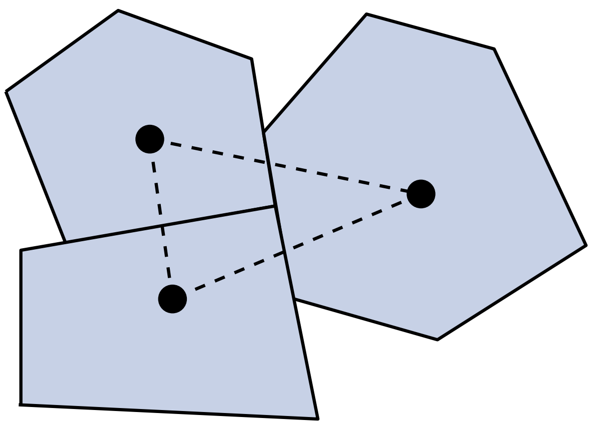
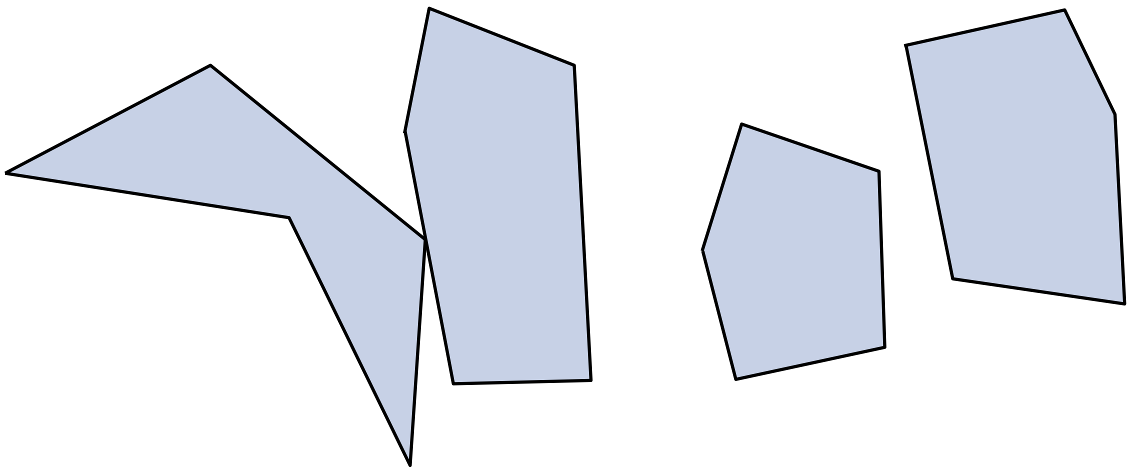


Touch



Do not touch

2 - 2

Contact representation

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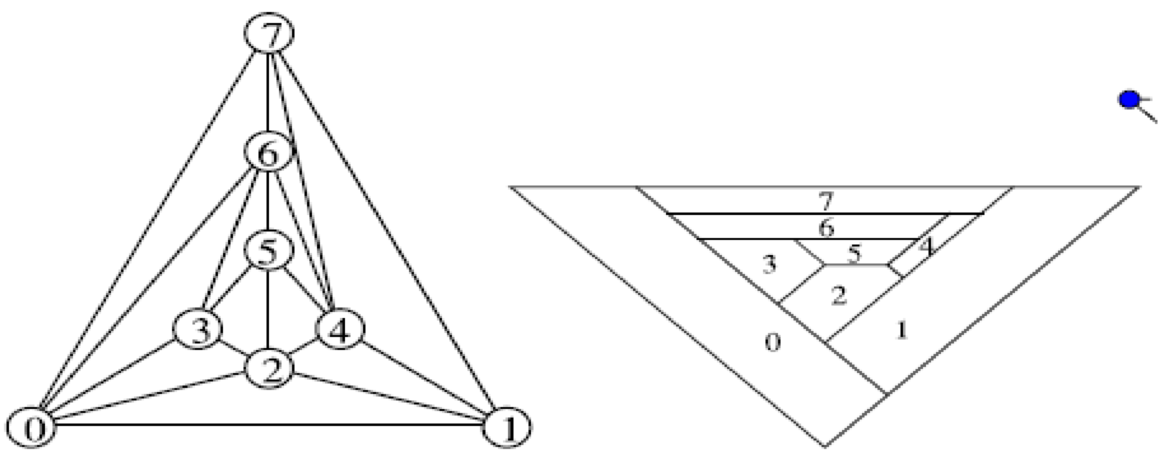

- 6-gons are necessary and suffisient for planar graphs! (Gansner et. al. 2010)

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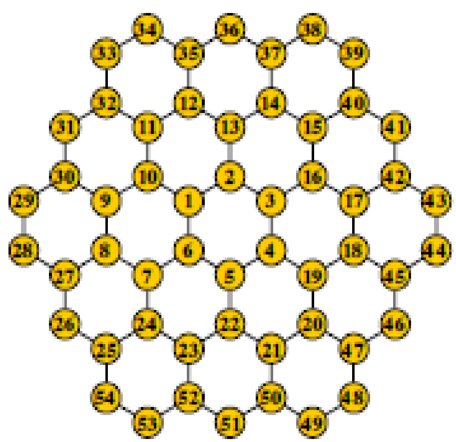
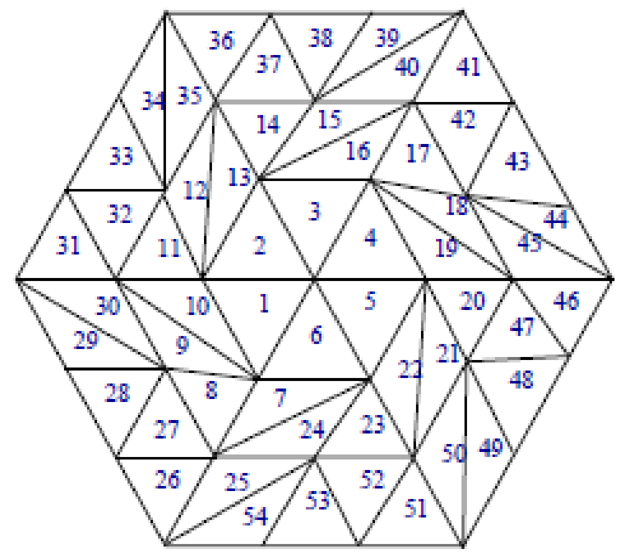
Touching triangle representation





- Every 3-connected cubic planar graph admits a contact representation with triangles (Kobourov et. al. 2012)

Contact representation

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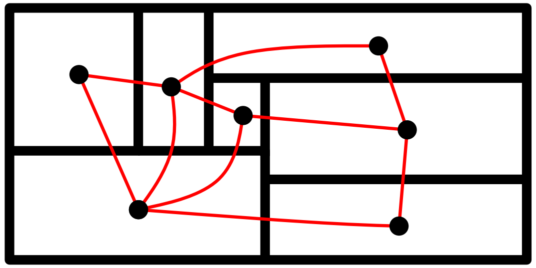
Contact representation with rectangles



- Every 4-connected planar graph* admits a contact representation with rectangles (Xin He 1993*)

Contact representation

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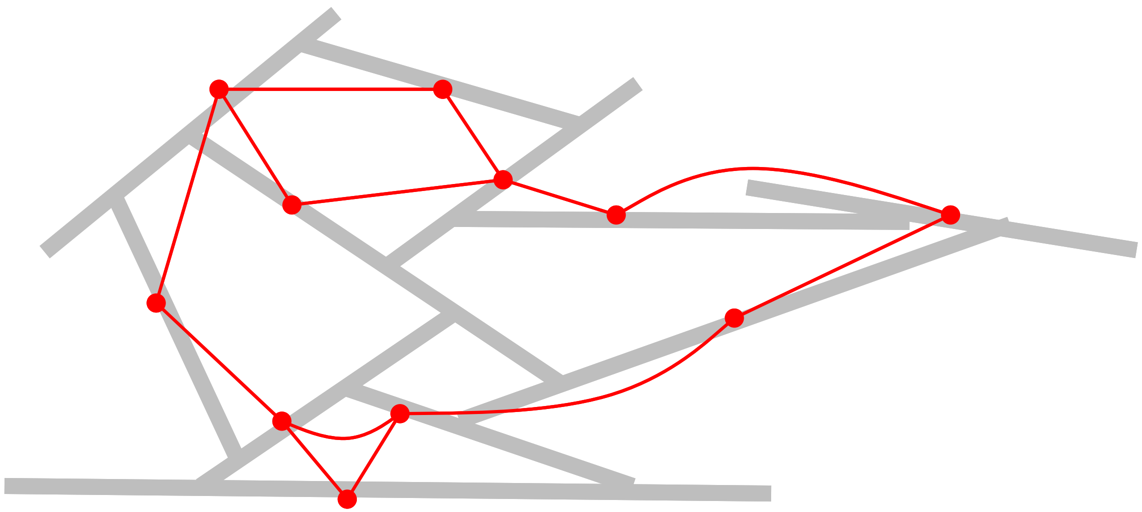
In a contact representation of a planar graph each vertex is represented as a geometrical object such that two object thouch if and only if the corresponding vertices are connected by an edge.

Touching segment representation



- Every triangle-free planar graph has a contact representation with line segments in just three directions (de Castro et. al. 1999)

Contact representation

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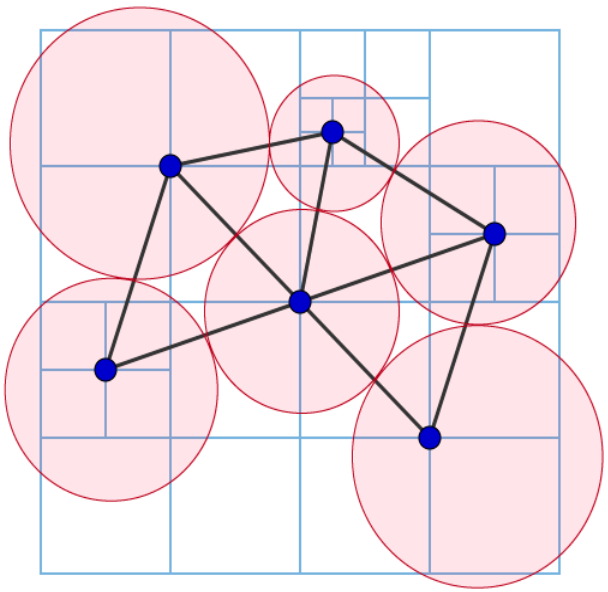
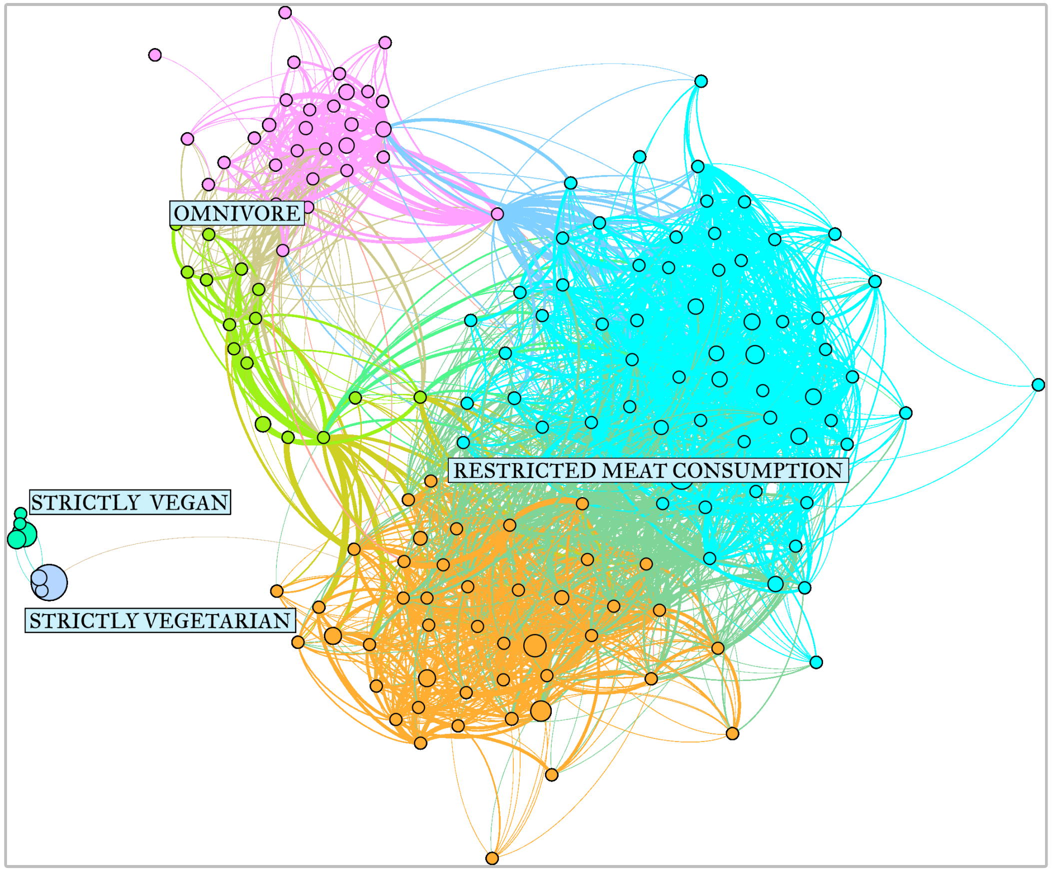
Touching disk representation



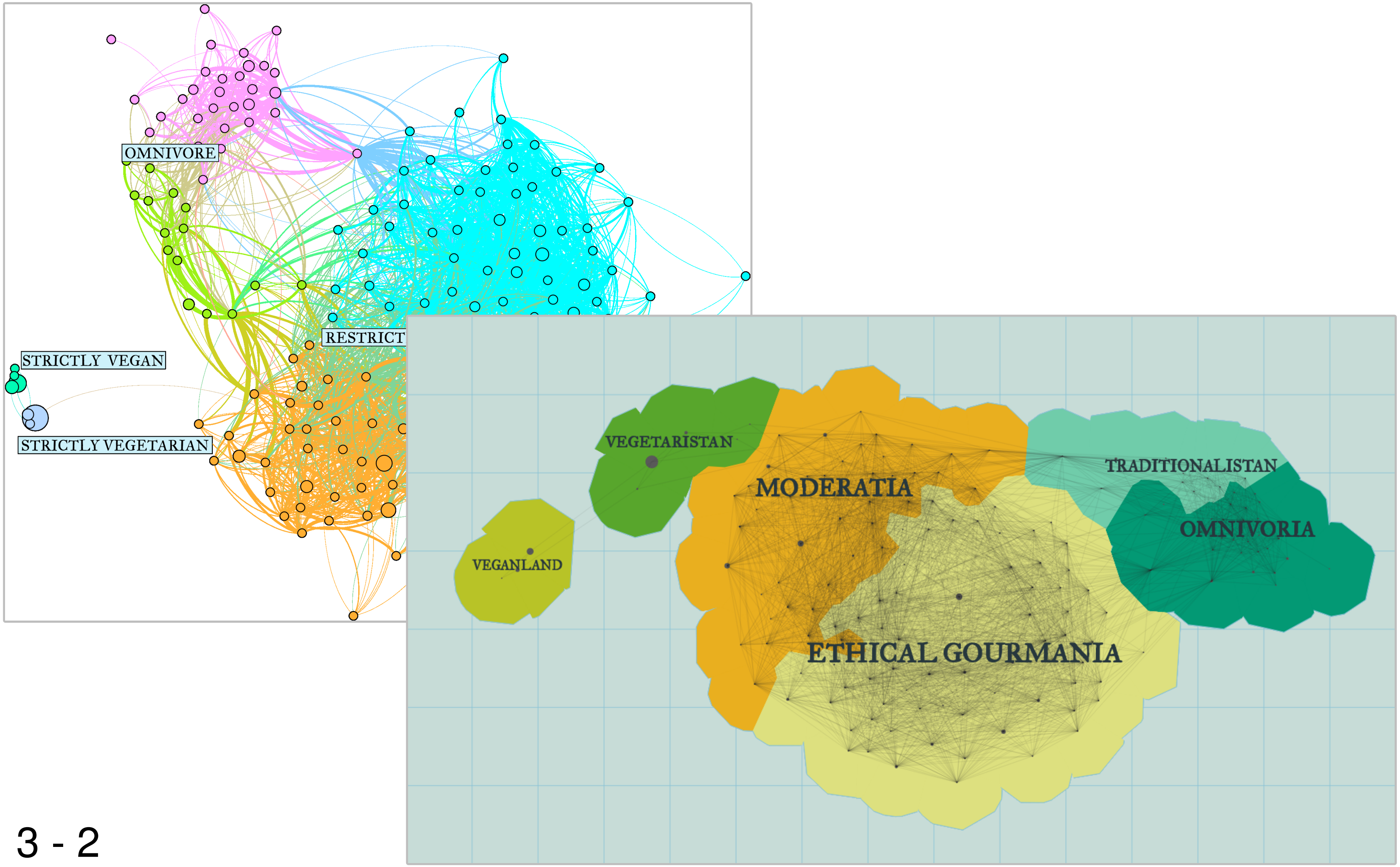
- Each planar graph has a touching disks representation (Koebe 1936)

Application: visualization of a clustering



3 - 1

Application: visualization of a clustering



Application: visualization of a clustering

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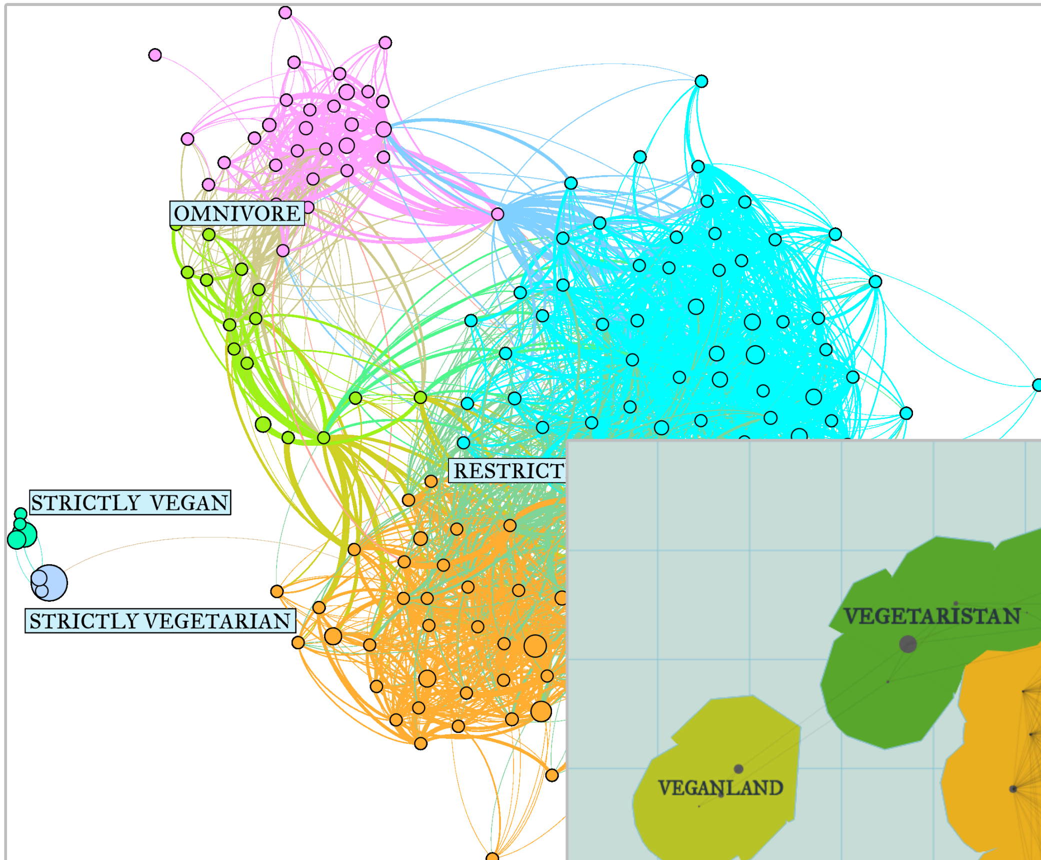

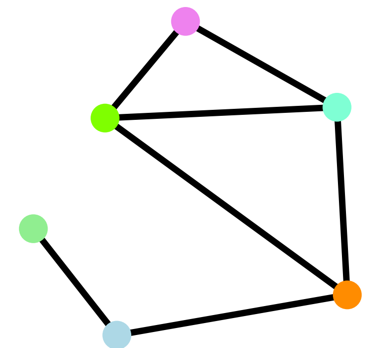

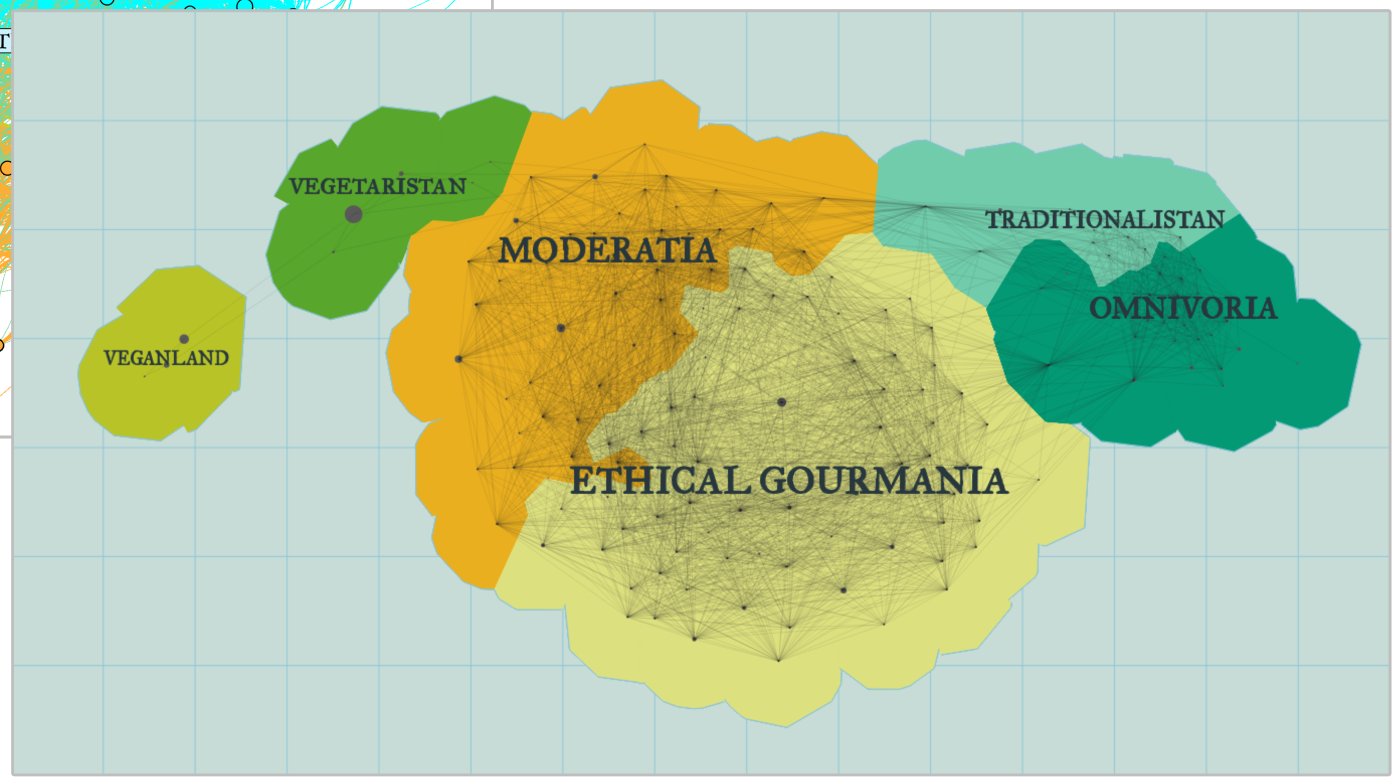

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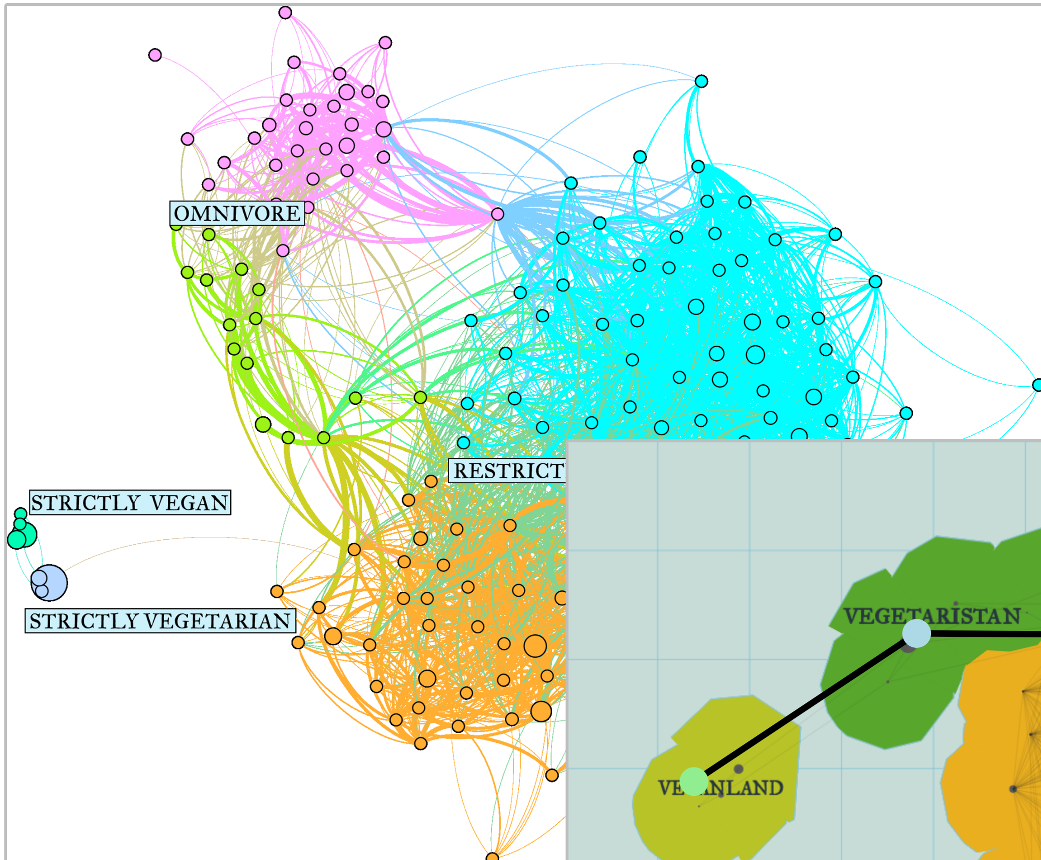

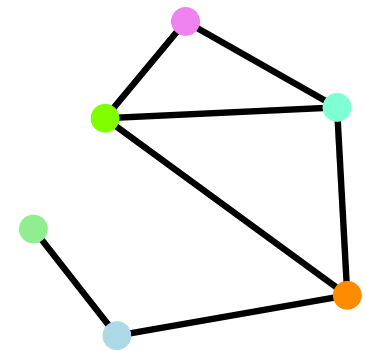

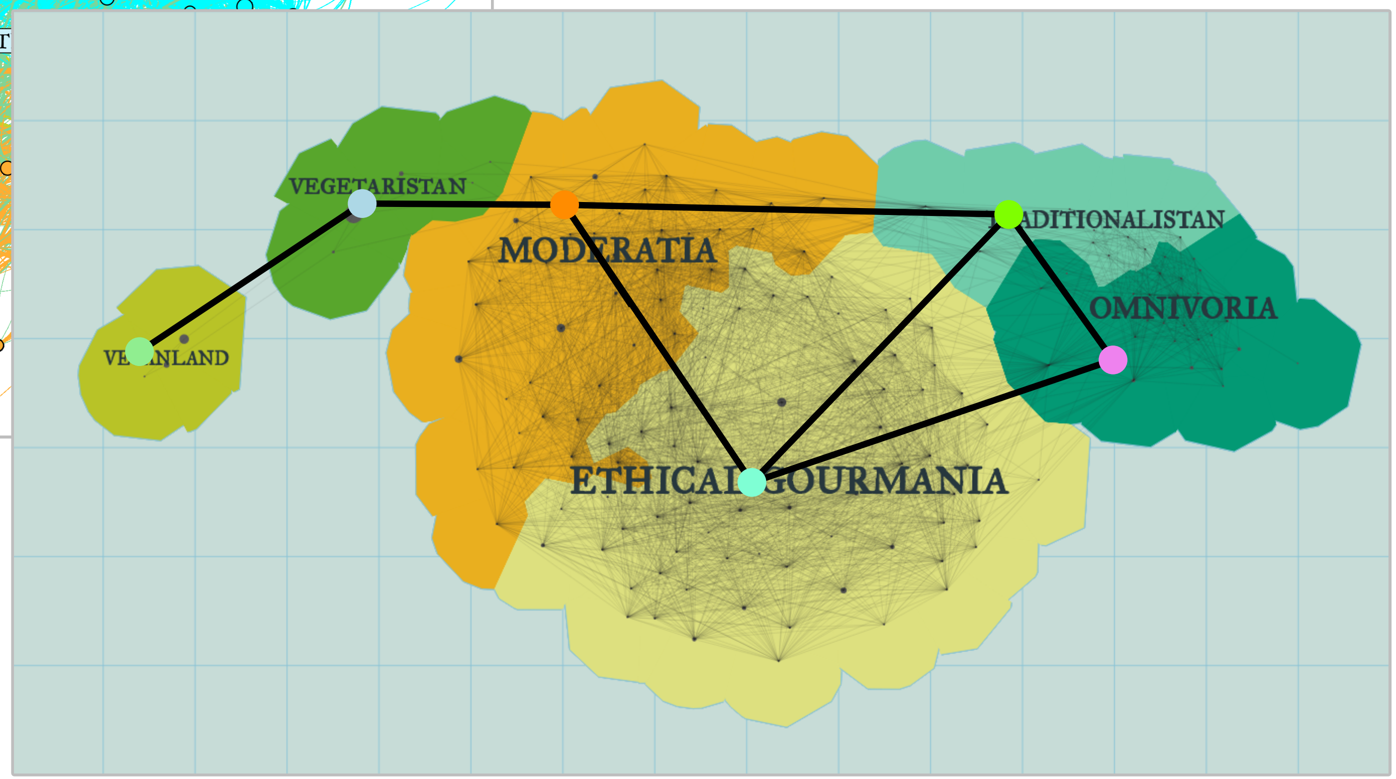

Today

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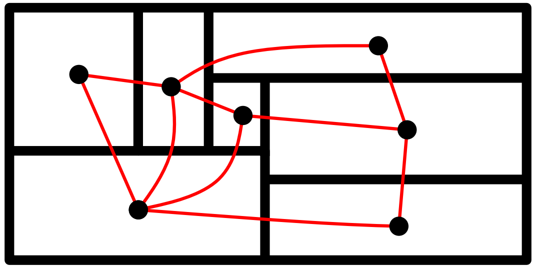
Contact representation with rectangles



- Every 4-connected planar graph* admits a contact representation with rectangles (Xin He 1993*)

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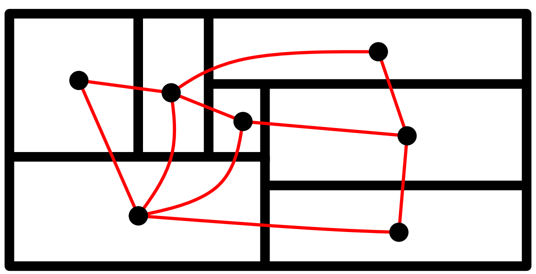


- Every 4-connected planar graph* admits a contact representation with rectangles (Xin He 1993*)
- A contact representation of G with rectangles, without holes and with rectangular outer boundary is called a **rectangular dual** of G

Rectangular Dual

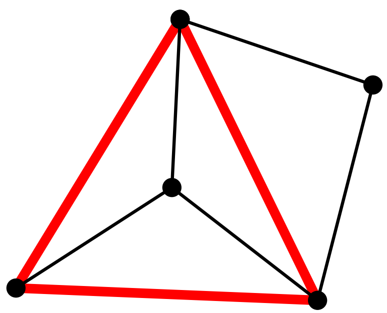
- Which graphs have a rectangular dual?

Rectangular Dual

- Which graphs have a rectangular dual?

Separating triangle

Let G be a graph. A triangle C of G whose removal results in at least two disconnected components is called a separating triangle of G.



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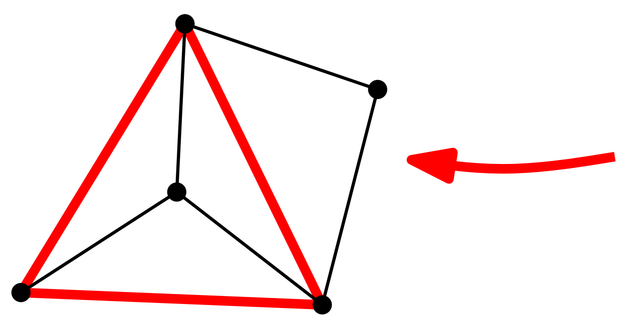

Does not have a rectangular dual!

(In order to enclose an area we need at least four boxes)

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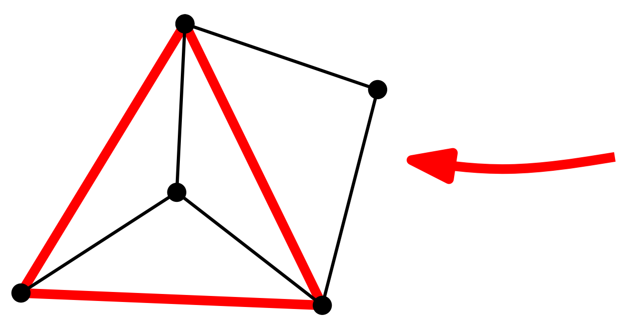

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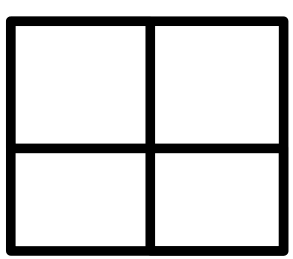
No four rectangles meet a a point!



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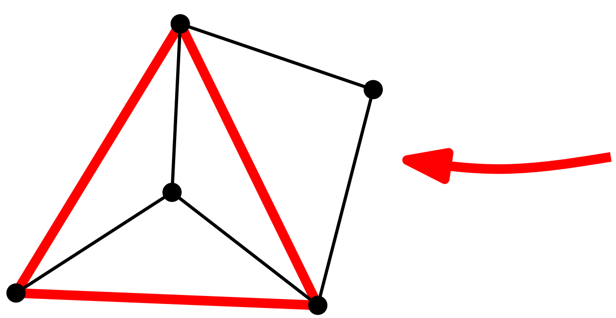

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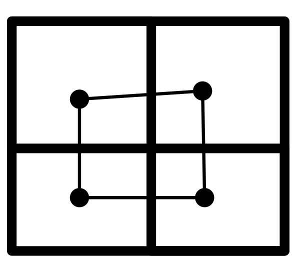
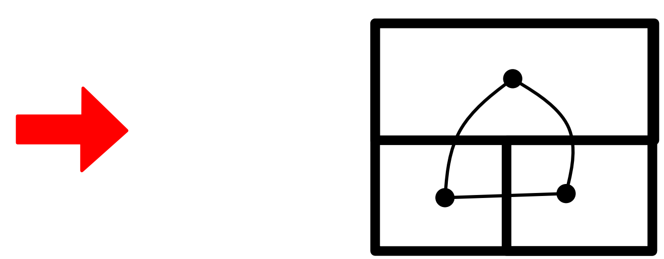


Each face of G must be a triangle!



