Algorithms for Graph Visualization

Layered Layout – Part 2

INSTITUT FÜR THEORETISCHE INFORMATIK · FAKULTÄT FÜR INFORMATIK

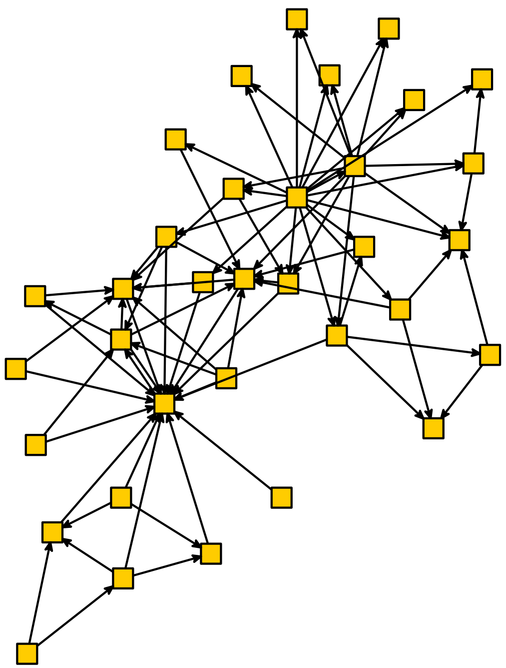
Tamara Mchedlidze

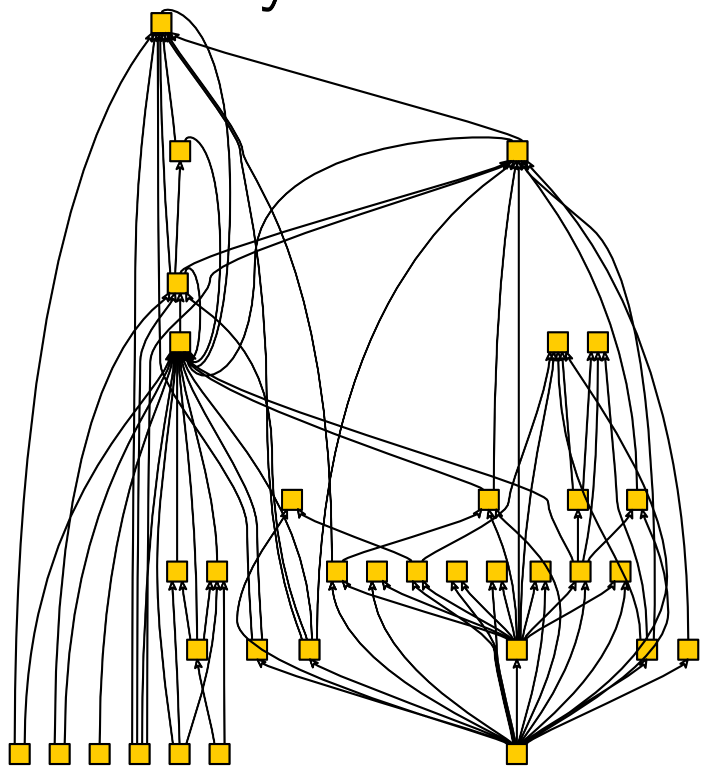
11.12.2018

Layered Layout

Given: directed graph $D = (V, A)$

Find: drawing of D that emphasizes the hierarchy by positioning nodes on horizontal layers





Layered Layout

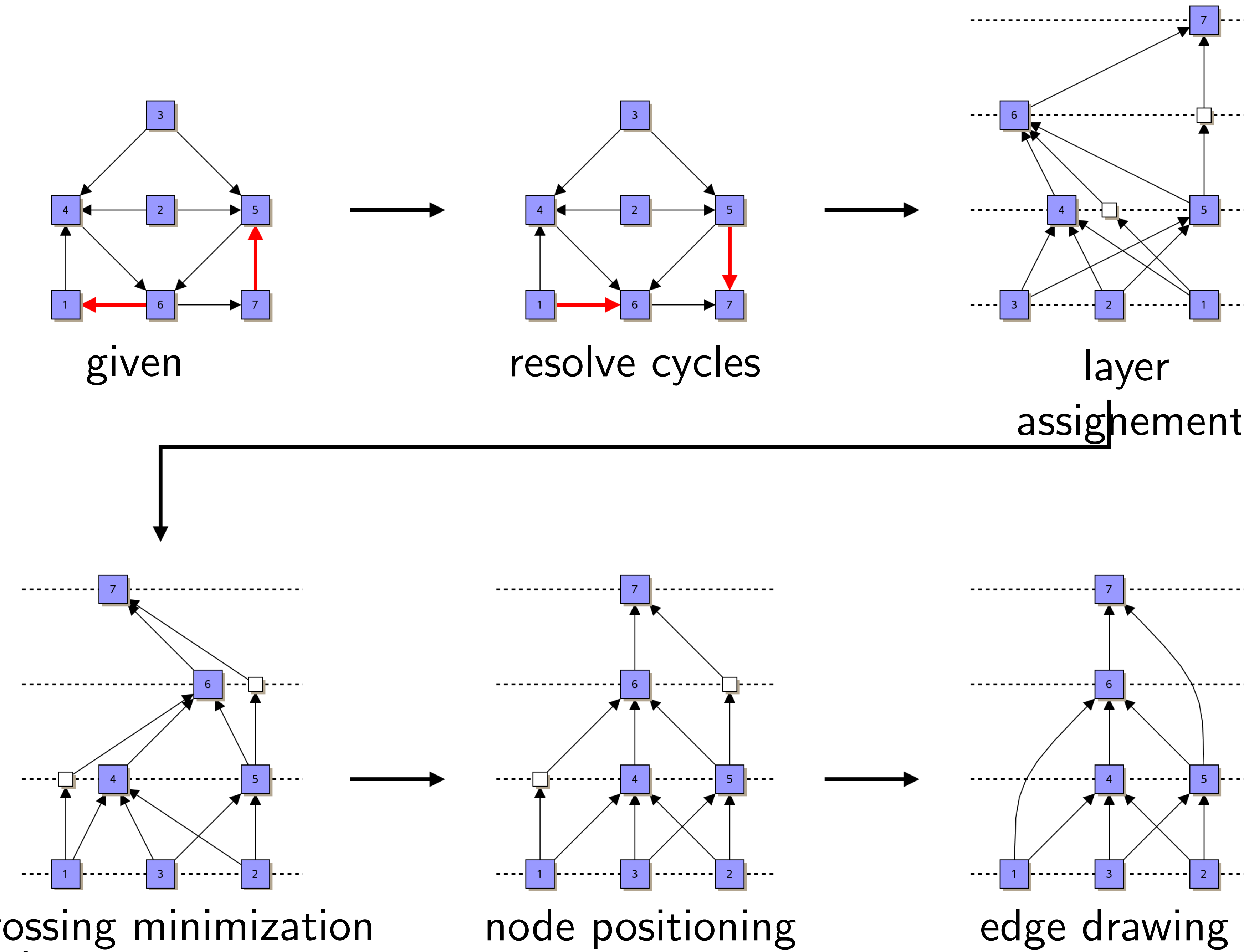
Given: directed graph $D = (V, A)$

Find: drawing of D that emphasizes the hierarchy by positioning nodes on horizontal layers

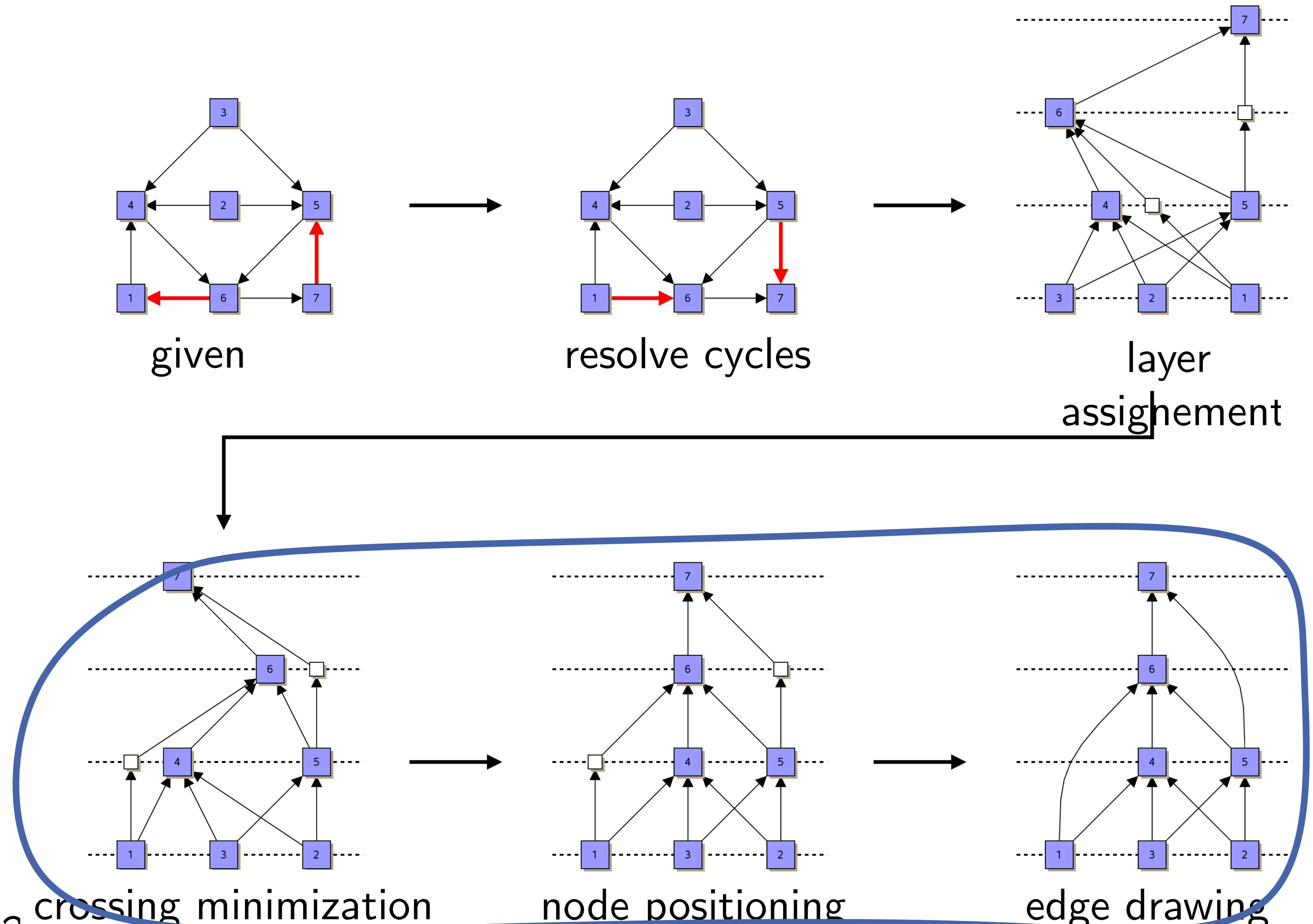
Criteria:

- many edges pointing to the same direction
- few layers or limited number of nodes per layer
- preferably few edge crossings
- nodes distributed evenly
- edges preferably straight and short

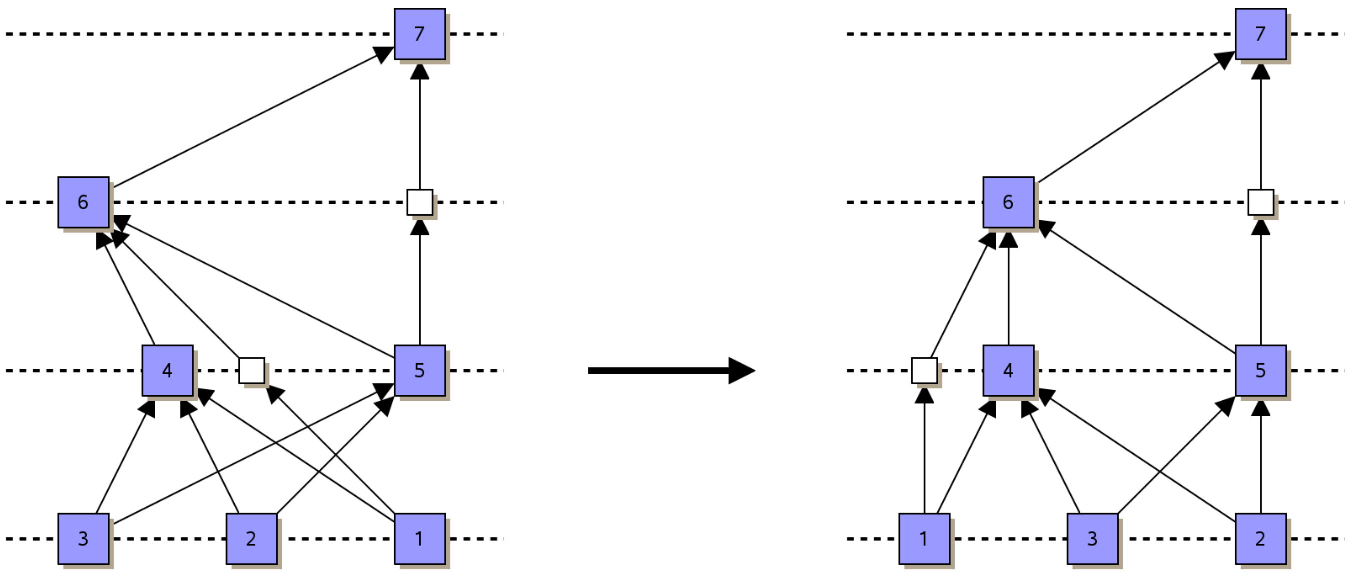
Sugiyama Framework (Sugiyama, Tagawa, Toda 1981)



Sugiyama Framework (Sugiyama, Tagawa, Toda 1981)



Step 3: Crossing Minimization



Problem Statement

Given: DAG $D = (V, A)$, nodes are partitioned in disjoint layers

Find: Order of the nodes on each layer, so that the number of crossing is minimized

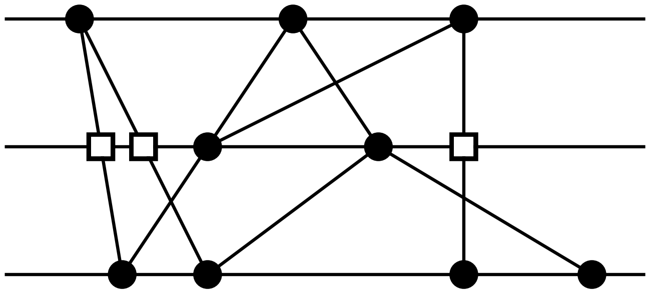
Problem Statement

Given: DAG $D = (V, A)$, nodes are partitioned in disjoint layers

Find: Order of the nodes on each layer, so that the number of crossing is minimized

Properties

- Problem is NP-hard even for two layers (Bipartite Crossing Number [Garey, Johnson '83])
- No approach over several layers simultaneously
- Usually iterative optimization for two adjacent layers
- For that: insert dummy nodes at the intersection of edges with layers



