Algorithms for Graph Visualization

Layered Layout – Part II

INSTITUT FÜR THEORETISCHE INFORMATIK · FAKULTÄT FÜR INFORMATIK

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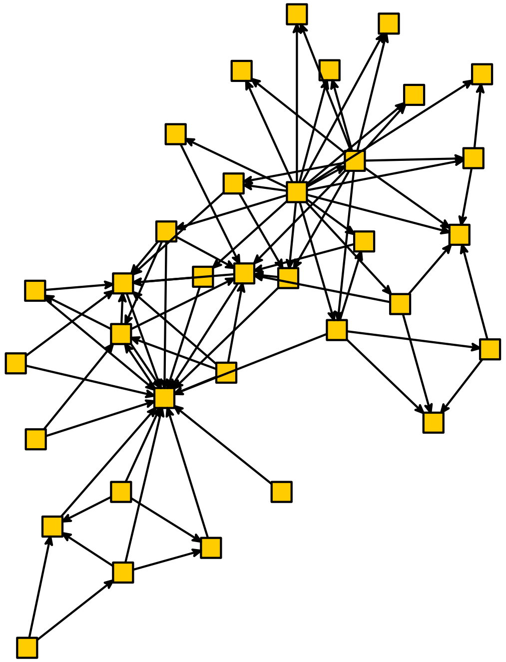
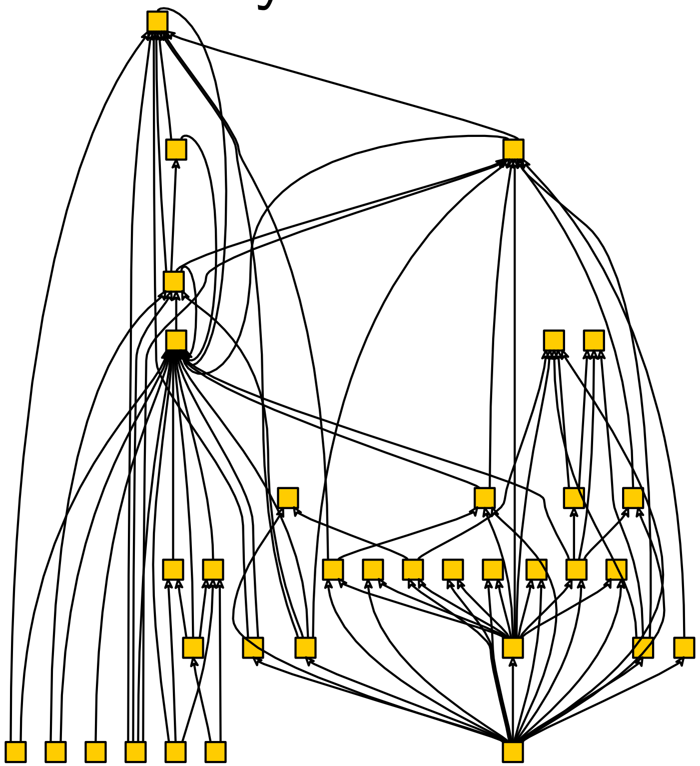
22.01.2020

Layered Layout

Given: directed graph $D = (V, A)$

Find: drawing of D that emphasizes the hierarchy by positioning nodes on horizontal layers





Layered Layout

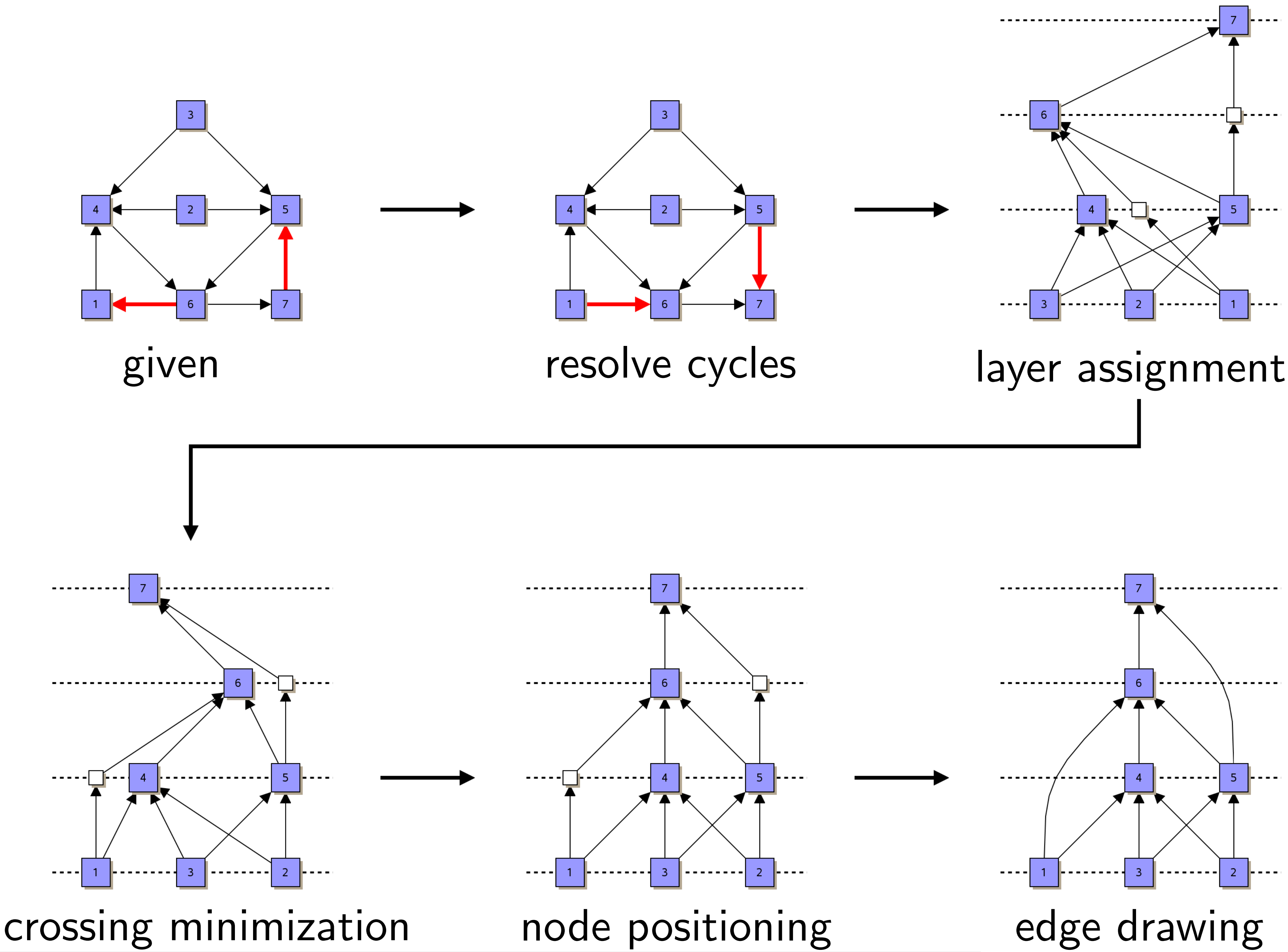
Given: directed graph $D = (V, A)$

Find: drawing of D that emphasizes the hierarchy by positioning nodes on horizontal layers

Criteria:

- many edges pointing to the same direction
- few layers or limited number of nodes per layer
- preferably few edge crossings
- nodes distributed evenly
- edges preferably straight and short

Sugiyama Framework (Sugiyama, Tagawa, Toda 1981)



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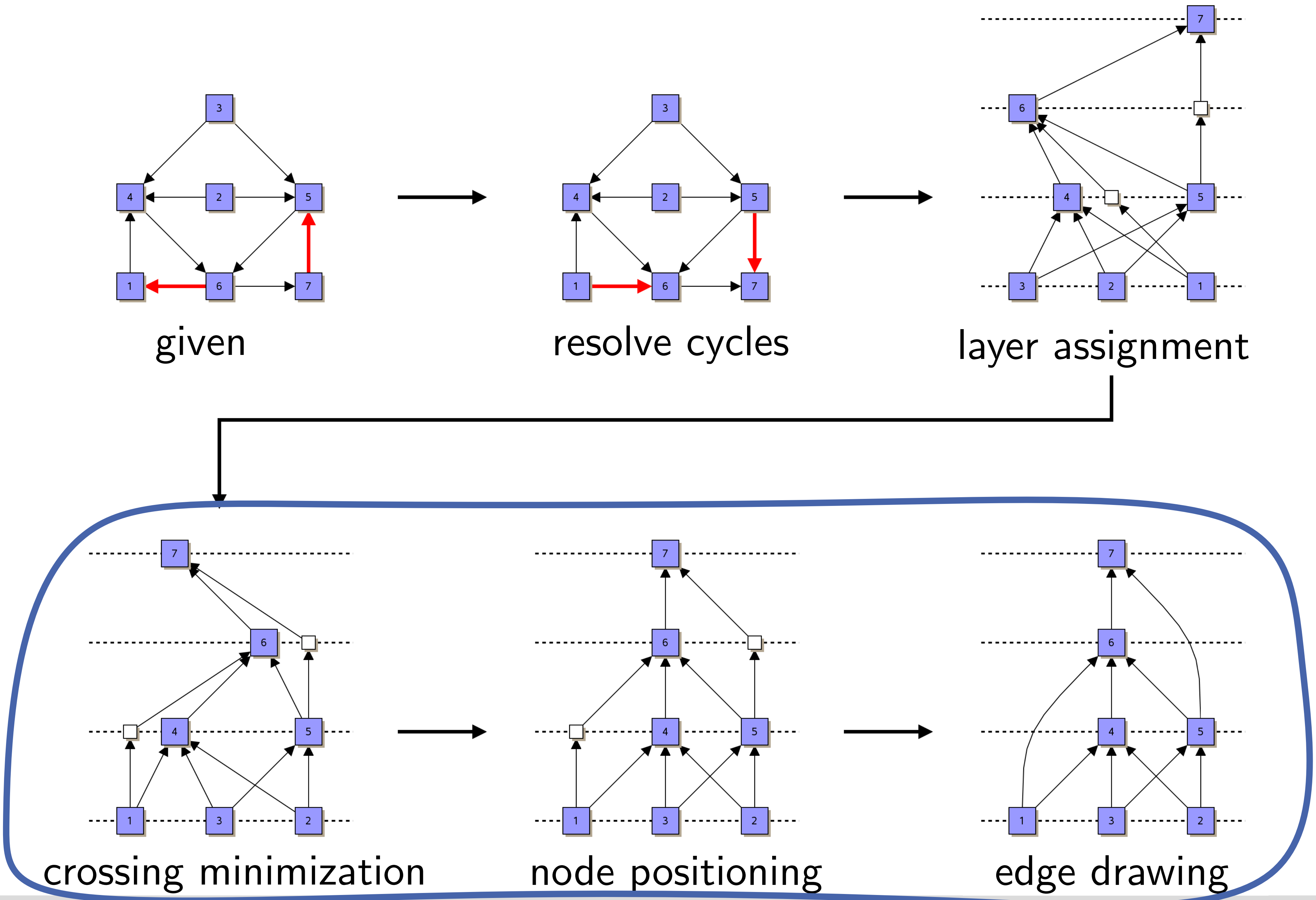
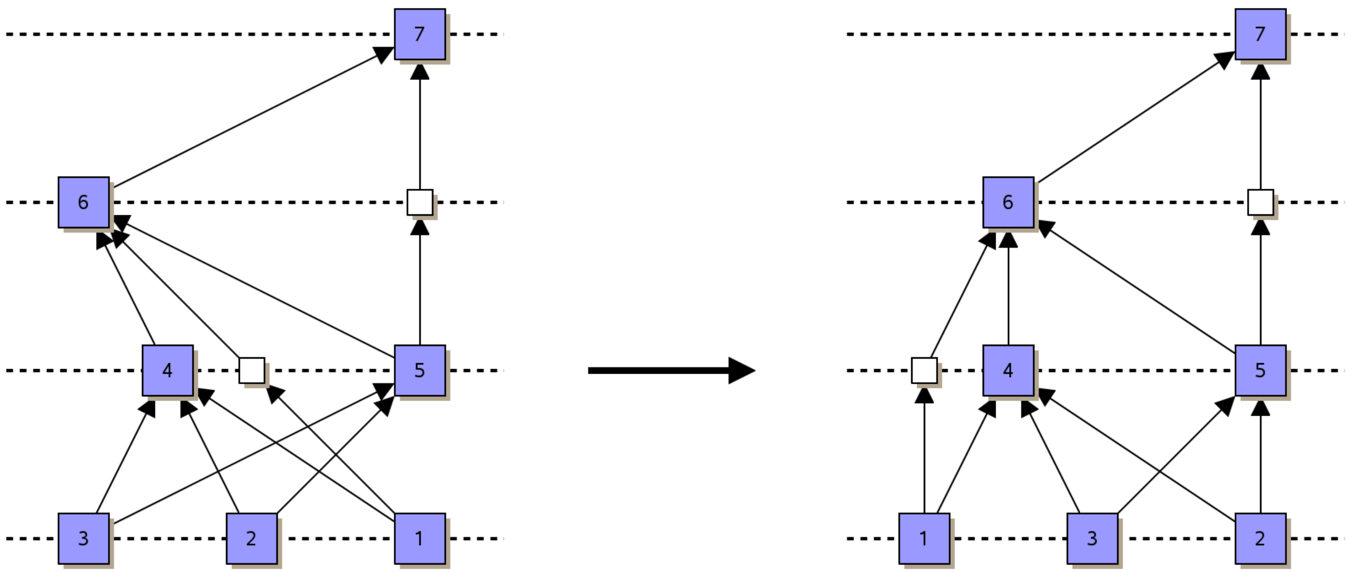


Step 3: Crossing Minimization



Problem Statement

Given: DAG $D = (V, A)$, nodes are partitioned in disjoint layers

Find: Order of the nodes on each layer, so that the number of crossing is minimized

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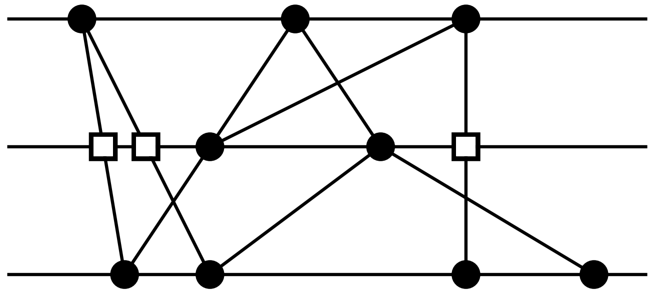
Properties