Rectangular Dual

Necessary conditions for a planar graph G to have a rectangular dual:

- G must have at least 4 vertices on the outer face
- G must have no separating triangle
- each internal face of G must be a triangle

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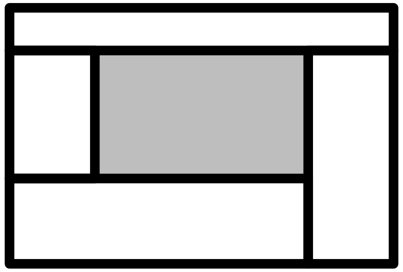
Theorem [1985,1986,1987,1988,1990,1993 by Xin He]

A planar graph $G = (V, E)$ has a rectangular dual R with four rectangles on the boundary of R if and only if the following conditions hold:

- Every interior face of G is a triangle and the exterior face of G is a quadrangle;
- G has no separating triangles



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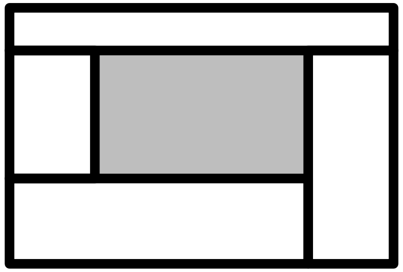
Proper Triangular Planar Graph (PTP)



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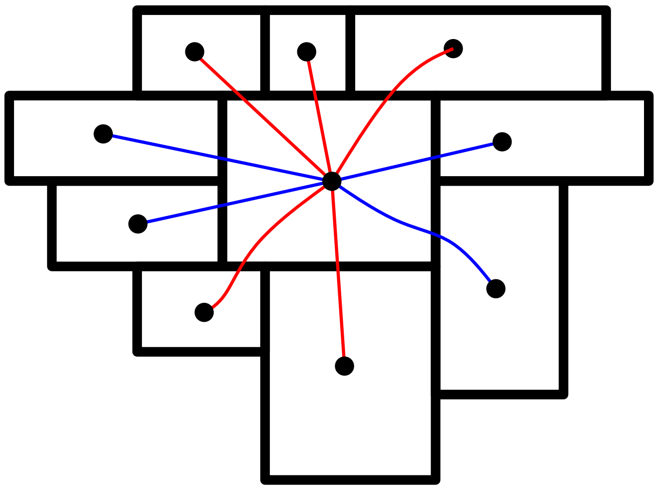
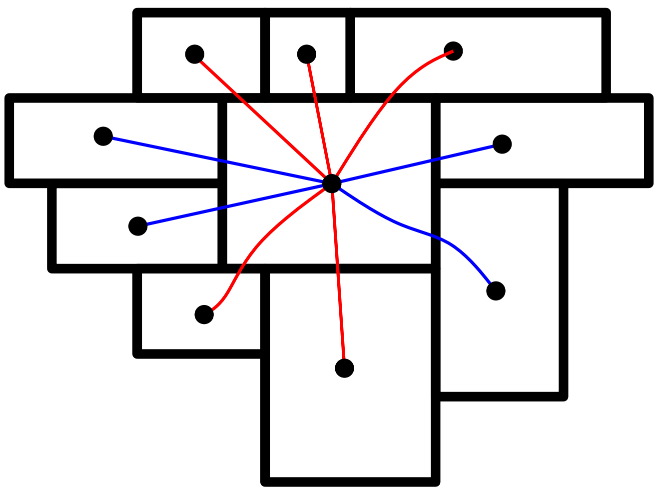


In order to construct a rectangular dual we need to partition our edges on vertical and horizontal. Regular edge labeling (REL, for short) is a tool for that.

Rectangular Dual



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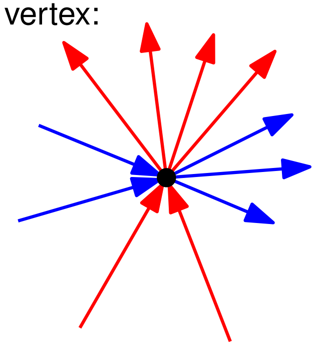
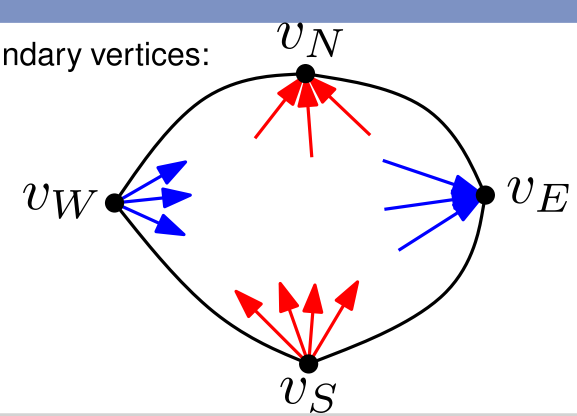
Regular edge labeling

For each internal vertex:



For the boundary vertices:

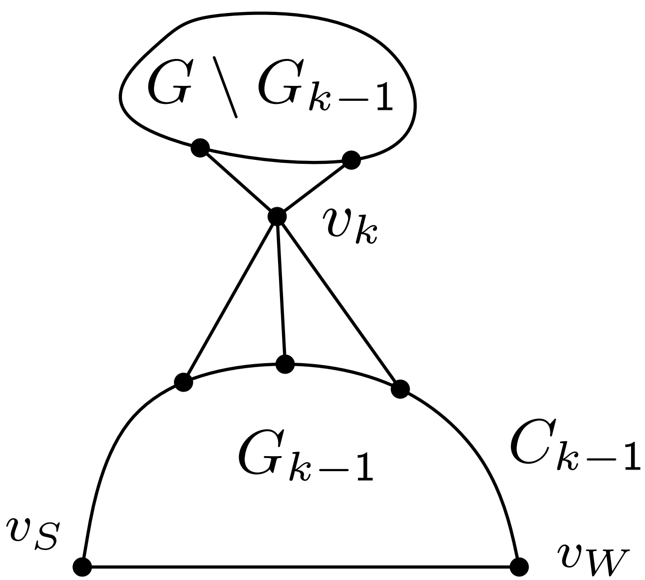


Rectangular Dual

Theorem

Let $G = (V, E)$ be a PTP graph. There exists a labeling of the vertices of G $v_1 = v_S, v_2 = v_W, v_3, \ldots, v_n = v_N$ such that for every $4 \leq k \leq n$:

- The subgraph G_{k-1} induced by $v_1, \ldots, v_{k-1}$ is biconnected and boundary C_{k-1} of G_{k-1} contains the edge (v_S, v_W).
- v_k is in exterior face of G_{k-1}, and its neighbors in G_{k-1} form (at least 2-element) subinterval of the path $C_{k-1} \setminus (v_S, v_W)$. If $k \leq k - 2$, v_k has at least 2 neighbors in $G \setminus G_{k-1}$



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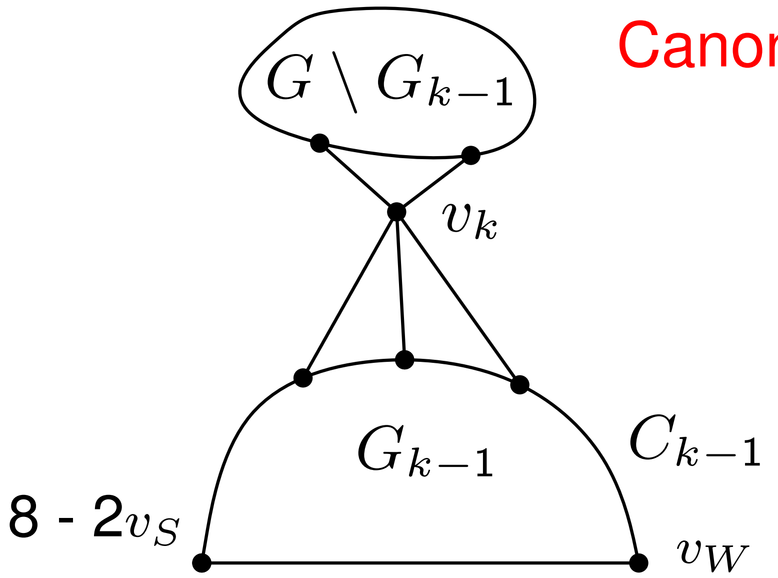

Canonical ordering with extra condition on v_k!

Rectangular Dual

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Theorem (Refined canonical ordering)

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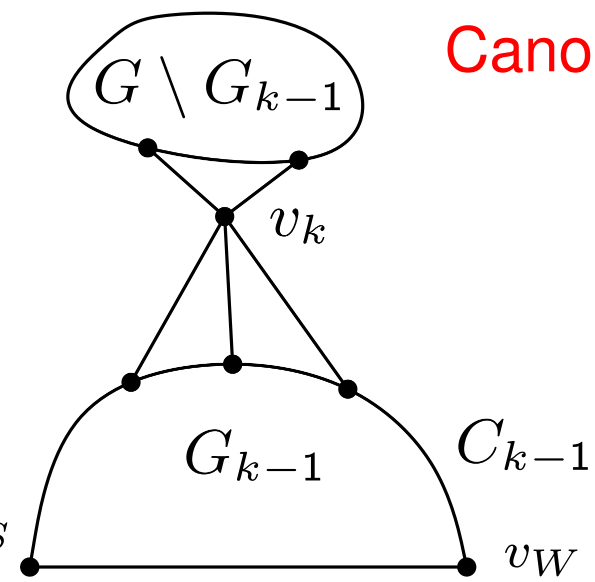
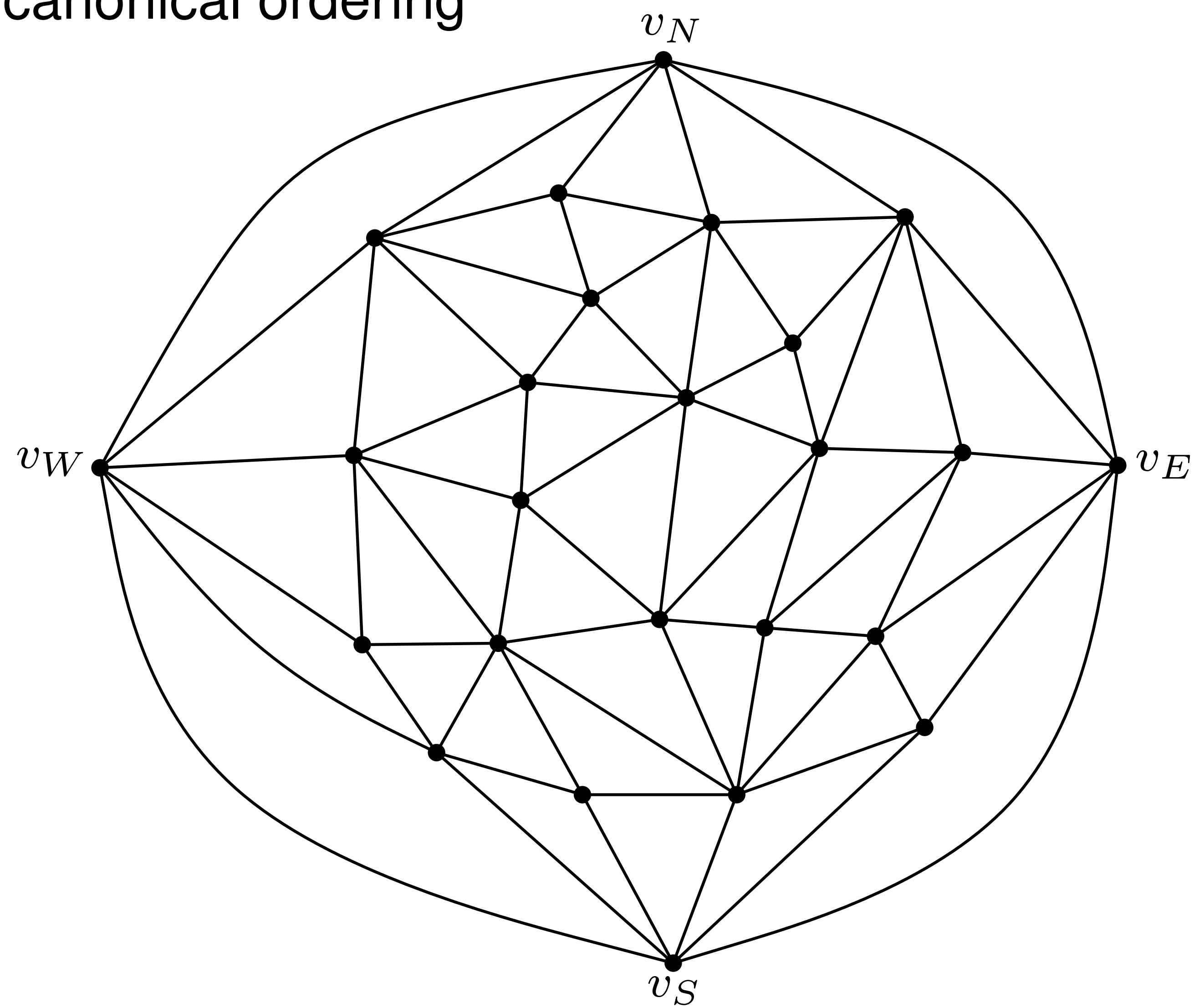
Canonical ordering with extra condition on v_k!

8 - 3



Rectangular Dual

Refined canonical ordering



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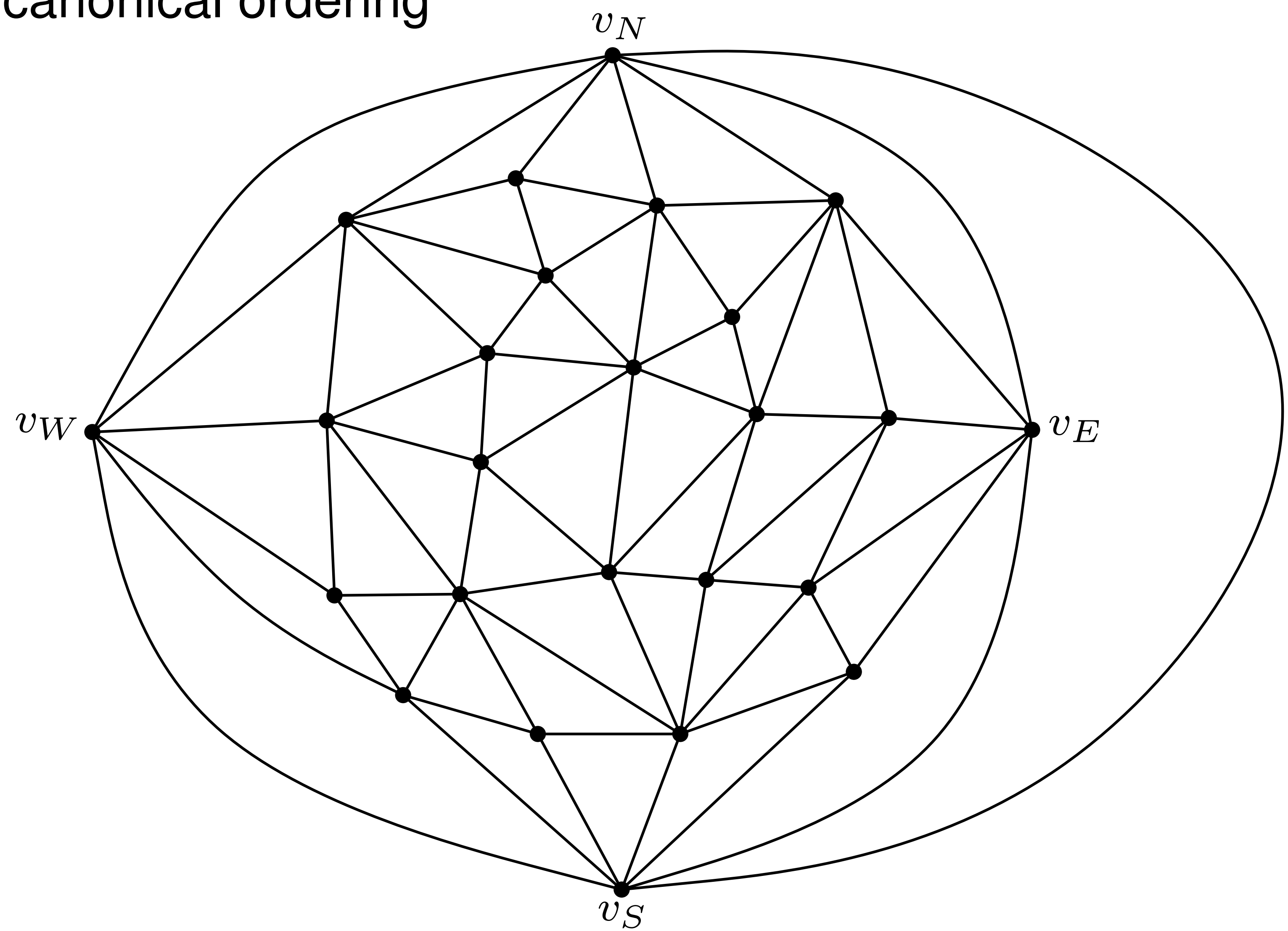

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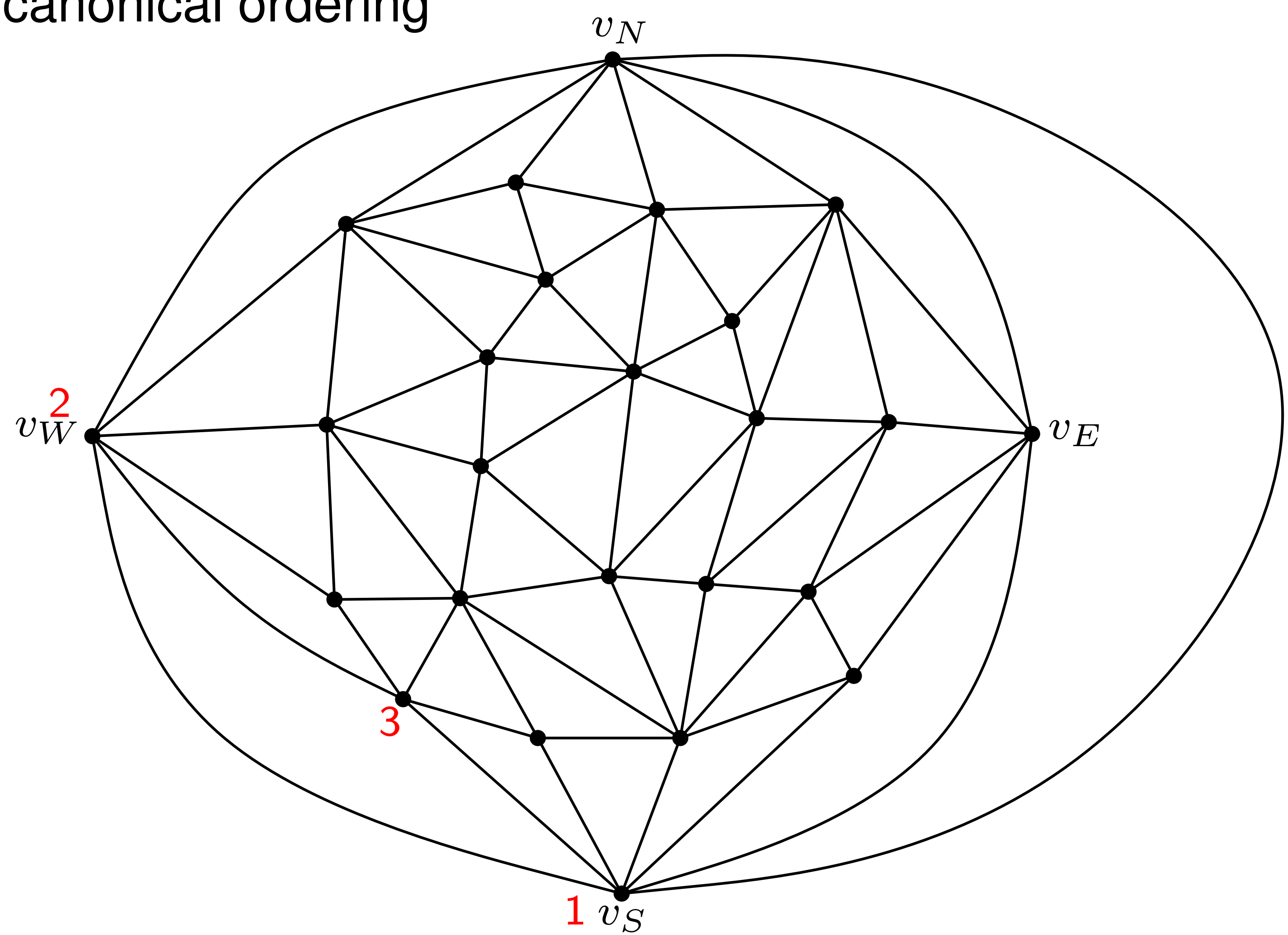

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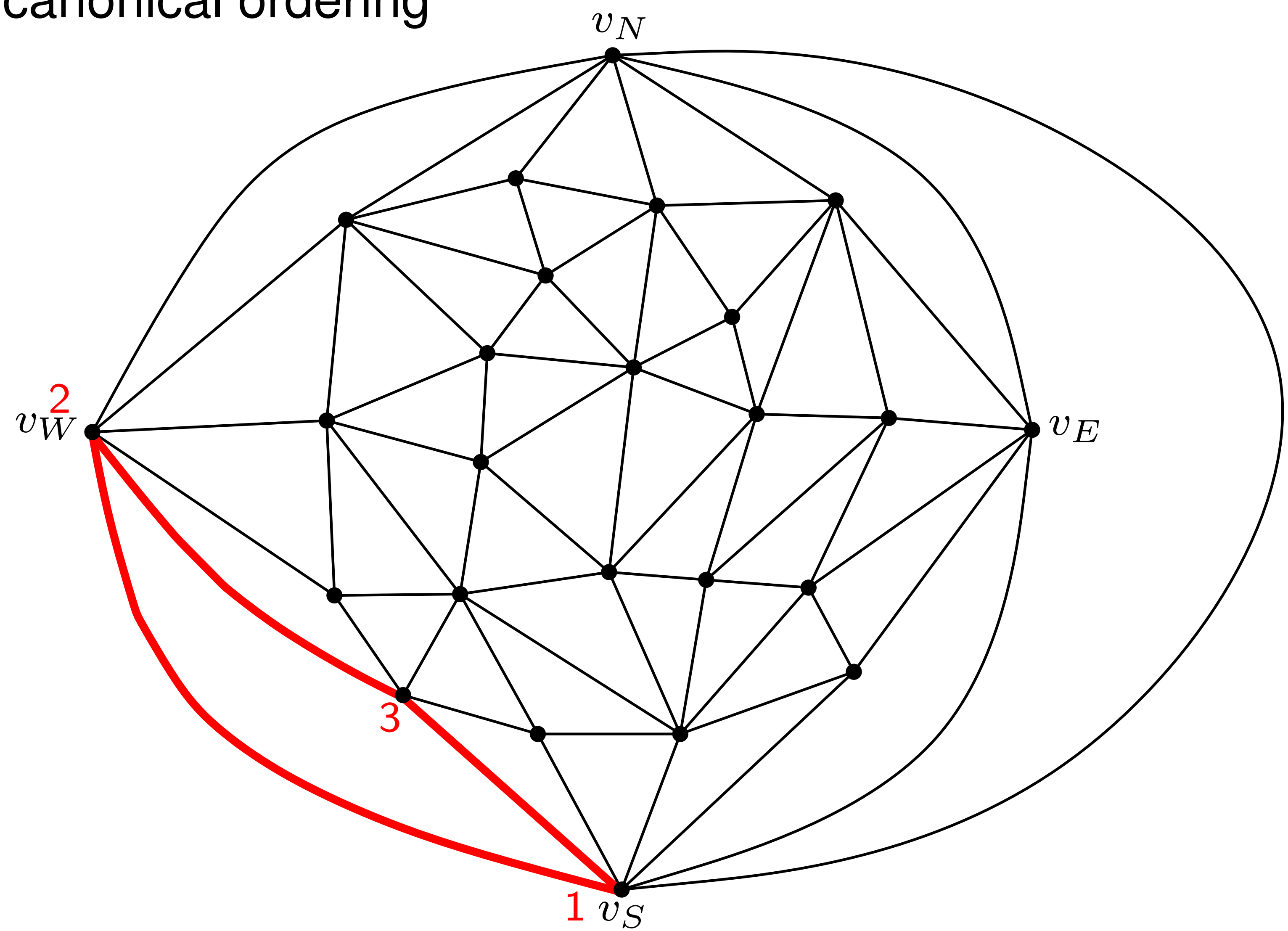

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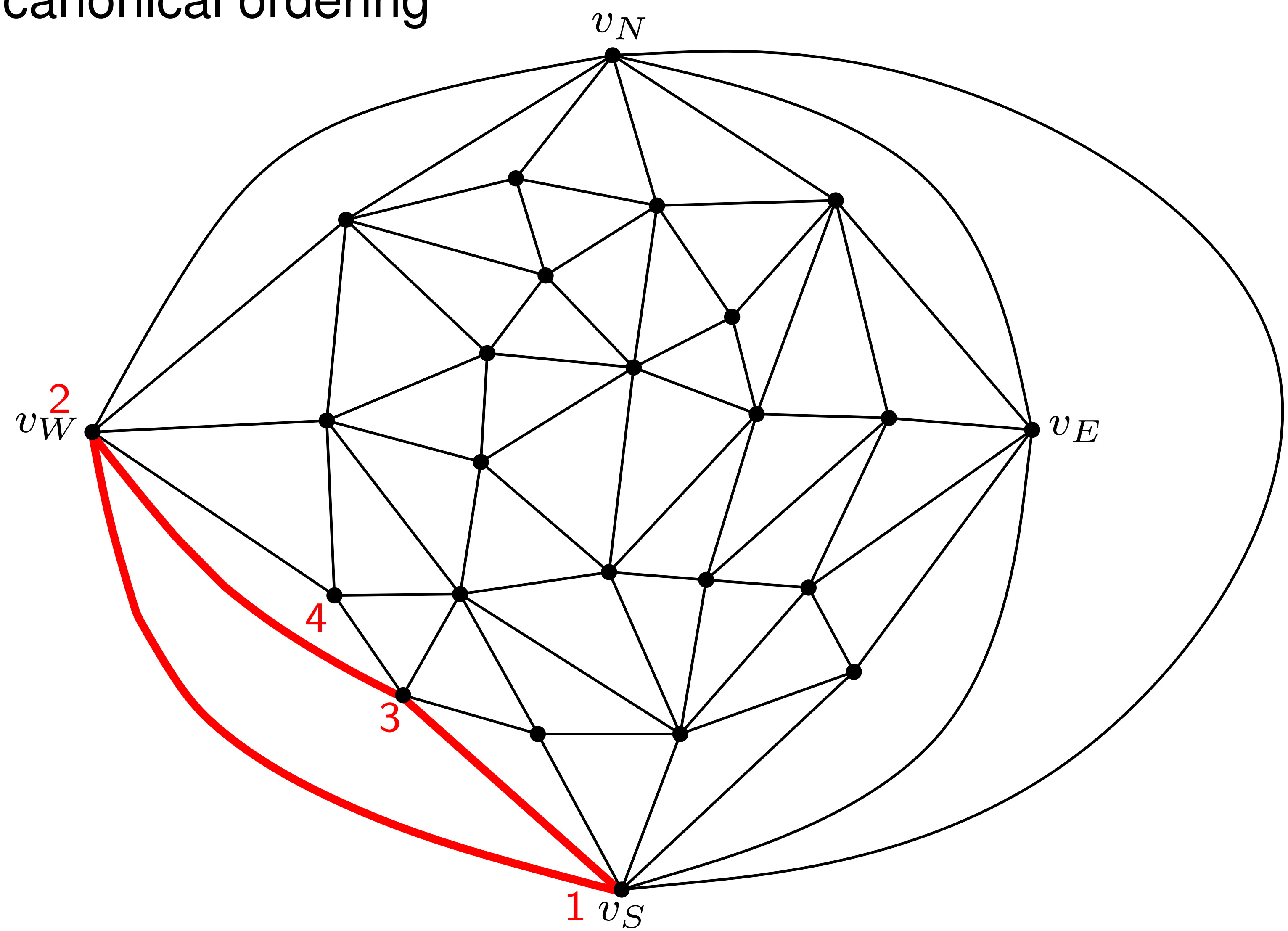
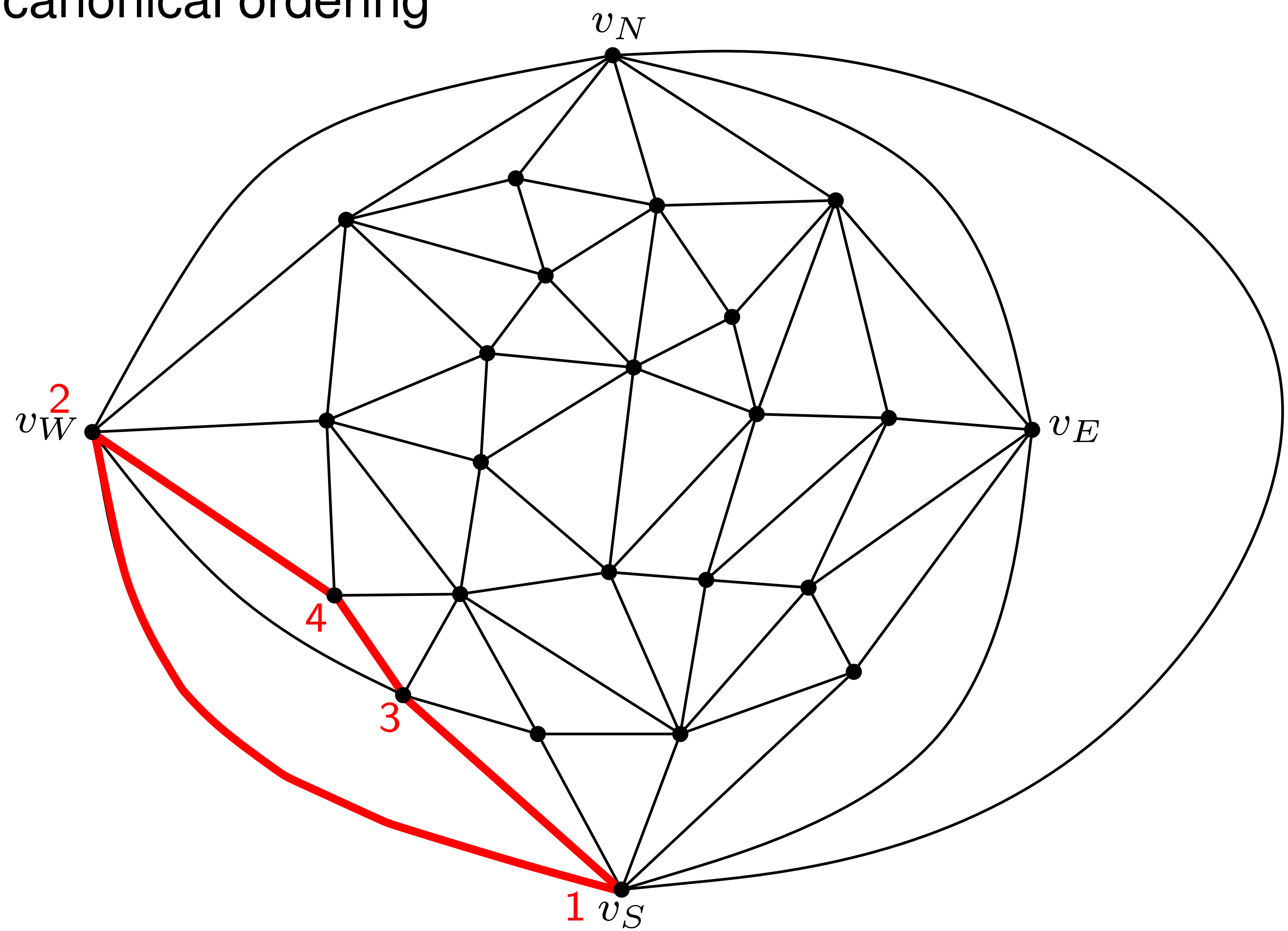


9 - 5

Rectangular Dual

Refined canonical ordering



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Rectangular Dual

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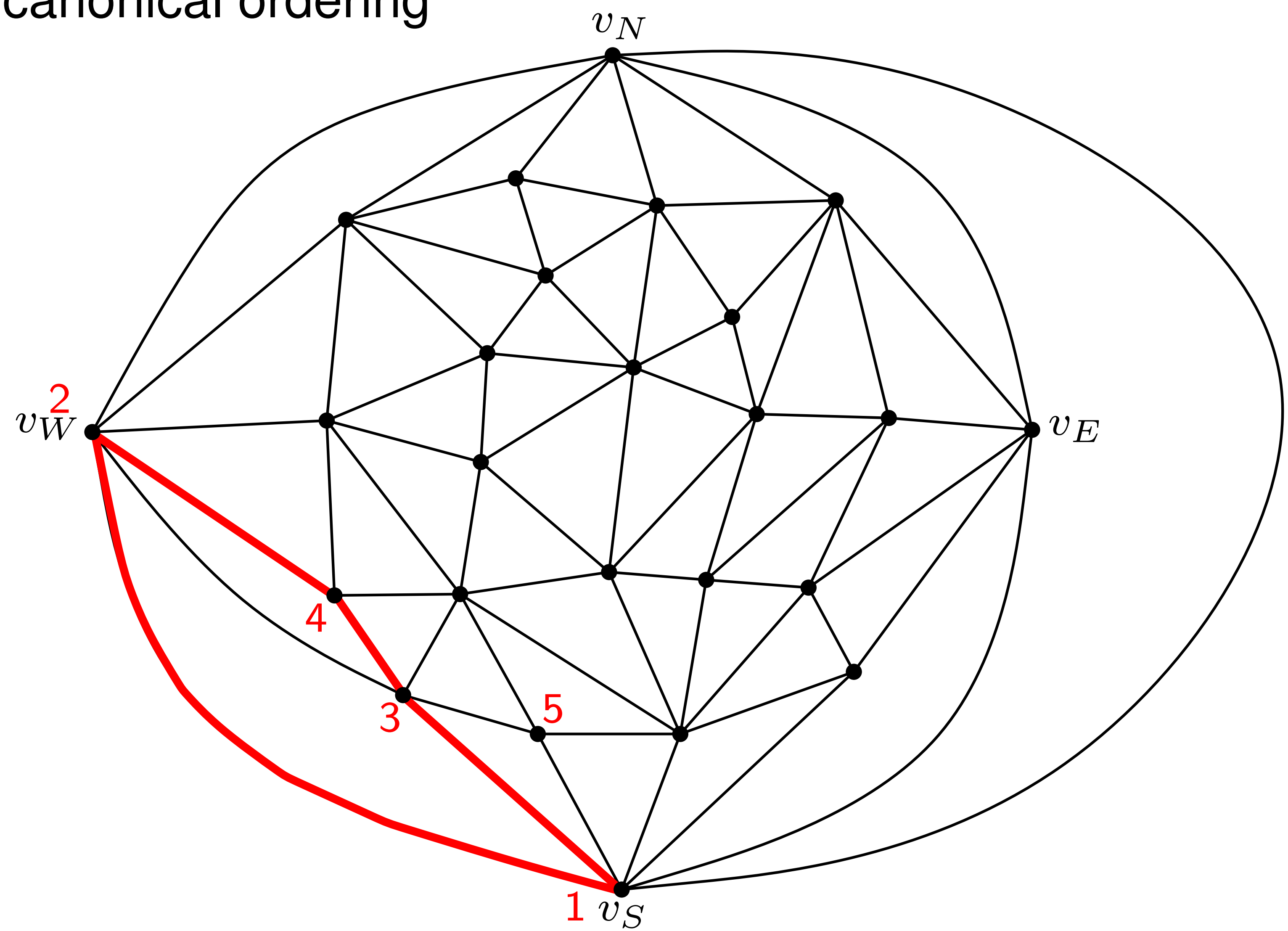