One-sided Crossing Minimization (OSCM)

Given: 2-Layered-Graph $G = (L_1, L_2, E)$ and ordering of the nodes x_1 of L_1

Find: Node ordering x_2 of L_2, such that the number of crossing among E is minumum

One-sided Crossing Minimization (OSCM)

Given: 2-Layered-Graph $G = (L_1, L_2, E)$ and ordering of the nodes x_1 of L_1

Find: Node ordering x_2 of L_2, such that the number of crossing among E is minumum

Observation:

- The number of crossing in 2-layered drawing of G depends only on ordering of the nodes, not from the exact positions
- for $u, v \in L_2$ the number of crossing among incident to them edges depends on whether $x_2(u) < x_2(v)$ or $x_2(v) < x_2(u)$ and not on the positions of other vertices

One-sided Crossing Minimization (OSCM)

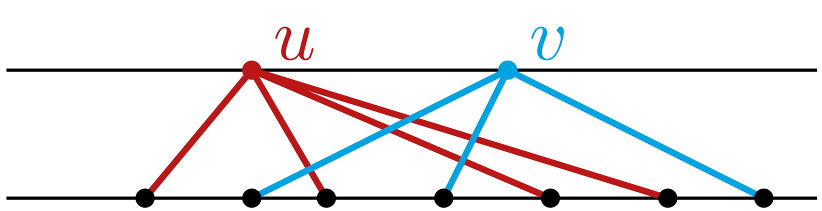
Given: 2-Layered-Graph $G = (L_1, L_2, E)$ and ordering of the nodes x_1 of L_1

Find: Node ordering x_2 of L_2, such that the number of crossing among E is minumum

Observation:

- The number of crossing in 2-layered drawing of G depends only on ordering of the nodes, not from the exact positions
- for $u, v \in L_2$ the number of crossing among incident to them edges depends on whether $x_2(u) < x_2(v)$ or $x_2(v) < x_2(u)$ and not on the positions of other vertices

Def: $c_{uv} := |\{(uw, vz) : w \in N(u), z \in N(v), x_1(z) < x_1(w)\}|$ for $x_2(u) < x_2(v)$



$c_{uv} = 5$

$c_{vu} = 7$

One-sided Crossing Minimization (OSCM)

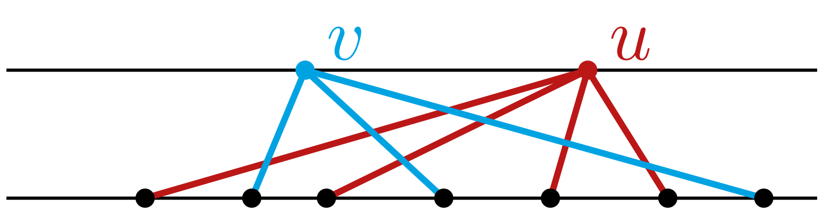
Given: 2-Layered-Graph $G = (L_1, L_2, E)$ and ordering of the nodes x_1 of L_1

Find: Node ordering x_2 of L_2, such that the number of crossing among E is minumum

Observation:

- The number of crossing in 2-layered drawing of G depends only on ordering of the nodes, not from the exact positions
- for $u, v \in L_2$ the number of crossing among incident to them edges depends on whether $x_2(u) < x_2(v)$ or $x_2(v) < x_2(u)$ and not on the positions of other vertices

Def: $c_{uv} := |\{(uw, vz) : w \in N(u), z \in N(v), x_1(z) < x_1(w)\}|$ for $x_2(u) < x_2(v)$



$c_{uv} = 5$
$c_{vu} = 7$

Further Properties

Def: Crossing number of G with orders x_1 and x_2 for L_1 and L_2 is denoted by cr(G, x_1, x_2);
for fixed x_1 then $\text{opt}(G, x_1) = \min_{x_2} \text{cr}(G, x_1, x_2)$

Lemma 1: The following equalities hold:

- $\text{cr}(G, x_1, x_2) = \sum_{x_2(u)<x_2(v)} c_{uv}$
- $\text{opt}(G, x_1) \geq \sum_{\{u,v\}} \min\{c_{uv}, c_{vu}\}$

Further Properties

Def: Crossing number of G with orders x_1 and x_2 for L_1 and L_2 is denoted by cr(G, x_1, x_2);
for fixed x_1 then $\mathrm{opt}(G, x_1) = \min_{x_2}$ cr(G, x_1, x_2)

Lemma 1: The following equalities hold:

- cr$(G, x_1, x_2) = \sum_{x_2(u) < x_2(v)} c_{uv}$
- $\mathrm{opt}(G, x_1) \geq \sum_{\{u,v\}} \min\{c_{uv}, c_{vu}\}$

Efficient computation of cr(G, x_1, x_2) see Exercise.

Further Properties

Def: Crossing number of G with orders x_1 and x_2 for L_1 and L_2 is denoted by cr(G, x_1, x_2);
for fixed x_1 then $\operatorname{opt}(G, x_1) = \min_{x_2} \text{cr}(G, x_1, x_2)$

Lemma 1: The following equalities hold:

- cr$(G, x_1, x_2) = \sum_{x_2(u) < x_2(v)} c_{uv}$
- $\operatorname{opt}(G, x_1) \geq \sum_{\{u,v\}} \min\{c_{uv}, c_{vu}\}$

Efficient computation of cr(G, x_1, x_2) see Exercise.

Think for a minute and then share

Why the second inequality is not an equality?

1 min

Iterative Crossing Minimization

Let $G = (V, E)$ be a DAG with layers $L_1, \dots, L_h$.

(1) compute a random ordering x_1 for layer L_1
(2) for $i = 1, \dots, h-1$ consider layers L_i and L_{i+1} and minimize cr(G, x_i, x_{i+1}) with fixed x_i ($\rightarrow$ **OSCM**)
(3) for $i = h-1, \dots, 1$ consider layers L_{i+1} and L_i and minimize cr(G, x_i, x_{i+1}) with fixed x_{i+1} ($\rightarrow$ **OSCM**)
(4) repeat (2) and (3) until no further improvement happens
(5) repeat steps (1)–(4) with another x_1
(6) return the best found solution

Iterative Crossing Minimization

Let $G = (V, E)$ be a DAG with layers $L_1, \dots, L_h$.

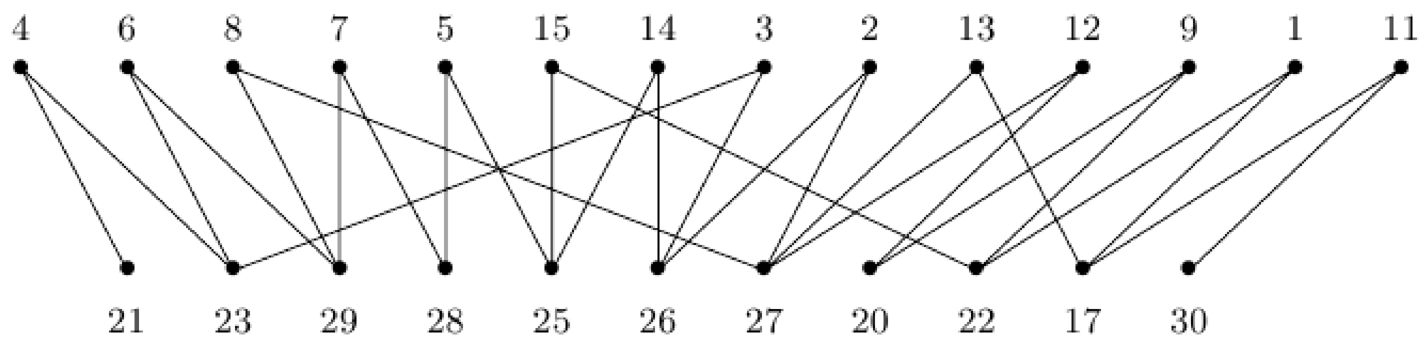
(1) compute a random ordering x_1 for layer L_1
(2) for $i = 1, \dots, h-1$ consider layers L_i and L_{i+1} and minimize $\mathrm{cr}(G, x_i, x_{i+1})$ with fixed x_i ($\rightarrow$ **OSCM**)
(3) for $i = h-1, \dots, 1$ consider layers L_{i+1} and L_i and minimize $\mathrm{cr}(G, x_i, x_{i+1})$ with fixed x_{i+1} ($\rightarrow$ **OSCM**)
(4) repeat (2) and (3) until no further improvement happens
(5) repeat steps (1)–(4) with another x_1
(6) return the best found solution

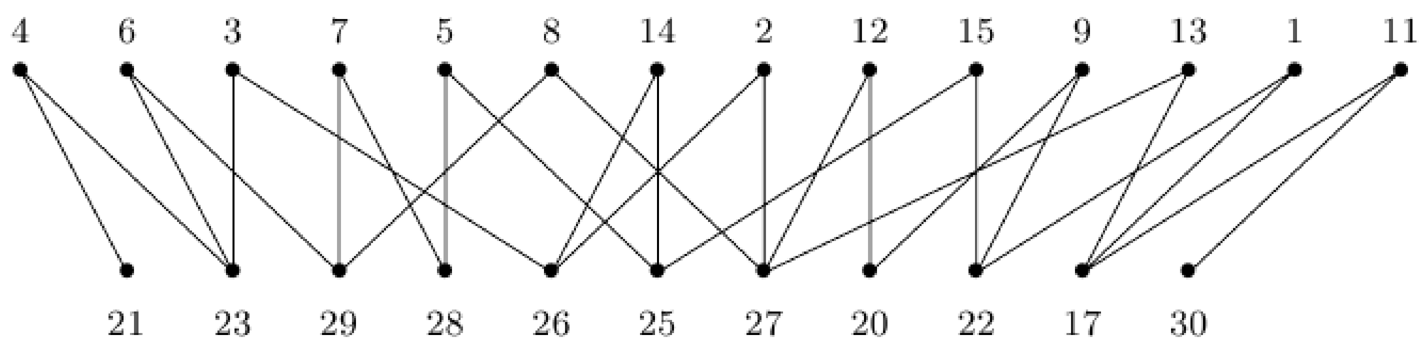
Theorem 1: The One-Sided Crossing Minimization (OSCM) problem is NP-hard (Eades, Wormald 1994).

Algorithms for OSCM

Heuristics:
- Barycenter
- Median

Exact:
- ILP Model





Barycenter Heuristic (Sugiyama, Tagawa, Toda 1981)

Idea: few crossing when nodes are close to their neighbours

- set

$$x_2(u) = \frac{1}{\deg(u)} \sum_{v \in N(u)} x_1(v)$$

- in case of equality introduce tiny gap

Barycenter Heuristic (Sugiyama, Tagawa, Toda 1981)

Idea: few crossing when nodes are close to their neighbours

- set

$$x_2(u) = \frac{1}{\deg(u)} \sum_{v \in N(u)} x_1(v)$$

- in case of equality introduce tiny gap

Properties:

- trivial implementation
- quick (exactly?)
- usually very good results
- finds optimum if $\mathrm{opt}(G, x_1) = 0$ (see Exercises)
- there are graphs on which it performs $\Omega(\sqrt{n})$ times worse than optimal

Barycenter Heuristic (Sugiyama, Tagawa, Toda 1981)

Idea: few crossing when nodes are close to their neighbours

- set

$$x_2(u) = \frac{1}{\deg(u)} \sum_{v \in N(u)} x_1(v)$$

- in case of equality introduce tiny gap

Properties:

- trivial implementation
- quick (exactly?)
- usually very good results
- finds optimum if $\operatorname{opt}(G, x_1) = 0$ (see Exercises)
- there are graphs on which it performs $\Omega(\sqrt{n})$ times worse than optimal

Work with your neighbour and then share

Construct an example proving that barycenter method works at least $\sqrt{n}$ times worse than optimal

5 min

Median-Heuristic (Eades, Wormald 1994)

Idea: use the median of the coordinates of neighbours

- for a node $v \in L_2$ with neighbours $v_1, \ldots, v_k$ set $x_2(v) = \mathrm{med}(v) = x_1(v_{\lceil k/2 \rceil})$ and $x_2(v) = 0$ if $N(v) = \emptyset$
- if $x_2(u) = x_2(v)$ and u, v have different parity, place the node with odd degree to the left
- if $x_2(u) = x_2(v)$ and u, v have the same parity, place an arbitrary of them to the left
- Runs in time $O(|E|)$

Median-Heuristic (Eades, Wormald 1994)

Idea: use the median of the coordinates of neighbours

- for a node $v \in L_2$ with neighbours $v_1, \ldots, v_k$ set $x_2(v) = \mathrm{med}(v) = x_1(v_{\lceil k/2 \rceil})$ and $x_2(v) = 0$ if $N(v) = \emptyset$
- if $x_2(u) = x_2(v)$ and u, v have different parity, place the node with odd degree to the left
- if $x_2(u) = x_2(v)$ and u, v have the same parity, place an arbitrary of them to the left
- Runs in time $O(|E|)$

Properties:

- trivial implementation
- fast
- mostly good performance
- finds optimum when $\mathrm{opt}(G, x_1) = 0$
- **Factor-3 Approximation**

Approximation Factor

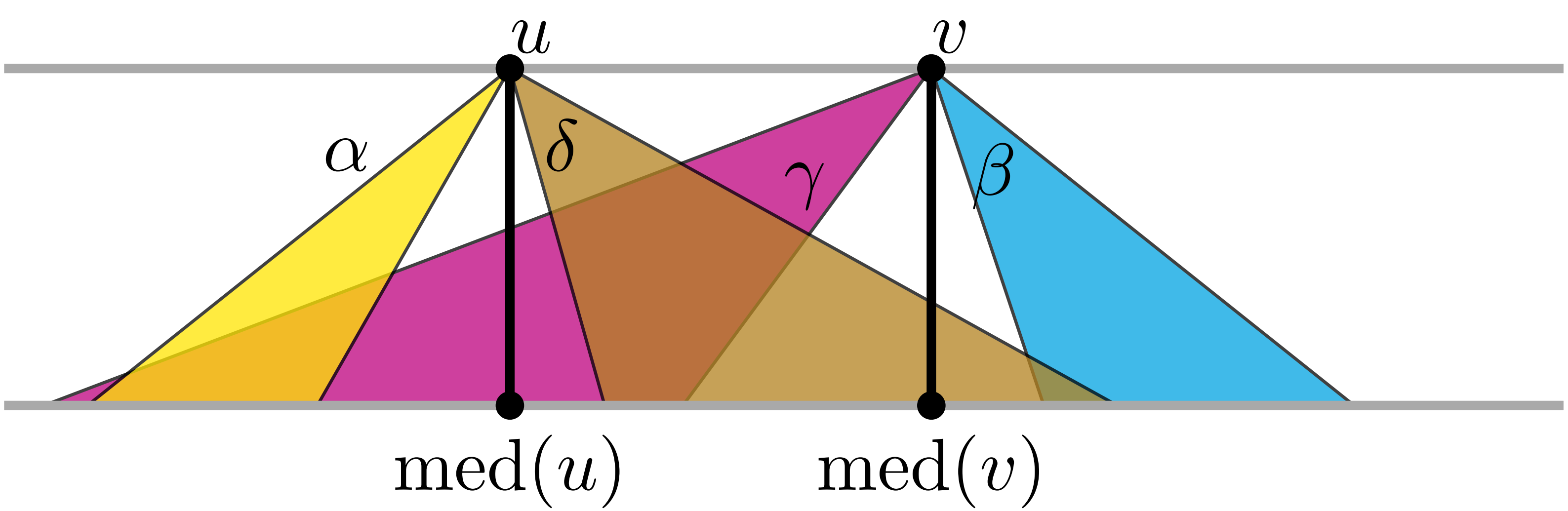
Theorem 2: Let $G = (L_1, L_2, E)$ be a 2-layered graph and x_1 an arbitrary ordering of L_1. Then it holds that

$$\mathrm{med}(G, x_1) \leq 3\,\mathrm{opt}(G, x_1).$$

Approximation Factor

Theorem 2: Let $G = (L_1, L_2, E)$ be a 2-layered graph and x_1 an arbitrary ordering of L_1. Then it holds that