- Problem is NP-hard even for two layers (Bipartite Crossing Number [Garey, Johnson '83])
- No approach over several layers simultaneously
- Usually iterative optimization for two adjacent layers
- For that: insert dummy nodes at the intersection of edges with layers



One-sided Crossing Minimization (OSCM)

Given: 2-layered graph $G = (L_1, L_2, E)$ and ordering of the nodes x_1 of L_1

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Observation:

- The number of crossings in a 2-layered drawing of G depends only on the ordering of the nodes, not on the exact positions
- for $u, v \in L_2$ the number of crossings among their incident edges depends on whether $x_2(u) < x_2(v)$ or $x_2(v) < x_2(u)$ and not on the ordering of other vertices

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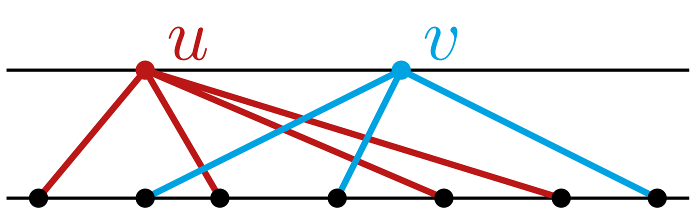
Def: $c_{uv} := |\{(uw, vz) : w \in N(u), z \in N(v), x_1(z) < x_1(w)\}|$



$c_{uv} = 5$

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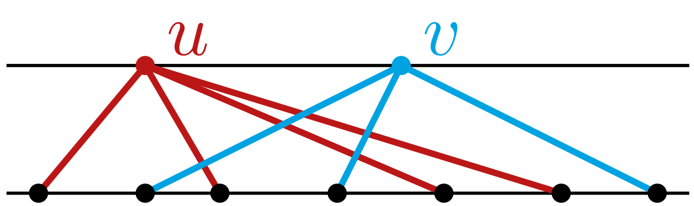

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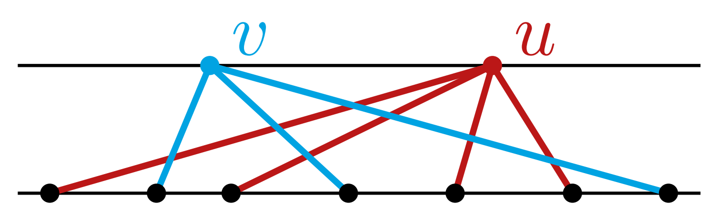
$c_{vu} = 7$



Further Properties

Def: Number of crossings in G with orders x_1 and x_2 for L_1 and L_2 is denoted by cr(G, x_1, x_2);
$\Rightarrow$ for fixed x_1 we have $\mathrm{opt}(G, x_1) = \min_{x_2} \mathrm{cr}(G, x_1, x_2)$

Lemma 1: Each of the following holds:

- $\mathrm{cr}(G, x_1, x_2) = \sum_{x_2(u)<x_2(v)} c_{uv}$
- $\mathrm{opt}(G, x_1) \geq \sum_{\{u,v\}} \min\{c_{uv}, c_{vu}\}$

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Efficient computation of cr(G, x_1, x_2) see Exercise.

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Efficient computation of $\mathrm{cr}(G, x_1, x_2)$ see Exercise.

Think for a minute and then share

Can you find an example where the second inequality is strict?

3 min

Iterative Crossing Minimization

Karlsruhe Institute of Technology

Let $G = (V, E)$ be a DAG with layers $L_1, \ldots, L_h$.

(1) compute an ordering x_1 for layer L_1

(2) for $i = 1, \ldots, h-1$ consider layers L_i and L_{i+1} and minimize cr(G, x_i, x_{i+1}) with fixed x_i ($\rightarrow$ **OSCM**)

(3) for $i = h-1, \ldots, 1$ consider layers L_{i+1} and L_i and minimize cr(G, x_i, x_{i+1}) with fixed x_{i+1} ($\rightarrow$ **OSCM**)

(4) repeat (2) and (3) until no further improvement happens

(5) repeat steps (1)–(4) with another x_1

(6) return the best found solution

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Theorem 1: The One-Sided Crossing Minimization (OSCM) problem is NP-hard (Eades, Wormald 1994).

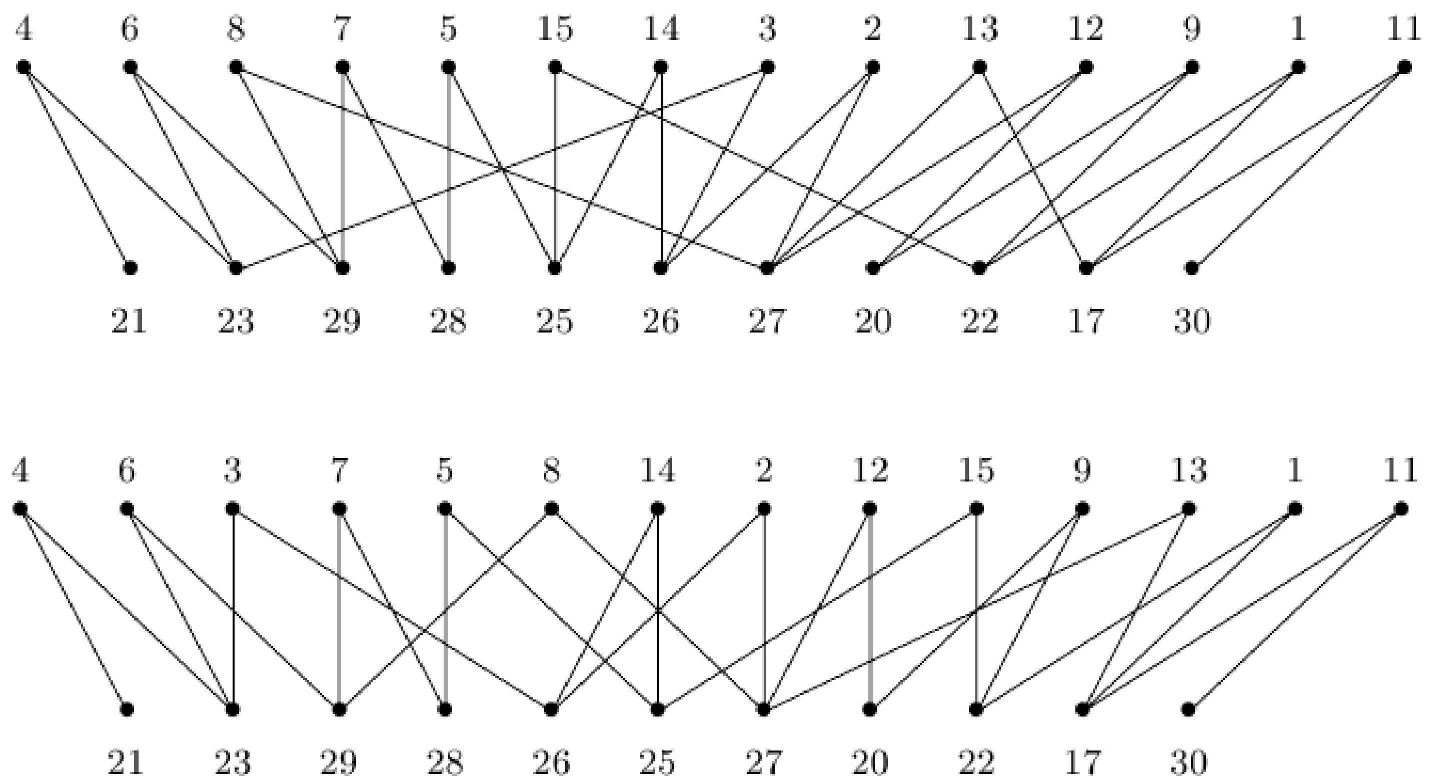
Algorithms for OSCM

Heuristics:

- Barycenter
- Median

Exact:

- ILP Model



Barycenter Heuristic (Sugiyama, Tagawa, Toda 1981)

Idea: Position nodes close to their neighbours.

- Set
$$x_2(u) = \frac{1}{\deg(u)} \sum_{v \in N(u)} x_1(v)$$
- In case of ties, break arbitrarily.

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Properties:

- trivial implementation
- quick (exactly?)
- usually very good results
- finds optimum if $\operatorname{opt}(G, x_1) = 0$ (see Exercises)
- there are graphs on which it performs $\Omega(\sqrt{n})$ times worse than optimal

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Work with your neighbour and then share

Construct an example where barycenter method produces very bad results.

3 min

Median Heuristic (Eades, Wormald 1994)

Idea: Use the median of the coordinates of the neighbours.

- For a node $v \in L_2$ with neighbours $v_1, \dots, v_k$ set $x_2(v) = \mathrm{med}(v) = x_1(v_{\lceil k/2 \rceil})$ and $x_2(v) = 0$ if $N(v) = \emptyset$.
- If $x_2(u) = x_2(v)$ and u, v have different parity, place the node with odd degree to the left.
- If $x_2(u) = x_2(v)$ and u, v have the same parity, break tie arbitrarily.
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- Runs in time $O(|E|)$.

Properties:

- trivial implementation
- fast
- mostly good performance
- finds optimum when $\mathrm{opt}(G, x_1) = 0$
- **Factor-3 Approximation**

Approximation Factor

Theorem 2: Let $G = (L_1, L_2, E)$ be a 2-layered graph and x_1 an arbitrary ordering of L_1. Then it holds that

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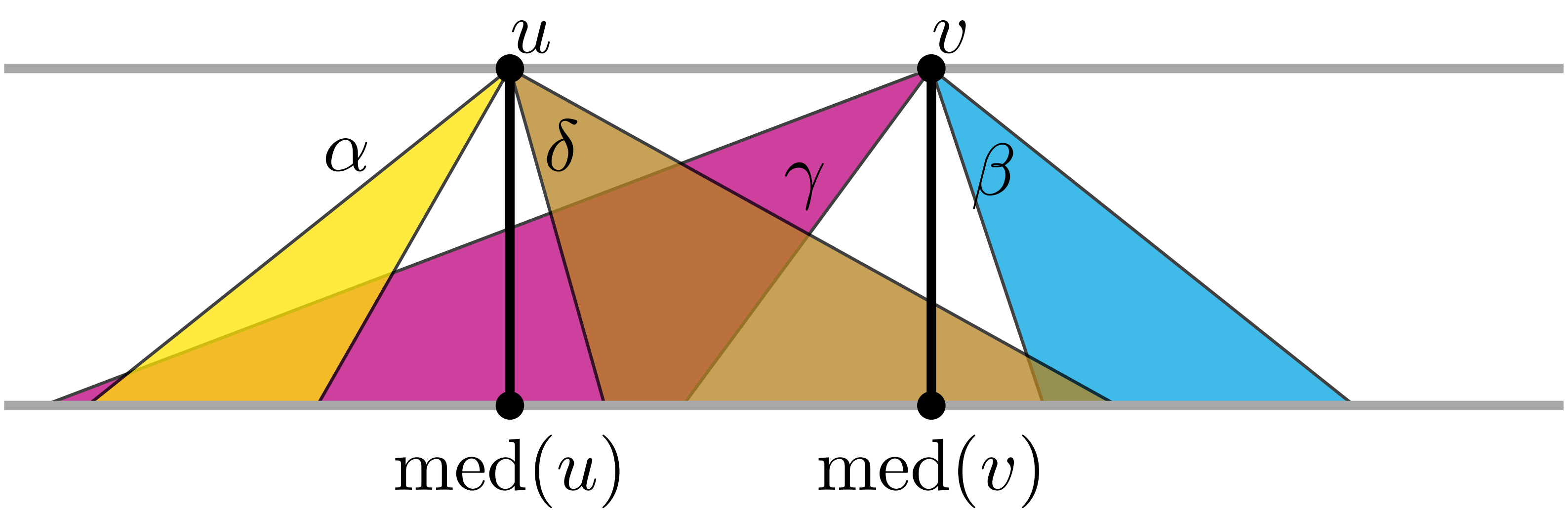

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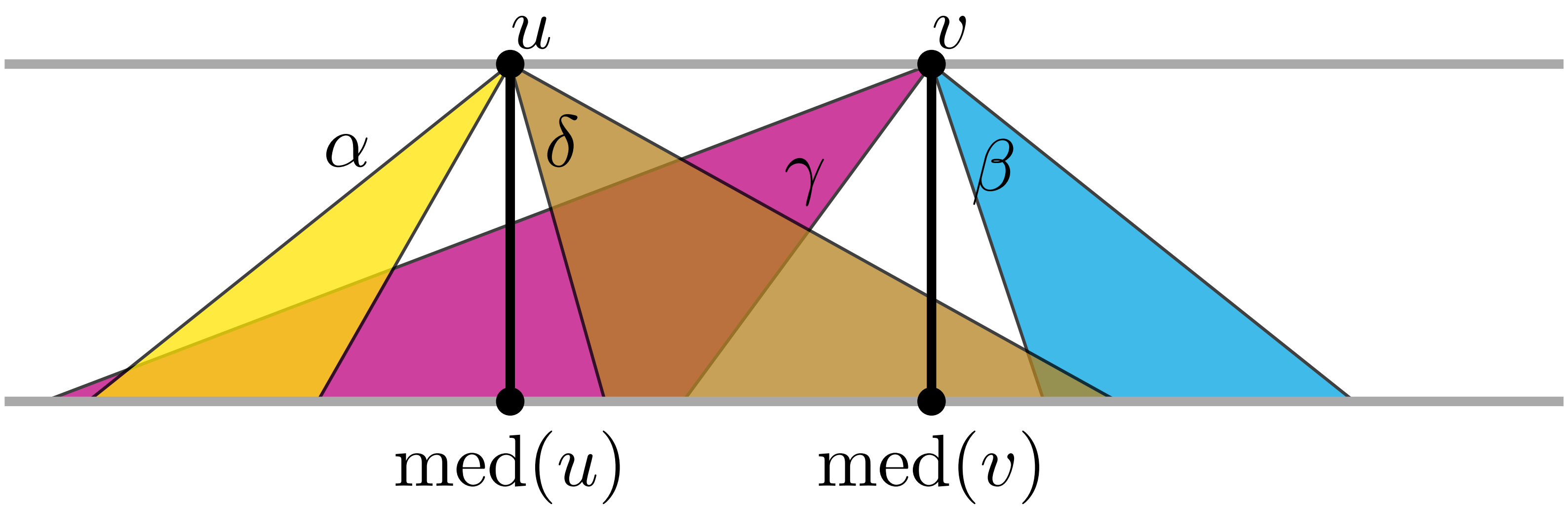