9 - 7

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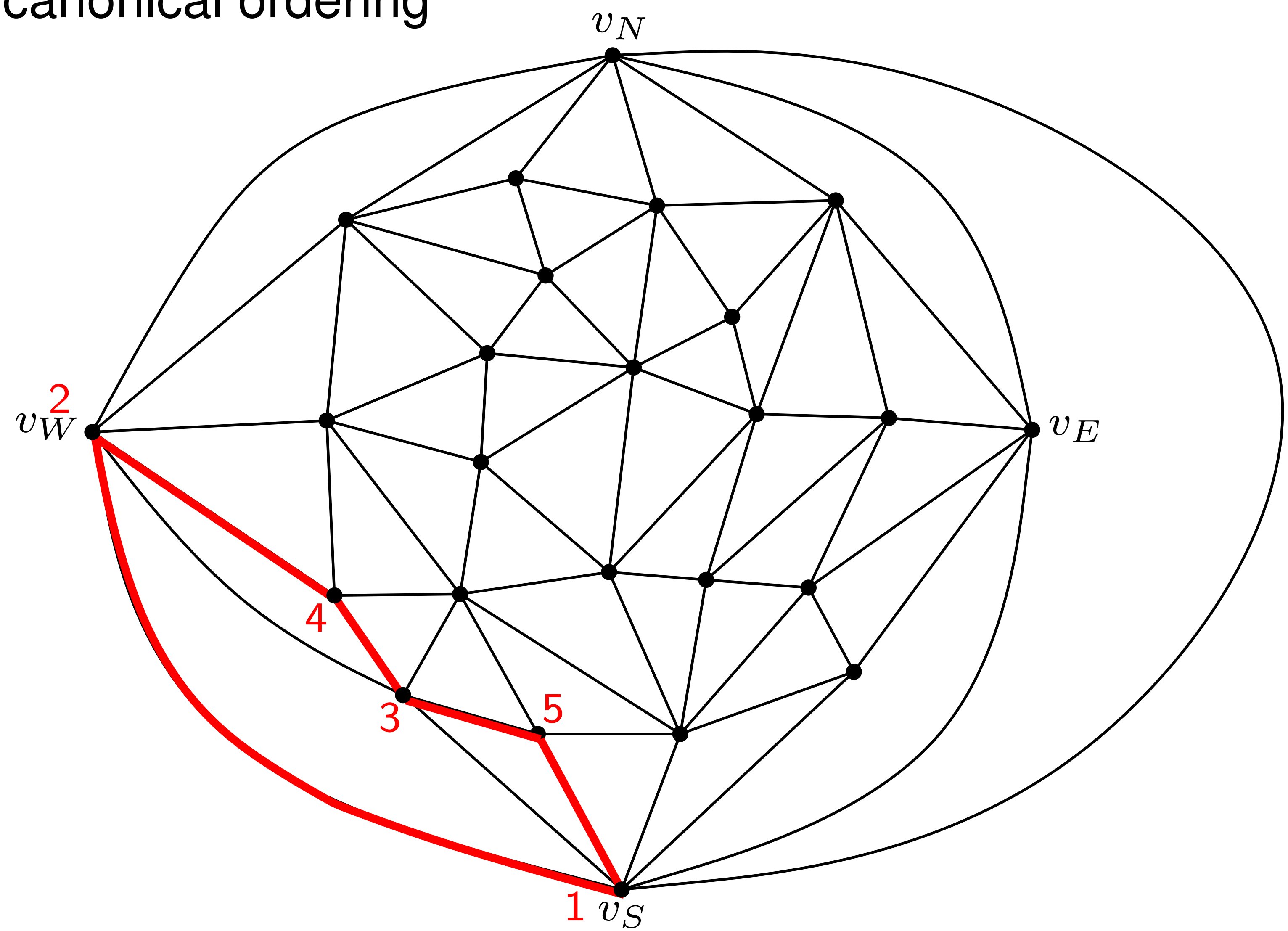

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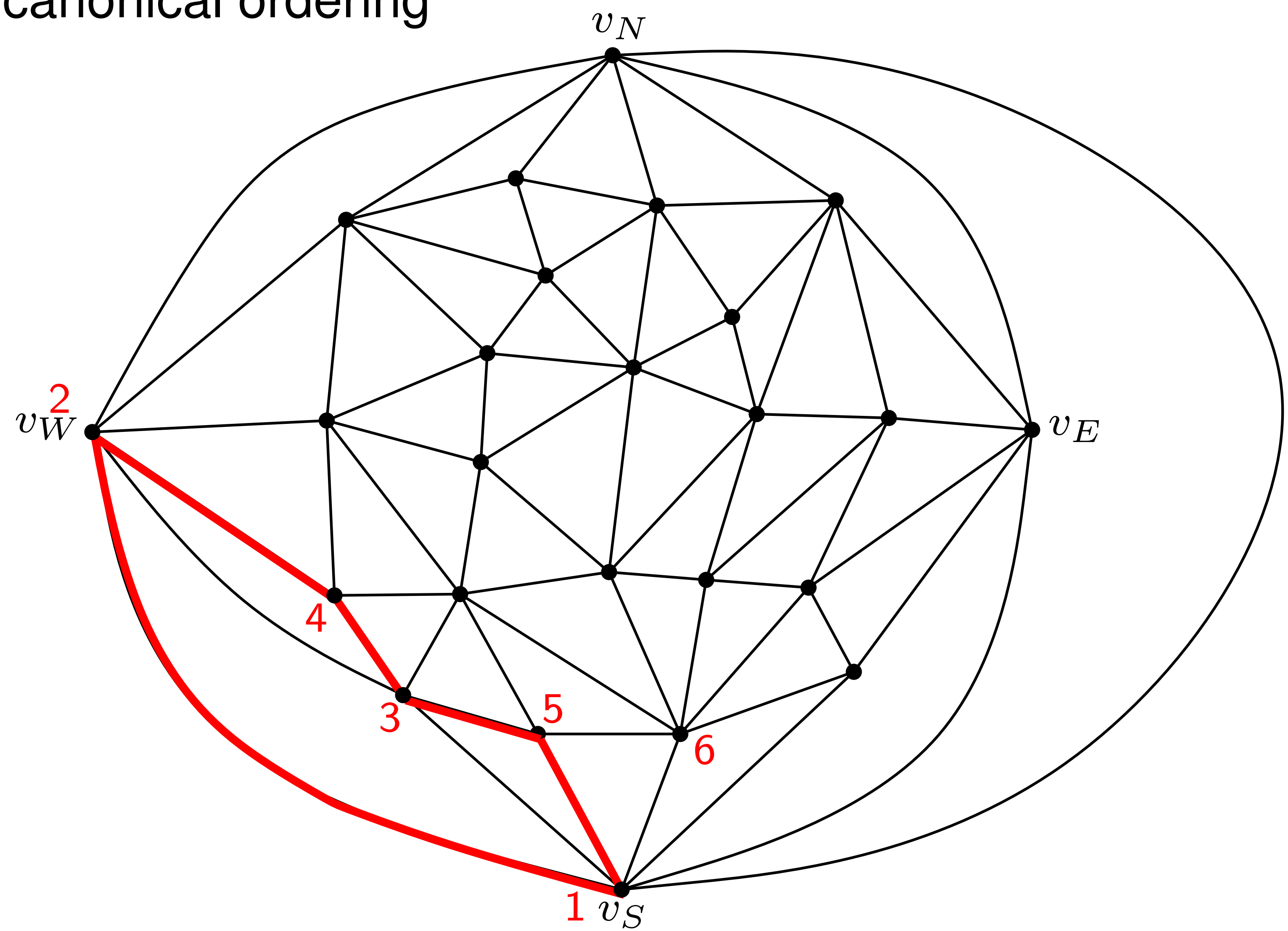

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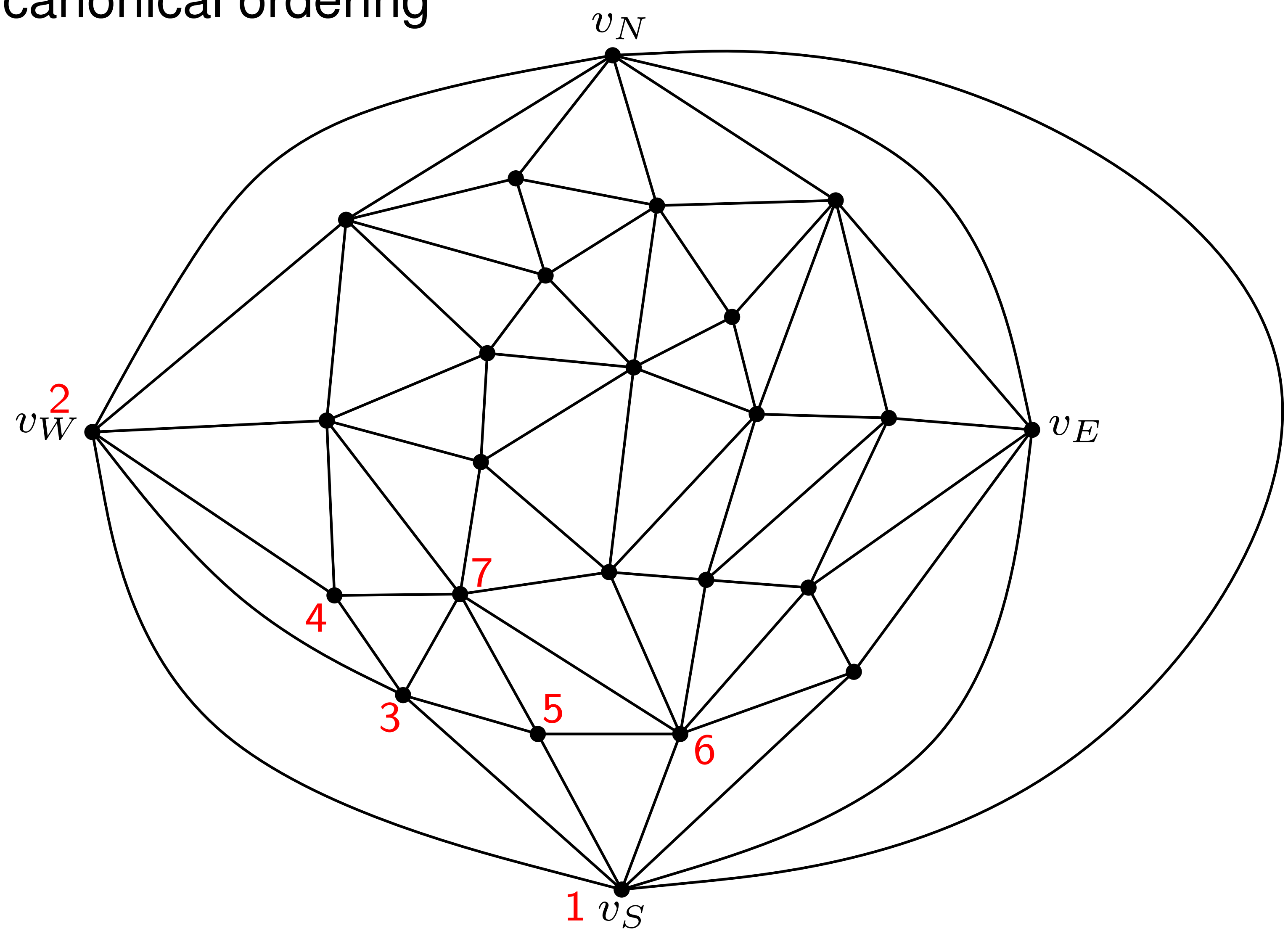

9 - 10

Rectangular Dual

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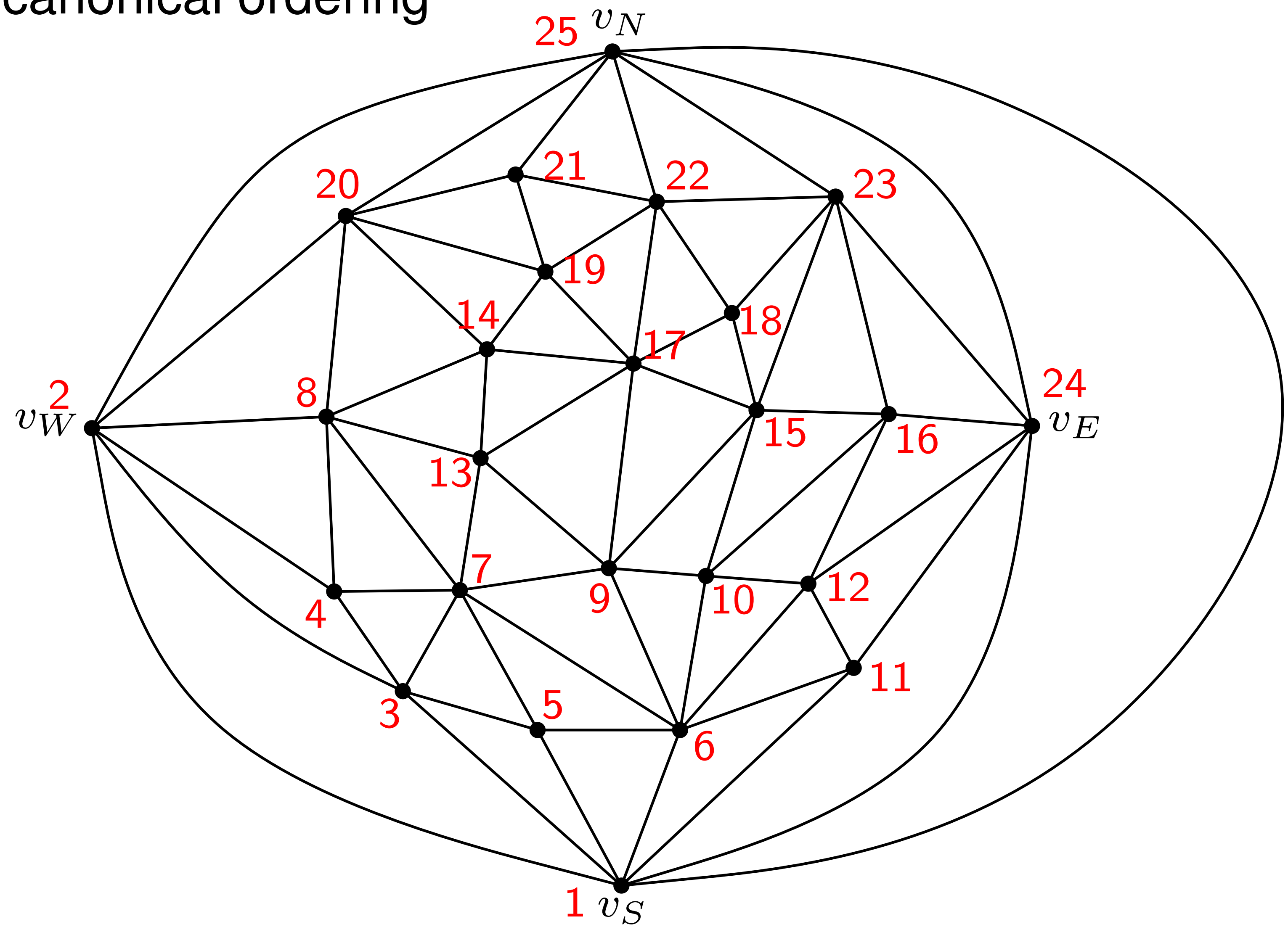
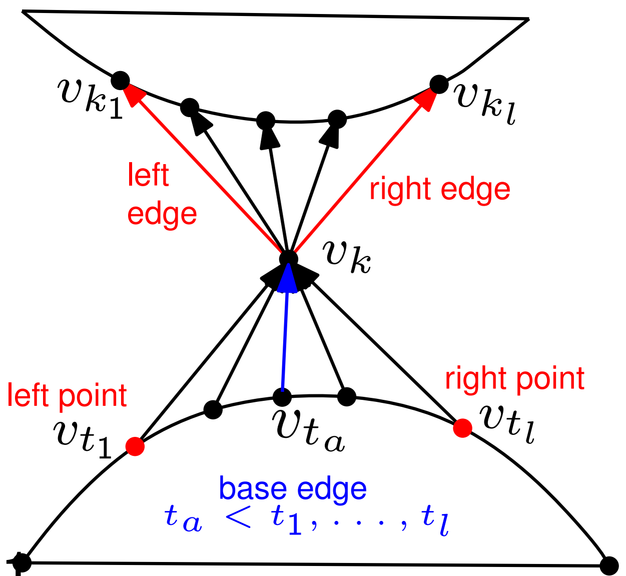


Rectangular Dual

Given a refined canonical ordering of G we construct a REL as follows:

- For each (v_i, v_j) orient it from v_i to v_j, for $i < j$;
- Base edge of v_k is (v_l, v_k), where $l < k$ is minimal.
- v_k has incoming edges from $v_{t_1}, \ldots, v_{t_l}$, we say that v_{t_1} is left point of v_k and v_{t_l} is right point of v_k.
- If $v_{k_1}, \ldots, v_{k_l}$ are higher numbered neighbors of v_k, we call (v_k, v_{k_1}) left edge and (v_k, v_{k_l}) right edge.



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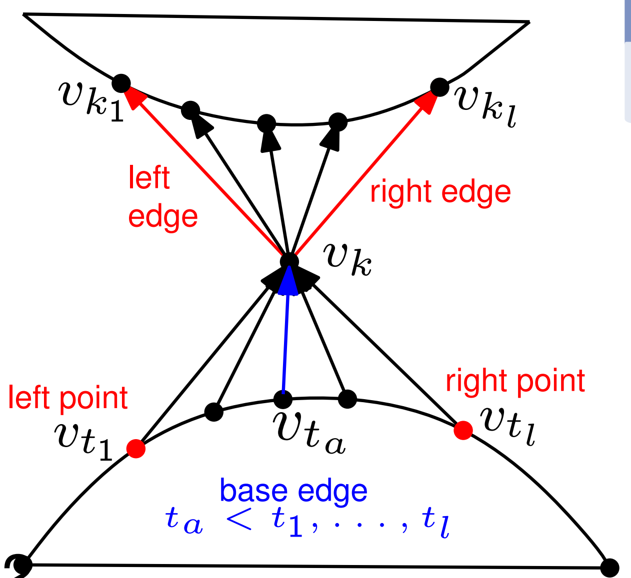

Lemma 1

Left edge or right edge can not be a base edge.

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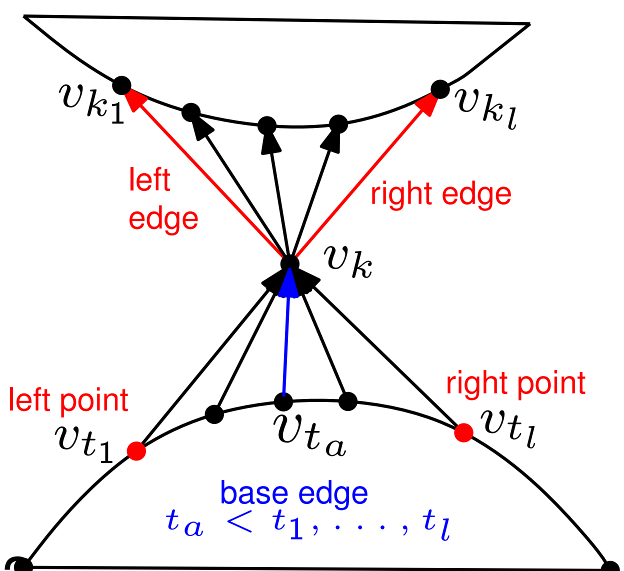

Lemma 1

Left edge or right edge can not be a base edge.

Proof: Assume that left edge (v_k, v_{k_1}) is the base edge of v_{k_1}.

Rectangular Dual

Given a refined canonical ordering of G we construct a REL as follows:

- For each (v_i, v_j) orient it from v_i to v_j, for $i < j$;
- Base edge of v_k is (v_l, v_k), where $l < k$ is minimal.
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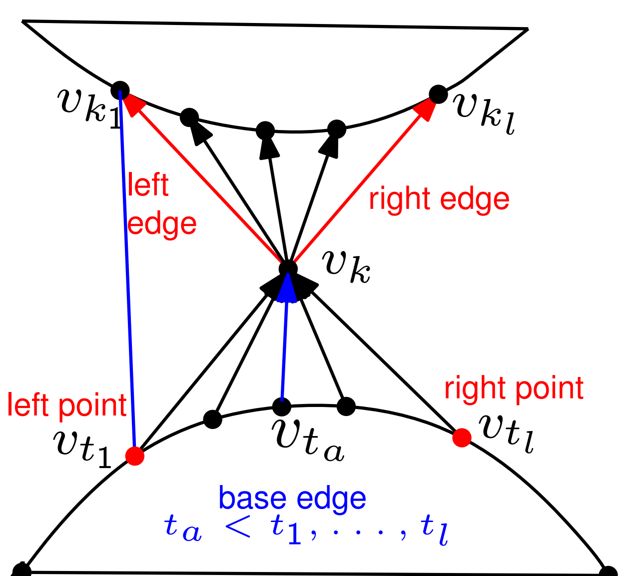

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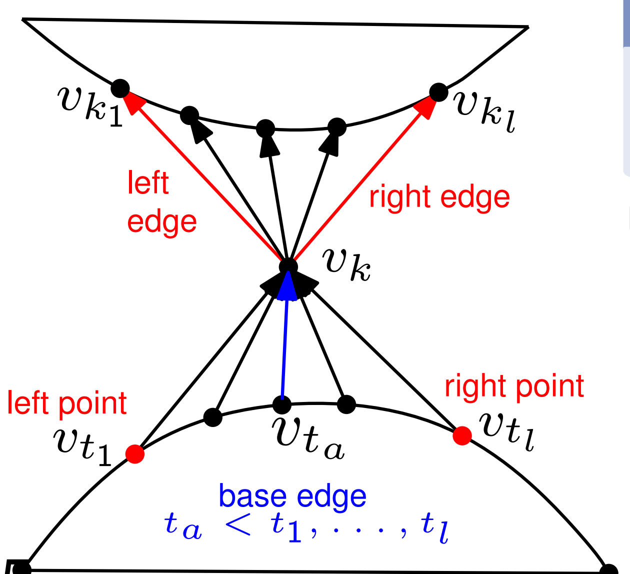

Lemma 2

An edge is either a left edge, a right edge or a base edge.

Proof:

- The exclusive "or" follows from Lemma 1.
- Let (v_{t_a}, v_k) be base edge of v_k.
- v_{t_a} is right point of $v_{t_{a-1}}$, $v_{t_{a-1}}$ is right point of $v_{t_{a-2}}$, generally $v_{t_{i+1}}$ is right point of v_{t_i}, $1 \leq i < a - 1$
- Edges (v_{t_i}, v_k), $1 \leq i < a - 1$, are right edges;
- Similarly we prove that edges (v_{t_i}, v_k), $a+1 \leq i < l$, are left edges;

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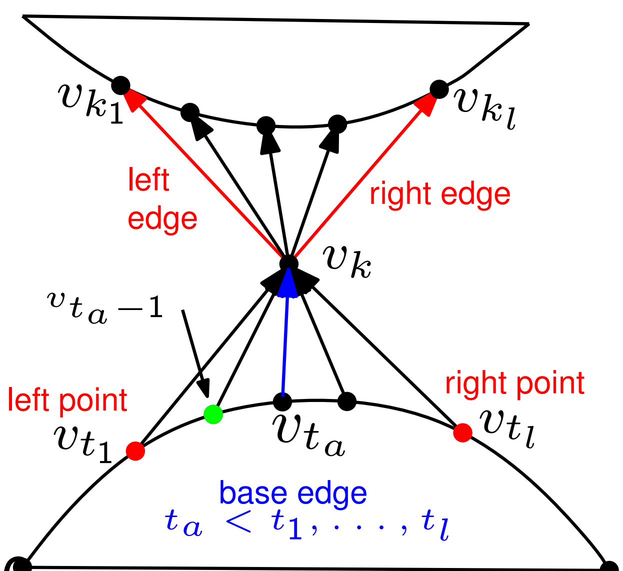

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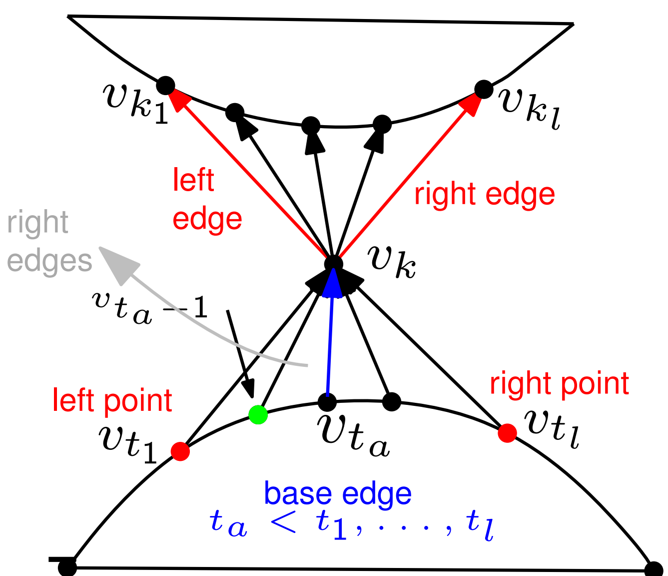

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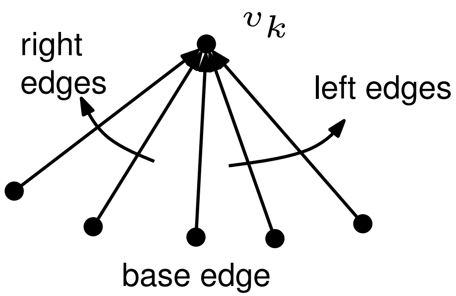
Rectangular Dual



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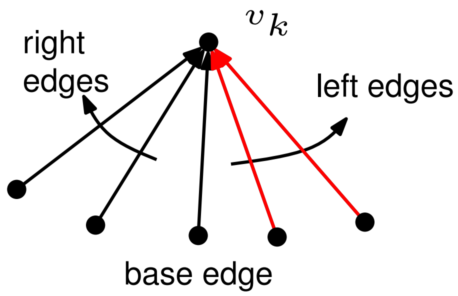
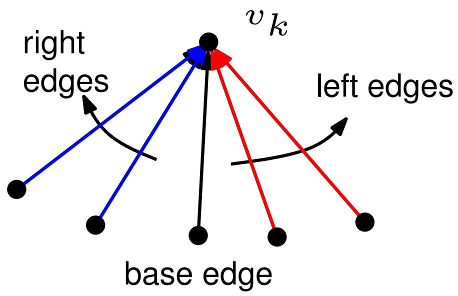


11 - 2

Rectangular Dual



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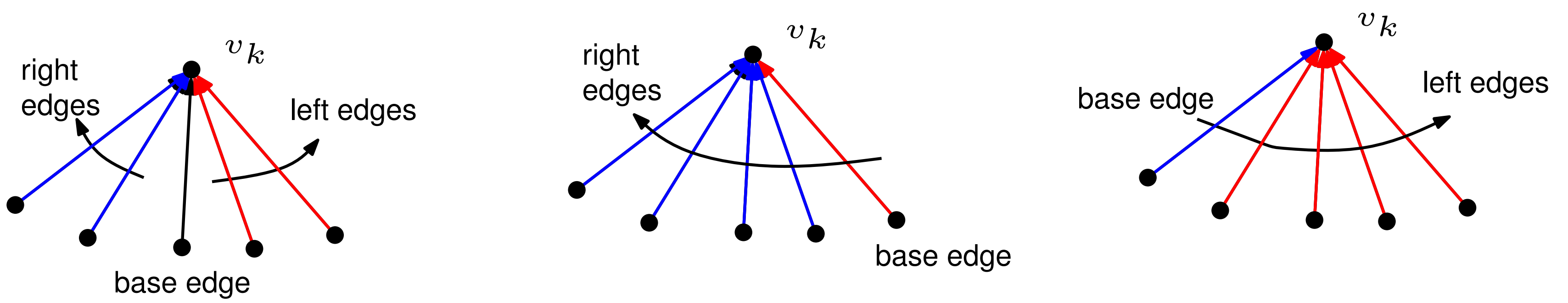

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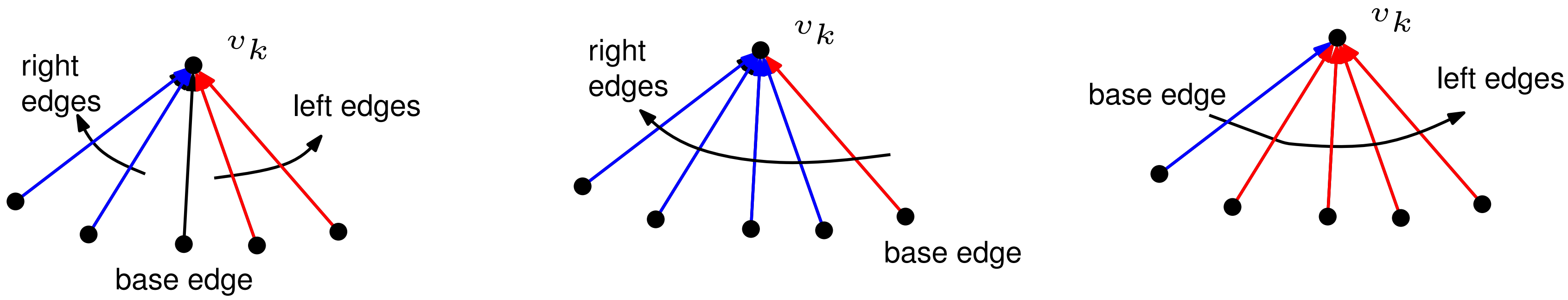
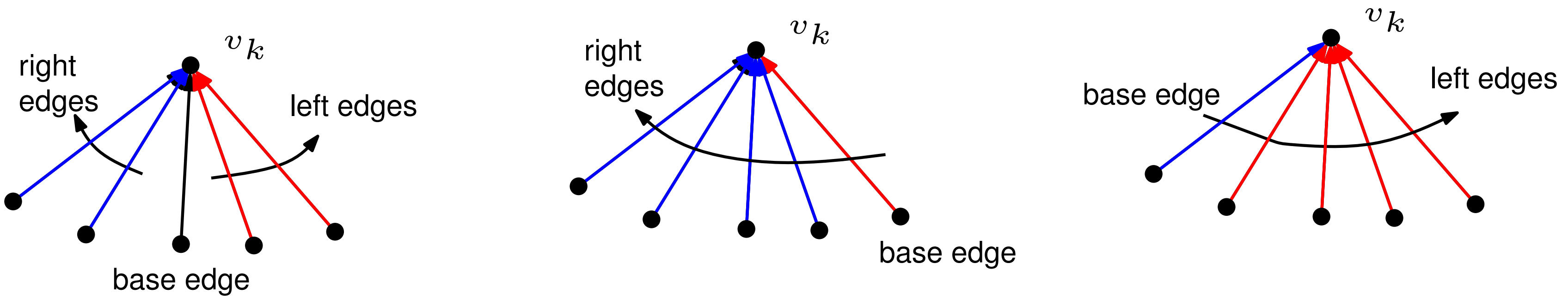


We call T_b blue edges and T_r red edges.