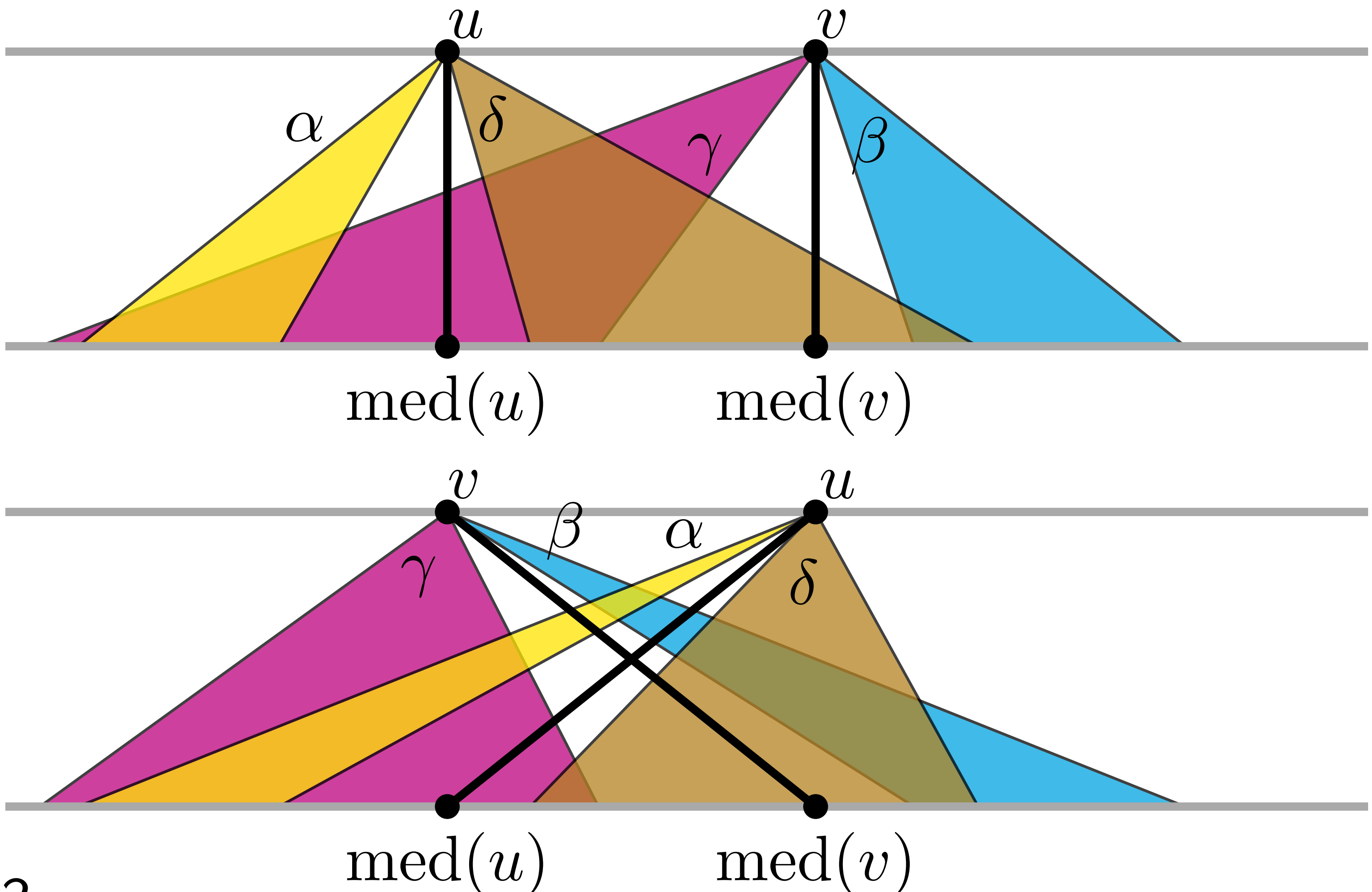
$$\mathrm{med}(G, x_1) \leq 3\,\mathrm{opt}(G, x_1).$$



Approximation Factor

Theorem 2: Let $G = (L_1, L_2, E)$ be a 2-layered graph and x_1 an arbitrary ordering of L_1. Then it holds that

$$\mathrm{med}(G, x_1) \leq 3\,\mathrm{opt}(G, x_1).$$



Integer Linear Programming

Properties:

- branch-and-cut technique for DAGS of limited size
- useful for graphs of small to medium size
- finds optimal solution
- solution in polynomial time is not guaranteed

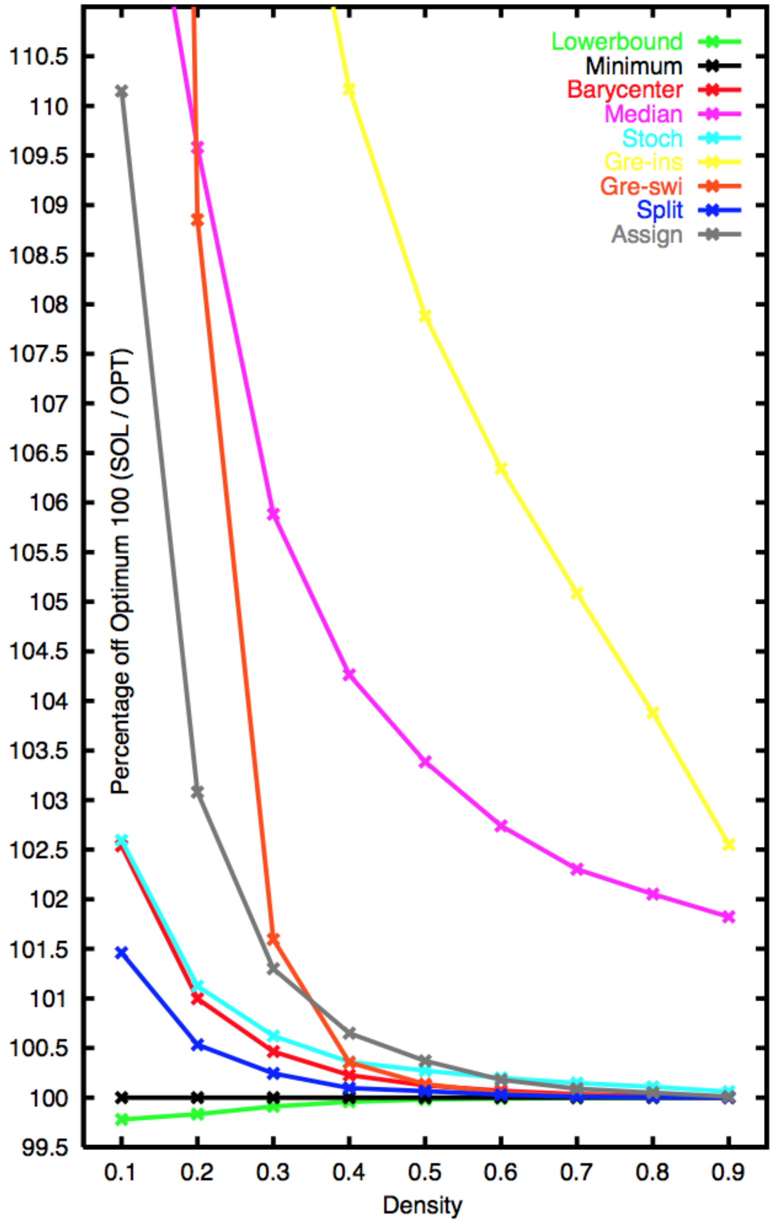
Integer Linear Programming

Properties:

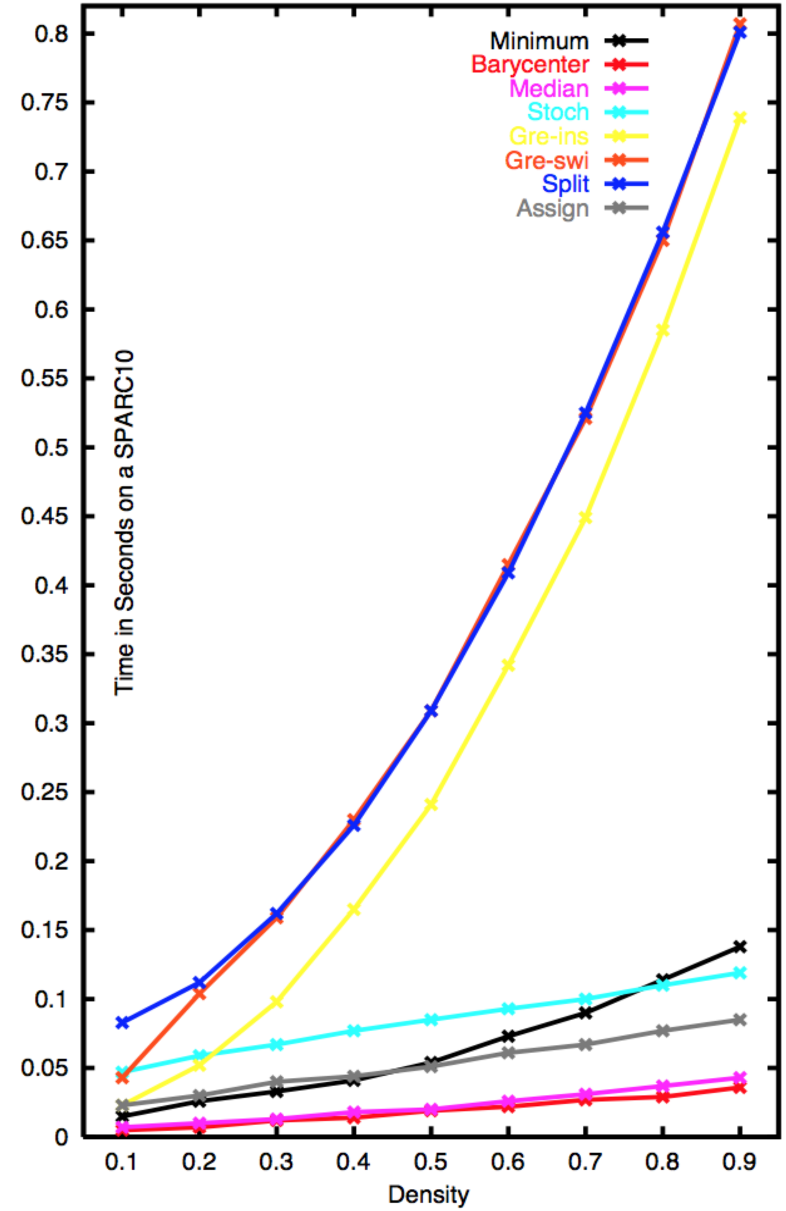
- branch-and-cut technique for DAGS of limited size
- useful for graphs of small to medium size
- finds optimal solution
- solution in polynomial time is not guaranteed

Modell: see Blackboard

Experimental Evaluation (Jünger, Mutzel 1997)