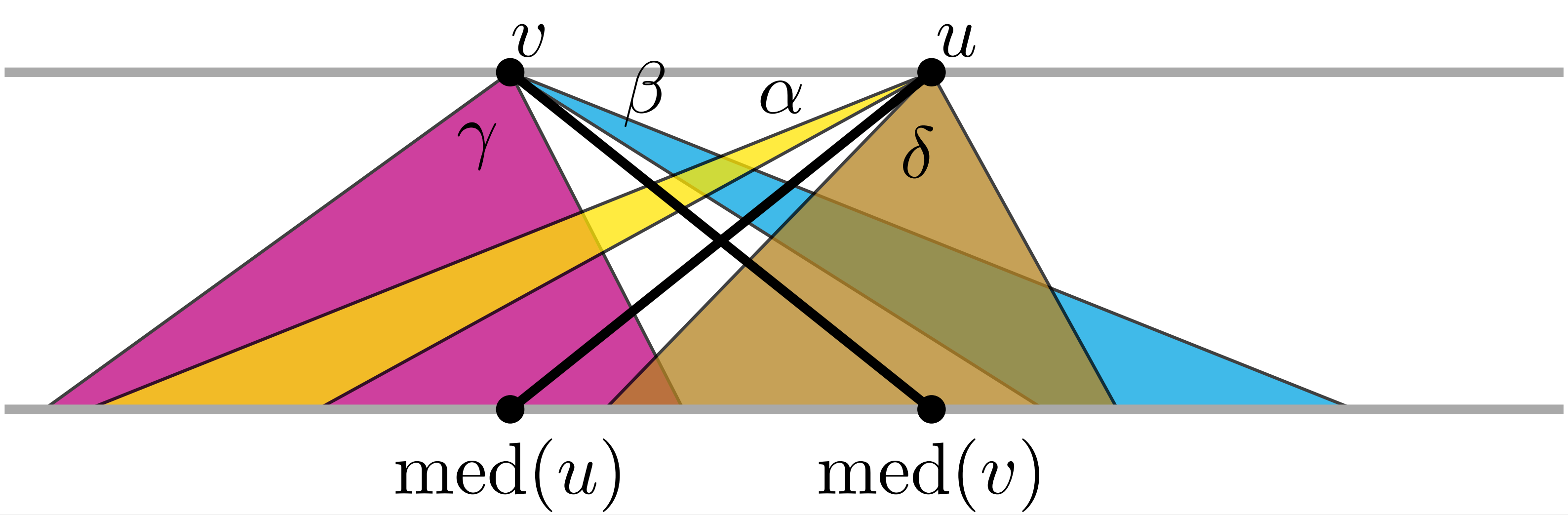

Integer Linear Programming

Properties:

- branch-and-cut technique for DAGs of limited size
- useful for graphs of small to medium size
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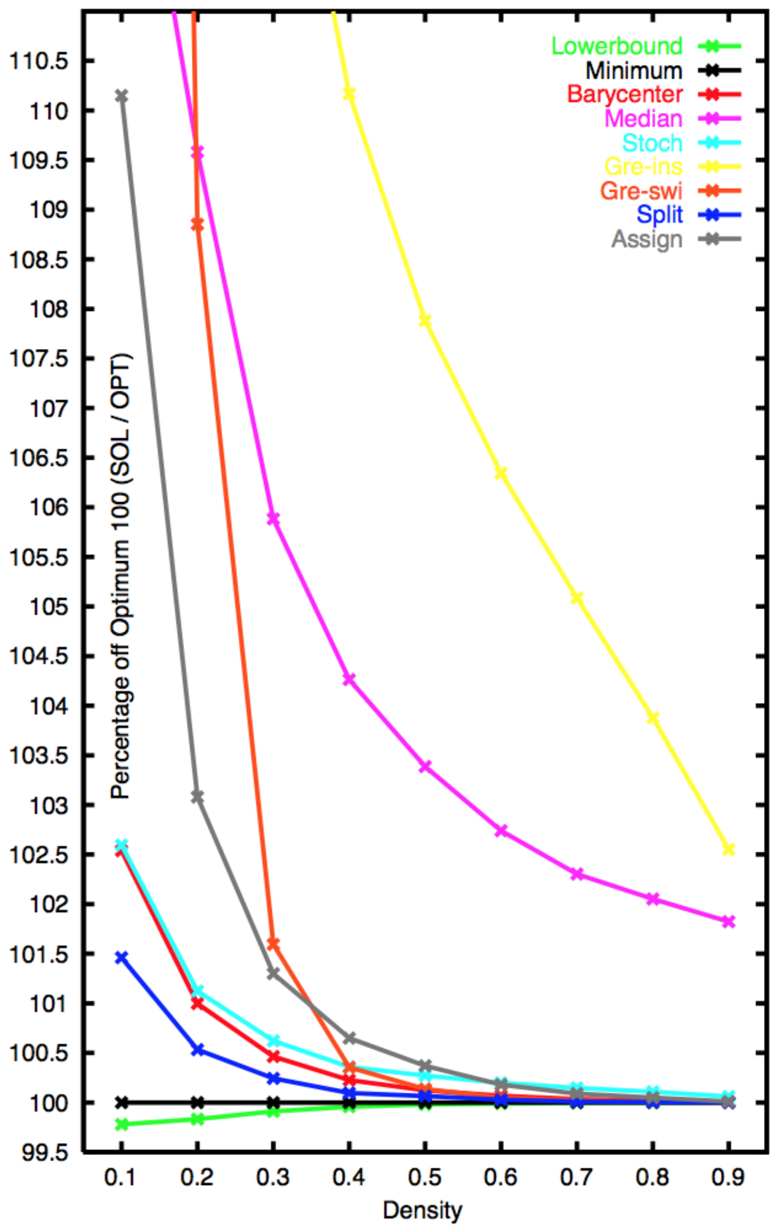
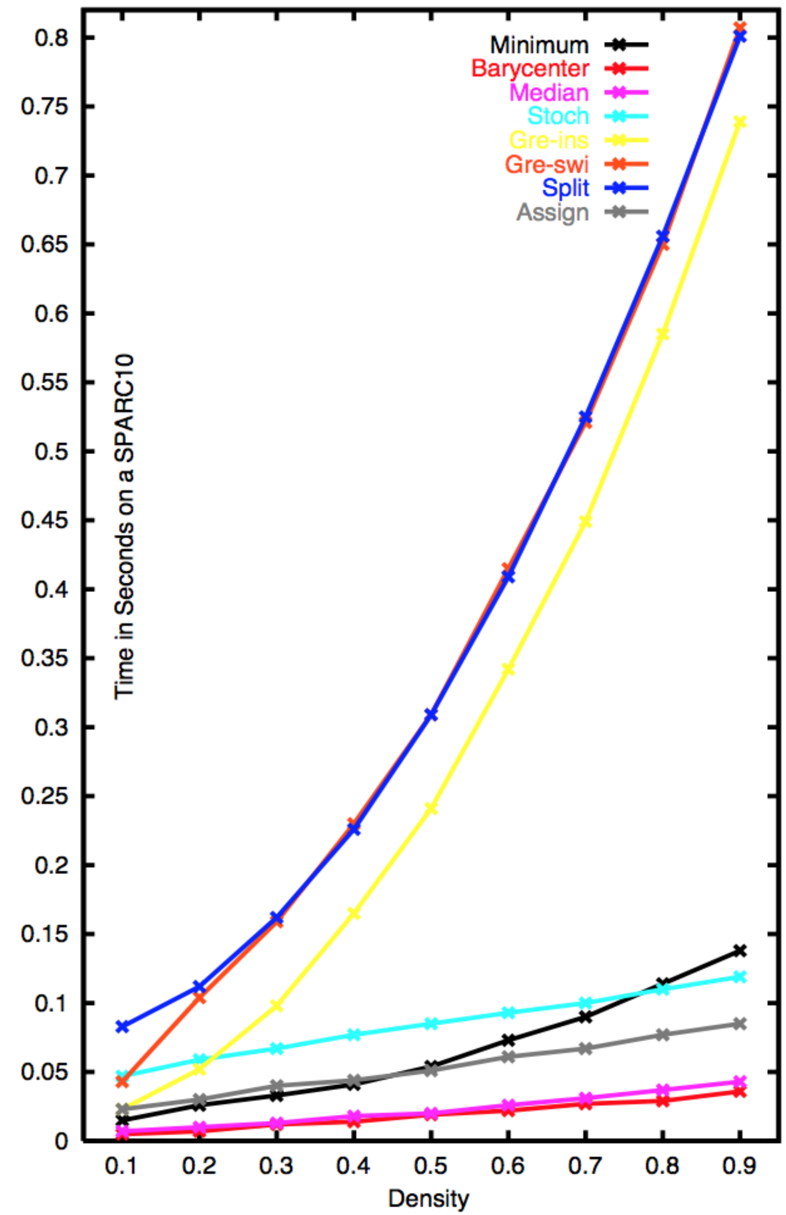
Model: see blackboard

Experimental Evaluation (Jünger, Mutzel 1997)



Results for 100 instances on 20 + 20 nodes with increasing density



Time for 100 instances on 20 + 20 nodes with increasing density

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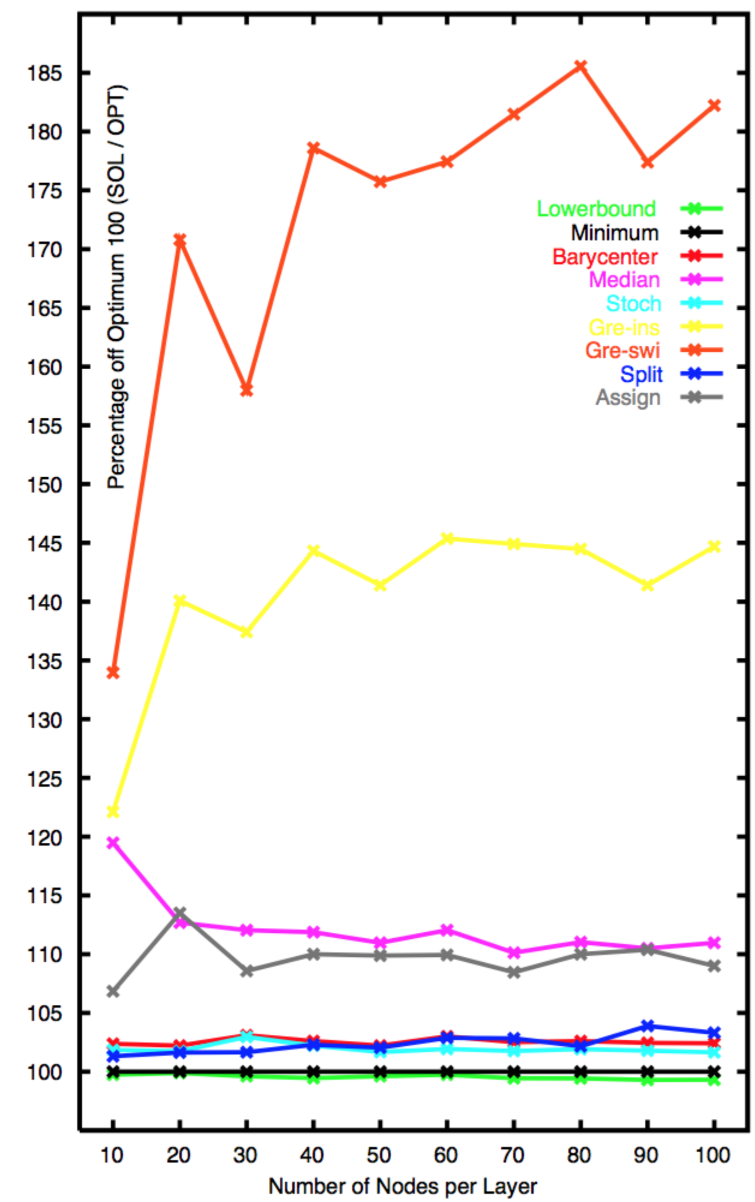
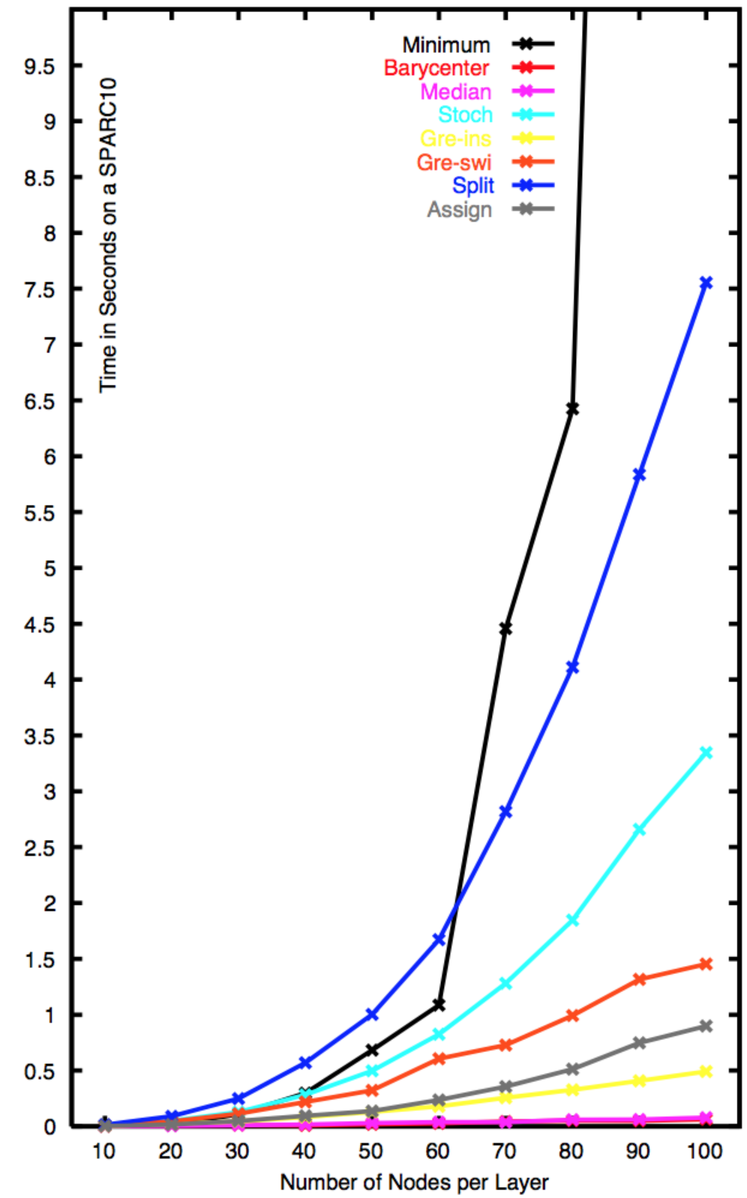