Rectangular Dual



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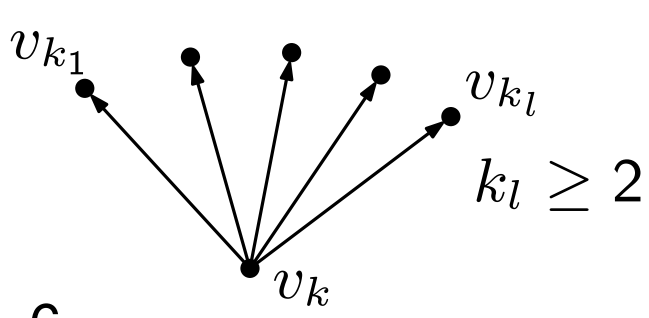
Proof:



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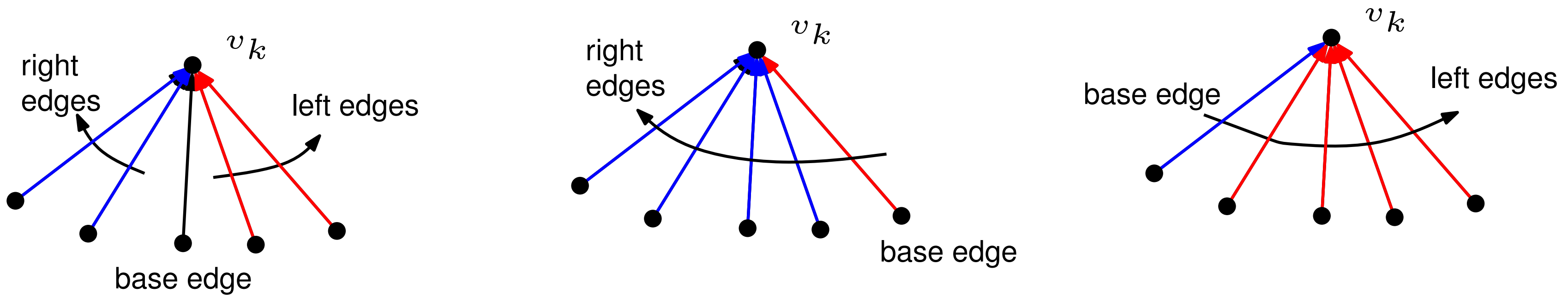

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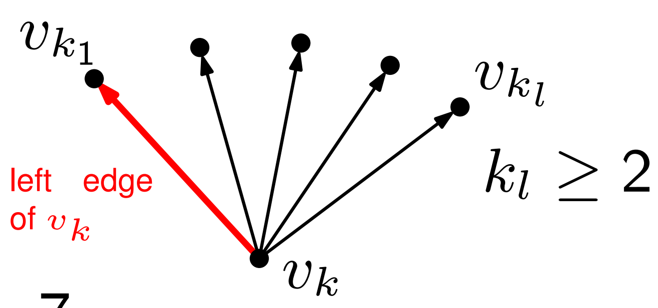

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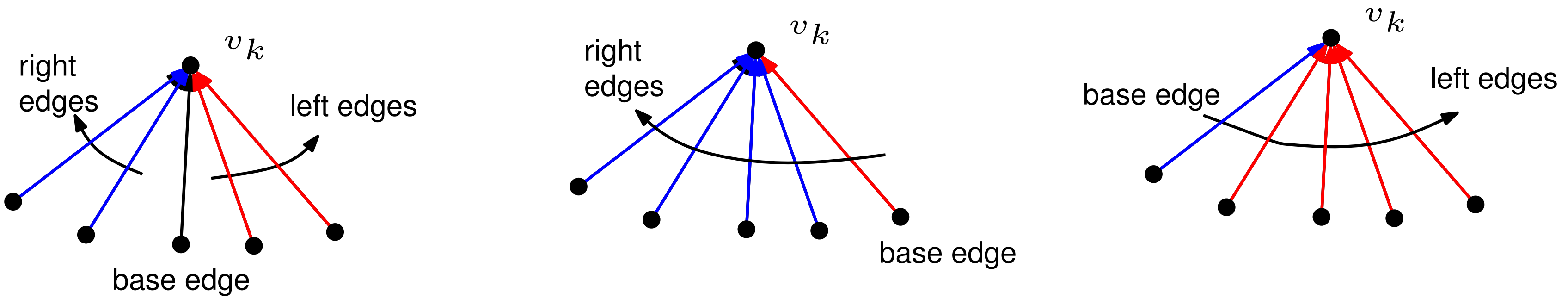

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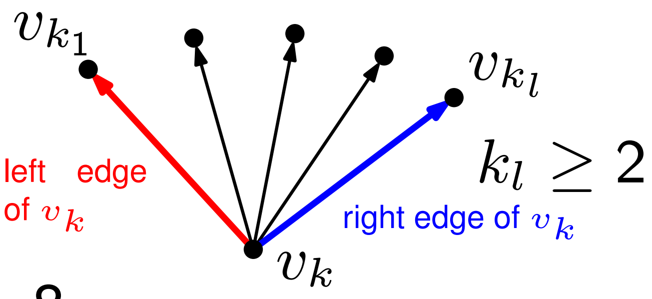

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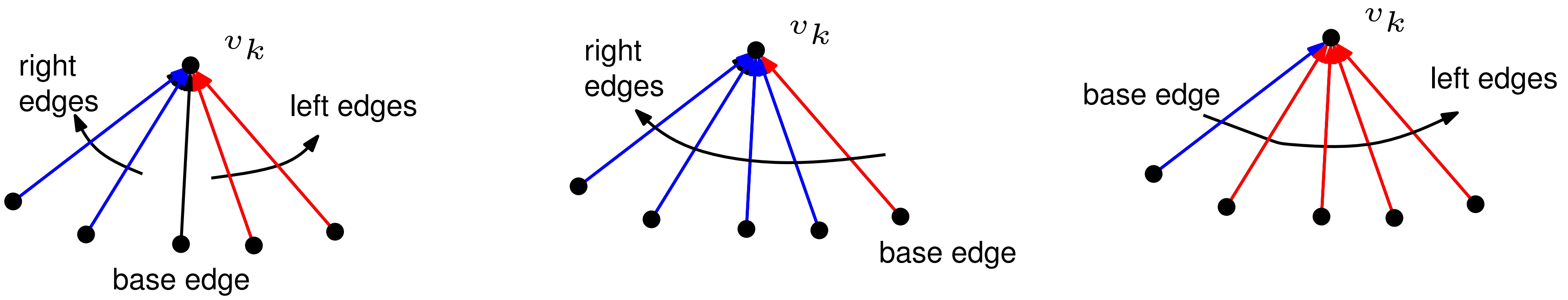

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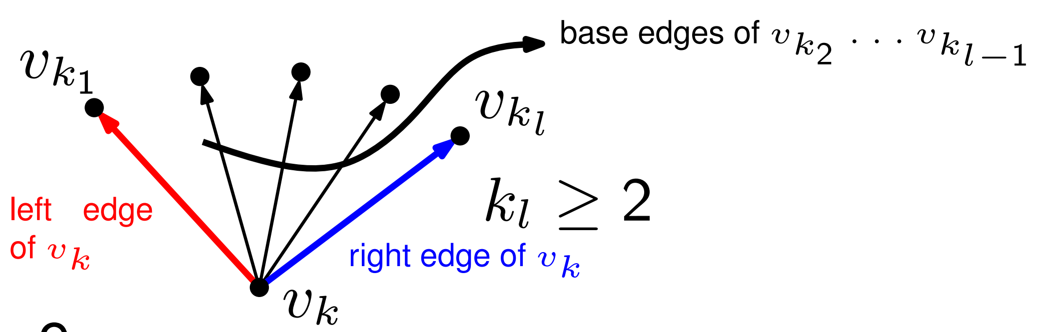

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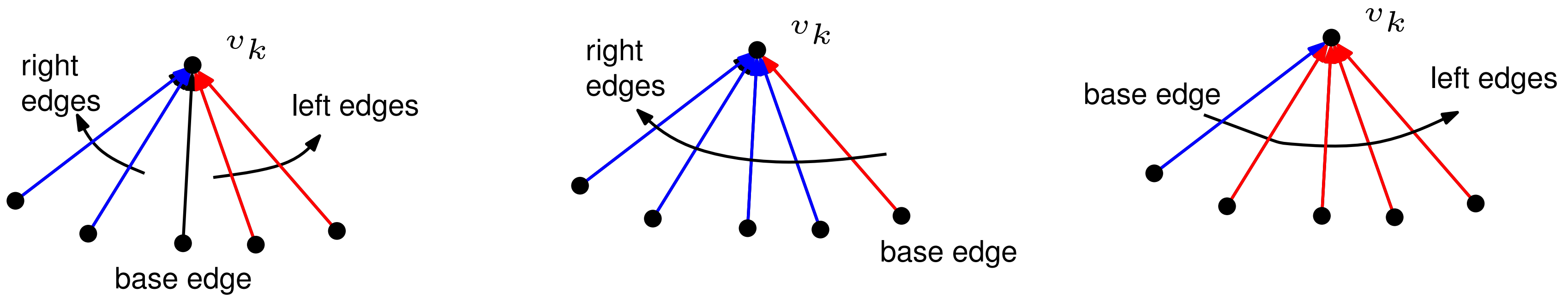

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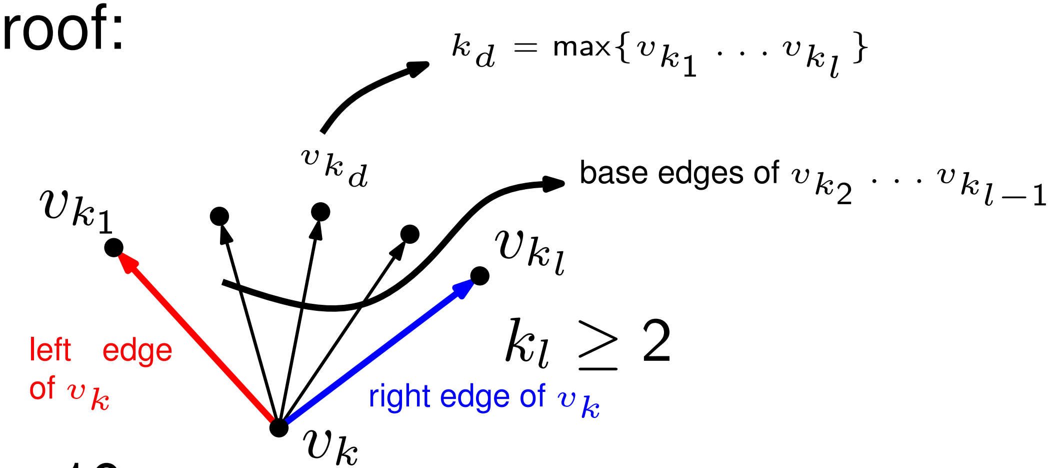

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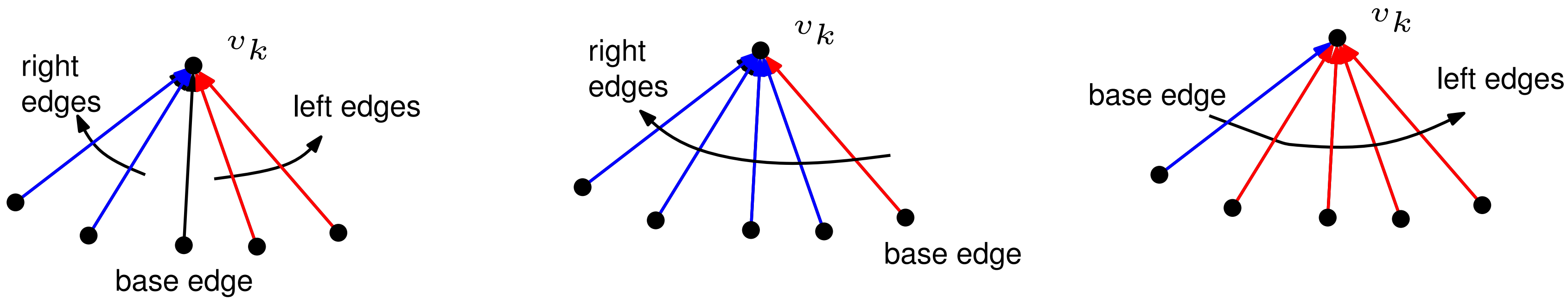

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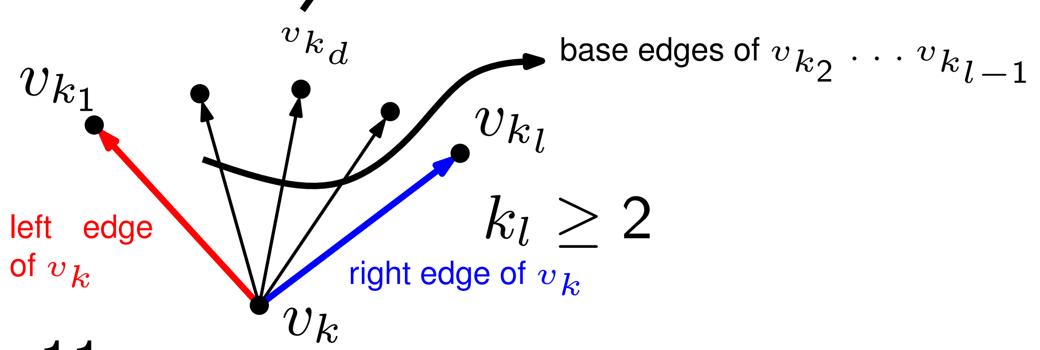
$k_d = \max\{v_{k_1} \dots v_{k_l}\}$ ⇒ $k_1 < k_2 < \dots < k_d$ and $k_d > k_{d+1} > \dots > k_l$



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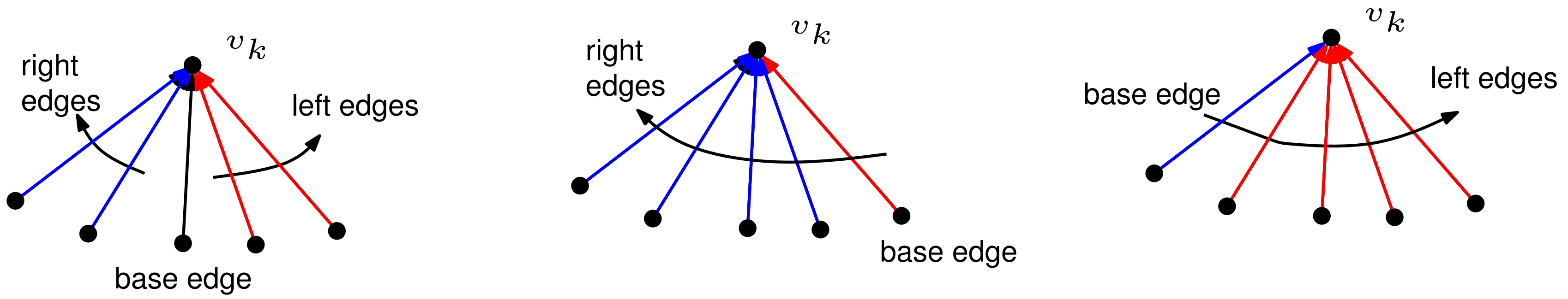

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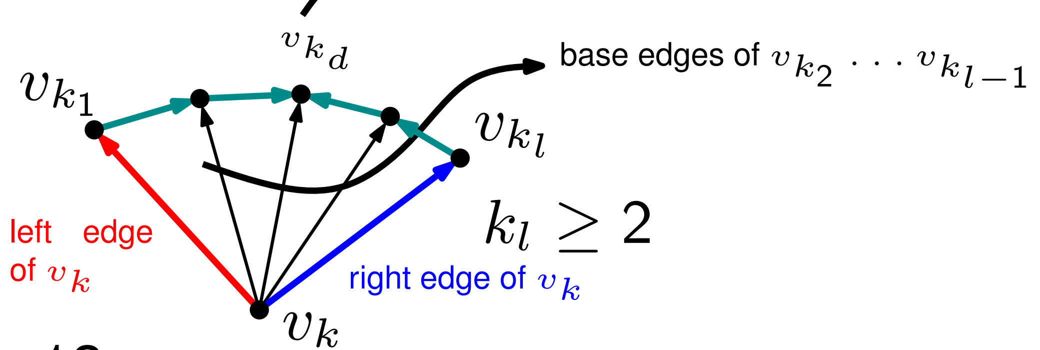

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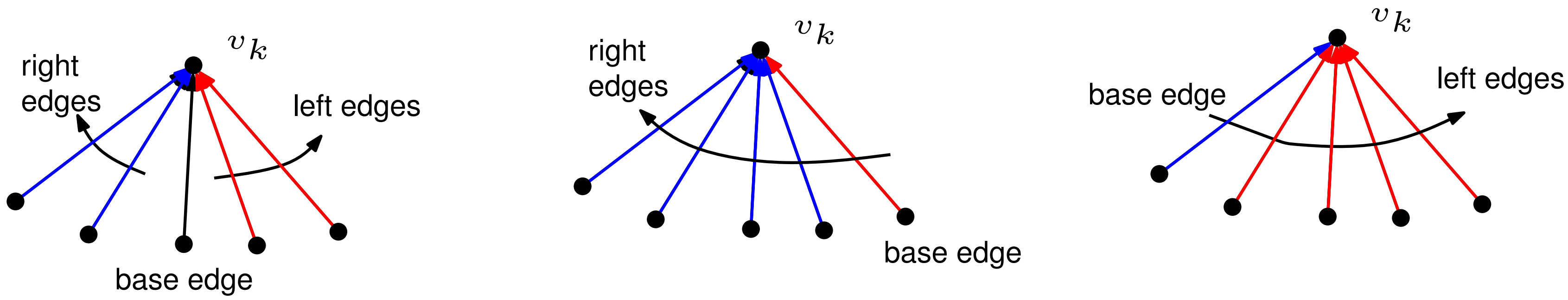

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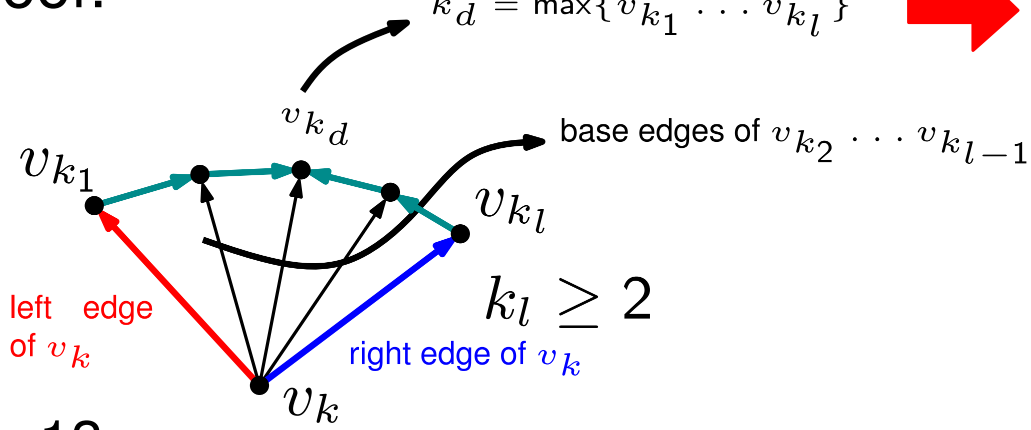

$k_1 < k_2 < \dots < k_d$ and
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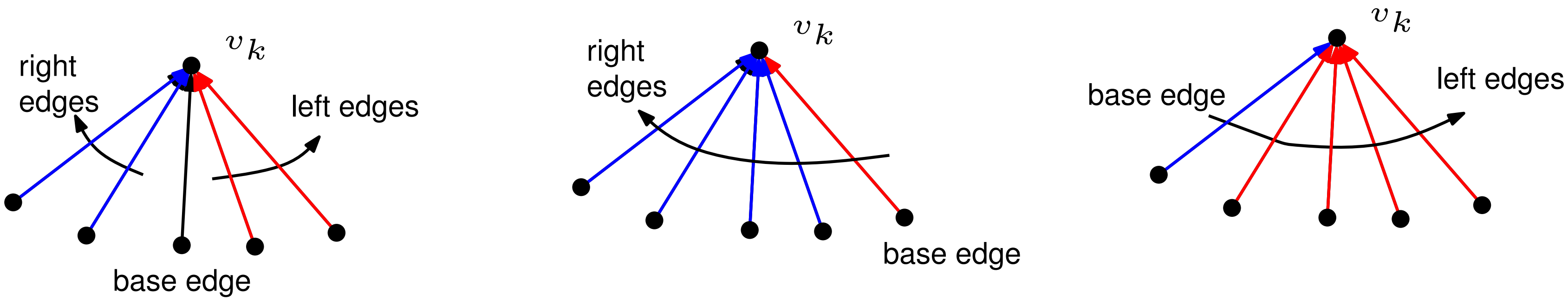
(v_k, v_{k_i}), $2 \leq i \leq d-1$ are red
(v_k, v_{k_i}), $d+1 \leq i \leq l-1$ are blue
edge (v_k, v_{k_d}) is either red or blue

Rectangular Dual



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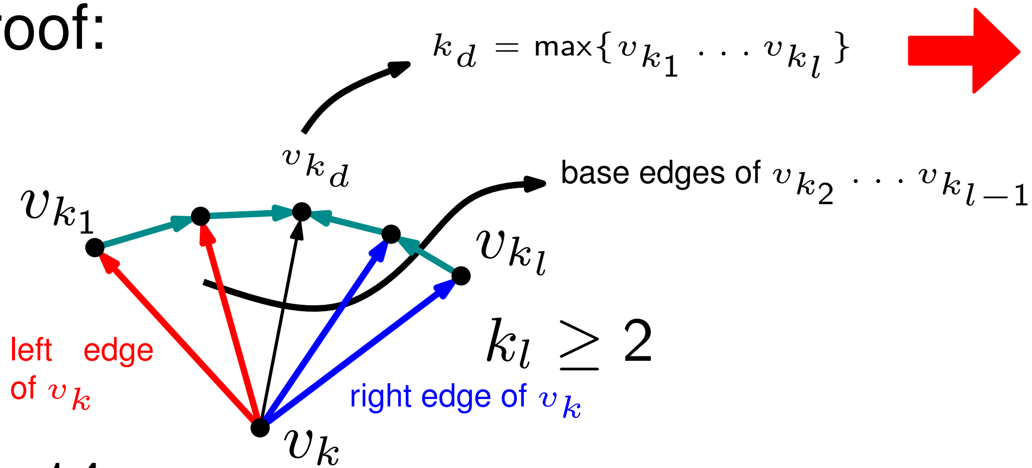

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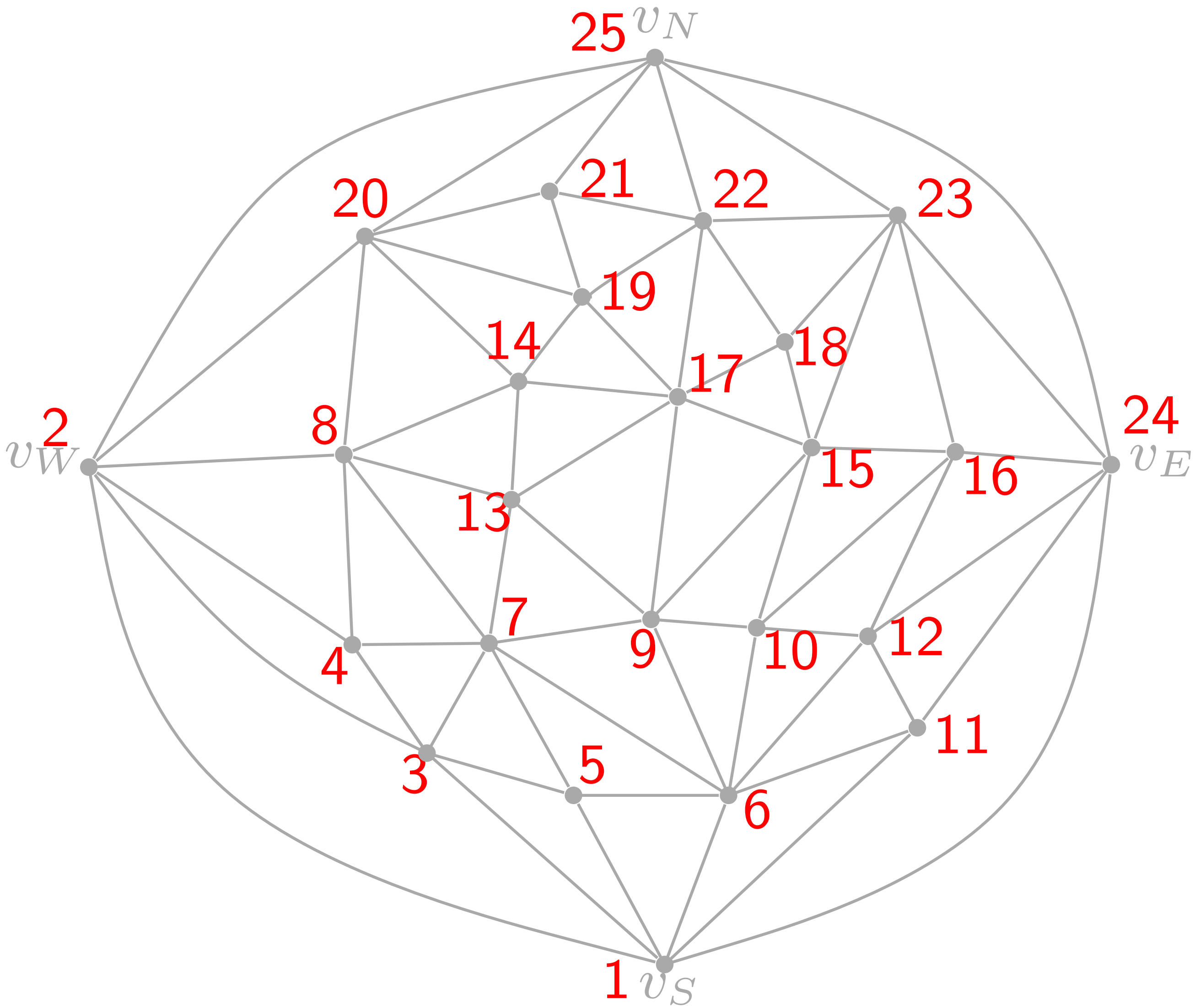

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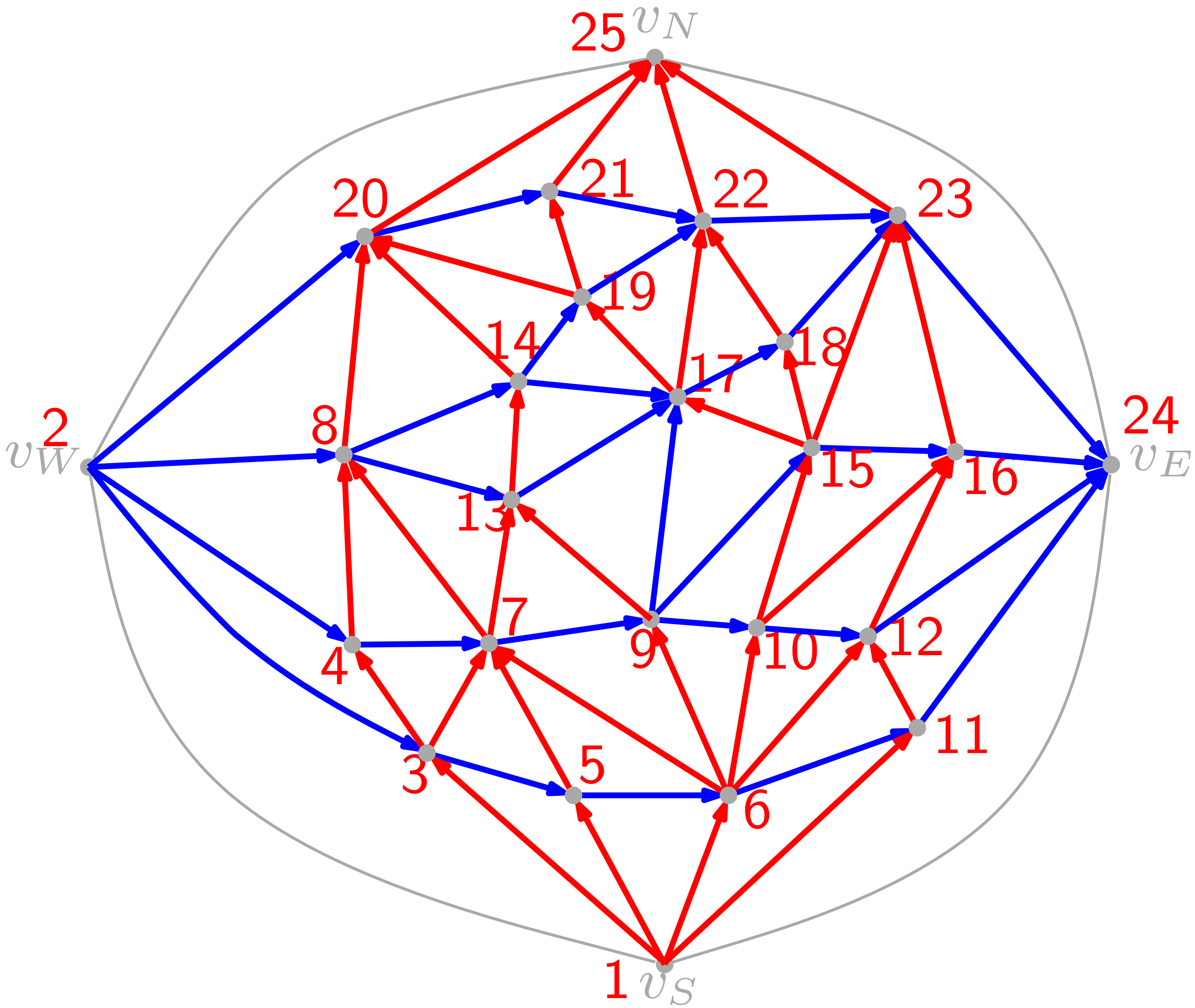
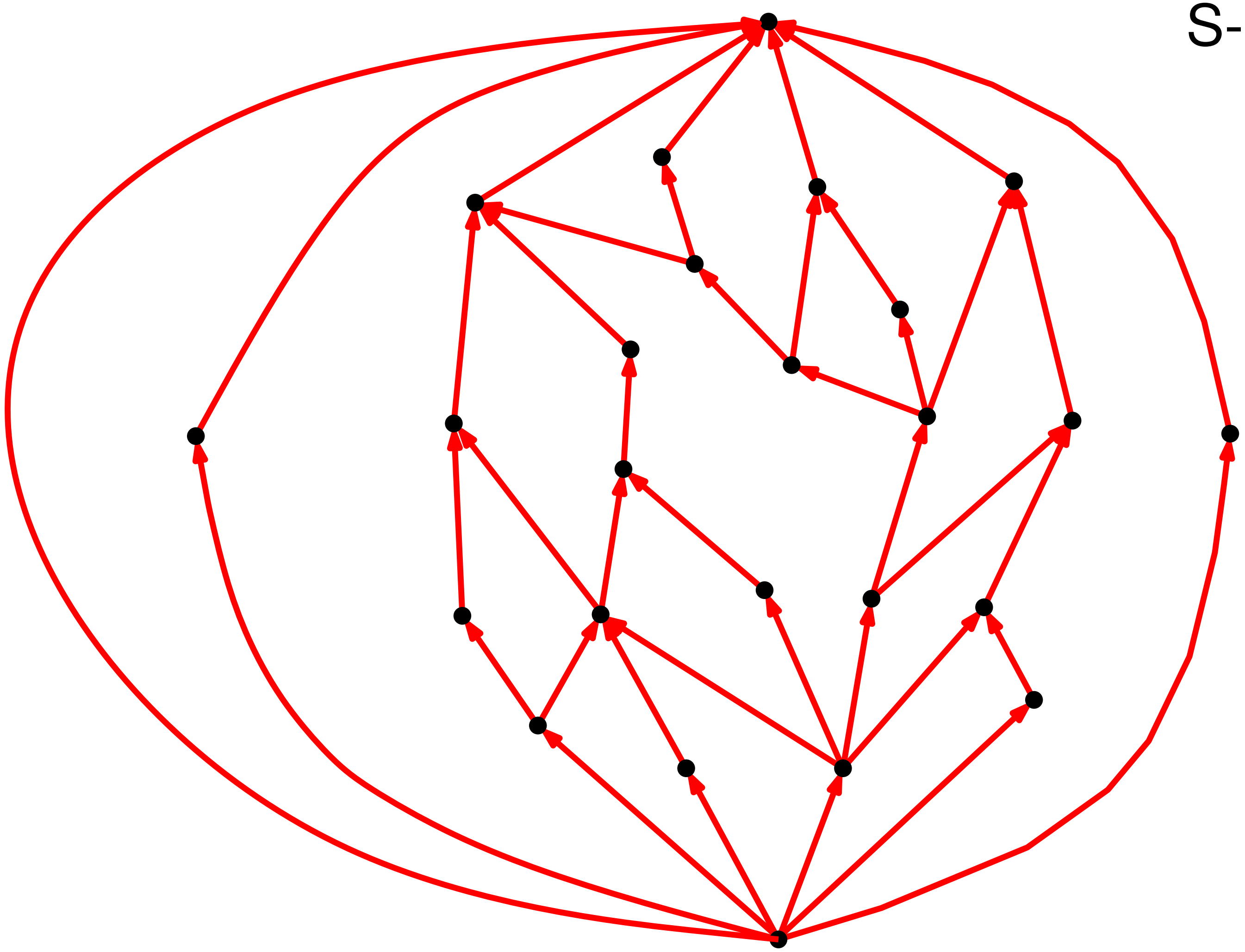


Rectangular Dual

S-N net G_{S-N}



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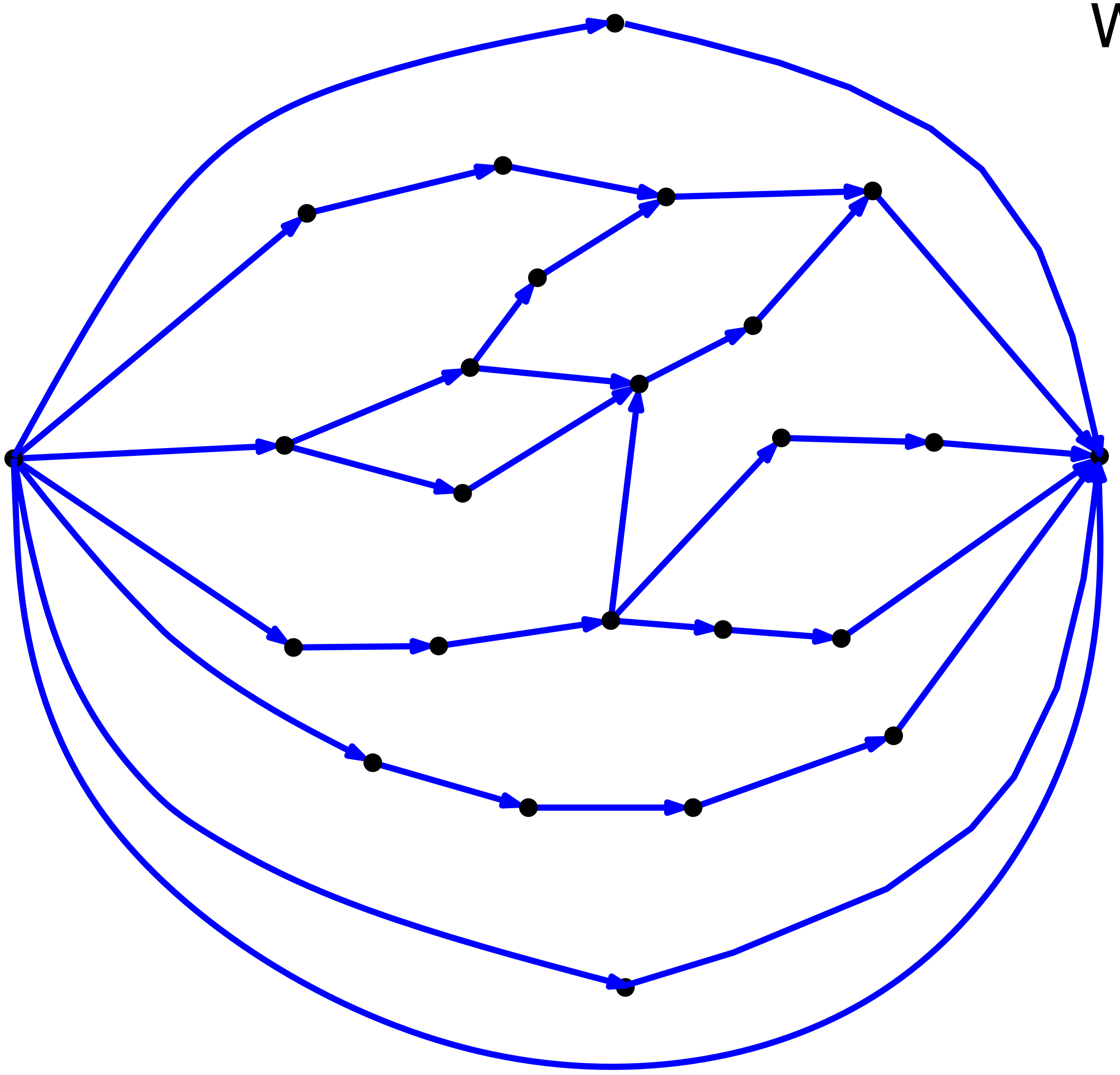
Rectangular Dual

