Exercise Sheet 2

Discussion: 12. November 2019

Exercise 1: Outerplanar and Series-Parallel Graphs ⋆

A graph G is called *outerplanar* if it has a planar drawing where all vertices lie on the boundary of the outer face. Prove the following lemma.

Lemma 1

1. *There is an outerplanar graph that is not series-parallel.*
2. *Every biconnected outerplanar graph is series-parallel.*

Exercise 2: Visibility Representation ⋆

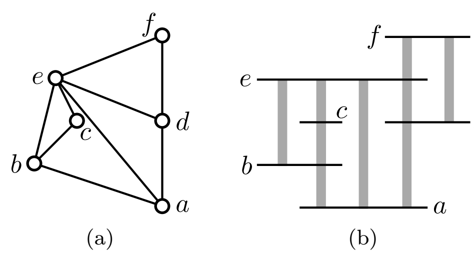


Figure 1: A visibility representation (b) of the graph G (a).

In a *visibility representation* of a graph $G = (V, E)$ the vertices are represented by horizontal segments (*vertex-segments*). We say that two vertices u and v *see* each other, if their vertex-segments can be connected by a vertical rectangle of non-zero width that does not cross any other vertex-segment. Thus, in a visibility representation of G, two vertices u, v see each other if and only if $(u, v) \in E$; see Figure 1. Prove the following lemma.

Lemma 2 *Every series-parallel graph has a visibility representation.*

Exercise 3: Canonical Orderings for Triconnected Planar Graphs ⋆⋆

Let $G = (V, E)$ be a triconnected plane graph with a vertex v_1 on the outer face. Further, let $\pi = (V_1, \dots, V_K)$ be an ordered partition of V, i.e., $V_1 \cup \dots \cup V_K = V$ and $V_i \cap V_j = \emptyset$ for $i \neq j$. We define G_k to be the subgraph of G induced by $V_1 \cup \dots \cup V_K$ and denote by C_k the outer face of G_k.

The sequence π is a *canonical ordering* of G, if

- V_1 consists of $\{v_1, v_2\}$, where v_2 lies on the outer face and $(v_1, v_2) \in E$.
- $V_K = \{v_n\}$ is a singleton, where v_n lies on the outer face, $\{v_1, v_n\} \in E$, and $v_n \neq v_2$.
- Each C_k $(k > 1)$ is a cycle containing $\{v_1, v_2\}$.
- Each G_k is biconnected and internally triconnected, that is, removing two interior vertices of G_k does not disconnect it.
- For each k with $2 \leq k \leq K - 1$, one of the following conditions holds:
    1. $V_k = \{z\}$, where z belongs to C_k and has at least one neighbor in $G - G_k$.
    2. $V_k = \{z_1, \dots, z_\ell\}$ is a chain, where each z_i has at least one neighbor in $G - G_k$ and where z_1 and z_ℓ each have one neighbor on C_{k-1}, and these are the only two neighbors of V_k in G_{k-1}.

Prove the following lemma.

Lemma 3 *Every triconnected planar graph admits a canonical ordering.*

Hint: Use reverse induction. For the induction step, consider the two cases that G_k *is triconnected* and G_k *is not triconnected.*

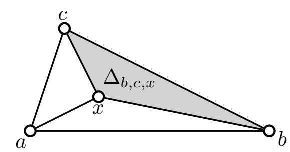
Exercise 4: Barycentric Coordinates ⋆⋆

Let $\Delta_{a,b,c}$ be a triangle on the plane on vertices a, b and c. For each point x laying inside triangle $\Delta_{a,b,c}$ there exists a triple (x_a, x_b, x_c) such that $x_a \cdot a + x_b \cdot b + x_c \cdot c = x$ and $x_a + x_b + x_c = 1$. The triple (x_a, x_b, x_c) is called *barycentric coordinates* of x with respect to $\Delta_{a,b,c}$.

Prove that:

(a) If $A(\Delta)$ denotes the area of the triangle A, then

$$x_a = \frac{A(\Delta_{b,c,x})}{A(\Delta_{a,b,c})},\ x_b = \frac{A(\Delta_{a,c,x})}{A(\Delta_{a,b,c})},\ x_c = \frac{A(\Delta_{a,b,x})}{A(\Delta_{a,b,c})}$$



(b) Equations $x_a = 0$, $x_b = 0$, $x_c = 0$ represent lines through bc, ab and ab, respectively.

(c) Let (x_a, x_b, x_c) be barycentric coordinates of point x in triangle Δ_{abc}. The set of points $\{(x_a, x'_b, x'_c) : x'_b, x'_c \in \mathbb{R}\}$ represents a line parallel to edge bc passing through point x. Similarly, sets of points $\{(x'_a, x_b, x'_c) : x'_a, x'_c \in \mathbb{R}\}$, $\{(x'_a, x'_b, x_c) : x'_a, x'_b \in \mathbb{R}\}$ represent lines parallel to edges ac, ab, respectively, passing through point x.