Results for 10 instances of sparse graphs with increasing size



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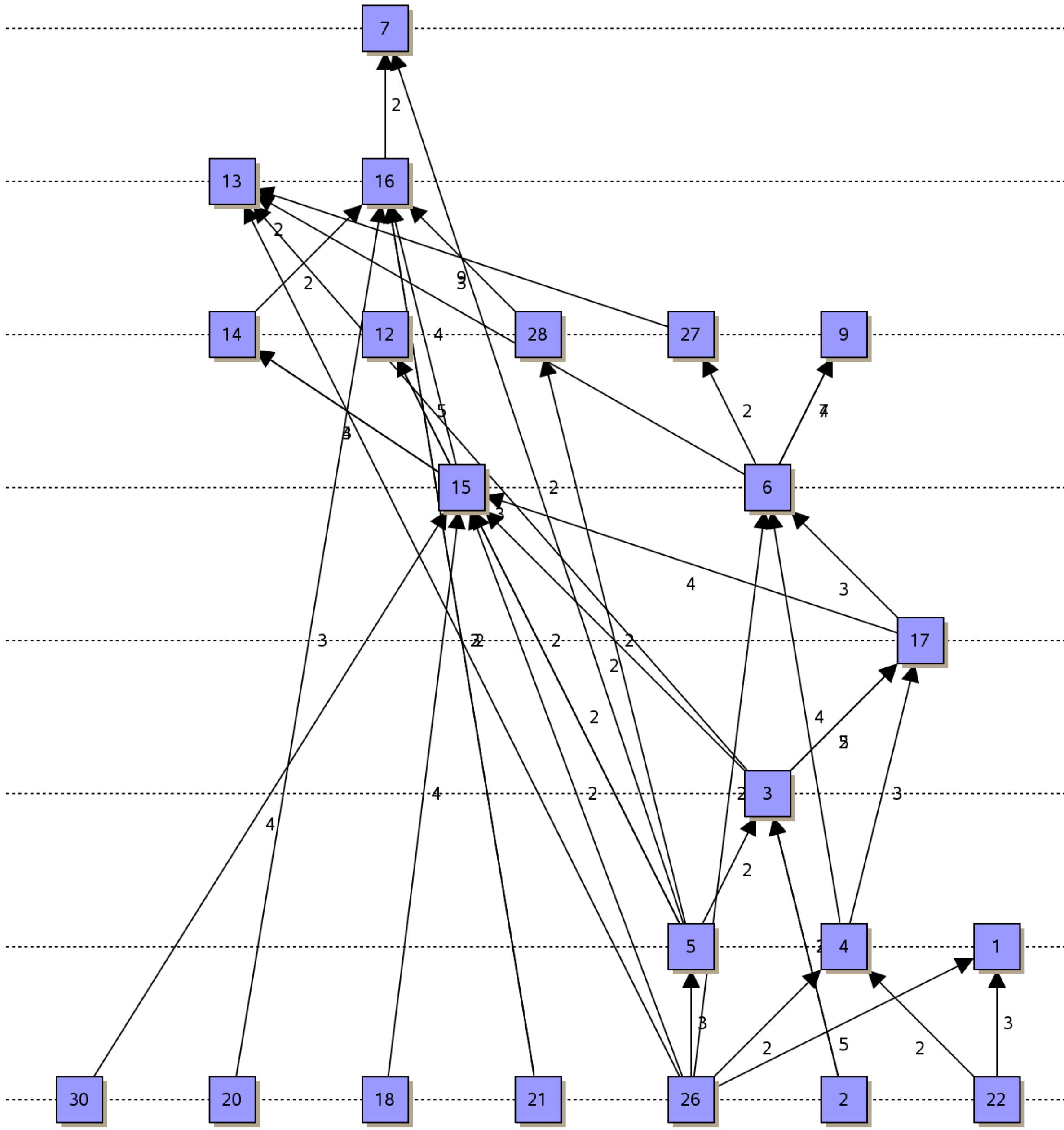
Example



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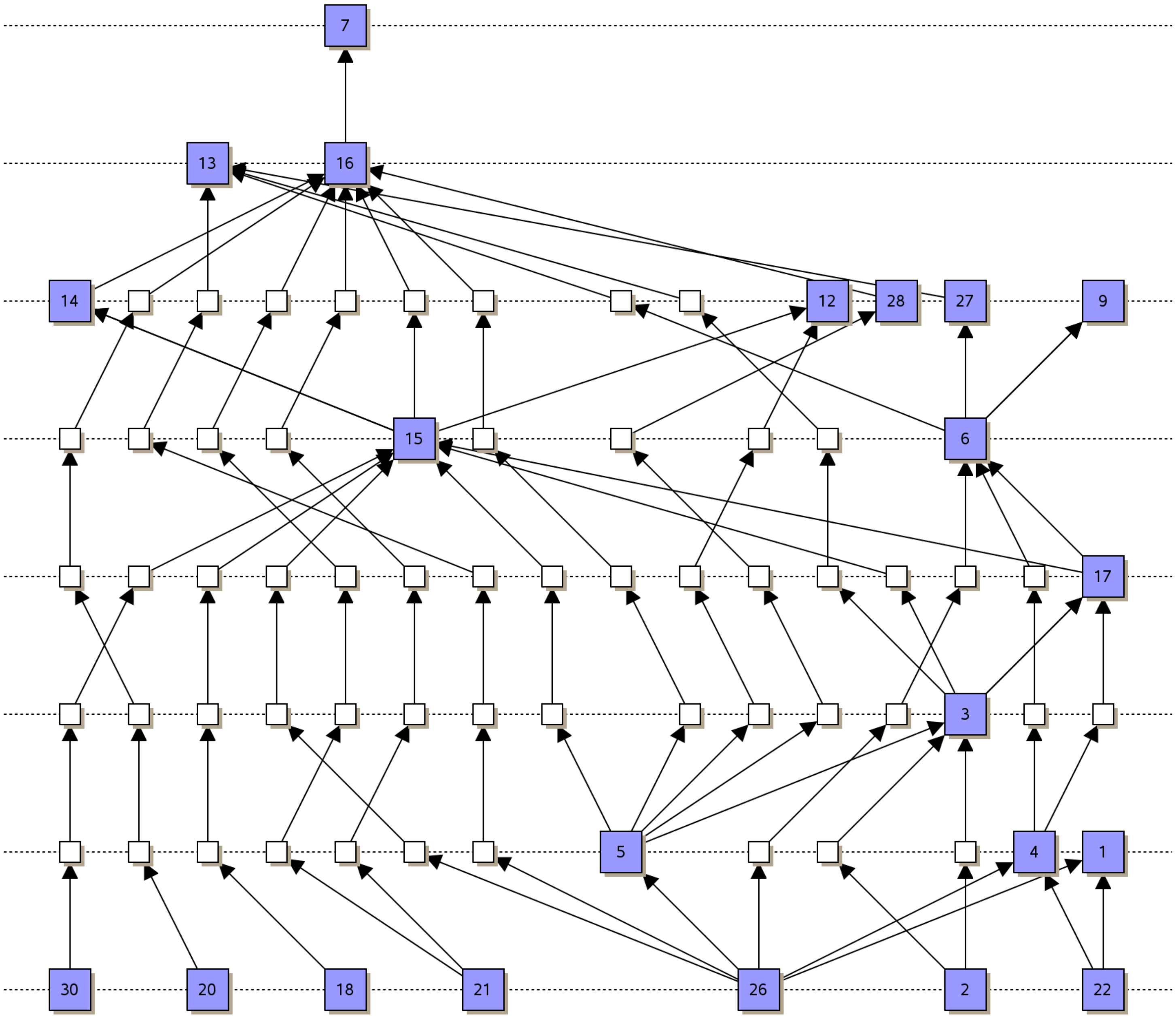

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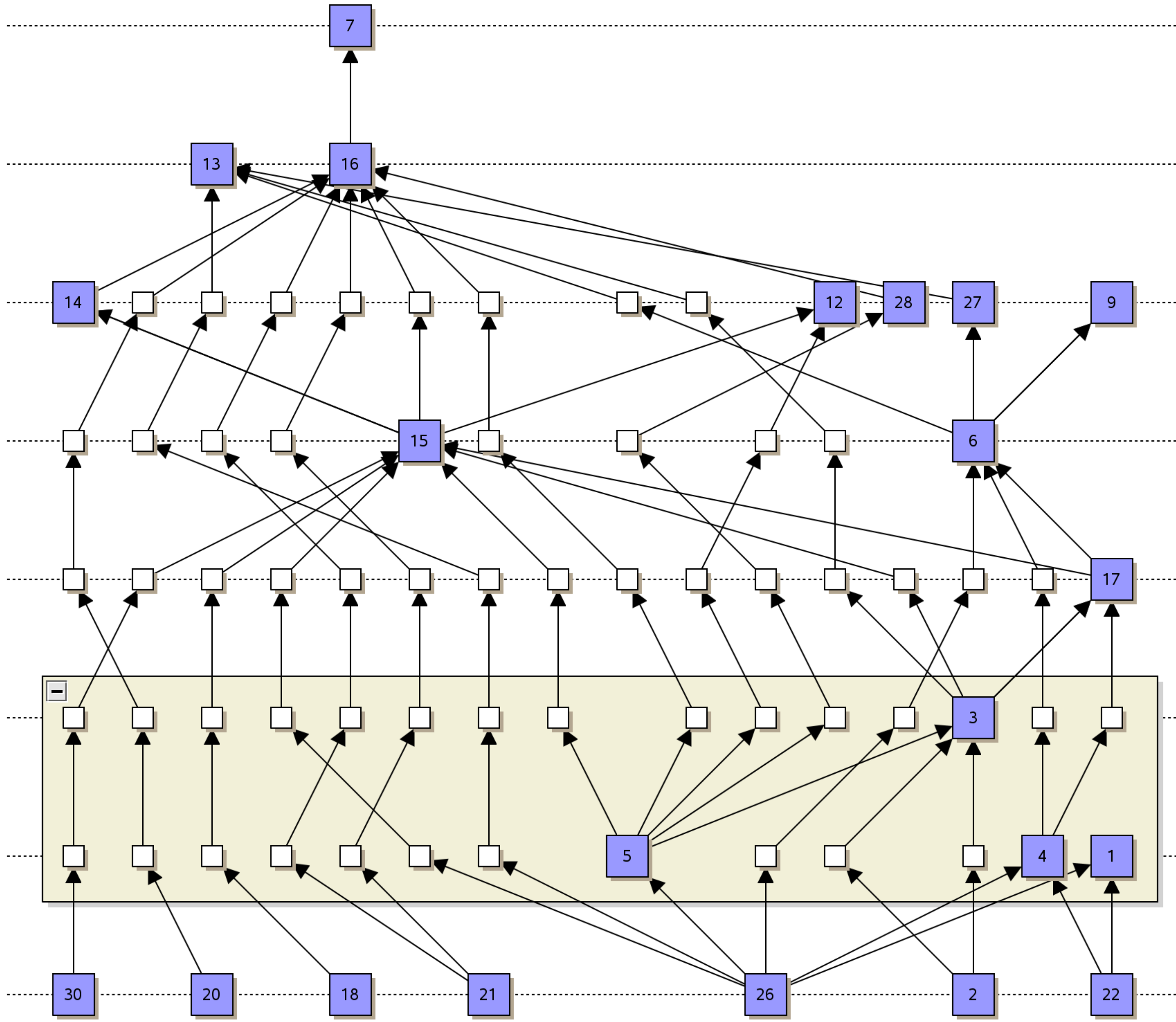

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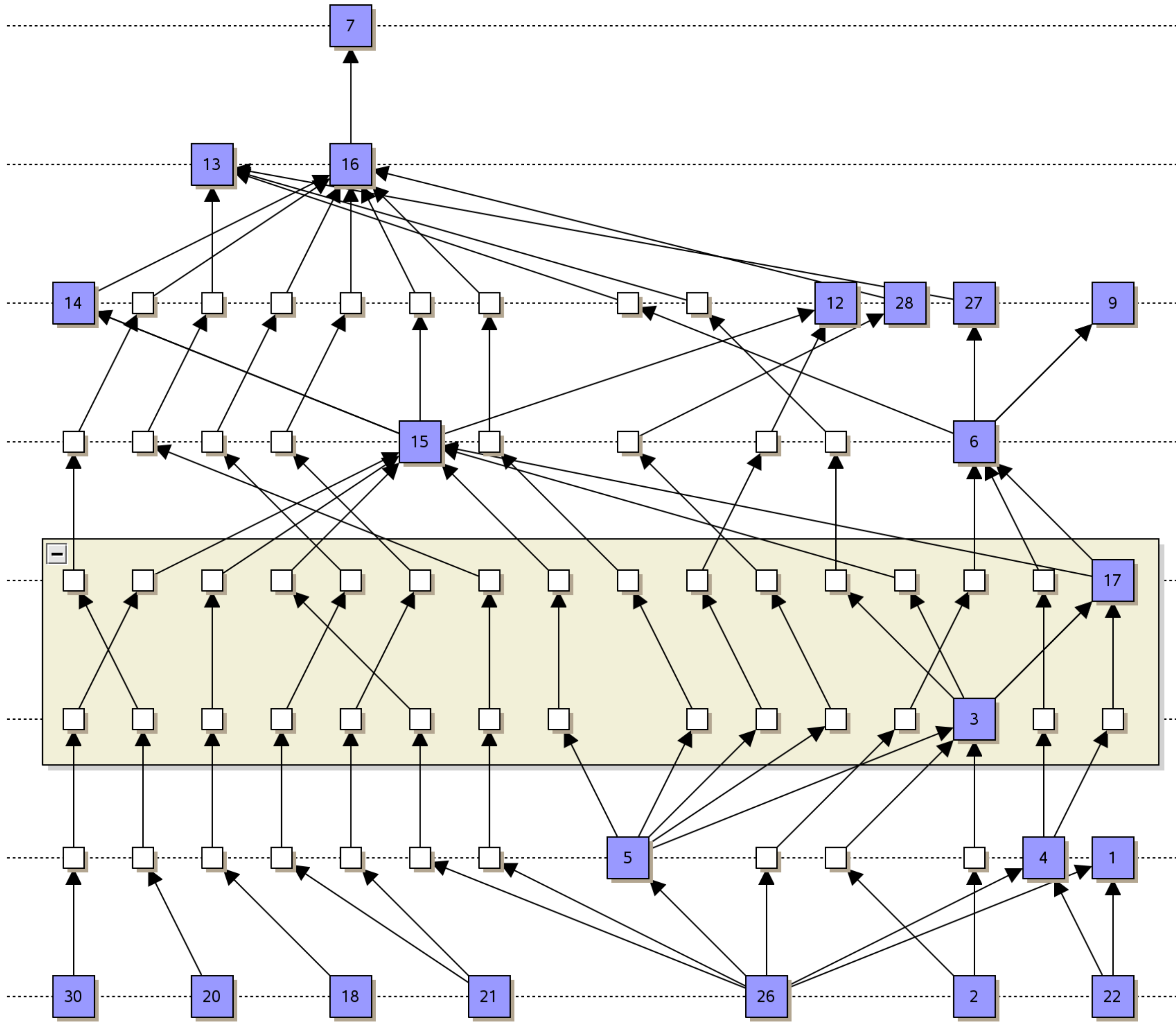