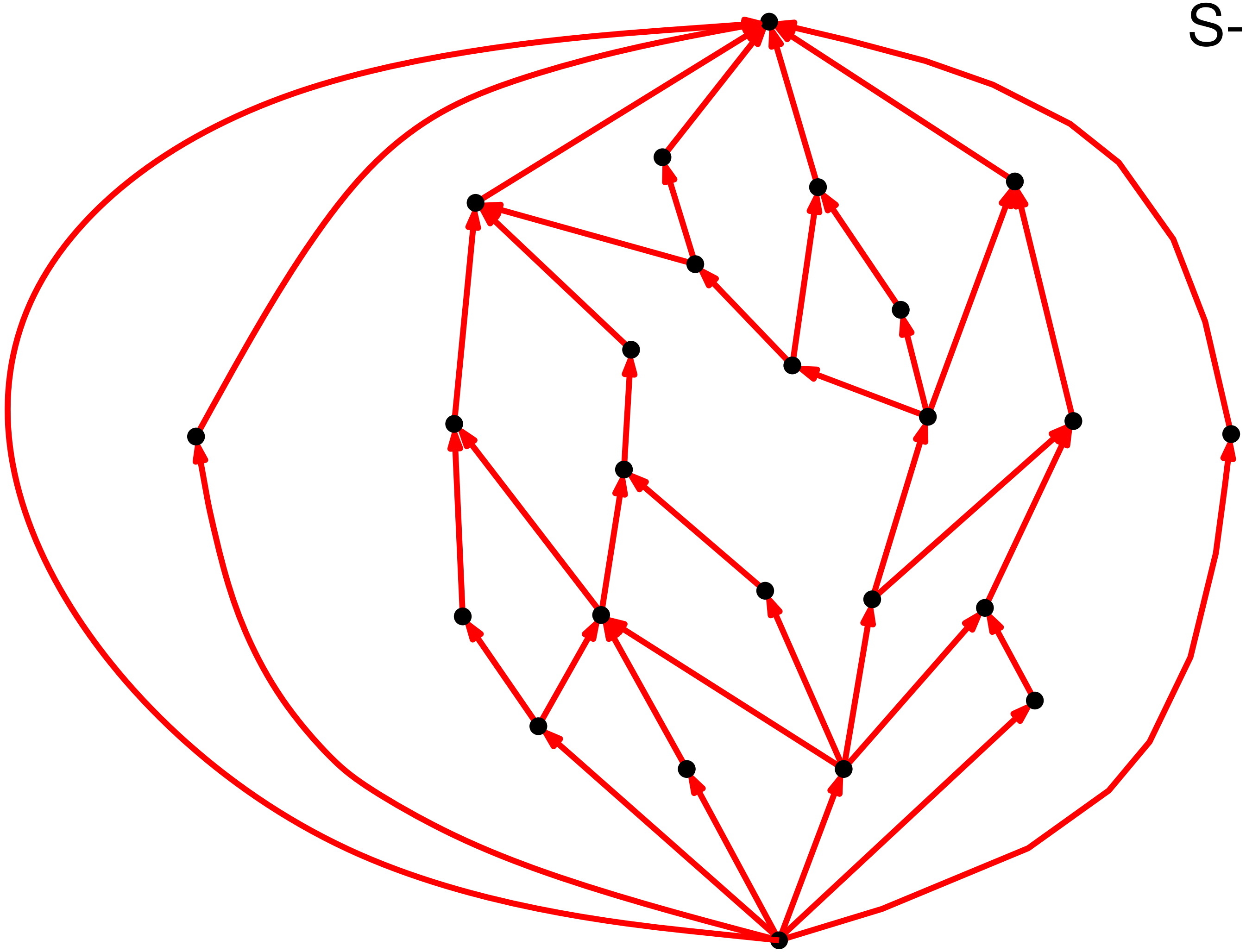
W-E net G_{W-E}



12 - 4

Rectangular Dual

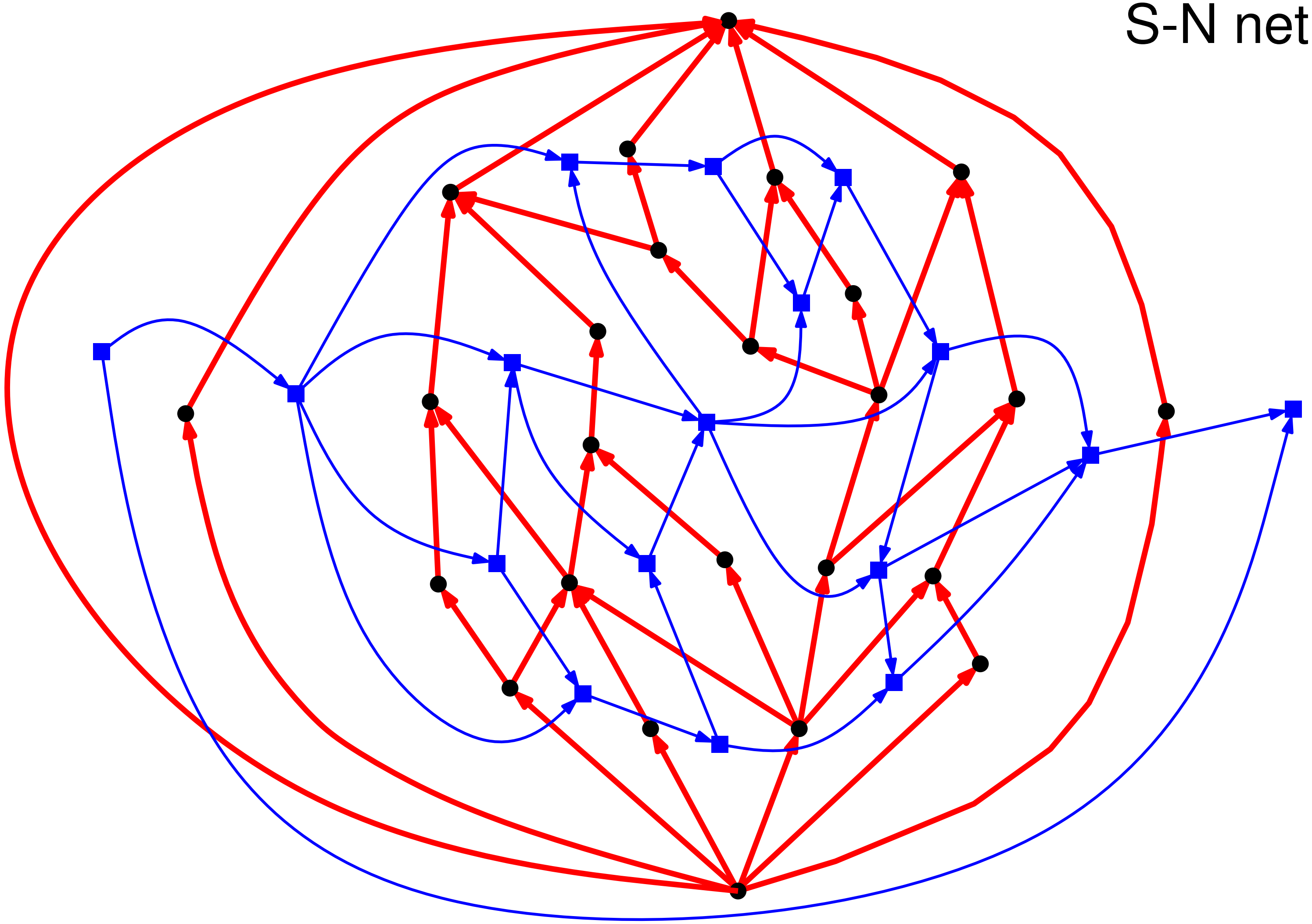
S-N net G_{S-N}



12 - 5

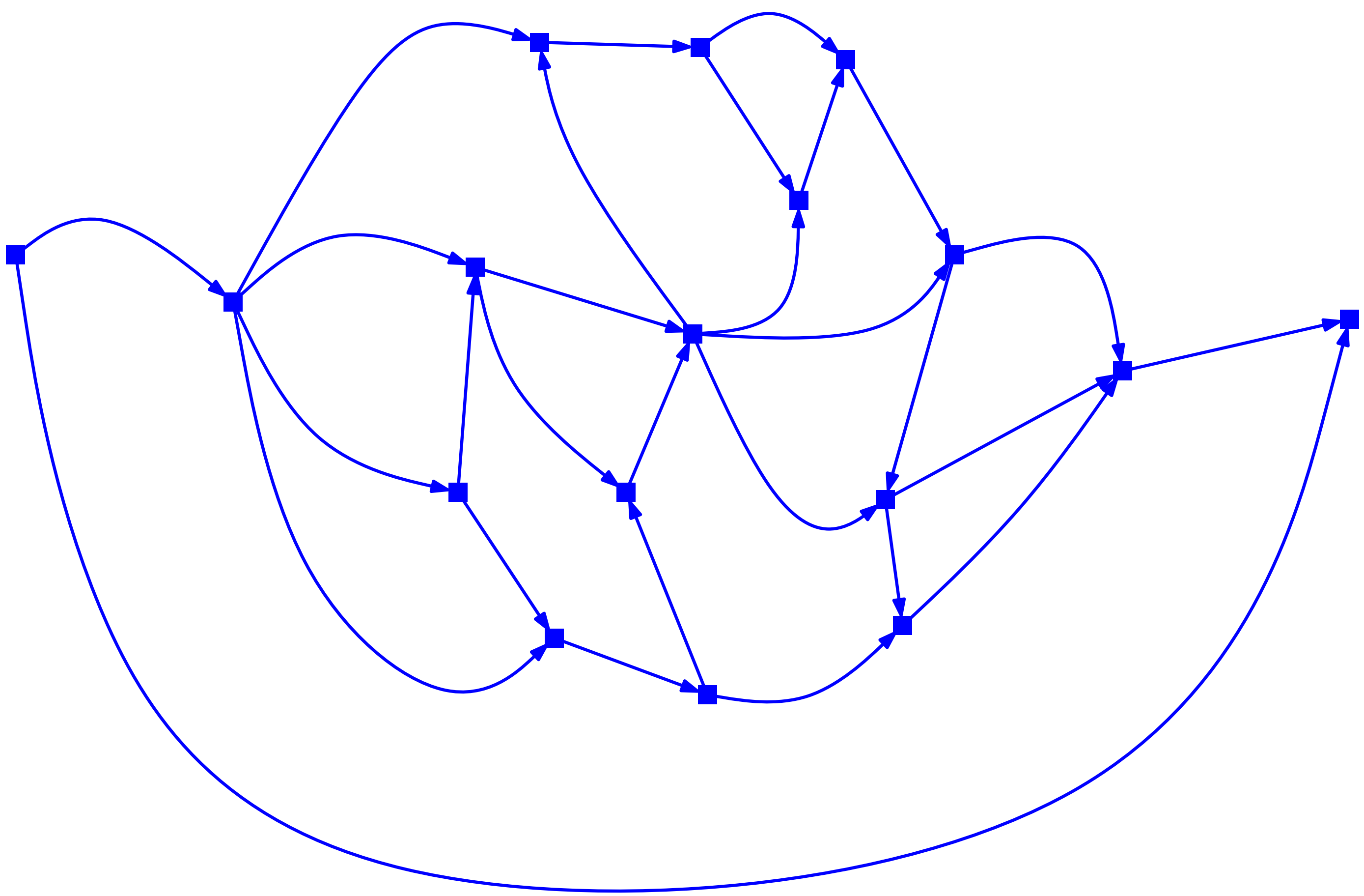
Rectangular Dual

S-N net G_{S-N}



12 - 6

Rectangular Dual



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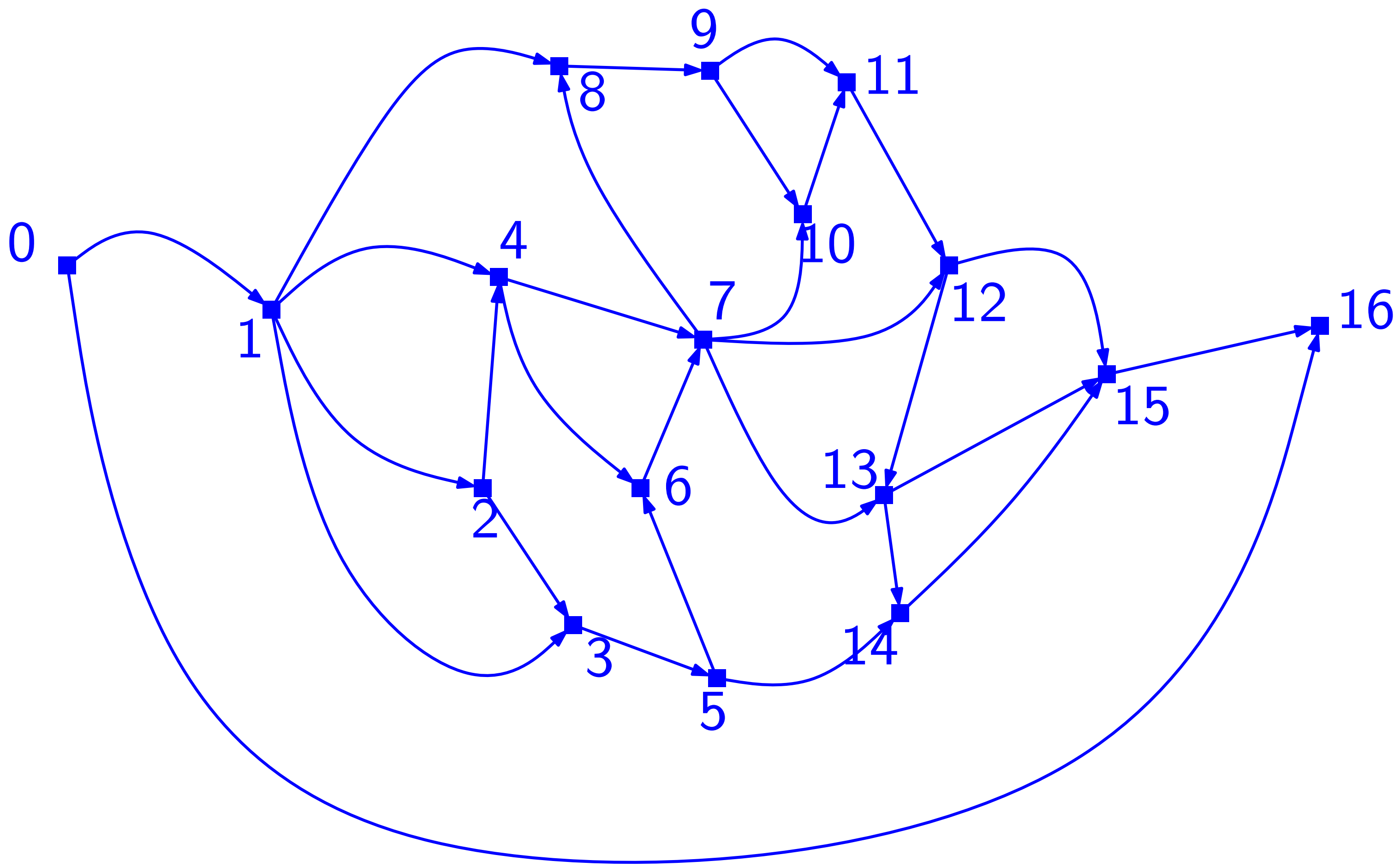

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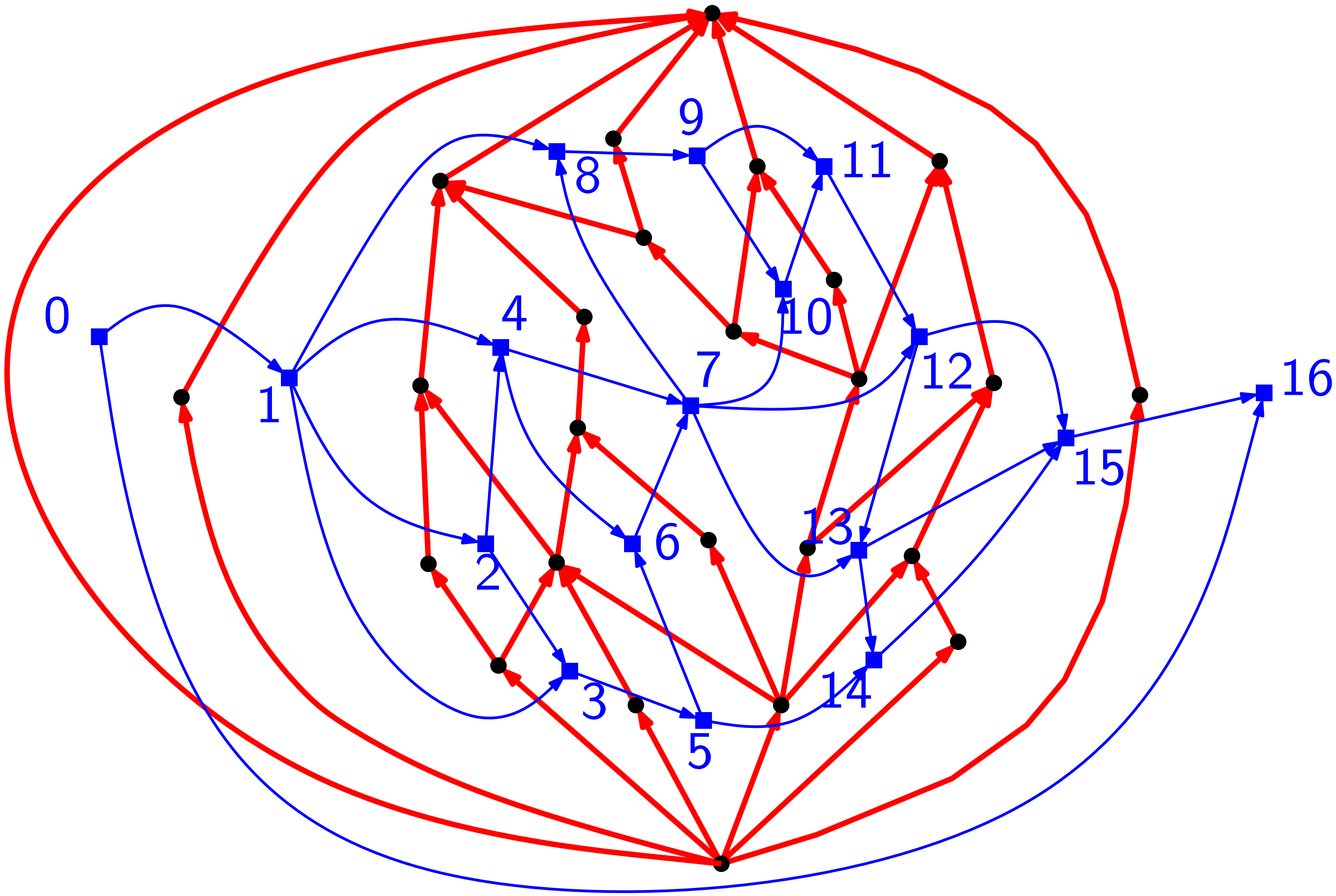

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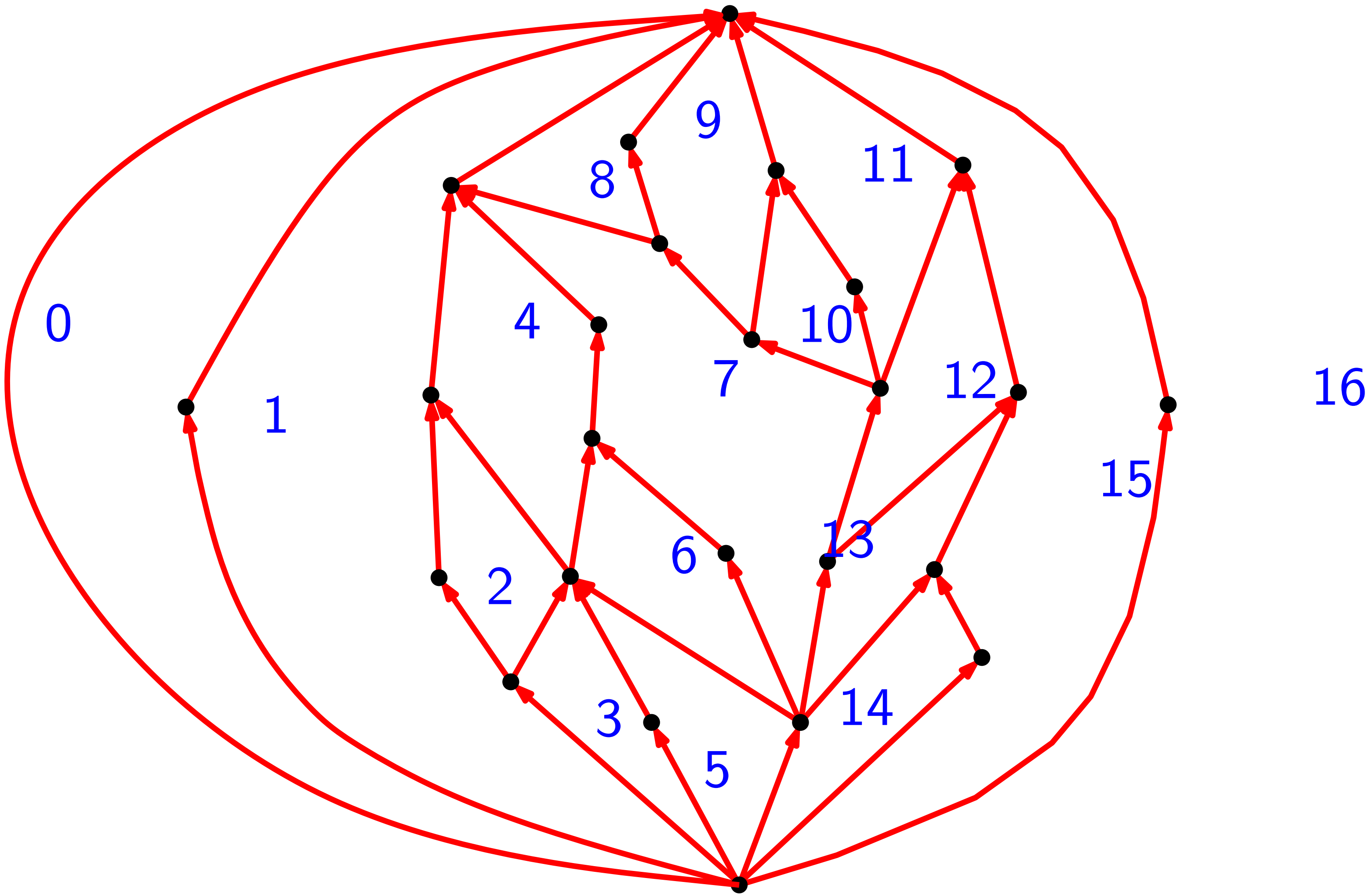
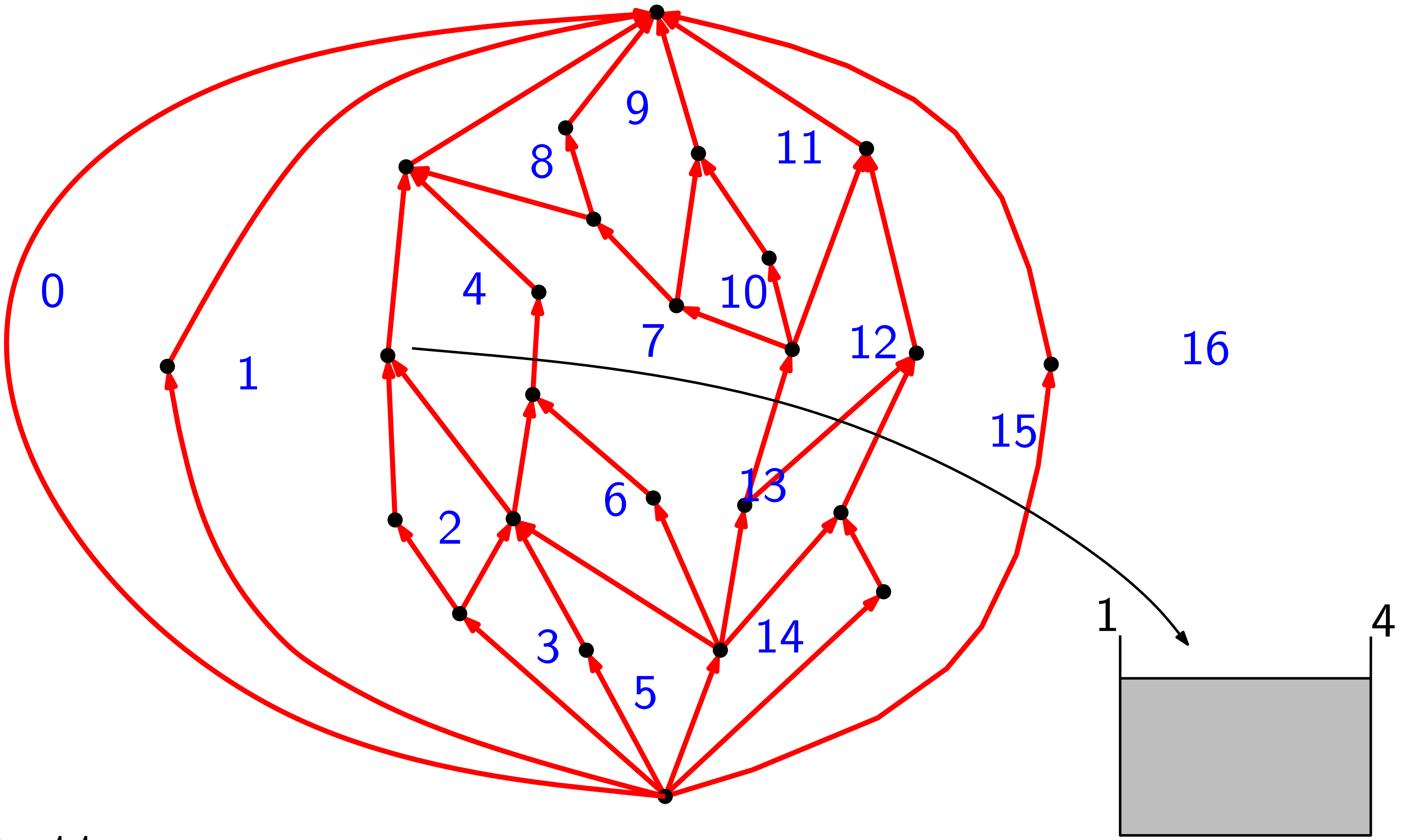
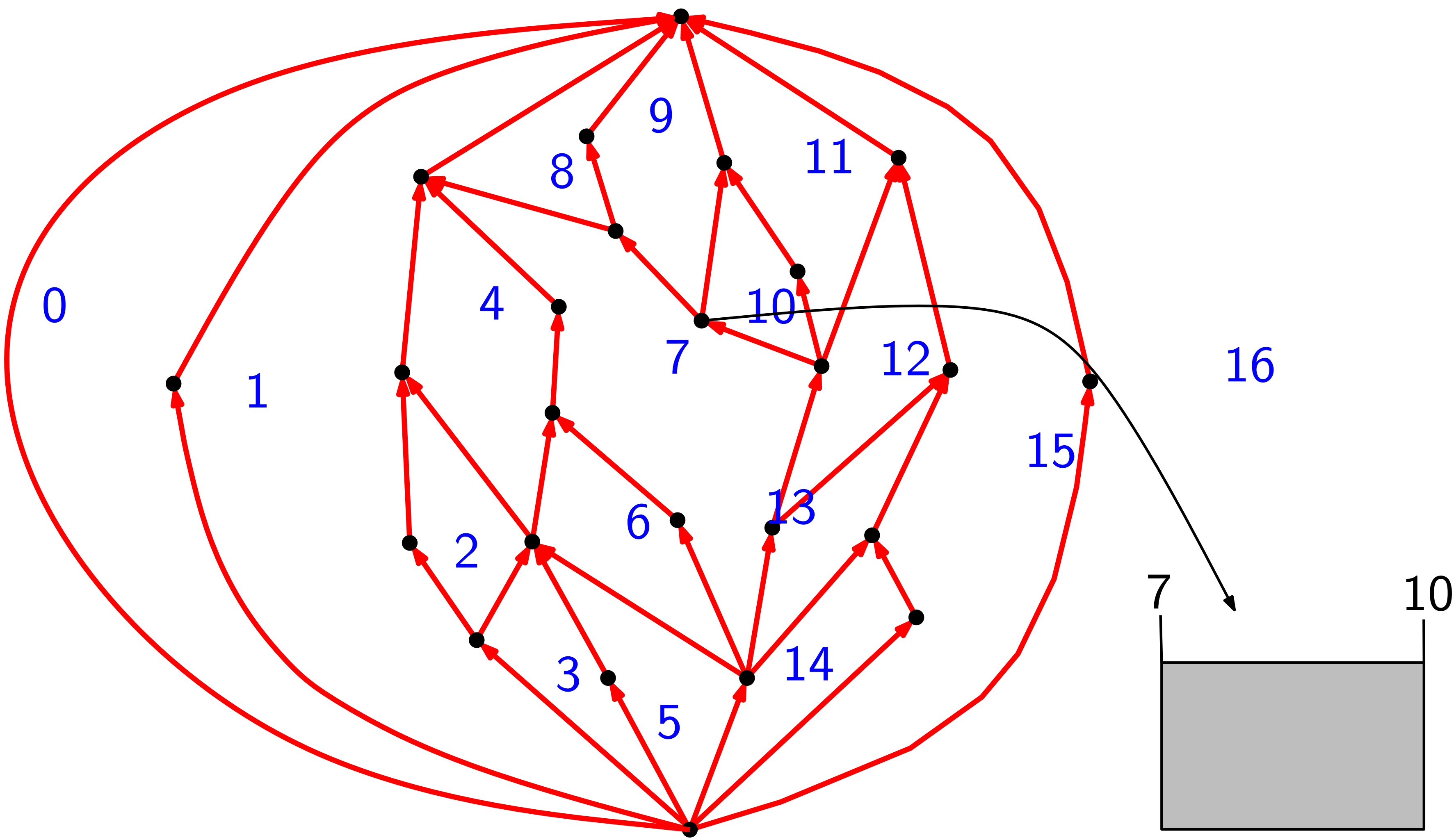


12 - 10

Rectangular Dual



Rectangular Dual



Rectangular Dual

Algorithm Rectangular dual
Input: A PTP graph $G = (V, E)$

- Find a REL T_r, T_b of G;
- Construct a S-N net G_{S-N} of G (consists of T_r plus outer edges)
- Construct the dual $G^{\star}_{S-N}$ of G_{S-N} and compute a topological ordering f_{sn} of $G^{\star}_{S-N}$
- For each vertex $v \in V$, let f and g be the face on the left and face on the right of v. Set $x_1(v) = f_{sn}(f)$ and $x_2(v) = f_{sn}(g)$.
- Define $x_1(v_N) = x_1(v_S) = 1$ and $x_2(v_N) = x_2(v_S) = \max f_{sn} - 1$

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Lehrstuhl Algorithmik I

Rectangular Dual

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Input: A PTP graph $G = (V, E)$

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Rectangular Dual

Algorithm Rectangular dual
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Karlsruhe Institute of Technology

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- Define $y_1(v_W) = y_1(s_E) = 0$ and $y_1(v_W) = y_1(s_E) = \max f_{we}$

Rectangular Dual

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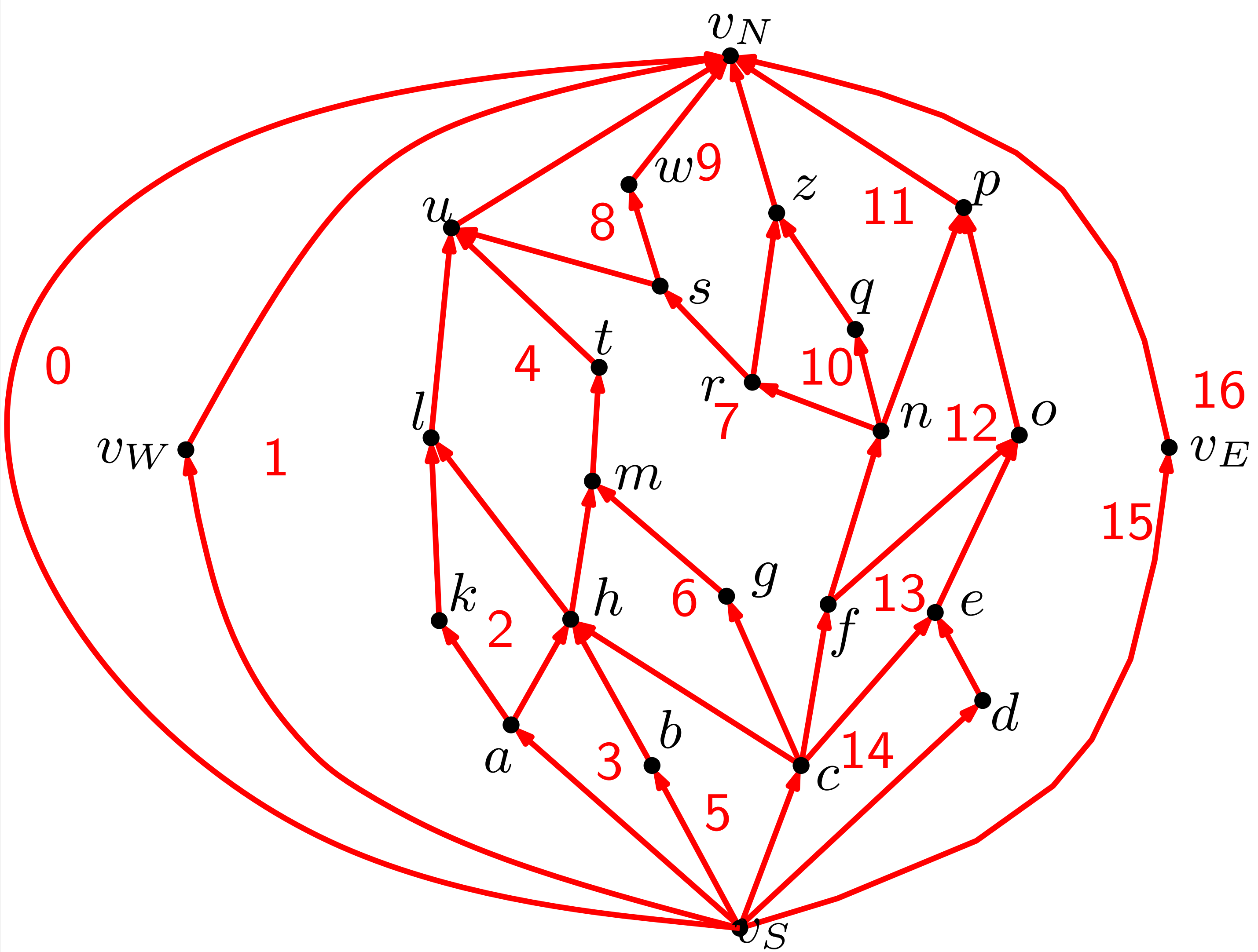
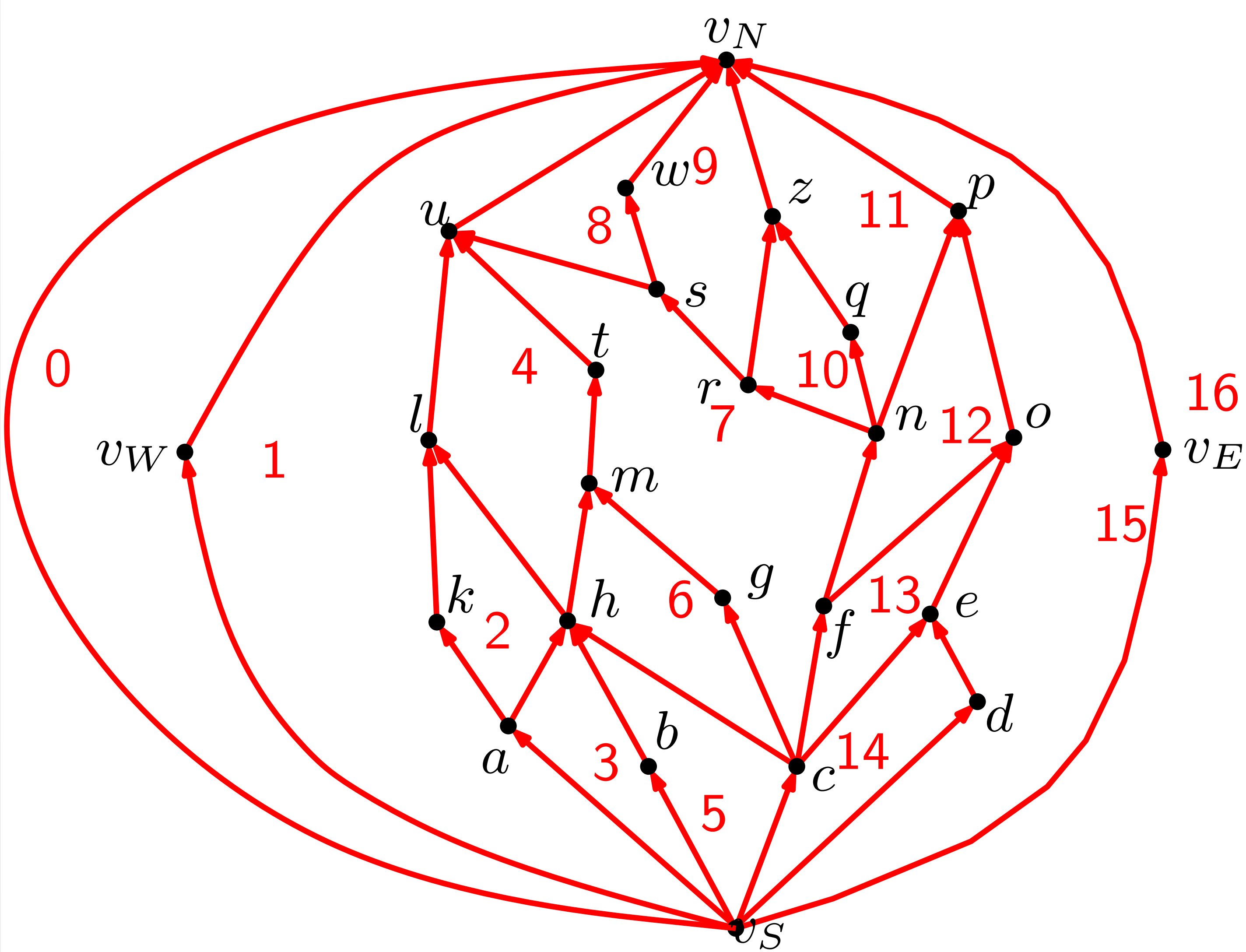
- Define $y_1(v_W) = y_1(s_E) = 0$ and $y_1(v_W) = y_1(s_E) = \max f_{we}$
- For each $v \in V$, assign a rectangle $R(v)$ bounded by x-coordinates $x_1(v),\ x_2(v)$ and y-coordinates $y_1(v),\ y_2(v)$.