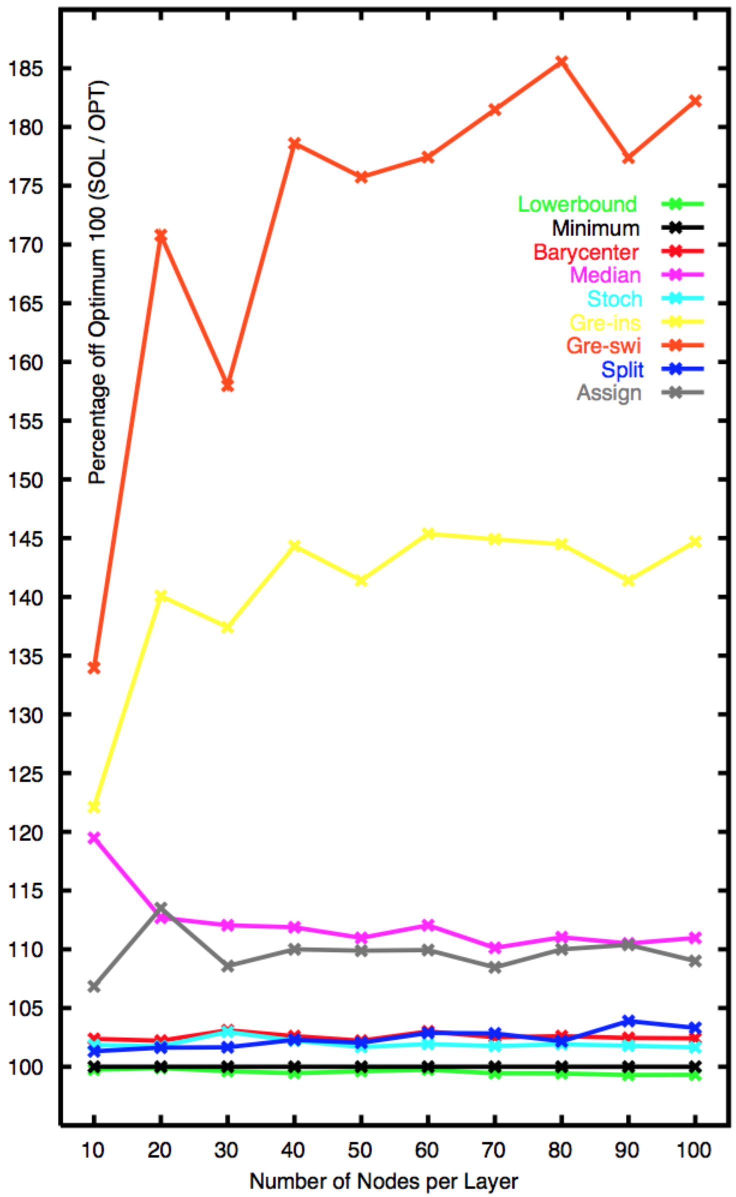


Results for 100 instances on 20 + 20 nodes with increasing density

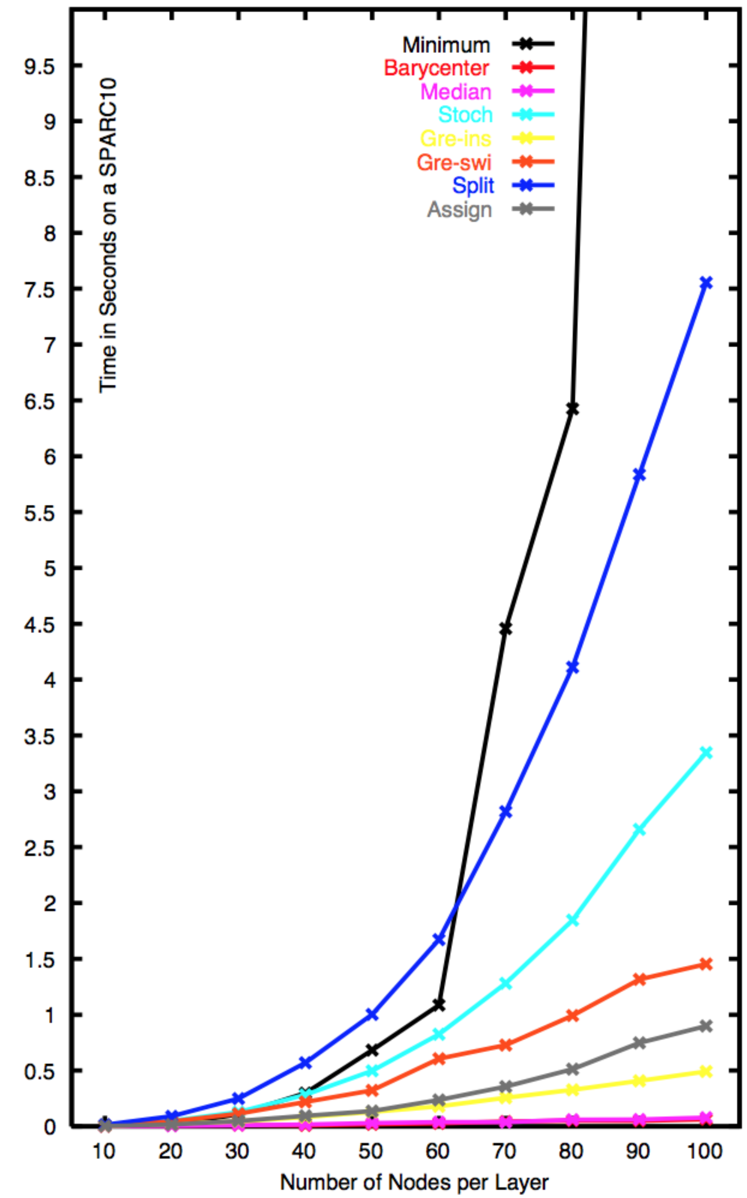


Time for 100 instances on 20 + 20 nodes with increasing density

Experimental Evaluation (Jünger, Mutzel 1997)



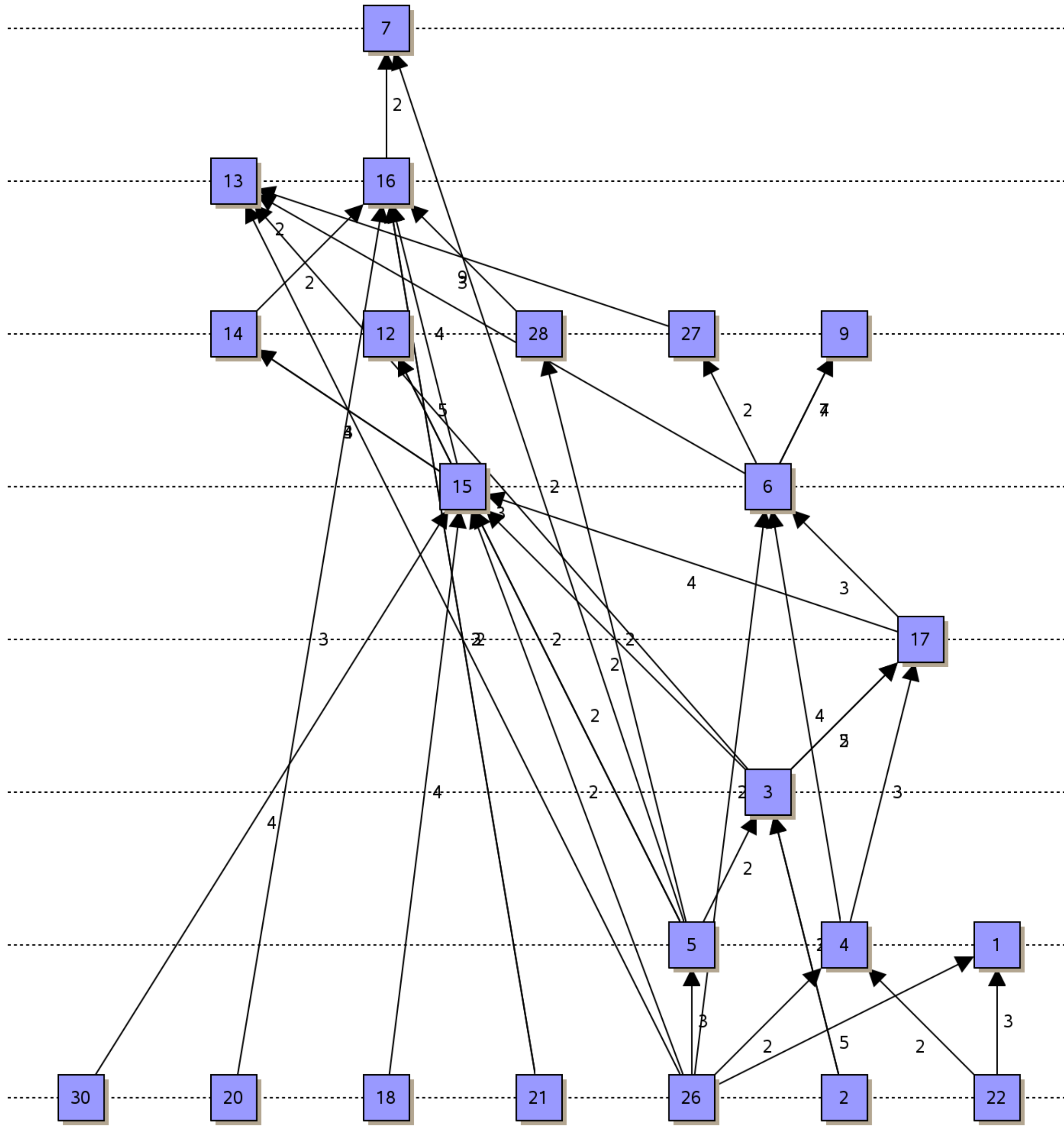
Results for 10 instances of sparse graphs with increasing size



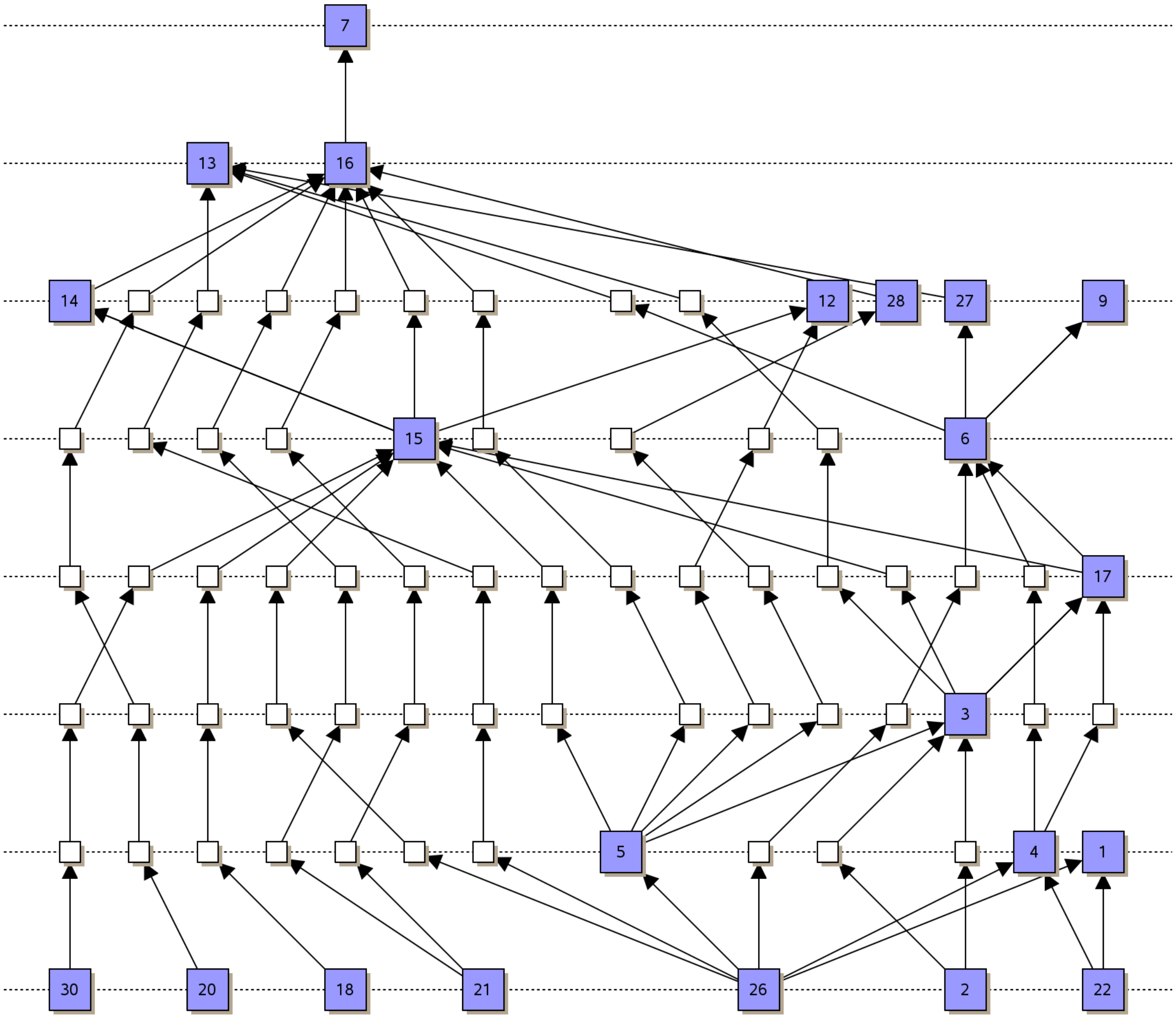
Time for 10 instances of sparse graphs with increasing size