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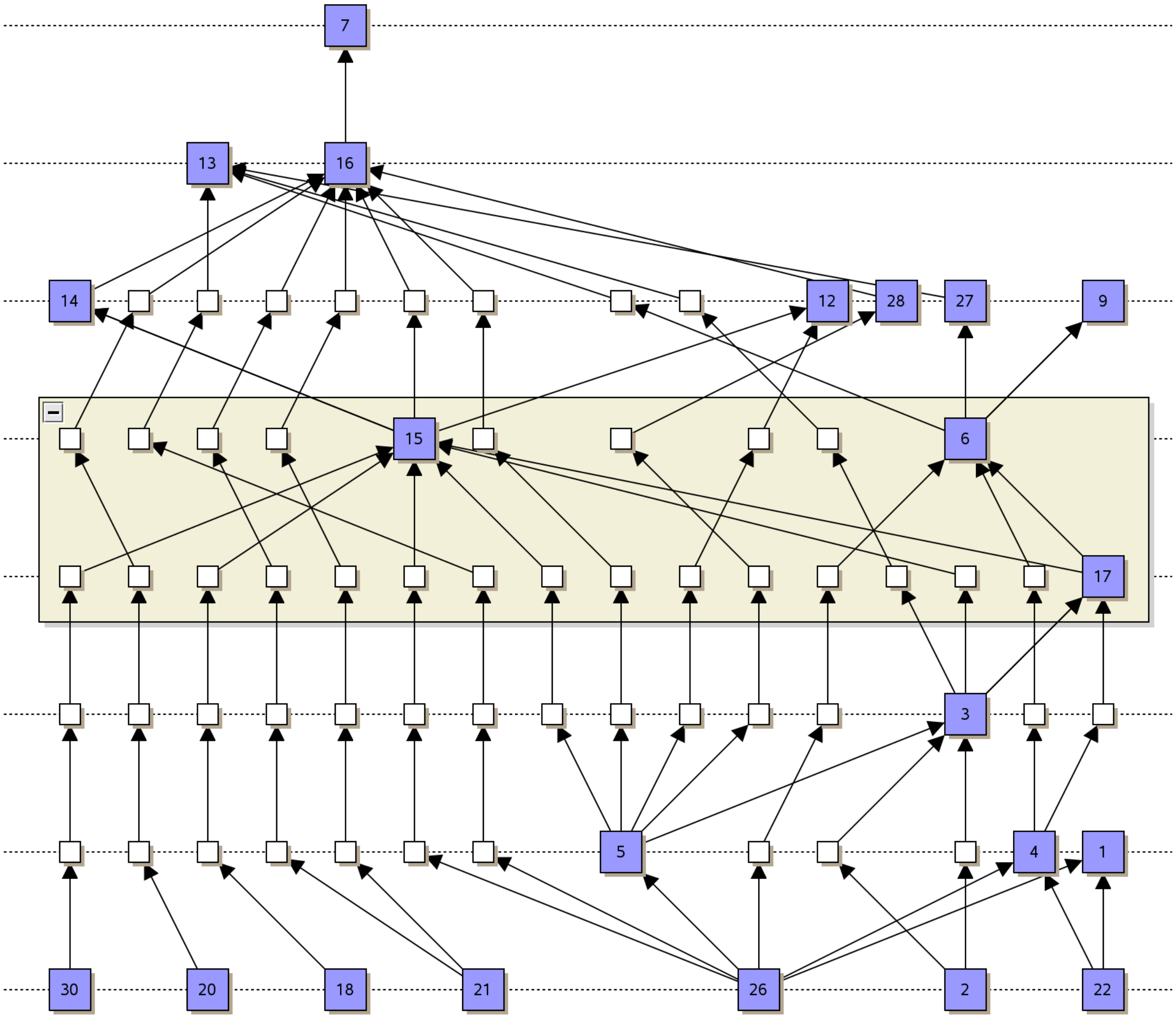

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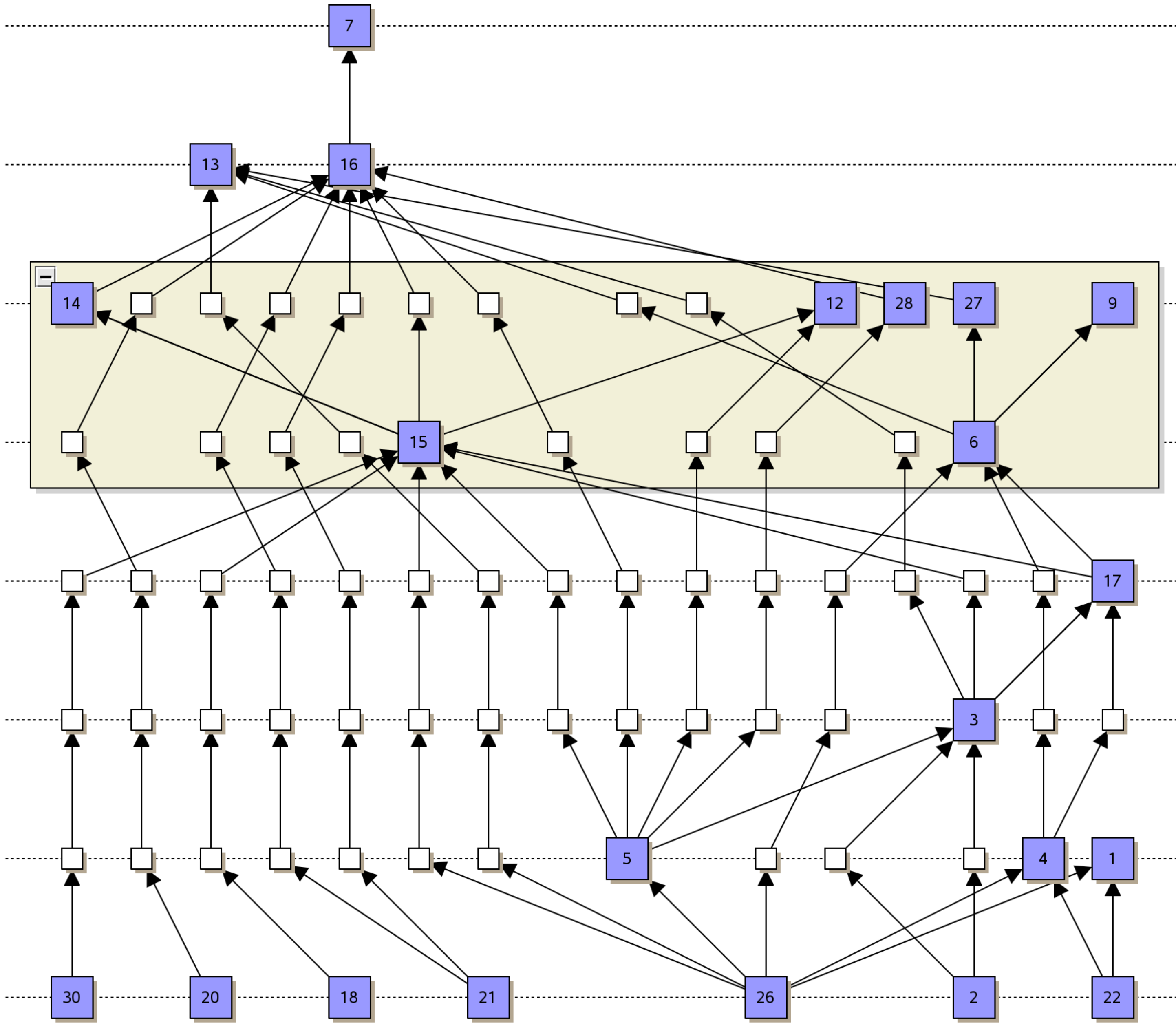

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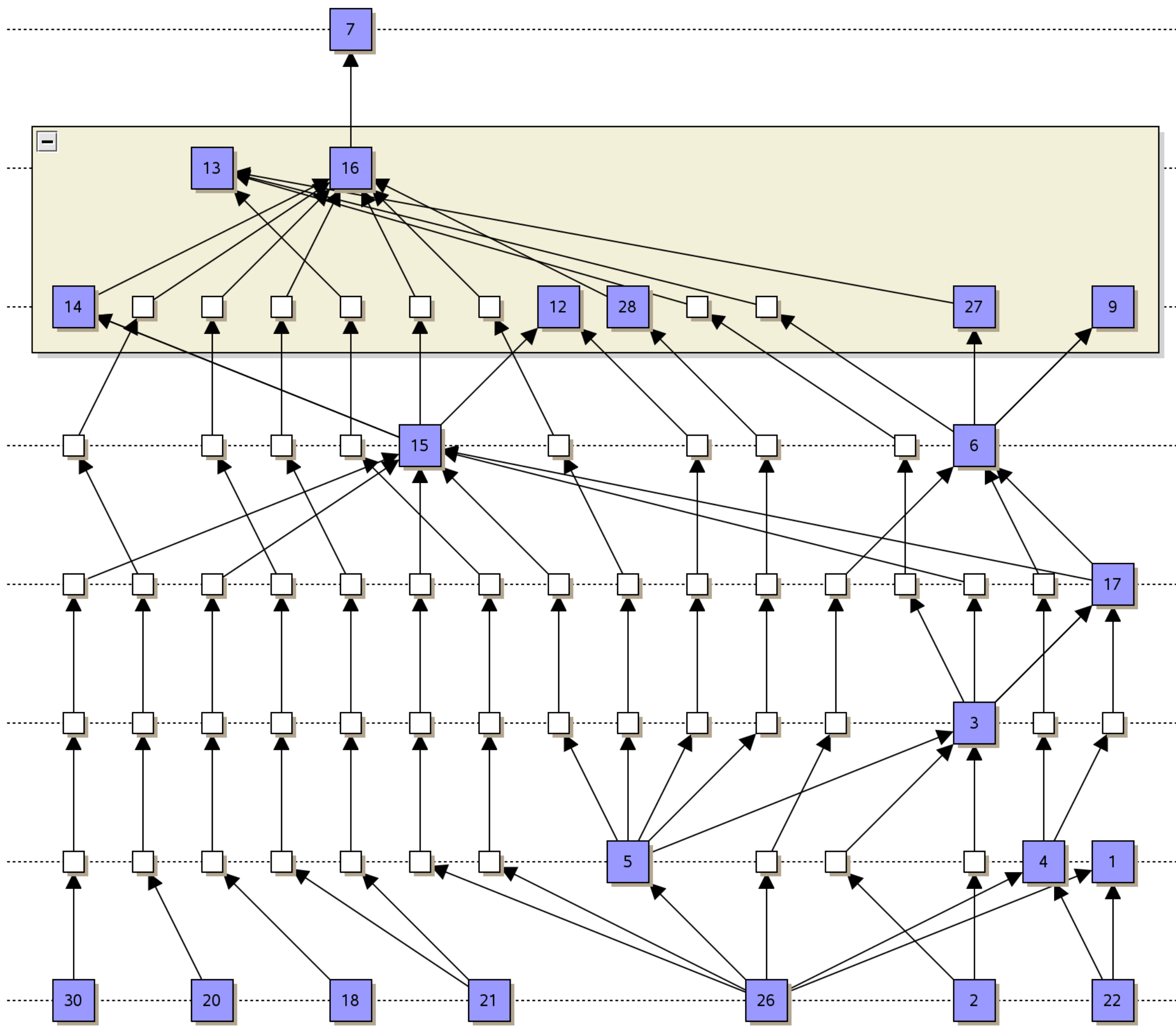