Rectangular Dual



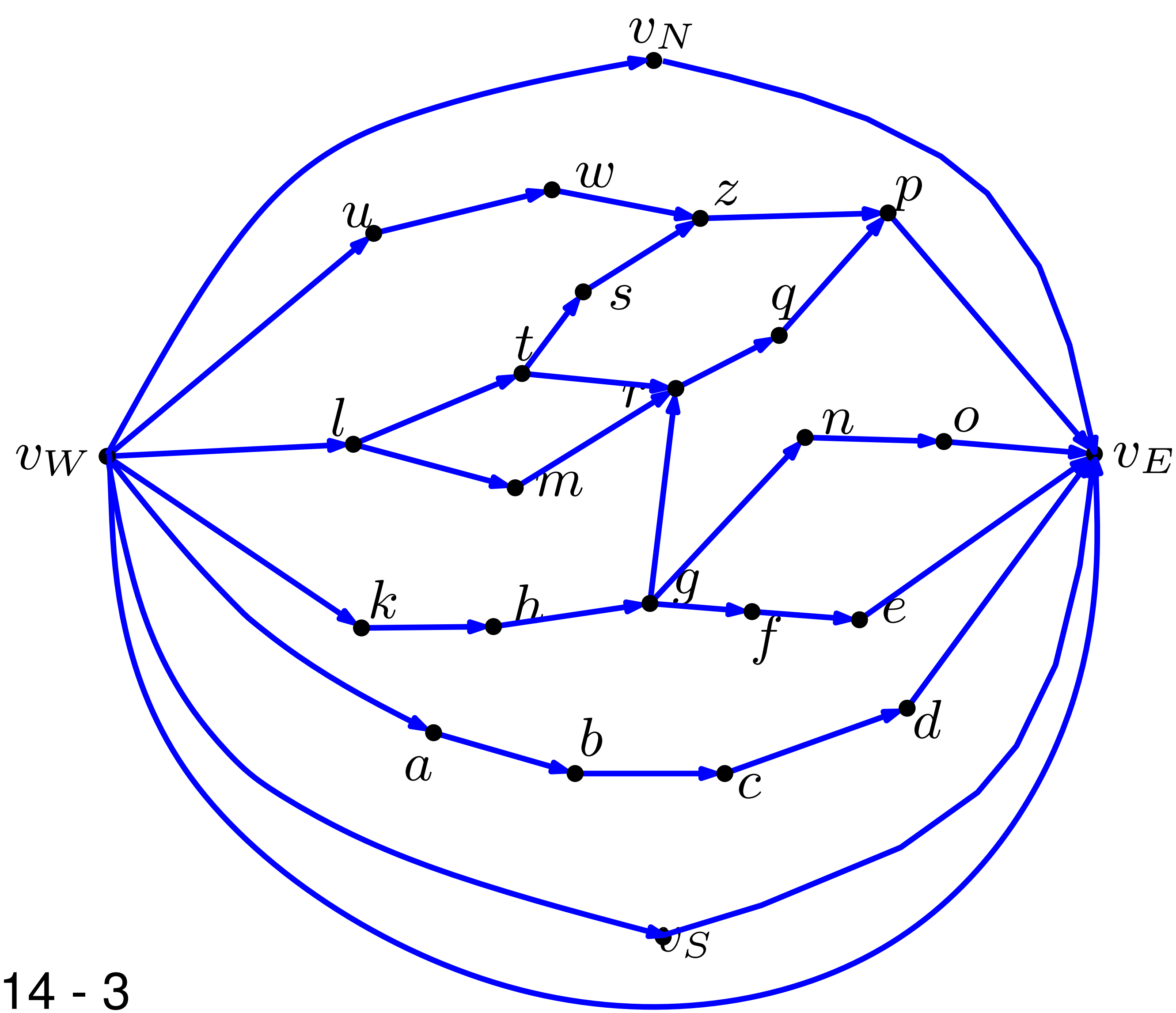
14 - 1

Rectangular Dual



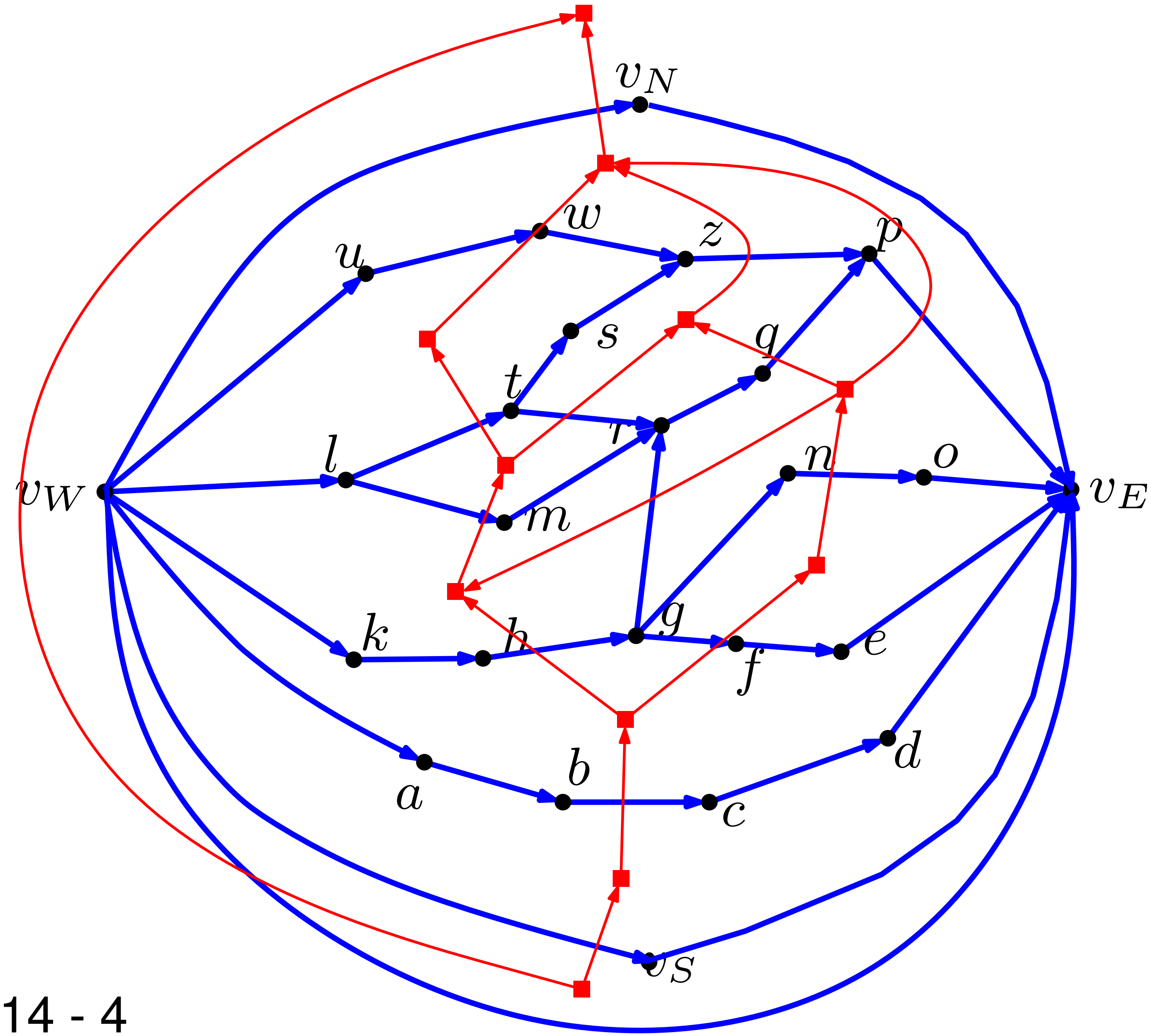
$x_1(v_N) = 1,\ x_2(v_N) = 15$
$x_1(v_S) = 1,\ x_2(v_S) = 15$
$x_1(v_W) = 0, x_2(v_W) = 1$
$x_1(v_E) = 15,\ x_2(v_E) = 16$
$x_1(a) = 1,\ x_2(a) = 3$
$x_1(b) = 3,\ x_2(b) = 5$
$x_1(c) = 5,\ x_2(c) = 14$
$x_1(d) = 14,\ x_2(d) = 15$
$x_1(e) = 13,\ x_2(e) = 15$

Rectangular Dual



$x_1(v_N) = 1,\ x_2(v_N) = 15$
$x_1(v_S) = 1,\ x_2(v_S) = 15$
$x_1(v_W) = 0, x_2(v_W) = 1$
$x_1(v_E) = 15,\ x_2(v_E) = 16$
$x_1(a) = 1,\ x_2(a) = 3$
$x_1(b) = 3,\ x_2(b) = 5$
$x_1(c) = 5,\ x_2(c) = 14$
$x_1(d) = 14,\ x_2(d) = 15$
$x_1(e) = 13,\ x_2(e) = 15$

Rectangular Dual



$x_1(v_N) = 1,\ x_2(v_N) = 15$

$x_1(v_S) = 1,\ x_2(v_S) = 15$

$x_1(v_W) = 0, x_2(v_W) = 1$

$x_1(v_E) = 15,\ x_2(v_E) = 16$

$x_1(a) = 1,\ x_2(a) = 3$

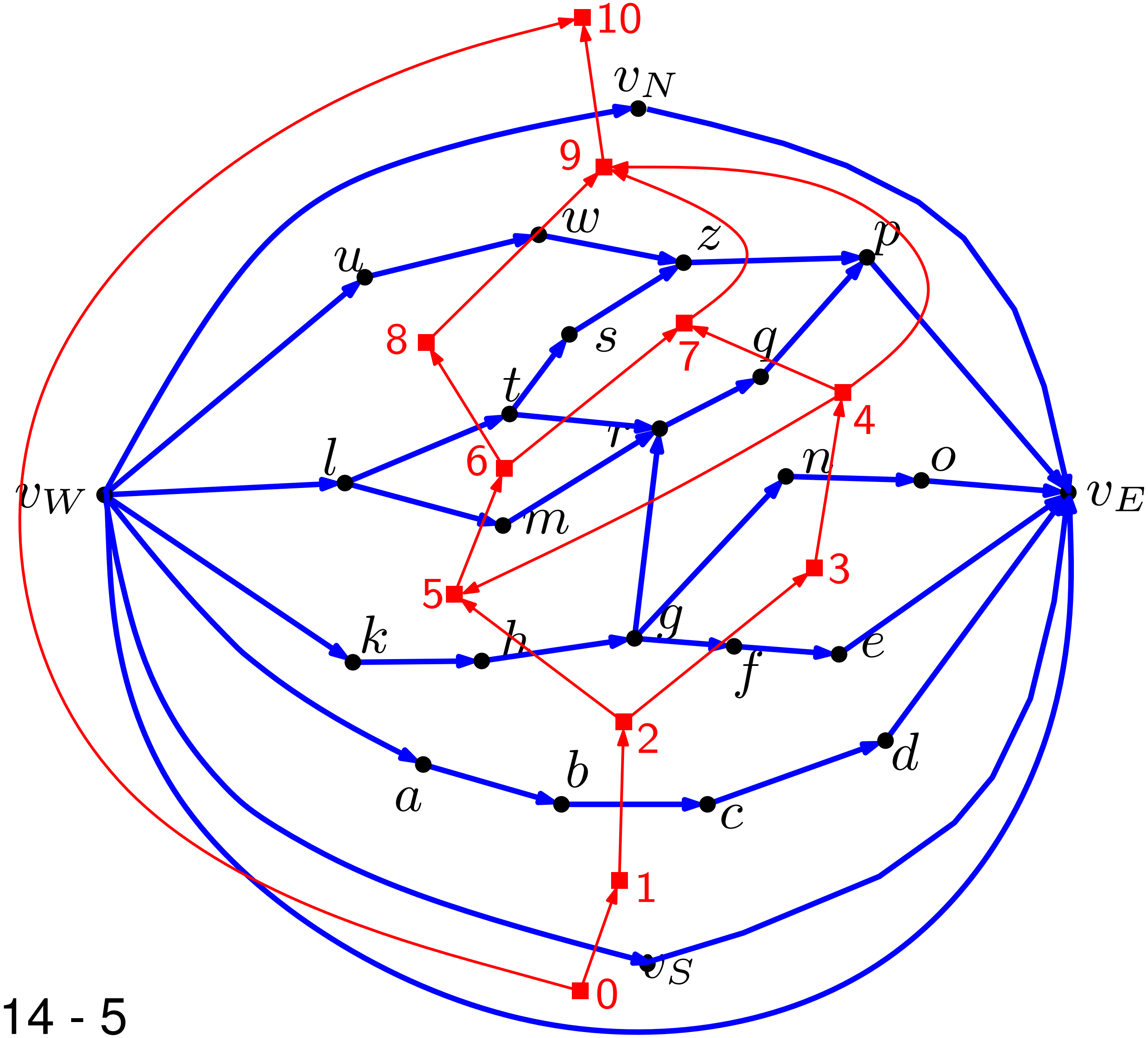
$x_1(b) = 3,\ x_2(b) = 5$

$x_1(c) = 5,\ x_2(c) = 14$

$x_1(d) = 14,\ x_2(d) = 15$

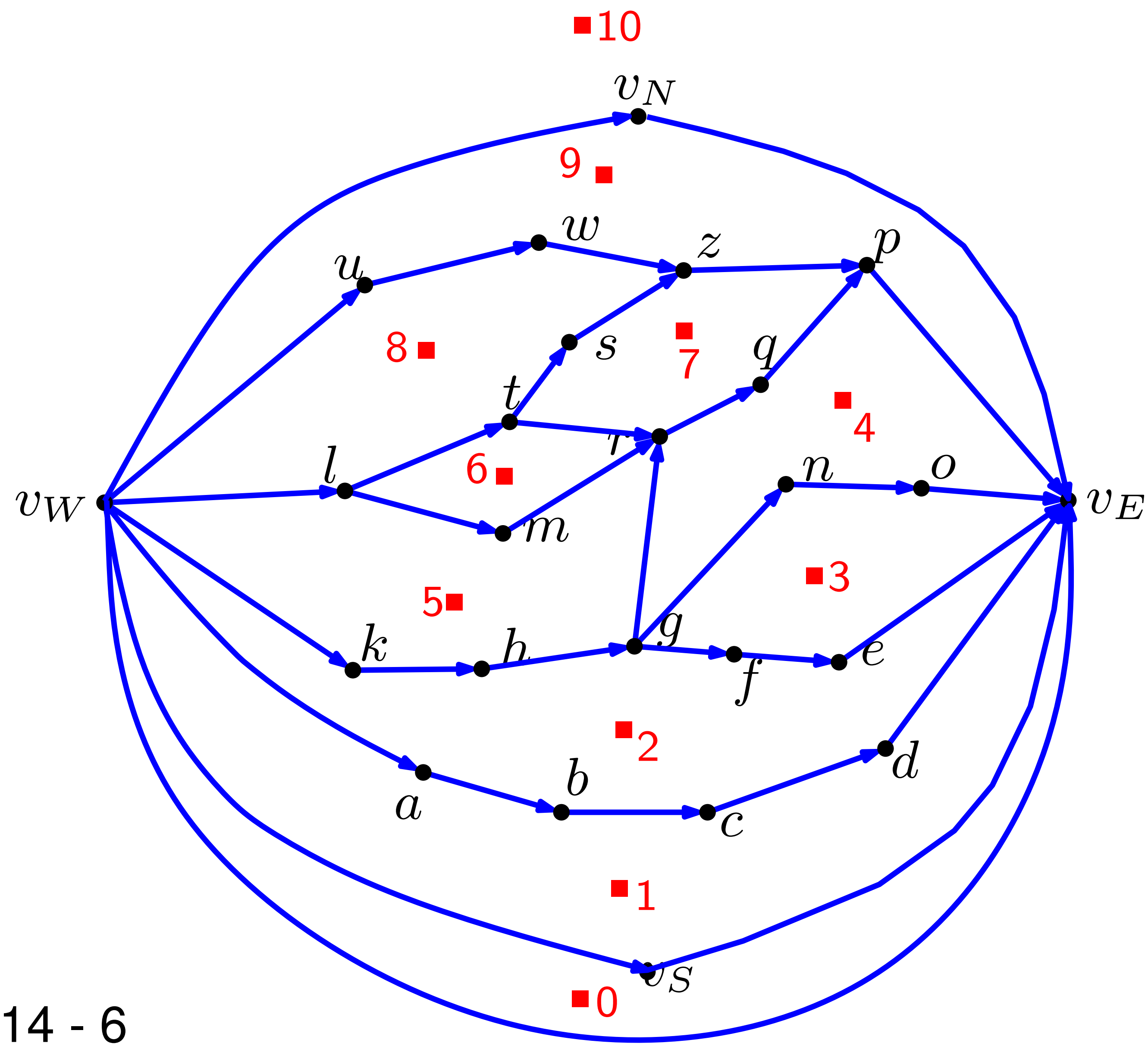
$x_1(e) = 13,\ x_2(e) = 15$

Rectangular Dual



$x_1(v_N) = 1,\ x_2(v_N) = 15$
$x_1(v_S) = 1,\ x_2(v_S) = 15$
$x_1(v_W) = 0, x_2(v_W) = 1$
$x_1(v_E) = 15,\ x_2(v_E) = 16$
$x_1(a) = 1,\ x_2(a) = 3$
$x_1(b) = 3,\ x_2(b) = 5$
$x_1(c) = 5,\ x_2(c) = 14$
$x_1(d) = 14,\ x_2(d) = 15$
$x_1(e) = 13,\ x_2(e) = 15$

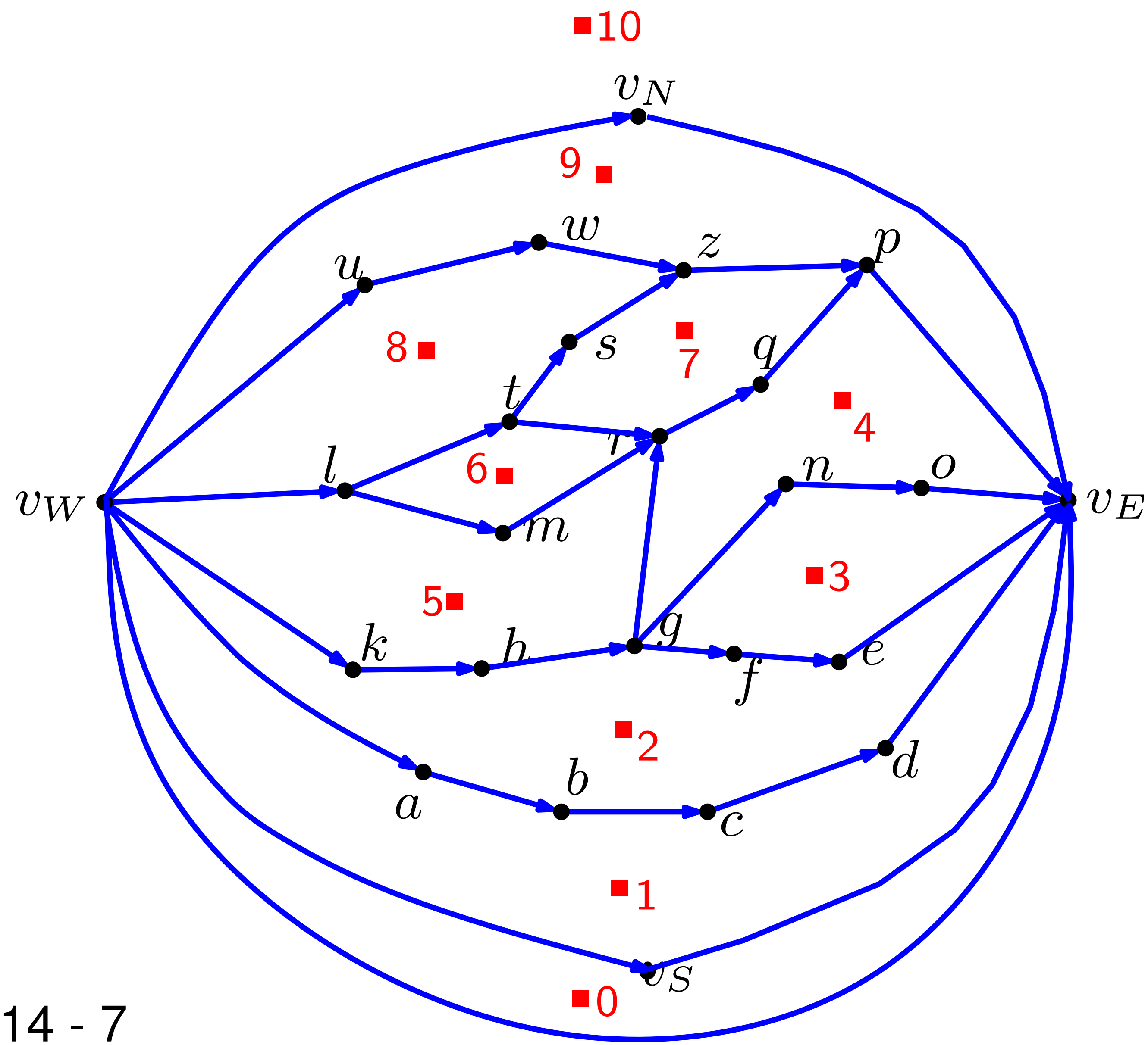
Rectangular Dual



$x_1(v_N) = 1,\ x_2(v_N) = 15$
$x_1(v_S) = 1,\ x_2(v_S) = 15$
$x_1(v_W) = 0, x_2(v_W) = 1$
$x_1(v_E) = 15,\ x_2(v_E) = 16$
$x_1(a) = 1,\ x_2(a) = 3$
$x_1(b) = 3,\ x_2(b) = 5$
$x_1(c) = 5,\ x_2(c) = 14$
$x_1(d) = 14,\ x_2(d) = 15$
$x_1(e) = 13,\ x_2(e) = 15$

14 - 6

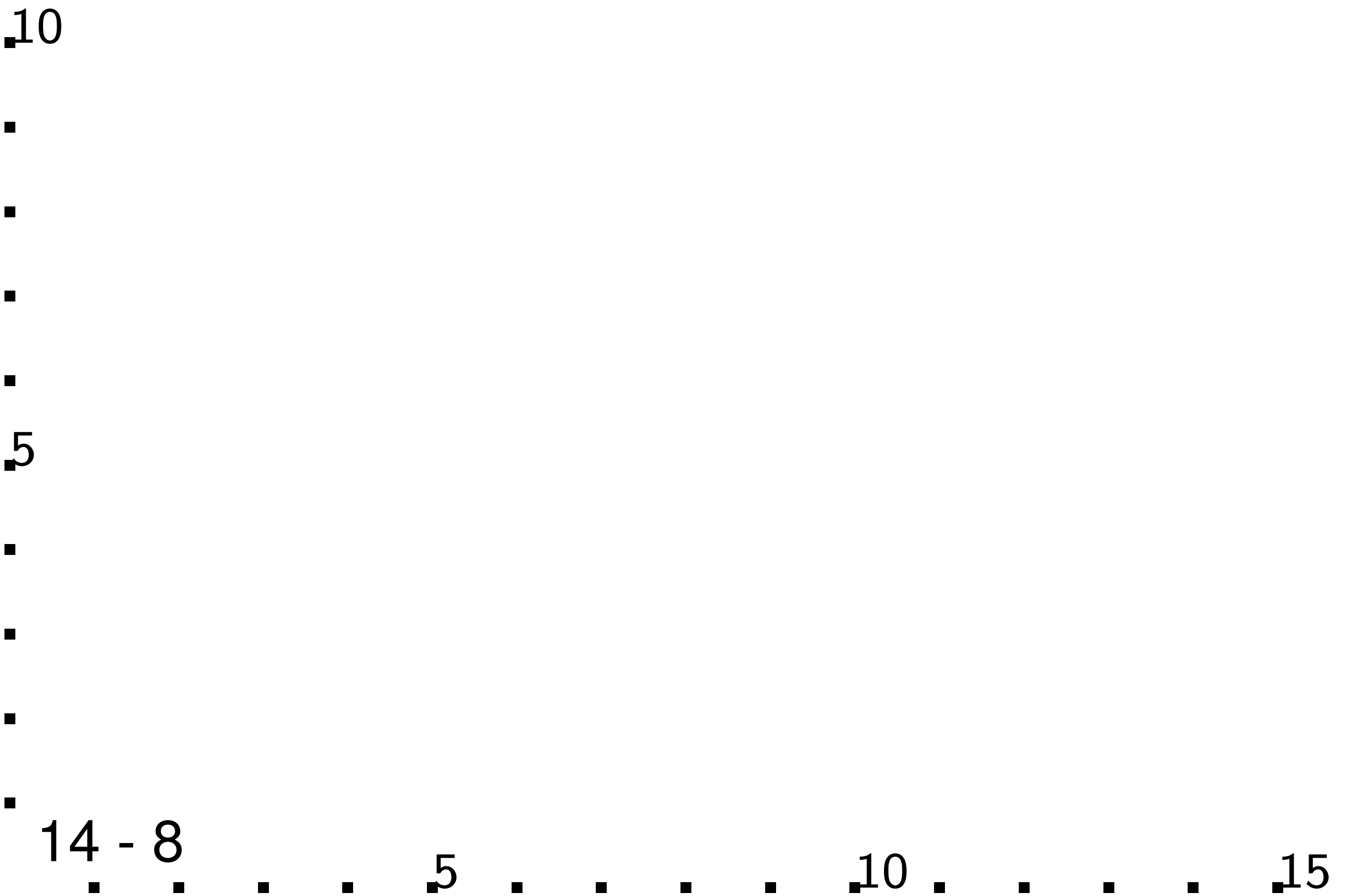
Rectangular Dual



$x_1(v_N) = 1,\ x_2(v_N) = 15$
$x_1(v_S) = 1,\ x_2(v_S) = 15$
$x_1(v_W) = 0, x_2(v_W) = 1$
$x_1(v_E) = 15,\ x_2(v_E) = 16$
$x_1(a) = 1,\ x_2(a) = 3$
$x_1(b) = 3,\ x_2(b) = 5$
$x_1(c) = 5,\ x_2(c) = 14$
$x_1(d) = 14,\ x_2(d) = 15$
$x_1(e) = 13,\ x_2(e) = 15$

$y_1(v_W) = 0, y_2(v_W) = 10$
$y_1(v_E) = 0,\ y_2(v_E) = 10$
$y_1(v_N) = 9,\ y_2(v_N) = 10$
$y_1(v_S) = 0,\ y_2(v_S) = 1$
$y_1(a) = 1,\ y_2(a) = 2$
$y_1(b) = 1,\ y_2(b) = 2$
$y_1(c) = 1,\ y_2(c) = 2$
$y_1(d) = 1,\ y_2(d) = 2$
$y_1(e) = 2,\ y_2(e) = 3$

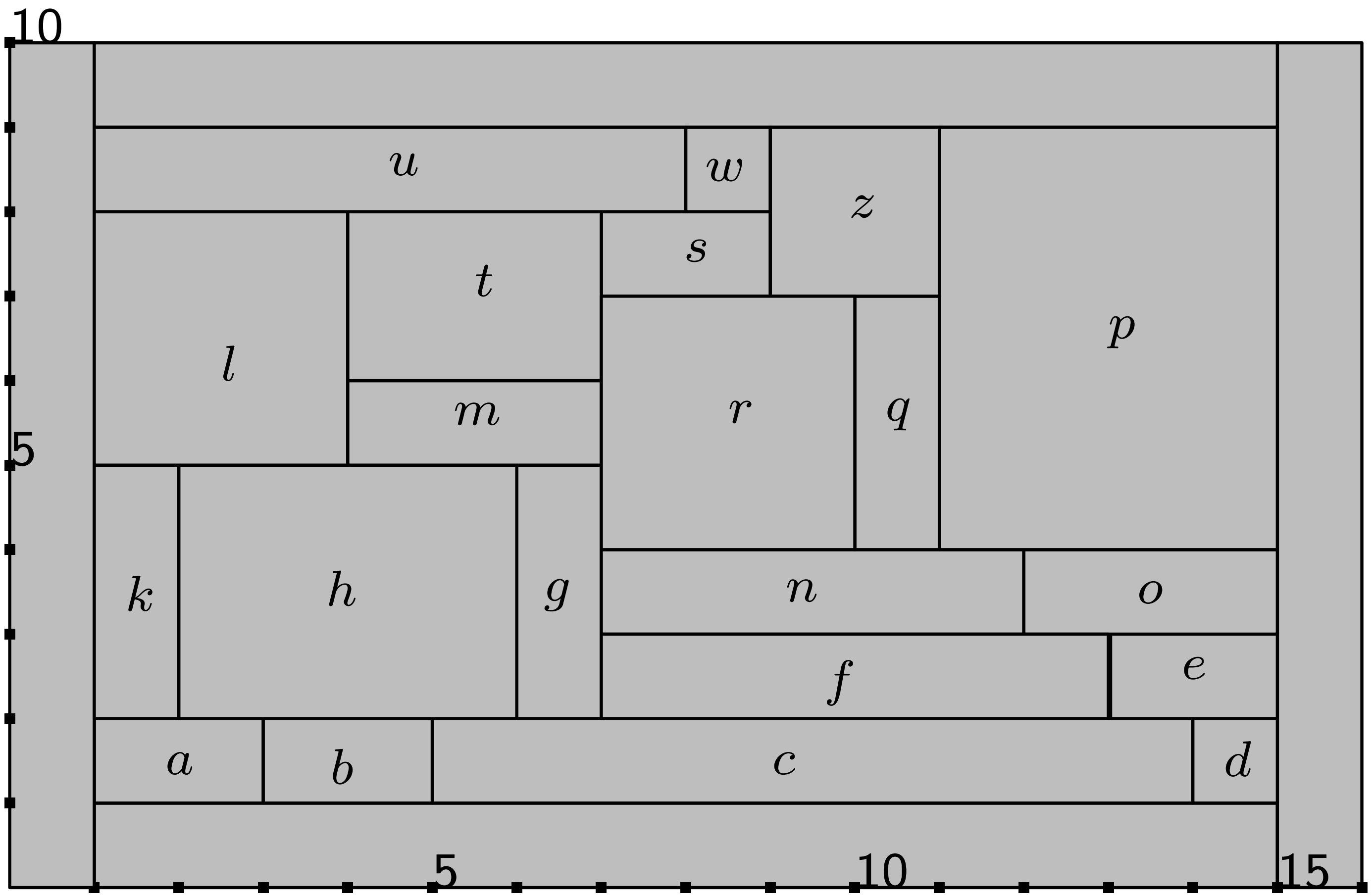
Rectangular Dual



$x_1(v_N) = 1,\ x_2(v_N) = 15$
$x_1(v_S) = 1,\ x_2(v_S) = 15$
$x_1(v_W) = 0, x_2(v_W) = 1$
$x_1(v_E) = 15,\ x_2(v_E) = 16$
$x_1(a) = 1,\ x_2(a) = 3$
$x_1(b) = 3,\ x_2(b) = 5$
$x_1(c) = 5,\ x_2(c) = 14$
$x_1(d) = 14,\ x_2(d) = 15$
$x_1(e) = 13,\ x_2(e) = 15$

$y_1(v_W) = 0, y_2(v_W) = 10$
$y_1(v_E) = 0,\ y_2(v_E) = 10$
$y_1(v_N) = 9,\ y_2(v_N) = 10$
$y_1(v_S) = 0,\ y_2(v_S) = 1$
$y_1(a) = 1,\ y_2(a) = 2$
$y_1(b) = 1,\ y_2(b) = 2$
$y_1(c) = 1,\ y_2(c) = 2$
$y_1(d) = 1,\ y_2(d) = 2$
$y_1(e) = 2,\ y_2(e) = 3$

Rectangular Dual



$x_1(v_N) = 1,\ x_2(v_N) = 15$
$x_1(v_S) = 1,\ x_2(v_S) = 15$
$x_1(v_W) = 0, x_2(v_W) = 1$
$x_1(v_E) = 15,\ x_2(v_E) = 16$
$x_1(a) = 1,\ x_2(a) = 3$
$x_1(b) = 3,\ x_2(b) = 5$
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$x_1(d) = 14,\ x_2(d) = 15$
$x_1(e) = 13,\ x_2(e) = 15$

$y_1(v_W) = 0, y_2(v_W) = 10$
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$y_1(v_N) = 9,\ y_2(v_N) = 10$
$y_1(v_S) = 0,\ y_2(v_S) = 1$
$y_1(a) = 1,\ y_2(a) = 2$
$y_1(b) = 1,\ y_2(b) = 2$
$y_1(c) = 1,\ y_2(c) = 2$
$y_1(d) = 1,\ y_2(d) = 2$
$y_1(e) = 2,\ y_2(e) = 3$

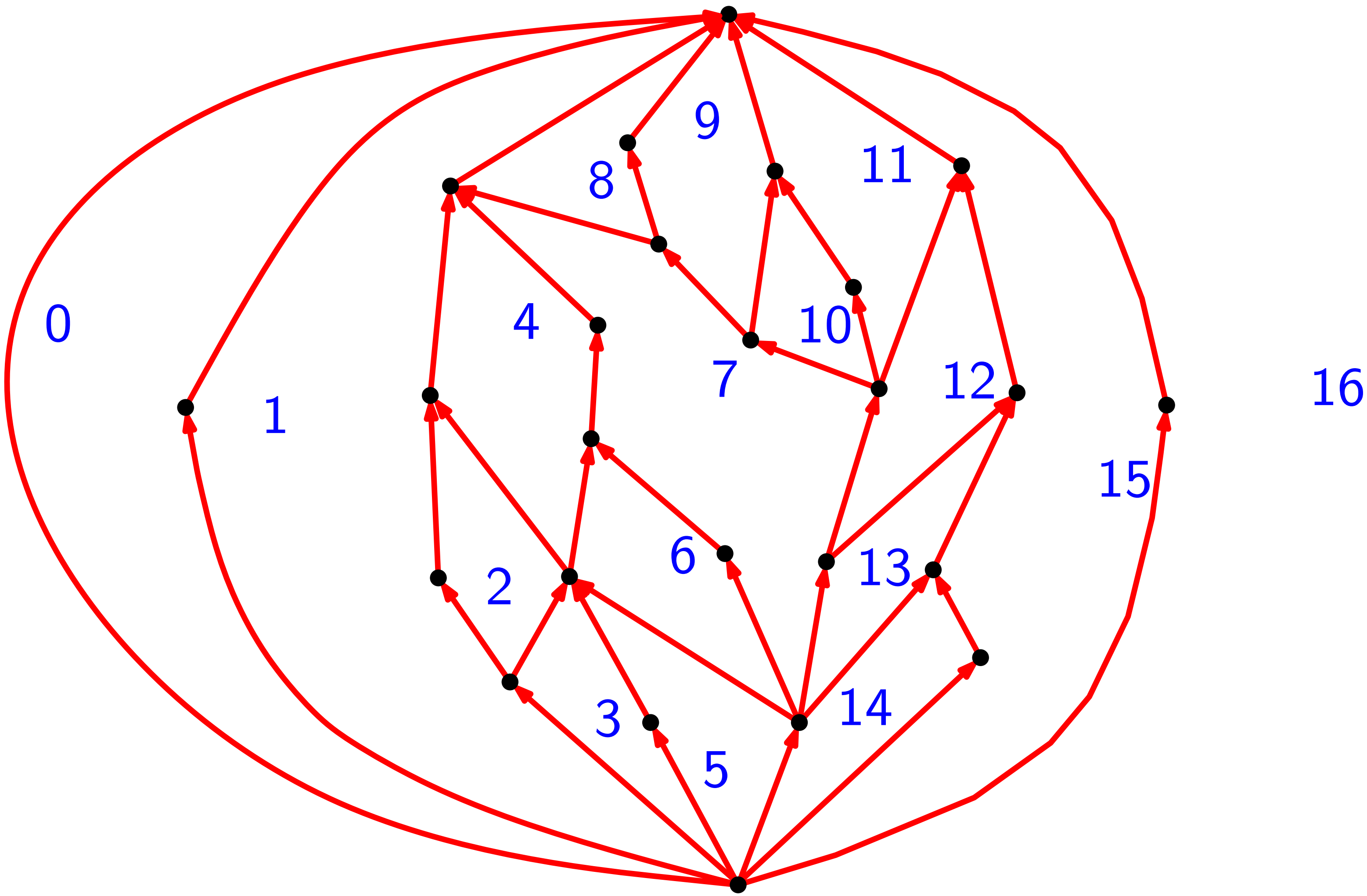
Rectangular Dual

In the following we prove that presented algorithm constructs a rectangular dual of G.