Example

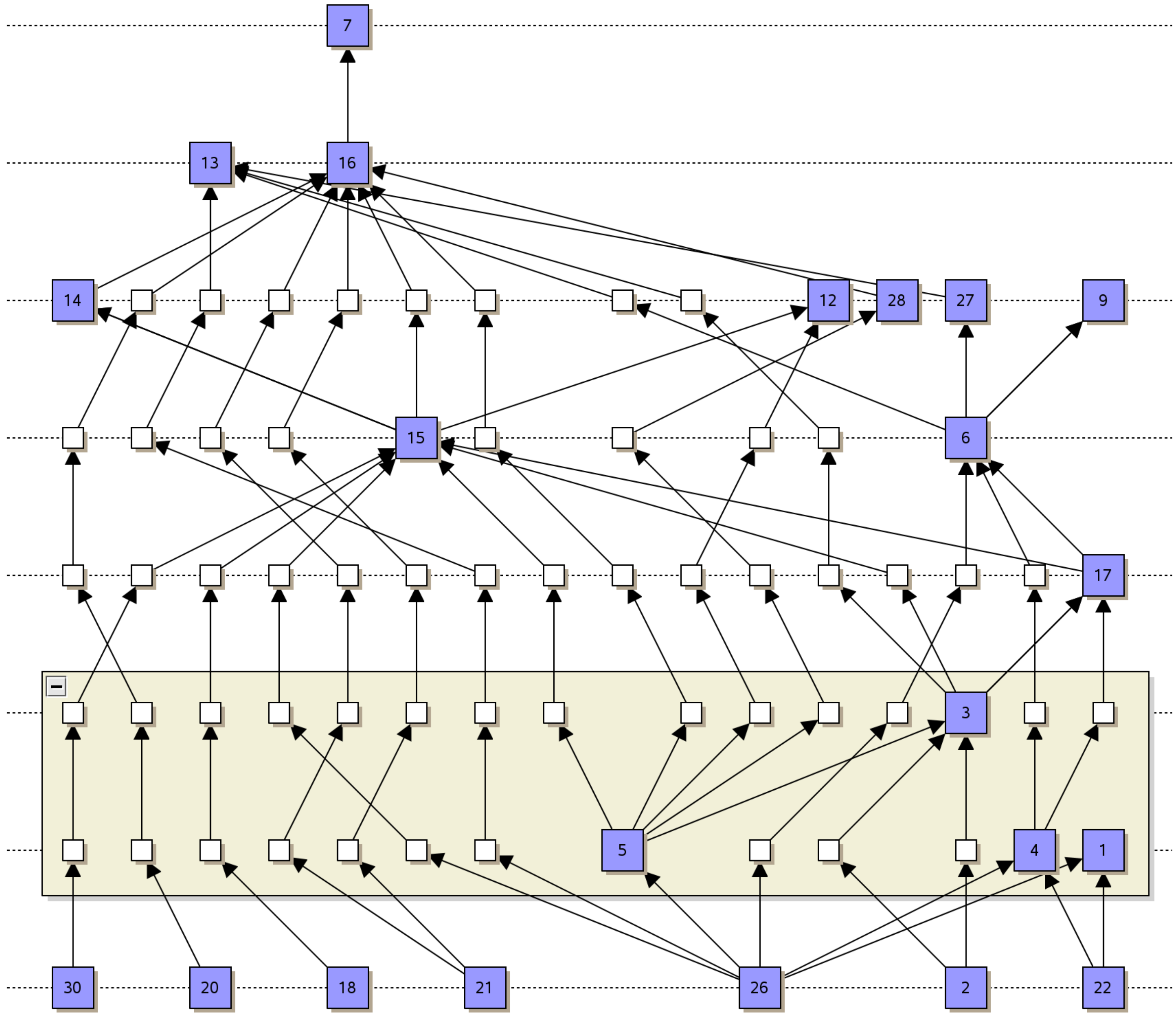
Karlsruhe Institute of Technology



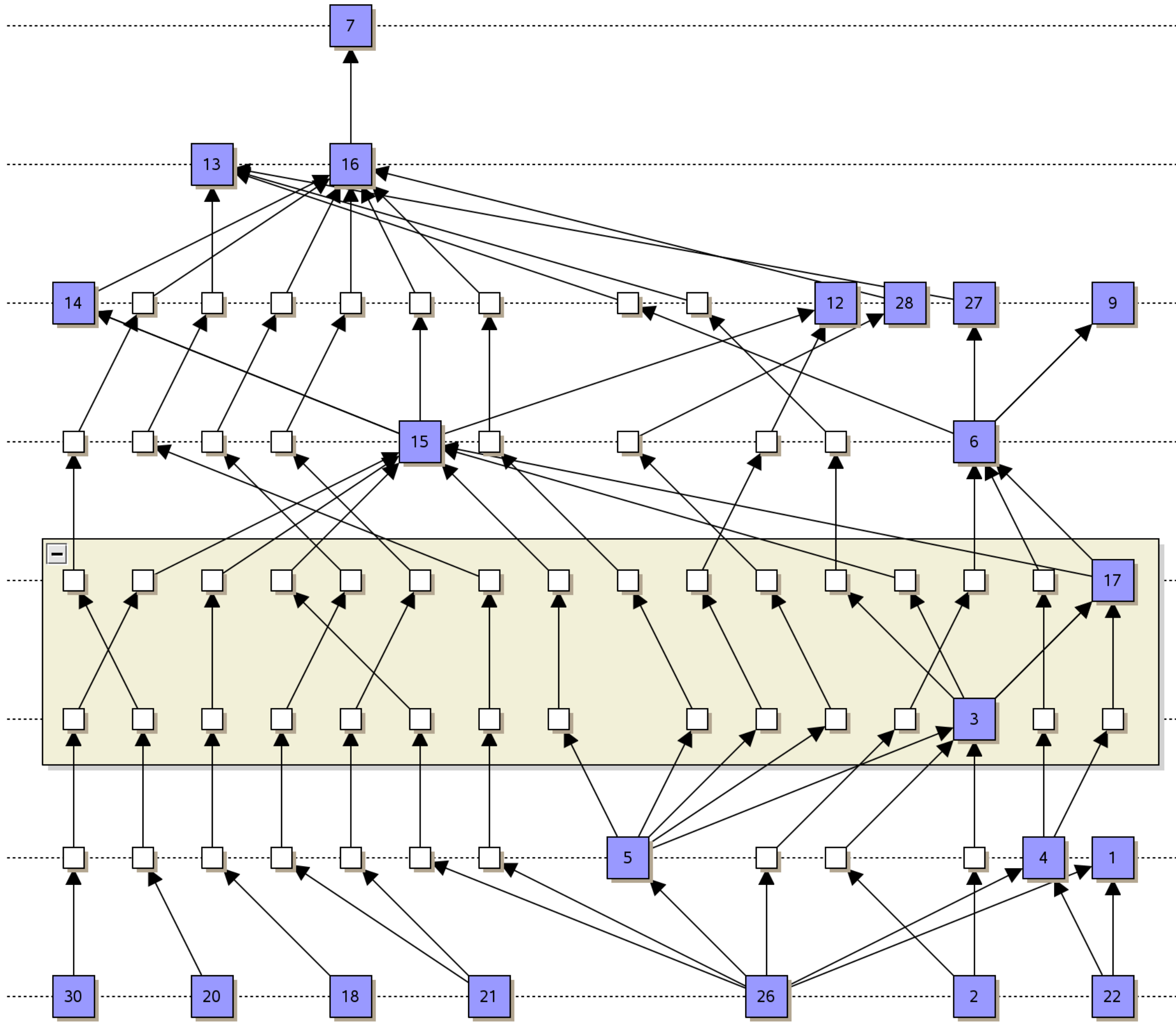
Example



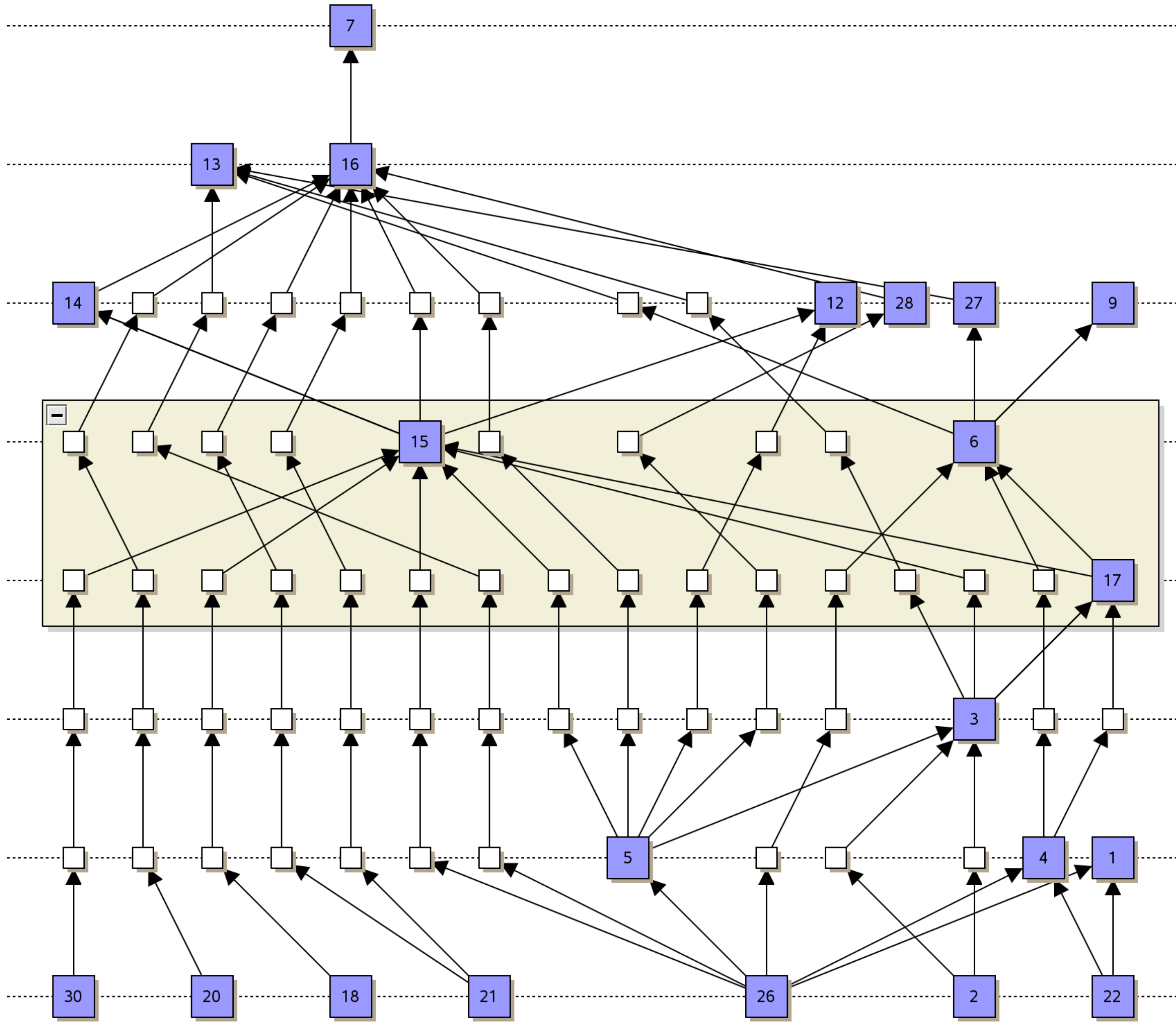
Example



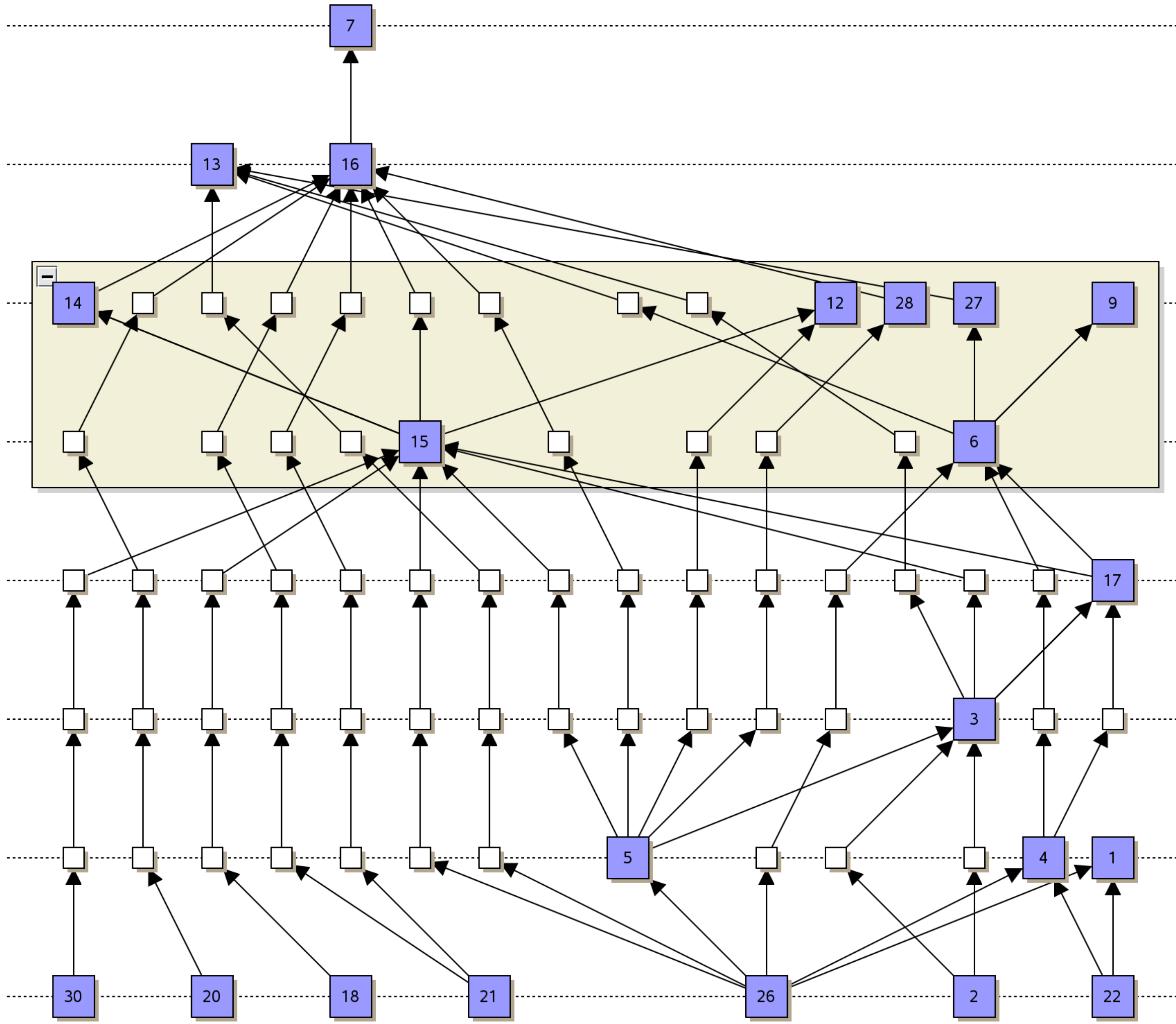
Example



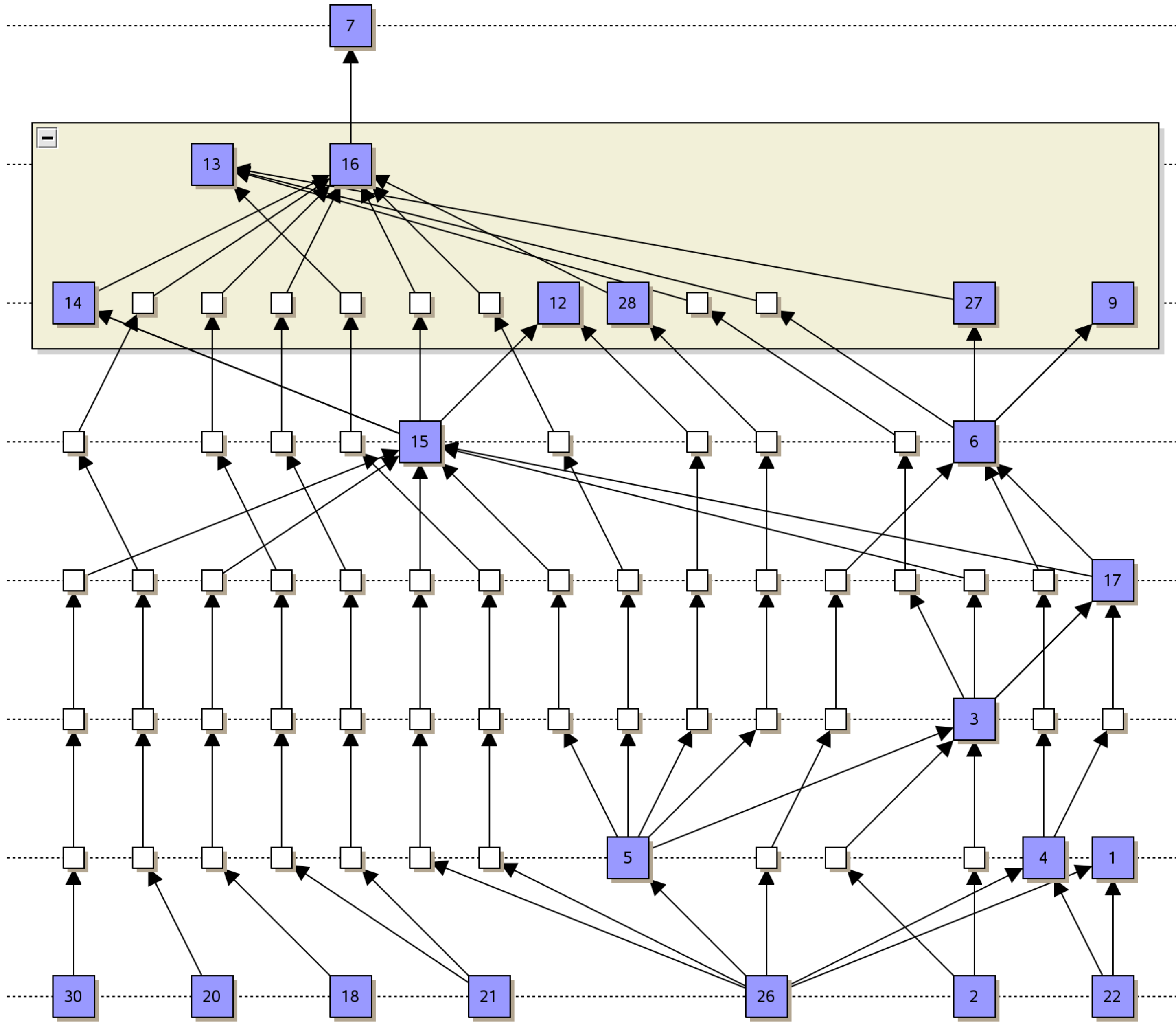
Example



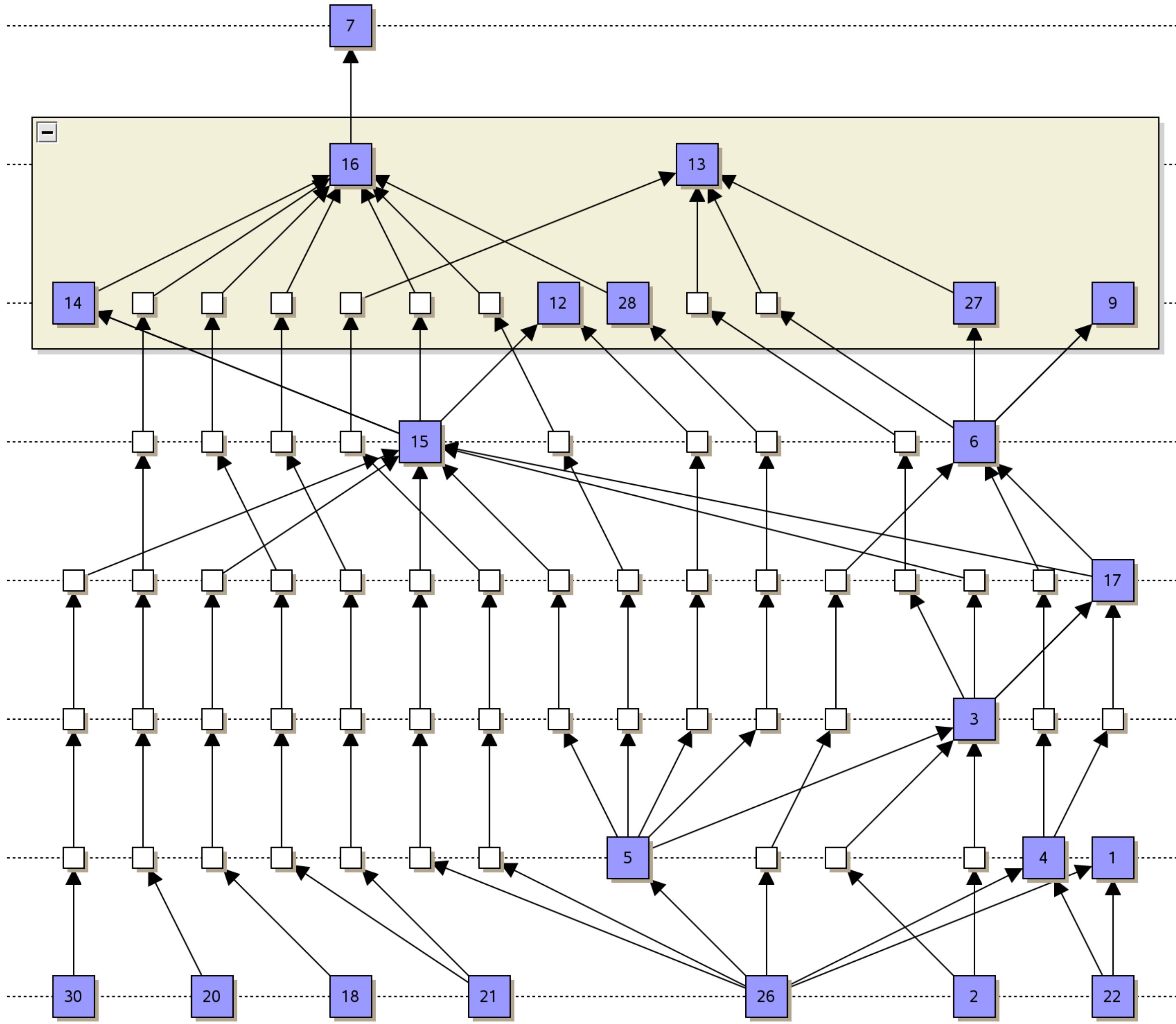
Example