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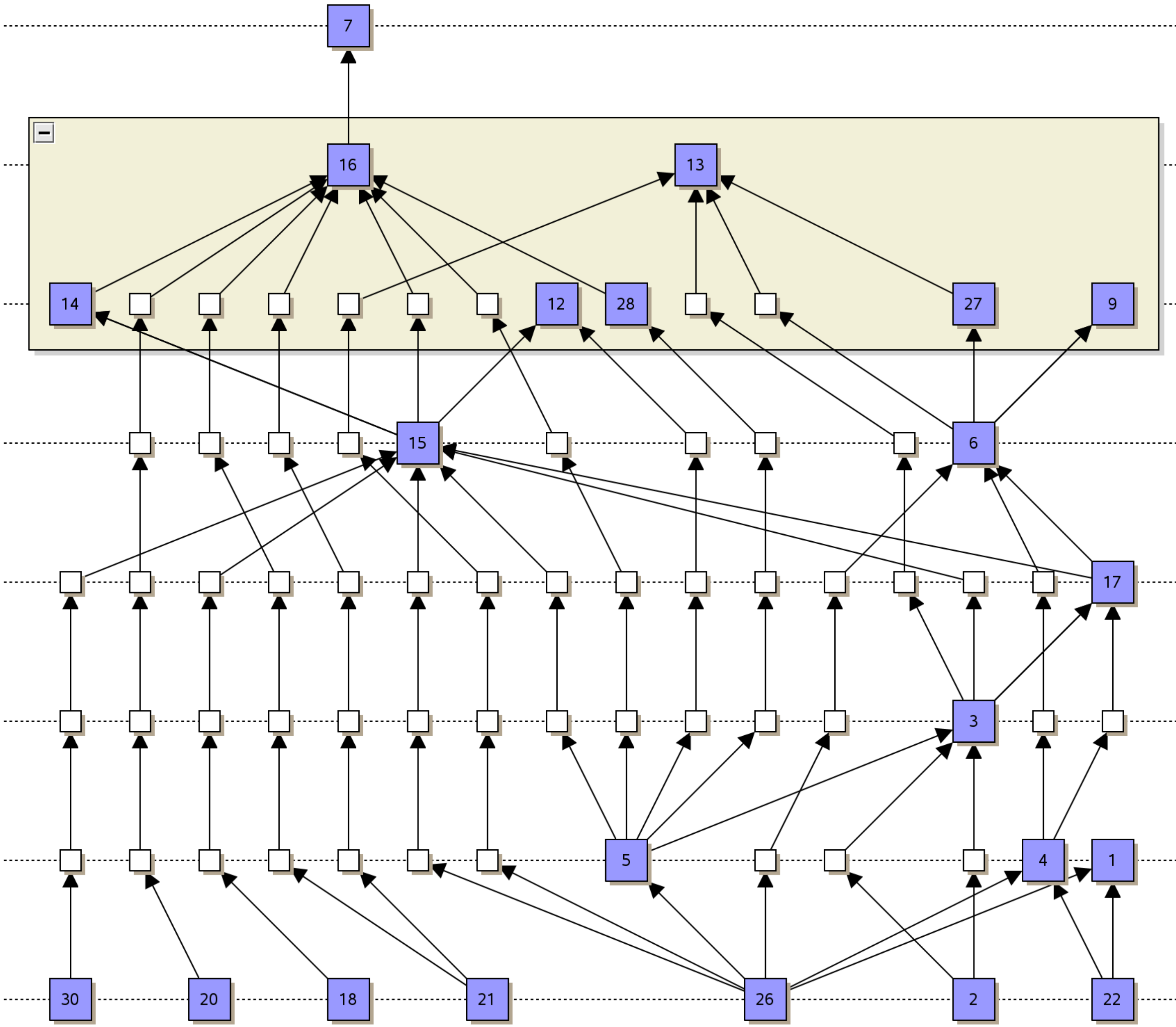

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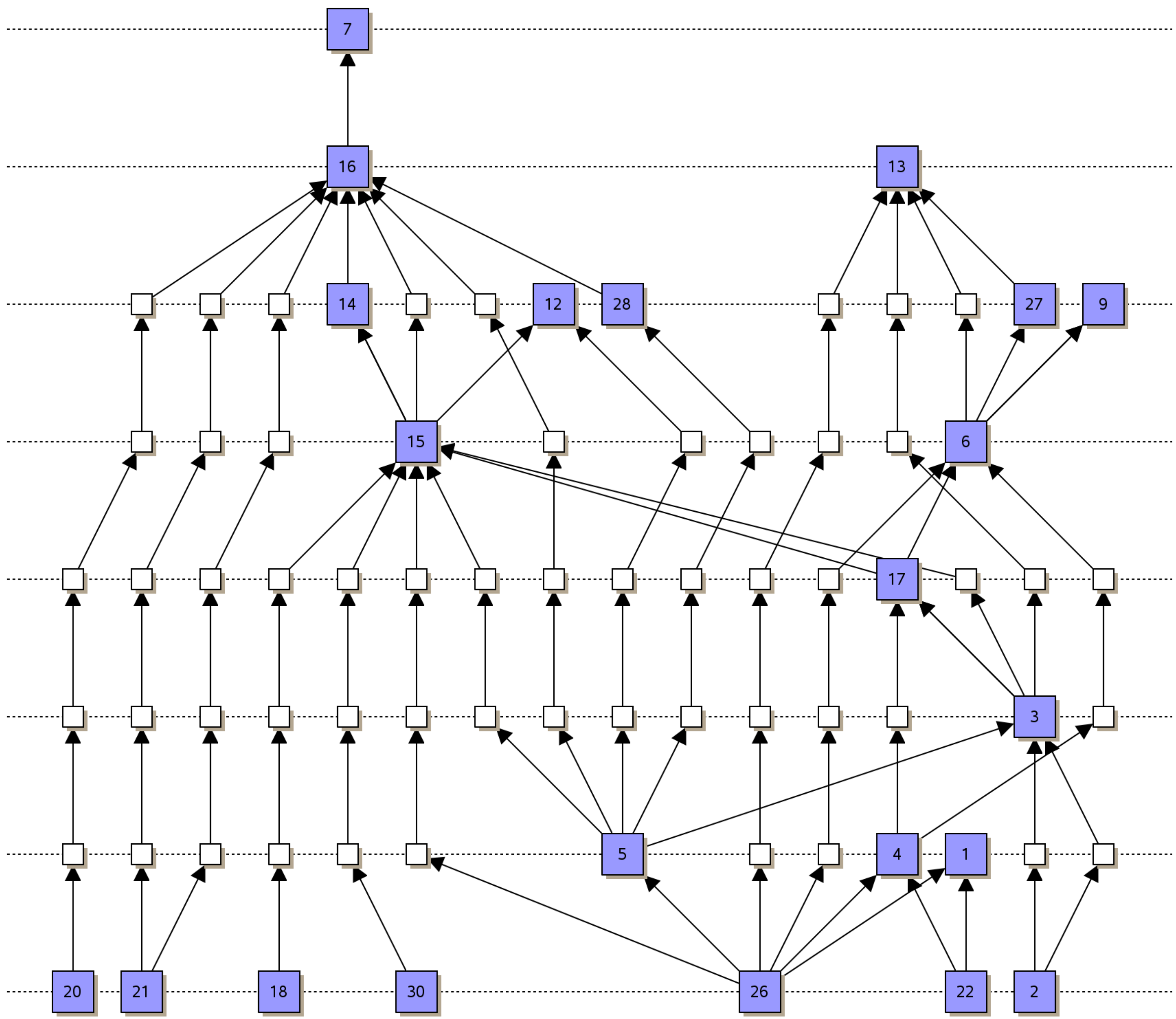
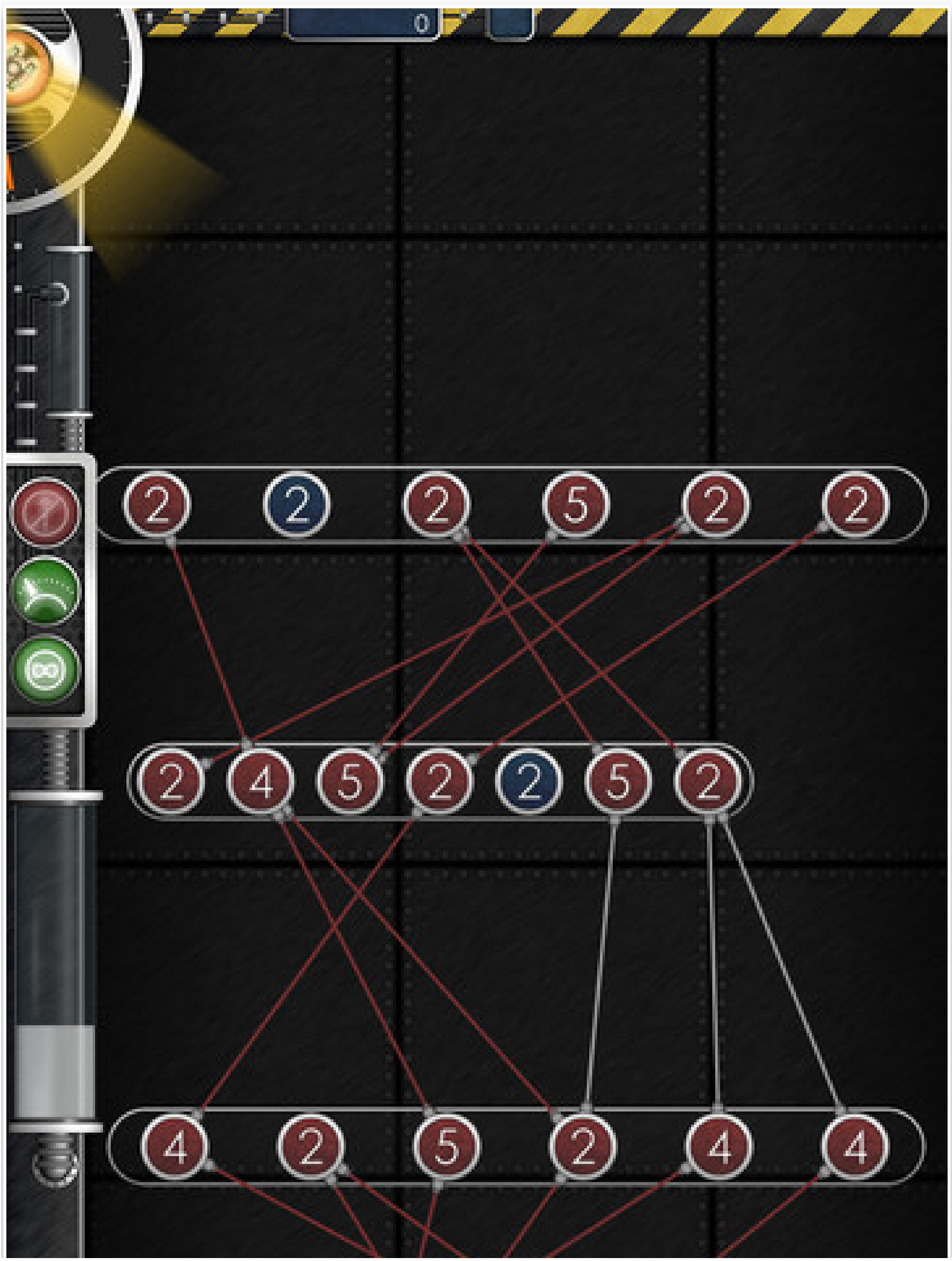


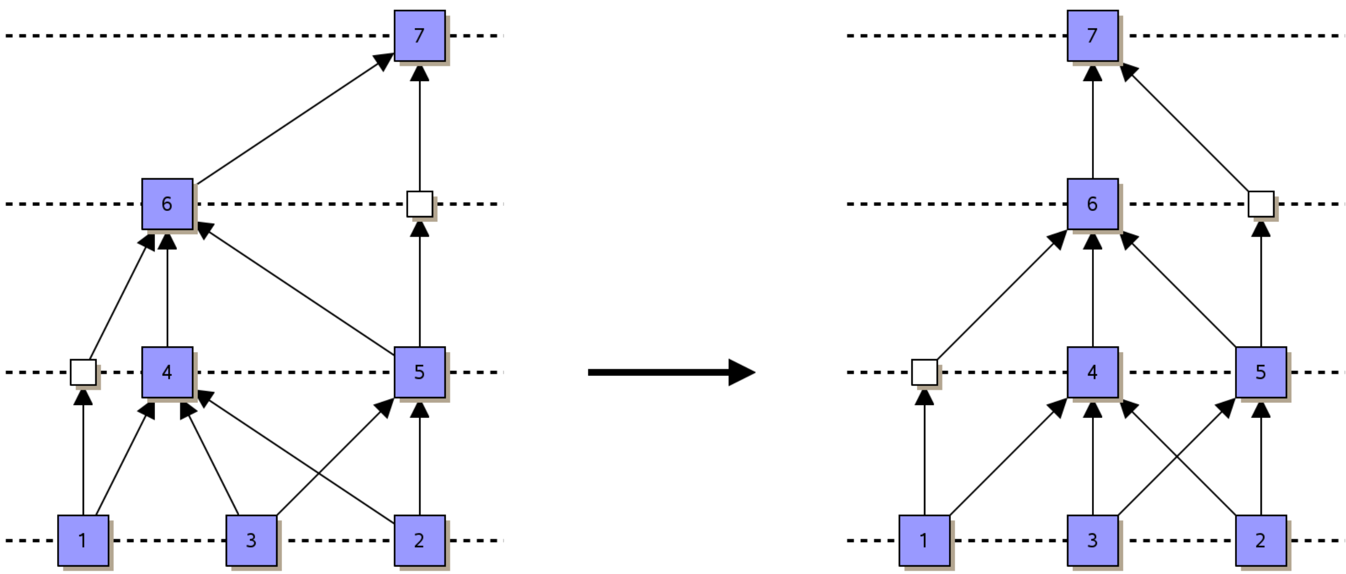
CrossingX



There was even an iPad game **CrossingX** for the OSCM problem!

Winner of Graph Drawing Game Contest 2012

Step 4: Coordinate Computation



What could be our goals?

Steightening Edges

Goal: For the edges with dummy nodes, minimize deviation from a straight line.

Idea: Use quadratic program.

- Let $p_{uv} = (u, d_1, \ldots, d_k, v)$ be $u - v$-path with k dummy nodes.
- Consider the x-coordinate of d_i when (u, v) would be straight: $a_i = x(u) + \frac{i}{k+1}(x(v) - x(u))$.
- Define the sum of deviations squared: $g(p_{uv}) = \sum_{i=1}^{k}(x(d_i) - a_i)^2$.
- Minimize $\sum_{uv \in E} g(p_{uv})$.
- Subject to: $x(w) - x(z) \geq \delta$ for consecutive nodes w, z on the same layer, w right from z for some distance parameter δ.