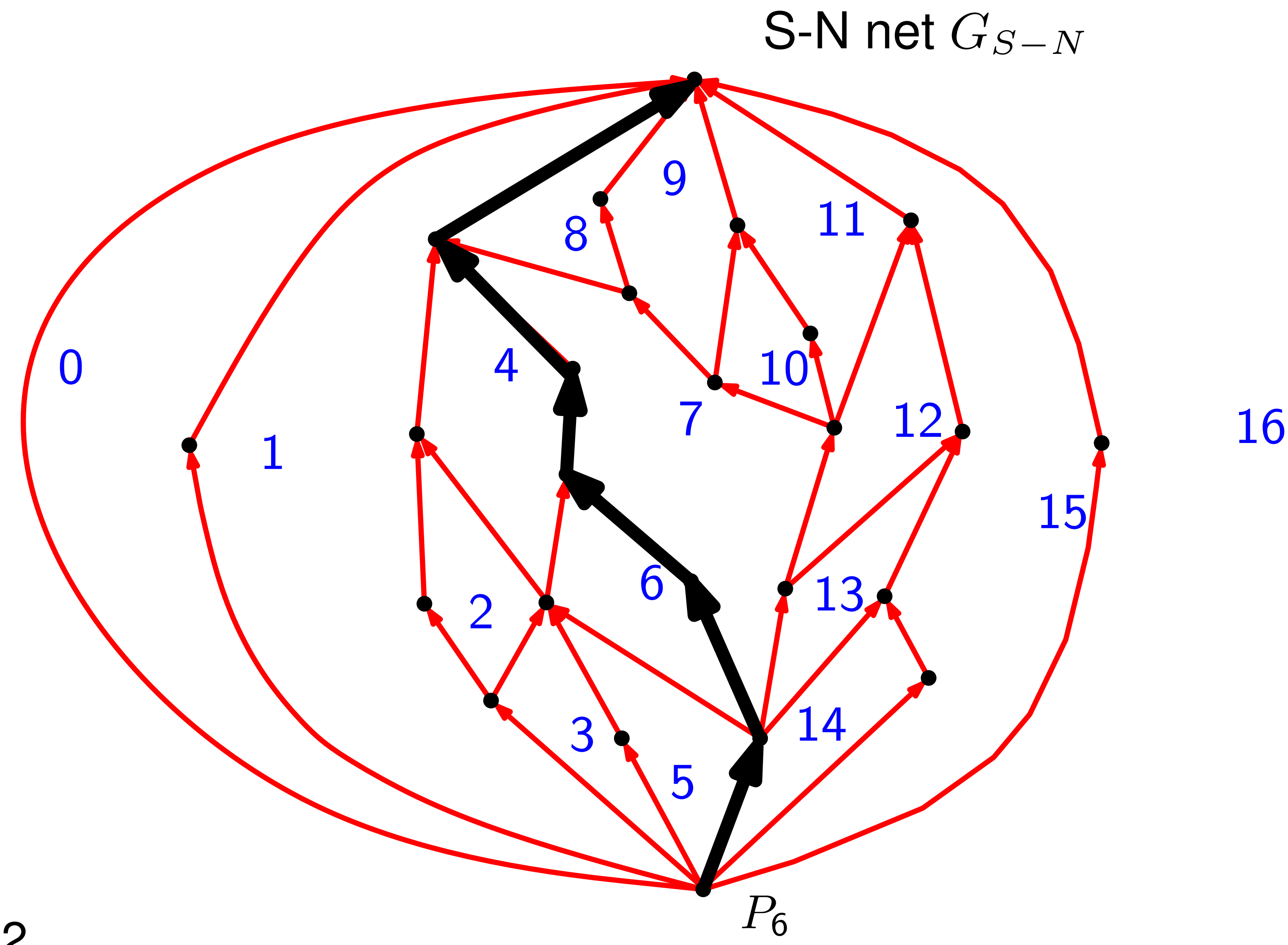
- Let G be a PTP graph and let G_{S-N} (resp. G_{W-E}) be its S-N net, let $G^{\star}_{S-N}$ (resp. $G^{\star}_{W-E}$) be the dual of G_{S-N} (resp. G_{W-E})
- Let $f_1, \ldots, f_k$ be the faces of $G^{\star}_{S-N}$ (resp. $G^{\star}_{W-E}$) , enumerated according to st-numbering f_{sn} (resp. f_{we})
- Let G^{i}_{S-N} (resp. G^{i}_{W-E}) denote the subgraph of G that is induced by vertices and edges of $f_1, \ldots, f_i$
- We denote P_i (reps. Q_i) the right (resp. top) boundary of G^{i}_{S-N} (resp. G^{i}_{W-E}) .

Rectangular Dual

S-N net G_{S-N}

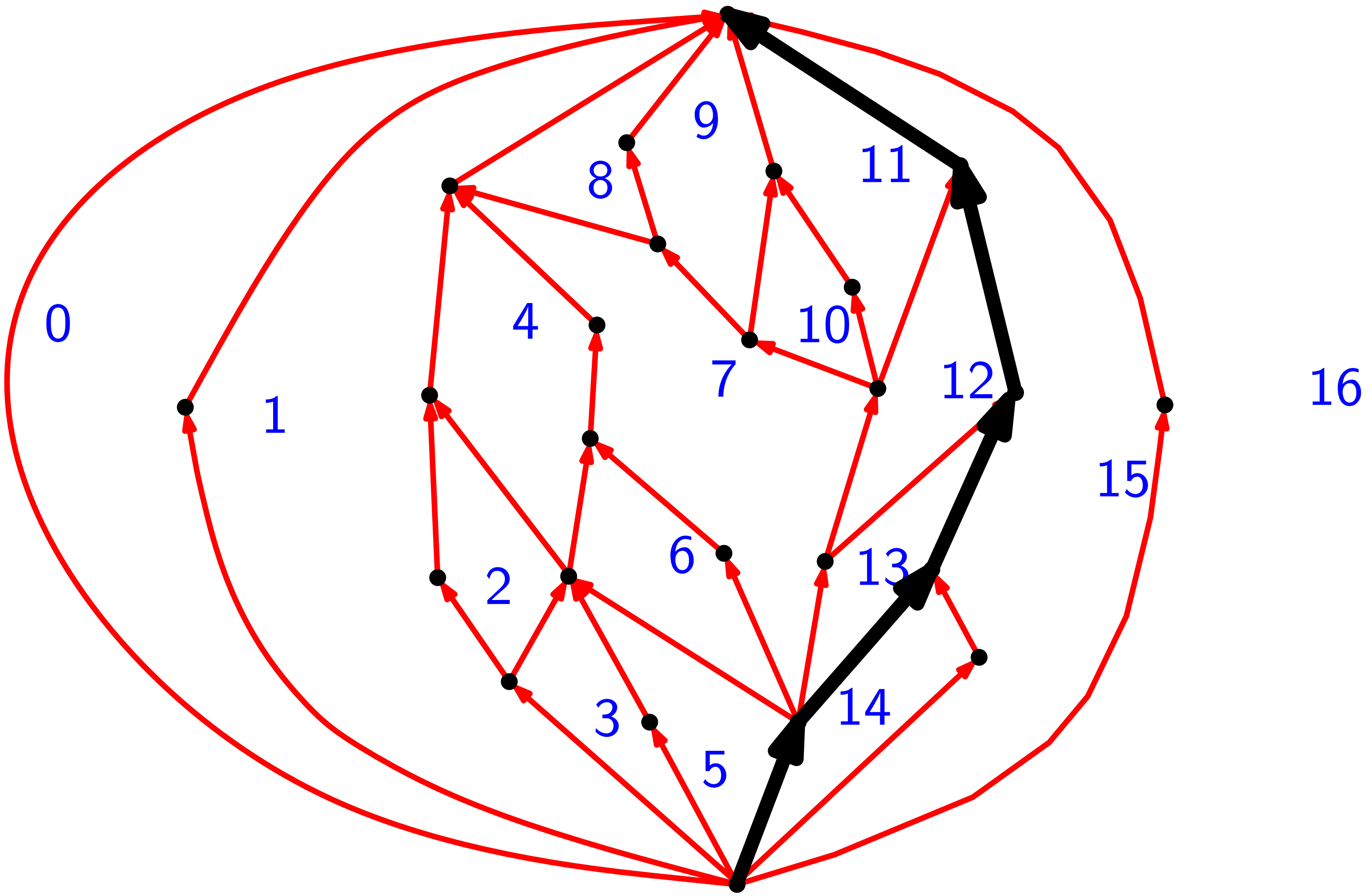


Rectangular Dual



Rectangular Dual

S-N net G_{S-N}



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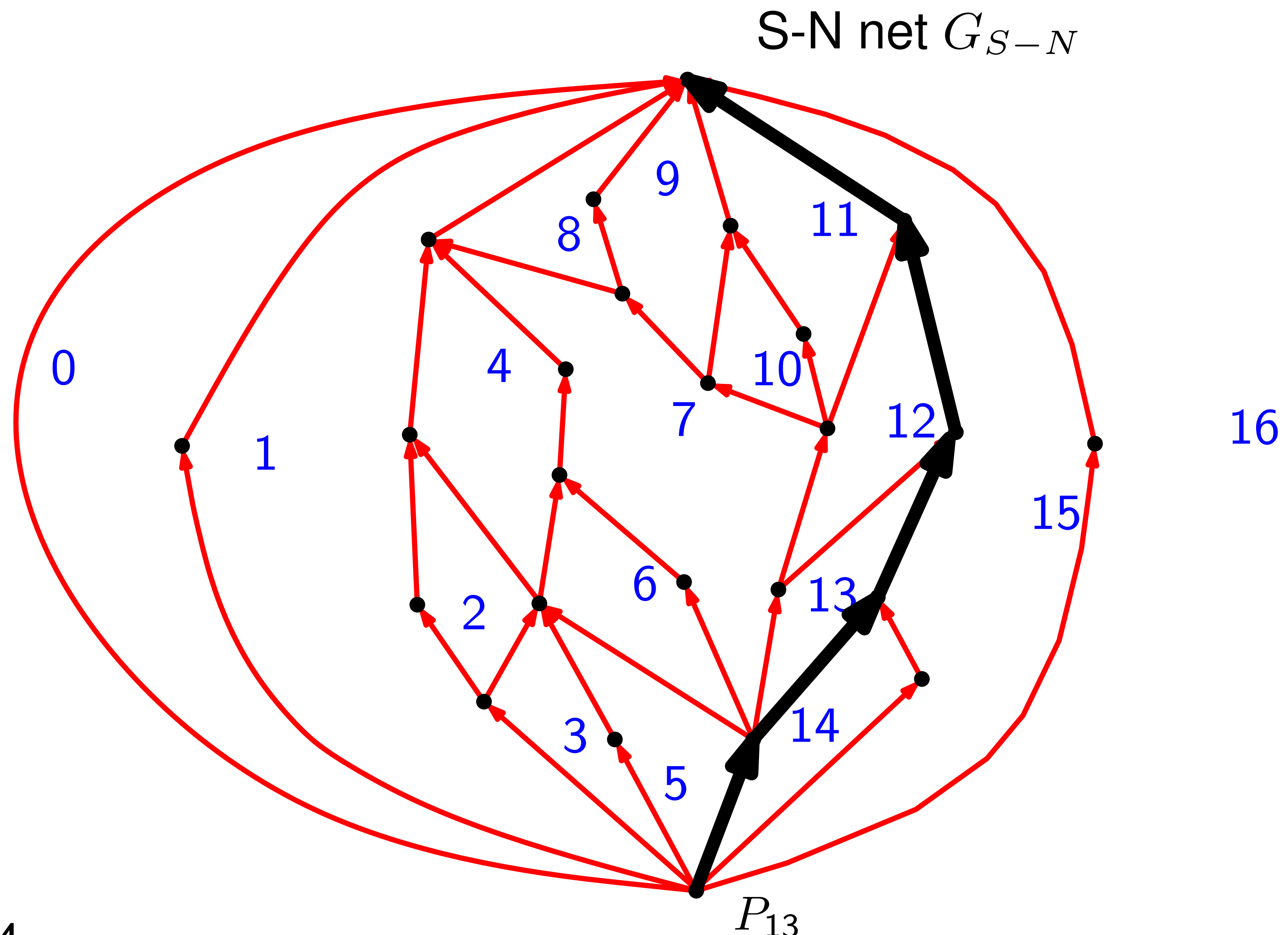
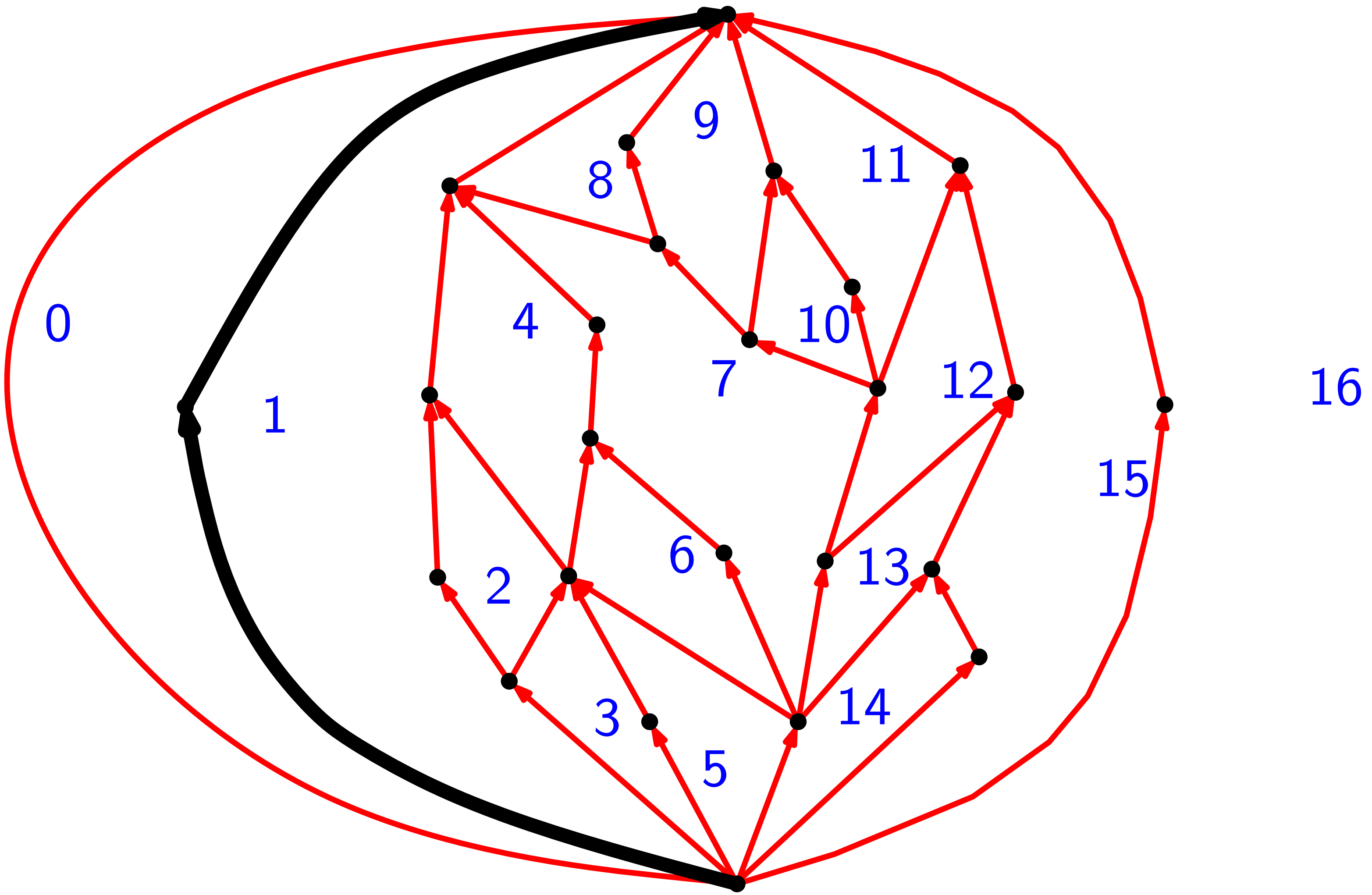
Rectangular Dual



Rectangular Dual

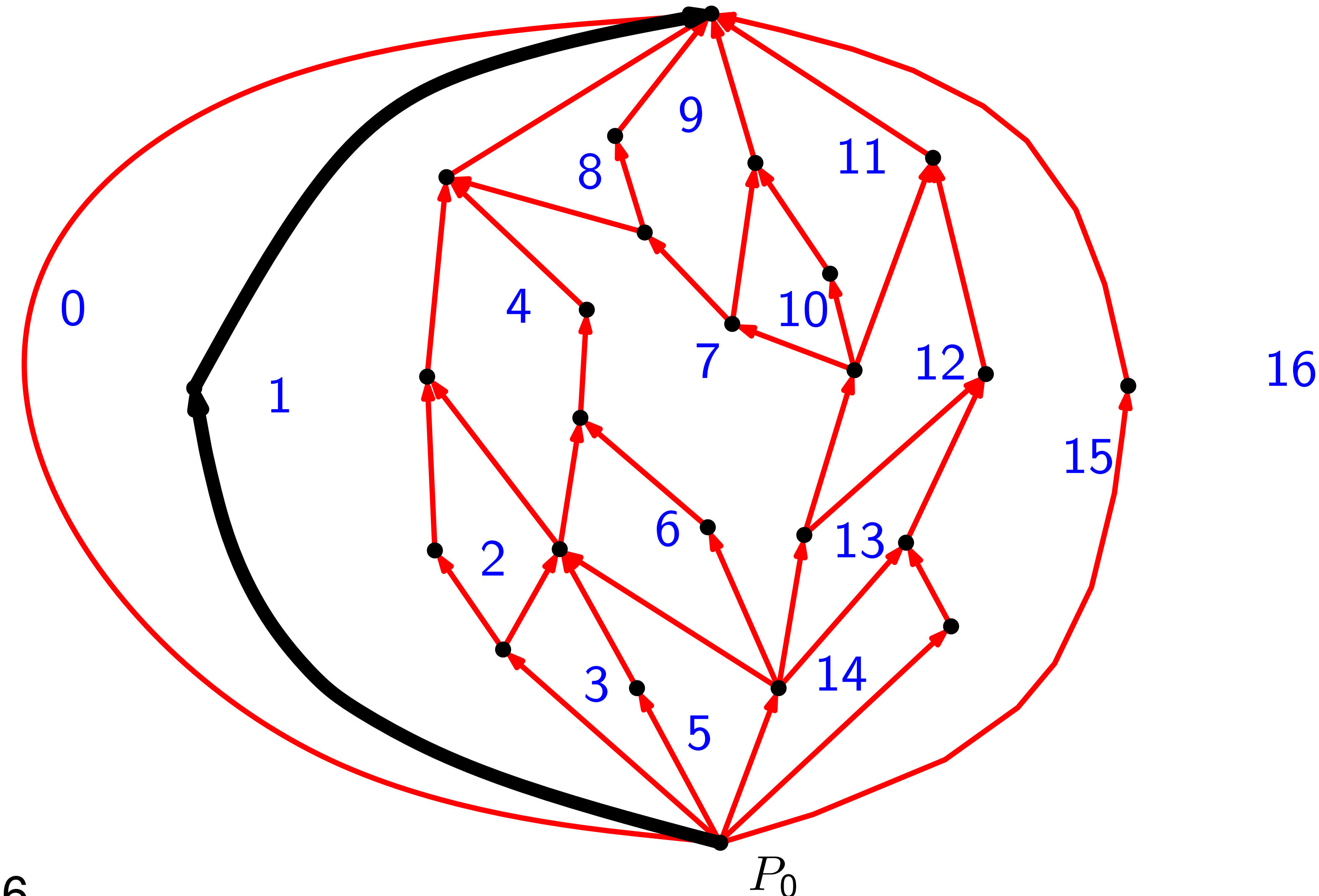
S-N net G_{S-N}



16 - 5

Rectangular Dual

S-N net G_{S-N}



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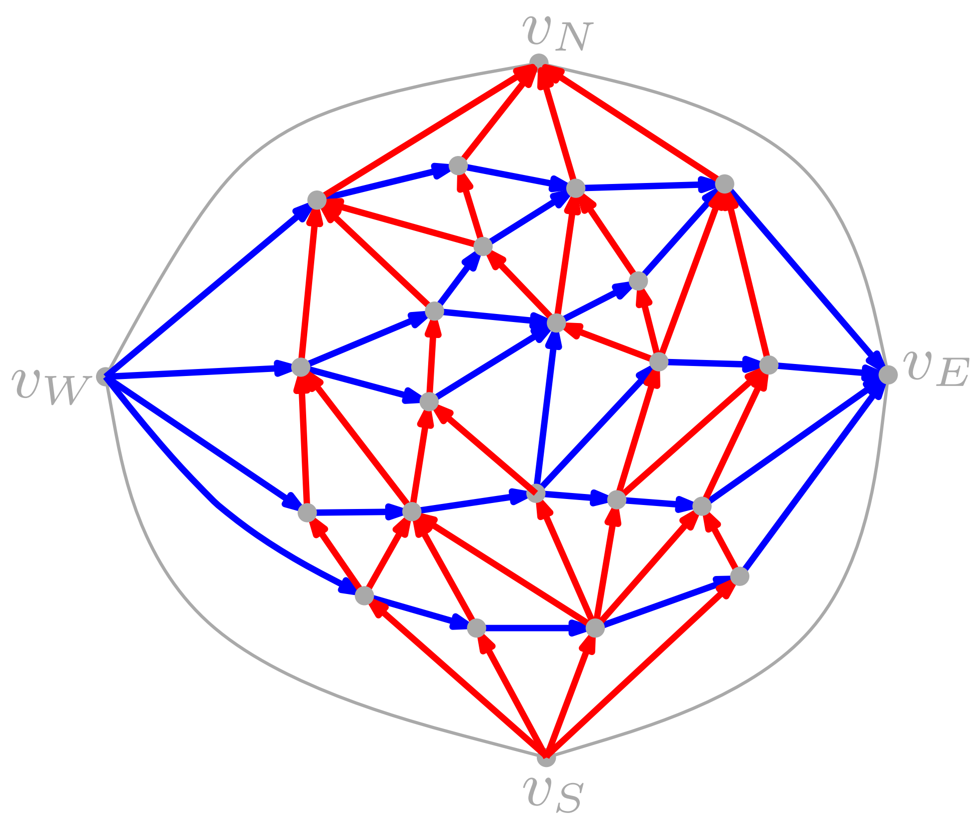
Lehrstuhl Algorithmik I

Rectangular Dual

- Paths P_i and Q_j for any i, j (except for (a) $i = 0, j = 0$, (b) $i = \max f_{sn} - 1, j = 0$, (c) $i = 0, j = \max f_{we} - 1$, (d) $i = \max f_{sn} - 1, j = \max f_{we} - 1$) **cross at exactly one vertex**.



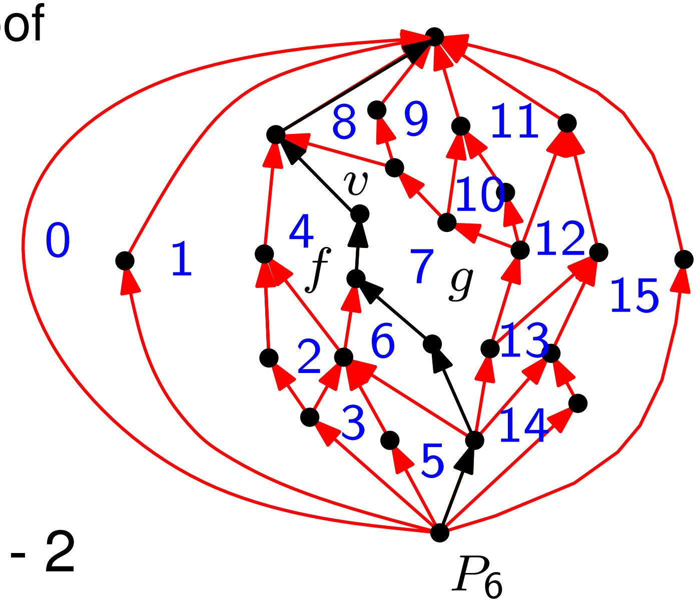
Rectangular Dual

- Paths P_i and Q_j for any i, j (except for (a) $i = 0, j = 0$, (b) $i = \max f_{sn} - 1, j = 0$, (c) $i = 0, j = \max f_{we} - 1$, (d) $i = \max f_{sn} - 1, j = \max f_{we} - 1$) **cross at exactly one vertex**.

Lemma 4

Let $v \in V$, f and g are the left and the right face of v. Let $x_1(v) = f_{sn}(f)$ and $x_2(v) = f_{sn}(g)$. Vertex v belongs to path P_i if and only if $x_1(v) \leq i \leq x_2(v) - 1$.

Proof



- $f_{sn}(f) \leq i$ and $f_{sn}(g) \geq i + 1$

Rectangular Dual

- Paths P_i and Q_j for any i, j (except for (a) $i = 0, j = 0$, (b) $i = \max f_{sn} - 1, j = 0$, (c) $i = 0, j = \max f_{we} - 1$, (d) $i = \max f_{sn} - 1, j = \max f_{we} - 1$) **cross at exactly one vertex**.

Lemma 4

Let $v \in V$, f and g are the left and the right face of v. Let $x_1(v) = f_{sn}(f)$ and $x_2(v) = f_{sn}(g)$. Vertex v belongs to path P_i if and only if $x_1(v) \leq i \leq x_2(v) - 1$.

Lemma 5

Let $v \in V$, f and g are the fases below and above v in G_{W-E}. Let $y_1(v) = f_{we}(f)$ and $y_2(v) = f_{we}(g)$. Vertex v belongs to path Q_j if and only if $y_1(v) \leq j \leq y_2(v) - 1$.

Proof (identical)

Rectangular Dual

Lemma 6

The assignment provided by the algorithm do not produce neither gaps nor overlapping rectangles.

Rectangular Dual

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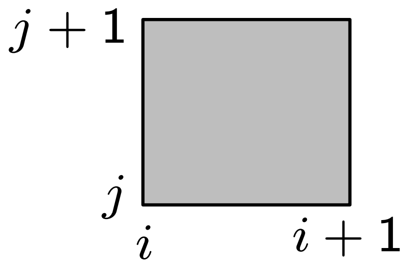
The assignment provided by the algorithm do not produce neither gaps nor overlapping rectangles.

Proof:

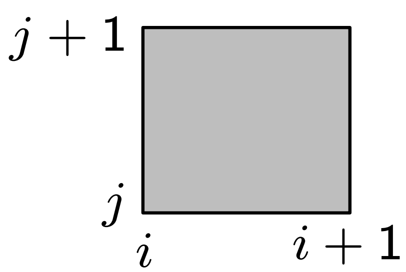


Rectangular Dual

Lemma 6

The assignment provided by the algorithm do not produce neither gaps nor overlapping rectangles.

Proof: Show that there **exists a vertex** over this box: $u \in P_i \cap Q_j$

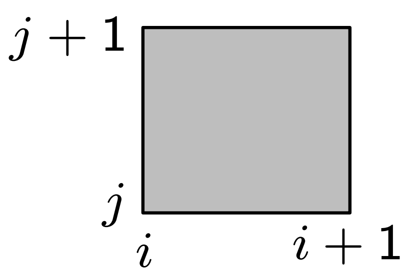


Rectangular Dual

Lemma 6

The assignment provided by the algorithm do not produce neither gaps nor overlapping rectangles.

Proof: Show that there is **at most one vertex** over this box



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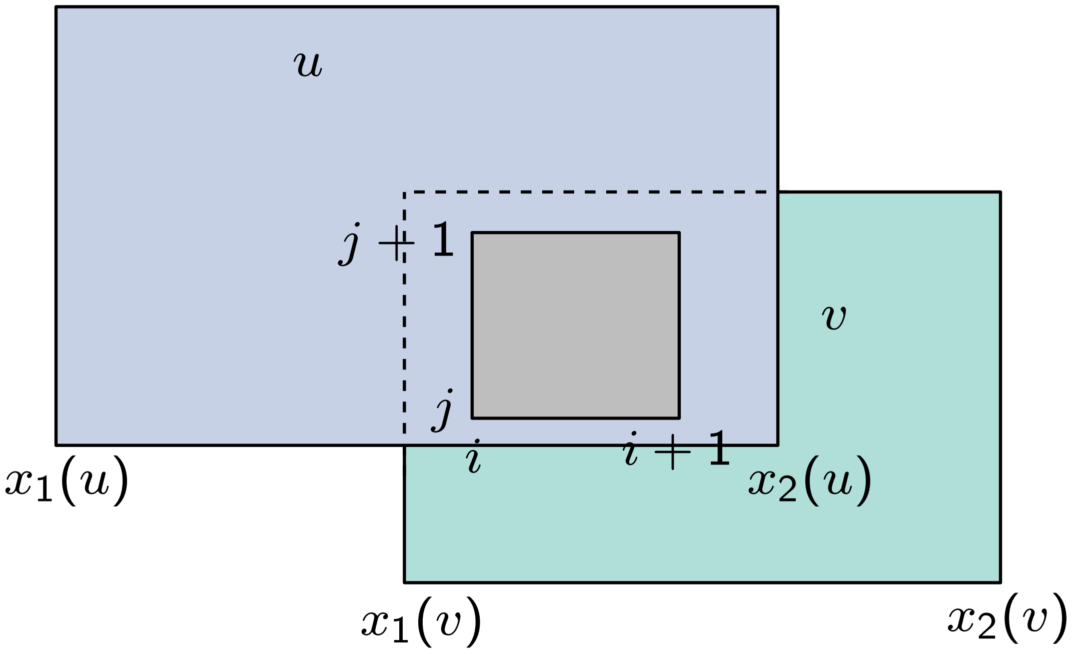

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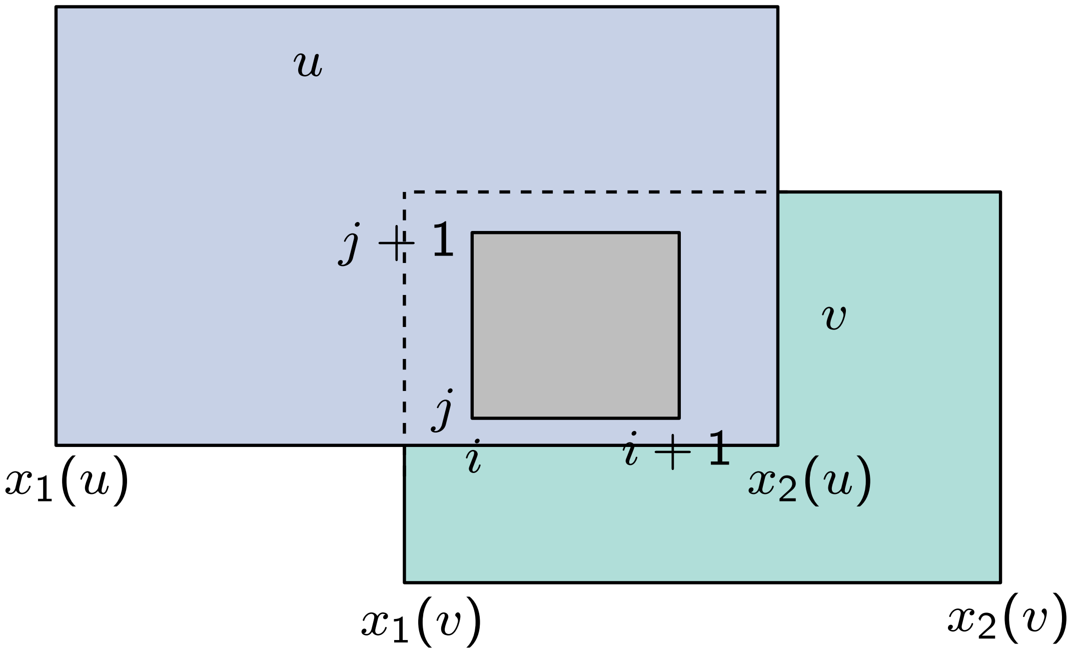

$x_1(u) \leq i$ and $i+1 \leq x_2(u)$

Rectangular Dual

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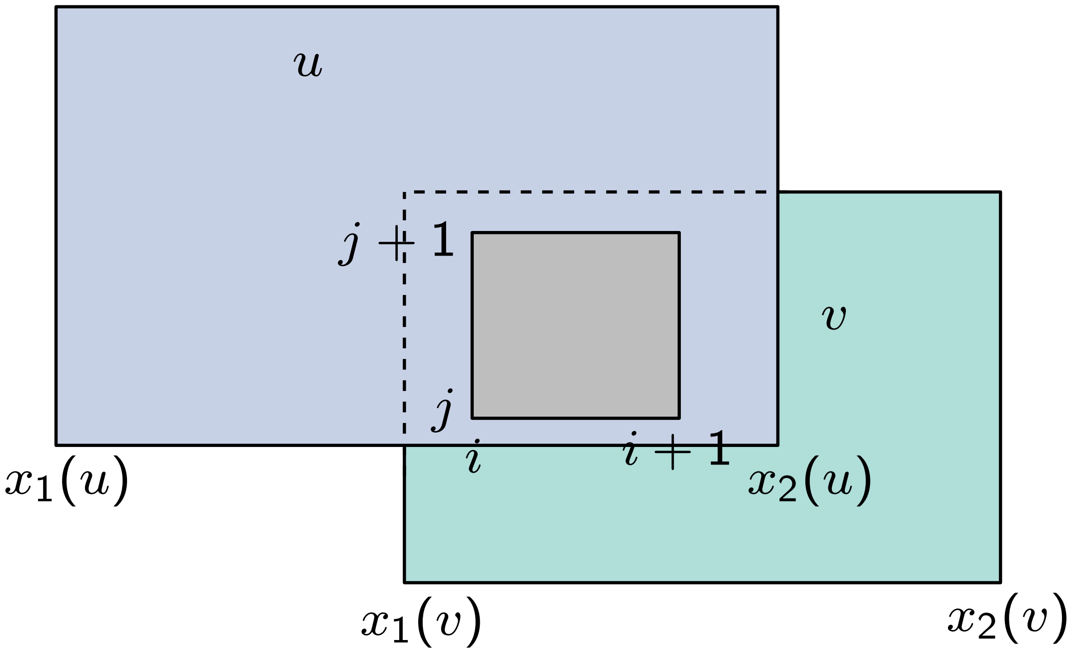
Proof: Show that there is **at most one vertex** over this box



$x_1(u) \leq i$ and $i+1 \leq x_2(u)$

⇓ (Lemma 4)

u belongs to P_i

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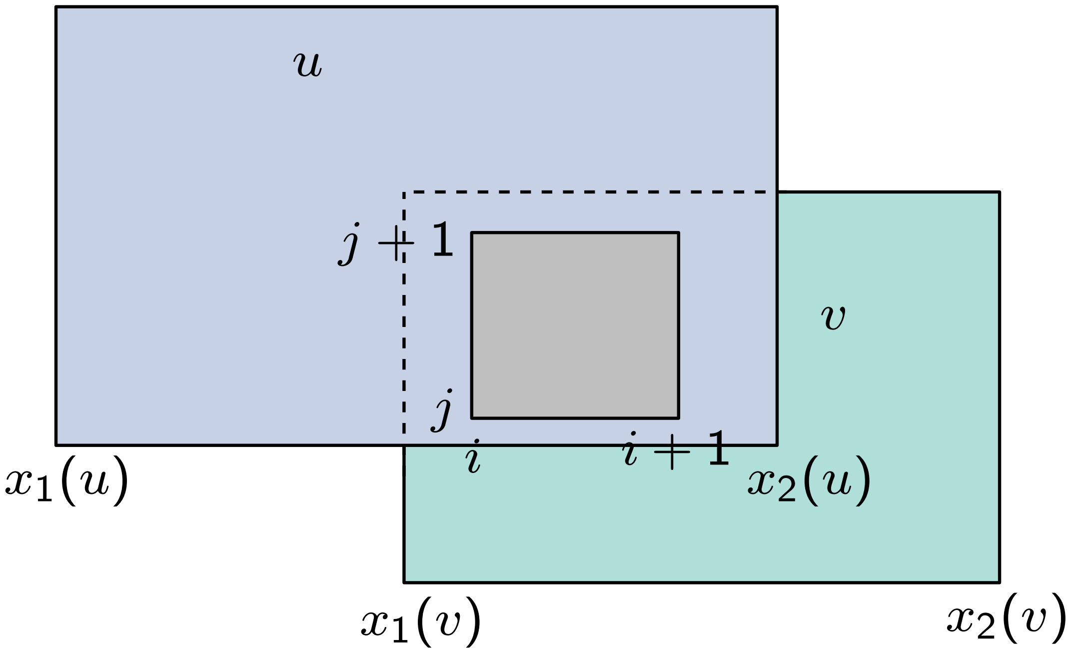
Proof: Show that there is **at most one vertex** over this box



$x_1(u) \leq i$ and $i+1 \leq x_2(u)$

(Lemma 4)

u belongs to P_i

Similarly: $v \in P_i$, $u \in Q_j$, $v \in Q_j$.