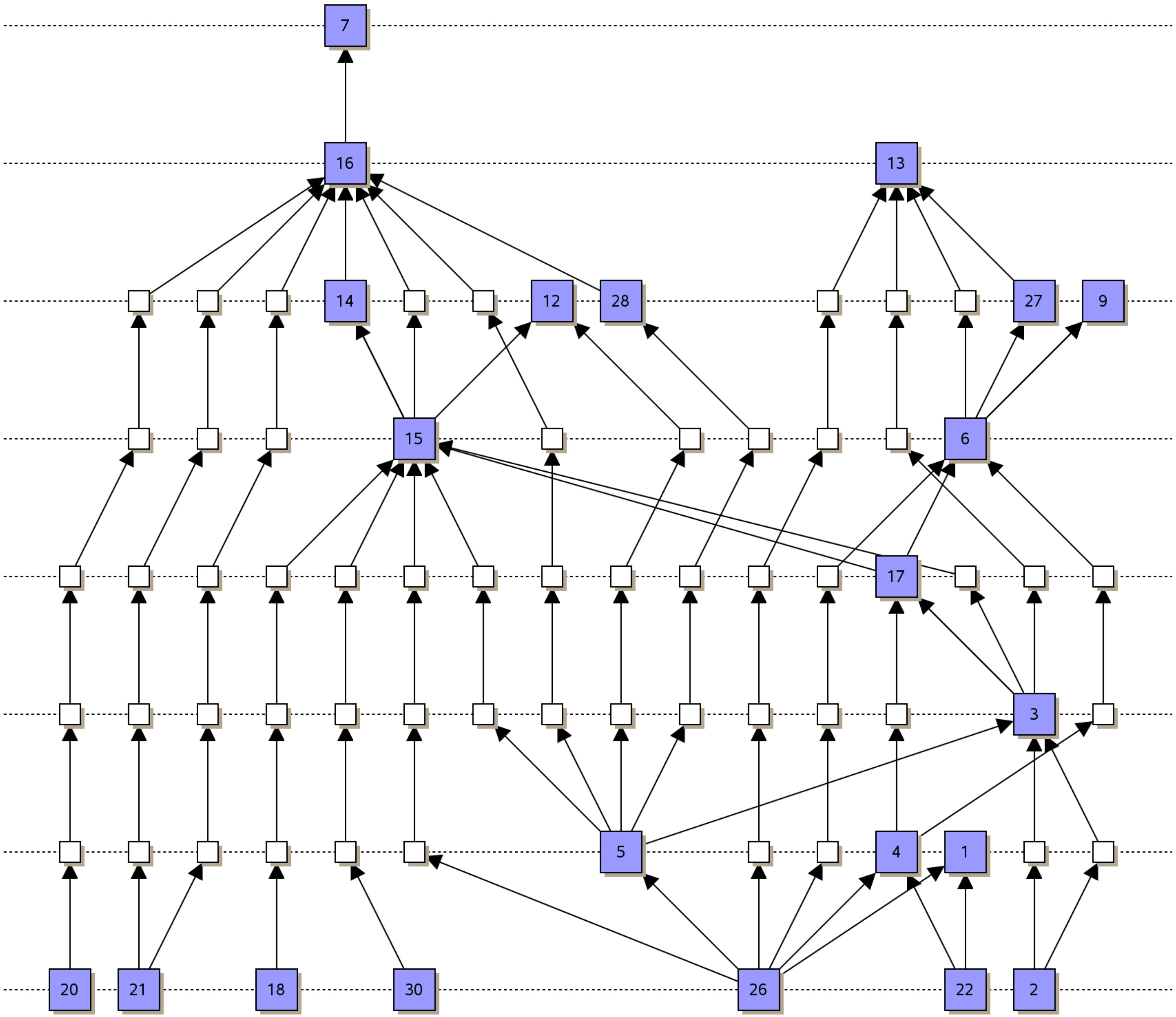
Example



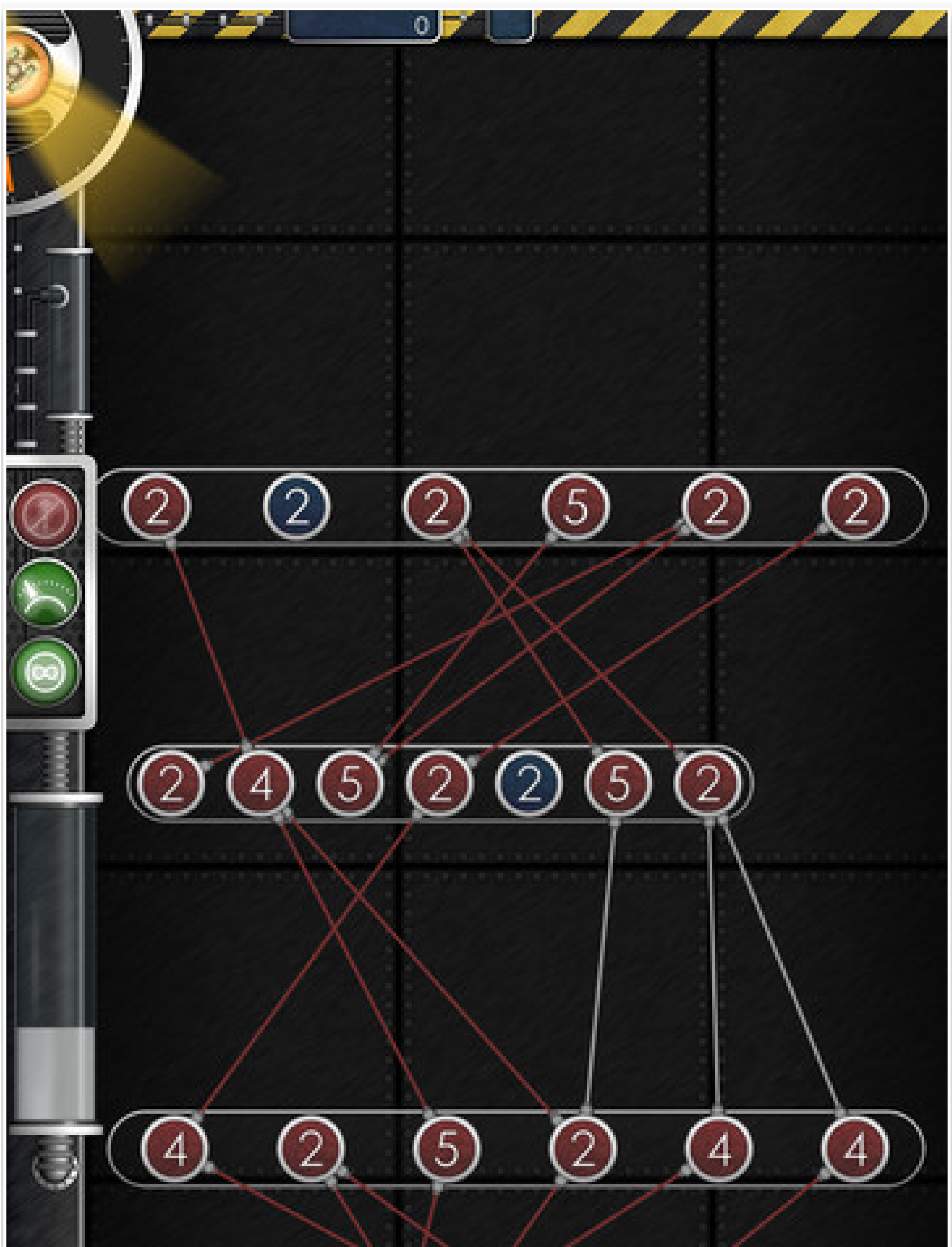
Example



Example



CrossingX

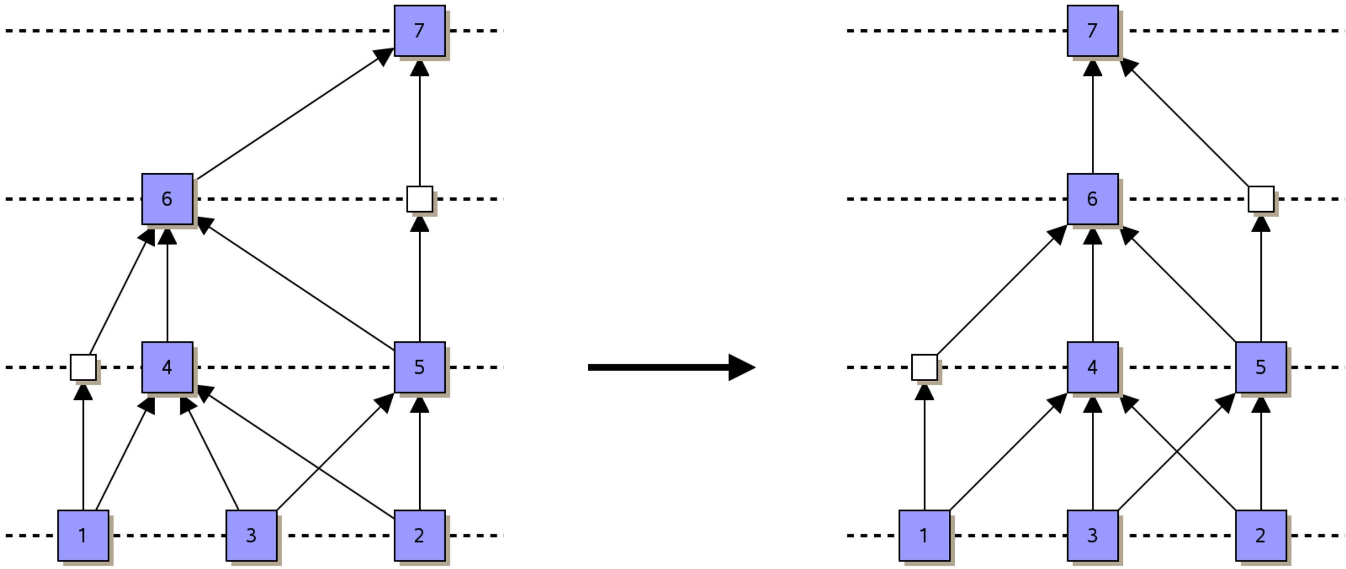


There was even an iPad game **CrossingX** for the OSCM Problem!

Winner of Graph Drawing Game Contest 2012

Step 4: Coordinate Computation

Karlsruhe Institute of Technology



Which could be the goals?