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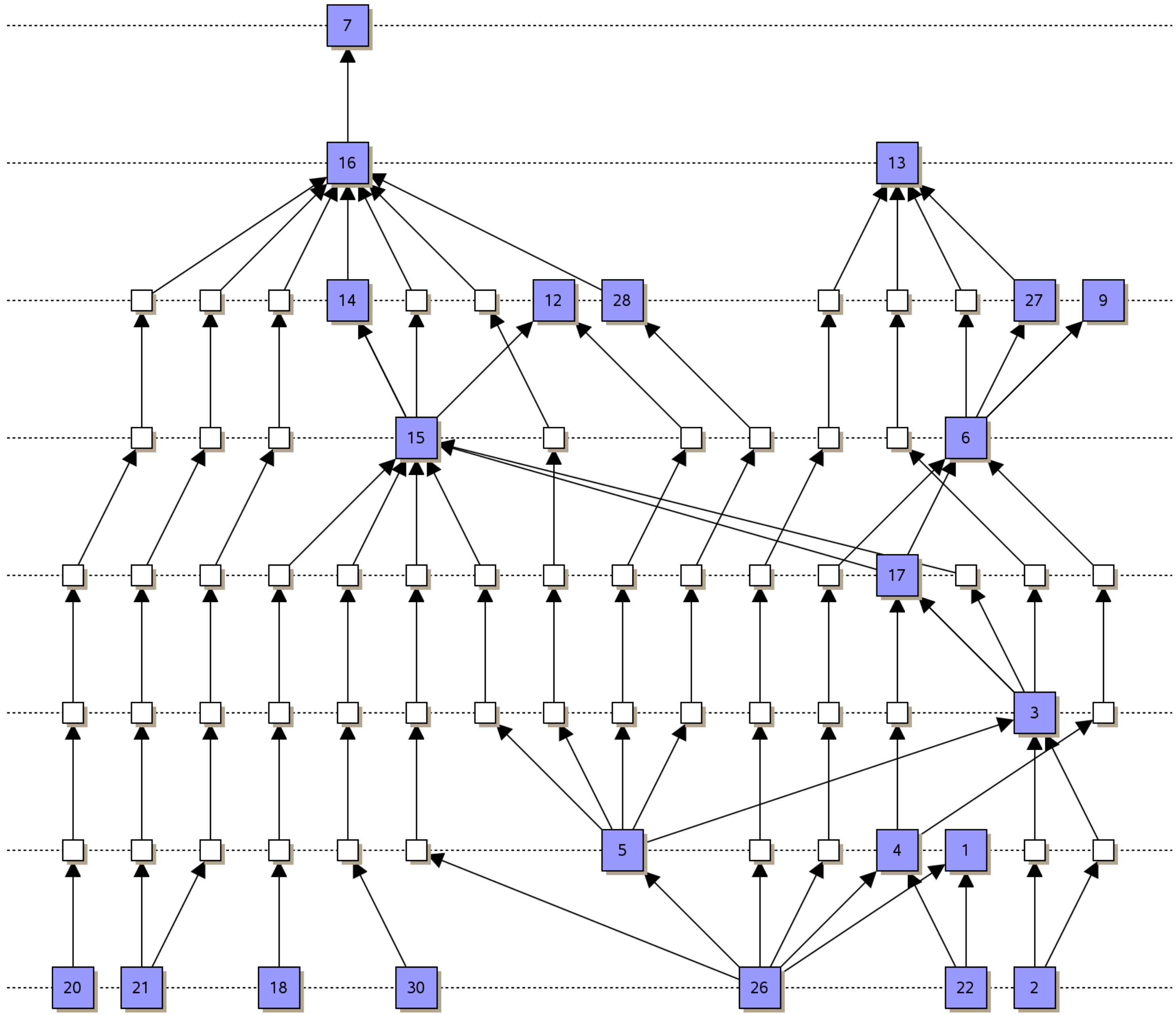
Properties:

- quadratic program is time-expensive
- width can be exponential
- optimization function can be adapted to optimize "verticality"

Example



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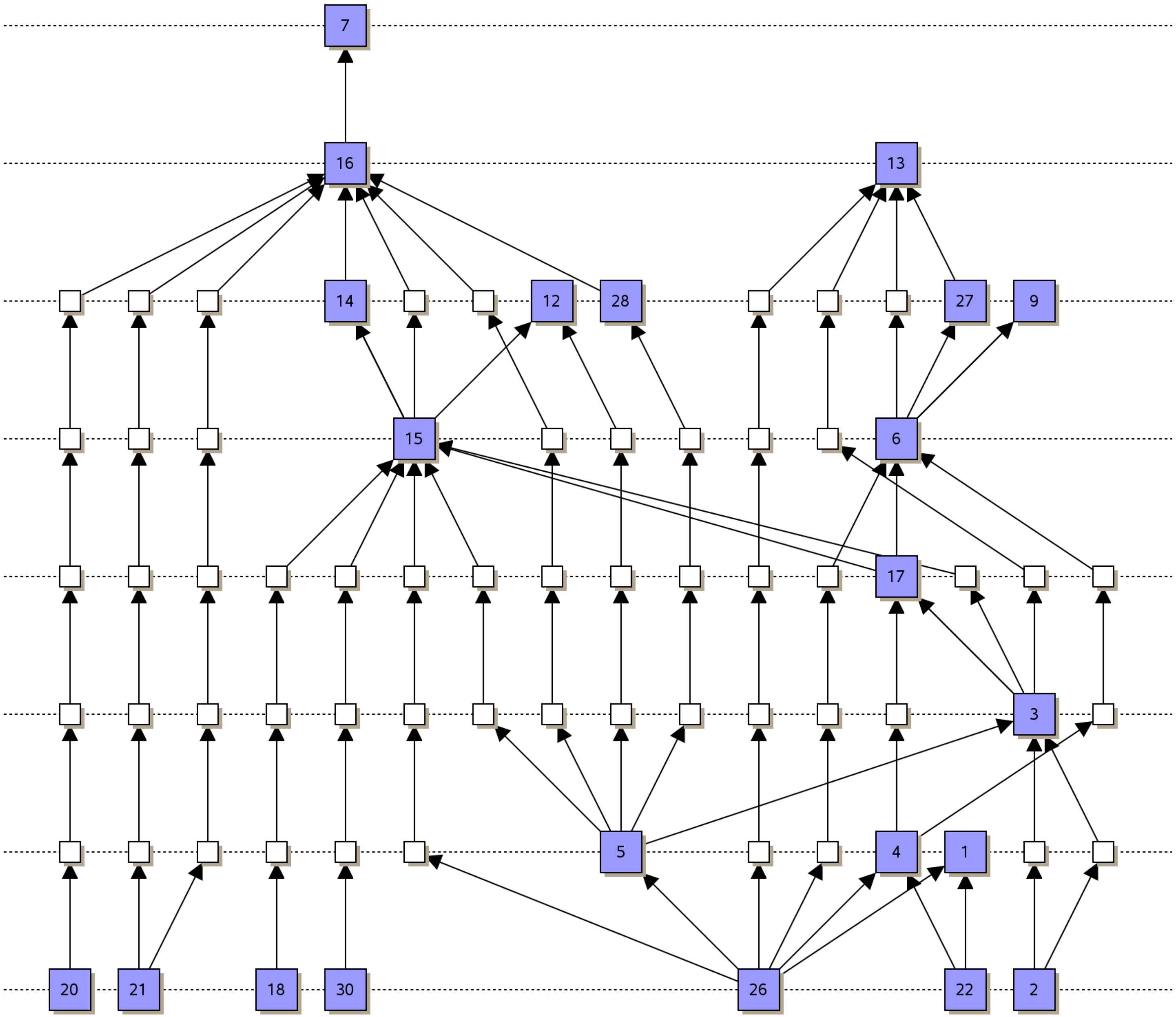
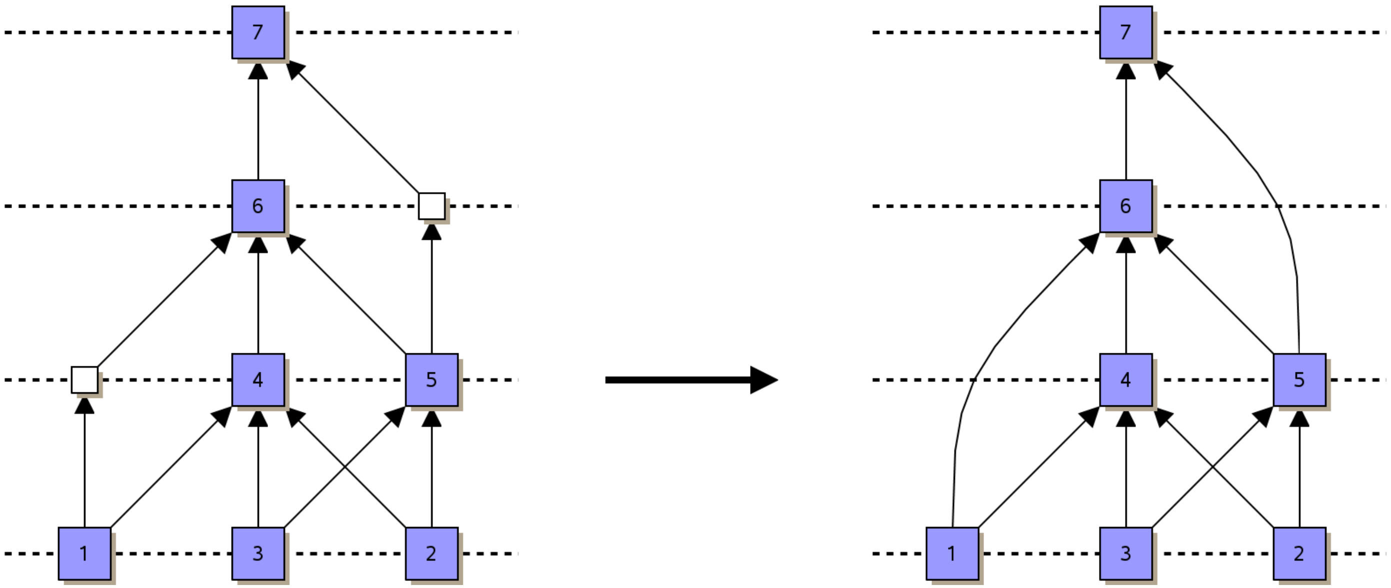
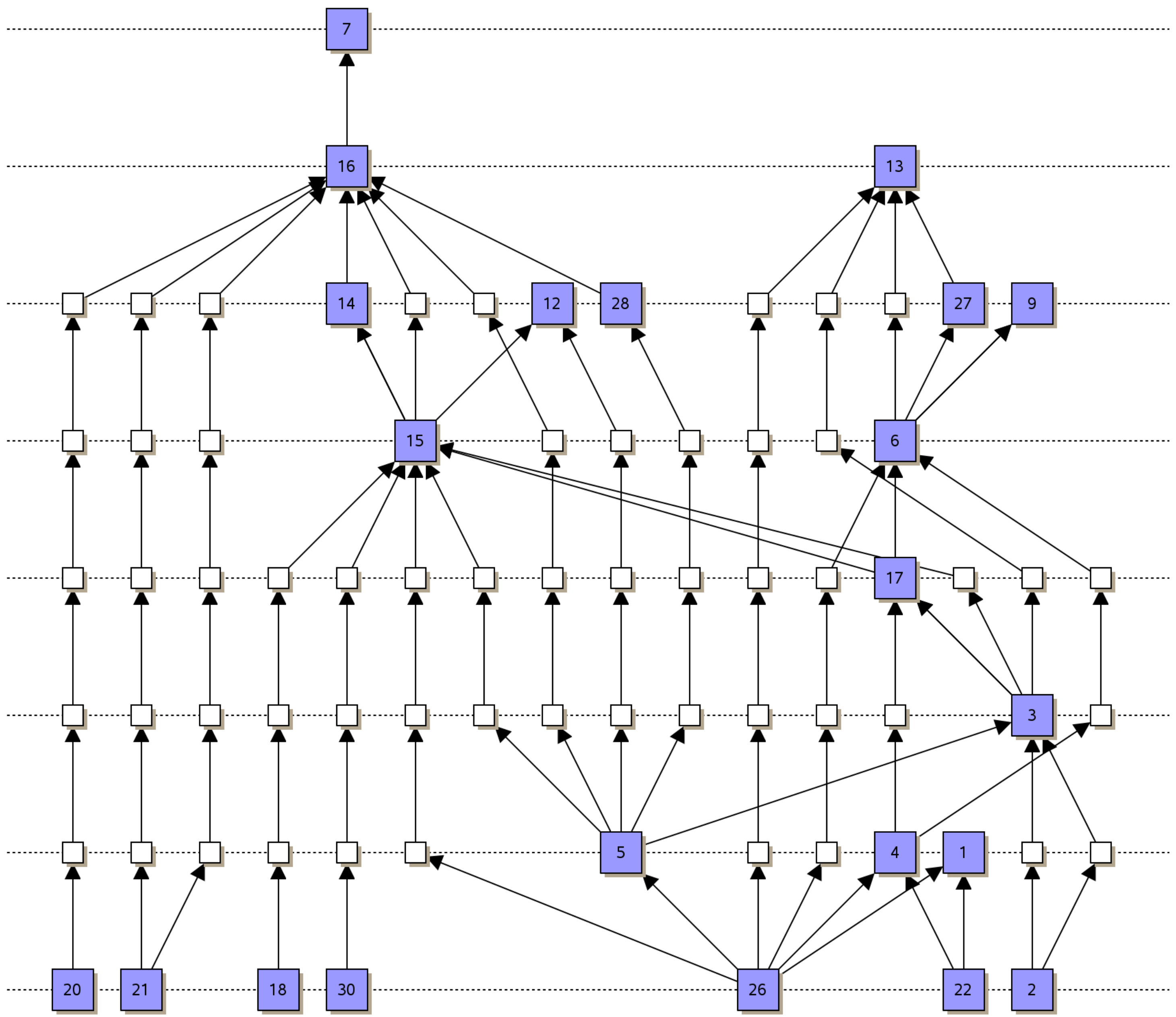


