Exercise Sheet 4

Discussion: 18. December 2019

Exercise 1: Sum of Angles ⋆

Let Γ be an orthogonal drawing of a planar embedded graph G. Denote by p_f the number of vertex-angles and bend-angles inside a face f in Γ. Prove the following lemma.

Lemma 1 *If f is an interior face, then the sum of all vertex-angles and bend-angles inside a face f in the orthogonal drawing Γ is $\pi(p_f - 2)$. If f is an exterior face, then the corresponding sum is $\pi(p_f + 2)$.*

Exercise 2: Bends in Octahedrons ⋆

A *octahedron* is a completely triangulated 4-regular graph on 6 vertices. Prove the following.

Lemma 2 *Every orthogonal drawing of a octahedron contains an edge with at least 3 bends.*

Hint: Count the number of bends of the edges incident to the exterior face.

Exercise 3: Bend Minimization with Additional Constraints ⋆⋆

Prove the following.

Lemma 3 *Let G be biconnected planar embedded graph G with maximal vertex degree of 4 and let $k_E : E \to \mathbb{N}_0, k_F : F \to \mathbb{N}_0$ be cost functions on the set of edges E and faces F, respectively. There is a polynomial time algorithm that decides whether G has an orthogonal layout that satisfies the following constraints*

(a) every edge e has at most $k_E(e)$ bends,

(b) every face f has at most $k_F(f)$ concave corners (interior angle of $3\pi/2$), and

(c) the number of concave corners of all interior faces is minimal.

Hint: Adapt the flow network introduced in the lecture.

Given an orthogonal drawing with a minimal number of bends and one edge e with many bends, what is the shape of e?

Exercise 4: Edges with Many Bends ⋆

Define a family of embedded planar graphs with maximum degree 4 and $O(n)$ vertices, such that for each bend-minimal orthogonal drawing of the given embedding there is an edge that has $\Omega(n)$ bends.

Hint: Consider spirals.

Exercise 5: Area of Grid Drawings ⋆⋆

Let Γ be an orthogonal drawing of a planar graph G and let $[x_{\min}, x_{\max}] \times [y_{\min}, y_{\max}]$ be the *bounding box* of Γ, i.e., the smallest axis-aligned rectangle that contains Γ. Rows or columns in Γ that do not contain a bend can be removed without changing the underlying orthogonal representation. Thus, it is natural to require that every column $x_i \in [x_{\min}, x_{\max}] \cap \mathbb{Z}$ and row $y_i \in [y_{\min}, y_{\max}] \cap \mathbb{Z}$ contains at least one bend or one vertex. We refer to drawings with this property as *compact*.

Prove the following lemmas.

Lemma 4 *Let G be a planar graph with minimum vertex degree 2 and let H be an orthogonal representation of G with b bends. Every compact orthogonal drawing of G that realizes H has an area of at most $\lfloor (n+b)/2 \rfloor \cdot \lceil (n+b)/2 \rceil$.*

Prove that this bound is tight.

Lemma 5 *There is a family of graphs with orthogonal representations such this family has compact orthogonal drawings with minimal area in $\Omega((n+b)^2)$.*