Steightening Edges

Goal: minimize deviation from a straight-line for the edges with dummy-nodes

Idea: use quadratic Program

- let $p_{uv} = (u, d_1, \ldots, d_k, v)$ path with k dummy nodes betwen u and v
- let $a_i = x(u) + \frac{i}{k+1}(x(v) - x(u))$ the x-coordinate of d_i when (u, v) is straight
- $g(p_{uv}) = \sum_{i=1}^{k}(x(d_i) - a_i)^2$
- minimize $\sum_{uv \in E} g(p_{uv})$
- constraints: $x(w) - x(z) \geq \delta$ for consecutive nodes on the same layer, w right from z (δ distance parameter)

Steightening Edges

Goal: minimize deviation from a straight-line for the edges with dummy-nodes

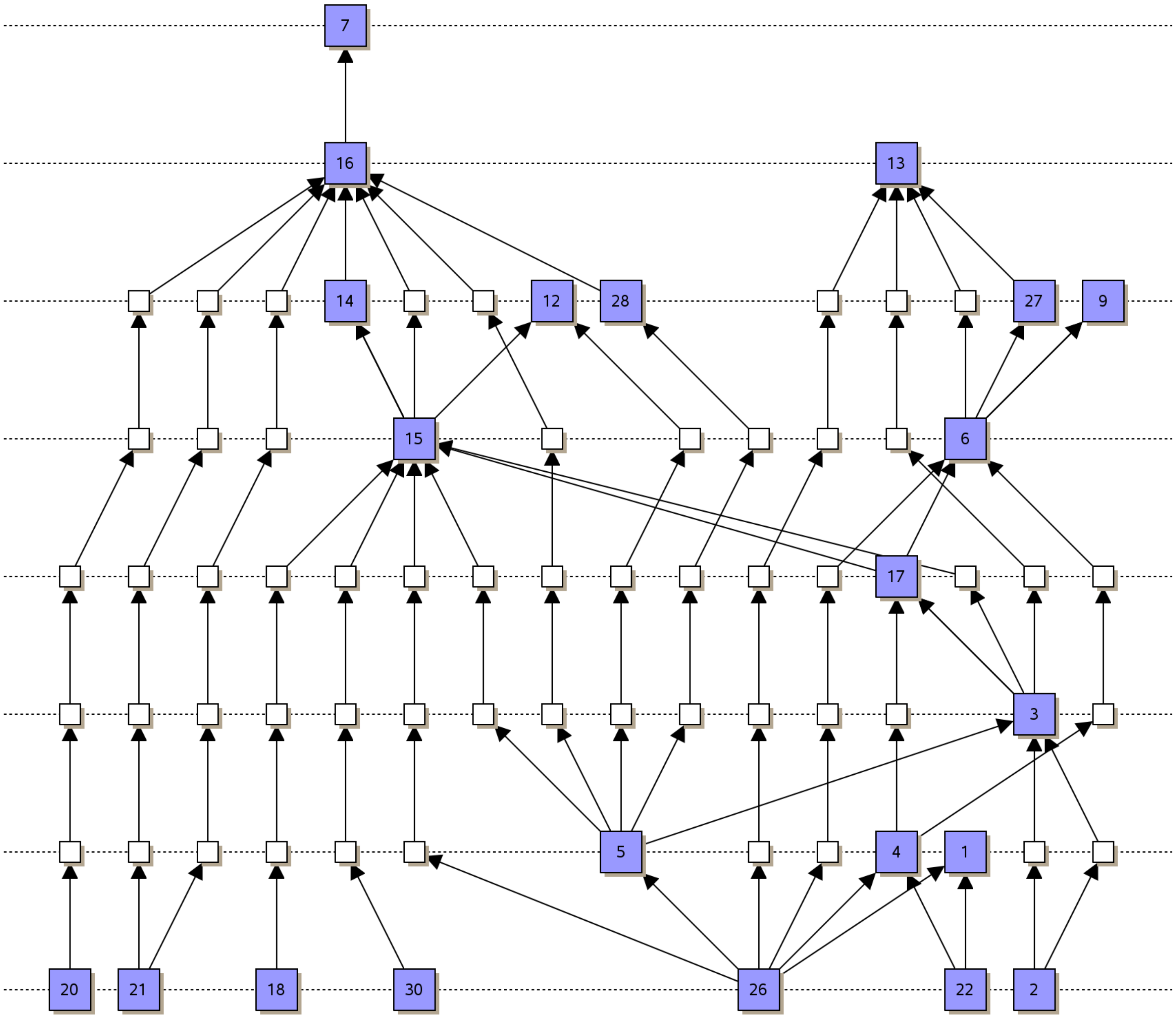
Idea: use quadratic Program

- let $p_{uv} = (u, d_1, \ldots, d_k, v)$ path with k dummy nodes betwen u and v
- let $a_i = x(u) + \frac{i}{k+1}(x(v) - x(u))$ the x-coordinate of d_i when (u, v) is straight
- $g(p_{uv}) = \sum_{i=1}^{k}(x(d_i) - a_i)^2$
- minimize $\sum_{uv \in E} g(p_{uv})$
- constraints: $x(w) - x(z) \geq \delta$ for consecutive nodes on the same layer, w right from z (δ distance parameter)

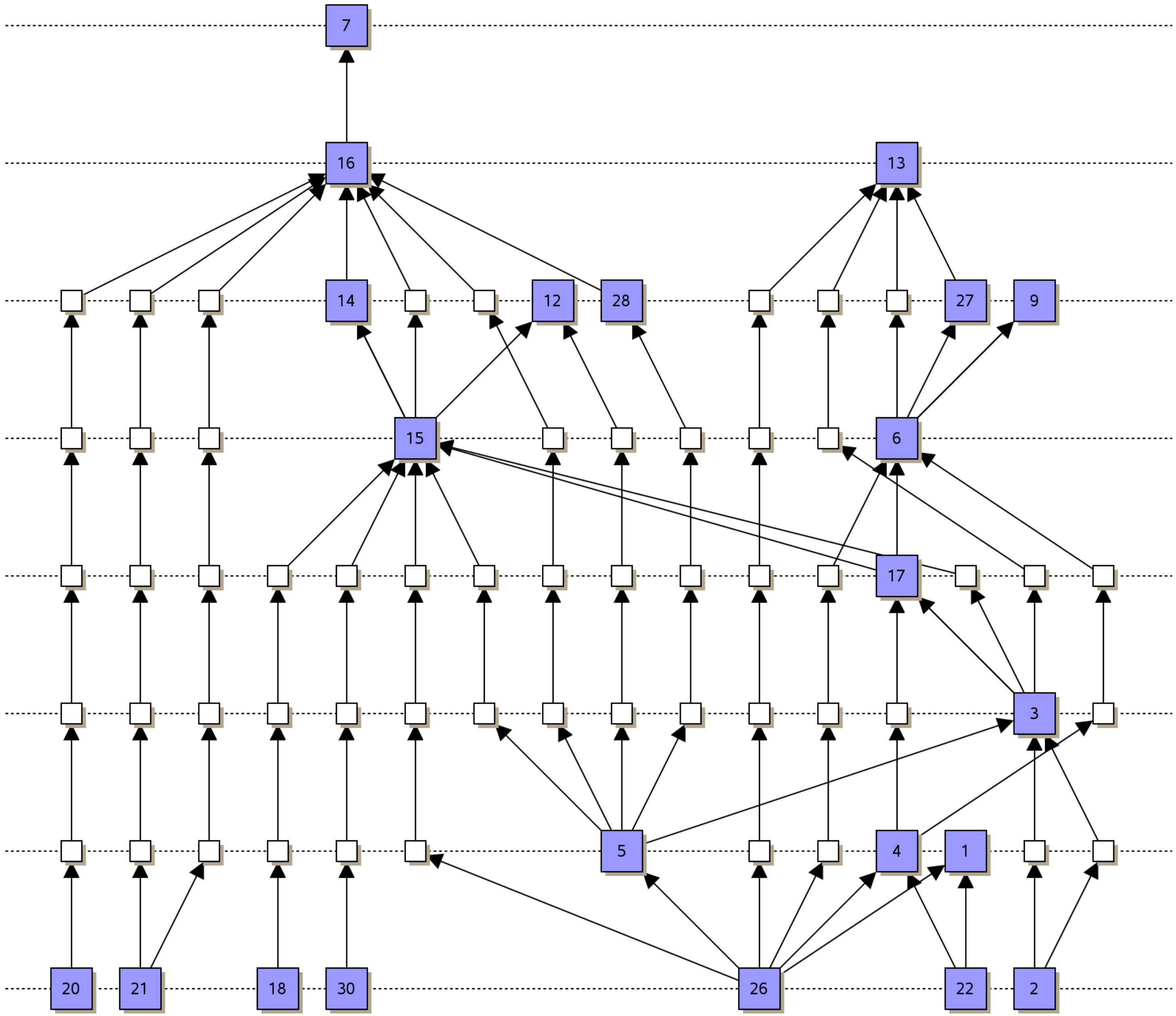
Properties:

- quadratic program is time-expensive
- width can be exponential
- optimization function can be adapted to optimize "verticality"