Step 5: Drawing edges



Possibility: Substitute polylines by Bézier curves

Example



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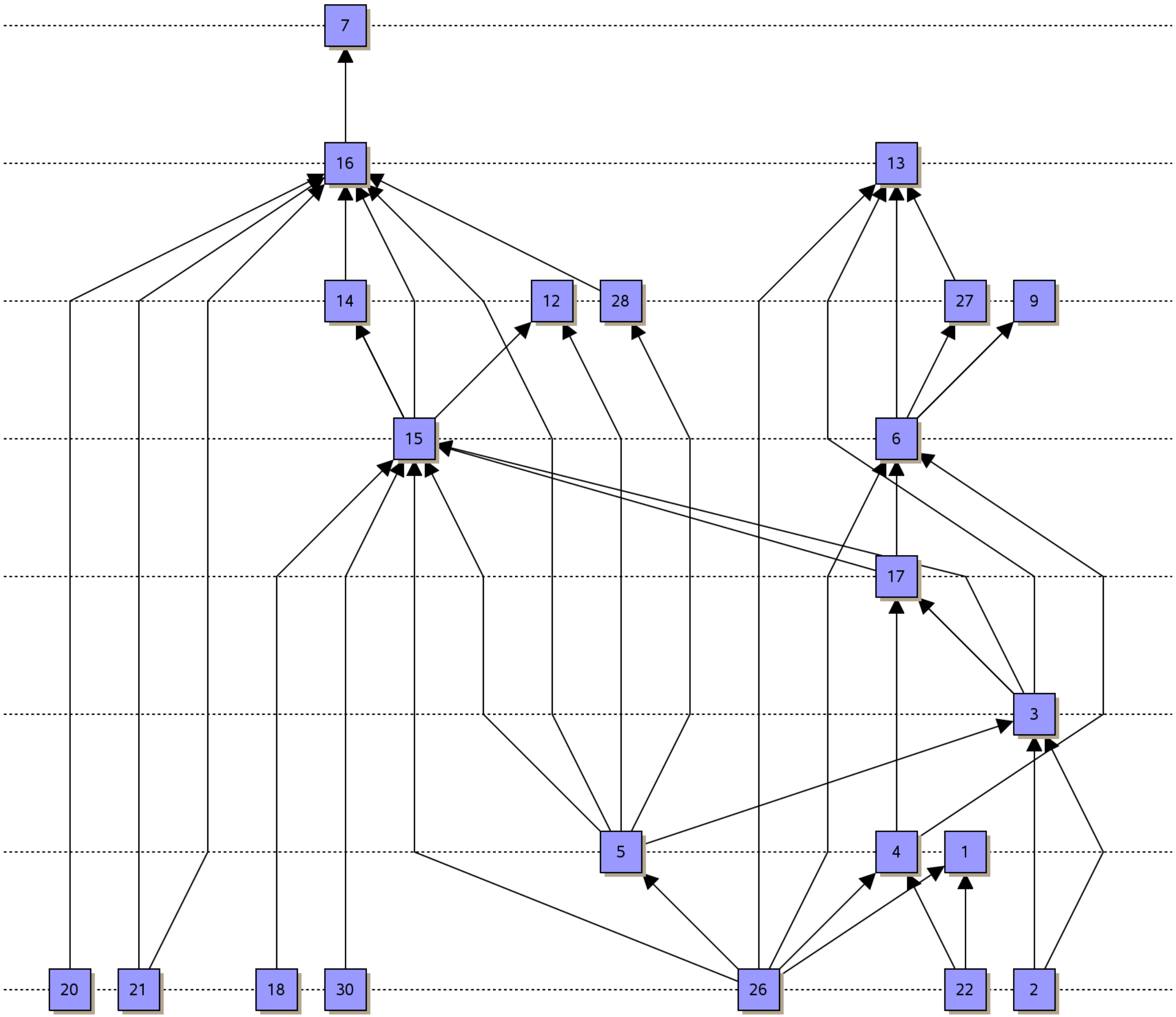

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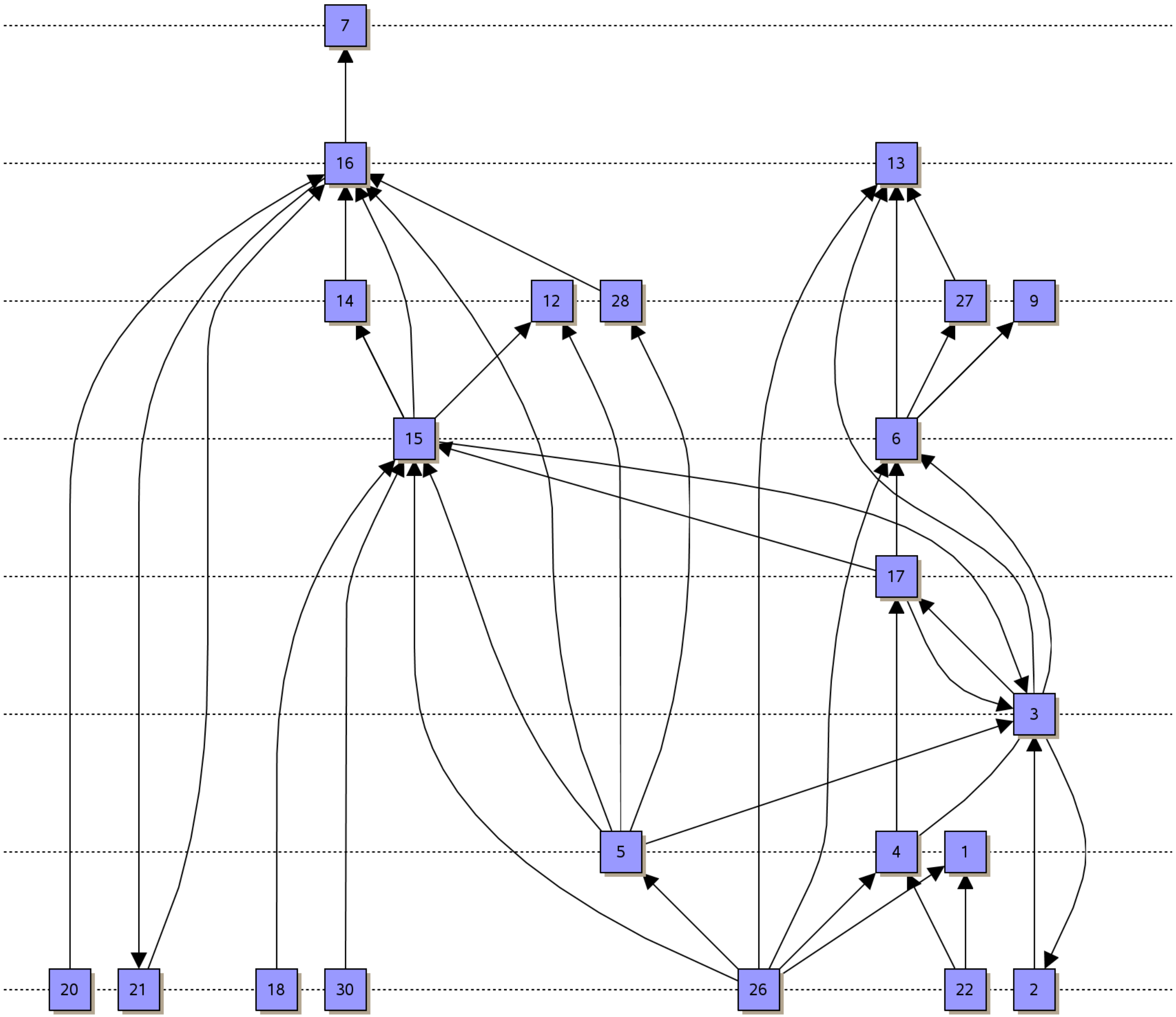
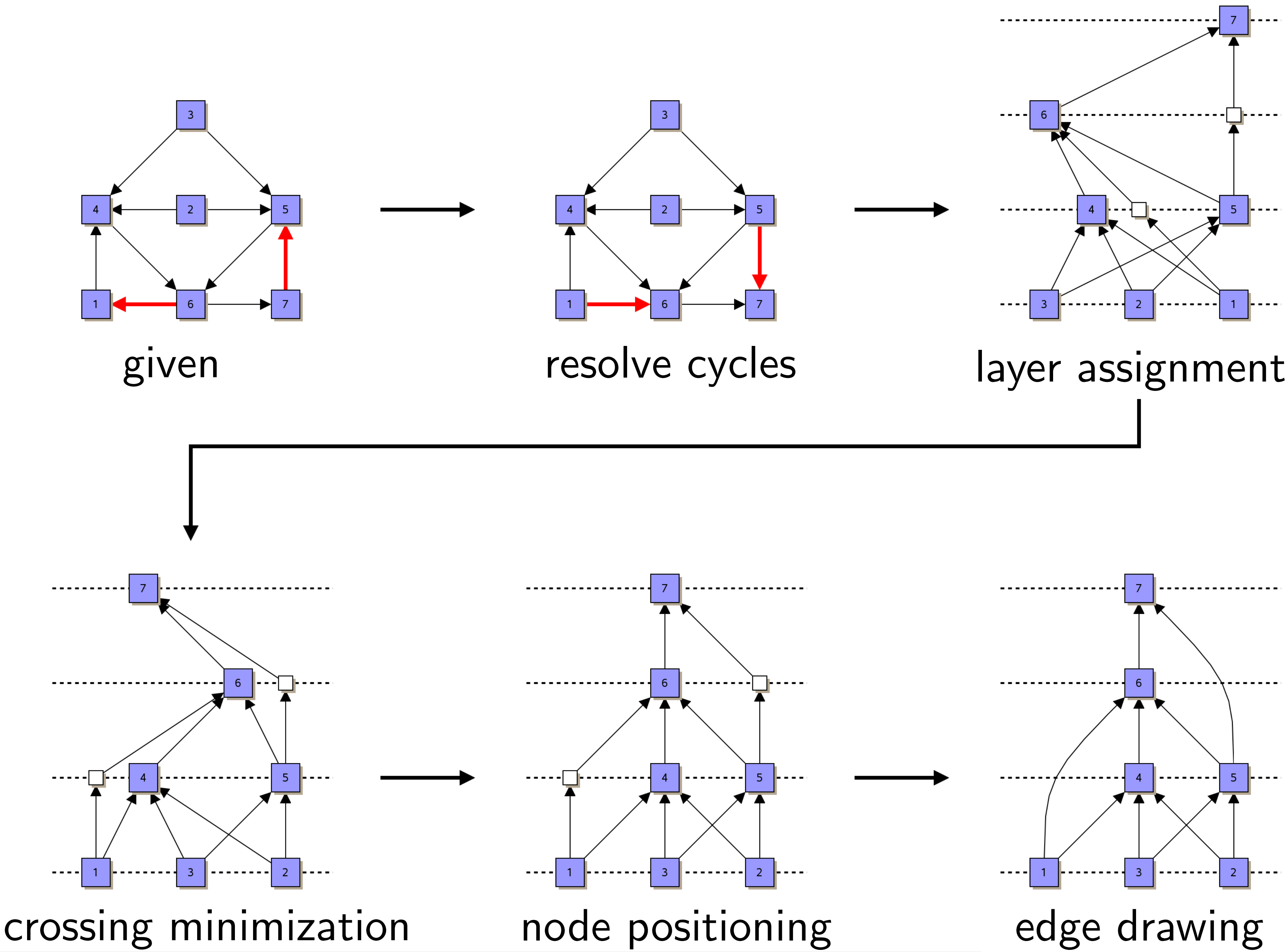
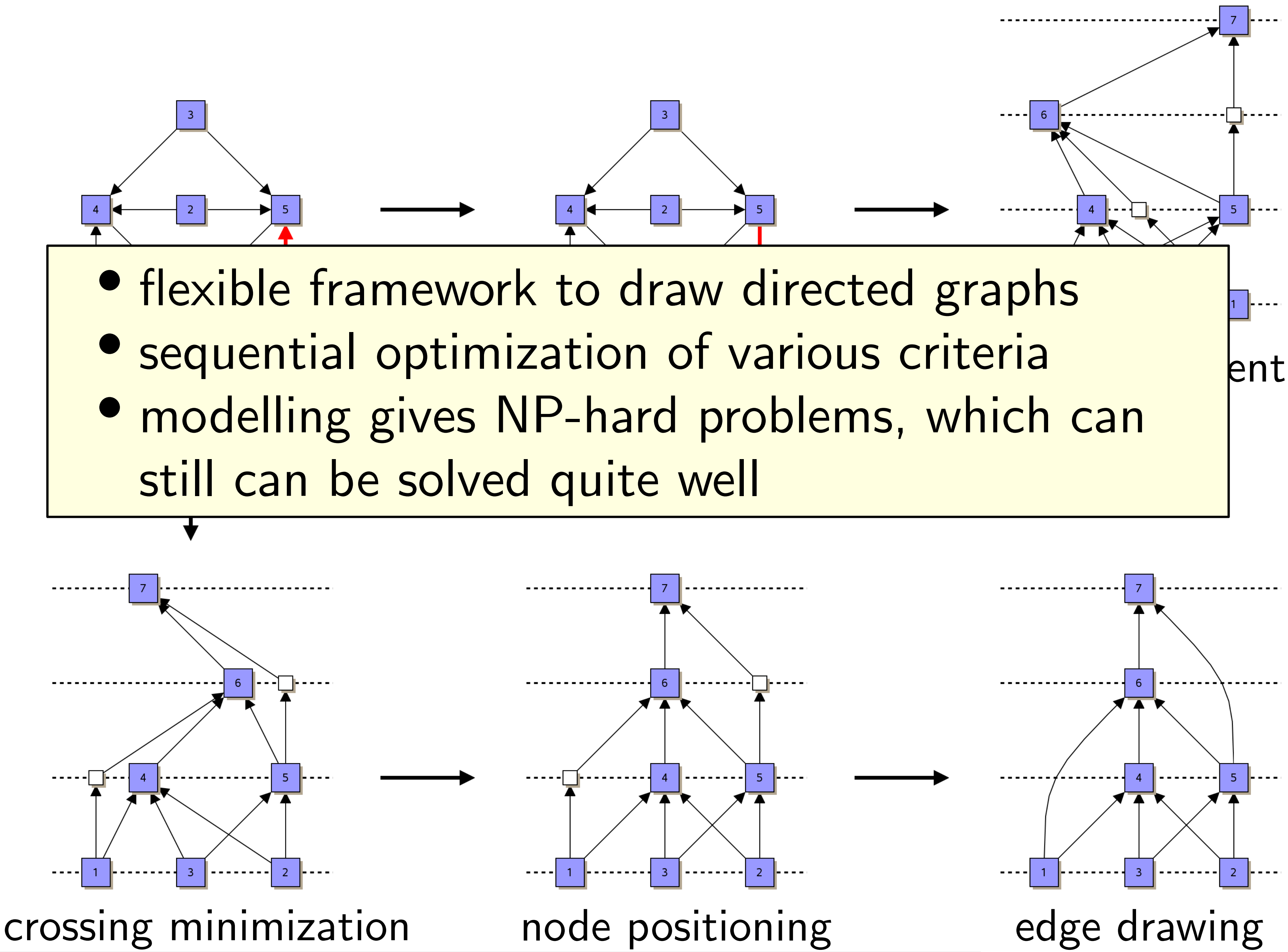


Summary


given
resolve cycles
layer assignment
crossing minimization
node positioning
edge drawing

Summary



- flexible framework to draw directed graphs
- sequential optimization of various criteria
- modelling gives NP-hard problems, which can still can be solved quite well