Rectangular Dual

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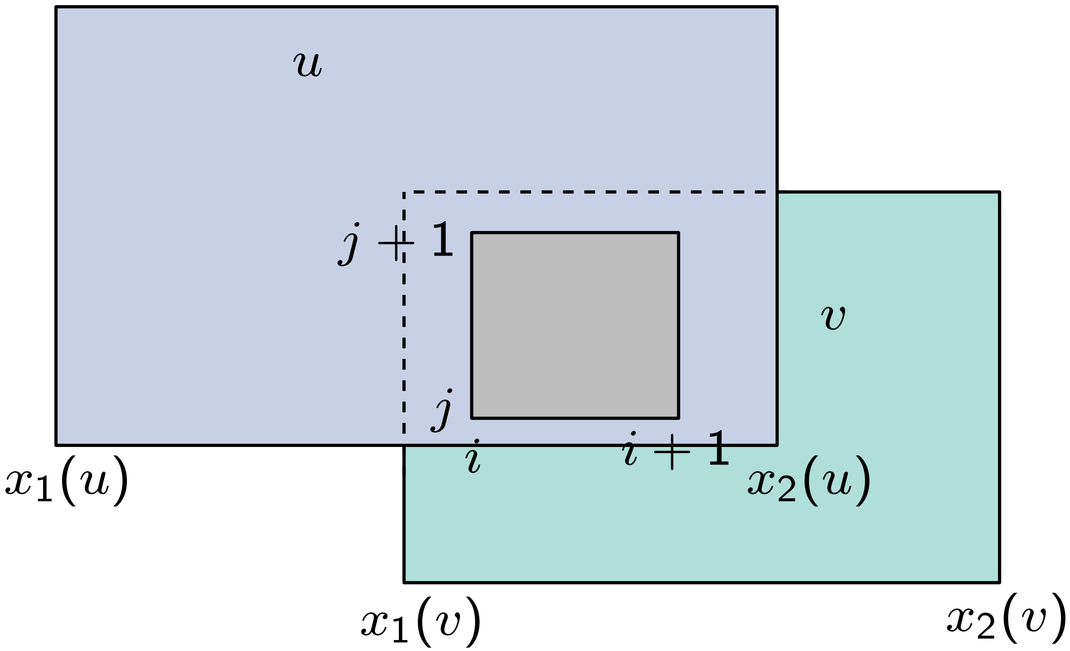
The assignment provided by the algorithm do not produce neither gaps nor overlapping rectangles.

Proof: Show that there is **at most one vertex** over this box



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Similarly: $v \in P_i$, $u \in Q_j$, $v \in Q_j$.

Paths P_i and Q_j intersect at two vertices u and v.

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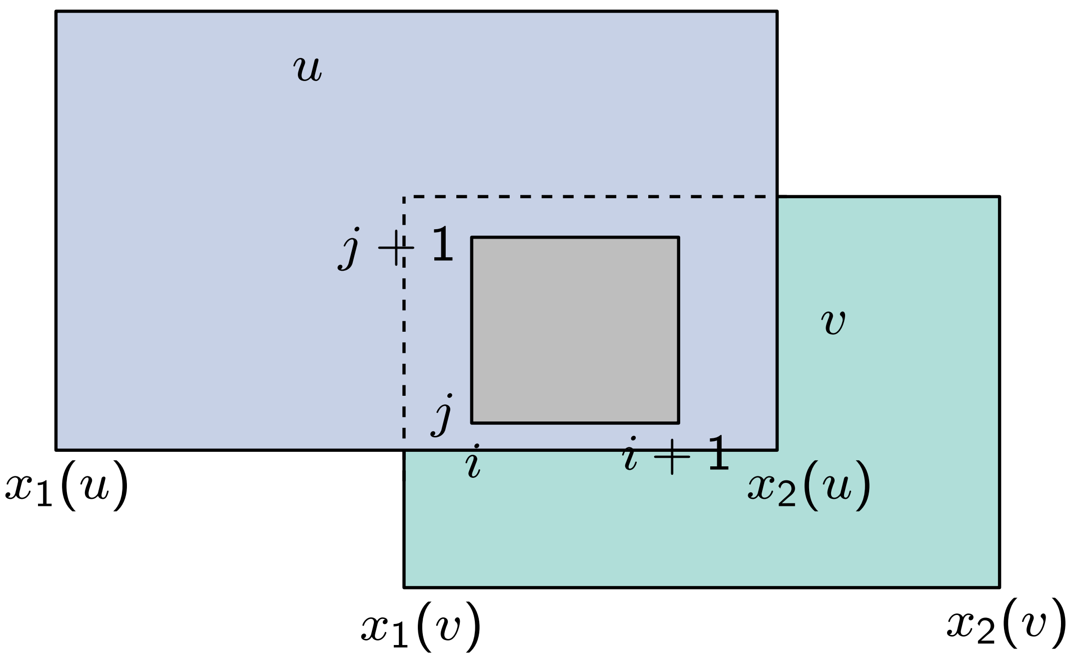

$x_1(u) \leq i$ and $i+1 \leq x_2(u)$

(Lemma 4)

u belongs to P_i

Similarly: $v \in P_i$, $u \in Q_j$, $v \in Q_j$.

Paths P_i and Q_j intersect at two vertices u and v.

Which is a contradiction to the property of paths P_i, Q_j except for the cases when: (a) $i = 0, j = 0$, (b) $i = \max f_{sn} - 1, j = 0$, (c) $i = 0, j = \max f_{we} - 1$, (d) $i = \max f_{sn} - 1, j = \max f_{we} - 1$ (corner boxes).

Rectangular Dual

Lemma 7

Let G_{S-N} and G_{W-E}. The following are true:

- If $(u, v) \in G_{W-E}$ then $x_2(u) = x_1(v)$;
- If there exist a directed path from u to v in G_{W-E} containing at least two edges, then $x_2(u) < x_1(v)$

Rectangular Dual

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Proof

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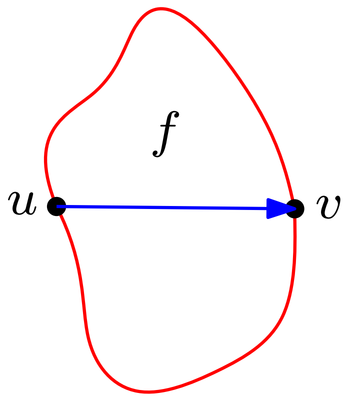
Proof



Rectangular Dual

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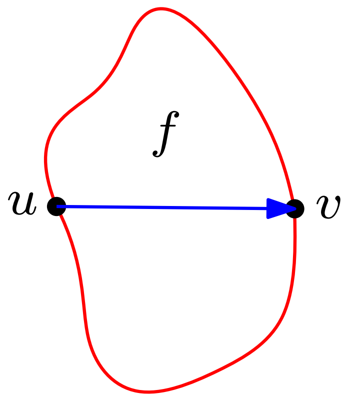
Proof

- $x_2(u) = f_{sn}(f) = x_1(v)$



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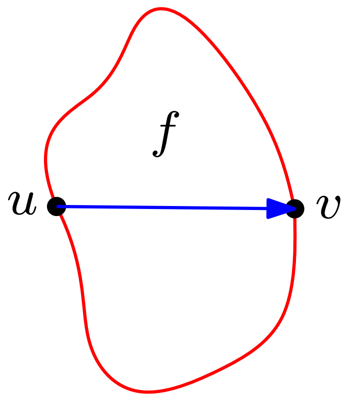

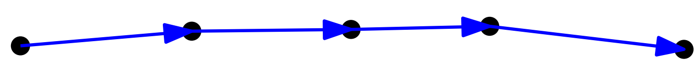

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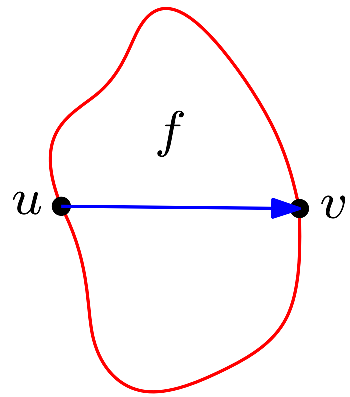

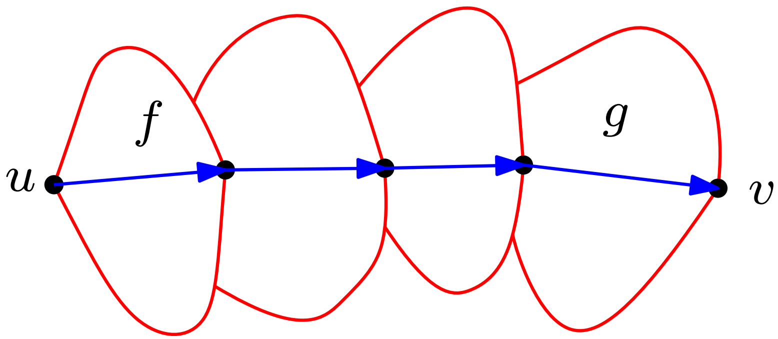

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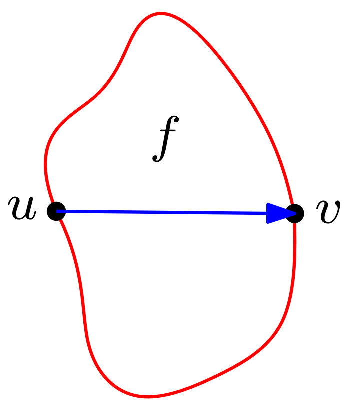
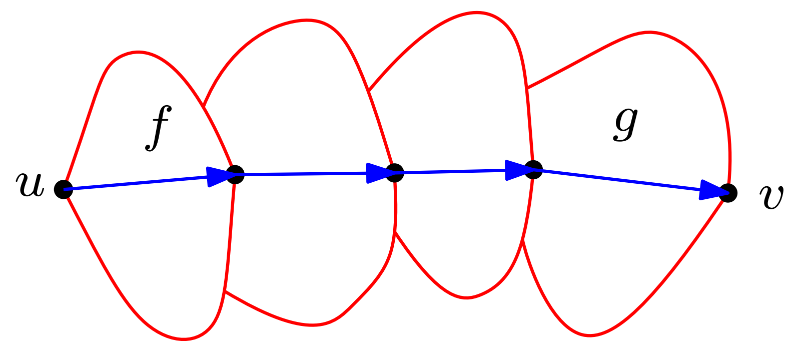
- If $(u, v) \in G_{W-E}$ then $x_2(u) = x_1(v)$;
- If there exist a directed path from u to v in G_{W-E} containing at least two edges, then $x_2(u) < x_1(v)$

Proof

- $x_2(u) = f_{sn}(f) = x_1(v)$



- $x_2(u) = f_{sn}(f)$
- $x_1(v) = f_{sn}(g)$



Rectangular Dual

Lemma 7

Let G_{S-N} and G_{W-E}. The following are true:

- If $(u, v) \in G_{W-E}$ then $x_2(u) = x_1(v)$;
- If there exist a directed path from u to v in G_{W-E} containing at least two edges, then $x_2(u) < x_1(v)$
- If $(u, v) \in G_{S-N}$ then $y_2(u) = y_1(v)$;
- If there exist a directed path from u to v in G_{S-N} containing at least two edges, then $y_2(u) < y_1(v)$

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Let G_{S-N} and G_{W-E}. The following are true:

- If $(u, v) \in G_{W-E}$ then $x_2(u) = x_1(v)$;
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- If $(u, v) \in G_{S-N}$ then $y_2(u) = y_1(v)$;
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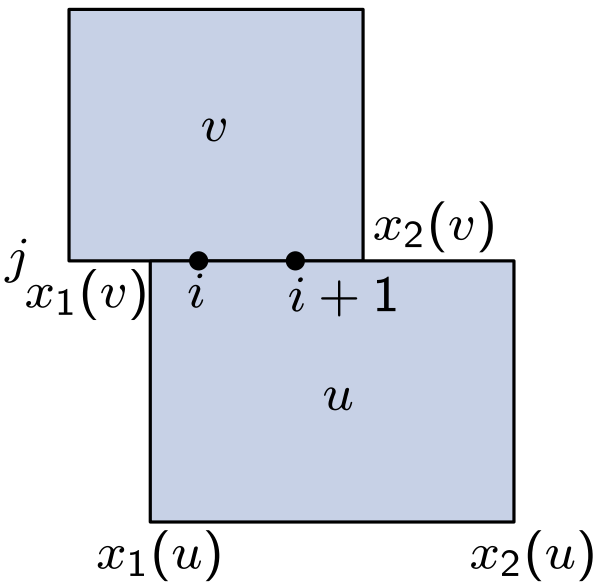
Lemma 8

The assignment provided by the algorithm has the following property: rectangles assigned to vertices u and v have a common segment if and only if there exists edge (u, v) in the graph.

Proof:

Rectangular Dual

- Assume $R(u)$ and $R(v)$ have a common boundary.



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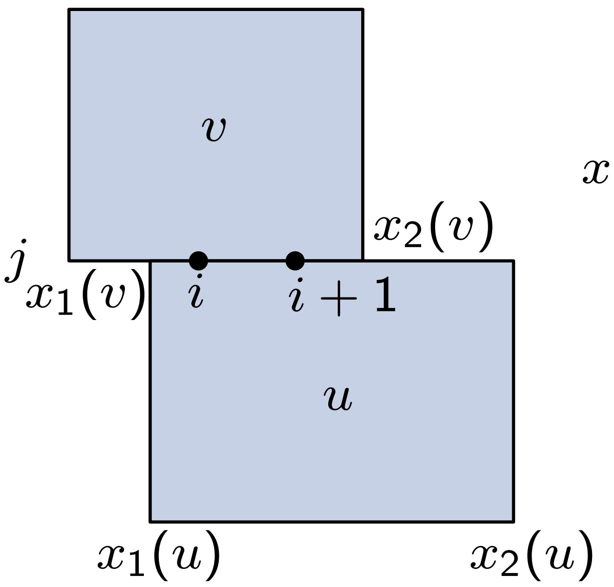

$x_1(v) \leq i$, $i+1 \leq x_2(v)$ and $x_1(u) \leq i$, $i+1 \leq x_2(u)$

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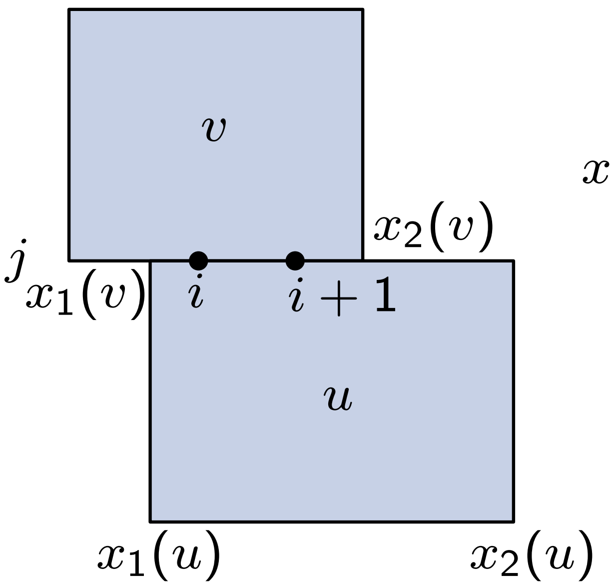

$\Downarrow$

$x_1(v) \leq i$, $i+1 \leq x_2(v)$ and $x_1(u) \leq i$, $i+1 \leq x_2(u)$

$\Downarrow$ (Lemma 4)

u, v belong to P_i

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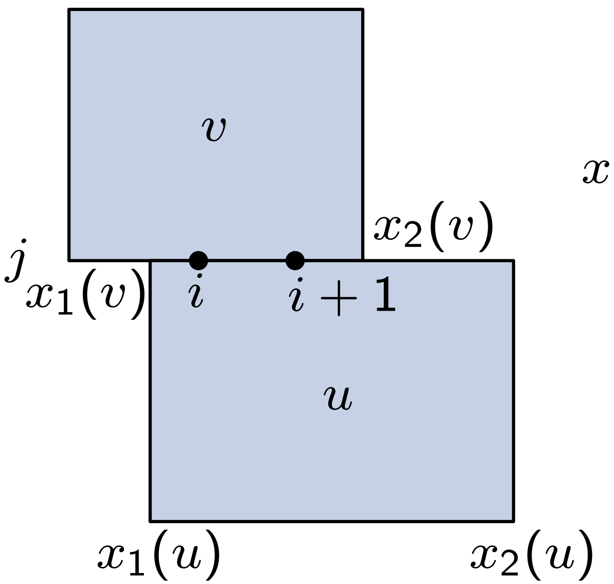

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If path between u and v has at least 2 edges, then by Lemma 7, $y_2(u) < y_1(v)$

Rectangular Dual

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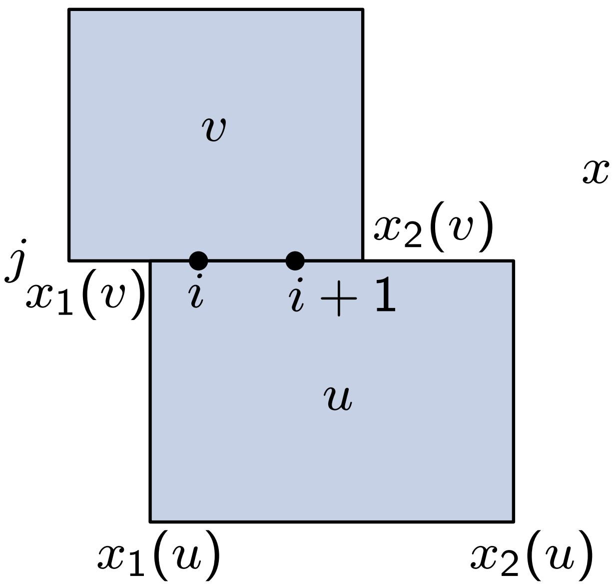

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(Lemma 4)

u, v belong to P_i

If path between u and v has at least 2 edges, then by Lemma 7, $y_2(u) < y_1(v)$

A contradiction to the hypothesis!

Rectangular Dual

- Assume there exists an edge $(u, v) \in G_{W-E}$.

$\Downarrow$

- Let Q_j be the path of G_{W-E} where (u, v) belongs. By Lemma 5, $y_1(u) \leq j$, $j + 1 \leq y_2(u)$ and $y_1(v) \leq j$, $j + 1 \leq y_2(v)$
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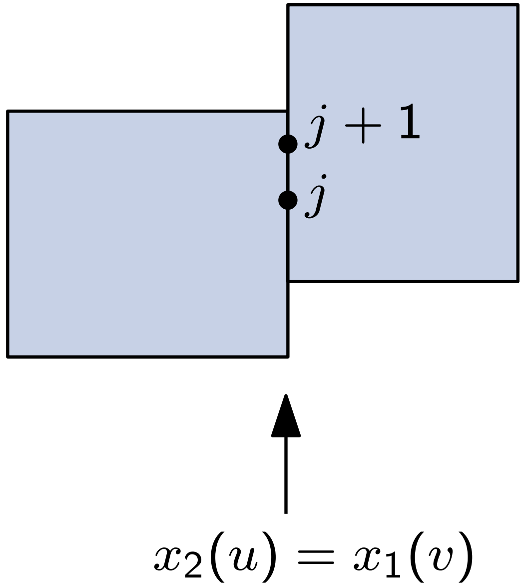

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↓

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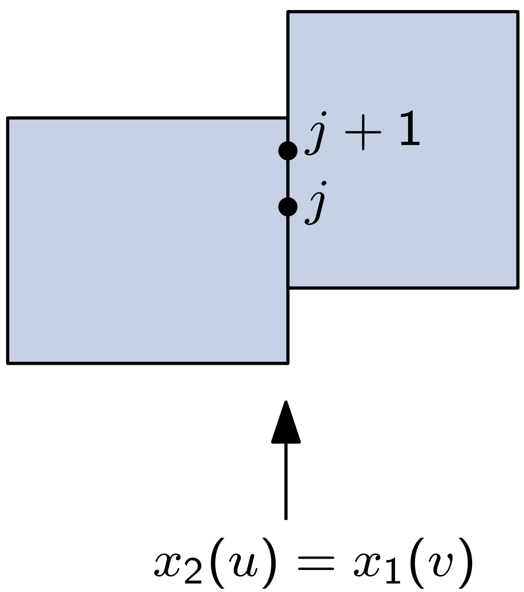


Lemma 8 is proved! ✓

Rectangular Dual

Theorem

Every PTP graph G has a rectangular dual which can be computed in linear time.

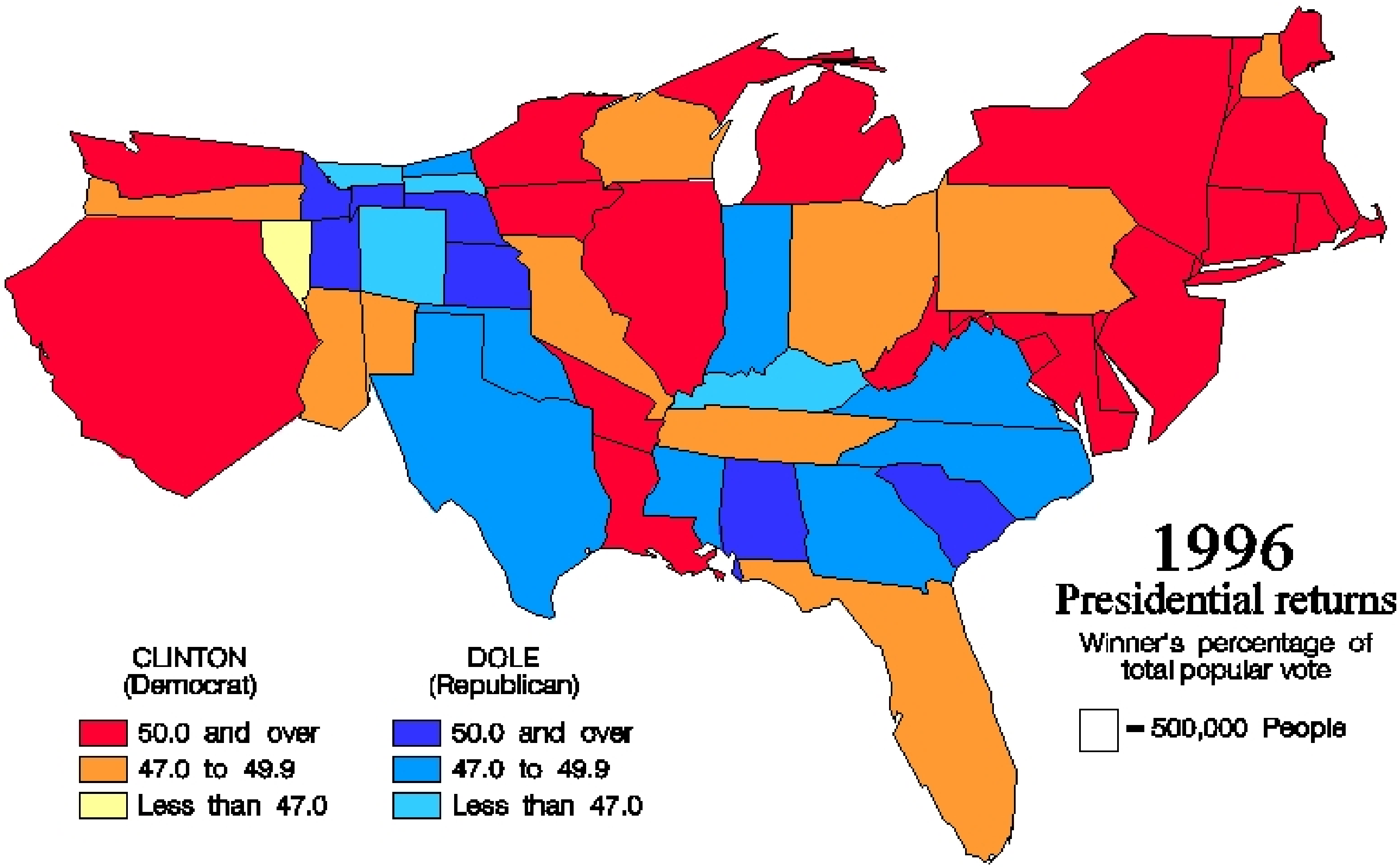
Rectangular Dual

Theorem

Every PTP graph G has a rectangular dual which can be computed in linear time.

- Compute a planar embedding of G
- Compute a revised canonical ordering of G
- Traverse the graph and color the edges, construct G_{S-N} and G_{E-W}
- Construct the duals $G^{\star}_{S-N}$ and $G^{\star}_{E-W}$ of G_{S-N} and G_{E-W}, respectively
- Compute a topological ordering of $G^{\star}_{S-N}$ and $G^{\star}_{E-W}$
- Assing coordinates to the rectangles representing vertices.

Discussion



Proportional Cartogram. Source: `http://www.ncgia.ucsb.edu`

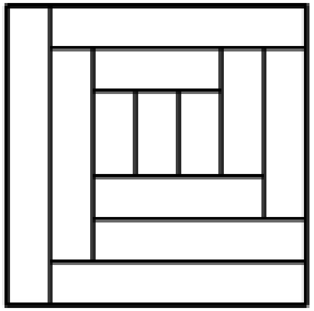
Discussion

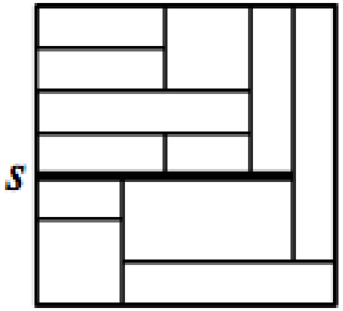
- A layout is area-universal if any assignment of areas to rectangles can be realized by a combinatorially equivalent rectangular layout.

- A rectangular layout is **area-universal** if and only if it is **one-sided**.

[Eppstein et al. SIAM J. Comp. 2012]

one-sided





not one-sided

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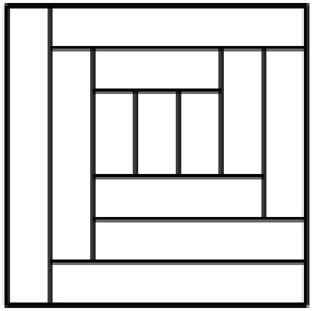
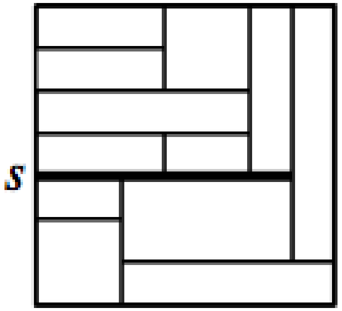
one-sided  

not one-sided

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  - De Berg et al. 2009: 40 sides

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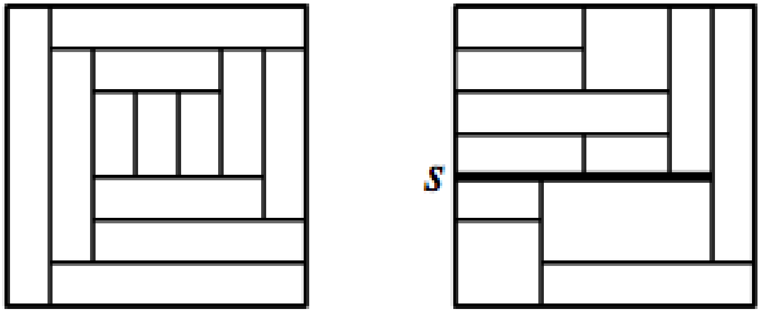

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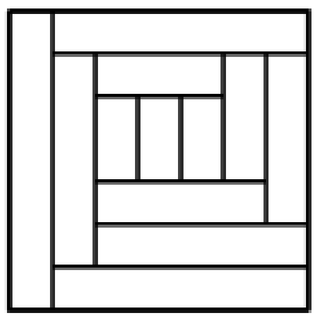

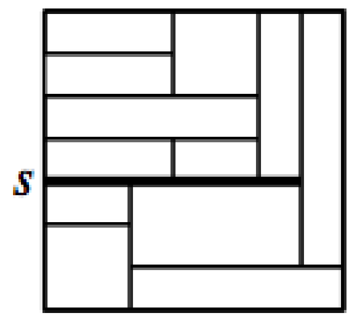

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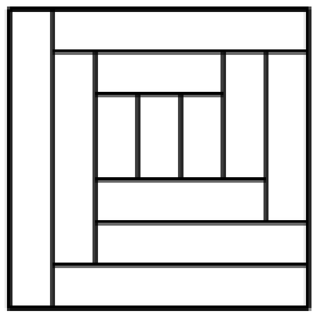
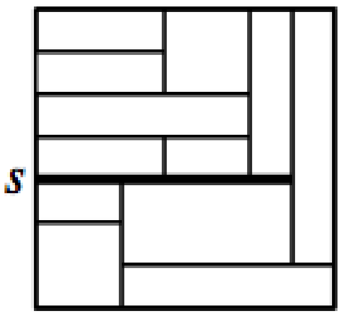
one-sided  

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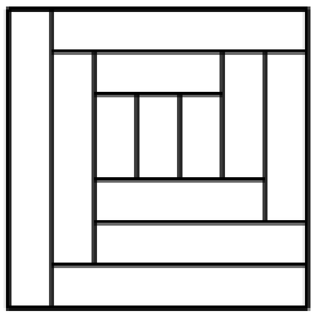

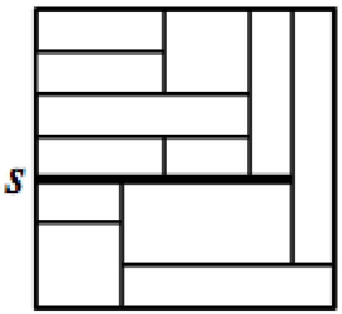

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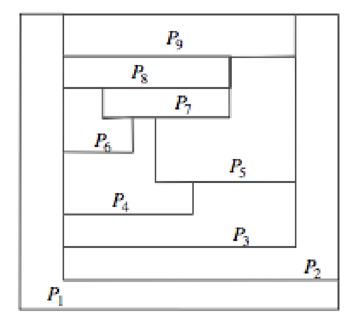
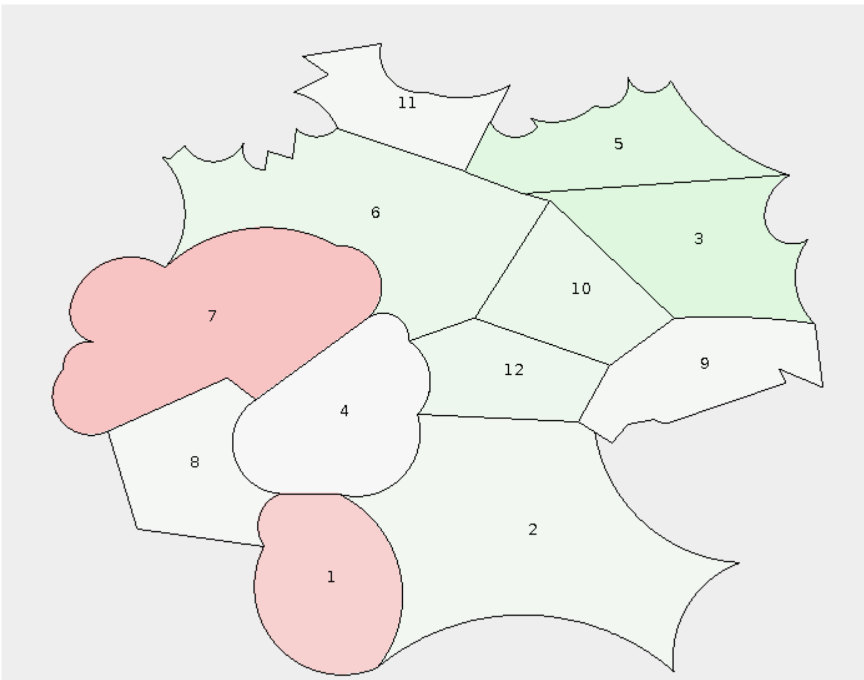
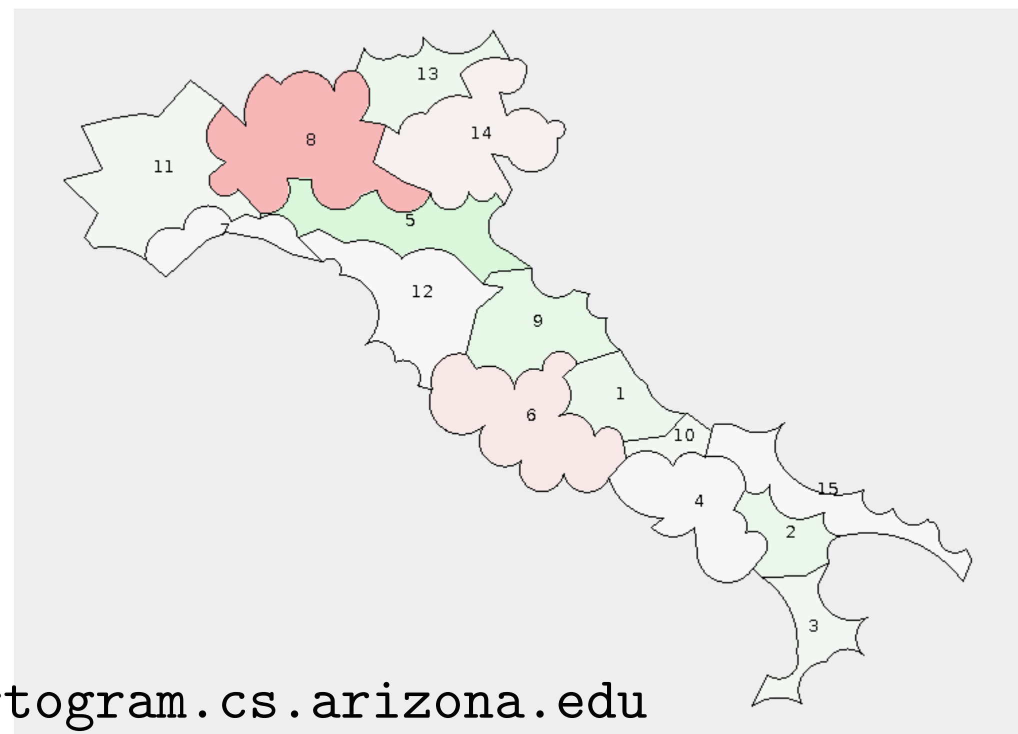
  - Alam et al. 2013: 8 sides (matches the lower bound)



Discussion

- Circular Arc Cartograms [Kämper, Kobourov, Nöllenburg. IEEE PasViz 2013]





Source: `http://cartogram.cs.arizona.edu`