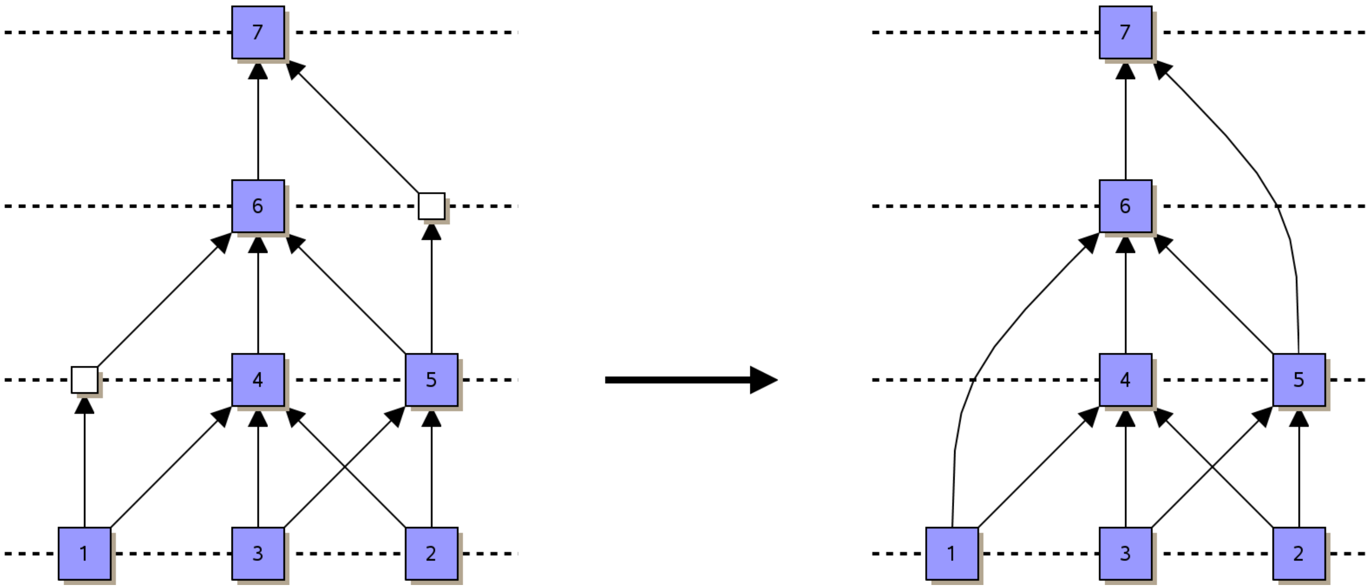
Example



Example

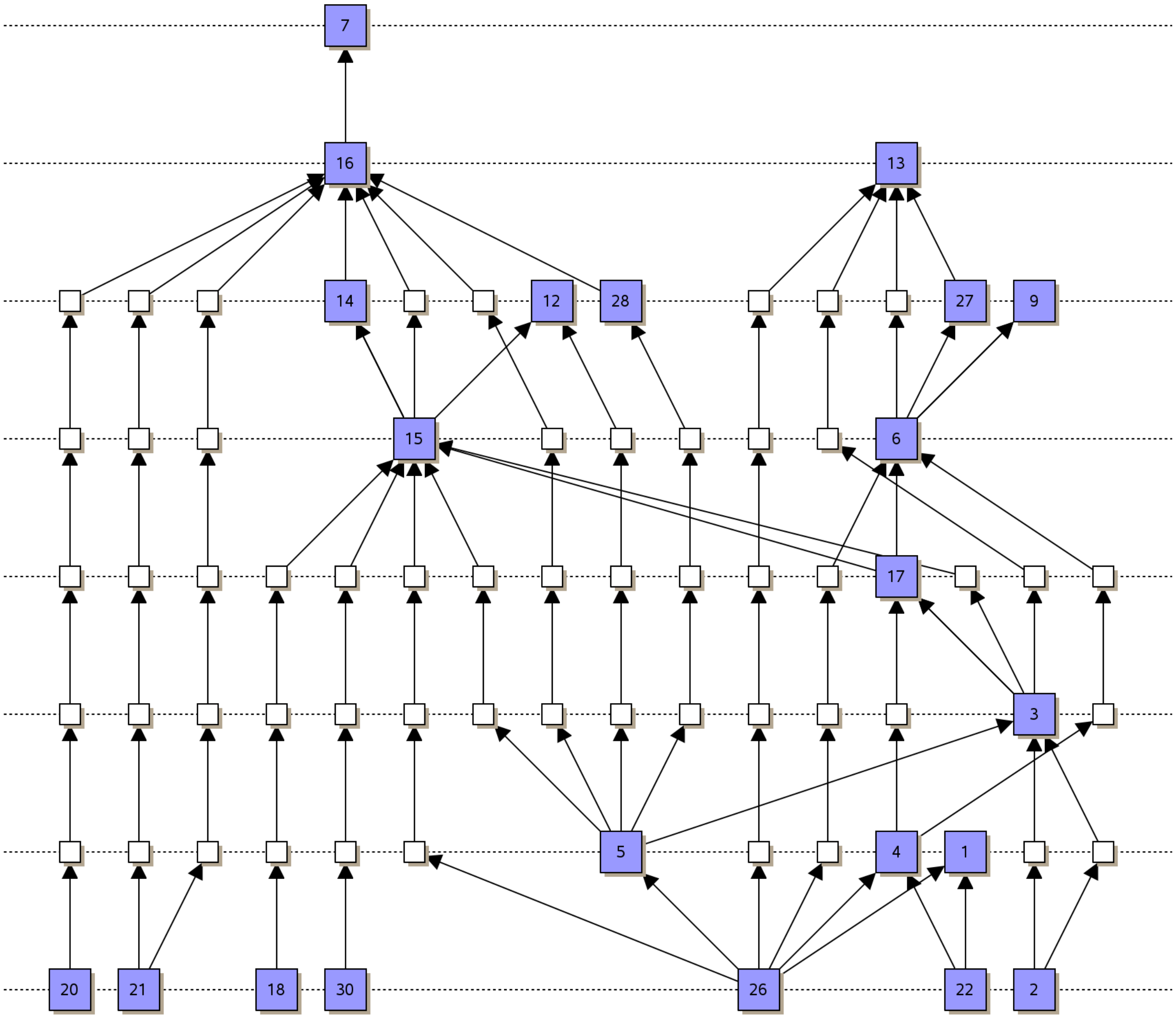


Step 5: Drawing edges

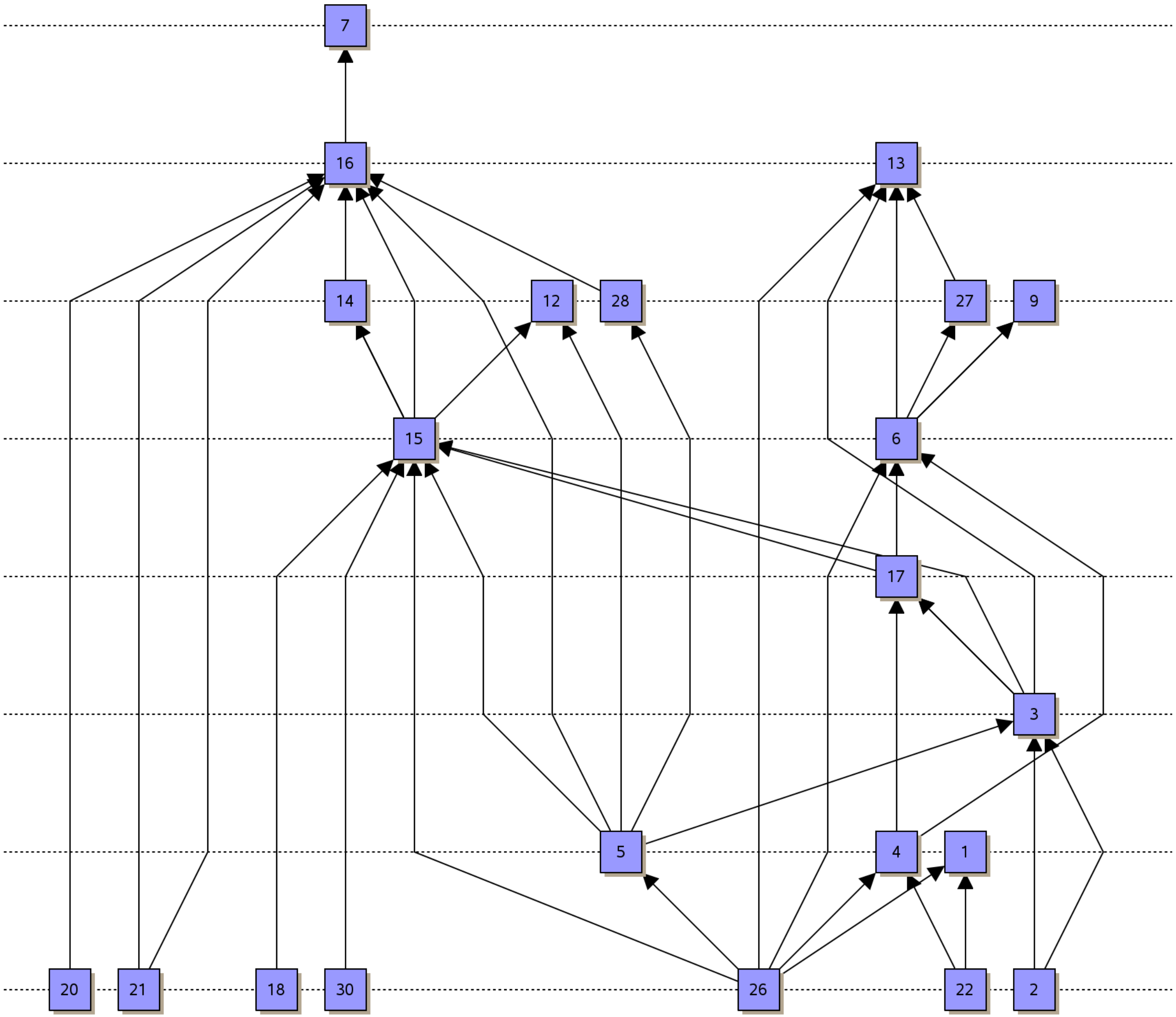


Possibility: Substitute polylines by Bézier curves

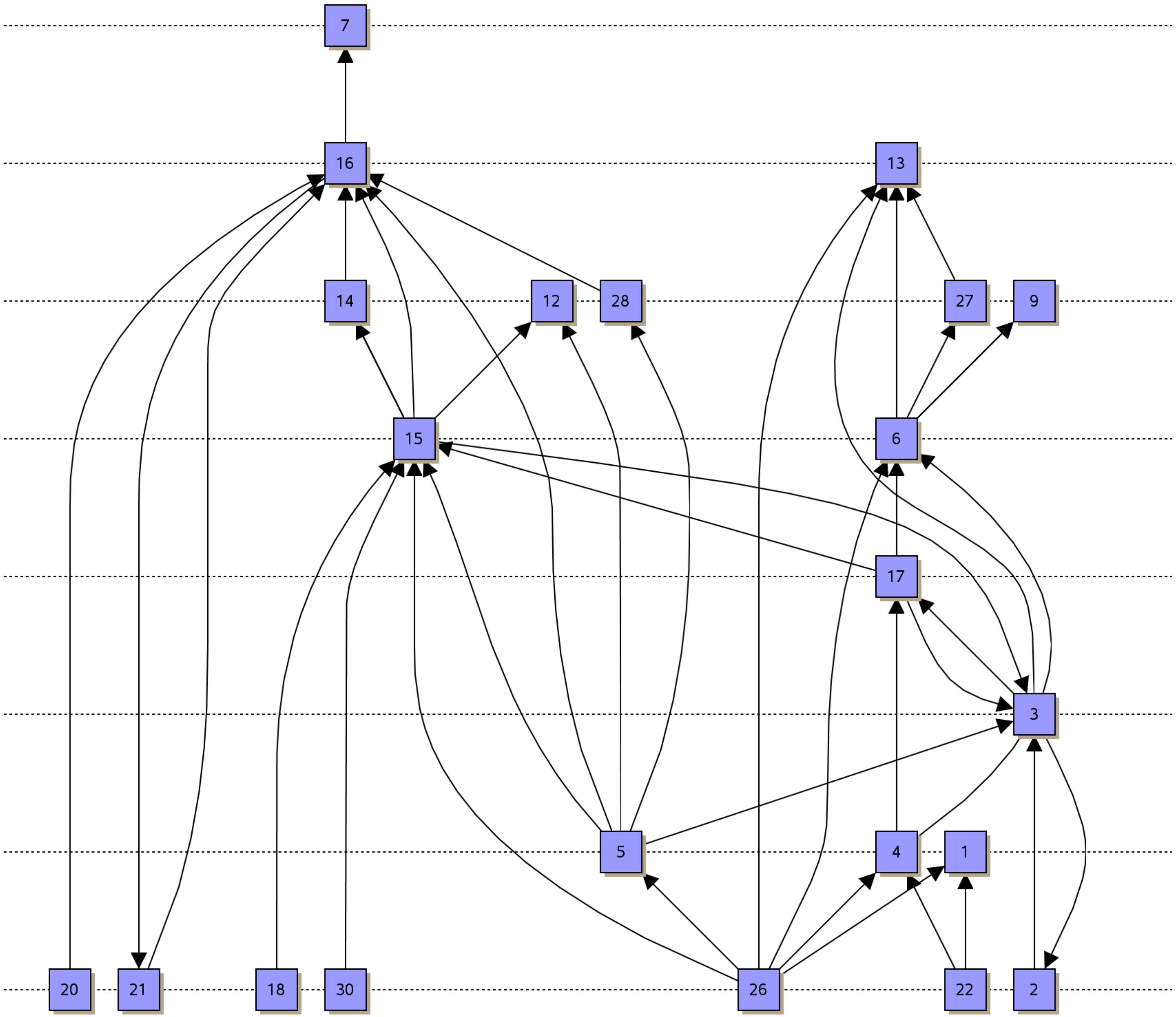
Example



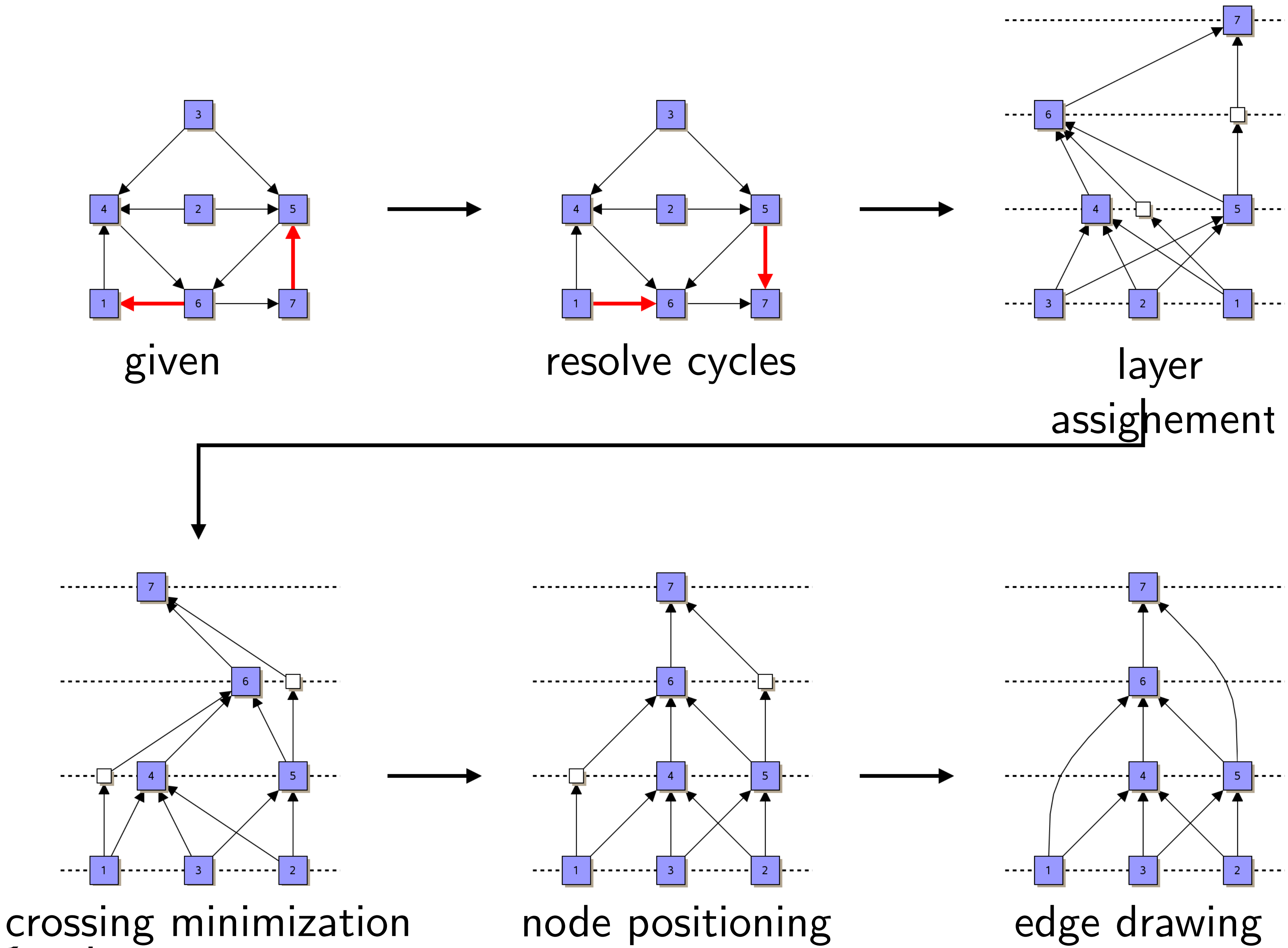
Example



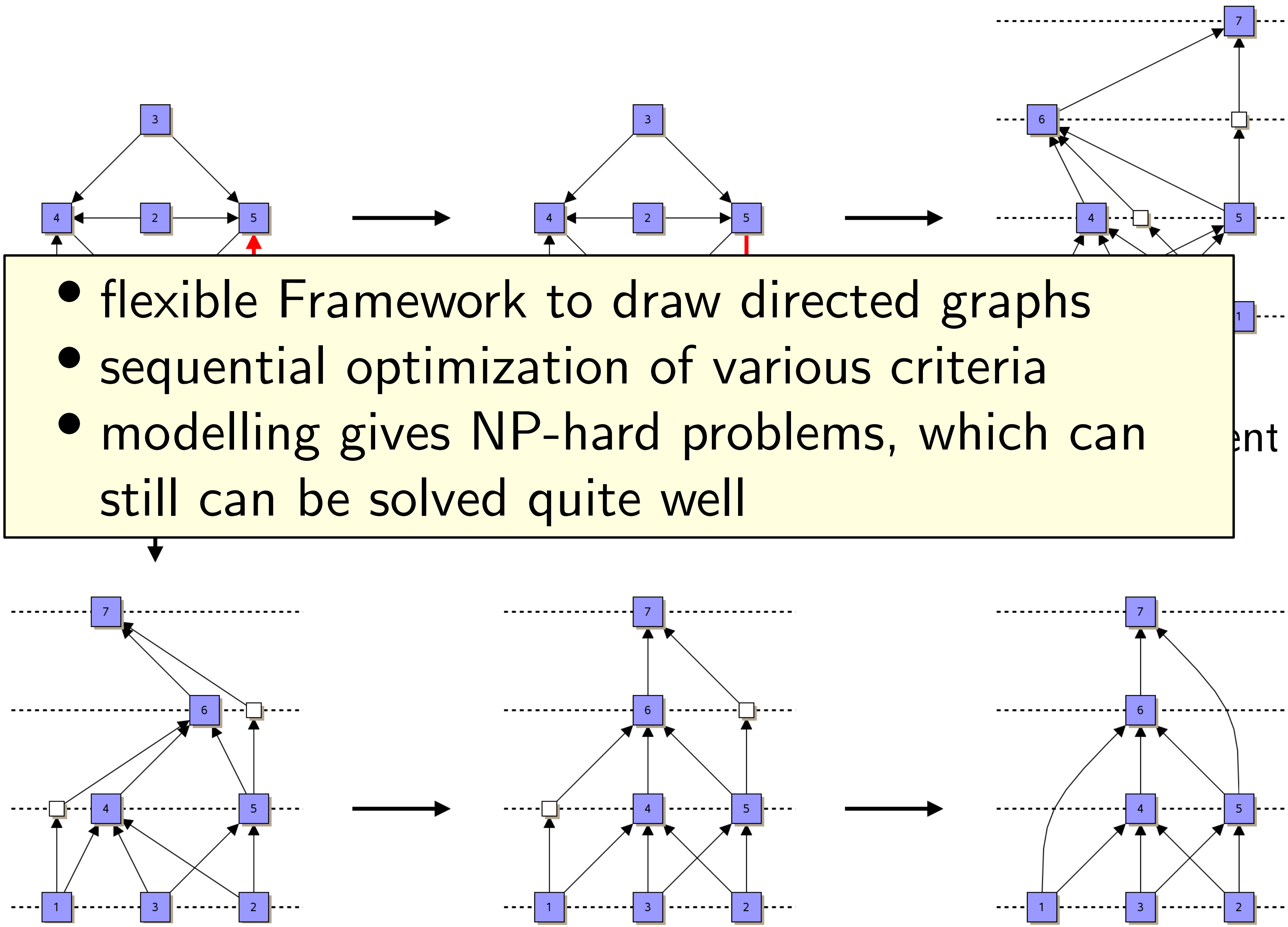
Example



Summary


given
resolve cycles
layer
assighement
crossing minimization
node positioning
edge drawing

Summary



- flexible Framework to draw directed graphs
- sequential optimization of various criteria
- modelling gives NP-hard problems, which can still can be solved quite well